CHAPTER 1

METRIC SPACES

1 Fundamental Definitions n' Stuff

Definition 1.1: Metric Space

A set X along with a function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ called distance, is said to be a metric space if:

1. $d(x, y) = 0 \iff x = y$ (Positivity)
2. $d(x, y) = d(y, x), \forall x, y \in X$ (Symmetric)
3. $\forall x, y, z \in X$ we have, $d(x, y) \leq d(x, z) + d(y, z)$ (Triangle Inequality)

Example : $\mathbb{R}^n$ ***as a metric space***
Note that $\mathbb{R}^n$, the set of all n-tuples of $\mathbb{R}$, is a metric space with

$$d(\vec{x}, \vec{y}) := |\vec{x} - \vec{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^+ \cdots + (x_n - y_n)^2}$$

Definition 1.2: Open and Closed Balls around x in X

Open ball is defined as:

$$B_r(x) := \{y \in X : d(y, x) < r\}$$

Closed ball is defined as:

$$B_{[r]}(x) := \{y \in X : d(y, x) \leq r\}$$

Definition 1.3: Convexity

A set S in $\mathbb{R}^n$ is said to be convex if $\forall x, y \in S, t \in [0,1]$, $x + t(y-x) \in S$

Example : Open and closed balls in $\mathbb{R}^n$ are convex
Consider $B_r(x) := \{z \in \mathbb{R}^n : |z-x| < r\}$. Consider arbitrary p and q in $B_r(x)$. We have that $d(p,x) < r$ and $d(q,x) < r$. Consider $p + t(q-p)$ and consider $d(p + t(q-p), x) = |p + t(q-p) - x| = |tq + (1-t)p - x + tx - tx| = |tq - tx + (1-t)p - (1-t)x| \leq t|q-x| + (1-t)|p-x| = td(q,x) + (1-t)d(p,x) < r$. Replacing $<$ with $\leq$ in the above proves the result for closed balls.

Definition 1.4: Sequences in Metric Spaces

A sequence $\{x_n : x_n \in X\}$ is a mapping from the naturals to X, where order is implicit. We say a sequence in a metric space X is convergent to $x \in X$ if:

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_0)(d(x_n, x) < \varepsilon)$$

Definition 1.5: Limit Point of a set E

We say p is a limit point of a set E if

$$(\forall \varepsilon > 0)(\exists q_\varepsilon \in E; q_\varepsilon \neq p)(d(q_\varepsilon, p) < \varepsilon)$$

In other words, in every ε-ball around p, there would exist a point q_ε in E, which is different from p.

Theorem 1.6

Every ball / neighbourhood of p which is a limit point of E, would contain infinitely many points q such that $q \in B_\varepsilon(p) \cap E \setminus \{p\}$

Proof for Theorem.

Suppose for some neighbourhood, there only exists finite points $q_1, q_2, \cdots q_k$ such that $q_j \in B_{\varepsilon_0} \cup E \setminus \{p\}$. Let $\delta < min\{d(p, q_j) : j \in [1, 2, ...k]\}$. We then have that, there exists no point $q \in E$ such that its distancce from p is less than δ, making p a non-limit point. Absurd.

Corollary 1.7

A finite set has no limit points

Theorem 1.8: Recharacterisation of Limit points

A point $p \in X$ is a limit point of $E \subset X$ if and only if there exists a sequence $x_n \in E$, $x_n \neq p \forall n \in \mathbb{N}$ such that $\{x_n\} \to p$

Proof for Theorem.

$\implies$) If p is a limit point, around every neighbourhood, there would exist a point $q_\varepsilon \in E$ such that $0 < d(q_\varepsilon, p) < \varepsilon$. Choose $\varepsilon_1 = 1$, and obtain x_1 such that $x_1 \in E$, $x_1 \neq p$ and $0 < d(x_1, p) < 1$. Choose $\varepsilon_2 = \frac{1}{2}(d(x_1, p))$. We find $x_2 \in E$, $x_2 \neq p$ such that $d(x_2, p) < \frac{d(x_1,p)}{2} < \frac{1}{2}$. Continue as such to obtain a sequence that converges to p.

