(Un)Real Analysis

Notes

IISER TVM

CONTENTS

Part I

Real Analysis of the Class Struggle

CHAPTER 1

SETS, REAL AND COMPLEX NUMBER SYSTEMS

1 Prelimnaries

Definition 1.1: Prelimnary definitions

1. (Cartesian Product): if A and B are non empty sets, the *Cartesian Product* $A \times B$ is defined as the set of ordered pairs a, b wherein $a \in A, b \in B$. i.e, $A \times B := \{(a, b) : a \in A, b \in B\}$
2. (Function): A function from A to B is a set $f \subseteq A \times B$ such that, $a, b \in f$ and $a, c \in f \implies b = c$. A is called the **Domain of** f. $Range(f) := f(A)$ (see next definition)
3. (Direct Image): Direct image $f(A) := \{y \in B : \exists x \in A$ such that $f(x) = y\}$
4. (Inverse Image):$f^{-1}(S \subseteq B) := \{x \in A : f(x) \in S\}$
5. (Relation): Any subset $R \subseteq A \times B$ is a relation from A to B.
   We say $x \in X$ is "related to" $y \in Y$ under the relation R, or simply xRy or $R(x) = y$ if $(x, y) \in R \subseteq X \times Y$.
6. (Injection): $f : A \to B$ is injective if $\forall x_1, x_2 \in A, (x_1, b) \in f$ and $(x_2, b) \in f \iff x_1 = x_2$
7. (Surjection): $f : A \to B$ is surjective if $\forall b \in B, \exists a \in A$ such that $f(a) = b$
8. (Bijection): $f : A \to B$ is bijective if its both surjective and injective.
9. (Identity function on a set): $I_A : A \to A$ defined by $\forall x \in A, I_A(x) = x$
10. (Permutation): Simply a bijection from A to itself is called a permutation.

Definition 1.2: (Left Inverse)

We say $f : A \to B$ has a left inverse if there is a function $g : B \to A$ such that $g \circ f = I_A$

Theorem 1.3

$f : A \to B$ has a left inverse if and only if it is injective.

Proof for Theorem.

$\implies$) If f has a left inverse g, Consider $x, y \in A$ such that $f(x) = f(y) = p$.
We have $g \circ f(x) = g(p) = x = g \circ f(y) = y$. Hence, $x = y$, Injective.
$\impliedby$) Given that $f : A \to B$ is injective, define $g : B \to A$ as:

$$g(z \in B) = \begin{cases} a, \text{ where } f(a) = z, \text{ if } z \in f(A) \\ \text{whatever}, \text{ if } z \notin f(A) \end{cases}$$

consider $g \circ f(x \in A) = g \circ (f(x))$.
Obviously, $f(x) \in f(A)$, therefore, $g(f(x)) =$ that a such that $f(a) = f(x)$.
That a is x. Hence, $g(f(x)) = x$ ■

Definition 1.4: (Right Inverse)

$f : A \to B$ is said to have a right inverse if there is a function $g : B \to A$ such that $f \circ g = I_B$

Theorem 1.5

$f : A \to B$ has a right inverse if and only if f is Surjective.

Proof for Theorem.

$\implies$) If f has a right inverse g, such that $f \circ g = I_B : B \to B$, then it is evident that the range of f is B, for if not, range of $f \circ g$ wouldn't be B either.
$\impliedby$) If f is surjective, then for all $b \in B$, there exists atleast one $a \in A$ such that $f(a) = b$ define g as:

$$g(x \in B) = \text{one of those a} \in A \text{ such that} f(a) = b$$

Consider $f \circ g(x \in B) = f($ one of the a such that $f(a) = b) = b, \forall b \in B$
Hence, $f \circ g = I_B$ ■

Theorem 1.6

If f has left inverse g_1 and right inverse g_2, then $g_1 = g_2$. *(True for anything that is Associative, and function composition is associative.)*

Proof for Theorem.

$g_1 \circ f = I_A$ and $f \circ g_2 = I_B$
$g_1 \circ (f \circ g_2) = g_1 \circ I_B = g_1$
$= (g_1 \circ f) \circ g_2 = I_A \circ g_2 = g_2$
Hence $g_1 = g_2$

Corollary 1.7

f is invertible (i.e, both left and right inverse exist) if and only if it is bijective.

Proof for Corollary.

Obvious

1.1 Operations on Relations

If R and S are binary relations over $X \times Y$:

1. $R \bigcup U := \{(x, y) | xRy \text{ or } xSy\}$
2. $R \bigcap S := \{(x, y) | xRy \text{ and } xSy\}$
3. Given $S : Y \to Z$ and $R : X \to Y$, $S \circ R := \{(x, z) | \exists y \text{ such that } ySz \ \& \ xRy\}$
4. If R is binary over $X \times Y$, $\bar{R} := \{(x, y) | \neg(xRy)\}$

1.2 Homogeneous Relations

If R is a binary relation over $X \times X$, it is Homogeneous.

Definition 1.8: Definitions Regarding Relations

1. (Reflexive): $\forall x \in X, xRx$
2. (Symmetric): $\forall x, y \in X, xRy \implies yRx$
3. (Transitive): $\forall x, y, z \in X,$ if xRy & $yRz \implies xRz$
4. (Dense): $\forall x, y \in X,$ if $xRy,$ then there is some $z \in X$ such that xRz & zRy
5. (**Equivalence Relation**): R is an equivalence relation if it is Reflexive, Symmetric and Transitive.
6. (Equivalence class of $a \in A$(where there is an equivalence relation defined)): Set of all $b \in A$ such that bRa.
7. (Partition of A): Any collection of sets $\{A_i : i \in I\}$ (where I is some indexing set) such that:
$$A = \bigcup_{i \in I} A_i$$
$$A_i \cap A_j = \phi \text{ if } \forall i, j \in I, i \neq j$$

Theorem 1.9

Let A be a non-empty set. If R defines an equivalence Relation on A, then the set of all equivalence classes of R form a partition of A

Proof for Theorem.

Define our collection $\{A_\alpha\}$ as the set of all equivalence classes of A. Clearly, $\bigcup_{\alpha \in I} A_\alpha = A$. If A only has one element, obviously, that singleton set makes up the partition. Let A_α and $A_{\alpha'}$ be equivalence classes of two elements a and a' in A. If aRa', then $A_\alpha = A_{\alpha'}$ since every element in the equivalence class of a will, from the transitive property, be in the equivalence class of a' . Suppose $\neg(aRa')$. If, then, $\exists x \in A_\alpha$ such that $x \in A_{\alpha'}$, this means that $xR\alpha$ and $xR\alpha'$, but from transitive property, this means $\alpha R\alpha'$, which is a contradiction. Therefore, the pairwise intersection is disjoint. ■

Theorem 1.10

If $\{A_i : i \in I\}$ is a partition of A, then there exists an equivalence relation R on A whose equivalence classes are $\{A_i : i \in I\}$.

Proof for Theorem.

Define $R(x, y)$ if and only if $\exists$ unique $m \in I$ such that $x \in A_m$ and $y \in A_m$.

$R(x, x)$ is obvious if non empty, hence R is reflexive.

Suppose $R(x, y)$ and $R(y, z)$. Then, there exists a unique $m \in I$ such that x, y are in A_m. Similarly, there exists a unique $n \in I$ such that y, z are in A_n . Obviously, if $n \neq m$, intersection of A_n and A_m would be non empty, hence, $n = m$. Hence, R is transitive.

Consider $R(x, y)$, which means $\exists$ unique $n \in I$ such that $x, y \in A_n \implies R(y, x)$. Hence, R is an equivalence relation. ■

2 Induction, Naturals, Rationals and the Axiom of Choice

Axiom 2.1: Peano Axioms, characterisation of $\mathbb{N}$

1. $1 \in \mathbb{N}$
2. every $n \in \mathbb{N}$ has a predecessor $n - 1 \in \mathbb{N}$ except 1
3. if $n \in \mathbb{N} \implies n + 1 \in \mathbb{N}$

Definition 2.2: (Sequence of something)

A sequence of some object is simply a collection of objects $\{O_l : l \in \mathbb{N}\}$ which can be counted.

Axiom 2.3: Well Ordering Property of $\mathbb{N}$

Every non empty subset of $\mathbb{N}$ has a least element.

Axiom 2.4: Weak Induction

For all subsets $S \subseteq \mathbb{N}$, $((1 \in S)\&((\forall k \in \mathbb{N})(k \in S \implies k + 1 \in S)) \iff S = \mathbb{N})$

Weak Induction's Negation:(One direction)
There exists subset $S_0 \subseteq \mathbb{N}$, $((1 \in S_0)\&((\forall k \in \mathbb{N})(k \in S_0 \implies k + 1 \in S_0))$ but $S_0 \neq \mathbb{N})$

Axiom 2.5: Strong Induction

For all subsets $S \subseteq \mathbb{N}$, $((1 \in S)\&((\forall k \in \mathbb{N})(1, 2, ...k \in S' \implies k + 1 \in S')) \iff S = \mathbb{N})$

Strong Induction's Negation:(One direction)
There exists subset $S' \subseteq \mathbb{N}$, $((1 \in S')\&((\forall k \in \mathbb{N})(1, 2, ...k \in S' \implies k + 1 \in S'))$ but $S' \neq \mathbb{N})$

Theorem 2.6

Weak Induction $\iff$ Strong Induction.

Proof for Theorem.

$\implies$) Suppose Weak induction is true, but not strong induction. Take our set to be that S' in the negation of the Strong Induction Statement. $S' \neq \mathbb{N}$ implies that, either $1 \notin S'$ or $\exists k \in \mathbb{N}$ such that $k \in S'$ but $k+1 \notin S'$. We know that $1 \in S'$, so it must be that $\exists k \in \mathbb{N}$ such that $k \in S'$ but $k+1 \notin S'$. $\{1\} \in S' \implies \{1,2\} \in S'$. Assume that for n, $\{1,2,..n\} \in S'$. This means that $\{1,2,...n+1\} \in S'$. This means that for every $n \in \mathbb{N}$, $\{1,2,...n\} \in S' \implies n \in S'$. Contradiction.
$\impliedby$)Suppose Strong Induction is true, but not weak induction. Take the set S_0 from the negation of Weak Induction. $S_0 \neq \mathbb{N}$. This means, from strong induction, either $1 \notin S_0$ or $\exists k \in \mathbb{N}$ such that $1,2,\ldots,k \in S_0$ but $k+1 \notin S_0$. $1 \in S_0$, hence, $2 \in S_0$ and $\{1,2\} \in S_0$. assume that $\{1,2,....n\} \in S_0$. This means, $n \in S_0 \implies n+1 \in S_0$, which means that $\forall k \in \mathbb{N}, \{1,2,..k\} \in S_0 \implies k+1 \in S_0$. Therefore, S_0 is $\mathbb{N}$. ■

Theorem 2.7

Weak Induction $\iff$ Strong Induction $\iff$ Well ordering.

Proof for Theorem.

$\implies$) Suppose that, on the contrary, S_0 is a non empty subset of $\mathbb{N}$, with no least element. Does 1 exist in S_0? No, for that will be the least element. Likewise, then, 2 does not belong in S_0. Assume that $\{1,2,...n\} \notin S_0$. Does $n+1$ exist in S_0? No, for that will become the least element then. From Strong Induction, $\mathbb{N} - S_0 = \mathbb{N} \implies S_0 = \phi$. Contradiction.
$\impliedby$)Suppose $\exists S_0 \subseteq \mathbb{N}$ such that $1 \in S_0$ and $\forall k \in \mathbb{N}$, $k \in S_0 \implies k+1 \in S_0$. Suppose on the contrary, S_0 is not $\mathbb{N}$. $\mathbb{N} - S_0$ is then, non-empty. From Well Ordering, there is a least element $q \in \mathbb{N} - S_0$. $\implies$, $q-1 \in S_0$. But this would imply $q-1+1 \in S_0$. Contradiction. $\mathbb{N} - S_0$ is empty. ■

Definition 2.8: (Finite Sets)

A set X is said to be finite, with n elements in it, if $\exists n \in \mathbb{N}$ such that there exists a bijection $f : \{1,2..,n\} \to X$. Set X is *infinite* if it is non-finite.

Theorem 2.9

If A and B are finite sets with m and n elements respectively, and $A \cap B = \phi$, then $A \cup B$ is finite, with $m+n$ elements.

Proof for Theorem.

$f : \mathbb{N}_m \to A$ and $g : \mathbb{N}_n \to B$.

Define $h : \mathbb{N}_{m+n} \to A \cup B$ given by:

$$h(i) = \begin{cases} f(i) \text{ if } i = 1, 2...m \\ g(i-1) \text{ if } i = m+1, m+2, ...m+n \end{cases}$$

If $i = 1, 2, ...m$,$h(i)$ covers all the elements in A through f. If $i = m+1, ...m+n$, $h(i)$ covers all the elements in B through g.

Moreover, $h(i) \neq h(j); i \in [1, m], j \in [m+1, m+n]$ since $A \cap B = \phi$ ■

Theorem 2.10

If C is infinite, and B is finite, then $C - B$ is infinite.

Proof for Theorem.

Suppose $C - B$ is finite. We have $B \cap (C - B) = \phi$ and $B \cup (C - B) = C \cup B$

$n(C \cup B) = n(B \cup (C - B)) = n(B) + n(C - B)$ This implies $C \cup B$ is finite. Contradiction. ■

Theorem 2.11

Theorem: Suppose T and S are sets such that $T \subseteq S$. Then:

a) If S is finite, T is finite.

b)If T is infinite, S is infinite.

Proof for Theorem.

Given that S is finite, there is a function $f : \mathbb{N}_m \to S$. Suppose that S has 1 element. Then either T is empty, or S itself, which means T is finite. Suppose that, upto n, it is true that, if S is finite with n elements, all its subsets are finite. Consider S with $n+1$ elements.

$f : \mathbb{N}_{n+1} \to S$.

If $f(n+1) \in T$, consider $T_1 := T - \{f(n+1)\}$ We have $T_1 \subseteq S - \{f(n+1)\}$, and since $S - \{f(n+1)\}$ is a finite set with $n-1$ elements, from induction hypothesis, T_1 is finite. Moreover, since $T = T \cup \{f(n+1)\}$, T is also finite with one more element than T_1.

If $f(n+1) \notin T$, then $T \subseteq S - \{f(n+1)\}$, we are done.

(b) is simply the contrapositive of (a). ■

Definition 2.12: (Countable Sets)

A set S is said to be *countable*,or *denumerable* if, either S is finite, or $\exists f : \mathbb{N} \to S$ which is a bijection. If S is *not countable*, S is said to be *uncountable*

Theorem 2.13

The set $\mathbb{N} \times \mathbb{N}$ is countable.

Proof for Theorem.

The number of points on diagonals $1, 2, ...l$ are: $\psi(k) = 1 + 2 + ...k = \frac{k(k+1)}{2}$

The point (m, n) occurs on the $(m + n - 1$ th diagonal, on which the number $m + n$ is an invariant. The (m, n) point occurs m points down the diagonal. So, to characterise a point, it is enough to specify the diagonal it falls in, and its ordinate (the "rank" of that point on that diagonal). Count the elements till the $m + n - 2$nd diagonal, then add m, and this would be the position of the point (m, n).
Define $r : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $r(m, n) = \psi(m + n - 2) + m$ That this is a bijection is pretty clear because we are counting the position of the point (m, n). For a given point (m, m), there can only be one unique diagnonal on which it exists, and on the diagonal, its rank is unique. Moreover, for every $q \in N$, there corresponds an (m, n) such that $r(m, n) = q$, for, we simply count along each diagonal in the "zig-zag" manner until we reach that (m, n) for which the position is given by q. Therefore, r is a bijection. (There are other explicit bijections too)

Alt Proof (Slicker): Define the explicit map $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as:

$$f(m, n) = 2^m(2n - 1)$$

For every $k \in \mathbb{N}$, there exists a unique prime decomposition. Let m be the power of 2 in that, and n be so that the product of the rest of the primes (which is odd) comes to be $2n - 1$. From here, it is easy to see a bijection.

■

Theorem 2.14

The following are equivalent:

1. S is countable
2. $\exists$ a surjective function from $\mathbb{N} \to S$
3. $\exists$ an injective function from $S \to \mathbb{N}$

Proof for Theorem.

$(1 \implies 2)$ is obvious
$(2 \implies 3)$ $f : \mathbb{N} \to S$, every element of of S has atleast one preimage in $\mathbb{N}$. Define a function from $S \to \mathbb{N}$ by taking for each $s \in S$ the least such $n \in \mathbb{N}$ such that $f(n) = s$. This defines an injection.
$(3 \implies 1)$ If there is an injection from $S \to \mathbb{N}$, then there is a bijection from $S \to$ a subset of $\mathbb{N}$, which implies S is countable. ■

Corollary 2.15

The set of Rational Numbers $\mathbb{Q}$ is countable.

Proof for Corollary.

We know that a surjection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{Q}$ exists (where $f(0,0) = 0$, and $f(m,n) = \frac{m}{n}$). We know that $\mathbb{N} \times \mathbb{N}$ is bijective to $\mathbb{N}$. This means $\mathbb{N}$ is surjective to $\mathbb{Q}$. We are Done. ■

Theorem 2.16

Every infinite subset of a countable set is countable.

Proof for Theorem.

Consider $N_s \subseteq \mathbb{N}$ which is infinite.
Define $g(1) =$ least number in N_s

Having defined $g(n)$, define $g(n+1)=$ least number in N_s which is larger than $g(n)$.

That it is an injection is obvious, for $g(m) > g(n)$ if $m > n$.

Suppose it is not a surjection, i.e, $g(\mathbb{N}) \neq N_s \implies g(\mathbb{N}) \subset N_s \implies N_s - g(\mathbb{N}) \neq \phi$
Therefore, $N_s - g(\mathbb{N})$ has a least element, k. This means that $k-1$ is in $g(\mathbb{N})$. Therefore, there exists q in $\mathbb{N}$ such that $g(q) = k-1$. But then, $g(q+1)=$ least number in N_s such that it is bigger than $g(q)$. This would, ofcourse be, k, which means $k = g(q+1)$, which puts k in $g(\mathbb{N})$. Contradicton. Hence, $g(\mathbb{N}) = N_s$, therefore, g is a bijection from $\mathbb{N} \to N_s$. Since every countable set is bijective to $\mathbb{N}$, and every infinite subset of a countable set is bijective to an infinite subset of $\mathbb{N}$, the theorem holds generally for countable sets. ■

Theorem 2.17

$\mathbb{N} \times \mathbb{N} \cdots \mathbb{N}$ is bijective to $\mathbb{N}$

Proof for Theorem.

$\mathbb{N} \times \mathbb{N}$ is bijective to $\mathbb{N}$ obviously. Assuume that $f : \mathbb{N} \to \mathbb{N} \cdots \mathbb{N}($ n times $)$ is bijective.

Consider $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cdots \mathbb{N}(n+1$ times $)$ given by $g(m,n) = (f(m), n)$. Clearly, this is bijective. ■

2.1 Axiom of Choice

Axiom 2.18: Axiom of Choice (AC)

For any collection of non empty sets $C = \{A_l : l \in L\}$, there exists a function f called the "counting function" which maps each set A_l to an element in A_l.
Formally: $f : C \to \bigcup_l A_l$ such that $\forall l \in L, f(A_l) \in A_l$

Theorem 2.19

Countable union of Countable sets is countable *(This theorem is an example of a theorem that requires* Axiom of Choice*)*

Proof for Theorem.

Suppose we are given a sequence of countable sets $\{S_n : n \in \mathbb{N}\}$. Since each S_j is countable, we have for each j, at least one bijective map $f_j : \mathbb{N} \to S_j$. Define $k : \mathbb{N} \times \mathbb{N} \to \bigcup_j S_j$ given by: $k(m, n) = f_m(n)$. Suppose $x \in \bigcup_j S_j$, i.e, $x \in S_j$ for some j. This means that, $f(n) = x$ for some n. Therefore, $k(j, n) = x$. Hence, k is surjective. From theorem 2.14, we are done.

(**Remark:** Keep in mind, for each S_j, there are a myriad of functions $(f_j)_k : \mathbb{N} \to S_j$. For each S_j, which is countably infinite, we have to choose one of the many functions that biject $\mathbb{N}$ to S_j. So we have a countable collection of sets $C = \{E_j : j \in \mathbb{N}\}$, where E_j denotes the set of all functions that biject $\mathbb{N}$ into S_j. So for every element in C, we need to choose one element in each element of C. This is where the Axiom of Choice comes into play.) ■

Theorem 2.20

If $f : A \to B$ is a surjection, then B is bijective to a subset of A

Proof for Theorem.

We are told that $f(A) = B$, i.e, for every $b \in B$, $\exists x_b$(many such x_b-s are possible) such that $f(x_b) = b$. Define a functino $g : B \to A$ as: $g(b) =$ one of those x_b such that $f(x_b) = b$. g is bijective to the set of all the chosen x_b for every b ■

Remark.

We make use of the Axiom of Choice in the previous theorem when we choose an x_b from a set of all possible x_b-s for b. Let A_b be the set of all possible x_b-s. Then the collection $\{A_b : b \in B\}$ is a collection of non-empty sets. And we are to select "one" element from each A_b. This requires AC.

Definition 2.21: (Power Set of a set)

Power set of A, denoted by $P(A)$ is the set of all subsets of A.

Definition 2.22: Partial ordering relation on a set P

A "partial order" on a set is a relation $\leq$ defined on a set that follows:

1. $\forall x \in P, x \leq x$ (reflexive)
2. If $x, y \in P$ such that $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry)
3. forall $x, y, z \in P$, if $x \leq z$ and $z \leq y$, then $x \leq y$

Any set P that is partially ordered is called a **poset**. A set C which is totally ordered and is a subset of a poset P is called a **chain** of P.

Definition 2.23: Total Order on a set C

An ordering relation (that obeys the above conditions), added with the condition that, if $x, y \in C$, then it definitely must be that one of $x \leq y$ or $y \leq x$ is true. Then the set C is *totally* ordered.

Remark.

For example, the order relation on R is a total ordering from the trichotomy property. Consider the ordering relation on $\mathbb{Z}$ as follows: If $a \in \mathbb{Z}$ divides $b \in \mathbb{Z}$, then $a \leq b$. This is a partial order, certainly. It is reflexive, anti symmetric and transitive. But 3 and 5 are unrelated in this definition. Whereas, consider the same ordering but in the space $1, p, p^2, \cdots$ where p is a prime, we then see that the set is totally ordered here.

Definition 2.24: Boundedness of a subset C of P, where a partial order is defined

We say set C in a poset P is bounded, or has an upper bound, if $\exists M \in P$ so that $\forall q \in C, q \leq M$.

Example :
We consider the power set of a set P, where inclusion is defined by $S \subseteq T$ if $\forall x \in S, x \in T$. This defines a partial ordering on the power set.

Axiom 2.25: Zorn's Lemma

Given a poset P, if for every chain C of P, there exists an upper bound M_c in P, then P has a maximal emement with respect to $\leq$ (i.e, there exists M in P so that $x \leq M, \forall\, x \in P$)

Theorem 2.26

AC $\iff$ Zorn's Lemma

Proof for Theorem.

hmm

Some Corollaries of AC:

Theorem 2.27

Every vector space has a basis

Proof for Theorem.

Theorem 2.28

$|S| = |\text{set of all finite subsets of } S|$

Proof for Theorem.

Theorem 2.29

For an arbitrary dimensioned vector space, every linearly independent set is injective to any spanning set.

Proof for Theorem.

Theorem 2.30: Cantor's Theorem

For any set A, there *does not exist* any surjection from A onto $P(A)$

Proof for Theorem.

Suppose, on the contrary, a surjection $\psi : A \to P(A)$ exists. For every subset A_s of A, there exists an element x of A such that $\psi(x) = A_s$. Either this x exists in A_s, or it doesnt. Conider $D := \{x \in A : x \notin \psi(x)\}$. D is a subset of A, so there must be some element $y \in A$ such that $\psi(y) = D$. does y belong in D? If so, $y \notin \psi(y) = D$. Which means $y \notin D$. If, though, $y \notin D$, that implies $y \notin \psi(y) \implies y \in D$. Contradictions left and right. ∎

3 The Real and Complex Fields

Definition 3.1: (Field $(F, +, \cdot)$)

set F, along with two functions $+ : F \times F \to F$ and $\cdot : F \times F \to F$ is called a field if:

1. $\forall x, y \in F$, $x + y \in F$ (closed under addition)
2. $\forall x, y \in F$, $x + y = y + x$ (commutative under addition)
3. $\forall x, y, z \in F$, $x + (y + Z) = (x + y) + z$ (assiciative under addition)
4. $\exists 0($ additive identity$)$ such that $\forall x \in F, x + 0 = 0 + x = x$ (Additive identity)
5. $\forall x \in F, \exists(-x)($ additive inverse$)$ such that $x + (-x) = (-x) + x = 0$ (Additive inverse)
6. $\forall x, y \in F, x \cdot y \in F$ (Multiplication is closed)
7. $\forall x, y \in F, x \cdot y = y \cdot x$ (Multiplication is commutative)
8. $\forall x, y, z \in F, x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (Multiplication is associative)
9. $\forall x \in F, x \neq 0, \exists(x^{-1})$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$ (Multiplicative inverse)
10. $\forall x \in F, 1 \cdot x = x \cdot 1 = x$ (Multiplicative identity)
11. $\forall a, b, c \in F, a \cdot (b + c) = a \cdot b + a \cdot c$ (Left Distributivity)
12. $\forall a, b, c \in F, (a + b) \cdot c = a \cdot c + b \cdot c$ (Right Distributivity)

3.1 The Reals $\mathbb{R}$

The real numbers are characterised by the following axioms:

Definition 3.2: (The Real Field)

1. F, field axioms listed above.
2. Order Axioms:There exists a subset $P \subset \mathbb{R}$ called "positive numbers" such that:
   (a) $\forall a \in \mathbb{R}$, only one of $a \in P$, $-a \in P$ or $a = 0$ are true. (Trichotomy Law)
   (b) $\forall a, b \in P, a + b \in P$ (Positive numbers are closed under addition)
   (c) $\forall x, y \in P, x \cdot y \in P$ (Positive numbers are closed under multiplication)
3. Completeness axiom*

Definition 3.3

1. (Max of a set) $M \in S \subseteq \mathbb{R}$ is said to be the maximum of S if $M \geq x \forall x \in S$
2. (Min of a set) $m \in S \subseteq \mathbb{R}$ is said to be the minimum of a set if $m \leq x \forall x \in S$
3. (Upper bound of a set) $L \in \mathbb{R}$ is said to be an upper bound of S if $L \geq x \forall x \in S$
4. (Lower bound of a set) $l \in \mathbb{R}$ is said to be a lower bound of S if $l \leq x \forall x \in S$
5. (Sup(S)) $\alpha \in \mathbb{R}$ is said to be the supremum of S if it is the minimum of the set of all Upper bounds of S.
6. (Inf(S)) $\beta \in \mathbb{R}$ is said to be the infimum of S if it is the maximum of the set of all lower bounds of S.

Axiom 3.4: Completeness Axiom for the Real Number Field

Every non empty subset of $\mathbb{R}$ that is bounded above has a supremum.

Corollary 3.5

Every non empty subset of $\mathbb{R}$ that is bounded below has an infimum.

Proof for Corollary.

$S \subseteq \mathbb{R}$ has a lower bound $\implies \exists L \in \mathbb{R}$ such that $L \leq x \forall x \in S \implies -L \geq -x \forall x \in S$
Define $S' := \{-x : x \in S\}$. Then, $-L$ is an upper bound of S'.
$\implies \exists q \in \mathbb{R}$ such that $q \geq y \forall y \in S'$ and $q \leq z \forall z$ such that $z \geq y \forall y \in S'$.
$\implies \exists q' = -q \in \mathbb{R}$ such that $q' \leq x \forall x \in S$ and $q' \geq z' = -z \forall z$ such that $z' \leq x \forall x \in S$
Cleaning up a bit: $\implies \exists q' \in \mathbb{R}$ such that $q' \leq x \forall x \in S$ and $q' \geq z' \forall z'$ such that $z' \leq x \forall x \in S$
Therefore q' is the greatest lower bound, i.e the infimum. ■

Lemma 3.6: The Lemma

Suppose $a \in \mathbb{R}^+$ and $0 \leq a < \varepsilon \ \forall \varepsilon \in \mathbb{R}^+$ then $a = 0$

Proof for Lemma

Suppose not, i.e, $a > 0$. Choose $\varepsilon = \frac{a}{2}$. Contradiction. ■

Proposition 3.7

Supremum of a set is unique.

Proof for Proposition.

Supremum is the "least" of the set of the upper bounds, it itself being part of the set of all the upper bounds. Since minima is unique, Supremum is unique.

Lemma 3.8

$U \in \mathbb{R}$ is the supremum of $S \subseteq \mathbb{R} \iff$

1. $s \leq U \forall s \in S$
2. if $v < U, \exists s_v \in S$ such that $v < s_v$

Proof for Lemma

$\implies$) Given that U is the supremum, (1) is pretty obvious since it is an upper bound. Suppose $v < U$, but for every $s \in S$, $s \leq v$. This would mean v is the supremum, and not U. Absurd.

Fact 3.9

Given that U is an upper bound of S, U is the supremum of $S \iff \forall \varepsilon > 0, \exists s_\epsilon \in S$ such that $U - \varepsilon < s_\epsilon$

Theorem 3.10: Archimedean Property of $\mathbb{R}$

Given $a, b \in \mathbb{R}^+, \exists n \in \mathbb{N}$ such that $an - b > 0$

Proof for Theorem.

Suppose not, i.e, $\forall n \in \mathbb{N}, an < b \implies \forall n \in \mathbb{N}, n < \frac{b}{a}$. Consider the set $S = \{an : n \in \mathbb{N}\}$. This has an upper bound b, and therefore, a supremum u. consider $u - n$. $\exists n_0$ such that $u - n < an_0 \implies u < a(n_0 + 1)$. Absurd.

Corollary 3.11

Alternate formulation of the previous statement: $\forall x \in \mathbb{R}, \exists n_0 \in \mathbb{N}$ such that $x < n_0$

Lemma 3.12: Useful Lemma

$\forall \varepsilon > 0, x < \varepsilon \iff \forall n \in \mathbb{N}, x < \frac{1}{n}$

Proof for Lemma

$\implies$)Contrapositive to prove would be: $\exists n_0 \in \mathbb{N}, x \geq \frac{1}{n_0} \implies \exists \varepsilon_0 > 0, x \geq \varepsilon_0$. Simply choose $\varepsilon_0 = \frac{1}{n_0}$
$\impliedby$)Contrapositive to prove would be: $\exists \varepsilon_0 > 0$ such that $x \geq \varepsilon \implies \exists n_0$ such that $x \geq \frac{1}{n_0}$. From Archimedean, $\exists n_0$ such that $n_0 \geq \frac{1}{\varepsilon_0} \implies \varepsilon_0 \geq \frac{1}{n_0} \implies x \geq \varepsilon_0 \geq \frac{1}{n_0}$

Theorem 3.13: Archimedean Properties of $\mathbb{R}$

1. $inf(\{\frac{1}{n} : n \in \mathbb{N}\}) = 0$
2. If $t > 0, \exists n_0 \in \mathbb{N}$ such that, $0 < \frac{1}{t} < n$
3. If $y > 0$, $\exists n_y \in \mathbb{N}$ such that $n_y - 1 \leq y < n_y$

Proof for Theorem.

