

Groups and Rings

Notes

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Part I

Groups and Rings

CHAPTER 1

PRELIMINARIES

1 Sets, functions and all that

Definition 1.1: Preliminary definitions

1. (Cartesian Product): if A and B are non empty sets, the *Cartesian Product* $A \times B$ is defined as the set of ordered pairs a, b wherein $a \in A, b \in B$. i.e, $A \times B := \{(a, b) : a \in A, b \in B\}$
2. (Function): A function from A to B is a set $f \subseteq A \times B$ such that, $a, b \in f$ and $a, c \in f \implies b = c$. A is called the **Domain of f** . $Range(f) := f(A)$ (see next definition)
3. (Direct Image): Direct image $f(A) := \{y \in B : \exists x \in A \text{ such that } f(x) = y\}$
4. (Inverse Image): $f^{-1}(S \subseteq B) := \{x \in A : f(x) \in S\}$
5. (Relation): Any subset $R \subseteq A \times B$ is a relation from A to B .
We say $x \in X$ is "related to" $y \in Y$ under the relation R , or simply xRy or $R(x) = y$ if $(x, y) \in R \subseteq X \times Y$.
6. (Injection): $f : A \rightarrow B$ is injective if $\forall x_1, x_2 \in A, (x_1, b) \in f \text{ and } (x_2, b) \in f \iff x_1 = x_2$
7. (Surjection): $f : A \rightarrow B$ is surjective if $\forall b \in B, \exists a \in A \text{ such that } f(a) = b$
8. (Bijection): $f : A \rightarrow B$ is bijective if its both surjective and injective.
9. (Identity function on a set): $I_A : A \rightarrow A$ defined by $\forall x \in A, I_A(x) = x$
10. (Permutation): Simply a bijection from A to itself is called a permutation.

Definition 1.2: (Left Inverse)

We say $f : A \rightarrow B$ has a left inverse if there is a function $g : B \rightarrow A$ such that $g \circ f = I_A$

Theorem 1.3

$f : A \rightarrow B$ has a left inverse if and only if it is injective.

Proof for Theorem.

\Rightarrow) If f has a left inverse g , Consider $x, y \in A$ such that $f(x) = f(y) = p$.
 We have $g \circ f(x) = g(p) = x = g \circ f(y) = y$. Hence, $x = y$, Injective.
 \Leftarrow) Given that $f : A \rightarrow B$ is injective, define $g : B \rightarrow A$ as:

$$g(z \in B) = \begin{cases} a, & \text{where } f(a) = z, \text{ if } z \in f(A) \\ \text{whatever,} & \text{if } z \notin f(A) \end{cases}$$

consider $g \circ f(x \in A) = g \circ (f(x))$.

Obviously, $f(x) \in f(A)$, therefore, $g(f(x)) =$ that a such that $f(a) = f(x)$.

That a is x . Hence, $g(f(x)) = x$ ■

Definition 1.4: (Right Inverse)

$f : A \rightarrow B$ is said to have a right inverse if there is a function $g : B \rightarrow A$ such that $f \circ g = I_B$

Theorem 1.5

$f : A \rightarrow B$ has a right inverse if and only if f is Surjective.

Proof for Theorem.

\Rightarrow) If f has a right inverse g , such that $f \circ g = I_B : B \rightarrow B$, then it is evident that the range of f is B , for if not, range of $f \circ g$ wouldn't be B either.

\Leftarrow) If f is surjective, then for all $b \in B$, there exists atleast one $a \in A$ such that $f(a) = b$ define g as:

$$g(x \in B) = \text{one of those } a \in A \text{ such that } f(a) = b$$

Consider $f \circ g(x \in B) = f(\text{one of the } a \text{ such that } f(a) = b) = b, \forall b \in B$

Hence, $f \circ g = I_B$ ■

Theorem 1.6

If f has left inverse g_1 and right inverse g_2 , then $g_1 = g_2$. *(True for anything that is Associative, and function composition is associative.)*

Proof for Theorem.

$$\begin{aligned}
 g_1 \circ f &= I_A \text{ and } f \circ g_2 = I_B \\
 g_1 \circ (f \circ g_2) &= g_1 \circ I_B = g \\
 &= (g_1 \circ f) \circ g_2 = I_A \circ g_2 = g_2 \\
 \text{Hence } g_1 &= g_2
 \end{aligned}$$

Corollary 1.7

f is invertible (i.e, both left and right inverse exist) if and only if it is bijective.