$\impliedby$) Suppose there is a sequence x_n such that $x_n \neq p \forall n \in \mathbb{N}$ and $\forall \varepsilon, \exists n_0(\varepsilon)$ such that $\forall n \geq n_0$ we have $d(x_n, p) < \varepsilon$ which means for a given ε, there exists a point x_{n_0+1} in E such that it is not equal to p and it is in the ε-ball of p. Hence, p would be a limit point. ■

Definition 1.9: Closed sets in X

A set E is closed in X if every limit point of E is contained in E

Definition 1.10: Equivalent definition of closed sets in X

A set E in X is closed if for every convergent sequence x_n in x such that $lim(x_n) \neq x_n$ for any n, we have $lim(x_n) \in E$.

Definition 1.11: Open sets in X

A set E is said to be open if $\forall x \in E$, $\exists \xi_x > 0$ such that $B_{\xi_x}(x) \subset E$

Theorem 1.12

Every open ball is an open set

Proof for Theorem.

Suppose a is a fixed point in X and $\delta > 0$ is given. $B = B_\delta(a) := \{y \in X : d(y, a) < \delta\}$.Consider arbitrary $z \in B$, for which we have $d(z, a) = t < \delta$. Therefore $\delta - t > 0$. Consider $0 < \xi_z = r < \delta - t$ from Density. Consider an arbitrary x such that $d(x, z) < \xi_z = r < \delta - t$. $d(x, a) \leq d(x, z) + d(a, z) = r + t \leq \delta - t + t = \delta$. We are done. ■

Definition 1.13: Compliment with respect to X

If $E \subseteq X$, we define compliment of E as

$$E^C := \{x \in X : x \notin E\}$$

Definition 1.14: Bounded

A set $E \subset X$ is bounded if $\exists$ a positive number $M > 0$ and $q \in E$ such that $d(x,q) < M$ $\forall x \in E$. i.e, all the points of E gets contained in some ball in X.

Theorem 1.15: De Morgan's Law

Let $\{E_\alpha : \alpha \in A\}$ where A is some arbitrary indexing set represent a collection of sets in X. Then

$$(\cup_\alpha E_\alpha)^C = \cap_\alpha E_\alpha^C$$

Proof for Theorem.

Consider $(\cup_\alpha E_\alpha)^c = \{x \in X : \exists \alpha \in A : x \in E_\alpha\}^c = \{x \in X : \forall \alpha \in A : x \notin E_\alpha\} = \{x \in X : \forall \alpha \in A : x \in E_\alpha^c\} = \cap_\alpha E_\alpha^c$ ∎

Theorem 1.16: The Big Equivalence

$E \subset X$ is open $\iff$ E^c is closed.

Proof for Theorem.

$\implies$) Suppose that E is open but E^c is not closed. This means that there exists a limit point of E^c that falls in E, i.e, outside E^c. Let this be q. This means for every ε-ball around q, a point of E^c exists. But since E is open and $q \in E$, we have for a particular ε-ball, inside which, no point of E^c resides. Contradiction.

$\impliedby$) Suppose E is closed but E^c is not open. This means that there is a point in E^c, p, such that for every ε-ball around p, some point in E falls into this ball. But this makes p a limit point of E, which is absurd since E is closed, limit points fall into the sets themselves. ∎

Theorem 1.17

For a collection of open sets $\{G_\alpha : \alpha \in A\}$, $\cup_\alpha G_\alpha$ is also an open set.

Proof for Theorem.

Consider $x \in \cup_\alpha G_\alpha$ which means $\exists \alpha_x \in A$ such that $x \in G_{\alpha_x}$ which means, there would exist an ξ-ball around x that is contained in G_{α_x} which is in turn contained in $\cup_\alpha G_\alpha$.

Corollary 1.18

For any collection of closed sets E_α, $\cap_\alpha E_\alpha$ is also closed.

Proof for Corollary.

$\{E_\alpha^c\}$ is a collection of open sets, and $\cup_\alpha E_\alpha^c$ is an open set, which means $\cup_\alpha E_\alpha^c = (\cap_\alpha E_\alpha)^c$ is an open set, from which we get that $(\cap_\alpha E_\alpha)$ is a closed set.