1)Obvious
2)Application of Archimedean
3) We know from archimedean that such an n_y exists such that $y < n_y$. Consider the set of all n such that $y < n$. Obviously, this is a non empty set. Therefore, from Well Ordering, this has a least element $n_0 \implies n_0 \leq y < n_0$ ■

Theorem 3.14: Density of $\mathbb{Q}$ in $\mathbb{R}$

$\forall x, y \in \mathbb{R}, x < y \implies \exists q \in \mathbb{Q}$ such that $x < q < y$

Proof for Theorem.

$y - x > 0 \implies$ from archimedean $\exists n_0$ such that $n_0(y - x) > 1 \implies n_0 y > 1 + n_0 x$. Form Archimedean Propery, $\exists m \in \mathbb{N}$ such that $m - 1 < n_0 y \leq m$.Since $m \geq n_0 y > 1 + n_0 x \implies m - 1 > n_0 x \implies n_0 y > m - 1 \leq n_0 x \implies y > \frac{(m-1)}{n_0} > x$ ■

Corollary 3.15

Given $x, y \in \mathbb{R}, y > x$, $\exists q \in \mathbb{R} - \mathbb{Q}$ such that $y > q > x$ (Assumed that $\sqrt{2}$ is irrational.)

Theorem 3.16: Existence of nth Roots in $\mathbb{R}^+$

let $y \in \mathbb{R}^+$ and $n \in \mathbb{N}$, then $\exists$ a unique $x \in \mathbb{R}^+$ such that $x^n = y$

Proof for Theorem.

Consider $E := \{t \in \mathbb{R} : t^n < y\}$. Is E bounded above? obviously, $1 + y$ is an upper bound. Is it non empty? Of course, consider $t = \frac{y}{1+y} < y$. Hence, E has a supremum u. Claim: $u^n = y$. Suppose not. Let $u^n < y$. We want to find an $h \in \mathbb{R}^+$ such that $(u + h)^n < y$ so that a contradiction can be raised ($u + h$ cannot be in the set). In effect we want to show that $(u+h)^n - u^n < y - u^n$. Recall the identity: $p^n - q^n = (p-q)(p^{n-1} + qp^{n-2} \cdots + q^{n-1})$. If $p > q$, we have $p^n - q^n < n(p-q)(p^{n-1})$. Therefore: $(u+h)^n - u^n \leq n(h)(u+h)^{n-1}$. We want h so that $n(h)(u + h)^{n-1} < y - u^n \implies h < \frac{y-u^n}{n(u+h)^{n-1}}$. Choose $h < 1$, which would mean $\frac{y-u^n}{n(u+1)^{n-1}} < \frac{y-u^n}{n(u+h)^{n-1}}$. Now simply choose such an h such that $h < \frac{y-u^n}{n(u+1)^{n-1}} < \frac{y-u^n}{n(u+h)^{n-1}}$ which is possible from density.
Suppose now that $u^n > y$, we need an h so that $(u - h)^n > y$ or $-(u - h)^n < -y$,

which would mean that $u-h$ is the actual supremum, contradicting the assumption. Therefore, we have to show that $u^n-(u-h)^n < u^n - y$. From the identity, we have that $u^n-(u-h)^n \leq n(h)(u)^n$. It would suffice if we find an h so that $nhu^{n-1} < u^n - y$ or $h < \frac{u^n-y}{nu^{n-1}}$. Again, from Density theorem this is possible.
Uniqueness, once existence is established, is trivial since, if $q_1 > q_2, (q_1)^n > (q_2)^n$.

Fact 3.17

1. $n > 0, q > 0$ and $r = \frac{m}{n} = \frac{p}{q}$, then $(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$
2. $x^{(p+q)} = x^p x^q$ for $p, q \in \mathbb{Q}$

Theorem 3.18: Results regarding powers

Suppose $b > 1$. If $x \in \mathbb{R}$, define $B(x) := \{b^t : t \in \mathbb{Q} : t \leq x\}$. Then $sup(B(r)) = b^r$ if r is a rational number.

Proof for Theorem.

Of course, the set is bounded and non-empty, hence, has a supremum. It is clear that b^r cannot be $< sup(B(r))$ because if so, there would exist $t \in \mathbb{Q}, t \leq r$ such that $b^r < b^t$. Absurd. So $b^r \geq sup(B(r))$. It also can't be strictly greater, since b^r is in the set itself, so it can't exceed its supremum. Hence,$Sup(B(r)) = b^r$.

With the previous result in mind as motivation, we define the following:

Definition 3.19: (Real "raised" to Reals)

Given $b > 1$, we define $b^x := sup(B(x) := \{b^t : t \in QQ : t \leq x \in \mathbb{R}\})$

Theorem 3.20

$b^x b^y = b^{x+y}$ for all $b > 1, x, y \in \mathbb{R}$

Proof for Theorem.

$$b^x := sup(\{b^p : p \in \mathbb{Q} : p \leq x\})$$
$$b^y := sup(\{b^q : q \in \mathbb{Q} : q \leq y\})$$
$$b^{x+y} := sup(\{b^t : t \in \mathbb{Q} : t \leq x+y\})$$

Suppose that $b^x b^y < b^{x+y}$. Then, $\exists q \in \mathbb{Q}, q < x+y$ such that $b^x b^y < b^q$. Suppose WLOG $x < y$. Choose a $t \in Q^+$ such that $q - x < t < y$. This means, $q < x+y$ as we know, but also, $q-t < x$ and $t < y$. We now have $b^x b^y \leq b^{q-t+t} = b^{q-t} b^t$ where $q-t < x$ and $t < y$. Absurd.
Now assume $b^x b^y > b^{x+y}$. This implies $b^x > \frac{b^{x+y}}{b^y} \implies \exists q < x$ such that $b^q > \frac{b^{x+y}}{b^y} \implies$

$b^y > \frac{b^{x+y}}{b^q} \implies \exists p < y$ such that $b^p > \frac{b^{x+y}}{b^q} \implies b^p b^q = b^{p+q} > b^{x+y}$ but $p < y$ and $q < x \implies p + q < x + y$. Absurd. So $b^x b^y = b^{x+y}$ ∎

Theorem 3.21: Existence of Log

Let $b > 1$, $y > 0$, then,$\exists$ a unique $x \in \mathbb{R}$ such that $b^x = y$

Proof for Theorem.

Consider $E := \{x \in \mathbb{R} : b^x \leq y\}$. The claim is that $Sup(E) = z$ exists and $b^z = y$.
Case 1-$y \geq b > 1$:
It is obvious that in this case E is non empty. Suppose that, it is unbounded. i.e, $\forall n \in \mathbb{N}, b^n \leq y$. Since $b > 1$,$b = 1 + \delta$ for some $\delta > 0$. $\implies b^n = (1+\delta)^n = 1 + n\delta + \frac{n(n-1)}{2}\delta^2 \cdots \leq y, \forall n \in \mathbb{N}$. i.e, $1 + n\delta < y, \forall n \in \mathbb{N}$. This would be absurd, obviously. Hence, $\exists n_0 \in \mathbb{N}$ such that $b^{n_0} > y$, and obviously $\forall n \geq n_0$. Therefore, in this case, E is bounded and non empty, hence has a supremum $z = sup(E)$.
Case 2- $b > y$:
Sub Case 1: $b > y > 1$:
Boundedness is clear here. We claim that $\exists n_0 \in \mathbb{N}$ such that $y^{n_0} \geq b$ or $y \geq b^{\frac{1}{n_0}}$. Suppose not, i.e $\forall n \in \mathbb{N}, y^n < b \implies (1+\delta)^n < b \implies 1 + n\delta < b \forall n \in \mathbb{N}$. Again, this is absurd. Hence, $\exists n_0$ such that $y > b^{\frac{1}{n_0}}$. Hence, it is bounded and non empty.
Sub Case 2: $b > 1 > y$:
Boundedness is clear here as well. Since $y < 1$, $y^{-1} > 1$, and say $z = y^{-1}$. Does $\exists r_0 \in \mathbb{Q}$ such that $b \geq z^{r_0}$? If $z \geq b > 1$, from the proof of case-1, $\exists r_0 \in \mathbb{Q}$ such that $b \geq z^{r_0} \implies b \geq y^{-r_0} \implies y \leq b^{-\frac{1}{r_0}}$. If $b > z > 1$, then that $r_0 = 1$. From here we see that $b > y^{-1} \implies b^{-1} < y$.

In all cases. Supremum exists for E. Call it s

Does $b^s = y$? suppose not, i.e, let $b^s < y$. We want to establish a number $s + z_0 > s$ such that $b^{s+z_0} < y$ which would lead to contradiction since s is supposed to be the supremum of E.
$\exists \delta \in \mathbb{R}^+$ such that $b^s + (\delta)b^s = b^s(1+\delta) < y$ from density. We need a $q \in \mathbb{Q}^+$ such that $b^q < 1 + \delta$. We know that $b > 1$ and $1 + \delta > 1$, so either from case 1 where $1 + \delta \geq b > 1$, or from case 2 subcase 1 where $b > 1 + \delta > 1$, we can find such a q. Hence, $b^s b^q = b^{s+q} < b^s(1+\delta) < y$. This would be absurd.
Consider the case where $b^s > y$. From density, $\exists \delta \in \mathbb{R}^+$ such that $b^s > y + \delta \implies b^s > y + \delta_0 y \implies b^s > y(1 + \delta_0$ for some δ_0. This means that $b^s \frac{1}{1+\delta_0} > y$. We need to find, again, a positive rational such that $b^q < 1 + \delta_0$. From the previous analysis, it can be done. Hence, $b^{s-q} > y$, which means that for every $z \in \mathbb{R}$ such that $z > s - q$, we have that $b^z > y$. This means that $s - q$ is an upper bound for E, which is absurd. Hence, $b^s = y$.

∎

3.2 The Complex field $\mathbb{C}$

Definition 3.22

We define $\mathbb{C}$ as the set of all ordered pairs in $\mathbb{R}^2$ with the following additional properties:

1. $x = (a_1, b_1), y = (a_2, b_2)$ with $a_1, b_1, a_2, b_2 \in \mathbb{R}$, then $x + y$ is defined as $(a_1 + a_2, b_1 + b_2)$
2. $x = (a_1, b_1), y = (a_2, b_2)$ with $a_1, b_1, a_2, b_2 \in \mathbb{R}$, we define multiplication xy as $(a_1a_2 - b_1b_2 - a_1b_2 + a_2b_1)$

This set $\mathbb{C}$ with $+$ and juxtaposition obey field axioms with $(0, 0)$ the additive identity, and $(1, 0)$ the multiplicative one.
For consistency, we define $\frac{1}{x} := (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$, where $x = (a, b)$.

Definition 3.23

1. (Conjugate) If $z = (a, b)$, the conjugate $\bar{z} =$ is defined as $\bar{z} = (a, -b)$.
2. (Mod) If $z = (a, b)$, the mod of z, $|z|$ is defined as $(z\bar{z})^{\frac{1}{2}}$

Fact 3.24

Some facts:

1. $\bar{(}x + y) = \bar{x} + \bar{y}$
2. $\bar{(}zw) = \bar{z}\bar{w}$
3. $z\bar{z} \geq 0$ and $= (a^2 + b^2, 0)$
4. We can identify $\mathbb{R}$ as a subset of $\mathbb{C}$ by setting $a \in \mathbb{R}$ to be $(a, 0)$ in $\mathbb{C}$.
5. $|zw| = |z||w|$
6. $|z + w| \leq |z| + |w|$
7. $|Re(z)| \leq |z|$

Theorem 3.25: Cauchy-Schwartz Inequality

If $a_1, a_2, \cdots a_n$ and $b_1, b_2, \cdots, b_n$ are numbers in $\mathbb{R}^+$, then

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \sum_{i=1}^{n} (a_i)^2 \sum_{j=1}^{n} (b_j)^2$$

Extending this theorem for a_j and b_j in the complex domain, we have

$$|\big(\sum_{i=1}^{n} a_i \bar{b}_i\big)|^2 \leq \sum_{i=1}^{n} |(a_i)|^2 \sum_{j=1}^{n} |(b_j)|^2$$

Proof for Theorem.

Consider $\alpha = (a_1b_1 + a_2b_2 \dots a_nb_n)^2 = (a_1b_1 + a_2b_2 \dots a_nb_n)(a_1b_1 + a_2b_2 \dots a_nb_n)$ which is $((a_1b_1)^2 + (a_2b_2)^2 \dots (a_nb_n)^2) + K$ where K is given by

$$\begin{array}{ccccc} 0 & +a_1b_1a_2b_2 & +a_1b_1a_3b_3 & \cdots & +a_1b_1a_nb_n \\ a_2b_2a_1b_1 & +0 & +a_2b_2a_3b_3 & \cdots & +a_2b_2a_nb_n \\ \vdots & & & & \\ a_nb_na_1b_1 & +a_nb_na_2n_2 & +a_nb_na_3b_3 & \cdots & +0 \end{array}$$

Consider $\beta = (a_1^2 + a_2^2 + \dots)(b_1^2 + b_2^2 \dots)$ This would be$(a_1b_1)^2 + (a_2b_2)^2 \cdots (a_nb_n)^2 + L$ where L is given by:

$$\begin{array}{ccccc} 0 & +a_1^2b_2^2 & +a_1^2b_3^2 & \cdots & +a_1^2b_n^2 \\ a_2^2b_1^2 & +0 & +a_2^2b_3^2 & \cdots & +a_2^2b_n^2 \\ \vdots & & & & \\ a_n^2b_1^2 & +a_n^2b_2^2 & +a_n^2b_3^2 & \cdots & +0 \end{array}$$

$$\beta - \alpha = \sum_{i=1}^{n} \sum_{j=i+1}^{n} (a_ib_j)^2 + (a_jb_i)^2 - \sum_{i=1}^{n} \sum_{j=i+1} 2a_ia_jb_ib_j = \sum_{i=1}^{n} \sum_{j=i+1} (a_ib_j - b_ia_j)^2$$

. Hence, $\beta = \alpha +$ some square term. Therefore

$$\beta - \alpha \geq 0$$

. ■

Theorem 3.26: Bernoulli's Inequality

Given $x > -1$, $(1 + x)^n \geq 1 + nx$

Proof for Theorem.

For $n = 1$, it's trivially true. Assume it's correct for $n = n$. Consider $(1+x)^n(1+x) \geq (1+nx)(1+x) = 1+x+nx+nx^2 = 1+x(n+1)+nx^2 \implies (1+x)^{n+1} \geq 1+(n+1)x$ ■

Theorem 3.27: AM-GM Inequality

Given $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$,

$$\left(\frac{S_n}{n}\right)^n = (\frac{a_1 + a_2 + \cdots + a_n}{n})^n \geq (a_1 a_2 \cdots a_n)$$

Proof for Theorem.

For a_1, it is trivially true. Assume for $n = n$, and consider

$$\left(\frac{S_{n+1}}{n+1}\right)^{n+1} = \left(\frac{S_n + a_{n+1}}{n+1}\right)^{n+1} \rightarrow$$

$$\left(\frac{\frac{nS_n}{n} + a_{n+1}}{n+1}\right)^{n+1} \rightarrow$$

$$\left(\frac{\frac{(n+1-1)S_n}{n} + a_{n+1}}{n+1}\right)^{n+1} = \left(\frac{\frac{(n+1)S_n - S_n}{n} + a_{n+1}}{n+1}\right)^{n+1} =$$

$$\left(\frac{\frac{(n+1)S_n}{n} - \frac{S_n}{n} + a_{n+1}}{n+1}\right)^{n+1} = \left(\frac{S_n}{n} + \frac{-\frac{S_n}{n} + a_{n+1}}{n+1}\right)^{n+1} =$$

$$\left(\frac{S_n}{n}\right)^{n+1}\left(1 + \frac{-1 + \frac{na_{n+1}}{S_n}}{n+1}\right)^{n+1}$$

From Bernoulli inequality,

$$\left(\frac{S_n}{n}\right)^{n+1}\left(1+\frac{-1 + \frac{na_{n+1}}{S_n}}{n+1}\right)^{n+1} \geq \left(\frac{S_n}{n}\right)^{n+1}\left(1+(n+1)\frac{-1 + \frac{na_{n+1}}{S_n}}{n+1}\right) = \left(\frac{S_n}{n}\right)^{n+1}\left(\frac{na_{n+1}}{S_n}\right)$$

$$\geq \left(\frac{S_n}{n}\right)^n \left(a_{n+1}\right) \geq a_1 a_2 \cdots a_n a_{n+1}$$

■

3.3 Some Inequalities

Theorem 3.28: Generalised AM-GM

If $x, y \in \mathbb{R}^+$ and $t \in (0,1)$, then

$$(1-t)x + ty \geq x^{1-t}y^t$$

Proof for Theorem.

We use the fact that a differentiable function f is convex if and only if its first derivative f' is monotone increasing. Under this criteria, e^x is certainly convex. That means $\forall x, y \in \mathbb{I}$, $t \in (0,1)$,

$$e(x) + t(e(y) - e(x)) \geq e(x + t(y-x))$$

or in other words:

$$e((1-t)x + ty) \leq (1-t)e(x) + te(y)$$

Making the substitution $x \mapsto ln(x)$ gives us

$$e((1-t)ln(x) + tln(y)) \leq (1-t)e(ln(x)) + tln(y)$$

which implies (since e is a monotone increasing function)

$$(1-t)x + ty \geq x^{1-t}y^t$$

■

Theorem 3.29: Concavity of $log_e : \mathbb{R}^+ \to \mathbb{R}$ (assumed knowledge of e, and existence of log.)

The function $f := log_b : \mathbb{R}^+ \to \mathbb{R}$ proven to be unique for a given $b > 1$ and any real number x, follows the following: For every $x, y \in \mathbb{R}^+$, and $t \in (0,1)$

$$f((1-t)x + ty) \geq (1-t)f(x) + tf(y)$$

Proof for Theorem.

We want to prove

$$log((1-t)x + ty) \geq (1-t)log(x) + tlog(y)$$

which amounts to showing that

$$log((1-t)x + ty) \geq log(x^{1-t}y^t)$$

Since log is an increasing function, from general AM-GM, we have $(1-t)x + t(y) \geq x^{1-t}y^t$ which gives (taking log on both sides)

$$log((1-t)x + ty) \geq log(x^{1-t}y^t)$$

which gives us the desired result. ■

Theorem 3.30: Young's Inequality

If a, b are non negative reals, with $p > 1, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

With equality holding if and only if $a^p = b^q$

Proof for Theorem.

Let $\frac{1}{p} = t < 1$ and $\frac{1}{q} = 1 - t < 1$. From generalised AM-GM, in place of $(1-t)x + ty \geq x^{1-t}y^t$, make the following transformations:

$$x \mapsto x^{\frac{1}{(1-t)}}$$

$$y \mapsto y^{\frac{1}{t}}$$

to arrive at the modified am-gm which is

$$(1-t)x^{\frac{1}{(1-t)}} + ty^{\frac{1}{t}} \geq xy$$

Plugging the values for $1 - t$ and t respectively we get

$$\frac{1}{q}x^q + \frac{1}{p}y^p \geq xy$$

■

Theorem 3.31: Hoelder's Inequality

Given $\{x_i\}_{i=1}^k, \{y_i\}_{i=1}^k$ in $\mathbb{R}$, we have (given positive integers p, q so that $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$)

$$\sum_{i=1}^{k} |x_i y_i| \leq (\sum_{i=1}^{k} |x_i|^p)^{1/p} (\sum_{i=1}^{k} |y_i|^q)^{1/q}$$

Proof for Theorem.

Consider the case when $\sum_{i=1}^k |x_i|^p = 1$ and $\sum_{i=1}^k |y_i|^q = 1$. Using Young's Inequality which is $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, we have

$$|x_i y_i| \leq \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q}$$

Taking the sum on both sides gives

$$\sum_{i=1}^{k} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Now suppose $z_i = \frac{|x_i|}{(\sum_{i=1}^{k} |x_i|^p)^{1/p}}$ and $w_i = \frac{|y_i|}{(\sum_{i=1}^{k} |y_i|^q)^{1/q}}$. It is easy to see that $\sum_{i=1}^{k} |z_i|^p = \sum_{i=1}^{k} |w_i|^p = 1$. This means, from Young's inequality,

$$\sum_{i=1}^{k} |z_i w_i| \leq 1$$

which implies

$$\sum_{i=1}^{k} |x_i y_i| \leq (\sum_{i=1}^{k} |x_i|^p)^{1/p} (\sum_{i=1}^{k} |y_i|^q)^{1/q}$$

■

Theorem 3.32: Minkowski's Inequality

Given $\{x_i\}_{i=1}^{k}$ and $\{y_i\}_{i=1}^{k}$ in $\mathbb{R}$, and an integer $p \geq 1$, we have:

$$(\sum_{i=1^k} (|x_i + y_i|^p))^{1/p} \leq (\sum_{i=1}^{k} |x_i|^p)^{1/p} + (\sum_{i=1}^{k} |y_i|^p)^{1/p}$$

Proof for Theorem.

Consider $(\sum_{i=1}^{k} |x_i + y_i|^p) \leq (\sum_{i=1}^{k} (|x_i| + |y_i|)^p) \leq (\sum_{i=1}^{k} (|x_i| + |y_i|)(|x_i| + |y_i|)^{p-1})$ This is

$$(\sum_{i=1}^{k} (|x_i|)(|x_i| + |y_i|)^{p-1} + \sum_{i=1}^{k} (|y_i|)(|x_i| + |y_i|)^{p-1})$$

Apply Holder inequality to each summand to get:

$$(\sum_{i=1}^{k} |x_i|^p)^{1/p} (\sum_{i=1}^{k} (||x_i| + |y_i||^{p-1})^{\frac{p}{p-1}})^{\frac{p-1}{p}}$$

$$+$$

$$(\sum_{i=1}^{k} |y_i|^p)^{1/p} (\sum_{i=1}^{k} (||x_i| + |y_i||^{p-1})^{\frac{p}{p-1}})^{\frac{p-1}{p}}$$

Which simplifies down to:

$$(\sum_{i=1}^{k} |x_i|^p)^{1/p} (\sum_{i=1}^{k} (|x_i| + |y_i|)^p)^{\frac{p-1}{p}} + (\sum_{i=1}^{k} |y_i|^p)^{1/p} (\sum_{i=1}^{k} (|x_i| + |y_i|)^p)^{\frac{p-1}{p}}$$

which again simplifies to

$$\{(\sum_{i=1}^{k}|x_i|^p)^{1/p}+(\sum_{i=1}^{k}|y_i|^p)^{1/p}\}(\sum_{i=1}^{k}(|x_i|+|y_i|)^p)^{\frac{p-1}{p}}$$

This gives us the equation:

$$\sum_{i=1}^{k}(|x_i|+|y_i|)^p \leq \{(\sum_{i=1}^{k}|x_i|^p)^{1/p}+(\sum_{i=1}^{k}|y_i|^p)^{1/p}\}(\sum_{i=1}^{k}(|x_i|+|y_i|)^p)^{\frac{p-1}{p}}$$

Which immediately gives us (after multiplying both sides)

$$(\sum_{i=1}^{k}(|x_i|+|y_i|)^p)^{1/p} \leq \{(\sum_{i=1}^{k}|x_i|^p)^{1/p}+(\sum_{i=1}^{k}|y_i|^p)^{1/p}\}$$

Which is the desired result

■

Theorem 3.33: Minkowski for Infinite Sums

Given sequences $\{x_n\}$ and $\{y_n\}$ we have (where it is assumed/ given that each of the right side term is finite)

$$(\sum_{i=1}^{\infty}(|x_i+y_i|^p))^{1/p} \leq (\sum_{i=1}^{\infty}|x_i|^p)^{1/p}+(\sum_{i=1}^{\infty}|y_i|^p)^{1/p}$$

o

Proof for Theorem.

Given $\{x_i\}_{i=1}^k$ and $\{y_i\}_{i=1}^k$, the good old Minkowski inequality reads:

$$(\sum_{i=1}^{k}(|x_i+y_i|^p))^{1/p} \leq (\sum_{i=1}^{k}|x_i|^p)^{1/p}+(\sum_{i=1}^{k}|y_i|^p)^{1/p}$$

For any $k \in \mathbb{N}$. Since the right side is monotone increasing (and assuming it is bounded above, i.e, convergent) we can take the limit to arrive at

$$(\sum_{i=1}^{k}(|x_i+y_i|^p))^{1/p} \leq (\sum_{i=1}^{\infty}|x_i|^p)^{1/p}+(\sum_{i=1}^{\infty}|y_i|^p)^{1/p}$$

Whence we see the inequality holding again for all $k \in \mathbb{N}$. Simply take the limit again th arrive at:

$$(\sum_{i=1}^{\infty}(|x_i+y_i|^p))^{1/p} \leq (\sum_{i=1}^{\infty}|x_i|^p)^{1/p}+(\sum_{i=1}^{\infty}|y_i|^p)^{1/p}$$

■

3.4 Intervals on the Real Line

Definition 3.34: Intervals

1. (Open Interval): $(a, b) \subset \mathbb{R} := \{x \in \mathbb{R} : a < x < b\}$
2. (Closed Interval): $[a, b] \subset \mathbb{R} := \{x \in \mathbb{R} : a \leq x \geq b\}$

Definition 3.35: Nested Intervals

If $I_n := [a_n, b_n] : n \in \mathbb{N}$ is a sequence of intervals such that $I_n \subseteq I_{n-1} \cdots \subseteq I_1$, then $\{I_n\}$ is said to be a sequence of nested intervals.

Theorem 3.36: Nested Interval Theorem

Given a sequence of closed and bounded, and non empty nested intervals $\{I_n : n \in \mathbb{N}\}$, $\exists \xi \in I_n \ \forall n \in \mathbb{N}$ or equivalently, $\xi \in \cap_{n=1}^{\infty} I_n$.

Proof for Theorem.

Let $I_n = [a_n, b_n]$. From the definition, it is clear that $\{a_n\}$ is an increasing sequence of reals, while $\{b_n\}$ is a decreasing sequence. Moreover, from the non empty property of each interval, we have that $a_m < b_n, \forall n \in \mathbb{N}, \forall m \in \mathbb{N}$. This implies that the set of $\{a_n\}$ has a supremum S_a, while the set $\{b_n\}$ has an infimum L_b. $a_n \leq L_b \forall n \in \mathbb{N}$ while $S_a \leq b_n \forall n \in \mathbb{N}$. $a_n \leq$. Moreover, $a_n \leq S_a \forall n \in \mathbb{N}$ while $L_b \leq b_n \forall n \in \mathbb{N}$. since a_n is a lower bound of $\{b_n\}$, $a_n \leq L_b$, and since L_b is an upper bound of $\{a_n\}$, $a_n \leq S_a \leq L_b \leq b_n : \forall n \in \mathbb{N}$. From Density, $\exists \xi \in [S_a, L_b]$ such that $\xi \in \cap_{i=1}^{\infty} I_i$ ■

Corollary 3.37

In the previous theorem, if the size of the nested intervals converges to 0, then the intersection contains only one point.

Proof for Corollary.

We see that $len(I_n) \to 0$ which means $\forall \varepsilon > 0 \ \exists n_0$ such that $\forall n \geq n_0$

$$len(I_n) < \varepsilon$$

Say two points survive the intersection, call them x and y. They have a finite, non zero distance d between them. Let $\varepsilon < d$. This means that after n_0, the length of the intervals go below $\varepsilon < d$. So, if x is in the intersection, y, by virtue of being d distance away, wont survive the intersection. ■

3.5 Decimal Expansions, and related results

Every $x \in \mathbb{R}$ can be written as an expansion in the following way:

Definition 3.38: Decimals

Let $z \in \mathbb{R}^+$ be given. Let n_0 be the "largest" integer such that $n_0 \leq z$. Let n_1 be the largest integer such that $n_0 + \frac{n_1}{10} \leq z$. As such, say n_k is defined for some k. Let n_{k+1} be the largest integer such that $n_0 + \frac{n_1}{10^1} + \frac{n_2}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$. Consider the set of all such "finite sums", i.e, the set of all

$$z_k = n_0 + \frac{n_1}{10^1} + \frac{n_2}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$$

. This set has a supremum and that is z itself. We symbolically write $z = n_0.n_1 n_2 \ldots$

Theorem 3.39

The set $K = \{z_n : n \in \mathbb{N}\}$ above, is bounded and non-empty, and Sup(K)=z

Proof for Theorem.

That it is non-empty and bounded is obvious. Suppose that $x = sup(K)$. Since z is an upper bound, let us assume $x < z \implies \exists \xi \in \mathbb{R}$ such that $\xi = z - x$. From Archimedean, choose a $k \in \mathbb{N}$ such that $\frac{1}{10^k} < \xi \implies -\frac{1}{10^k} > -\xi$. This means that $z - \frac{1}{10^k} > z - \xi = x$. Consider one such $z_k \in K$, we can see that $n_0 + \frac{n_1}{10^1} + \cdots + \frac{n_k}{10^k} < x < z - \frac{1}{10^k} \implies n_0 + \frac{n_1}{10^1} + \cdots + \frac{n_k+1}{10^k} < z$. But this would mean that for some $q \leq k \in \mathbb{N}$, n_k isn't the largest integer such that $z_k \leq z$.

■

Fact 3.40

The above definition is special in that, it ensures that decimal expansions are unique, since supremum's are unique. But the caveat is that, not all series' correspond to any real number as a decimal expansion. For example, in this definition: $0.999999\ldots \neq 1$ since the unique decimal expansion for $1 = 1.00000$. So $0.9999\ldots$ doesn't really correspond to any real number but we know obviously that it is 1.

Theorem 3.41

$x \in \mathbb{R}$ is rational $\iff$ x has either terminating, or repeating decimal expansion

Proof for Theorem.