Proof for Corollary.

Obvious

1.1 Operations on Relations

If R and S are binary relations over $X \times Y$:

1. $R \cup S := \{(x, y) | xRy \text{ or } xSy\}$
2. $R \cap S := \{(x, y) | xRy \text{ and } xSy\}$
3. Given $S : Y \rightarrow Z$ and $R : X \rightarrow Y$, $S \circ R := \{(x, z) | \exists y \text{ such that } ySz \text{ \& } xRy\}$
4. If R is binary over $X \times Y$, $\bar{R} := \{(x, y) | \neg(xRy)\}$

1.2 Homogeneous Relations

If R is a binary relation over $X \times X$, it is Homogeneous.

Definition 1.8: Definitions Regarding Relations

1. (Reflexive): $\forall x \in X, xRx$
2. (Symmetric): $\forall x, y \in X, xRy \implies yRx$
3. (Transitive): $\forall x, y, z \in X, \text{ if } xRy \ \& \ yRz \implies xRz$
4. (Dense): $\forall x, y \in X, \text{ if } xRy, \text{ then there is some } z \in X \text{ such that } xRz \ \& \ zRy$
5. (**Equivalence Relation**): R is an equivalence relation if it is Reflexive, Symmetric and Transitive.
6. (Equivalence class of $a \in A$ (where there is an equivalence relation defined)): Set of all $b \in A$ such that bRa .
7. (Partition of A): Any collection of sets $\{A_i : i \in I\}$ (where I is some indexing set) such that:

$$A = \bigcup_{i \in I} A_i$$

$$A_i \cap A_j = \phi \text{ if } \forall i, j \in I, i \neq j$$

Theorem 1.9

Let A be a non-empty set. If R defines an equivalence Relation on A , then the set of all equivalence classes of R form a partition of A

Proof for Theorem.

Define our collection $\{A_\alpha\}$ as the set of all equivalence classes of A . Clearly, $\bigcup_{\alpha \in I} A_\alpha = A$. If A only has one element, obviously, that singleton set makes up the partition. Let A_α and $A_{\alpha'}$ be equivalence classes of two elements a and a' in A . If aRa' , then $A_\alpha = A_{\alpha'}$ since every element in the equivalence class of a will, from the transitive property, be in the equivalence class of a' . Suppose $\neg(aRa')$. If, then, $\exists x \in A_\alpha$ such that $x \in A_{\alpha'}$, this means that $xR\alpha$ and $xR\alpha'$, but from transitive property, this means $\alpha R\alpha'$, which is a contradiction. Therefore, the pairwise intersection is disjoint. ■

Theorem 1.10

If $\{A_i : i \in I\}$ is a partition of A , then there exists an equivalence relation R on A whose equivalence classes are $\{A_i : i \in I\}$.

Proof for Theorem.

Define $R(x, y)$ if and only if \exists unique $m \in I$ such that $x \in A_m$ and $y \in A_m$.
 $R(x, x)$ is obvious if non empty, hence R is reflexive.

Suppose $R(x, y)$ and $R(y, z)$. Then, there exists a unique $m \in I$ such that x, y are in A_m . Similarly, there exists a unique $n \in I$ such that y, z are in A_n . Obviously, if $n \neq m$, intersection of A_n and A_m would be non empty, hence, $n = m$. Hence, R is transitive.

Consider $R(x, y)$, which means \exists unique $n \in I$ such that $x, y \in A_n \implies R(y, x)$. Hence, R is an equivalence relation. ■

2 Induction, Naturals, Rationals and the Axiom of Choice

Axiom 2.1: Peano Axioms, characterisation of \mathbb{N}

1. $1 \in \mathbb{N}$
2. every $n \in \mathbb{N}$ has a predecessor $n - 1 \in \mathbb{N}$ except 1
3. if $n \in \mathbb{N} \implies n + 1 \in \mathbb{N}$

Definition 2.2: (Sequence of something)

A sequence of some object is simply a collection of objects $\{O_l : l \in \mathbb{N}\}$ which can be counted.