Theorem 1.19

For any finite collection of open sets $\{E_1, E_2, \cdots E_k\}$, $\cap_{i=1}^k E_i$ is also open.

Proof for Theorem.

Suppose $x \in \cap_{j=1}^k E_j$, which means $\forall j \in [1, k], x \in E_j$. We have $\varepsilon_1, \varepsilon_2, \cdots \varepsilon_k$ such that, the ε_j-ball around x is fully contained in E_j. Choose $0 < \delta < min\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_\}$ (the minimum exists by virtue of being a finite set). We see that the δ-ball around x is a subset of every ε_j-ball around x, which means that the δ-ball around x is in every E_j, which proves the theorem.

Corollary 1.20

For any finite collection of closed sets $\{G_1, G_2, \cdots G_k\}$ we have $\cup_{j=1}^k G_j$ to be closed

Remark.

In the above theorem and corollary, we require that the collection be finite. The reason is that, we were able to get a minimal ε_j in the proof due to the finiteness of the set. It may not be possible to find a number δ that is both larger than 0 but smaller than a given infinite collection of ε-s. For example, consider the sequence of open sets $(-\frac{1}{n}, \frac{1}{n})$. The infinite intersection of these yields $\{0\}$ which is a closed set by virtue of being finite.

Definition 1.21: Closure of a set

Let E' be the set of all limit points of E. Then, the closure of E is :

$$\bar{E} := E \cup E'$$

Theorem 1.22

Closure of a set is closed

Proof for Theorem.

Let p be a limit point of $E \cup E'$. That means that $\forall \varepsilon > 0\ \exists q_\varepsilon \in (E \cup E'), q_\varepsilon \neq p$ such that $q_\varepsilon \in B_\varepsilon(p)$. If p is in $E \cup E'$, we are done (especially if p is in E). Suppose p is not in E. $\forall \varepsilon > 0\ \exists q_\varepsilon \in (E \cup E'), q_\varepsilon \neq p$ such that $q_\varepsilon \in B_\varepsilon(p)$. If the q_ε we recieve falls in E we are ok. Suppose q_ε falls in E'. That means: $\forall \delta > 0$, $\exists r_\delta \in E$, $r \neq q_\varepsilon$ such that $d(r_\delta, q_\varepsilon) < \delta$ $\implies d(r_\delta, p) \leq d(r_\delta, q_\varepsilon) + d(q_\varepsilon, p) < \delta + d(q_\varepsilon, p) < \delta + \varepsilon$ If we choose $\delta_0 < \varepsilon - d(q_\varepsilon, p)$ we get: $d(r_\delta, p) \leq d(r_\delta, q_\varepsilon) + d(q_\varepsilon, p) < \delta + d(q_\varepsilon, p) < \varepsilon$

Summarising we have: $\forall \varepsilon > 0$, $\exists q_\varepsilon \in E$ or E' where: $q_\varepsilon \in E$ and $q_\varepsilon \in B_\varepsilon(p)$

or

$\exists \delta(\varepsilon) > 0$ such that $\exists r_\delta \in E$ such that $r_\delta \neq p$ and $r_\delta \in B_\varepsilon(p)$. In either case, there would exist a point dependent on ε, in E such that the point itself is different from p, and exists in the ε-ball around p. Hence, we see that p is a limit point of E. Therefore, we see that all the limit points of E either are points of E or points of E'. Hence, $\bar{E}$ is closed. ∎

Theorem 1.23

$\bar{E} = E \iff E$ is closed

Proof for Theorem.

$\implies)\bar{E}$ is closed, so E would be too.

$\impliedby)$ if E is closed, $E' \subseteq E \implies E' \cup E = E = \bar{E}$ ∎

Theorem 1.24: $\bar{E}$ is the smallest closed set that contains E

If F_α is the collection of all closed sets such that $E \subseteq F_\alpha$, then $\bar{E} \subseteq F_\alpha$ for all α.

Proof for Theorem.