$\impliedby$)Obvious
$\implies$) Suppose $x = \frac{p}{q}$ for p, q integers. Then $xq = p$. Let k_0 be the smallest integer such

that $10^{k_0}p \geq q$. From Euclid's algorithm, we have

$$10^{k_0}p = z_0q + r_0 \implies \frac{p}{q} = \frac{z_0}{10^{k_0}} + \frac{\frac{r_0}{q}}{10^{k_0}}$$

with $|r_0| < q$. Also note that $z_0 < 10$ for if not, $10^{k_0}p = z_0q + r_0 \geq qz_0 \implies 10^{k_0-1}p \geq qz_0/10 \geq q$ which is contradictory. Next, choose the smallest $k_1 \in \mathbb{N}$ such that $10^{k_1}r_0 > q$ Now consider $10^{k_1}r_0$, again we have $10^{k_1}r_0 = z_1q+r_1$ with $|r_1| < q$. Thus $\frac{r_0}{q} = \frac{z_1}{10^{k_1}} + \frac{r_1}{10^{k_1}q}$. This implies

$$\frac{p}{q} = \frac{z_0}{10^{k_0}} + \frac{z_1}{10^{k_0+k_1}} + \frac{r_1}{q10^{k_0+k_1}}$$

. We can keep going on as such, finding k_n, and applying Euclid's algorithm so that

$$\frac{p}{q} = \frac{z_0}{10^{k_0}} + \frac{z_1}{10^{k_0+k_1}} + \frac{z_2}{10^{k_1+k_2+k_3}} \cdots + \frac{z_n}{10^{k_1+k_2 \cdots +k_n}} + \frac{r_n}{q10^{k_1+k_2 \ldots +k_n}}$$

Since for every $n \in \mathbb{N}, |r_n| < q$, and $q \in \mathbb{N}$, only finite amount of remainders are possible when dividing by q. Hence, at some point $p \in \mathbb{N}$, $r_n = r_p$ for a previous $n \in \mathbb{N}$. This means that, $10^{k_{p+1}}p = z_{p+1}q + r_{p+1} \implies \frac{r_p}{q} = \frac{z_{p+1}}{10^{k^{p+1}}} + \frac{r_{p+1}}{10^{k_{p+1}}q}$

$$\implies \frac{r_n}{q} = \frac{r_p}{q} \implies z_{n+1} = z_{p+1}$$

Hence, we can see that it is recurring. ■

Lemma 3.42

Given $p \geq 2$ and $a_n \leq p-1 : a_n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to some x in $[0, 1]$.

Proof for Lemma

We note that this is a monotone increasing sequence. Note that, since $a_n \leq p-1$, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq (p-1) \sum_{n=1}^{n} \frac{1}{p^n}$$

which is 1. Therefore, this sequence is bounded, and from monotone convergence theorem, is convergent to $x \in [0, 1]$. ■

Lemma 3.43

Conversely, if x is a number in $[0,1]$ and p is an integer ≥ 2, we have that, there exists $a_1, a_2, \cdots \in \mathbb{N}$ that are less than $p-1$ so that x can be written as the limit of

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

Proof for Lemma

The construction is very similar to the decimal expansion proof. Take a_1 to be the largest integer such that $\frac{a_1}{p} < x$. Find a_2 so that $\frac{a_1}{p} + \frac{a_2}{p^2} < x$. These can be achieved by Archimedean property. Note that $a_1 \leq p-1$, as is a_2. Inductively define a_n as the largest integer $a_n \leq p-1$ such that $\sum_{i=1}^{n} \frac{a_i}{p^i} < x$. Now, the decimal expansions proof proves that x is the supremum of the set of all

$$\sum_{i=1}^{n} \frac{a_i}{p^i}$$

. An alternate approach is to see that, by asserting that a_n is the *largest such* integer following the above property, we note that:

$$\sum_{i=1}^{n} \frac{a_i}{p^i} < x \leq \sum_{i=1}^{n} \frac{a_i}{p^i} + \frac{1}{p^n}$$

Taking limit on both sides proves the result. ■

4 Misc Results

Theorem 4.1: (A perhaps useful theorem for LimSup LimInf)

If $A + B := \{a + b : a \in A, b \in B\}$, then $\gamma = sup(A+B) = sup(A) + sup(B) = \alpha + \beta$.

Proof for Theorem.

$\alpha \geq a, \forall a \in A$ and $\beta \geq b, \forall b \in B$. We hence see that $\alpha + \beta \geq a + b \forall a \in A, b \in B$. Therefore, $\alpha + \beta \geq \gamma$.

Now consider $a + b \leq \gamma \ \forall a \in A, b \in B$. This means that $\alpha + b \leq \gamma \ \forall b \in B$. From here we see that $\alpha + \beta \leq \gamma$ whence we see that $\gamma = \alpha + \beta$

■

Theorem 4.2

The set of all $0,1-$sequences (i.e, sequences containing only 1 or 2) are atleast as large as $\mathbb{R}$ (**p-adic proof**)

Proof for Theorem.

Consider a point x in $[0,1]$. By virtue of the above decmial expansion results, we can find a sequence a_n consisting of just 0 and 1 so that $\sum_{i=1}^{\infty} \frac{a_n}{2^n}$ converges to x. For every $0,1$ sequence, we can map a unique number in $[0,1]$ so that that sequence becomes the $2-$adic or binary representation of that number. This is a well defined surjection from the set of all $0,1$ sequences to $[0,1]$. ■

CHAPTER 2

INTRODUCTION TO SEQUENCES AND SERIES OF REAL NUMBERS

Covers some of the elementary results regarding sequences and series, more of which will be explored after the section on metric spaces.

1 On Sequences (Introduction)

Definition 1.1: (Sequence of Real numbers)

$X := (x_n : n \in \mathbb{N})$ is a function $x : \mathbb{N} \to \mathbb{R}$. The mapping from N allows us natural ordering.

Definition 1.2: (Limit of a sequence)

A sequence (x_n) in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$ if:

$$(\forall \varepsilon > 0)(\exists\ n_0 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_0)(|x_n - x| < \varepsilon)$$

Whose negation reads: A sequence is *not convergent* to $x \in \mathbb{R}$ if:

$$(\exists \varepsilon_0 > 0)(\forall\ k \in \mathbb{N})(\exists n_k \in \mathbb{N}, n_k \geq k)(|x_n - x| \geq \varepsilon_0)$$

Theorem 1.3: Uniqueness of Limits

If $x_n \to x$, then its limit is unique.

Proof for Theorem.

Suppose two limits exist, x and x'.

$$(\forall \varepsilon > 0)(\exists\ n_1 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_1)(|x_n - x| < \frac{\varepsilon}{2})$$

$$(\forall \varepsilon > 0)(\exists\ n_2 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_2)(|x_n - x'| < \frac{\varepsilon}{2})$$

Choosing $j = maxn_1, n_2$ we have:

$$(\forall \varepsilon > 0)(\exists\ j \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_0)(|x_n - x| < \frac{\varepsilon}{2} \& |x_n - x'| < \frac{\varepsilon}{2}) \implies$$

$$(\forall \varepsilon > 0)(\exists\ j \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_0)(|x - x'| < \varepsilon)$$

From "The Lemma", $x = x'$

■

Theorem 1.4

Convergence $\implies$ Boundedness

Proof for Theorem.

$$(\forall \varepsilon > 0)(\exists\ n_1 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_1)(-\varepsilon < x_n - x < \varepsilon)$$

Fix $\varepsilon = 1$, and let the corresponding n we get, be n_1. We see that for $n \geq n_1$, the set is bounded. For the numbers x_1 to $x_{n_1 - 1}$, by virtue of being a finite set, it is readily bounded. Hence, the whole sequence is bounded. ■

Fact 1.5

Elementary Results:

1. If $\{x_n\}$ such that $x_n \geq 0 \forall n \in \mathbb{N}$, then, if limit exists, limit(x_n)≥ 0.
2. Given $\{x_n\}$ and $\{y_n\}$ such that $y_n > x_n \forall n \in \mathbb{N}$, then $lim(y_n) \geq lim(x_n)$
3. If $x_n \to x$ and $a \leq x_n \leq b$, then $a \leq x \leq b$.

Theorem 1.6: Squeeze Play

Given sequences x_n, y_n, z_n such that $\forall n \geq n_l, y_n \leq x_n \leq z_n$, and $z_n \to a, y_n \to a$, then $x_n \to a$.

Proof for Theorem.

$$(\forall \varepsilon > 0)(\exists\ n_y \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_y)(a - \varepsilon < y_n \leq x_n)$$

$$(\forall \varepsilon > 0)(\exists\ n_z \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_z)(x_n \geq a + \varepsilon)$$

Combining the two and setting $j = max\{n_y, n_z, n_l\}$ we get

$$(\forall \varepsilon > 0)(\exists\ j \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq j)(a - \varepsilon < x_n < a + \varepsilon)$$

■

Definition 1.7: (Unbounded sequence)

A sequence x_n is unbounded if it is neither bounded below, nor bounded above. It is not bounded above if:

$$\forall \xi \in \mathbb{R}, \exists n_k, x_{n_k} > \xi$$

It is not bounded below if:

$$\forall \xi \in \mathbb{R}, \exists n_k, x_{n_k} < \xi$$

Definition 1.8: (Divergence to infinity)

A sequence is said to diverge to +infinity if $\forall \xi \in \mathbb{R}, \exists n_0$ such that $x_n > \xi \forall n \geq n_0$. It is divergent to -infinity if $\forall \xi \in \mathbb{R}, \exists n_0$ such that $x_n < \xi \forall n \geq n_0$.

Theorem 1.9: Multiplication and division of sequences

Multiplication of sequences $x_n \to x$ and $y_n \to y$ gives a sequence $x_n y_n$ that converges to xy.
If $x_n \to x$ and $y_n \to y$ with $y_n \neq 0 \forall n \in \mathbb{N}$ and $y \neq 0$, we have: $\frac{x_n}{y_n} \to \frac{x}{y}$

Proof for Theorem.

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0$ we have $|x_n - x| < \varepsilon$. Similarly, $\forall \varepsilon > 0$, $\exists n_1 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_1$ we have $|y_n - y| < \varepsilon$. Consider $|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy| \leq |y_n||(x_n - x)| + |(x)||(y_n - y)|$. For $\varepsilon = 1$ we have n_{y1} such that $\forall n \geq n_{y1}$, $y - 1 < y_n < y + 1$. And obviously, for $n < n_{y1}$, there exists maxima M. Since y_n is bounded, forall $n \geq n_{y1}$, we have $|y_n||x_n - x| + |x||y_n - y| \leq (Max)|x_n - x| + |x||y_n - y|$. This means that, $\forall \varepsilon > 0, \exists n_0$ such that $|x_n y_n - xy| \leq |y_n||x_n - x| + |x||y_n - y| \leq (Max)|x_n - x| + |x||y_n - y| \leq Max(\varepsilon) + |x|(\varepsilon)$. Hence, we are done.

$$(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_\varepsilon)(|x_n - x| < \varepsilon)$$

$$(\forall \varepsilon > 0)(\exists n'_\varepsilon \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n'_\varepsilon)(|y_n - y| < \varepsilon)$$

Choose $\varepsilon = 1$. After some $k_1 \in \mathbb{N}$, we have $y - 1 < y_n$, which implies $\{|y_n|\}$ has a lower bound. Call this L. Cosider the difference $|\frac{x_n}{y_n} - \frac{x}{y}| = |\frac{x_n y - x y_n}{y y_n}| \leq \frac{|x||y_n - y| + |y||x_n - x|}{|y_n||y|} \implies$

$$\forall n \geq k_1, |\frac{x_n}{y_n} - \frac{x}{y}| \leq \frac{|x||y_n - y| + |y||x_n - x|}{|L||y|}$$

$$\forall \varepsilon, \exists n_i \geq n_0, n_1, k_1, \forall n \in \mathbb{N}, n \geq n_i, |\frac{x_n}{y_n} - \frac{x}{y}| < \frac{1}{|L||y|}|x|\varepsilon + |y|\varepsilon$$

Whence, we are done. ∎

Theorem 1.10: Some Results

1. if $a > 1$, then $a^n \to \infty$
2. if $a > 0$, then $a^{\frac{1}{n}} \to 1$

Proof for Theorem.

$a = 1 + \delta \implies a^n = 1 + n\delta + \frac{n(n-1)}{2}\delta^2 \cdots > 1 + n\delta$. This implies a^n diverges to infinity. Given $a > 0$, if $a > 1$, $a^{1/n} > 1$. We have $a^{1/n} = 1 + \delta_n$. $a = (1+\delta_n)^n = 1 + n\delta_n + \frac{n(n-1)}{2}\delta_n^2 \cdots \implies \frac{a}{n} \leq \delta_n$. This means that δ_n converges to 0, which means that $a^{1/n}$ converges to 1.
Suppose $0 < a < 1$, then $\frac{1}{a} > 1$. Therefore $(\frac{1}{a})^{\frac{1}{n}} = (a^{\frac{1}{n}})^{-1}$ converges to 1. This implies $a^{\frac{1}{n}}$ converges to 1 aswell (from the previous theorem on division) ■

Theorem 1.11: Slick Theorem

Let $\{x_n\}$ be a given sequence and $\{a_n\}; a_n \geq 0$ be a sequence converging to 0. Suppose, also, that for some $C > 0$, we have

$$|x_n - x| \leq C a_n \forall n \geq n_0$$

, then the sequence x_n converges to x.

Proof for Theorem.

$\forall \varepsilon \exists n_1 : \forall n \in \mathbb{N}, n \geq n_1, C a_n < \varepsilon \implies |x_n - x| < \varepsilon \forall n \geq max\{n_0, n_1\}$ ■

Definition 1.12: (Monotone Sequence)

A sequence is said to be *monotone increasing* if $\forall n \in \mathbb{N}, x_n \geq x_{n-1}$. It would be "ultimately" monotone increasing if $\exists n_0 \in \mathbb{N} \forall n \in \mathbb{N}, n \geq n_0, x_n \geq x_{n-1}$

Likewise, a sequence is said to be *monotone decreasing* if $\forall n \in \mathbb{N}, x_n \leq x_{n-1}$. It would be "ultimately" monotone decreasing if $\exists n_0 \in \mathbb{N} \forall n \in \mathbb{N}, n \geq n_0, x_n \leq x_{n-1}$

A sequence is "monotone" if its either monotone increasing or decreasing.

Theorem 1.13: Monotone Convergence Theorem

A monotone sequence is convergent $\iff$ it is bounded

Proof for Theorem.

$\implies$) Every convergent sequence is bounded.
$\impliedby$) We take the case of monotone increasing sequence that is bounded above. $\exists M \in \mathbb{R}$ such that $x_n \leq M \forall n \in \mathbb{N}$. Consider the set $\{x_n : n \in \mathbb{N}\}$, which is bounded and non

empty. Let z be the supremum of this set. Consider an arbitrary $\varepsilon > 0$. $\exists x_{n_0}$ such that $z - \varepsilon \leq x_{n_0} \leq x_n \forall n \geq n_0$. This means: $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, n \in \mathbb{N}$ we have $z - \varepsilon < x_n < z + \varepsilon$. Hence, $x_n \to sup(\{x_n\})$ ■

Euler's Number

Theorem 1.14

Consider the sequence

$$e_n := (1 + \frac{1}{n})^n$$

This sequences is convergent, and $\lim(e_n)$:=e is called the Napier's Constant or Euler's Number.

Proof for Theorem.

$$\begin{aligned} e_n &= (1 + \frac{1}{n})^n = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2}(\frac{1}{n^2}) + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}\cdots \\ &= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \frac{1}{4!}(1 - \frac{1}{n})(1 - \frac{2}{n})(1 - \frac{3}{n})\cdots \end{aligned}$$

And e_n has $n+1$ terms from the binomial theorem.
Consider

$$e_{n+1} = (1 + \frac{1}{n+1})^{n+1} = 1 + 1 + \frac{(n+1)(n)}{2}\frac{1}{(n+1)^2} + \frac{(n+1)(n)(n-1)}{3!}\frac{1}{(n+1)^3}\cdots =$$

$$2 + \frac{1}{2!}(1 - \frac{1}{n+1}) + \frac{1}{3!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1}) + \frac{1}{4!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1})(1 - \frac{3}{n+1}) + \cdots$$

and e_{n+1} has $n+2$ terms from Binomial. Notice that every term in e_{n+1} is greater than (or equal to) every term of e_n, with there being more terms in e_{n+1}. Therefore, e_n is monotone increasing.

$$\begin{aligned} e_n &= 2 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \frac{1}{4!}(1 - \frac{1}{n})(1 - \frac{2}{n})(1 - \frac{3}{n})\cdots \\ &\leq 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\cdots \end{aligned}$$

Since for $n \geq 3$ we have $2^n \leq n!$, this means $\frac{1}{2^n} \geq \frac{1}{n!}$ and hence,

$$e_n \leq 2 + \frac{1}{2} + \frac{1}{2^2}\cdots \leq 2 + \frac{1}{1 - \frac{1}{2}} \leq 4$$

e_n is bounded, hence convergent. ■

Theorem 1.15: Three Beauties

1. (Ratio Test) Let $\{a_n\}$ such that $a_n > 0 \forall n \in \mathbb{N}$. Let $lim(\frac{a_{n+1}}{a_n}) = L$. If $L > 1$, then $lim(a_n) = \infty$. If $L < 1$, $lim(a_n) = 0$ (Test fails if $L = 1$, with the example of $a_n = n$)

2. (Average Convergence Theorem) If $\{a_n\} \to L$, then $\frac{a_1+a_2+\cdots+a_n}{n} \to L$

3. (Cauchy's 2nd) $a_n > 0$, then $lim(a_n)^{\frac{1}{n}} = lim(\frac{a_{n+1}}{a_n})$ provided either $\frac{a_{n+1}}{a_n}$ converges or properly diverges.

Proof for Theorem.

1) Let $\frac{a_{n+1}}{a_n}$ converge to L.

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N}, n \geq n_0, L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon$$

This means that:

$$L - \varepsilon < \frac{a_{n_0+1}}{a_{n_0}} < L + \varepsilon$$
$$L - \varepsilon < \frac{a_{n_0+2}}{a_{n_0+1}} < L + \varepsilon$$
$$\vdots$$
$$L - \varepsilon < \frac{a_m}{a_{m-1}} < L + \varepsilon$$

Multiplying throughout we have:

$$a_{n_0}(L - \varepsilon)^{m-n_0} < a_m < a_{n_0}(L + \varepsilon)^{m-n_0} \qquad ((x))$$

If $L < 1$ choose a number ε such that $L+\varepsilon < 1$. Therefore, there exists a corresponding n_k such that

$$a_{n_k}(L - \varepsilon)^{m-n_k} < a_m < a_{n_k}(L + \varepsilon)^{m-n_k} < a_{n_k}(z)^{m-n_k} : \forall m \geq n_k, n_0$$

where $z < 1$. Therefore, $a_{n_k}(z)^{m-n_k}$ converges to 0. From squeeze play $a_m \to 0$.
If $L > 1$, choose a number ε such that $L - \varepsilon > 1$. Therefore, there exists a corresponding n_l and a number v such that

$$a_{n_l}(v)^{m-n_l} < a_{n_l}(L - \varepsilon)^{m-n_l} < a_m < a_{n_l}(L + \varepsilon)^{m-n_l} : \forall m \geq n_k, n_0$$

where $v > 1$. Hence, we see that a_m is properly divergent.

2)$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0, L - \varepsilon < a_n < L + \varepsilon \implies$

$$L - \varepsilon < a_{n_0+1} < L + \varepsilon$$

$$L - \varepsilon < a_{n_0+2} < L + \varepsilon$$

$$\vdots$$

$$L - \varepsilon < a_n < L + \varepsilon$$

Adding all these we get:

$$(n - n_0)(L - \varepsilon) < a_{n_0+1} + a_{n_0+2} + \cdots + a_n < (n - n_0)(L + \varepsilon)$$

Consider the set $\{a_1, a_2, \ldots, a_{n_0}\}$, this is a finite set, hence, it has a maximum and a minimum M and m respectively, which means $\forall n \leq n_0, m \leq a_n \leq M$. Therefore, for every $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ and a maxima and minima m and M so that

$$(n_0)m + (n - n_0)(L - \varepsilon) < a_1 + a_2 + \cdots + a_n < (n - n_0)(L + \varepsilon) + (n_0)M \implies$$

$$\mathfrak{L}(\varepsilon, n) = \frac{n_0 m}{n} + \frac{(n - n_0)}{n}(L - \varepsilon) < \frac{a_1 + a_2 \cdots + a_n}{n} < \frac{n_0 M}{n} + \frac{(n - n_0)}{n}(L + \varepsilon) = \mathfrak{U}(\varepsilon, n)$$

It is clear that $\mathfrak{L}(\varepsilon, n)$ and $\mathfrak{U}(\varepsilon, n)$ converge to $L - \varepsilon$ and $L + \varepsilon$ respectively. Therefore:

$$(\forall \varepsilon')(\exists j_l \in \mathbb{N})(\forall n \in \mathbb{N} : n \geq j_l)(L - \varepsilon - \varepsilon' < \mathfrak{L}(\varepsilon, n) \leq \frac{a_1 + a_2 + \cdots_a + n}{n})$$

and

$$(\forall \varepsilon')(\exists j_u \in \mathbb{N})(\forall n \in \mathbb{N} : n \geq j_u)(\frac{a_1 + a_2 + \cdots_a + n}{n} \leq \mathfrak{U}(\varepsilon, n) < L + \varepsilon + \varepsilon')$$

Whence we see that $\forall \varepsilon \forall \varepsilon$ there is some $n_p \geq j_l, j_u, n_0$ such that $\forall n \in \mathbb{N} : n \geq n_p$,

$$L - \varepsilon - \varepsilon' < \frac{a_1 + a_2 + \cdots_a + n}{n} < L + \varepsilon + \varepsilon'$$

Therefore, $\frac{a_1 + a_2 \cdots + a_n}{n}$ converges to L

3)From equation (x), we have that $\forall \varepsilon, \exists n_0(\varepsilon)$ such that

$$a_{n_0}(L - \varepsilon)^{m - n_0} < a_m < a_{n_0}(L + \varepsilon)^{m - n_0} : \forall m \geq n_0(\varepsilon)$$

Either $(L - \varepsilon)$ is positive, whence it is possible to take $m - th$ root which gives:

$$a_{n_0}^{\frac{1}{m}}(L - \varepsilon)^{1 - \frac{n_0}{m}} < a_m^{\frac{1}{m}} < a_{n_0}^{\frac{1}{m}}(L + \varepsilon)^{1 - \frac{n_0}{m}}$$

Or it is negative, where it is pretty obvious that

$$a_{n_0}^{\frac{1}{m}}(L - \varepsilon) < a_m^{\frac{1}{m}} < a_{n_0}^{\frac{1}{m}}(L + \varepsilon)^{1 - \frac{n_0}{m}}$$

Hence, $\forall \varepsilon > 0$, $\exists n_0(\varepsilon) \in \mathbb{N}$ such that $\forall m \geq n_0(\varepsilon)$

$$\begin{cases} a_{n_0}^{\frac{1}{m}}(L - \varepsilon)^{1 - \frac{n_0}{m}} < a_m^{\frac{1}{m}} < a_{n_0}^{\frac{1}{m}}(L + \varepsilon)^{1 - \frac{n_0}{m}} & \text{if } L - \varepsilon > 0 \\ a_{n_0}^{\frac{1}{m}}(L - \varepsilon) < a_m^{\frac{1}{m}} < a_{n_0}^{\frac{1}{m}}(L + \varepsilon)^{1 - \frac{n_0}{m}} & \text{if } L - \varepsilon < 0 \end{cases}$$

Call $a_{n_0}^{\frac{1}{m}}(L+\varepsilon)^{1-\frac{n_0}{m}} = \mathfrak{U}(\varepsilon, m)$
Call $a_{n_0}^{\frac{1}{m}}(L-\varepsilon)^{1-\frac{n_0}{m}} = \mathfrak{L}(\varepsilon, m)$

It is clear to see that both $\mathfrak{U}(\varepsilon, m)$ and $\mathfrak{L}(\varepsilon, m)$ converge, and do so to $(L+\varepsilon)$ and $(L-\varepsilon)$ respectively. Therefore: $\forall \varepsilon' > 0\ \exists j \in \mathbb{N}$ such that

$$L - \varepsilon - \varepsilon' < \mathfrak{L}(\varepsilon, n) \forall n \geq j$$

and

$$\mathfrak{U}(\varepsilon, n) < L + \varepsilon + \varepsilon' \forall n \geq j \implies$$

$$L - \varepsilon - \varepsilon' < a_m^{\frac{1}{m}} < L + \varepsilon + \varepsilon' : \forall m \geq t = \max\{j, n_0\}$$

This means that $\forall \varepsilon \forall \varepsilon' > 0$, $\exists t \in \mathbb{N}$ such that $\forall m \in \mathbb{N}, m \geq t$ we have:

$$L - \varepsilon - \varepsilon' < a_m^{\frac{1}{m}} < L + \varepsilon + \varepsilon'$$

The same argument can be applied for $\varepsilon' s$ where $(L-\varepsilon)$ is negative. Hence, $a_m^{\frac{1}{m}} \to L$
Suppose that $\frac{a_{m+1}}{a_m} \to \infty$, then $\frac{a_m}{a_{m+1}} \to 0$. Therefore, applying the previous result to this, we have $(\frac{1}{a_m})^{\frac{1}{m}} \to 0 \implies (a_m)^{\frac{1}{m}} \to \infty$ (Proof left as an exercise to the reader) ■

Definition 1.16: (Subsequences)

Given a sequence $\{x_n\}$, we define the subsequence $\{x_{n_k}\}$ as the sequence within $\{x_n\}$ generated through the increasing sequence of natural numbers $n_1 < n_2 < n_3 < \ldots$ with $k \geq n_k$

.

Theorem 1.17

Given $x_n \to x$, then all subsequences of x_n converge to x.

Proof for Theorem.

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N} : n \geq n_0)(|x_n - x| < \varepsilon) \implies$$

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n_k \in \mathbb{N} : n_k \geq k \geq n_0)(|x_{n_k} - x| < \varepsilon)$$

■

Theorem 1.18

Let $A \subseteq \mathbb{R}$ be an infinite subset that is bounded, and non empty, with supremum S. Then, $\exists \{x_n\}$ in A such that $x_n \to S$, with x_n being monotone increasing.

Proof for Theorem.

Choose $\delta_1 = 1$. There would exist some $x_1 \in A$ such that $S - 1 \leq x_1 < S$. Choose $\delta_2 = \frac{d(x_1,S)}{2}$ where $d(x_1, S)$ is the Euclidean distance from S to x_1. There would exist, again, some x_2 such that $S - \delta_2 = S - \frac{d(x_1,S)}{2} \leq x_2 < S$. It is easy to see here that $x_1 \leq x_2$. Having found x_n using $\delta_n = \frac{d(x_{n-1},S)}{n}$, now choose x_{n+1} using $\delta_{n+1} = \frac{d(x_n,S)}{n+1}$. Via this compression, we see that the sequence converges to S through squeeze play. Moreover, by construction this sequence is increasing. ■

Theorem 1.19: Equivalent statements pertaining to Divergence

1. $X_n \not\to x$
2. $\exists \varepsilon > 0$ such that $\forall k \in \mathbb{N}$ $\exists n_k \geq k$ $|x_{n_k} - x| \geq \varepsilon$
3. $\exists \varepsilon > 0$ and a subsequence x_{n_k} such that $\forall k \in \mathbb{N}$ $|x_{n_k} - x| \geq \varepsilon$

Proof for Theorem.

(1) $\implies$ (2)) Comes directly from the negation of convergence, with tweaks to the notation.
(2) $\implies$ (3)) For $k = 1$, $\exists n_1$ such that $|x_{n_1} - x| \geq \varepsilon$, likewise, $fork = j$, $\exists n_j$ such that $|x_{n_j} - x| \geq \varepsilon$. Note that $n_k \geq k$ which means that x_{n_k} would form a subsequence of x_n.
(3) $\implies$ (1)) If x_n on the contrary, converged to x, then all its subsequences converge to x as well, but for the subsequence x_{n_k} from (2), we have that $\exists \varepsilon > 0$ such that $\forall k \in \mathbb{N}$ $\exists n_k \in \mathbb{N}, n_k \geq k$ such that $|x_{n_k} - x| \geq \varepsilon$ which is the definition of divergence. Absurd. ■

Theorem 1.20: Monotone Subsequence Theorem

Every sequence x_n has a monotone subsequence

Proof for Theorem.

First Proof:

Call a point x_n a "peak" if $x_n \geq x_m \forall m \geq n$. i.e, it is larger than all the terms that come after it. Consider the case where there are finite peak points. List them as $x_{n_1}, x_{n_2}, x_{n_3} ....x_{n_k}$. The term x_{n_k+1} is not a peak, which means after $n_k + 1$-th term, there exists another term such that it is larger than this one. Since that term is not a peak either, there must exist another term with an index larger than both the previous terms, such that it is larger than both. As such, keep choosing such non peak terms to generate a monotone increasing sequence. If there are infinite peak points, $x_{n_1}, x_{n_2} \cdots x_{n_k} \cdots$, simply set the sequence to be these peak points. This would be a monotone decreasing sequence

Alternate Proof:

Case 1: The sequence is unbounded:
Call $S_0 := \{x_1, x_2, \cdots x_n \cdots\}$. For $\varepsilon = 1, \exists x_{n_0}$ such that $x_{n_0} \geq 1$. Consider the set $S_1 :=$

$S_0 - \{x_1, x_2, \cdots, x_{n_0}\}$. This is still unbounded, hence, choose $\varepsilon = x_{n_0}$ $\exists x_{n_1} \geq x_{n_0}$ with $n_1 > n_2$. Having chosen x_{n_k}, choose $x_{n_{k+1}}$ by taking the set $S_{k+1} := S_0 - \{x_1, x_2, \cdots x_{n_k}\}$. This set is unbounded. Therefore, for $\varepsilon_{k+1} = x_{n_k}$, $\exists x_{n_{k+1}} \geq x_{n_k}$ with $n_{k+1} > n_k$. This forms a subsequence that is monotonic increasing.

Case 2: The sequence is bounded:

Consider the set S_k to be defined as $S_k := \{x_n : n \geq m\} = \{x_m, x_{m+1}, x_{m+2}, \cdots, x_n \cdots\}$ and define the supremum's of each of these sets as $U_k = sup(S_k)$ (easy to see that they exist) Notice that $U_{k+1} \leq U_k$.

If only finite sets $S_{n_1}, S_{n_2} \cdots S_{n_j}$ has its own supremum, then for the sets $\{S_{n_j+1}, S_{n_j+2} \cdots\}$, the supremum of these sets are not within themselves. This would mean that all the sets $\{S_{n_j+1}, S_{n_j+2} \cdots\}$ contain the same supremum. To see this, suppose that S_{n_j+1} and S_{n_j+2} have different supremums. The fact is that these sets differ by just one element x_{n_j}. If, due to removing this one element, the supremum changes, that would imply that element is the supremum. But since supremums don't exist in these sets, we conclude they share the same supremum U. We deal with the set $S_{n_j+1} = \{x_{n_j+1}, x_{n_j+2}, \cdots\}$ whose supremum is U, lying outside the set. For $\varepsilon = 1$, $\exists x_{a_1} \in S_{n_j+1}$ such that $U - 1 \leq x_{a_1} < U$. Take the set $S_{a_1+1} := \{x_{a_1+1}, x_{a_1+2} \cdots\}$ Whose supremum is also U. Choose $\varepsilon = U - x_{a_1}$. We have then $U - \varepsilon = U - (U - x_{a_1} \leq x_{a_2} < U$. Having found x_{a_k}, consider the set $S_{a_k+1} = \{x_{a_k+1}, x_{a_k+2} \cdots\}$ whose supremum is U. Now we take ε to be $u - x_{a_k}$ which would imply $\exists x_{a_{k+1}} \in S_{a_k+1}$ such that $U - \varepsilon = U - (U - x_{a_k}) \leq x_{a_{k+1}} < U$. Hence we see that $x_{a_k} \leq x_{a_{k+1}} \forall k \in \mathbb{N}$ and by construction, $a_k < a_{k+1}$. This is then, a monotone increasing subsequence.