Axiom 2.3: Well Ordering Property of \mathbb{N}

Every non empty subset of \mathbb{N} has a least element.

Axiom 2.4: Weak Induction

For all subsets $S \subseteq \mathbb{N}$, $((1 \in S) \& ((\forall k \in \mathbb{N})(k \in S \implies k + 1 \in S))) \iff S = \mathbb{N}$

Weak Induction's Negation:(One direction)

There exists subset $S_0 \subseteq \mathbb{N}$, $((1 \in S_0) \& ((\forall k \in \mathbb{N})(k \in S_0 \implies k + 1 \in S_0)))$ but $S_0 \neq \mathbb{N}$

Axiom 2.5: Strong Induction

For all subsets $S \subseteq \mathbb{N}$, $((1 \in S) \& ((\forall k \in \mathbb{N})(1, 2, \dots, k \in S' \implies k + 1 \in S'))) \iff S = \mathbb{N}$

Strong Induction's Negation:(One direction)

There exists subset $S' \subseteq \mathbb{N}$, $((1 \in S') \& ((\forall k \in \mathbb{N})(1, 2, \dots, k \in S' \implies k + 1 \in S'))) but $S' \neq \mathbb{N}$$

Theorem 2.6

Weak Induction \iff Strong Induction.

Proof for Theorem.

\implies) Suppose Weak induction is true, but not strong induction. Take our set to be that S' in the negation of the Strong Induction Statement. $S' \neq \mathbb{N}$ implies that, either $1 \notin S'$ or $\exists k \in \mathbb{N}$ such that $k \in S'$ but $k + 1 \notin S'$. We know that $1 \in S'$, so it must be that $\exists k \in \mathbb{N}$ such that $k \in S'$ but $k + 1 \notin S'$. $\{1\} \in S' \implies \{1, 2\} \in S'$. Assume that for n , $\{1, 2, \dots, n\} \in S'$. This means that $\{1, 2, \dots, n + 1\} \in S'$. This means that for every $n \in \mathbb{N}$, $\{1, 2, \dots, n\} \in S' \implies n \in S'$. Contradiction.

\impliedby) Suppose Strong Induction is true, but not weak induction. Take the set S_0 from the negation of Weak Induction. $S_0 \neq \mathbb{N}$. This means, from strong induction, either $1 \notin S_0$ or $\exists k \in \mathbb{N}$ such that $1, 2, \dots, k \in S_0$ but $k + 1 \notin S_0$. $1 \in S_0$, hence, $2 \in S_0$ and $\{1, 2\} \in S_0$. assume that $\{1, 2, \dots, n\} \in S_0$. This means, $n \in S_0 \implies n + 1 \in S_0$, which means that $\forall k \in \mathbb{N}, \{1, 2, \dots, k\} \in S_0 \implies k + 1 \in S_0$. Therefore, S_0 is \mathbb{N} . ■

Theorem 2.7

Weak Induction \iff Strong Induction \iff Well ordering.

Proof for Theorem.

\implies) Suppose that, on the contrary, S_0 is a non empty subset of \mathbb{N} , with no least element. Does 1 exist in S_0 ? No, for that will be the least element. Likewise, then, 2 does not belong in S_0 . Assume that $\{1, 2, \dots, n\} \notin S_0$. Does $n + 1$ exist in S_0 ? No, for that will become the least element then. From Strong Induction, $\mathbb{N} - S_0 = \mathbb{N} \implies S_0 = \emptyset$. Contradiction.