Consider an arbitrary closed set F_α that contains E. It would obviously contain all the limit points of E among other things. Therefore, we can easily see that it contains $E \cup E' = \bar{E}$. ∎

Lemma 1.25: An equivalent definition for Closure.

An equivalent definition for closure is:

$$\bar{A} := \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \phi\}$$

Proof for Lemma

We see that obviously, if $x \in \bar{A}$, then either it is a point of A, or if not, it happens to be a limit point of A. And the back implication: If q is a point of A or if it is a limit point o A, it obviously falls into $\bar{A}$. ■

Example : If $E \subseteq \mathbb{R}$ ***is bounded (and non empty), with*** $s = sup(E)$***, then*** $s \in \bar{E}$
If $s \in E$ we are done. If not, then $\forall \varepsilon > 0$, $\exists \varepsilon > \delta(\varepsilon) > 0$, and a point $x_\varepsilon \in E$ such that $s - \varepsilon < s - \delta(\varepsilon) \leq x_\varepsilon < s + \delta < s + \varepsilon$ where $x_\varepsilon \neq s$. Hence, s is a limit point of E and hence, is a point in the closure. ■

Definition 1.26: Open Relative

Say $E \subseteq Y \subseteq X$, where X is a metric space. Y is also a metric space. We say E is open relative to Y if $\forall x \in E$, $\exists \varepsilon > 0$ such that if $y \in Y$ and $y \in B_\varepsilon(x)$ then $y \in E$. Formally:

$$(\forall x \in E)(\exists \varepsilon_x > 0)((y \in Y \cap B_\varepsilon(x)) \implies y \in E)$$

Remark.

> A set which is open relative to Y need not be open relative to X. For example, consider $\mathbb{R}$ as a subset of $\mathbb{R}^n$. An interval in $\mathbb{R}$ is open relative to $\mathbb{R}$, but it is not open relative to $\mathbb{R}^n$.

Theorem 1.27

A set $E \subseteq Y \subseteq X$ is open relative to $Y \iff \exists G \subset X$ that is open relative to X, such that $E = G \cap Y$

Proof for Theorem.

$\implies$) Say E is open relative to Y. This means that $\forall x \in E$, $\exists \varepsilon_x > 0$ such that if $y \in B_{\varepsilon_x}(x)$ and $y \in Y$, then $y \in E$. Call $G = \cup_{x \in E} B_{\varepsilon_x}(x)$ which is an open set. If $z \in E$, then $z \in G$ obviously, and hence $z \in G \cap Y$. Hence, $E \subseteq G \cap Y$. Consider a point $z \in G \cap Y$ which means z falls in one of the ε-balls around a point of E, and z is in Y. From definition of open relativeness, we see that $z \in E$. Hence, $E = G \cap Y$

$\impliedby$) Say $E = G \cap Y$ where G is an open set relative to X. Then, for every point in G, there would exist an ε-ball around that point that is completely contained in G. Let $x \in E$ be arbitrary. $\exists \varepsilon_x > 0$ such that $B_\varepsilon(x) \subset G$. Suppose $y \in Y$ and $y \in B_\varepsilon(x)$. This would mean that $y \in G \cap Y = E$. Hence, $\forall x \in E$ $\exists \varepsilon > 0$ such that if $y \in B_\varepsilon(x)$ and $y \in Y$, then $y \in E$, which is the definition of open relativeness. ■

2 Compactness

Definition 2.1: Open Conver

A collection of open sets $G_\alpha \subset X$ is an open cover of a set E if $E \subset \cup_\alpha G_\alpha$

Definition 2.2: Compact Set

A set $E \subset X$ is said to be ***Compact*** if Every open cover has a finite subcover. i.e, for every collection of open sets G_α, if $E \subseteq \cup_\alpha G_\alpha$, then there would exist a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2} \cdots G_{\alpha_k}\}$ of $\{G_\alpha\}$ such that $E \subseteq \cup_{i=1}^k G_{\alpha_i}$

Remark.

> The notion of *Being open* depends largely on the metric space one is talking about. For example, we see that certain sets may be open relative to $Y \subset X$, but not X in itself. This is not the case for compactness though, as shall be seen.