If there are infinitely many sets S_{k_n} that contain their own supremum, then simply create a sequence of these supremums $U_{k_1}, U_{k_2}, \cdots$. This is obviously monotone decreasing, and it is a subsequence since, by construction, $k_n \leq k_{n+1}$. ■

Theorem 1.21: Bolzano Weierstrass

Every bounded sequence has a convergent subsequence.

Proof for Theorem.

From monotone subsequence theorem, every sequence has a monotone subsequence. If the main sequence is bounded, every subsequence is bounded. Hence, this monotone sequence is bounded, hence, convergent. ■

Theorem 1.22

If x_n is a bounded sequence such that every convergent subsequence converges to x, then the main sequence converges to x.

Proof for Theorem.

Suppose that the main sequence *does not* converge to x, which means that there exists $\varepsilon > 0 \in \mathbb{R}$ and a subsequce x_{n_k} such that $\forall k \in \mathbb{N}$, $|x_{n_k} - x| \geq \varepsilon$. This subsequence is bounded, hence it has a sub-subsequence $x_{n_{k_j}}$ that is convergent. This sub-subsequence converges to x. But this raises a contradiction since for a particular ε, every term in this subsequence, and by extention, the sub-subsequence, falls outside the ε neighbourhood of x. ■

Definition 1.23: (LimSup and LimInf)

Given a sequence that is bounded (hence forth, all theorems involving limsup and liminf assumes a bounded sequence as given):

1. $Limsup(x_n) := inf(V := \{v \in \mathbb{R} : \exists n_v \in \mathbb{N} \text{ such that } \forall n \geq n_v, x_n \leq v\})$
2. $Liminf(x_n) := sup(U = \{u \in \mathbb{R} : \exists n_u \in \mathbb{N} \text{ such that } \forall n \geq n_u, x_n \geq v\})$

Theorem 1.24

The following are equivalent:

1. x is the LimSup(x_n)
2. if $\varepsilon > 0$, then $\exists$ utmost finite $n \in \mathbb{N}$ such that $x + \varepsilon < x_n$ but infinite $n \in \mathbb{N}$ such that $x - \varepsilon < x_n$. This implies $x + \varepsilon \in V$ but $x - \varepsilon \notin V$
3. If $S_m : \{x_m, x_{m+1}, \cdots\}$ and $U_m = sup(S_m)$, ,then $lim(U_m) = inf(U_m) = x$
4. If S is the set of all subsequential limits of x_n, then $sup(S) = x$

Proof for Theorem.

(1) $\implies$ (2)) Since x is the infimum of V, $\forall \varepsilon > 0$, $\exists z \in V$ such that $z \leq x + \varepsilon$. We see that, $\exists n_z$ such that $\forall n \geq n_z, x_n < z \leq x + \varepsilon$. Hence, $x + \varepsilon \in V$, or rather, there exists utmost finite n such that $x_n > x + \varepsilon$. $x - \varepsilon$ cannot belong in V since x is the infimum, therefore, $\forall k \in \mathbb{N}, \exists n_k \geq k$ such that $x_{n_k} > x - \varepsilon$, or rather, there would exist infinite n such that $x - \varepsilon < x_n$.

(2) $\implies$ (3)) We know that $U_m \geq U_{m+1}$, is a monotone decreasing sequence that is bounded below. Hence, from monotone convergence theorem, we have $lim(U_m) = inf(\{U_m\})$. From (2), we know that $\forall \varepsilon > 0$, $\exists n_\varepsilon$ such that $\forall n \geq n_z$ we have $x_n \leq x + \varepsilon$. Therefore, $U_{n_\varepsilon} \leq x+\varepsilon$. Hence, $\forall \varepsilon > 0, inf(U_n) = lim(U_n) \leq x+\varepsilon$. There exists infinite x_n such that $x - \varepsilon < x_n$ which means that $x - \varepsilon < U_n \forall n \in \mathbb{N}$. This implies $x - \varepsilon \leq inf(U_n)$. Therefore means $\forall \varepsilon > -, |inf(U_n) - x| \leq \varepsilon$. From the lemma, $inf(U_n) = x$.

(3) $\implies$ (4)) Since $inf(U_n) = x$, $\forall \varepsilon, \exists n_0(\varepsilon) \in \mathbb{N}$ such that $U_{n_0(\varepsilon)} \leq x + \varepsilon \implies \forall n \geq n_0(\varepsilon), x_n \leq x + \varepsilon$ so for every convergent subsequence, x_{n_k}, $lim(x_{n_k}) \leq x + \varepsilon$. Since the set of all subsequential limits is bounded (and non empty from Bolzano Weierstrass Theorem), $sup(S = \text{set of all subsequential limits}) \leq x + \varepsilon$. $\forall \varepsilon > 0, x - \varepsilon < inf(U_n) \implies \forall \varepsilon \forall n \in \mathbb{N}, x - \varepsilon < U_n$.
Choose $\varepsilon = 1$ and the set S_1, for which $\exists x_{n_1} \in S_1$ such that $U_1 - 1 \leq x_{n_1} < U_1$. Choose $\varepsilon = \frac{1}{2}$ and the set S_{n_1} for which $\exists x_{n_2} \in S_{n_1}$ such that $U_{n_1} - \frac{1}{2} \leq x_{n_2} < U_{n_1}$. From

construction, $n_2 > n_1$. Having chosen $\varepsilon = \frac{1}{j}$ and the set $S_{n_{j-1}}$ and obtaining n_j such that $\exists x_{n_j} \in S_{n_{j-1}}$ so that $U_{n_{j-1}} - \frac{1}{j} \leq x_{n_j} < U_{n_{j-1}}$ such that $n_j > n_{j-1}$, we choose $\varepsilon = \frac{1}{j+1}$ and the set S_{n_j}. The construction continues and we create a sequence x_{n_j} which from squeeze play, converges to x.To see this, we have that $\forall j \in \mathbb{N}$

$$U_{n_{j-1}} - \frac{1}{j} \leq x_{n_j} < U_{n_{j-1}}$$

Taking limit on both LHS and RHS we see that x_{n_j} converges to x. Therefore x itself is a subsequential limit, which means $x \leq sup(S)$. We already had $\forall \varepsilon > 0, sup(S) \leq x + \varepsilon$, which gives, $\forall \varepsilon > 0, x \leq sup(S) \leq x + \varepsilon$, which means $sup(S) = x$.

(4) $\implies$ (1)) Consider the set $V := \{v \in \mathbb{R} : \exists n_v \in \mathbb{N}$ such that $\forall n \geq n_v, x_n \leq v\}$. if $z \in V$, it means that every subsequential limit of x_n goes below z. Therefore, $sup(S) = x \leq z \forall z \in V$. This means $x \leq limsup(x_n)$. Suppose $sup(S) = x < limsup(x_n)$. This means $Sup(S) + \delta = limsup(x_n)$ or $x = limsup(x_n) - \delta$. $limsup(x_n)$ is an upper bound to the set of all subsequential limits S. Consider an arbitrary subsequential limit y. $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_\varepsilon$ we have $y - \varepsilon < x_n < y + \varepsilon < limsup(x_n) - \delta + \varepsilon$. Choose an ε slightly larger than δ, which would make $limsup(x_n) - (\delta - \varepsilon)$ slightly smaller than $limsup(x_n)$. This gives: for the chosen $\varepsilon \exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_\varepsilon$ we have $x_n < limsup(x_n) - \delta + \varepsilon < limsup(x_n)$. This means that a number slightly smaller than $inf(V)$ exists in V. This is absurd. Hence, $limsup(x_n) = sup(S =$ set of all subsequential limits $)$.

■

Theorem 1.25

The following are equivalent:

1. y is the Liminf(x_n)
2. if $\varepsilon > 0$, then $\exists$ utmost finite $n \in \mathbb{N}$ such that $x_n < y - \varepsilon$ but infinite $n \in \mathbb{N}$ such that $x_n < y + \varepsilon$. This implies $y + \varepsilon \notin U$ but $y - \varepsilon \in U$
3. If $S_m : \{x_m, x_{m+1}, \cdots\}$ and $L_m = inf(S_m)$, ,then $lim(L_m) = sup(L_m) = y$
4. If S is the set of all subsequential limits of x_n, then $inf(S) = y$

Proof for Theorem.

(1) $\implies$ (2)) Given $y = Liminf(x_n) := sup(U = \{u \in \mathbb{R} : \exists n_u \in \mathbb{N}$ such that $\forall n \geq n_u, x_n \geq v\})$. If $\varepsilon > 0$ we have a $z \in U$ such that $y - \varepsilon \leq z$. There exists only finite n such that $x_n < z$ which means there exists only finite n such that $x_n < y - \varepsilon$. Therefore, $y - \varepsilon \in U$. Consider $y + \varepsilon$. Since $y + \varepsilon \notin U$, we have that $\forall k \in \mathbb{N}\ \exists n_k \geq k$ such that $x_n < y + \varepsilon$ or infinite n_k such that x_{n_k} such that $x_{n_k} < y + \varepsilon$.

(2) $\implies$ (3)) We can see that, if $S_m := \{x_n : n \geq m\}$, and $L_m := inf(S_m)$, then $L_m \leq L_{m-1}$, which is a monotone increasing sequence, which is bounded, hence is convergent to $sup(\{L_m\}) = lim(L_m)$. Since from (2), $\exists$ infinite n_k such that $x_n < y + \varepsilon$, we see that $\forall m, inf(S_m) = L_m \leq y + \varepsilon \implies lim(L_m) \leq y + \varepsilon \forall \varepsilon > 0$. Since there exists only finite n such that $x_n < y - \varepsilon \implies \forall n \geq n_y(\varepsilon), x_n \geq y - \varepsilon$. This means that $y - \varepsilon \leq L_{n_y} \implies y - \varepsilon \leq lim(L_m) = sup(L_m)$. Hence $\forall \varepsilon > 0$, $y - \varepsilon \leq lim(L_m) \leq y + \varepsilon$, hence $y = lim(L_m)$.

(3) $\implies$ (4)) Sunce $y = sup(L_m) = lim(L_m)$. For an $\varepsilon > 0$, we have an L_{n_1} such that $y - \varepsilon \leq L_{n_l}$. Since L_{n_1} is the infimum of S_{n_1}, we have $y - \varepsilon < x_n, \forall n \geq n_l$. This would mean that every subsequence converges to a point larger than $y - \varepsilon$. Therefore $y - \varepsilon < t\ \forall t \in S$ where S is the set of all subsequential limits (This set is non empty from Bolzano Weierstrass, and is bounded, hence has a supremum and infinmum). Hence $\forall \varepsilon > 0$, $y - \varepsilon \leq inf(S)$. $y + \varepsilon$ is an upper bound for $\{L_m : m \in \mathbb{N}\}$. Choose $\varepsilon = 1$ and the set S_1. $L_1 + 1 \geq L_1$. Since L_1 is infimum of S_1, then $\exists x_{n_1} \in S_1$ such that $L_1 \leq x_{n_1} \leq L_1 + 1$. Choose $\varepsilon = \frac{1}{2}$, and the set S_{n_1}. $\exists x_{n_2} \in S_{n_1}$ such that $L_{n_1} \leq x_{n_2} \leq L_{n_1} + \frac{1}{2}$. Having chosen $\varepsilon = \frac{1}{j}$ and the set $S_{n_{j-1}}$, we have an $x_{n_j} \in S_{n_{j-1}}$ such that

$$L_{n_{j-1}} \leq x_{n_j} \leq L_{n_{j-1}} + \frac{1}{j}$$

Notice that, by construction of our sets, $n_j > n_{j-1}$. Hence, we have a subsequence of x_n which is x_{n_j} which, from squeeze play theorem, converges to y. Therefore, $y \in S$, which means $inf(S) \leq y$. We therefore have $\forall \varepsilon > 0$, $y - \varepsilon \leq inf(S) \leq y$. This means $inf(S) = y$.

(4) $\implies$ (1)) Given y is the infimum of the set of all subsequential limits. Say $\alpha =$

$Liminf(x_n) := sup(U = \{u \in \mathbb{R} : \exists n_u \in \mathbb{N}$ such that $\forall n \geq n_u, x_n \geq v\})$. If $z \in U$, it means that after some n_z, every $x_n \geq z$ which means every subsequence converges above z. Therefore, z is a lowerbound for the set of all subsequential limits S. $z \leq inf(S) = y, \forall z \in U$. We can see that $sup(U) = \alpha \leq y$ from this. Suppose $Sup(U) = \alpha < y \implies \alpha = y - \delta$ for some δ. Consider an arbitrary subsequential limit q. $\forall \varepsilon, \exists n_q(\varepsilon) \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_q(\varepsilon)$ we have $y - \varepsilon + \delta - \delta \leq q - \varepsilon < x_n < q + \varepsilon$. $(y - \delta) - (\varepsilon - \delta) = \alpha + (\delta - \varepsilon) \leq q - \varepsilon < x_n < q + \varepsilon$. This means that $\exists n_0$ such that $\forall n \geq n_0$, $\alpha + (\delta - \varepsilon) < x_n$. This means that, if we choose ε smaller than δ, we would have a number larger than $sup(U) = \alpha$ being inside U. Absurd. Hence $\alpha = y$. ■

Theorem 1.26

A bounded sequence is convergent if and only if $limsup(x_n) = liminf(x_n)$

Proof for Theorem.

$\implies$) If a bounded sequence is convergent, all its subsequences converge to the same limit x. Therefore x is both the supremum and the infimum of the set of all subsequential limits, which is also the limsup and liminf.

$\impliedby$) If limsup=liminf, then the set of all subsequential limits has infimum and the supremum equal. This means that the set of all subsequential limits is singleton, with $x \in S$. If x_n is bounded and all its convergent subsequences converge to x, then x_n converges to x. ■

Theorem 1.27: Shuffle Lemma

If x_n and y_n are sequences in $\mathbb{R}$, let the shuffled sequence z_n be defined as $z_{2n} = y_n$ and $z_{2n-1} = x_n$. Then, z_n is convergent $\iff$ x_n and y_n are convergent, and $lim(x_n) = lim(y_n)$.

Proof for Theorem.

$\implies$) If z_n converges, then for all $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that after n_0 every term of z_n lies in the ε neighbourhood of some limit z. This means that beyond some $n_{\text{something else}}$, every term of x_n and y_n - the sequences that make up z_n - falls into the ε neighbourhood of z. Therefore, both x_n and y_n are convergent, and to z.

$\impliedby$) If x_n and y_n are convergent to z, then $\forall \varepsilon > 0$,$\exists n_x, n_y \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_x$, $z - \varepsilon < x_n < z + \varepsilon$ and $z - \varepsilon < y_n < z + \varepsilon \implies z - \varepsilon < z_{2n} < z + \varepsilon$ and $z - \varepsilon < z_{2n-1} < z + \varepsilon$. Hence z_n is also convergent, and converges to z. ■

Definition 1.28: Cauchy Sequences

A sequence is said to be **Cauchy** if $\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_0$ we have $|x_n - x_m| < \varepsilon$

Theorem 1.29

Cauchy Sequences are bounded

Proof for Theorem.

Suppose x_n is cauchy. In the definition, fix $\varepsilon = 1$ and fix one element $x_j \geq n_0$. We then have $\forall n \in N, n \geq n_0$ $|x_n - x_j < 1 \implies x_j - 1 < x_n < x_j + 1$. We can see that $\forall n \geq n_0$, it is bounded, Since $x_1, x_2, \cdots x_{n_0}$ is a finite set, it too is bounded. Therefore, Cauchy sequences are bounded. ■

Definition 1.30: Contractive sequence

A sequence x_n is contractive if $\exists C > 0$ and $n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0$ we have $|x_{n+2} - x_{n+1}| < C|x_{n+1} - x_n|$

Theorem 1.31

A sequence is cauchy in $\mathbb{R}$ if and only if it is convergent in $\mathbb{R}$

Proof for Theorem.

$\Longleftarrow$) $\forall \varepsilon > 0$ $\exists n_1 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_1$

$$|x_n - x| < \frac{\varepsilon}{2}$$

$\forall \varepsilon > 0$ $\exists n_1 \in \mathbb{N}$ such that $\forall m \in \mathbb{N}, m \geq n_1$

$$|x - x_m| < \frac{\varepsilon}{2}$$

Adding these two we get: $\forall \varepsilon > 0$ $\exists n_1 \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_1$

$$|x_n - x_m| \leq |x_n - x| + |x - x_m| < 2\frac{\varepsilon}{2} = \varepsilon$$

$\Longrightarrow$)Say x_n is Cauchy, but not convergent. Since x_n is bounded, it has a convergent subsequence $x_{n_k} \to x$. Suppose x_n doesn't converge to x. This means that $\exists \varepsilon > 0$ and a subsequence x_{n_j} such that $\forall j \in \mathbb{N}$, we have $|x_{n_j} - x| \geq \varepsilon$. Since x_n is Cauchy,$\forall \varepsilon$, $\exists n_0$ such that $\forall j, k \in \mathbb{N}, j, k \geq n_0$

$$|x_{n_j} - x_{n_k}| < \frac{\varepsilon}{2}$$

Since $x_{n_k} \to x$, we have $\forall \varepsilon > 0$ $\exists l \in \mathbb{N}$ such that $\forall n_k \in \mathbb{N}, n_k \geq l$

$$|x_{n_k} - x| < \frac{\varepsilon}{2}$$

Adding those two we have $\forall \varepsilon$ $\exists n_{\text{something}} \in \mathbb{N}$ such that $\forall n_j, n_k \geq n_{\text{something}}$ we have

$$|x_{n_j} - x| \leq |x_{n_j} - x_{n_k}| + |x_{n_k} - x| < 2\frac{\varepsilon}{2} = \varepsilon$$

We see that x_{n_j} actually converges to x, contrary to the divergence criteria. ■

Theorem 1.32

Contractive sequences are Cauchy.

Proof for Theorem.

$\forall n \geq n_0$ we have $\forall n \geq n_0$, $|x_n - x_{n-1}| \leq C|x_{n-1} - x_{n-2}|$ where $C < 1$. $|x_n - x_{n-1}| \leq C|x_{n-1} - x_{n-2}| \leq C^2|x_{n-2} - x_{n-3}| \leq C^3|x_{n-3} - x_{n-4}| \cdots \leq C^{n-n_0-1}|x_{n_0+1} - x_{n_0}|$

Consider $|x_m - x_n|$ where WLOG $m > n$. $|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - x_{m-3} + x_{m-3} - \cdots x_{n+1} - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \leq (C^{m-n_0-1} + C^{m-n_0-2} + C^{m-n_0-3} \cdots + C^{n-n_0})|x_{n_0+1} - x_{n_0}|$. The term in the brackets can be made smaller than any ε for large enough m, n. This means that, x_n is Cauchy, hence convergent. ■

Example : (Applying Contraction) If $x_n = \frac{x_{n-1}+x_{n-2}}{2}$ ***Then*** x_n ***is cauchy.***
We see that $2x_n = x_{n-1} + x_{n-2} \implies 2x_n - 2x_{n-1} = -(x_{n-1} - x_{n-2})$. This implies: $2(|x_n - x_{n-1}|) = |x_{n-1} - x_{n-2}| \implies |x_n - x_{n-1}| \leq \frac{2}{3}|x_{n-1} - x_{n-2}|$. Therefore, x_n is Cauchy by virtue of being contractive. There are two methods to find the limit of this sequence.

Method 1 (Courtesy of TYS Arjun):

$$x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$$
$$x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$$
$$x_{n-1} = \frac{1}{2}(x_{n-2} + x_{n-3})$$
$$\vdots$$
$$x_4 = \frac{1}{2}(x_3 + x_2)$$
$$x_3 = \frac{1}{2}(x_2 + x_1)$$

Add all these to get:

$$x_3 + x_4 + \cdots x_{n-1} + x_n + x_{n+1} = \frac{1}{2}(x_n) + x_{n-1} + x_{n-2} \cdots + x_3 + x_2 + \frac{1}{2}(x_1) \implies$$

$$\frac{1}{2}(x_n) + x_{n+1} = x_2 + \frac{1}{2}(x_1)$$

Passing to the limit which was shown to exist we get:

$$\frac{1}{2}x + x = x_2 + \frac{1}{2}(x_1)$$

Method 2:

$$(x_n - x_{n-1}) = -\frac{1}{2}(x_{n-1} - x_{n-2}) = \frac{1}{2^2}(x_{n-2} - x_{n-3}) = -\frac{1}{2^3}(x_{n-3} - x_{n-4})\cdots$$
$$= (-1)^j\frac{1}{2^j}(x_{n-j} - x_{n-j-1}) = (-1)^{n-n_0+1}\frac{1}{2^{n-n_0+1}}(x_{n-(n-n_0+1)} - x_{n_0})$$

$\Longrightarrow$

$$x_n - x_1 = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + (x_{n-2} - x_{n-3})\cdots(x_2 - x_1) \Longrightarrow$$

$$x_n - x_1 = \sum_{k=1}^{n-1}(x_{k+1} - x_k) = \sum_{k=1}^{n}(-1)^{-n_0}(-1)^k((-2)^{n_0-1})(\frac{1}{2^k}(x_{n_0+1} - x_{n_0})) =$$

$$x_n - x_1 = (-1)^{-n_0}(-2)^{n_0-1}(x_{n_0+1} - x_{n_0})\sum_{k=1}^{n}(\frac{-1}{2})^k =$$

$$(-1)^{-n_0}(-2)^{n_0-1}(x_{n_0+1} - x_{n_0})(\frac{1}{3})((\frac{-1}{2})^n - 1)$$

Therefore,

$$lim(x_n - x_1) = (-1)^{-n_0}(-2)^{n_0-1}(x_{n_0+1} - x_{n_0})\sum_{k=1}^{n}(\frac{-1}{2})^k =$$

$$(-1)^{-n_0}(-2)^{n_0-1}(x_{n_0+1} - x_{n_0})(\frac{1}{3})lim((\frac{-1}{2})^n - 1)$$

■

Example : (Fibonacci) $f_1 = 1$***,*** $f_2 = 1$ ***and*** $f_n = f_{n-1} + f_{n-2}$ ***characterises the fibonacci sequence. The sequence*** $x_n = \frac{f_n}{f_{n+1}}$ ***is convergent.***

$x_n = \frac{f_n}{f_{n+1}} = \frac{f_n}{f_n+f_{n-1}} = \frac{1}{1+\frac{f_{n-1}}{f_n}} = \frac{1}{1+x_{n-1}}$. Since $x_{n-1} = \frac{1}{1+x_{n-2}}$, we have $x_n = \frac{1}{1+\frac{1}{1+x_{n-2}}}$. Notice that $x_1 > x_3 > x_5$ and $x_2 < x_4 < x_6$. Suppose, till $n = n_0$, we have $x_{2n-1} < x_{2n-3}$ and $x_{2n} > x_{2n-2}$. Consider x_{2n_0+1} and x_{2n_0-1}. $x_{2n_0+1} = \frac{1}{1+x_{2n_0}}$ and $x_{2n_0-1} = \frac{1}{1+x_{2n_0-2}}$. Since $1 + x_{2n_0-2} < 1 + x_{2n_0}$ we have $\frac{1}{1+x_{2n_0-2}} > \frac{1}{1+x_{2n_0}} \Longrightarrow x_{2n_0-1} > x_{2n_0+1}$. Hence, it is true for all $n \in \mathbb{N}$ from induction.

In a similar fashion, we can show via induction that the even subsequences are monotone increasing. Both the odd subsequences and even subsequences are monotone decreasing and increasing respectively, whilst being bounded. Hence, they are convergent. From the fact that $x_n = \frac{1}{1+x_{n-1}}$, we can see that both of these converge to the same number.

■

Definition 1.33: (Proper Divergence)

A sequence is said to diverge to +infinity if $\forall\xi \in \mathbb{R}, \exists n_0$ such that $x_n > \xi\forall n \geq n_0$.

It is divergent to -infinity if $\forall\xi \in \mathbb{R}, \exists n_0$ such that $x_n < \xi\forall n \geq n_0$.

Theorem 1.34

If x_n is monotone, then it is unbounded $\iff$ it is properly divergent.

Proof for Theorem.

$\implies$) $\forall \varepsilon > 0$, $\exists n(\varepsilon) \in \mathbb{N}$ such that $x_{n(\varepsilon)} > \varepsilon \implies x_n > \varepsilon \forall n \geq n(\varepsilon)$.

$\impliedby$)Properly Divergent is stronger than unboundedness. ■

Theorem 1.35: Comparision test #1

If $lim(x_n) = \infty$ and $x_n \leq y_n$, then $y_n \to \infty$. Similarly, if $lim(y_n) \to -\infty$, then $x_n \to -\infty$

Proof for Theorem.

Obvious ■

Theorem 1.36: Comparision test #2

If x_n and y_n and Positive sequences, and if $L > 0$, and if $lim(\frac{x_n}{y_n}) = L$, then $x_n \to \infty$ $\iff$ $y_n \to \infty$.

Proof for Theorem.

$\forall \varepsilon \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0$ we have $y_n(L - \varepsilon) < x_n < (L + \varepsilon)y_n$. Choose $\varepsilon = \frac{L}{2}$ so that we have $\forall n \geq n_0$, $\frac{L}{2}y_n < x_n < \frac{3}{2}y_n$ whence we see from test #1 that $x_n \to \infty \iff y_n \to \infty$. If $L = 0$. ■

Lemma 1.37: Useful Lemma

A monotone sequence is bounded if one of its subsequences is bounded

Proof for Lemma

Say the main sequence is properly divergent (which essentially means unboundede), then after some n_0 dependent on ε, all terms of all the subsequences coming after the index n_0 will be greater than ε. This is true for every ε, which means that all subsequences are unbounded. Therefore, the contrapositive gives that if one subsequence is bounded, the main sequence is bounded. ■

2 On Series (Introduction)

Definition 2.1: Series

Given a sequence x_n, we say the series generated by x_n is s_n if $s_n =\sim_{i=1}^{n} x_i$. (Sequence of partial sums defined inductively)

Lemma 2.2: n-th Term Test

A series $s_n = \sum_{i=1}^{n} x_i$ is convergent $\implies lim(x_n) = 0$

Proof for Lemma

$s_n = x_n + s_{n-1} \implies x_n = s_n - s_{n-1}$, and passing to the limit gives $lim(x_n) = 0$ ■

Theorem 2.3: Cauchy Criterion for Series

A series $s_n = \sum_{i=1}^{n} x_i$ is convergent $\iff \forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}, m, n \geq n_0$ we have $|s_m - s_n| = |x_{n+1} + x_{n+2} \cdots + x_m| < \varepsilon$

Example : The 1 harmonic: $\sum_{i=1}^{n} \frac{1}{i}$ is divergent.
Method 1:

Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$, and consider the subsequence of H_n which is $H_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n-1}} + \cdots \frac{1}{2^n}$

$$H_{2^1} = 1 + \frac{1}{2} \geq 1 + \frac{1}{2}$$

$$H_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) = 1 + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{2}{2}$$

$$H_{2^n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) \cdots + (\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}+1} \cdots \frac{1}{2^n}$$

$$\geq 1 + \frac{n}{2}$$

Hence, we see that H_{2^n} is properly divergent, which means that the main sequence is properly divergent.

Method 2:

Consider $|H_m - H_n| = |\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$ with the assumption that $m > n$. Note that $H_m - H_n$ has $m - n$ terms. $|H_m - H_n| > \frac{m-n}{m}$. Suppose $m = 2n$. We then have $|H_m - H_n| > \frac{n}{2n} = \frac{1}{2}$. Choose $\varepsilon = \frac{1}{2}$. We now have: $\exists \varepsilon = \frac{1}{2}$ such that $\forall k \in \mathbb{N}$, $\exists m(k), n(k) \in \mathbb{N}, m(k), n(k) \geq k$ with $m(k) = 2n(k)$ such that $|H_{m(k)} - H_{n(k)}| \geq \varepsilon = \frac{1}{2}$.

Hence, from negation of cauchy criteria, the 1 harmonic properly diverges.

Method 3:

Suppose that H_n is actually convergent. Consider H_{2n}. We have

$$H_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} \cdots \frac{1}{2n}$$

$$H_{2n} > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{6} + \frac{1}{6}) + (\frac{1}{8} + \frac{1}{8}) + \cdots + (\frac{1}{2n} + \frac{1}{2n}) =$$

$$H_{2n} = \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{1}{2} + H_n$$

Passing to the limit we have:

$$H \geq \frac{1}{2} + H$$

which is absurd.

Lemma 2.4

A positive termed series either converges or properly diverges

Proof for Lemma

Obvious

Example : The 2 harmonic $S_n = \sum_{i=1}^{n} \frac{1}{i^2}$ is convergent
Method 1:

$$S_n = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq \frac{1}{1} + \frac{1}{(2)(1)} + \frac{1}{(3)(2)} + \cdots + \frac{1}{(n)(n-1)} \implies$$

$$S_n \leq 1 + \frac{2-1}{(2)(1)} + \frac{3-2}{(3)(2)} + \cdots + \frac{n-(n-1)}{n(n-1)} =$$

$$1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots \frac{1}{n-1} - \frac{1}{n} \implies$$

$1 \leq S_n \leq 2 - \frac{1}{n} \leq 2$ which means this monotone increasing sequence is bounded above.

Method 2:

Consider the subsequence of S_n, S_{2^n-1}.

$$S_{2^1-1} = S_1 = \frac{1}{1}$$

$$S_{2^2-1} = S_3 = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} \leq 1 + \frac{1}{2}$$

$$S_{2^3-1} = S_7 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots \frac{1}{7^2} \leq 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}$$
$$\leq 1 + \frac{1}{2} + \frac{1}{2^2}$$

We can likewise easily see that $S_{2^n-1} \leq 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^{n-1}} \leq 2$. Which means S_n is convergent. ■

Theorem 2.5: Comparision test

Given X_n and Y_n, and $S_n = \sum X_i$ and $T_n = \sum Y_i$, and $\forall n \geq k_{\text{something}}, 0 \leq x_n \leq y_n$, then, if T_n converges, then S_n converges.

Proof for Theorem.