\impliedby) Suppose $\exists S_0 \subseteq \mathbb{N}$ such that $1 \in S_0$ and $\forall k \in \mathbb{N}, k \in S_0 \implies k + 1 \in S_0$. Suppose on the contrary, S_0 is not \mathbb{N} . $\mathbb{N} - S_0$ is then, non-empty. From Well Ordering, there is a least element $q \in \mathbb{N} - S_0$. $\implies, q - 1 \in S_0$. But this would imply $q - 1 + 1 \in S_0$. Contradiction. $\mathbb{N} - S_0$ is empty. ■

Definition 2.8: (Finite Sets)

A set X is said to be finite, with n elements in it, if $\exists n \in \mathbb{N}$ such that there exists a bijection $f : \{1, 2, \dots, n\} \rightarrow X$. Set X is *infinite* if it is non-finite.

Theorem 2.9

If A and B are finite sets with m and n elements respectively, and $A \cap B = \emptyset$, then $A \cup B$ is finite, with $m + n$ elements.

Proof for Theorem.

$f : \mathbb{N}_m \rightarrow A$ and $g : \mathbb{N}_n \rightarrow B$.

Define $h : \mathbb{N}_{m+n} \rightarrow A \cup B$ given by:

$$h(i) = \begin{cases} f(i) & \text{if } i = 1, 2, \dots, m \\ g(i - m) & \text{if } i = m + 1, m + 2, \dots, m + n \end{cases}$$

If $i = 1, 2, \dots, m$, $h(i)$ covers all the elements in A through f . If $i = m + 1, \dots, m + n$, $h(i)$ covers all the elements in B through g .

Moreover, $h(i) \neq h(j)$; $i \in [1, m]$, $j \in [m + 1, m + n]$ since $A \cap B = \emptyset$ ■

Theorem 2.10

If C is infinite, and B is finite, then $C - B$ is infinite.

Proof for Theorem.

Suppose $C - B$ is finite. We have $B \cap (C - B) = \emptyset$ and $B \cup (C - B) = C \cup B$

$n(C \cup B) = n(B \cup (C - B)) = n(B) + n(C - B)$ This implies $C \cup B$ is finite. Contradiction. ■

Theorem 2.11

Theorem: Suppose T and S are sets such that $T \subseteq S$. Then:

- a) If S is finite, T is finite.
- b) If T is infinite, S is infinite.

Proof for Theorem.

Given that S is finite, there is a function $f : \mathbb{N}_m \rightarrow S$. Suppose that S has 1 element. Then either T is empty, or S itself, which means T is finite. Suppose that, upto n , it is true that, if S is finite with n elements, all its subsets are finite. Consider S with $n + 1$ elements.

$f : \mathbb{N}_{n+1} \rightarrow S$.

If $f(n + 1) \in T$, consider $T_1 := T - \{f(n + 1)\}$. We have $T_1 \subseteq S - \{f(n + 1)\}$, and since $S - \{f(n + 1)\}$ is a finite set with n elements, from induction hypothesis, T_1 is finite.

Moreover, since $T = T_1 \cup \{f(n + 1)\}$, T is also finite with one more element than T_1 .

If $f(n + 1) \notin T$, then $T \subseteq S - \{f(n + 1)\}$, we are done.

(b) is simply the contrapositive of (a). ■

Definition 2.12: (Countable Sets)

A set S is said to be *countable*, or *denumerable* if, either S is finite, or $\exists f : \mathbb{N} \rightarrow S$ which is a bijection. If S is *not countable*, S is said to be *uncountable*

Theorem 2.13

The set $\mathbb{N} \times \mathbb{N}$ is countable.

Proof for Theorem.

The number of points on diagonals $1, 2, \dots, l$ are: $\psi(k) = 1 + 2 + \dots + k = \frac{k(k+1)}{2}$

The point (m, n) occurs on the $(m + n - 1)$ th diagonal, on which the number $m + n$ is an invariant. The (m, n) point occurs m points down the diagonal. So, to characterise a point, it is enough to specify the diagonal it falls in, and its ordinate (the "rank" of that point on that diagonal). Count the elements till the $m + n - 2$ nd diagonal, then add m , and this would be the position of the point (m, n) .