Theorem 2.3: "Compact Relativeness" is conserved.

Definition: We say $E \subseteq Y \subseteq X$ is compact relative to Y if for every open cover G_α open relative to Y we have a finite sub collection G_{α_k} of G_α such that $E \subseteq \cup_{j=1}^k G_{\alpha_j}$.

Theorem: $E \subseteq Y \subseteq X$ is compact relative to $Y \iff E$ is compact relative to X

Proof for Theorem.

$\implies$)Suppose E is compact relative to Y. This means that, for any collection of sets F_α which are open relative to Y (i.e, $F_\alpha = G_\alpha \cap Y$ where G_α is an open set in X), there exists a finite sub collection $F_{\alpha_1}, F_{\alpha_2} \cdots F_{\alpha_k}$ such that $E \subseteq \cup_{i=1}^k F_{\alpha_i}$. Consider an open cover H_α of E open relative to X. $E \subseteq \cup_\alpha H_\alpha$, but also, $E \subseteq (\cup_\alpha H_\alpha) \cap (Y)$ since E is subset of Y as well. This implies $E \subseteq \cup_\alpha (H_\alpha \cap (Y))$. $\{H_\alpha \cap Y\}$ is an open cover of E open relative to Y which means there would be a finite sub collection $\{H_{\alpha_j} \cap Y : j \in [1, k]\}$ such that $E \subseteq \cup_{j=1}^k (H_{\alpha_j} \cap Y) = (\cup_{j=1}^k H_{\alpha_j}) \cap Y$. Since E is a subset of Y, we then have $E \subseteq (\cup_{j=1}^k H_{\alpha_j})$ which proves that for an arbitrary open cover open relative to X, we have a finite subcover.

$\impliedby$) Suppose E is open relative to X. Consider an open cover of E open relative to Y, which is $\{F_\alpha\}$. This means that $F_\alpha = G_\alpha \cap Y$ for G_α open relative to X. $E \subseteq \cup_\alpha F_\alpha = (\cup_\alpha G_\alpha) \cap Y$. Since E is a subset of Y, we have $E \subseteq (\cup_\alpha G_\alpha)$. Therefore, there would be a finite subcollection of $\{G_\alpha\}$, $\{G_{\alpha_1}, G_{\alpha_2} \cdots G_{\alpha_k}\}$ such that $E \subseteq \cup_{i=1}^k G_{\alpha_i}$. This means, $E \subseteq \cup_{i=1}^k G_{\alpha_i} \cap Y = \cup_{i=1}^k F_{\alpha_i}$. Hence, for every open cover open relative to Y, there exists a finite subcover.

■

Fact 2.4

Every finite set in X is compact

Proof. Consider an open cover G_α for finite set E. This means that, for every point $x_1, x_2, \cdots, x_k$ in E, there would exist some $\{\alpha_1, \alpha_2, \cdots\}$ collection of "α-s" that is utmost finite, such that $x_j \in G_{\alpha_j}$. Simply take the union of G_{α_j} to get a finite subcover. □

Theorem 2.5: Alternate definition for compactness

A set E is compact if for every closed collection of sets K_α such that $\cap_\alpha K_\alpha \subset E^c$, we have a finite subcollection $\{K_{\alpha_1}, K_{\alpha_2} \cdots K_{\alpha_p}\}$ such that $\cap_{i=1}^{p} K_{\alpha_i} \subset E^c$

Theorem 2.6

Closed balls in X are closed

Proof for Theorem.

Consider $B_{[\varepsilon]}(p) := \{x \in X : d(x,p) \leq \varepsilon\}$. $B^c = C := \{x \in X, d(x,p) = t_{xp} > \varepsilon\}$. Consider an arbitrary point $x \in C$. We have $d(x,p) = t_{xp} > \varepsilon$. Find, from density, a δ such that $t_{xp} > \delta > \varepsilon$. Let $d(x,y) < \delta - \varepsilon$. We then have from triangle, $d(y,p) \geq d(x,p) - d(x,y) > t_{xp} - (\delta - \varepsilon) > \varepsilon$. Hence, y is also in C. Therefore, C is open, which means B is closed. ■