If T_n converges, we have from cauchy criteria that $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}; m, n \geq n_0$ we have $|y_{n+1} + y_{n+2} \cdots y_m| < \varepsilon$. After $n \geq max\{n_0, k_{\text{something}}\}$ we have $|x_{n+1} + x_{n+2} \cdots + x_m| < \varepsilon$ which fulfils the cauchy criteria for S_n. ■

Example : The alternating harmonic series: $S_n = \frac{(-1)^{1+1}}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)^{n+1}}{n}$ ***is convergent.***
Consider the odd subsequences:

$$S_{2n+1} = \frac{(-1)^{1+1}}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)^{2n+1}}{2n} + \frac{(-1)^{2n+2}}{2n+1}$$

$$S_{2n+1} = \frac{(-1)^2}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)}{2n} + \frac{(1)}{2n+1} = S_{2n} + \frac{1}{2n+1} = S_{2n-1} - (\frac{1}{2n} - \frac{1}{2n+1})$$

We see that odd subsequences are decreasing.

$$S_{2n} = \frac{(-1)^2}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)}{2n} = S_{2n-1} + \frac{-1}{2n} = S_{2n-2} + (\frac{1}{2n-1} - \frac{1}{2n})$$

We see that even subsequences are increasing. S_{2n+1} has $2n+1$ terms, with

$$S_{2n+1} = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) \cdots + (\frac{1}{2n-1} - \frac{1}{2n}) + \frac{1}{2n+1}$$

Hence, odd subsequences are bounded below by 0. Therefore, $S_{2n+1} > 0, \forall n \in \mathbb{N}$. For even subsequences, S_{2n} has $2n$ terms, and

$$S_{2n} = 1 - (\frac{1}{2} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{5}) \cdots - (\frac{1}{2n-1} - \frac{1}{2n})$$

We see that even terms are bounded above by 1. Hence, both even and odd subsequences converge.

we have $S_{2n+2} - S_{2n} = \frac{1}{2n+1} - \frac{1}{2n+2} \implies S_{2n+2} = S_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2}$. Hence, even limit= odd limit. From shuffle play theorem, the alternating harmonic series converges. ■

Theorem 2.6: Limit Comparision Test

Given that x_n and y_n are such that $x_n > 0$ and $y_n > 0$ $\forall n \in \mathbb{N}$, and $\exists r \in \mathbb{R}^+ \cup \{U\}$ such that

$$r = lim(\frac{x_n}{y_n})$$

Then:

1. if $r \neq 0$, $\sum x_n$ converges iff $\sum y_n$ converges.
2. if $r = 0$, then $\sum y_n$ converges $\implies$ $\sum x_n$ converges.

Proof for Theorem.

$\forall \varepsilon > 0$, $\exists n_0 > 0$ such that $\forall n \geq n_0$, we have

$$y_n(r - \varepsilon) < x_n < y_n(r + \varepsilon)$$

If we choose ε appropriately, we would have $\forall n \geq n_0$

$$y_n(\frac{r}{2}) < x_n < y_n(\frac{3r}{2})$$

Whence we can see that the $\iff$ statement is true from the first comparision test for series'.

If $r = 0$, then we would have $y_n(-\varepsilon) < x_n < y_n(\varepsilon)$, where we see that from the first comparision test, if $\sum y_n$ converges, we have $\sum x_n$ converges. To see that the forward implication does not hold, consider $x_n = \frac{1}{n^2}$ and $y_n = \frac{1}{n}$. $lim(\frac{x_n}{y_n}) = 0$, but $\sum x_n$ converges, whilst $\sum y_n$ diverges. ∎

Theorem 2.7: Addition of Series

If $\sum x_n$ and $\sum y_n$ converge, then the series $\sum x_n + y_n$ also converges.

Proof for Theorem.

$\forall \varepsilon > 0$, $\exists n_x \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_x$ we have

$$|x_{n+1} + x_{n+2} \cdots + x_m| < \frac{\varepsilon}{2}$$

And $\forall \varepsilon > 0$, $\exists n_y \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_y$ we have

$$|y_{n+1} + y_{n+2} \cdots + y_m| < \frac{\varepsilon}{2}$$

Hence, $\forall \varepsilon > 0$, $\exists j = max\{n_x, n_y\} \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq j$ we have

$$|(x_{n+1} + y_{n+1}) + \cdots (x_m + y_m)| \leq |x_{n+1} + x_{n+2} \cdots + x_m| + |y_{n+1} + y_{n+2} \cdots + y_m| < \varepsilon$$

Which is the cauchy criteria for $\sum x_n + y_n$.

∎

Theorem 2.8

Let $S_n = \sum_{j=1}^{n} a_j$ be a given series constructed from $\{a_n\}$, and suppose $T_n := \sum_{i=1}^{n} b_i$ constructed from the non-zero terms of $\{a_n\}$, maintaining order. Then $lim(S_n) = a \iff lim(T_n) = a$

Proof for Theorem.

$\implies$)

$$a_1 + a_2 + \cdots a_n$$

is the same as

$$b_1 + b_2 + \cdots b_k$$

WLOG, assume infinite terms exists (non-zero). This means, $\forall k \in \mathbb{N}$, $\exists n(k) \geq k \in \mathbb{N}$ such that

$$\sum_{j=1}^{k} b_j = \sum_{i=1}^{n(k)} a_i$$

We see that $\sum_{i=1}^{n(k)}$ is a subsequence of $\sum a_j$, hence is convergent to a.

$\impliedby$) Suppose that (and assume WLOG that there are infinite non zero terms) $lim(\sum_{j=1}^{k} b_j) = a$ $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall k \geq n_0, k \in \mathbb{N}$ we have

$$|\sum_{j=1}^{k} b_j - a| < \varepsilon$$

$\forall k \in \mathbb{N}$, $\exists n(k) \in \mathbb{N}, n(k) \geq k$ such that

$$\sum_{i=1}^{n(k)} a_i = \sum_{j=1}^{k} b_j$$

Note that $n(k) \geq k, n(k+1) > n(k)$ and $\forall n \notin \{n_1, n_2, \cdots\}$ we have $a_n = 0$. If our n in consideration falls on some n_k, then for such an n, we already have

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n(k)} a_i = \sum_{j=1}^{k} b_j$$

and as such, for sufficiently large n in this consideration,

$$|\sum_{i=1}^{n} a_i - a| < \varepsilon$$

Suppose our n doesn't fall on some n_k. Then it must belong between some n_{k_0} and n_{k_0+1}. Therefore, $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n_{k_0}} a_i = \sum_{j=1}^{k_0} b_j$ which means, for sufficiently large n (sufficiency governed by the ε), we have

$$|\sum_{i=1}^{n} -a| < \varepsilon$$

Therefore, we have covered all the n-s. We finally have: $\forall\varepsilon\ \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0$

$$|S_n - a| < \varepsilon$$

■

Theorem 2.9

Convergence of a series is not affected by altering a finite number of terms. The limit, ofcourse, can change.

Proof for Theorem.

Let S_n be the given series, and S'_n be the altered series, altering the terms $\{a_{n_1}, a_{n_2} \cdots a_{n_k}\}$. We have $\forall\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_0$,

$$|S_m - S_n| < \varepsilon$$

Let $j(\varepsilon) = max(\{n_0, k\})$. We are done. ■

Theorem 2.10: Cauchy Condensation Test

Suppose $a(n)$ is a monotone decreasing, positive termed sequence. Then $\sum_{i=1}^{n} a(i)$ converges $\iff$ $\sum_{j=1}^{n} 2^j a(2^j)$ converges.

Proof for Theorem.

We are told $a_1 \geq a_2 \geq a_3 \cdots$. Consider

$$2S_{2^n} = 2a_1 + 2a_2 + \cdots 2a_{2^{n-1}} + 2a_{2^{n-1}+1} + \cdots + 2a_{2^n}$$

$$2S_{2^n} \geq a_1 + 2a_2 + 2a_4 + 2a_4 + 2a_8 + 2a_8 + 2a_8 + 2a_8 + \cdots + \underbrace{2a_{2^n} + \cdots 2a_{2^n}}_{2^n-1-2^{n-1}-1+1=\text{ terms}} + 2a_{2^n}$$

We therefore have:

$$2S_{2^n} \geq a_1 + 2a_2 + 4a_4 + 8a_8 \cdots + 2a_{2^n}(2^n - 1 - \frac{2^n}{2}) + 2a_{2^n} =$$

$$a_1 + 2a_2 + 4a_4 + \cdots 2^n a_{2^n}$$

Or

$$\frac{1}{2}(a(1) + 2a(2) + 4a(4) + \cdots 2^n a(2^n)) \leq S_{2^n}$$

Consider another distribution scheme:

$$2S_{2^n} = 2a_1 + 2a_2 + 2a_3 + 2a_4 \cdots 2a_{2^{n-1}} + 2a_{2^{n-1}+1} + \cdots 2a_{2^n-1} + 2a_{2^n}$$

$$2S_{2^n} \leq 2a_1 + 2a_2 + 2a_2 + 2a_4 + 2a_4 + 2a_4 + 2a_4 + \cdots 2a_{2^{n-1}} + 2(2^n - 1 - 2^{n-1})a_{2^{n-1}} + 2a_{2^n} \implies$$

$$2S_{2^n} \leq 2a_1 + 4a_2 + 8a_4 + \cdots 2^n a_{2^{n-1}} + 2a_{2^n} \implies$$

$$S_{2^n} \leq a_1 + 2a_2 + 4a_4 + \cdots 2^{n-1}a_{2^{n-1}} + a_{2^n}$$

Finally we have

$$\sum_{j=1}^{n} \frac{1}{2} 2^j a(2^j) \leq S_{2^n} \leq \sum_{i=1}^{n-1} 2^j a(2^j) + a_{2^n}$$

And from Limit comparision test, the result is obvious.

Example : The p-harmonic series: $H_n^p = \sum_{i=1}^{n} \frac{1}{i^p}$ diverges if $p \leq 1$ and Converges if $p > 1$

We already know from limit comparision test that the p series, by virtue of the 1 series diverging, diverges for $p \leq 1$. Consider the case of $p > 1$.

Method 1:

Consider the subsequence $H^p(2^n - 1)$.

$$H^p(1) = 1$$

$$H^p(3) = 1 + \frac{1}{2^p} + \frac{1}{3^p} \leq 1 + \frac{1}{2^{p-1}}$$

$$H^p(7) = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \frac{1}{7^p}$$

$$\leq 1 + (\frac{1}{2^p} + \frac{1}{2^p}) + (\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}) \leq 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}}$$

Likewise, we can see that $H^p(2^n - 1) \leq 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \cdots \frac{1}{(2^{p-1})^{n-1}}$. Hence, the sequence $H^p(n)$ converges by virtue of being bounded.

Method 2:

Applying cauchy condensation:

$$\sum_{j=1}^{n} 2^j a(2^j) = \sum_{j=1}^{n} 2^j \frac{1}{(2^j)^p} = \sum_{j=1}^{n} (\frac{1}{2^j})^{p-1}$$

This series converges (geometric series), and hence, from Cauchy Condensation, the main series converges.

CHAPTER 3

METRIC SPACES

1 Fundamental Definitions n' Stuff

Definition 1.1: Metric Space

A set X along with a function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ called distance, is said to be a metric space if:

1. $d(x, y) = 0 \iff x = y$ (Positivity)
2. $d(x, y) = d(y, x), \forall x, y \in X$ (Symmetric)
3. $\forall x, y, z \in X$ we have, $d(x, y) \leq d(x, z) + d(y, z)$ (Triangle Inequality)

Example : $\mathbb{R}^n$ as a metric space
Note that $\mathbb{R}^n$, the set of all n-tuples of $\mathbb{R}$, is a metric space with

$$d(\vec{x}, \vec{y}) := |\vec{x} - \vec{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^+ \cdots + (x_n - y_n)^2}$$

Definition 1.2: Open and Closed Balls around x in X

Open ball is defined as:

$$B_r(x) := \{y \in X : d(y, x) < r\}$$

Closed ball is defined as:

$$B_{[r]}(x) := \{y \in X : d(y, x) \leq r\}$$

Definition 1.3: Convexity

A set S in $\mathbb{R}^n$ is said to be convex if $\forall x, y \in S, t \in [0,1]$, $x + t(y - x) \in S$

Example : Open and closed balls in $\mathbb{R}^n$ are convex
Consider $B_r(x) := \{z \in \mathbb{R}^n : |z - x| < r\}$. Consider arbitrary p and q in $B_r(x)$. We have that $d(p, x) < r$ and $d(q, x) < r$. Consider $p + t(q - p)$ and consider $d(p + t(q - p), x) = |p + t(q - p) - x| = |tq + (1 - t)p - x + tx - tx| = |tq - tx + (1 - t)p - (1 - t)x| \leq t|q - x| + (1 - t)|p - x| = td(q, x) + (1 - t)d(p, x) < r$. Replacing $<$ with $\leq$ in the above proves the result for closed balls.

Definition 1.4: Sequences in Metric Spaces

A sequence $\{x_n : x_n \in X\}$ is a mapping from the naturals to X, where order is implicit. We say a sequence in a metric space X is convergent to $x \in X$ if:

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_0)(d(x_n, x) < \varepsilon)$$

Definition 1.5: Cauchy Sequence in X

A sequence $\{x_n\}$ in X is said to be cauchy if

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n, m \in \mathbb{N})(n, m \geq n_0 \implies d(x_n, x_m) < \varepsilon)$$

Theorem 1.6

Convergence $\implies$ cauchy

Proof for Theorem.

Say x_n converges to x in the metric space.

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n, m \geq n_0)(d(x_n, x) < \varepsilon, d(x_m, x) < \varepsilon)$$

which implies (from triangle inequality)

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n, m \geq n_0)(d(x_n, x) < \varepsilon, d(x_m, x_n) \leq d(x_m, x) < \varepsilon)$$

Which is the cauchy condition.

1.1 Inner products, Normz and some common metrics

Definition 1.7: Inner Product

Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$, and V a vector space over $\mathbb{F}$. An **inner product** on V is a function that assigns to each ordered pair of vectors $v, u \in V$ a scalar $\langle v|u\rangle \in \mathbb{F}$ so that the following holds:

1. $\langle v + cw|u\rangle = \langle v|u\rangle + c\langle w|u\rangle$
2. $\langle u|v\rangle = \overline{\langle v|u\rangle}$ where $\bar{\cdot}$ is just complex conjugation
3. $\langle u|u\rangle > 0$ if $u \neq 0$

A space $V(\mathbb{F}), \langle\cdot\rangle$ is called an *inner product space*.

Definition 1.8: Norm (From the inner product)

Given an inner product space $V(\mathbb{F}), \langle\cdot\rangle$, we define the *norm* or *length* of a vector x, given by $||x||$ as $\sqrt{\langle x|x\rangle}$

Example : p-norm is a metric
The p-norm is defined on $\mathbb{R}^n$ as:

$$||\vec{x}||_p := (|x_1|^p + |x_2|^p \cdots |x_n|^p)^{1/p} = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$$

We say $d_p(\vec{x}, \vec{y})$, the distance between $\vec{x}$ and $\vec{y}$ to be $||x - y||_p = (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p}$. From this definition, unique 0 distance property and reflexivity property are obvious. Consider $d_p(x, z)$ and $d_p(z, y)$. $d_p(x, y) = (\sum_{i=1}^{k} |(x_i - z_i) + (z_i - y_i)|^p)^{1/p}$ whence from Minkowski we see that

$$(\sum_{i=1}^{k} |(x_i - z_i) + (z_i - y_i)|^p)^{1/p} \leq (\sum_{i=1}^{n} |x_i - z_i|^p)^{1/p} + (\sum_{i=1}^{n} |z_i - y_i|^p)^{1/p} = d_p(x, z) + d_p(y, z)$$

Therefore, p-norm is a metric ■

Example : The Max Norm
Suppose $x \in \mathbb{R}^n$, the max norm $||x||_\infty$ is defined as:

$$||\vec{x}||_\infty := \max_{1\leq i\leq n}(|x_i|)$$

Clearly, $d_\infty(x, y) = 0$ if and only if $x = y$. Moreover, from the properties of $|\cdot|$, it is reflexive. Consider $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2| \cdots, |x_n - y_n|\}, d_\infty(x, z) = \max\{|x_1 - z_1|, |x_2 - z_2| \cdots, |x_n - z_n|\}$ and $d_\infty(y, z) = \max\{|y_1 - z_1|, |y_2 - z_2|, \cdots |y_n - z_n|\}$.

$$d_\infty(x, y) = \max_{1\leq 1\leq n}\{|(x_i - z_i) + (z_i - y_i|)\} \leq \max_{1\leq 1\leq n}\{|(x_i - z_i)\} + \max_{1\leq 1\leq n}\{|(y_i - z_i)\} = d_\infty(x, z) + d_\infty(z, y)$$

Hence the max norm is a metric.

Example : Norm on function spaces
Let X be a non empty set. Define $\mathfrak{B}(X)$ as the space of all bounded, real functions. Then $||f||_\infty := \sup_{x\in X}|f(x)|$ defines a norm on $\mathfrak{B}(X)$ since $|f(x)-g(x)| \le |f(x)|+|g(x)| \le \sup_{x\in X}|f(x)| + \sup_{x\in X}|g(x)| = ||f||_\infty + ||g||_\infty$ which gives us $||f+g||_\infty \le ||f||_\infty + ||g||_\infty$

Example : the $\frac{d}{1+d}$ metric
Say X, d is a metric space, then $g(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric.

pfexercise to the reader

Continuity (Read definition of sequence in metric space, and limit points)

Definition 1.9: Continuous functions

A function $f : X \to Y$ that maps a metric space X, d_x to a metric space Y, d_y is said to be continuous at $a \in X$ (which is a limit point of X) if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p \in X)(d_x(p,a) < \delta \implies d_y(f(p), f(a)) < \varepsilon)$$

Theorem 1.10: Sequential Criteria for continuity

Let X, d_x and Y, d_y be metric spaces and $f : X \to Y$. Let a be a limit point of X. Then f is continuous at a if and only if for every sequence $(q_n) \in X$, $q_n \neq a$ with $\lim(q_n) = a$, we have $lim(f(q_n)) = f(a)$

Proof for Theorem.

$\implies$) We have that for every $\varepsilon > 0$, there exists a $\delta > 0$ so that for all $z \in X$ such that $z \in B_\delta(a)$, we have $f(z) \in B_\varepsilon(f(a))$. Consider an arbitrary sequence in X that is so that $(q_n) \in X$, $q_n \neq a$ and $\lim(q_n) = a$. Which reads $\forall \delta > 0, \exists n_0 \in \mathbb{N}$ so that $\forall n \ge n_0$, we have $q_n \in B_\delta(a)$. Let $\varepsilon > 0$ be arbitrary. From the definition of continuity, there exists a corresponding $\delta > 0$ so that $\forall z \in X$, $z \in B_\delta(a)$ would imply $f(z) \in B_\varepsilon(f(a))$. For this δ_ε, there exists an $n_0(\delta)(\varepsilon) \in \mathbb{N}$ so that $\forall n \ge n_0$, $q_n \in B_\delta(a)$ which would imply $f(q_n) \in B_\varepsilon(f(a))$. This means that $\forall \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ so that $\forall n \ge n_0$, we have $f(q_n) \in B_\varepsilon(f(a))$ which means that $\lim(f(q_n)) = f(a)$.

$\impliedby$) Suppose that every sequence $q_n \in X$ so that $q_n \neq a$ with $\lim(q_n) = a$, we have that $\lim(f(q_n)) = f(a)$ but that the function f is *not* continuous. i.e, $\exists \varepsilon > 0$ so that $\forall \delta > 0$, $\exists q_\delta \in X$ so that $d_x(q_\delta, a) < \delta$ but $d_y(f(q_\delta), a) \ge \varepsilon$. Choose $\delta = 1$ and get the corresponding $x_1 \in B_1(a)$ for which we have $d_y(f(x_1), f(a)) \ge \varepsilon$. Choose $\delta = 1/2$ and get $x_2 \in X$ so that $d_x(x_2, a) < 1/2$ but $d_y(f(x_2), f(a)) \ge \varepsilon$. As such, keep going with $\delta = 1/n$ to generate a sequence $x_n \in X$ so that $d(x_n, a) < 1/n$ but $d(f(x_n), f(a)) \ge \varepsilon$. Note that x_n converges by definition, to a, but there exists an $\varepsilon > 0$ so that no matter what n_0 we take, there exists some $n \ge n_0$ so that $d_y(f(x_n), f(a)) \ge \varepsilon$ which means that the sequence

$f(x_n)$ does not converge to $f(a)$, which contradicts hypothesis. ■

Corollary 1.11

Composition of continuous functions (at a point a) is finally a continuous function

Proof for Corollary.

Suppose X, d_x, Y, d_y and Z, d_z are metric spaces with $f : X \to Y$ and $g : Y \to Z$ two continuous functions. $g \circ f : X \to Z$. Consider a sequence $(x_n) \in X$ that converges to a. From sequential criteria, $f(x_n) \to f(a)$. We have $f(x_n)$, a sequence in Y, that converges to $f(a)$. From the continuity of g, we note that $g \circ (f(x_n))$ converges to $g \circ (f(a))$, which ultimately tells that for every sequence $\{x_n\}$ in X that converges to a, $g \circ f(x_n)$ is a sequence in Z that converges to $g \circ f(a)$ whence we are done ■

Example : Dirichlet Function
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

This function is discontinuous at every point of $\mathbb{R}$. Bear in mind that to show that a function is discontinuous at a point z, we need only show that *one* sequence in A which converges to z is such that $\lim(f(x_n)) \neq f(\lim(x_n))$. Consider $a \in \mathbb{R}$ an irrational point. Consider the rational sequence x_n that converges to a via density theorem. $f(x_n) = 1$ for every $n \in \mathbb{N}$ which means $\lim f(x_n) = 1$. But $f(\lim(x_n)) = 0$. Hence, discontinuous at $a \in \mathbb{R}$, irrational. In fact, the same argument can be re-used to show that f is irrational at every point. ■

Example : Thomae Function
We define the Thomae function as follows: Let $f : \mathbb{R}^+ \to \mathbb{R}$,

$$f(x) := \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/n & \text{if } x \text{ is rational, with } x = m/n, gcd(m, n) = 1 \end{cases}$$

This function is discontinuous at rational points quite obviously since, consider a rational number m/n where $f(m/n) = 1/n > 0$. Consider an irrational sequence in $\mathbb{R}^+$ converging to m/n. Clearly, $f(x_n) = 0$ for every $n \in \mathbb{N}$ whence, it is clear.

Consider z an irrational point where $f(z) = 0$. Consider a $\varepsilon > 0$ and an n_0 so that $1/n_0 < \varepsilon$. Consider the subinterval $(z - \gamma, z + \gamma)$ for some γ. We note that in this subinterval, only finite rational points have their denominator smaller than n_0 (because $(z - \gamma)n < m < (z + \gamma)n$ and this interval has size $2n\gamma$. For a fixed n and γ, m can pick values only from this interval. If $n < n_0$, then the interval size is smaller than $2n_0(\gamma)$. If we make γ small enough, m would be restricted to just one value. say, m_0. With this γ (and m_0 fixed) the only values n can take that are less than n_0 are certainly finite now since there are only finite ns smaller than n_0. Summarising, for a given ε, we choose an n_0 so that $1/n_0 < \varepsilon$, where we concluded that for small enough γ, only finite rational points in the interval $(z-\gamma, z+\gamma)$ have their denominator smaller than n_0). Now, we can eliminate all these rational points by picking an appropriate δ so that no rational points with denominator smaller than n_0 exist in $(z-\delta, z+\delta)$. Therefore, for a given ε $(\exists n_0 : 1/n_0 < \varepsilon)$ there exists δ so that either a point q in $(z-\delta, z+\delta)$ is irrational, whence $f(q) = 0 < \varepsilon$, or it has its denominator larger than n_0, which means $f(q) = 1/n < 1/n_0 < \varepsilon$ which leads to $|f(q) - f(z)| = |f(q)| = f(q) < \varepsilon$ which is the "$\varepsilon - \delta$" criterion for Continuity. Hence, at all irrational points z, the Thomae function is continuous. ■

Now we deal with, primarily, continuous functions on closed, bounded intervals of the kind $\mathbb{I} = [a, b] \subset \mathbb{R}$.

Theorem 1.12: Boundedness Theorem

Continuous functions on closed bounded intervals are bounded

Proof for Theorem.

Suppose $f : \mathbb{I} \to \mathbb{R}$ is actually unbounded, i.e, for any $M \in \mathbb{R}$ we take, there exists an $x_M \in \mathbb{I}$ so that $f(x_M) > M$. Let $M_1 = 1$ and get the corresponding x_1 so that $f(x_1) > 1$. Do the same for $M_n = n$ to get x_n so that $f(x_n) > n$. This sequence $f(x_n)$ is divergent. But the sequence $\{x_n\}$ is bounded, hence from Bolzano, there is a subsequence x_{n_k} that is convergent, say to p. But $f(x_{n_k})$ is a subsequence of $f(x_n)$, which means it diverges. But according to continuity, $f(x_{n_k})$ converges to $f(p)$, which is a contradiction. Hence, f is bounded. ■

Theorem 1.13: Maxima-Minima Theorem

Let $f : \mathbb{I} \to \mathbb{R}$ be a continuous function on closed bounded interval $\mathbb{I}$. Then f attains its maxima and minima.

Proof for Theorem.

We know that $f(\mathbb{I})$ is actually bounded, which means it has supremum and an infimum U and L. Suppose it *does not* attain bounds, i.e, (talking about upper bound) $\forall x \in \mathbb{I}$, $f(x) < U$. Consider $U - 1$. There exists $f(x_1)$ so that $U - 1 < f(x_1) < U$. Choose x_2 likewise so that $U - 1/2 < f(x_2) < U$. Keep going as such, to find x_n so that $U - 1/n < f(x_n) < U$. This means from squeeze theorem that $f(x_n)$ converges to U. Note that x_n that we have collected is a sequence that is in $\mathbb{I}$, a bounded, closed interval. Hence, it has a convergent subsequence $x_{n_k} \to x$. Also, $f(x_{n_k})$ is a subsequence of $f(x_n)$ which means $f(x_{n_k}) \to U$. But from continuity, $f(x_{n_k}) \to f(x)$, which implies $f(x) = U$, where $x \in \mathbb{I}$. This means that the function actually *does* attain upper bound (similar argument for the lower bound can be performed). ■

Theorem 1.14: Location of roots theorem

Let $f : \mathbb{I} \to \mathbb{R}$ be a continuous function on a closed bonunded interval $\mathbb{I}$. Let $a < b$ with $f(a) < 0$ and $f(b) > 0$ (or the other way around). Then there exists $c \in (a, b)$ so that $f(c) = 0$

Proof for Theorem.

Let $a_0 = a$ and $b_0 = b$ with $a < b$. a_0, b_0 forms an interval of size $\xi = b_0 - a_0$. Look at $f(a_0 + b_0/2)$. Is it greater than 0? If so, make $a_1 = a_0$, and $b_1 = a + b/2$, a new interval a_1, b_1 of size $\xi/2$. This new interval obeys the property that $f(a_1) < 0$ and $f(b_1) > 0$. Is $f(a_0 + b_0/2) < 0$? If so, make $b_1 = b_0$ and $a_1 = a_0 + b_0/2$ to make a_1, b_1 a new interval of size $\xi/2$. Again, $f(a_1) < 0$ and $f(b_1) > 0$ Suppose you have made the $n - th$ interval a_n, b_n so that $f(a_n) < 0$ and $f(b_n) > 0$ of size $\xi/(2^n)$. Make a_{n+1}, b_{n+1} by asking similar questions as above. We thus have a sequence of nested intervals $[a_n, b_n]$, for which $f(a_n) < 0$ and $f(b_n) > 0$. From Nested interval theorem, we have $\gamma \in \cap_n [a_n, b_n]$, moreover, since the size of these intervals are converging to 0, this is a unique point in the intersection. $a_n \to \gamma$ and $b_n \to \gamma$. From continuity, then, we have $f(a_n) \to f(\gamma) \leq 0$ and $f(b_n) \to f(\gamma) \geq 0$ which means that $f(\gamma) = 0$. ■

Theorem 1.15: Bolzano's IVT

Let $\mathbb{I}$ be a closed, bounded interval and let $f : \mathbb{I} \to \mathbb{R}$ be a continuous function. If $a, b \in \mathbb{I}$ with $k \in \mathbb{R}$ satisfying $f(a) < k < f(b)$, then there is a point c in $\mathbb{I}$ so that $f(c) = k$

Proof for Theorem.

Let $a < b$ and define $g(x) = f(x) - k$. This is a continuous mapping on $\mathbb{I}$. We then have $g(a) < 0 < g(b)$ whence from location or roots theorem, we find $c \in \mathbb{I}$ so that $g(c) = 0$ or $f(c) = k$. ■

Theorem 1.16: Preservation of intervals Theorem

Let $\mathbb{I}$ be a closed, bounded interval. Let $f : \mathbb{I} \to \mathbb{R}$ be a continuous map. Then, $f(I)$ is also a closed bounded interval.

Proof for Theorem.

Consider $f(\mathbb{I}) := \{z \in \mathbb{R} : \exists x \in \mathbb{I} \text{ such that } f(x) = z\}$. We know that continuous functions attain bounds, i.e, $f(\mathbb{I})$ has a maxima and a minima that will (eventually) be the end points of our closed bounded interval. If the function is constant, we are done. Consider a point $\min(f(\mathbb{I})) < z < \max(f(\mathbb{I}))$. By Bolzano's IVT, we know that a pre-image exists for z. This means that z is in the image. We are, therefore, done. The requisite interval is $[\min(f), \max(f)]$. ■

Definition 1.17: Uniform Continuity

A function $f : A \to \mathbb{R}$ is said to be uniformly continuous on A if for every $\varepsilon > 0$, $\exists \delta > 0$ so that for all $x, u \in A$ such that $0 < |x - u| < \delta$, we have $|f(x) - f(u)| < \varepsilon$.

We see that this definition is similar to the definition of continuity on A except for the fact that δ is independent of the x, the point at which we speak of continuity.

Theorem 1.18

Let f be a function defined on $A \subseteq \mathbb{R}$. The following are equivalent:

1. f is not uniformly continuous on A
2. $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists x_\delta, u_\delta \in A$ so that $0 < |x_\delta - u_\delta| < \delta$ but $|f(x_\delta) - f(u_\delta)| \geq \varepsilon$
3. $\exists \varepsilon$ and two sequences x_n and y_n so that $lim(x_n - u_n) = 0$ but $|f(x_n) - f(u_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$

Proof for Theorem.

(1) $\implies$ (2)) Obvious negation of the definition

(2) $\implies$ (3)) Choose $\delta = 1$, and get corresponding x_1, u_1 so that $0 < |x_1 - u_1| < 1$ and $|f(x_1) - f(u_1)| \geq \varepsilon$. Choose $\delta = 1/2$ and get the corresponding x_2 and u_2 so that $0 < |x_2 - u_2| < 1/2$ and $|f(x_2) - f(u_2)| \geq \varepsilon$. Keep going as such to get x_n, u_n so that $0 < |x_n - u_n| < 1/n$ $\forall n \in \mathbb{N}$ and $|f(x_n) - f(u_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$. This means that $\lim(x_n - u_n) = 0$ but $|f(x_n) - f(u_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$.

(3) $\implies$ (2)) $\forall \delta > 0$, $\exists n_0(\delta)$ so that $\forall n \geq n_0$, we have $0 < |x_n - u_n| < \delta$ but $|f(x_n) - f(u_n)| \geq \varepsilon$, choose one of these x_n−s for n greater than $n_0(\delta)$ as our x_δ(and likewise for u_δ). We are then done. ■

Theorem 1.19: Uniform Continuity Theorem

Let $\mathbb{I}$ be a closed bounded interval and let $f : \mathbb{I} \to \mathbb{R}$ be continuous on $\mathbb{I}$. Then f is uniformly continuous on I

Proof for Theorem.