Define $r : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ given by $r(m, n) = \psi(m + n - 2) + m$. That this is a bijection is pretty clear because we are counting the position of the point (m, n) . For a given point (m, m) , there can only be one unique diagonal on which it exists, and on the diagonal, its rank is unique. Moreover, for every $q \in \mathbb{N}$, there corresponds an (m, n) such that $r(m, n) = q$, for, we simply count along each diagonal in the "zig-zag" manner until we reach that (m, n) for which the position is given by q . Therefore, r is a bijection. (There are other explicit bijections too) ■

Theorem 2.14

The following are equivalent:

1. S is countable
2. \exists a surjective function from $\mathbb{N} \rightarrow S$
3. \exists an injective function from $S \rightarrow \mathbb{N}$

Proof for Theorem.

(1 \implies 2) is obvious

(2 \implies 3) $f : \mathbb{N} \rightarrow S$, every element of S has atleast one preimage in \mathbb{N} . Define a function from $S \rightarrow \mathbb{N}$ by taking for each $s \in S$ the least such $n \in \mathbb{N}$ such that $f(n) = s$. This defines an injection.

(3 \implies 1) If there is an injection from $S \rightarrow \mathbb{N}$, then there is a bijection from $S \rightarrow$ a subset of \mathbb{N} , which implies S is countable. ■

Corollary 2.15

The set of Rational Numbers \mathbb{Q} is countable.

Proof for Corollary.

We know that a surjection from $\mathbb{N} \times \mathbb{N}$ to \mathbb{Q} exists (where $f(0, 0) = 0$, and $f(m, n) = \frac{m}{n}$). We know that $\mathbb{N} \times \mathbb{N}$ is bijective to \mathbb{N} . This means \mathbb{N} is surjective to \mathbb{Q} . We are Done. ■

Theorem 2.16

Every infinite subset of a countable set is countable.

Proof for Theorem.

Consider $N_s \subseteq \mathbb{N}$ which is infinite.

Define $g(1) = \text{least number in } N_s$

Having defined $g(n)$, define $g(n+1) = \text{least number in } N_s \text{ which is larger than } g(n)$.

That it is an injection is obvious, for $g(m) > g(n)$ if $m > n$.

Suppose it is not a surjection, i.e, $g(\mathbb{N}) \neq N_s \implies g(\mathbb{N}) \subset N_s \implies N_s - g(\mathbb{N}) \neq \emptyset$. Therefore, $N_s - g(\mathbb{N})$ has a least element, k . This means that $k-1$ is in $g(\mathbb{N})$. Therefore, there exists q in \mathbb{N} such that $g(q) = k-1$. But then, $g(q+1) = \text{least number in } N_s \text{ such that it is bigger than } g(q)$. This would, ofcourse be, k , which means $k = g(q+1)$, which puts k in $g(\mathbb{N})$. Contradiction. Hence, $g(\mathbb{N}) = N_s$, therefore, g is a bijection from $\mathbb{N} \rightarrow N_s$. Since every countable set is bijective to \mathbb{N} , and every infinite subset of a countable set is bijective to an infinite subset of \mathbb{N} , the theorem holds generally for countable sets. ■

Theorem 2.17

$\mathbb{N} \times \mathbb{N} \cdots \mathbb{N}$ is bijective to \mathbb{N}

Proof for Theorem.

$\mathbb{N} \times \mathbb{N}$ is bijective to \mathbb{N} obviously. Assume that $f : \mathbb{N} \rightarrow \mathbb{N} \cdots \mathbb{N}$ (n times) is bijective.

Consider $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \cdots \mathbb{N}$ ($n+1$ times) given by $g(m, n) = (f(m), n)$. Clearly, this is bijective. ■

2.1 Axiom of Choice**Axiom 2.18: Axiom of Choice (AC)**

For any collection of non empty sets $C = \{A_l : l \in L\}$, there exists a function f called the "counting function" which maps each set A_l to an element in A_l .

Formally: $f : C \rightarrow \bigcup_l A_l$ such that $\forall l \in L, f(A_l) \in A_l$

Theorem 2.19

Countable union of Countable sets is countable (This theorem is an example of a theorem that requires Axiom of Choice)

Proof for Theorem.