Suppose f is not uniformly continuous on $\mathbb{I}$. From the previous *non uniform* criteria, we have that $\exists \varepsilon$ and two sequences in $\mathbb{I}$, x_n, u_n so that $\lim(x_n - u_n) = 0$ (or $\lim(x_n) = \lim(u_n)$) but $|f(x_n) - f(u_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$. Since x_n and u_n are sequences in $\mathbb{I}$ which is closed and bounded, x_n has a subsequence x_{n_k} converging to x. Since $\lim(x_n - u_n) = 0$, $\lim(x_{n_k} - u_{n_k}) = 0$ Which means that $\lim(u_{n_k}) = x$ as well. We have $x_{n_k} \to x$ and $u_{n_k} \to x$, which means from continuity that $f(x_{n_k}) \to f(x)$ and $f(u_{n_k}) \to f(x)$, whence we find that

$\lim(f(x_{n_k}) - f(u_{n_k})) = 0$ necessarily. But this contradicts the assumption that f is not uniform continuous. Hence, we are done. ■

Sequences of Functions

Given a set $A \subset \mathbb{R}$, we primarily work with a (countably) infinite collection of functions $\{f_n\}$ with $f_n : A \to \mathbb{R}$. We generate a sequence of numbers by evaluating f_n at a fixed point $x \in A$. It could be, then, that $\{f_n(x)\}$, treated as a sequence of numbers in $\mathbb{R}$, either converges or does not. For a subset $A_0 \subseteq A$ (which can possibly be empty), $f_n(x)$ converges for every $x \in A_0$. If this happens, there is a uniquely determined value that we would like to call "$f(x)$". Thus, there arises naturally a function $f : A_0 \to \mathbb{R}$ that we call the "limit" of $\{f_n\}$.

Definition 1.20: Convergence of sequence of functions

Let $\{f_n\}$ be a sequence of functions defined on $A \subseteq \mathbb{R}$. Let $A_0 \subset A$, and let $f : A_0 \to \mathbb{R}$. We say the sequence (f_n) converges to f **pointwise** if for each $x \in A_0$, The sequence (of real numbers) $f_n(x)$ converges to $f(x)$.

In other words, $\{f_n : A \to \mathbb{R}\}$ converges to f pointwise if for every $x \in A_0$, $\forall \varepsilon > 0$, $\exists n_0(x, \varepsilon) \in \mathbb{N}$ such that $\forall n \geq n_0(x, \varepsilon)$, we have $|f_n(x) - f(x)| < \varepsilon$

If we remove the dependence of n_0 above on the x, and leave it just to depend on ε, we arrive at the definition of *Uniform Convergence.*

Definition 1.21: Uniform Convergence of a sequence of functions

Let $\{f_n\}$ be a sequence of functions defined on $A \subset \mathbb{R}$. We say $\{f_n\}$ converges **uniformly** to $f : A_0 \to \mathbb{R}$ if $\forall x \in A_0$, $\forall \varepsilon > 0$, $\exists n_0(\varepsilon) \in \mathbb{N}$ so that $\forall n \geq n_0(\varepsilon)$, $|f_n(x) - f(x)| < \varepsilon$.

Since the dependence of n on x is non-existent, we can rewrite the above definition to read: $\{f_n\}$ converges **uniformly** to $f : A_0 \to \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists n_0(\varepsilon) \in \mathbb{N}$ so that $\forall n \geq n_0(\varepsilon)$, $\forall x \in A_0$, $|f_n(x) - f(x)| < \varepsilon$.

Theorem 1.22: Non uniform convergence criteria

A sequence of functions $\{f_n\}$ defined on A **does not** converge **uniformly** to $f : A_0 \to \mathbb{R}$ if and only if $\exists \varepsilon > 0$ such that $\forall k \in \mathbb{N}$, $\exists n_k \geq k$ and $x_k \in A$ so that $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon$, which means for some $\varepsilon > 0$, there exists a subsequence f_{n_k} of f_n and a sequence in x_k in A_0 so that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon \ \forall k \in \mathbb{N}$$

Proof for Theorem.

Obvious ■

Definition 1.23: The Uniform Norm

Say $A \subseteq \mathbb{R}$ and $\phi : A \to \mathbb{R}$. We say ϕ is bounded on A if $\phi(A)$, the set, is bounded in $\mathbb{R}$. If it is bounded, we can define what is **the uniform norm of ϕ on A** by:

$$||\phi||_A := \sup\{|\phi(x)| : x \in A\}$$

It follows that $||\phi||_A \leq \varepsilon \iff |\phi(x)| \leq \varepsilon, \forall x \in A$

Lemma 1.24

A sequence f_n of bounded functions on A converges uniformly to $f : A \to \mathbb{R}$ if and only if $||f_n - f||_A \to 0$

Proof for Lemma

$\implies$) Say $f_n \rightrightarrows f$, this means that $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ so that $\forall n \geq n_0$, $\forall x \in A$, we have $|(f_n - f)(x)| < \varepsilon$, which means that for every $n \geq n_0$, we have $||(f_n - f)||_A \leq \varepsilon$ which concludes the forward direction.

$\impliedby$) Say $||f_n - f||_A \to 0$, which means that $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ so that $\forall n \geq n_0$, we have $||f_n - f||_A < \varepsilon$ which means $\sup\{|(f_n - f)(x)| : x \in A\} < \varepsilon$ which means $\forall x \in A$, $|f_n(x) - f(x)| < \varepsilon$ whence the back implication is also done.

■

Theorem 1.25: Cauchy criteria for uniform convergence

A sequence f_n of bounded functions on A is uniformly convergent to $f : A \to \mathbb{R}$ (a bounded function) if and only if for every $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ so that $\forall m, n \geq n_0$, $||f_m - f_n||_A \leq \varepsilon$

Proof for Theorem.

$\implies$) Say $f_n \rightrightarrows f$. This means that $\forall \varepsilon > 0$, $\exists n_0$ so that $\forall n, m \geq n_0$, we have, for every $x \in A$, $|f_n(x) - f(x)| < \varepsilon/2$ and $|f(x) - f_m(x)| < \varepsilon/2$. Simply add them. We then have $|(f_m - f_n)(x)| < \varepsilon$ for every $x \in A$. In terms of the uniform norm. we see that $\forall \varepsilon > 0$,$\exists n_0 \in \mathbb{N}$ so that for all $m, n \geq n_0$, we have $||f_m - f_n||_A \leq \varepsilon$.

Suppose that $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ so that $\forall m, n \geq n_0$, $||f_m - f_n||_A \leq \varepsilon$ which means that for every $x \in A$, we have $|f_m(x) - f_n(x)| < \varepsilon$. This means that for every x, $f_n(x)$ is a cauchy sequence, hence convergent to some $f(x)$. More can be said for the sequence at each x. Fix some $x \in A$. We have that $\forall \varepsilon > 0$, $\exists n_0(\varepsilon) \in \mathbb{N}$ so that $\forall n, m \geq n_0$, we have $|f_n(x) - f_m(x)| < \varepsilon$ which means that $\forall n \geq n_0$, $|f_n(x) - f(x)| < \varepsilon$ which means that, all in all, $\forall \varepsilon > 0$, $\exists n_0(\varepsilon) \in \mathbb{N}$ so that $\forall n \geq n_0$, $\forall x \in A$, we have $|f_n(x) - f(x)| < \varepsilon$ which is the uniform convergence criteria. ■

l_p Space

Definition 1.26: l_p-Space

Define l_p as the space of all sequences $(a_n)_{n=1}^{\infty}$ such that $\sum_{i=1}^{\infty} |a_i|^p$ exists.

It is easily seen thet l_p is a vector space over $\mathbb{R}$. We defne the metric on l_p as $d_p(\{x\}, \{y\}) = (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{1/p}$

Definition 1.27: Limit Point of a set E

We say p is a limit point of a set E if

$$(\forall \varepsilon > 0)(\exists q_\varepsilon \in E; q_\varepsilon \neq p)(d(q_\varepsilon, p) < \varepsilon)$$

In other words, in every ε-ball around p, there would exist a point q_ε in E, which is different from p.

Theorem 1.28

Every ball / neighbourhood of p which is a limit point of E, would contain infinitely many points q such that $q \in B_\varepsilon(p) \cap E \setminus \{p\}$

Proof for Theorem.

Suppose for some neighbourhood, there only exists finite points $q_1, q_2, \cdots q_k$ such that $q_j \in B_{\varepsilon_0} \cup E \setminus \{p\}$. Let $\delta < min\{d(p, q_j) : j \in [1, 2, ...k]\}$. We then have that, there exists no point $q \in E$ such that its distancce from p is less than δ, making p a non-limit point. Absurd. ∎

Corollary 1.29

A finite set has no limit points

Theorem 1.30: Recharacterisation of Limit points

A point $p \in X$ is a limit point of $E \subset X$ if and only if there exists a sequence $x_n \in E$, $x_n \neq p \forall n \in \mathbb{N}$ such that $\{x_n\} \to p$

Proof for Theorem.

$\Longrightarrow$) If p is a limit point, around every neighbourhood, there would exist a point $q_\varepsilon \in E$ such that $0 < d(q_\varepsilon, p) < \varepsilon$. Choose $\varepsilon_1 = 1$, and obtain x_1 such that $x_1 \in E$, $x_1 \neq p$ and $0 < d(x_1, p) < 1$. Choose $\varepsilon_2 = \frac{1}{2}(d(x_1, p))$. We find $x_2 \in E$, $x_2 \neq p$ such that $d(x_2, p) < \frac{d(x_1,p)}{2} < \frac{1}{2}$. Continue as such to obtain a sequence that converges to p.

$\Longleftarrow$) Suppose there is a sequence x_n such that $x_n \neq p \forall n \in \mathbb{N}$ and $\forall \varepsilon, \exists n_0(\varepsilon)$ such that $\forall n \geq n_0$ we have $d(x_n, p) < \varepsilon$ which means for a given ε, there exists a point x_{n0+1} in E such that it is not equal to p and it is in the ε-ball of p. Hence, p would be a limit point. ∎

Definition 1.31: Closed sets in X

A set E is closed in X if every limit point of E is contained in E

Definition 1.32: Equivalent definition of closed sets in X

A set E in X is closed if for every convergent sequence x_n in x such that $lim(x_n) \neq x_n$ for any n, we have $lim(x_n) \in E$.

Definition 1.33: Open sets in X

A set E is said to be open if $\forall x \in E$, $\exists \xi_x > 0$ such that $B_{\xi_x}(x) \subset E$

Theorem 1.34

Every open ball is an open set

Proof for Theorem.

Suppose a is a fixed point in X and $\delta > 0$ is given. $B = B_\delta(a) := \{y \in X : d(y,a) < \delta\}$.Consider arbitrary $z \in B$, for which we have $d(z,a) = t < \delta$. Therefore $\delta - t > 0$. Consider $0 < \xi_z = r < \delta - t$ from Density. Consider an arbitrary x such that $d(x,z) < \xi_z = r < \delta - t$. $d(x,a) \leq d(x,z) + d(a,z) = r + t \leq \delta - t + t = \delta$. We are done. ■

Definition 1.35: Compliment with respect to X

If $E \subseteq X$, we define compliment of E as

$$E^C := \{x \in X : x \notin E\}$$

Definition 1.36: Bounded

A set $E \subset X$ is bounded if $\exists$ a positive number $M > 0$ and $q \in E$ such that $d(x,q) < M$ $\forall x \in E$. i.e, all the points of E gets contained in some ball in X.

Theorem 1.37: De Morgan's Law

Let $\{E_\alpha : \alpha \in A\}$ where A is some arbitrary indexing set represent a collection of sets in X. Then

$$\big(\cup_\alpha E_\alpha\big)^C = \cap_\alpha E_\alpha^C$$

Proof for Theorem.

Consider $(\cup_\alpha E_\alpha)^c = \{x \in X : \exists \alpha \in A : x \in E_\alpha\}^c = \{x \in X : \forall \alpha \in A : x \notin E_\alpha\} = \{x \in X : \forall \alpha \in A : x \in E_\alpha^c\} = \cap_\alpha E_\alpha^c$ ■

Theorem 1.38: The Big Equivalence

$E \subset X$ is open $\iff$ E^c is closed.

Proof for Theorem.

$\implies$) Suppose that E is open but E^c is not closed. This means that there exists a limit point of E^c that falls in E, i.e, outside E^c. Let this be q. This means for every ε-ball around q, a point of E^c exists. But since E is open and $q \in E$, we have for a particular ε-ball, inside which, no point of E^c resides. Contradiction.

$\impliedby$) Suppose E is closed but E^c is not open. This means that there is a point in E^c, p, such that for every ε-ball around p, some point in E falls into this ball. But this makes p a limit point of E, which is absurd since E is closed, limit points fall into the sets themselves. ■

Theorem 1.39

For a collection of open sets $\{G_\alpha : \alpha \in A\}$, $\cup_\alpha G_\alpha$ is also an open set.

Proof for Theorem.

Consider $x \in \cup_\alpha G_\alpha$ which means $\exists \alpha_x \in A$ such that $x \in G_{\alpha_x}$ which means, there would exist an ξ-ball around x that is contained in G_{α_x} which is in turn contained in $\cup_\alpha G_\alpha$. ■

Corollary 1.40

For any collection of closed sets E_α, $\cap_\alpha E_\alpha$ is also closed.

Proof for Corollary.

$\{E_\alpha^c\}$ is a collection of open sets, and $\cup_\alpha E_\alpha^c$ is an open set, which means $\cup_\alpha E_\alpha^c = (\cap_\alpha E_\alpha)^c$ is an open set, from which we get that $(\cap_\alpha E_\alpha)$ is a closed set. ■

Theorem 1.41

For any finite collection of open sets $\{E_1, E_2, \cdots E_k\}$, $\cap_{i=1}^k E_i$ is also open.

Proof for Theorem.

Suppose $x \in \cap_{j=1}^k E_j$, which means $\forall j \in [1, k], x \in E_j$. We have $\varepsilon_1, \varepsilon_2, \cdots \varepsilon_k$ such that, the ε_j-ball around x is fully contained in E_j. Choose $0 < \delta < min\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_\}$ (the minimum exists by virtue of being a finite set). We see that the δ-ball around x is a subset of every ε_j-ball around x, which means that the δ-ball around x is in every E_j, which proves the theorem. ■

Corollary 1.42

For any finite collection of closed sets $\{G_1, G_2, \cdots G_k\}$ we have $\cup_{j=1}^{k} G_j$ to be closed

Remark.

> In the above theorem and corollary, we require that the collection be finite. The reason is that, we were able to get a minimal ε_j in the proof due to the finiteness of the set. It may not be possible to find a number δ that is both larger than 0 but smaller than a given infinite collection of ε-s. For example, consider the sequence of open sets $(-\frac{1}{n}, \frac{1}{n})$. The infinite intersection of these yields $\{0\}$ which is a closed set by virtue of being finite.

Definition 1.43: Closure of a set

Let E' be the set of all limit points of E. Then, the closure of E is :

$$\bar{E} := E \cup E'$$

Theorem 1.44

Closure of a set is closed

Proof for Theorem.

Let p be a limit point of $E \cup E'$. That means that $\forall \varepsilon > 0\ \exists q_\varepsilon \in (E \cup E'), q_\varepsilon \neq p$ such that $q_\varepsilon \in B_\varepsilon(p)$. If p is in $E \cup E'$, we are done (especially if p is in E). Suppose p is not in E. $\forall \varepsilon > 0\ \exists q_\varepsilon \in (E \cup E'), q_\varepsilon \neq p$ such that $q_\varepsilon \in B_\varepsilon(p)$. If the q_ε we recieve falls in E we are ok. Suppose q_ε falls in E'. That means: $\forall \delta > 0, \exists r_\delta \in E, r \neq q_\varepsilon$ such that $d(r_\delta, q_\varepsilon) < \delta \implies d(r_\delta, p) \leq d(r_\delta, q_\varepsilon) + d(q_\varepsilon, p) < \delta + d(q_\varepsilon, p) < \delta + \varepsilon$ If we choose $\delta_0 < \varepsilon - d(q_\varepsilon, p)$ we get: $d(r_\delta, p) \leq d(r_\delta, q_\varepsilon) + d(q_\varepsilon, p) < \delta + d(q_\varepsilon, p) < \varepsilon$

Summarising we have: $\forall \varepsilon > 0$, $\exists q_\varepsilon \in E$ or E' where: $q_\varepsilon \in E$ and $q_\varepsilon \in B_\varepsilon(p)$

or

$\exists \delta(\varepsilon) > 0$ such that $\exists r_\delta \in E$ such that $r_\delta \neq p$ and $r_\delta \in B_\varepsilon(p)$. In either case, there would exist a point dependent on ε, in E such that the point itself is different from p, and exists in the ε-ball around p. Hence, we see that p is a limit point of E. Therefore, we see that all the limit points of E either are points of E or points of E'. Hence, $\bar{E}$ is closed. ■

Theorem 1.45

$\bar{E} = E \iff E$ is closed

Proof for Theorem.

$\implies$)$\bar{E}$ is closed, so E would be too.

$\impliedby$) if E is closed, $E' \subseteq E \implies E' \cup E = E = \bar{E}$

Theorem 1.46: $\bar{E}$ is the smallest closed set that contains E

If F_α is the collection of all closed sets such that $E \subseteq F_\alpha$, then $\bar{E} \subseteq F_\alpha$ for all α.

Proof for Theorem.

Consider an arbitrary closed set F_α that contains E. It would obviously contain all the limit points of E among other things. Therefore, we can easily see that it contains $E \cup E' = \bar{E}$.

Lemma 1.47: An equivalent definition for Closure.

An equivalent definition for closure is:

$$\bar{A} := \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \phi\}$$

Proof for Lemma

We see that obviously, if $x \in \bar{A}$, then either it is a point of A, or if not, it happens to be a limit point of A. And the back implification: If q is a point of A or if it is a limit point o A, it obviously falls into $\bar{A}$.

Example : If $E \subseteq \mathbb{R}$ is bounded (and non empty), with $s = sup(E)$, then $s \in \bar{E}$
If $s \in E$ we are done. If not, then $\forall \varepsilon > 0$, $\exists \varepsilon > \delta(\varepsilon) > 0$, and a point $x_\varepsilon \in E$ such that $s - \varepsilon < s - \delta(\varepsilon) \leq x_\varepsilon < s + \delta < s + \varepsilon$ where $x_\varepsilon \neq s$. Hence, s is a limit point of E and hence, is a point in the closure.

Definition 1.48: Open Relative

Say $E \subseteq Y \subseteq X$, where X is a metric space. Y is also a metric space. We say E is open relative to Y if $\forall x \in E$, $\exists \varepsilon > 0$ such that if $y \in Y$ and $y \in B_\varepsilon(x)$ then $y \in E$. Formally:

$$(\forall x \in E)(\exists \varepsilon_x > 0)((y \in Y \cap B_\varepsilon(x)) \implies y \in E)$$

Remark.

A set which is open relative to Y need not be open relative to X. For example, consider $\mathbb{R}$ as a subset of $\mathbb{R}^n$. An interval in $\mathbb{R}$ is open relative to $\mathbb{R}$, but it is not open relative to $\mathbb{R}^n$.

Theorem 1.49

A set $E \subseteq Y \subseteq X$ is open relative to $Y \iff \exists G \subset X$ that is open relative to X, such that $E = G \cap Y$

Proof for Theorem.

$\implies$) Say E is open relative to Y. This means that $\forall x \in E$, $\exists \varepsilon_x > 0$ such that if $y \in B_{\varepsilon_x}(x)$ and $y \in Y$, then $y \in E$. Call $G = \cup_{x \in E} B_{\varepsilon_x}(x)$ which is an open set. If $z \in E$, then $z \in G$ obviously, and hence $z \in G \cap Y$. Hence, $E \subseteq G \cap Y$. Consider a point $z \in G \cap Y$ which means z falls in one of the ε-balls around a point of E, and z is in Y. From definition of open relativeness, we see that $z \in E$. Hence, $E = G \cap Y$

$\impliedby$) Say $E = G \cap Y$ where G is an open set relative to X. Then, for every point in G, there would exist an ε-ball around that point that is completely contained in G. Let $x \in E$ be arbitrary. $\exists \varepsilon_x > 0$ such that $B_\varepsilon(x) \subset G$. Suppose $y \in Y$ and $y \in B_\varepsilon(x)$. This would mean that $y \in G \cap Y = E$. Hence, $\forall x \in E$ $\exists \varepsilon > 0$ such that if $y \in B_\varepsilon(x)$ and $y \in Y$, then $y \in E$, which is the definition of open relativeness. ■

2 Compactness

Definition 2.1: Open Conver

A collection of open sets $G_\alpha \subset X$ is an open cover of a set E if $E \subset \cup_\alpha G_\alpha$

Definition 2.2: Compact Set

A set $E \subset X$ is said to be ***Compact*** if Every open cover has a finite subcover. i.e, for every collection of open sets G_α, if $E \subseteq \cup_\alpha G_\alpha$, then there would exist a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2} \cdots G_{\alpha_k}\}$ of $\{G_\alpha\}$ such that $E \subseteq \cup_{i=1}^k G_{\alpha_i}$

Remark.

> The notion of *Being open* depends largely on the metric space one is talking about. For example, we see that certain sets may be open relative to $Y \subset X$, but not X in itself. This is not the case for compactness though, as shall be seen.

Theorem 2.3: "Compact Relativeness" is conserved.

Definition: We say $E \subseteq Y \subseteq X$ is compact relative to Y if for every open cover G_α open relative to Y we have a finite sub collection G_{α_k} of G_α such that $E \subseteq \cup_{j=1}^k G_{\alpha_j}$.

Theorem: $E \subseteq Y \subseteq X$ is compact relative to $Y \iff E$ is compact relative to X

Proof for Theorem.

$\implies$)Suppose E is compact relative to Y. This means that, for any collection of sets F_α which are open relative to Y (i.e, $F_\alpha = G_\alpha \cap Y$ where G_α is an open set in X), there exists a finite sub collection $F_{\alpha_1}, F_{\alpha_2} \cdots F_{\alpha_k}$ such that $E \subseteq \cup_{i=1}^k F_{\alpha_i}$. Consider an open cover H_α of E open relative to X. $E \subseteq \cup_\alpha H_\alpha$, but also, $E \subseteq (\cup_\alpha H_\alpha) \cap (Y)$ since E is subset of Y as well. This implies $E \subseteq \cup_\alpha (H_\alpha \cap (Y))$. $\{H_\alpha \cap Y\}$ is an open cover of E open relative to Y which means there would be a finite sub collection $\{H_{\alpha_j} \cap Y : j \in [1, k]\}$ such that $E \subseteq \cup_{j=1}^k (H_{\alpha_j} \cap Y) = (\cup_{j=1}^k H_{\alpha_j}) \cap Y$. Since E is a subset of Y, we then have $E \subseteq (\cup_{j=1}^k H_{\alpha_j})$ which proves that for an arbitrary open cover open relative to X, we have a finite subcover.

$\impliedby$) Suppose E is open relative to X. Consider an open cover of E open relative to Y, which is $\{F_\alpha\}$. This means that $F_\alpha = G_\alpha \cap Y$ for G_α open relative to X. $E \subseteq \cup_\alpha F_\alpha = (\cup_\alpha G_\alpha) \cap Y$. Since E is a subset of Y, we have $E \subseteq (\cup_\alpha G_\alpha)$. Therefore, there would be a finite subcollection of $\{G_\alpha\}$, $\{G_{\alpha_1}, G_{\alpha_2} \cdots G_{\alpha_k}\}$ such that $E \subseteq \cup_{i=1}^k G_{\alpha_i}$. This means, $E \subseteq \cup_{i=1}^k G_{\alpha_i} \cap Y = \cup_{i=1}^k F_{\alpha_i}$. Hence, for every open cover open relative to Y, there exists a finite subcover.

■

Fact 2.4

Every finite set in X is compact

Proof. Consider an open cover G_α for finite set E. This means that, for every point $x_1, x_2, \cdots, x_k$ in E, there would exist some $\{\alpha_1, \alpha_2, \cdots\}$ collection of "α-s" that is utmost finite, such that $x_j \in G_{\alpha_j}$. Simply take the union of G_{α_j} to get a finite subcover. $\square$

Theorem 2.5: Alternate definition for compactness

A set E is compact if for every closed collection of sets K_α such that $\cap_\alpha K_\alpha \subset E^c$, we have a finite subcollection $\{K_{\alpha_1}, K_{\alpha_2} \cdots K_{\alpha_p}\}$ such that $\cap_{i=1}^p K_{\alpha_i} \subset E^c$

Theorem 2.6

Closed balls in X are closed

Proof for Theorem.

> Consider $B_{[\varepsilon]}(p) := \{x \in X : d(x,p) \leq \varepsilon\}$. $B^c = C := \{x \in X, d(x,p) = t_{xp} > \varepsilon\}$. Consider an arbitrary point $x \in C$. We have $d(x,p) = t_{xp} > \varepsilon$. Find, from density, a δ such that $t_{xp} > \delta > \varepsilon$. Let $d(x,y) < \delta - \varepsilon$. We then have from triangle, $d(y,p) \geq d(x,p) - d(x,y) > t_{xp} - (\delta - \varepsilon) > \varepsilon$. Hence, y is also in C. Therefore, C is open, which means B is closed. ■

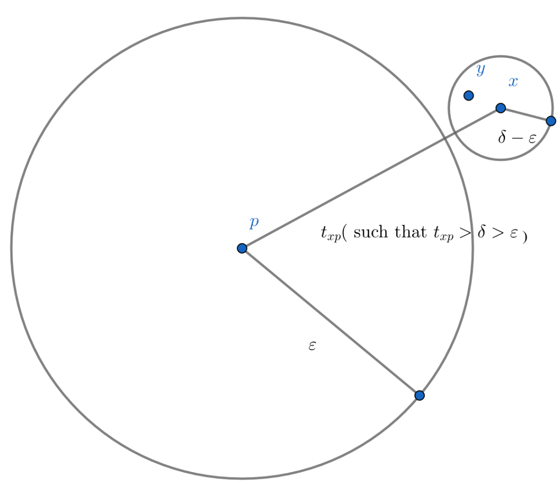

Figure 3.1: Figure for the proof: Closed balls are closed

Theorem 2.7

Compact sets are closed.

Proof for Theorem.

Method 1:

Let E be compact. Consider a point $p \in E^c$. Let ε_x be the "half" distance between a point $x \in E$ and p. Therefore, $B_{\varepsilon_x}(x)$ is completely outside $B_{\varepsilon_x}(p)$. Consider $\cup_{x \in E} B_\varepsilon(x)$ which is an open cover for E. This means there is a finite subcover

$\{B_{\varepsilon_{x_1}}(x_1), B_{\varepsilon_{x_2}}(x_2), B_{\varepsilon_{x_3}}(x_3) \cdots, B_{\varepsilon_{x_k}}(x_k)\}$ such that $E \subset \cup_{i=1}^k B_{\varepsilon_{x_i}}(x_i)$. $B_{\varepsilon_{x_i}}(p)$ does not intersect with $B_{\varepsilon_{x_i}}(x_i)$. Therefore, $\cap_{i=1}^k B_{\varepsilon_{x_i}}(p)$ does not intersect with any $B_{\varepsilon_{x_i}}(x_i)$ for any i. Hence, it does not intersect with $\cup_{i=1}^k B_{\varepsilon_{x_i}}(x_i)$ which means $\cap_{i=1}^k B_{\varepsilon_{x_i}}(p)$ lies completely outside E. If we choose $\delta < min\{\varepsilon_{x_1}, \varepsilon_{x_2} \cdots \varepsilon_{x_k}\}$, we would have $B_\delta(p) \subseteq \cap_{i=1}^k B_{\varepsilon_{x_i}}(p)$. This means that, for p outside E, there would exist a δ such that the δ-ball around p is fully contained in E^c. This means that E^c is open, hence, E is closed.

Method 2:

Consider E to be compact, i.e, for every closed collection $\{F_\alpha\}$ such that $\cap_\alpha F_\alpha \subset E^c$, there exists a finite sub collection $\{F_{\alpha_1}, F_{\alpha_2} \cdots F_{\alpha_k}\}$ such that $\cap_{j=1}^k F_{\alpha_j} \subset E^c$. Consider a point p outside E, i.e, in E^c. Notice that $\cap_{\varepsilon \in \mathbb{R}^+} B_{[\varepsilon]}(p) = \{p\}$ which is in E^c. This would be a collection of closed sets whose intersection falls completely inside E^c. Hence, there would exist a finite subcollection such that $\cap_{j=1}^k B_{[\varepsilon_j]}(p) \subset E^c$ which means there would exist a neighbourhood around p which is completely in E^c. Hence, E^c is open, and E is closed.

■

Fact 2.8

ϕ and X are both open and closed.

Theorem 2.9

Closed subsets of compact sets are compact

Proof for Theorem.

Consider $K \subset E$ where E is compact and K is closed. K^C is, therefore, open. Consider an arbitrary open cover $\{F_\alpha\}$ for K. since $K \subseteq \cup_\alpha F_\alpha$, and K^c is open, we have $X = \cup_\alpha F_\alpha \cup K^c$ which means $E \subset \cup_\alpha F_\alpha \cup K^c$. Since E is compact, there would exist a finite subcover such that $E \subset \cup_{j=1}^n F_{\alpha_j} \cup K^c$. We then have $K \subset \cup_{j=1}^n F_{\alpha_j} \cup K^c$, which would mean $K \subset \cup_{j=1}^n F_{\alpha_j}$, whence, we see that K is compact. ■

Corollary 2.10

If F is closed, and K is compact, then $F \cap K$ is compact

Fact 2.11

A compact set is bounded

Proof. Consider (WLOG, a non empty compact set E) and an arbitrary point q in X. $B_\varepsilon(x)$ for every $\varepsilon > 0$ forms an open cover for E (since it is basically X). Which means there is a finite subcover, i.e, a number $\varepsilon_0 > 0$ such that $E \subset B_{\varepsilon_0}(p)$ which makes E bounded. □

Theorem 2.12

Finite union of compact sets is compact

Proof for Theorem.

Let $K_1, K_2, \cdots K_r$ be r compact sets. Let $K = \cup_{i=1}^r K_i$. Consider an open cover F_α whose union subsumes K. We have that, for every $i \leq r$, $K_i \subset \cup_\alpha F_\alpha$. Since each K_i is compact, there exists a finite of F_α whose union subsumes K_i. For each $i = 1$, to r, we have a finite subcollection, therefore, taking the union of all these finite subcollections gives us a finite subcollection which subsumes whole of K. Hence K is compact.

■

Remark.

Note that finiteness in the above theorem is important. This is because, each compact may have finite subcollection, but at the end, the union of all these finite collections will be countable, not finite.

Theorem 2.13

If $\{K_\alpha\}$ is a collection of compact sets such that for every finite subcollection $\{K_{\alpha_j} : 1 \leq j \leq k\}$ we have that $\cap_{j=1}^k K_{\alpha_j} \neq \phi$. Then $\cap_\alpha K_\alpha \neq \phi$. In pithy words:

"If you have a collection of compact sets for which every finite subcollection's intersection is non-empty, the intersection of the whole collection is non empty"
-Krishna, to Arjuna

Proof for Theorem.

Suppose, on the contrary, let $\cap_\alpha K_\alpha = \phi$ which means $\cup_\alpha K_\alpha^c = X$ which means for every for some α_0, we have $K_{\alpha_0} \subset \cup_\alpha K_\alpha^c$, where $\cup_\alpha K_\alpha^c$ is an open cover of K_{α_0}. This implies that there exists a finite subcollection $\{K_{\alpha_1}^C, K_{\alpha_2}^C \cdots K_{\alpha_r}^C\}$ such that $K_{\alpha_0} \subset \cup_{j=1}^r K_{\alpha_j}^C \implies \cap_{j=1}^r K_{\alpha_j} \subset K_{\alpha_0}^C$. But this means $\cap_{j=1}^r K_{\alpha_j} \cap K\alpha_0 = \phi$, which is absurd since all finite intersection is non empty. ■

Corollary 2.14

If $K_1, K_2, \cdots$ is a sequence of non-empty compact sets such that $\cdots K_n \subset K_{n-1} \cdots K_3 \subset K_2 \subset K_1$, then $\cap_{i=1}^{\infty} K_i$ is non empty.

Theorem 2.15: Compactness $\implies$ Limit point Compact

If K is a compact set and E is an infinite subset of K, then E has a limit point in K

Proof for Theorem.

Suppose that E has no limit point in K. Since K is closed, E must have no limit points. Hence, E is closed. Since closed subsets of compact sets are compact, E is compact. If no point of E is a limit point of E, then $\forall x \in E$, $\exists \varepsilon_x > 0$ such that no point of E apart form x itself falls into the ε_x-ball of x. Consider the open cover $\{B_{\varepsilon_x}(x) : x \in E\}$ of E. This has a finite subcover $\{B_{\varepsilon_{x_1}}(x_1), B_{\varepsilon_{x_2}}(x_2), \cdots B_{\varepsilon_{x_l}}(x_l)\}$. We see that $E \subseteq \cup_{j=1}^{l} B_{\varepsilon_{x_j}}(x_j)$. But since for every ε_{x_j}-ball around x_j, no point in E except x_j resides, $\cup_{j=1}^{l} B_{\varepsilon_{x_j}}(x_j)$ will have utmost finite points. Since an infinite set E cannot be the subset of a finite set, we have a contradiction. ■

Definition 2.16: k-cell

a k-cell, E is a set in $\mathbb{R}^k$ such that $E := \{\vec{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^n : a_j \leq x_j \leq b_j$ for given a_j and b_j for every $1 \leq j \leq k\}$

A k-cell is basically a k dimensional cuboid.

Theorem 2.17

k-cells are closed

Proof for Theorem.

Consider a k-cell E. Consider a point z not in E, i.e, $\exists j_0$ such that either $z_j < a_j$ or $z_j > b_j$. WLOG, take the case of $z_j < a_j$. Let $0 < \delta < (a_j - z_j)$. Consider a point q in the δ-ball around z. i.e, $d(z,q) < \delta \implies \sqrt{(z_1 - q_1)^2 + (z_2 - q_2)^2 + \cdots + (z_k - q_k)^2} < \delta \implies (z_1 - q_1)^2 + (z_2 - q_2)^2 + \cdots + (z_k - q_k)^2 < \delta^2 < (a_j - z_j)^2 \implies 0 < (q_j - z_j)^2 < (a_j - z_j)^2 \implies q_j < a_j$. Hence $q \notin E$, which implies there exists, for every x in E^c, a δ for which the δ-ball around x is fully contained in E^c which means E^c is open. This implies E is closed. Same argument applies for the case where $z_j > b_j$. ■

Theorem 2.18

Closed intervals in $\mathbb{R}$ are compact

Proof for Theorem.

Let $\mathbb{I}$ be, WLOG, $[-a,a]$. Suppose it is not compact. i.e, There is an open cover G_α of $\mathbb{I}$ such that there exists no finite subcover. $\forall x \in \mathbb{I}$, $\exists \alpha_x$ such that $x \in G_{\alpha_x}$ and $\exists \varepsilon_x$ such that $B_{\varepsilon_x}(x) \subset G_{\alpha_x}$. $\cup_x B_{\varepsilon_x}(x) \subset \cup_\alpha G_\alpha$ is an open cover for $\mathbb{I}$. note that, if no finite subcover for G_α exists, then no finite subcover for $B_{\varepsilon_x}(x)$ exists either. So we can safely work with $B_{\varepsilon_x}(x)$. Split the interval into two halves, $[-a,0]$ and $[0,a]$. One of these intervals is not finitely covered by $B_{\varepsilon_x}(x)$, for if not, the whole thing would be finitely covered. let that interval which is not finitely covered be $\mathbb{I}_1$. This interval's size is a. Split this interval into two again. Yet again, one of the halves must not be finitely covered, for if not, $\mathbb{I}_1$ would be finitely covered, which is contradictory. Let this interval be $\mathbb{I}_2$. This is of size $\frac{a}{2}$. Yet again, keep doing this process to obtain a sequence of intervals $\mathbb{I}_j$, sized $\frac{a}{j}$, which are not finitely covered. These are nested intervals, non empty, and closed. From nested intervals theorem, we see that a point ξ exists in $\cap_{j=1}^{\infty}\mathbb{I}_j$. ξ is a point in $\mathbb{I}$, and there is a corresponding ε_ξ. Consider that j_0 for which $\frac{a}{j_0} < \frac{\varepsilon_\xi}{2}$. We know from Archimedean such a j_0 exists. This means that the interval $\mathbb{I}_{j_0}$ containing ξ, sized $\frac{a}{j_0}$, is completely inside the ε_ξ-ball around ξ, which means it is finitely covered. Contradiction. Hence, $\mathbb{I}$ is compact. ■

Corollary 2.19

Since intervals of the form $[-a,a]$ are compact, every closed interval of the form $[a,b]$ is compact since it would be a closed subset of an interval of the form $[-x,x]$.

Generalisation:

Theorem 2.20

n-cells are compact

Proof for Theorem.

Consider $K := \{\vec{x} \in \mathbb{R}^n : -a \leq x_j \leq a; \forall j \leq n\}$ to be non-compact. There is an open cover G_α of K such that there exists no finite subcover. $\forall x \in K$, $\exists \alpha_x$ such that $x \in G_{\alpha_x}$ and $\exists \varepsilon_x$ such that $B_{\varepsilon_x}(x) \subset G_{\alpha_x}$. $\cup_x B_{\varepsilon_x}(x) \subset \cup_\alpha G_\alpha$ is an open cover for K. note that, if no finite subcover for G_α exists, then no finite subcover for $B_{\varepsilon_x}(x)$ exists either. So we can safely work with $B_{\varepsilon_x}(x)$. Till here, everything is the same as the 1-d case. Note that here, the n-cell is constructed by taking the cartesian product of n- intervals in $\mathbb{R}$ of the kind $[-a,a]$. Construct 2^n subdivisions of K by halving each interval $[-a,a]$ in the construction of K. The total number of subdivisions we make would be $2 \times 2 \times \cdots \times 2$, n times (simple combinatorial argument: for each i, there exists 2 choices, the two half intervals, for crossing. From $i = 1$, you have 2 choices, likewise, $j = 2, 3, \cdots n$). We assume that atleast one of these 2^n subdivisions are not finitely covered by $\{B_{\varepsilon_x}(x)\}$. We let this one be K_1, whose each interval size is now a. Subdivide this yet again into 2^n subsets, and assert that one of these subdivisions is not finitely covered. Call this K_2, whose each interval is of size $\frac{a}{2}$. Construct a sequence of sets K_j, each of whose intervals are sized $\frac{a}{j}$. Each K_j is closed and non empty, hence compact, and are nested. Therefore, $\cap_{j=1}^{\infty} K_j \neq \phi$. Let $p \in \cap_{j=1}^{\infty} K_j$. For this p, there would exist a ε_p and the corresponding ball $B_{\varepsilon_p}(p)$. We require one of our K_j n-cell to fall into this ε_p ball. Let δ be smaller than $\frac{\varepsilon_p}{2}$. Let p be the centre of the $\delta-$ball. Let p be the centre of the n-cube H in the following construction: Consider w to be the side length of H. We require the diagonal length $w\sqrt{n} = \delta$, which gives us $w = \frac{\delta}{\sqrt{n}}$. Consider H such that each side is the interval $[p_j - \frac{w}{2}, p_j + \frac{w}{2}]$. This would force p to fall in the centre of the n-cube H. This cube is fully contained in the δ ball of p which is contained in the ε_p ball of p. We consider that n cell K_j for which each side $\frac{a}{j} < w$. This can be found, and hence, the this K_j cell is finitely covered by $B_{\varepsilon_p}(p)$, which is absurd. Hence, K is compact. ∎

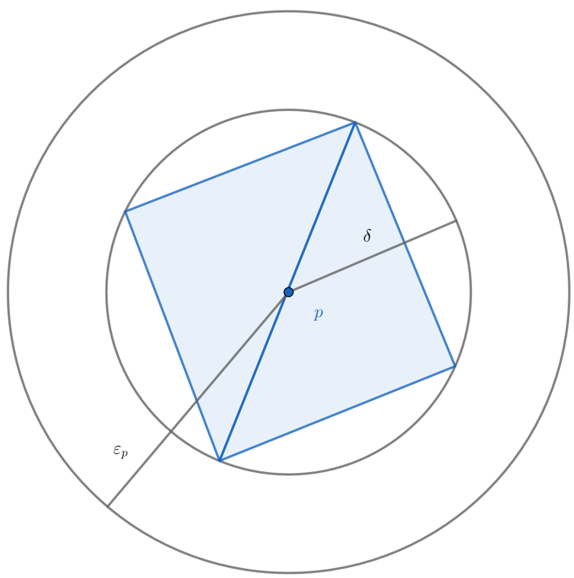

Figure 3.2: Figure for the proof: n-cells are compact.(The ε_p-ball around p, and the n-cell construction)

Remark.

We proved the result for k-cells of the kind $[-a, a]^k$, but it is easily generalised by noting that arbitrary k-cells are contained in some k-cell of the above kind. By virtue of being closed, they also are compact.

Theorem 2.21: Heine-Borel

Given a set $E \subset \mathbb{R}^n$, the following are equivalent:

1. E is closed and bounded
2. E is compact

Proof for Theorem.

$\Longleftarrow$) We know that all compact sets are closed.

$\Longrightarrow$) If $E \subset \mathbb{R}^n$ is closed and bounded, it is contained in some n-cell, which is compact. By virtue of being a closed subset of a compact set, E is also compact. ■

Theorem 2.22

If $\{x_n\}$ is a sequence in X convergent to $x \in X$, then the set $\{x_n\}$ has only one limit point, which is x.

Proof for Theorem.

That x is a limit point is clear. We note that $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \in \mathbb{N}, n \geq n_0$ we have $d(x_n, x) < \varepsilon$. i.e, beyond a particular n_0, every point of $\{X_n\}$ falls in the ε-ball of x. Therefore, only finite points lie outside this ε-ball of x. Suppose it has another limit point y, other than x. Therefore, there would exist a δ such that the δ-ball around y lies completely outside the δ-ball around x. This means that, only finite points of x_n lie in the δ ball of y, making it unviable to be a limit point. ■

Extending Heine Borel we have:

Theorem 2.23: (Extension)

For a subset $E \subset \mathbb{R}^n$, the following are equivalent:

1. E is closed and bounded
2. E is compact
3. every infinite subset K of E has a limit point in E

Proof for Theorem.

(1) $\Longrightarrow$ (2)) Heine Borel

(2) $\Longrightarrow$ (3)) Already seen

(3) $\Longrightarrow$ (1)) Let us assume that E is either not closed, or not bounded. We start by assuming it is not closed. Which means that $\exists q$ outside E such that there exists a

sequence in E that converges to q. We take this sequence $\{x_n\}$ as our infinite set, and we see that, from the previous theorem, this has only one limit point q, which lies outside E. Hence, there exists an infinite set $\{x_n\}$ which has no limit point in E.

Suppose that E is unbounded. We then have that, $\forall x \in X$, $\forall \varepsilon > 0$, $\exists y \in E$ such that $d(x, y) > \varepsilon$. Fix some x_0 in X. Choose some y_0 that is a distance $z_{00} = d(y_0, x_0)$ away from x_0. Look if there is a point y_1 so that its distance from x_0 is more than z_{00} but less than $2(z_{00})$. If it doesn't exist, check for less than $3(z_{00})$. Find some $k_1(z_{00})$ so that distance of y_1 to x_0 is more than z_{00} but less than $k_1(z_{00})$. Same way, for y_2, find y_2 so that its distance from x_0 is more than $k_1(z_{00})$ but less than some other $k_2(z_{00})$. Inductively, find y_j whose distance is more than $k_{j-1}(z_{00})$ but less than $k_j(z_{00})$. Note that $1 < k_1 < k_2 < \cdots$. Hence, for any ε-ball around x_0, only finite y_j exists in that ball, since there would exist some $k_q(z_{00})$ and $k_{q-1}(z_{00})$ between which ε lies. And inside $k_{q-1}(z_{00})$ ball around x_0, utmost finite points y_j exists. Hence, x_0 is clearly not a limit point for the set of y_j-s. Consider any other point $a \in X$. For some every ε-ball around x_0, only finite points exists. For some, perhaps larger δ-ball around a, a chosen ε-ball around x_0 gets subsumed into the δ-ball around a. This implies that only finite points of y_j-s exists in the δ-ball around a as well, making a a non viable limit point. We see that, for this infinite subset $\{y_j\}$ of E, no limit point exists. ■

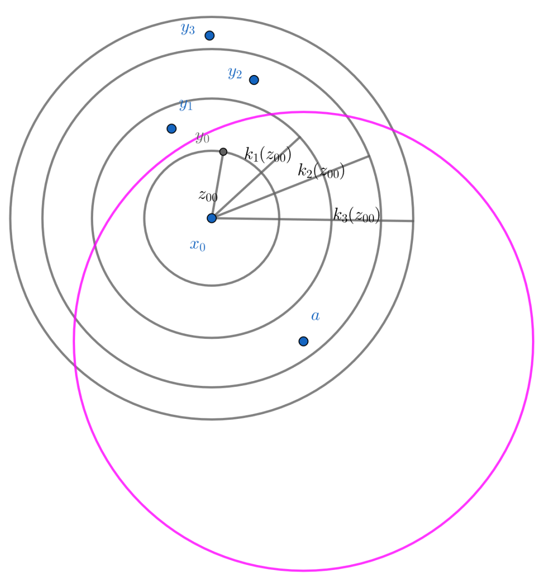

Figure 3.3: Figure for proof: Lim point compact $\implies$ closed+bounded. (The construction of an unbounded sequence)

Remark.

> In the previous proof, we note that (3), which is called Limit Point Compactness, implies (1) Closed and Bounded, in any metric space, not just $\mathbb{R}^n$, as we see in the proof, no property of $\mathbb{R}^n$ was used.
>
> **Spoiler Alert:** In any metric space, Limit Point Compact $\iff$ Compact

Theorem 2.24: Weierstrass Theorem

Every Bounded, infinite set in $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.

Proof for Theorem.

> If a set is bounded in $\mathbb{R}^n$, it is the subset of a compact set (i.e, a closed and bounded set). From the previous equivalence, an infinite subset E of a compact set has a limit point in the compact set, which means the bounded, infinite set we have has a limit point in the compact set that contains it, hence, it has a limit point in $\mathbb{R}^n$. ■

Remark.

> The above "Weierstrass Theorem" is just the "Bolzano-Weierstrass" Theorem we saw in sequences. Actually, the "Bolzano-Weierstrass" Theorem is a direct corollary of the more general "Weierstrass Theorem". Let $\{x_n\}$ be any sequence in $\mathbb{R}$ that is bounded. This means that this sequence is the subset of a compact set, hence, has a limit point in $\mathbb{R}$. This implies, a subsequence of $\{x_n\}$ converges in $\mathbb{R}$. Hence, every bounded sequence has a convergent subsequence.

Fact 2.25

Let $X = \mathbb{R}^n$. The closure of any open ball is the corresponding closed ball.

Proof. Consider $B := B_\delta(x_0) := \{y \in \mathbb{R}^n : ||y - x_0|| < \delta\}$. Let z_0 be a point on the rim of B, i.e $d(x_0, z_0) = \delta$. Such a point obviously exists. Consider $\vec{\gamma}(t) = t\vec{z}_0 + (1-t)\vec{x}_0$ with $t \in (0,1)$. For every $t \in (0,1)$, $\vec{\gamma}(t)$ belongs in B. To see this, consider $||\vec{\gamma}(t) - \vec{x}_0|| = ||t\vec{z}_0 + (1-t)\vec{x}_0 - \vec{x}_0|| = ||t(\vec{z}_0 - \vec{x}_0)|| = t||\vec{z}_0 - \vec{x}_0|| < t(\delta) < \delta$. Suppose we are given an arbitrary $\eta > 0$. Does there exist a $t \in (0,1)$ so that $\vec{\gamma}(t)$ belongs in the η-ball of $\vec{z}_0$? i.e, we need a t so that $||\vec{\gamma}(t) - \vec{z}_0|| = ||t\vec{z}_0 + (1-t)\vec{x}_0 - \vec{z}_0|| = ||(1-t)(\vec{z}_0 - \vec{x}_0)|| < \eta \implies (1-t)||\vec{z}_0 - \vec{x}_0|| < \eta \implies 1 - t < \frac{\eta}{\delta} \implies 1 - \frac{\eta}{\delta} < t < 1$. Such a t exists for every η. Hence, $\vec{z}_0$ is a limit point of B (by virtue of there existing a sequence of $\vec{\gamma}(t_j)$ that converges to $\vec{z}_0$). Hence, every point on the rim is a limit point. Moreover, no point w so that $d(w, x_0) > \delta$ is a limit point of B, since there would exist an ε-ball around w so that no point of B falls into it (from openness). Hence, closure of B is the corresponding closed ball, in $\mathbb{R}^n$.

□

3 Perfect Sets

Definition 3.1: Perfect Set

A set $E \subset X$ is perfect if every point of E is a limit point of E, and E is closed

Theorem 3.2

Perfect subsets in $\mathbb{R}^n$ are uncountable.

Proof for Theorem.

Suppose E is a perfect set in $\mathbb{R}^n$ but is countable. i.e, it can be enumerated as $E = \{x_1, x_2, \cdots\}$.

Choose x_1, and $\varepsilon_0 = 1$. Let V_0 denote the ε_0-ball around x_1. This ball is non empty, moreover, $\bar{V_0} \cap E$(which is the corresponding closed ball of V_0) is non empty, and is compact by virtue of being closed and bounded. Inside, $V_0 \cap E$, there exists infinite points of E, since x_1 is a limit point of E.

Choose an arbitrary point z_1 in V_0 that is not x_1. Now let $\varepsilon_1 < d(x_1, z_1)$. Let V_1 be the ε_1-ball around z_1. Notice the following: z_1 is a limit point of E, hence, there are infinite points of E in V_1. x_1 is not in $\bar{V_1}$. $\bar{V_1} \cap E$ is closed, bounded and non empty, hence Compact.

Choose a point z_2 in V_1 that is not x_2, and let $\varepsilon_2 < min\{\varepsilon_1, d(x_2, z_2)\}$. Let V_2 be the ε_2-ball around z_2. Note that, x_2 is not in $\bar{V_2}$. Also note yet again that there are infinitely many points of E in V_2. It is crucial to note now that $\bar{V_2} \cap E \subset \bar{V_1} \cap E \subset \bar{V_0} \cap E$.

Suppose you have already constructed V_k by finding z_k in V_{k-1} that is not x_k and an $\varepsilon_k < mind(z_k, x_k), \varepsilon_{k-1}$ such that $x_k \notin \bar{V_k}$, $\bar{V_k} \cap E$ is compact, non empty and $\bar{V_k} \cap E \subset \bar{V_{k-1}} \cap E \cdots$.

Now, choose $z_{k+1} \neq x_{k+1}$, inside V_k. Choose $\varepsilon_{k+1} < min\{d(z_{k+1}, x_{k+1}), \varepsilon_k\}$. Let V_{k+1} be the ε_{k+1}-ball around z_{k+1}. Yet again, we see that $\bar{V_{k+1}} \cap E$ is non empty, x_{k+1} is not in $\bar{V_{k+1}}$, and $\bar{V_{k+1}} \cap E \subset \bar{V_k} \cap E$. Hence, we have a sequence of non empty, nested compact sets. This implies that $\exists \xi \in E \subset \mathbb{R}^n$ such that $\xi \in \cap_{i=1}^{\infty}(\bar{V_i} \cap E)$. Is ξ any one of x_j enumerated? No, because if it was, from the construction, x_j would not belong in V_j. Hence, ξ is not in the enumeration of E. Contradiction. ■

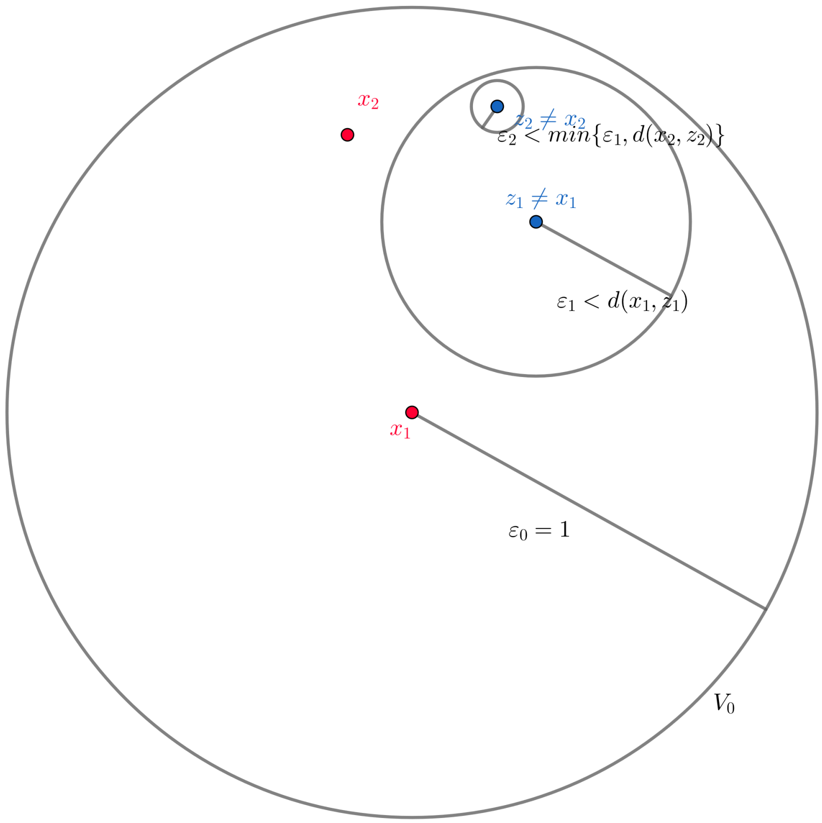

Figure 3.4: Figure: Perfect sets are uncountable. Construction of the nested sequence of compact sets by choosing $z_k \neq x_k \in V_{k-1}$.

Remark.

> It is easily seen that, closed intervals in $\mathbb{R}$ are perfect: From density theorem, for every point in I, there would exist a sequence of rationals converging to that point. Moreover, closed intervals in $\mathbb{R}$ are closed since closed balls in metric spaces are closed. Therefore, we see that intervals are uncountable.

3.1 The Cantor Set

The following is the construction of an uncountable, perfect set that contains no intervals: The Cantor Set.

Let $I_0 = [0, 1]$. size of the interval(s) in I_0 is 1, and there are $2^0 = 1$ intervals.

Let $I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ be constructed by trisecting I_0 and tossing the middle one. Here, we have each interval sized $\frac{1}{3^1}$, and there are $2^1 = 2$ intervals total.

Let $I_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ be generated by taking each of the two sub intervals in I_1, trisecting them, and tossing the middle one, and joining them finally. We have each interval sized $\frac{1}{3^2}$ and there are $2^2 = 4$ intervals total.

Inductively keep making these trisections+tossings to make a sequence of closed, nested intervals (Compact, too) I_k, each containing 2^k intervals each of size $\frac{1}{3^k}$.

Finally, define the Cantor set P as

$$P := \cap_{i=1}^{\infty} I_i$$

Note that P is compact since it is the closed subset of a compact set. It is also non empty by virtue of being the intersection of a sequence of nested, non empty, compact sets.

Note that, no interval of the kind $[a, b]$ exists in the Cantor Set. The size of each interval in I_j is $\frac{1}{3^j}$. We can find j so that $\frac{1}{3^j} < b - a \implies \frac{1}{b-a} < 3^j \implies \log_3(\frac{1}{b-a}) < j$. For such I_j, we notice that $[a, b]$ has "inbetween" points that doesn't exist in any of I_j's intervals. Hence, taking the intersection, these "inbetween" terms don't survive. Hence, no intervals exist.

Theorem 3.3

The Cantor set P is perfect.

Proof for Theorem.

> We already know that the Cantor set is closed. We need to show that every point in the cantor set is a limit point. First, observe that, for any I_k, if z is the end point of any of the sub interval of I_k, it survives the ∞-intersection. This is because, after I_k-s trisection, the end points still stay endpoints. Let ξ be any point in the cantor set, which means it is a point in every I_k. Let $\delta > 0$ be given. Consider the interval $(\xi - \delta, \xi + \delta)$. This interval is sized 2δ. ξ exists in one of the sub intervals of I_k for all $K \geq k_0$ for some k_0. Choose j so that $\frac{1}{3^j} < \delta$. Then, the interval in I_j containing ξ would fall completely inside $(\xi - \delta, \xi + \delta)$. Choose q as one of the end points of this sub interval of I_j. Therefore, $\forall \xi \in P$, $\forall \delta > 0$, $\exists q \in P, q \neq \xi$ so that $q \in (\xi - \delta, \xi + \delta)$. Therefore, every $\xi \in P$ is a limit point of P. Hence, P is perfect. ■

4 Connected Sets

Definition 4.1: Separated Sets

$A \subset X$ and $B \subset X$ are said to be *separated* if $\bar{A} \cap B = \bar{B} \cap A = \phi$, i.e, they are disjoint and no point of one, is the limit point of the other.

Definition 4.2: Connected Set

A set $E \subset X$ is said to be *connected* if it is *not* the union of two non-empty separated sets. In other words, for every "split" of E into two non empty sets, none of them are separated. Even if one split of E is separated, then E is *not connected.*

Example : Separated $\implies$ Disjoint, but Disjoint $\not\Longrightarrow$ Separated.
$[0,1]$and $(1,2)$ are disjoint, but are not connected since a sequence in $(1,2)$ converges to 1 in $[0,1]$.

Theorem 4.3

$E \subset \mathbb{R}$ is connected $\iff \forall x, y \in E,\ x < z < y \implies z \in E$.

Proof for Theorem.

$\implies$) Suppose $\exists x_0, y_0 \in E$ so that $\exists z, x_0 < z < y_0$, but $z \notin E$. Consider $A := (-\infty, z)$ and $B := (z, \infty)$. A and B are seen to be separated, and E is a subset of $A \cup B$, which makes it disconnected.

$\impliedby$) Suppose that we have E disconnected, which means it is the union of two separated sets A and B that are non-empty. $x_0 \in A$ and $y_0 \in B$. Consider $z(t) = x_0 + t(y_0 - x_0)$ for $t \in [0,1]$. Note that $z(0) = x_0$ and $z(1) = y_0$.

Conjecture: There exists a $t_B \in (0,1)$ so that for every $t < t_B$, $z(t)$ does not belong in B. If it is not true, then for every $t \in (0,1)$, there exists a point $t_B < t$ so that $z(t_B)$ is in B. Choose $t = 1$ to get $z(t_1)$ in B. Chooe $t = \frac{t_1}{2}$ to get $z(t_2)$ in B with $t_2 < t_1$ and $t_2 < \frac{1}{2}$. Keep going with $t = \frac{t_{n-1}}{2^{n-1}}$ to get $z(t_n)$ in B with $t_n < t_{n-1}$ and $t_n < \frac{1}{2^{n-1}}$. This gives us a sequence $z(t_k)$ which we can see is monotone decreasing assuming $x_0 < y_0$. This sequence converges to $z(0)$ which is in A which means that there exists a sequence in B, $z(t_n)$ that converges to A. Absurd.

In a similar vein, we can show that there exists $t_A \in (0,1)$ so that for every $t > t_A$, $z(t)$ is not in A. Consider

$$S_A := \{t \in [0,1] : z(t) \in A \cap [x_0, y_0]\}$$

and

$$S_B := \{t \in [0,1] : z(t) \in B \cap [x_0, y_0]\}$$

It is easy to see that S_A and S_B are disjoint. If a sequence in one converges in another, say $t_n \in S_B$ converges to $t_0 \in S_A$. Then $z(t_n) \in B$ by definition, for every n. But then by definition, $z(t_n) = x_0 + t_n(y_0 - x_0) \in B$ such that $lim(z(t_n)) = x_0 + t_0(y_0 - x_0) \in A$, which means a sequence in B converges in A. Absurd. So S_A and S_B are separated.

Note that, for $t > t_A$, no $z(t)$ is in A. Hence, we see that for every t so that $z(t)$ falls in A, there is an upperbound. Likewise, for every t such that $z(t)$ falls in B, there is a lowerbound. Hence, S_A has a supremum $sup(S_A)$ and S_B has an infimum $inf(S_B)$.

At this point, we may as well assume that for every $t < inf(S_B)$, $t \in S_A$ for if not, what we wanted to prove would get proved. Suppose then, for argument sake, that for every $t > Sup(S_A)$, $t \in S_B$, and likewise, for every $t < Inf(S_B)$. $t \in S_A$. Now then, does $Sup(S_A)$ belong in S_A? we see that for every $t > sup(S_A)$, $t \in S_B$ which means we can construct a sequence in S_B using those ts, which converge to $Sup(S_A)$ in S_A. So that is ruled out. So is $Sup(S_A)$ in S_B? That is not possible either, since S_A is a bounded, infinite set (mainly because supremum isn't in the set), we know that there is a monotone subsequence in S_A converging to $Sup(S_A)$ which is in S_B. We therefore conclude that, there exists a point $t \in (0, 1)$ so that t is neither in S_A, nor in S_B. This translates to there being a point $z = z(t)$, between x_0 and y_0 so that $z(t) \notin A \cup B \implies \notin E$. We are, therefore, done.

Slicker Argument: Suppose $E = A \cup B$ with $\bar{A} \cap B = \bar{B} \cap A = \phi$. Consider $x_0 \in A$ and $y_0 \in B$ and WLOG assume $x_0 < y_0$. Define $z = sup(A \cap [x_0, y_0])$. There would be a sequence in A that converges to z, by virtue of being the supremum. $z \in \bar{A} \implies z \notin B$. This means $x_0 \leq z < y_0$. If $z \notin A$, we would be done. If $z \in A$, then $z \notin \bar{B}$. Therefore, z is in an open set $\bar{B}^C$. There would exist an ε_z-ball around z so that it is fully contained outside $\bar{B}$. Choose $z + \frac{\varepsilon_z}{2}$ as your z'. Note that z' is greater than the supremum of A. We see that z' is not in B, and not in A either. Hence, we are done. ■

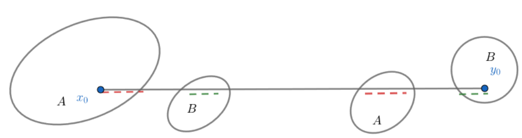

Figure 3.5: Figure: Proof for the equivalence for connectedness for sets in $\mathbb{R}$. A look at $A \cap [x_0, y_0]$ and $B \cap [x_0, y_0]$.