Suppose we are given a sequence of countable sets $\{S_n : n \in \mathbb{N}\}$. Since each S_j is countable, we have for each j , at least one bijective map $f_j : \mathbb{N} \rightarrow S_j$. Define $k : \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_j S_j$ given by: $k(m, n) = f_m(n)$. Suppose $x \in \bigcup_j S_j$, i.e. $x \in S_j$ for some j . This means that, $f(n) = x$ for some n . Therefore, $k(j, n) = x$. Hence, k is surjective. From theorem 2.14, we are done.

(Remark: Keep in mind, for each S_j , there are a myriad of functions $(f_j)_k : \mathbb{N} \rightarrow S_j$. For each S_j , which is countably infinite, we have to choose one of the many functions that biject \mathbb{N} to S_j . So we have a countable collection of sets $C = \{E_j : j \in \mathbb{N}\}$, where E_j denotes the set of all functions that biject \mathbb{N} into S_j . So for every element in C , we need to choose one element in each element of C . This is where the Axiom of Choice comes into play.) ■

Theorem 2.20

If $f : A \rightarrow B$ is a surjection, then B is bijective to a subset of A

Proof for Theorem.

We are told that $f(A) = B$, i.e. for every $b \in B$, $\exists x_b$ (many such x_b -s are possible) such that $f(x_b) = b$. Define a function $g : B \rightarrow A$ as: $g(b) =$ one of those x_b such that $f(x_b) = b$. g is bijective to the set of all the chosen x_b for every b ■

Remark.

We make use of the Axiom of Choice in the previous theorem when we choose an x_b from a set of all possible x_b -s for b . Let A_b be the set of all possible x_b -s. Then the collection $\{A_b : b \in B\}$ is a collection of non-empty sets. And we are to select "one" element from each A_b . This requires AC.

Definition 2.21: (Power Set of a set)

Power set of A , denoted by $P(A)$ is the set of all subsets of A .

Theorem 2.22: Cantor's Theorem

For any set A , there *does not exist* any surjection from A onto $P(A)$

Proof for Theorem.

Suppose, on the contrary, a surjection $\psi : A \rightarrow P(A)$ exists. For every subset A_s of A , there exists an element x of A such that $\psi(x) = A_s$. Either this x exists in A_s , or it doesn't. Consider $D := \{x \in A : x \notin \psi(x)\}$. D is a subset of A , so there must be some element $y \in A$ such that $\psi(y) = D$. Does y belong in D ? If so, $y \notin \psi(y) = D$. Which means $y \notin D$. If, though, $y \notin D$, that implies $y \notin \psi(y) \implies y \in D$. Contradictions left and right. ■

3 Elementary Results regarding Integers

Definition 3.1

1. (Divides) We say $a \in \mathbb{Z} \setminus \{0\}$ divides $b \in \mathbb{Z}$ if there exists an integer δ such that $a\delta = b$. We denote it by $a|b$.
2. (GCD) We call a number d the "Greatest Common Divisor" of two integers a and b if $d|a$ and $d|b$, and d is the largest such number that divides both a and b (that the largest such number exists is clear, since divisors are finite).
3. (LCM) We call a number l the "Least Common Multiple" of two integers a and b if $a|l$ and $b|l$ and l is the smallest such integer.

Theorem 3.2

1. The GCD d of $a, b \in \mathbb{Z}$ is unique, and has the property that, if any other integer q is a divisor of a and b , then q divides d .
2. The LCM l of $a, b \in \mathbb{Z}$ is unique, and has the property that, if another integer p is a multiple of a and b , then l divides p .
3. If LCM of a, b and GCD of a, b are l and d respectively, then $dl = ab$.

Proof for Theorem.

(1) If d is the GCD of a, b , obviously $d|a$ and $d|b \implies dz_a = a, dz_b = b$ for some $z_a, z_b \in \mathbb{Z}$. Consider an arbitrary divisor q such that $\exists r_a, r_b \in \mathbb{Z}$ such that $qr_a = a, qr_b = b$. ■

CHAPTER 2

TBD

CHAPTER 3

TBD