5 Misc Knowledge

Theorem 5.1

Suppose $A_1, A_2 \cdots, A_n \cdots \in X$. Then,

1. If $B_n = \cup_{i=1}^{n} A_i$, then $\bar{B_n} = \cup_{i=1}^{n} \bar{A_i}$
2. $B = \cup_{i=1}^{\infty} A_i$, $\bar{B} \supseteq \cup_{i=1}^{\infty} \bar{A_i}$ with possibility of strict inequality.

Proof for Theorem.

Suppose $x \in \cup_{i=1}^{n} \bar{A}_i$. Which means $x \in \bar{A}_i$ for some i. It is clear that x is either a point of A_i or a limit point of A_i. We have that, whatever maybe the case either x is a point in B_n or a limit point of B_n. Therfore $\bar{B_n} \supset \cup_{i=1}^{n} \bar{A}_i$.

Suppose x is a point in $\bar{B_n}$. If it is a point of B_n, we are done. Suppose it is the limit point of B_n, but not a point. Also suppose that x is not a limit point of any $A_i : i = 1 \to n$. This means that,

$$\exists \varepsilon_1 \text{ such that } \forall q \in A_1, q \neq x, \text{ we have } d(q, x) \geq \varepsilon_1$$

$$\exists \varepsilon_2 \text{ such that } \forall q \in A_2, q \neq x, \text{ we have } d(q, x) \geq \varepsilon_2$$

$$\vdots$$

$$\exists \varepsilon_n \text{ such that } \forall q \in A_n, q \neq x, \text{ we have } d(q, x) \geq \varepsilon_n$$

If we choose $0 < \varepsilon_0 < min\{\varepsilon_i : i = 1 \to n\}$, we would have that, for every point q in $A_1 \cup A_2 \cdots A_n$, $q \neq x$(which is needless to say), we have $d(q, x) \geq \varepsilon_0$ which makes x a non-limit point of B_n, which is absurd. Hence we see that if x is point of B_n or a limit point of B_n, then it is a point or the limit point of some A_j.

For a good counterexample, we look to $\mathbb{Q} := \{q : \text{ q is rational}\}$. This set is countable. Let $\{q_1, q_2, \cdots\}$ be the enumeration of $\mathbb{Q}$. Consider $\mathbb{Q} := \cup_{j=1}^{n} \{q_j\}$. The closure of $\mathbb{Q}$ is $\mathbb{R}$ but since these singleton sets are by definition closed, the union of them only gives you $\mathbb{Q}$. ■

Definition 5.2: Interior of a Set

Given $S \in X$, the interior $\underline{\mathrm{S}}$ is defined as:

$$\underline{\mathrm{S}} := \{x \in X : \exists \varepsilon_x > 0 \text{ such that } B_{\varepsilon_x}(x) \subset S\}$$

Theorem 5.3

The Interior is an open set

Proof for Theorem.

Consider $(\underline{\mathrm{S}})^C := \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap S^C \neq \phi\}$. It is possible that x is a point of S^C, if it is in $\underline{\mathrm{S}}^C$. Suppose it is a point of $\underline{\mathrm{S}}^C$ but not a point of ^{C}. From the definition, we see that $\forall \varepsilon > 0, \exists q \in S^C, q \neq x$ such that $d(q, x) < \varepsilon$. This makes x a limit point of S^C, which means, for every x in $\underline{\mathrm{S}}^C$, x is either a point of S^C or a limit point of S^C. Hence,

$$\underline{\mathrm{S}}^C \subseteq \bar{S^C}$$

Suppose x is a point of $\bar{S^C}$. Say it is a point of S^C, then obviously, it is a point of $(\underline{\mathrm{S}})^C$. Suppose x is not a point of S^C, but a limit point of S^C. This means $\forall \varepsilon > 0, \exists q \in S^C, q \neq x$ so that $d(q, x) < \varepsilon$. This is precisely the condition for which x is a point of $(\underline{\mathrm{S}})^C$. Hence we see $(\underline{S})^C \supseteq (\bar{S^C})$. Therefore, $(\underline{S})^C = (\bar{S^C})$. From here we see that $\underline{S}$ is an open set.

Alternate Argument (similar): Consider $(\underline{S})^C := \{x \in X : \forall\varepsilon > 0, B_\varepsilon(x) \cap S^C \neq \phi\}$ Consider a limit point p of $(\underline{S})^C$. $\forall\varepsilon > 0, \exists q_\varepsilon \in \underline{S}^C$ such that $d(q_\varepsilon, p) < \frac{\varepsilon}{2}$. Since q_ε is in $(\underline{S})^C$, we have that: $\forall\delta > 0, \exists r_\delta \in S^C, r_\delta \neq q_\varepsilon$ such that $d(r_\delta, q_\varepsilon) < \delta$

Combining these we have:

$$(\forall\varepsilon > 0)(\exists q_\varepsilon \in \underline{S}^C)(\exists\delta > 0)(\exists q_\delta \in S^C)$$

$$(d(q_\delta, p) \leq d(q_\delta, q_\varepsilon) + d(q_\varepsilon, p) < \frac{\varepsilon}{2} + \delta < \varepsilon)$$

This means p is a limit point of S^C. Hence, $\underline{S}^C$ is closed. ■

Theorem 5.4

$\underline{S} = S \iff S$ is open

Proof for Theorem.

$\Longrightarrow$) if $\underline{S} = S$, obviously S is open.

$\Longleftarrow$) If S is open, then by definition $\underline{S}$ = set of all points in S so that there's an ε-ball of x in S. But that is every point of S. ■

Theorem 5.5

$\underline{S}$ is the largest open set contained in S

Proof for Theorem.

Consider an open subset of S. These are subsets of S whose each point has an ε-ball around it so that the ball is contained in the subset, which is contained in S. So by definition, these points in these subsets are contained in $\underline{S}$. ■

Theorem 5.6

$$(\underline{S})^C = \overline{(S^C)}$$

"The compliment of the interior is the closure of the compliment"

Proof for Theorem.

Refer to the proof of "Interiors of sets are Open", to see this construction.

Alternate method(slicker):

$\underline{S} \subseteq S \implies S^C \subset (\underline{S})^C$ where $(\underline{S})^C$ is a closed set containing S^C. Since $\overline{S^C}$ is the smallest closed set that contains S^C, we have $\overline{S^C} \subseteq (\underline{S})^C$.

Note that $S^C \subseteq \overline{S^C} \implies (\overline{S^C})^C \subseteq S$ where $(\overline{S^C})^C$ is an open set inside S. Since $\underline{S}$ is the largest open set containing S, we have that $(\overline{S^C})^C \subseteq \underline{S} \implies \underline{S}^C \subseteq \overline{S^C}$. Combining these two set inequalities, we are done.

Theorem 5.7

1. If A and B are closed, disjoint subsets of X, then A and B are separated.
2. If A and B are open, disjoint sets, then A and B are separated.

Proof for Theorem.

(1) A B closed implies $A = \overline{A}$ and $B = \overline{B}$ which are disjoint. From here it is obvious.

(2) A and B are open disjoint sets, then we see that $A \subseteq B^C$ and $B \subseteq A^C$ where A^C and B^C are closed by definition. Since closure is the smallest closed set containing A (and B), we see that $\overline{A} \subseteq B^C$ and $\overline{B} \subseteq A^C$. It is now trivial to see that $\overline{A} \cap B \subseteq B^C \cap B = \phi = \overline{B} \cap A$ which is the definition of separated.

Corollary 5.8

Let $p \in X$ and $\delta > 0$. Define $A := B_\delta(p)$ and $B = (B_{[\delta]}(p))^C$. A and B are, then, separeted.

Proof for Corollary.

Easy to see that they are both open sets that are disjoint.

Theorem 5.9

Every connected metric space with atleast two points is uncountable.

Proof for Theorem.

Let a and b be in X. Let $\xi \leq d(a, b)$. Note that, $P = B_\xi(a)$ and $Q = (B_{[\xi]}(a))^C$ are non empty, separated sets (from the previous corollary). If X is not the union of P and Q, then there is a point z_ξ in X so that it is neither in P not in Q. That means that it is exactly ξ distance away from p. For every $\xi < d(a, b)$, there exists a point z_ξ so that its distance from p is exactly ξ. Therefore, every z_ξ is unique (from positivity property of metric spaces) which means there are uncountable z_ξ-s.

Theorem 5.10

If P and Q are connected such that $P \cap Q \neq \phi$, then $P \cup Q$ is also connected.

Proof for Theorem.

Suppose $P \cup Q$ is actually not connected. This means $P \cup Q = A \cup B$ for non empty, separated sets A and B. Suppose P is fully contained in A. This means that Q has intersection with A and intersection with B which are non empty. Obviously $Q \subseteq A \cup B$ which means $Q = (A \cap Q) \cup (B \cap Q)$ where $(A \cap Q)$ and $(B \cap Q)$ are separated and non empty. Since Q is connected, this is absurd. Suppose then that P is not fully contained in A. This means that $P \cap A$ and $P \cap B$ is non empty each. This means $P = (P \cap A) \cup (P \cap B)$. From the same reasoning, this is absurd. ■

Lemma 5.11

Given two balls B_1 and B_2 in $\mathbb{R}^n$ that are closed, with $B_1 \cap B_2 = \{z\}$ with $z \in \mathbb{R}^n$, then the interior of $B_1 \cup B_2$, i.e, $\underline{B_1 \cup B_2}$, is $\underline{B_1} \cup \underline{B_2}$, which are their respective open ball counterparts.

Proof for Lemma

We understand that $\underline{B_1} \cup \underline{B_2} \subseteq \underline{B_1 \cup B_2}$. Note that none of the "rim" points of B_1 or B_2, are in the interior. This would conclude the result. ■

Corollary 5.12

If $A \subset X$ is connected, it needn't be true that $\underline{A}$ is connected.

Proof for Corollary.

Consider the set $B_1 \cup B_2$ from the previous lemma. We note that, its interior is the disjoint union of two non empty open balls. These two sets are separated, which makes $\underline{B_1 \cup B_2}$ a separated set. ■

Theorem 5.13

Let E be the set of all $x \in [0, 1] \subseteq \mathbb{R}$ so that the decimal expansion of x only contains 4 and 7. Then:

1. E is uncountable
2. E is not dense in $[0, 1]$
3. E is compact
4. E is prefect

Proof for Theorem.

(1) E is uncountable via the diagonal argument. If $\{x_1, x_2, \cdots\}$ is the enumeration of E, simply take the first decimal place of x_1 and flip it (i.e. to 4 if 7 or vice versa). Likewise for x_2 and so on to get a new decimal expansion that is unlike all $x_1, x_2 \cdots x_n \cdots$ which is a contradiction.

(2) Obviously, since $0.4 \leq x \leq 0.8$ for any $x \in E$.

(3) We already know E is bounded. Consider E^c. This is the set of all numbers in $[0,1]$ so that not all points in the decimal expansion is 4 or 7. i.e, there would be a point in the expansion that is neither 4, nor 7. Suppose we take one such arbitrary $x \in E^c$. Let the first non $4,7$ number occur at the $j-$th place. $0.z_1 z_2 z_3 \cdots x_j \cdots$. Look for another non $4,7$ after the $j-$th place (if it exists). If it doesn't exist, then all the numbers after x_j would be 4 or 7 so safely add $10^{-(j+1)}$ as our ε. This ε range around x would contain only points of E^c. Suppose another point exists that is non $4,7$ after j, perhaps at $k > j$-th index. Then it would look something like: $0.z_1 z_2 \cdots z_j x_{j+1} x_{j+2} \cdots z_k \cdots$. Here we simply take $\varepsilon = 10^{-(k+1)}$ so that all points in the ε-neighbourhood of x is in E^c. Therefore, E^c is open, which means E is closed. Closed and bounded implies compact in $\mathbb{R}$.

We saw that E is closed. We need only show that every point of E is a limit point of E. This can be done easily for any ε-ball, using the technique that follows: Choose a k so that $\frac{1}{10^k} < \varepsilon$. Look at the interval $x - \frac{1}{10^k}, x + \frac{1}{10^k}$ that is contained in E. Just find some $n > k$, and flip the 4 to a 7 or vice versa to land in a "different" element from x, yet within the neighbourhood in consideration. Hence, every point is a limit point, making E a perfect set.

Theorem 5.14: Existence of a compact set in $\mathbb{R}$ with countable limit points.

Title

Proof for Theorem.

Consider the points $x_1 = 1, x_0 = 0, x_2 = \frac{1}{2}, \cdots x_n = \frac{1}{n} \cdots$. Let $x_{11} < x_{12} \cdots < x_{1n}$ be a sequence that converges to $x_1 = 1$. Let $x_{21} < x_{22} < x_{23} \cdots < x_{2n} \cdots$ be a sequence convergent to $x_2 = \frac{1}{2}$, with the added condition that $x_{2j} < x_{1k}$ for every $j, k \in \mathbb{N}$. Likewise for every x_n, create a sequence x_{nk} that converges to x_n. Make sure that $x_{an} < x_{bm}$ if $a > b$, for every m, n. We therefore have:

$$x_{11} < x_{12} \cdots \to x_1$$

$$x_{21} < x_{22} \cdots \to x_2$$

$$\vdots$$

We claim that the set $\Phi = x_0, x_1, x_2, \cdots$ along with $x_{11}, x_{12} \cdots, x_{21}, x_{22} \cdots x_{n1}$ etc. forms a Compact set in $\mathbb{R}$ that has countable limit points.

Suppose that $q \in \mathbb{R} \neq 0$ and $q \neq \frac{1}{j}$ for any j be a limit point of Φ. This means a subsequence z_n in Φ converges to q. Does infinite points of x_{1j} exist in the subsequence z_n convergent to q? Obviously not, since that would make a subsequence of z_n convergent to one of 1. So only utmost finite elements of x_{1j} are in z_n. Same way one can argue that utmost finite elements of x_{kj} are in z_n for every k. If we establish a subsequence of z_n that converges to 0, then the only limit point of $\{z_n\}$ would be 0 which would mean 0 is where z_n would converge. Enough wishful thinking; Does any point of x_{1j} exist in z_n? If yes, choose that point. If not, find the next n_1 so that a point in x_{n_1j} is in z_n. Find, then $n_2 > n_1$ so that some point in x_{n_2j} is in z_n. As such keep going, making $n_1 < n_2 < \cdots$ and a sequence that is monotone decreasing by construction, that converges to 0. Hence, this is a subsequence in $\{z_n\}$ that converges to 0. If z_n converged to q, then the only limit point would be q. Hence, $q = 0$. This means that the only limit points of Φ other than $\frac{1}{j}$ is 0. Hence, we have a countable limit point. Hence, Φ is a closed set and bounded obviously, with countable limit points. ■

Theorem 5.15: technique weve already seen

If A and B are separated sets in $\mathbb{R}^n$ (that are non empty), and $\vec{x_0} \in A$ and $\vec{y_0} \in B$, define $p(t) = \vec{x_0} + t(\vec{y_0} - \vec{x}_0)$ for $t \in [-\infty, \infty]$ and

$$S_A := \{t \in \mathbb{R} : p(t) \in A\}$$

$$S_B := \{t \in \mathbb{R} : p(t) \in B\}$$

Then:

1. S_A and S_B are separated sets
2. $\exists t_0 \in (0, 1)$ so that $t_0 \notin S_A \cup S_B$

Proof for Theorem.

(1) If S_A and S_B weren't disjoint, then obviously A and B wont be. If there is a sequencei n S_A converging in S_B or vice-versa, it is easy to see that this would lead to there exiting a sequence in A converging to B (or vice-versa). Therefore, S_A and S_B are separated.

(2) Since S_A and S_B are separated and non-empty, there are two points x_0 and y_0 in S_A and S_B respectively. Define $z(l) = x_0 + l(y_0 - x_0)$ for $l \in [0, 1]$. Now, for some l_A, we have that for every $l > l_A$, $z(l) \notin S_A$, the set $G_A := \{l : z(l) \in S_A\}$ is therefore bounded above with supremum u_A. Likewise for some l_B, we have that for every $l < l_B$, $z(l) \notin S_B$, the set $G_B := \{l : z(l) \in S_B\}$ is therefore bounded below with infimum v_B. Note that G_A and G_B are separated sets. We may as well assume that for every $l < l_B$, l is in G_A, or $z(l)$ falls in S_A. If not, we would be done. We now ask: does v_B fall in G_A or G_B? If it falls in G_B, there exists a sequence in G_A that would converge to v_B, which is absurd. If it falls in G_A, then by virtue of being the infimum of G_B, there is a sequence in G_B converging in G_A. Hence, there exists a point l between x_0 and y_0 $l \notin G_A$ or G_B which means $z(l) \notin S_A$ or S_B. This again means that $p(z(l)) \notin A$ or B. Phew. ■

Corollary 5.16

Every convex set in $\mathbb{R}^n$ is connected.

Proof for Corollary.

If they were not connected, then there would exist sets A and B, non empty, disjoint and separated so that our convex set C would be $A \cup B$. Suppose $x_0 \in A$ and $y_0 \in B$. From the previous theorem we see that, if $z(t) = x_0 + t(y_0 - x_0)$, then there would exist t' so that $z(t') \notin A$ or B, which means it wont be in C. But if x_0 and y_0 are in C, by definition of convexity, $z(t)$ for any t must exist in C. Contradiction.

Example : A Pedagogical Example.
If $E := \{q \in \mathbb{Q} : 2 < q^2 < 3\}$ is considered a set in the metric subspace of $\mathbb{Q}$ with the usual distance, then we have:

1. E is closed and bounded (wrt $\mathbb{Q}$ obviously)
2. E is *non compact.*
3. E is open with respect to $\mathbb{Q}$

Proof. Consider an arbitrary convergent sequence p_n in E. We see that if p_n is contained in E, then $2 < p_n^2 < 3$. If we pass to the limit, we would have $2 \leq q^2 \leq 3$ but the limit q would either not exist in $\mathbb{Q}$ (whence the sequence p_n wouldn't be convergent anyway) or it does, in which case it follows $2 < q^2 < 3$ (since no rational number has its square as 2 or 3). Therefore, we can see that every convergent sequence in E converges in E. Hence, E is closed. Boundedness is obvious.

One way to see that E is non compact is simply by making use of the "conservation" of compactness going from one space to a bigger space or vice-versa. Since E is clearly not compact in $\mathbb{R}$, it wont be compact in the metric subspace $\mathbb{Q}$. Another way to see that it is non compact is to consider the following construction: Look at the union of the two split intervals $(-\sqrt{3}, -\sqrt{2})$ and $(\sqrt{2}, \sqrt{3})$. Let $h = \sqrt{3} - \sqrt{2}$. We construct it for the right side interval, and simply copy it to the left one. Choose the point $\sqrt{3}$ and a $h/2$-ball around $\sqrt{3}$ and call it V_0. This would intersect the right interval at a distance $h/2$ from $\sqrt{3}$. Choose this intersection point and a $h/4$-ball around this called V_2. Both V_1 and V_2 together covers $3h/4$ of the interval. Choose this intersection point and let the $h/8$-ball around this point be V_3. V_1, V_2 and V_3 together cover $7h/8$ of the interval. Keep going as such to construct V_n so that $(V_i)_{i=1}^n$ covers $\frac{(n-1)h}{n}$ of the whole interval. As n tends to ∞, the "coverage" converges to h. This means that for every ε distance away, there exists a finite n_0 so that $V_1 \to V_{n_0}$ covers up to that ε distance. Every point e in the strict interval would therefore be covered up by some V_k by our construction. But obviously, this "open cover" has no finite subcover, for no finite "coverage" covers all the way till h distance of the interval. Some ε-gap is always left, hence missing points.

E can also be written as the disjoint union of the set A of all p so that $\sqrt{2} < p < \sqrt{3}$ and the set B of all p so that $-\sqrt{3} < p < -\sqrt{2}$. WLOG say $q \in A$. Obviously $\sqrt{2} < q < \sqrt{3}$. Choose $\delta < min\{\sqrt{3} - q, q - \sqrt{2}\}$ and the $\delta-$ball called V around q. Easy to see that this ball V is fully contained in A. Likewise, it can be shown for B as well. Hence, E is open with respect to Q. □

Theorem 5.17: Existence of a non-empty, perfect set in $\mathbb{R}$ that contains no rational points

Title

Proof for Theorem.

Definition 5.18: Separability

A metric space X is said to be separable if it has a countable dense subset S.

Example : $\mathbb{R}^n$ ***is separable***
Consider the set of all points $z \in \mathbb{R}^n$ with coordinates in $\mathbb{Q}$. Basically $\mathbb{Q}^n$. This would be a countable set, and every point $x \in \mathbb{R}^n$ is a limit point of this set (since coordinate we can have a rational sequence converging to that coordinate)

Definition 5.19: Basis

A collection of open sets $\{V_\alpha : \alpha \in A\}$ is said to be a basis for X, a metric space, if for every point $x \in X$, and for every open set G so that $x \in G$, we have a V_α in the basis so that $x \in V_\alpha \subset G$. In other words, every open set G can be "covered" by a subcollection of $\{V_\alpha\}$

Theorem 5.20

For a metric space X, separability $\iff$ countable basis

Proof for Theorem.

$\implies$)Suppose X is separable, i.e, has a countably dense set $\{x_1, x_2, \cdots, x_n, \cdots\}$. Choose all the $1-$balls of each of these points to form a countable set. Choose all the $1/2-$ball of each for another countable set. As such keep choosing $1/n$-balls for each x_j in the countably dense set. We claim that this collection of open sets are my basis. Say z is an arbitrary point in X, and G an arbitrary open set containing z. There would exist ε_z^G so that ε_z^G-ball of z would be contained in G. Choose $\delta < \frac{\varepsilon_z^G}{4}$. There would be an n so that $\frac{1}{n} < \delta$. In this $1/n$ ball around z, there would exist a point x_k in the dense set. Choose now, the $1/n$ ball around x_k which contains z, and would be completely contained in the ε_z^G ball, which would be contained in G. Hence, we are done.

$\Longleftarrow$) Say X has a countable basis $\{V_1, V_2, \cdots\}$ that are non empty. Choose $x_1 \in V_1$, $x_2 \in V_2 \cdots$. We claim that this set $\{x_1, x_2 \cdots\}$ forms our countable dense set. Say a point $z \in X$, choose any arbitrary $\varepsilon > 0$. The $\varepsilon-$ball around z would be an open set, which means a set V_k is inside $\varepsilon-$ball around z. This would mean that x_k would be in the set, which makes z a limit point of this supposed to be dense set $\{x_1, x_2 \cdots\}$. We are done. ■

Theorem 5.21

$$"(\exists\{V_\alpha : \alpha \in A\})(\forall x \in X)(\forall G \subset X : G \text{ open} : x \in G)(\exists \alpha \in A)(x \in V_\alpha \subset G)"$$

$$\Longleftrightarrow$$

$$\exists\{V_\alpha : \alpha \in A\}(\forall G \subset X : G \text{ open in } X)(\exists A' \subseteq A)(G = \cup_\alpha V_\alpha : \alpha \in A')$$

Or in pithy words,

V_α is basis if and only if every open set is the union of a subcollection of $\{V_\alpha\}$

Proof for Theorem.

$\Longrightarrow$) Consider an open set G (arbitrary) and an arbitrary point $x \in G$. There exists, for every point x in G a corresponding base V_{α_x} so that $x \in V_{\alpha_x} \subset G$. This means, if we collect all the x in G and its corresponding bases, the union of that basis would give us G.

$\Longleftarrow$) Suppose for any given $G \subset X$ that is open, there is a subcollection of a (fixed) $\{V_\alpha\}$ so that $G = \cup_{\alpha'} V_{\alpha'}$. Consider an arbitrary point $x \in X$ and an arbitrary $G : x \in G$. This means there is an epsilon ball around x that is fully contained in G. But since open balls are open, there is a subcollection $\{V_{\alpha''}\}$ so that "this ball"$= \cup V_{\alpha''}$. Since $x \in$ this ball, $x \in V_k$ for some k. But this v_k is in G. Hence, we are done. ■

Fact 5.22

$$"(\exists\{V_\alpha : \alpha \in A\})(\forall x \in X)(\forall G \subset X : G \text{ open} : x \in G)(\exists \alpha \in A)(x \in V_\alpha \subset G)"$$

is the same thing as

$$"(\exists\{V_\alpha : \alpha \in A\})(\forall G \subset X : G \text{ open})(\forall x \in X : x \in G)(\exists \alpha \in A)(x \in V_\alpha \subset G)"$$

Theorem 5.23

Let X be a metric space in which every infinite subset E has a limit point in X. Then X has countable basis (or equivalently is separable).

Proof for Theorem.

Note that no infinite subset of X can be "unbounded" since we have seen before that Limit point compact implies closed and bounded. Consider an arbitrary δ_0. Choose an initial point x_1. Is there a point x_2 that is outside the $\delta-$ball around x_1? If no, the only $\delta-$ball of x_1 covers the whole space. If yes, then again ask the question, is there a point x_3 that is both outside the $\delta-$ball of x_1 and the $\delta-$ball of x_2? If not, then these two balls will cover the whole space, if yes, again ask the question, is there a point x_3 so that it is not in x_1, x_2 and x_3's $\delta-$balls. Keep going, as such. We claim that it must terminate after a finite number of steps. Suppose not, i.e for every x_n you can find, you can find an x_{n+1} to form an infinite set of x_j such that x_j is not in the δ-ball of any of its preceeding elements $x_{j-1}, x_{j-2} \cdots$. This is true for every x_j. From the hypothesis, this set has a limit point q. Choose $\varepsilon < \delta/4$, within which exists a point x_k in this infinite set. But since no point exists within the $\delta-$ball of x_k apart from itself, no other point exists in the $\varepsilon-$ball around x_k as well. This would contradict the proposition that q is a limit point. Hence, this procedure must terminate in a finite amount of steps. So for a given $\delta = d$, there exists n_δ balls $V_1, V_2 \cdots V_{n_d}$ with centres $x_1^d, x_2^d \cdots x_{n_d}^d$. Choose $\delta = 1$, and create a collection of 1-balls (by the procedure mentioned above) around the points $x_1^1, x_2^1 \cdots x_{n_1}^1$. Likewise, for $\delta = 1/2$, find points $x_2^1, x_2^2 \cdots x_{n_2}^2$, and for $\delta = 1/k$, find points $x_2^k, x_2^k \cdots x_{n_k}^k$. Let z be a point in X. Choose a ε ball around z. Let $1/n < \varepsilon$. There must be some x_j^n so that z is in the $1/n$-ball of x_j^n. Hence, $\{x_j^k : j \in \mathbb{N}, k \in \mathbb{N}\}$ is the dense set we need. ■

Theorem 5.24

A compact metric space has a countable base (or equivalently, is separable- has a countable dense set in it).

Proof for Theorem.

Choose $\delta_1 = 1$ and all the δ_1-balls in X (centred at every point of X). This makes an open cover of X that has a finite subcover. i.e, finite points $x_1^1, x_2^1 \cdots x_{n_1}^1$ whose $\delta_1 = 1$ balls cover the whole space X. Next, choose $\delta_2 = 1/2$ and every δ_2 ball in X. This likewise, would have a finite subcover i.e, δ_2-balls around $x_1^2, x_2^2 \cdots x_{n_2}^2$. As such, for every $\delta_k = 1/k$, we have points $x_1^k, x_2^k, \cdots x_{n_k}^k$ whose δ_k balls cover the whole space.

That this space is separable is obvious now. Choose $x_j^i : i, j \in \mathcal{N}$. Let z be a point in X. Choose any arbitrary ε. Find an n so that $\frac{1}{n} < \varepsilon$. From the result we have, we can always find a x_j^k in our set so that it is in the $\frac{1}{n}$ ball of z. This means that in every ε ball of z, there exists a point from our stipulated set. Hence, our set is dense in X. ■

Theorem 5.25

If X is a metric space so that every infinite set in X has a limit point in X, then every open cover of X (open relative to X) has a finite subcover. Hence:

"If X is a limit point compact metric space, then X is a compact metric space (compact relative to itself, but it also applies to all spaces it lives in, due to preservation of compactness)"
-Krishna, to Arjuna, at Kurukshetra

Proof for Theorem.

Suppose that X is limit point compact. This means that it has a countable basis. Consider an open cover $\{G_\alpha : \alpha \in A\}$ so that $X = \cup_{\alpha \in A} G_\alpha$. Let $\{V_1, V_2 \cdots\}$ be the countable basis for X. This means that for every open set G and every point $x \in X$ so that $x \in G$, we have an $n \in \mathbb{N}$ so that $x \in V_n \subset G$. Let $P := \{n \in \mathbb{N} : \exists \alpha \in A \text{ such that } V_n \subseteq G_\alpha\}$. Consider any point $x \in X$. For this x, there exists some G_α so that $x \in G_\alpha$ which would imply from the conutable bases property that $\exists V_n$ so that $x \in V_n \subseteq G_\alpha$. For every point, there exists a V_n and a corresponding G_α so that $x \in V_n \subset G_\alpha$. Therefore, X is countably "subcovered" by G_{α_j} (those G_α that correspond to the V_n, $n \in P$).

Let this countable subcover be $G_1, G_2 \cdots G_n \cdots$. Suppose to this, there does not exist a finite subcover. i.e,

$$\exists x_1 \in G_1^C$$

,

$$\exists x_2 \in (G_1 \cup G_2)^C = C_1^c \cap G_2^C$$

.

$$\vdots$$

$$\exists x_n \in (G_1 \cup G_2 \cup G_3 \cup \cdots G_n)^C = \cap_{i=1}^n G_i^C$$

, for every $n \in \mathbb{N}$. Collect all such x_ns to form an infinite set in X. By hypothesis, this has a limit point, call it z. Note that, outside a given G_k^C, only finite points of our stipulated set exists, which are possibly $x_1, x_2 \cdots x_{n-1}$. Suppose there exists a G_j^C so that $z \notin G_j^C$. This means that z is in G_j which is an open set. Therefore, there exists an ε ball that is completely inside G_j, or rather completely outside G_j^C. This means that only finitely many $x_1, x_2 \cdots$ make into this ε-ball of z which is absurd since z is supposedly a limit point. Hence, we cannot have a countable cover that has no finite subcover. Hence, $G_j : j \in \mathbb{N}$ has a finite subcover, which means X is a compact metric space. ■

Remark.

> We showed that every limit point compact *metric space* is compact, i.e, every open cover, open relative to X, has a finite subcover. Suppose X is actually a subset of a bigger set H and X inherits the metric from H. From conservation of compact relativeness, we see that X must be compact relative to H as well. Hence, every limit point compact set in X is compact in X.