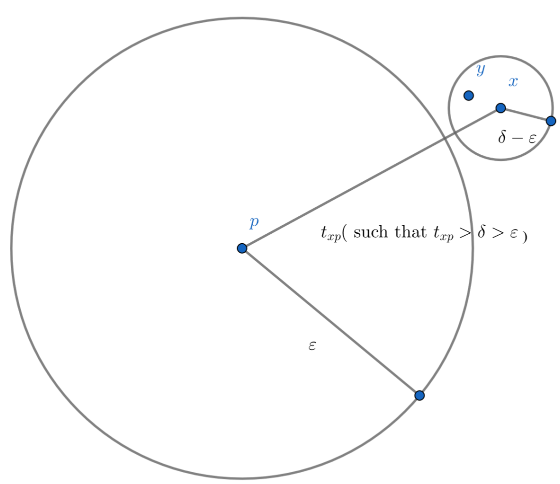

Figure 1.1: Figure for the proof: Closed balls are closed

Theorem 2.7

Compact sets are closed.

Proof for Theorem.

Method 1:

Let E be compact. Consider a point $p \in E^c$. Let ε_x be the "half" distance between a point $x \in E$ and p. Therefore, $B_{\varepsilon_x}(x)$ is completely outside $B_{\varepsilon_x}(p)$. Consider $\cup_{x \in E} B_\varepsilon(x)$ which is an open cover for E. This means there is a finite subcover

$\{B_{\varepsilon_{x_1}}(x_1), B_{\varepsilon_{x_2}}(x_2), B_{\varepsilon_{x_3}}(x_3) \cdots, B_{\varepsilon_{x_k}}(x_k)\}$ such that $E \subset \cup_{i=1}^k B_{\varepsilon_{x_i}}(x_i)$. $B_{\varepsilon_{x_i}}(p)$ does not intersect with $B_{\varepsilon_{x_i}}(x_i)$. Therefore, $\cap_{i=1}^k B_{\varepsilon_{x_i}}(p)$ does not intersect with any $B_{\varepsilon_{x_i}}(x_i)$ for any i. Hence, it does not intersect with $\cup_{i=1}^k B_{\varepsilon_{x_i}}(x_i)$ which means $\cap_{i=1}^k B_{\varepsilon_{x_i}}(p)$ lies completely outside E. If we choose $\delta < min\{\varepsilon_{x_1}, \varepsilon_{x_2} \cdots \varepsilon_{x_k}\}$, we would have $B_\delta(p) \subseteq \cap_{i=1}^k B_{\varepsilon_{x_i}}(p)$. This means that, for p outside E, there would exist a δ such that the δ-ball around p is fully contained in E^c. This means that E^c is open, hence, E is closed.

Method 2:

Consider E to be compact, i.e, for every closed collection $\{F_\alpha\}$ such that $\cap_\alpha F_\alpha \subset E^c$, there exists a finite sub collection $\{F_{\alpha_1}, F_{\alpha_2} \cdots F_{\alpha_k}\}$ such that $\cap_{j=1}^k F_{\alpha_j} \subset E^c$. Consider a point p outside E, i.e, in E^c. Notice that $\cap_{\varepsilon \in \mathbb{R}^+} B_{[\varepsilon]}(p) = \{p\}$ which is in E^c. This would be a collection of closed sets whose intersection falls completely inside E^c. Hence, there would exist a finite subcollection such that $\cap_{j=1}^k B_{[\varepsilon_j]}(p) \subset E^c$ which means there would exist a neighbourhood around p which is completely in E^c. Hence, E^c is open, and E is closed.

■

Fact 2.8

ϕ and X are both open and closed.

Theorem 2.9

Closed subsets of compact sets are compact

Proof for Theorem.

Consider $K \subset E$ where E is compact and K is closed. K^C is, therefore, open. Consider an arbitrary open cover $\{F_\alpha\}$ for K. since $K \subseteq \cup_\alpha F_\alpha$, and K^c is open, we have $X = \cup_\alpha F_\alpha \cup K^c$ which means $E \subset \cup_\alpha F_\alpha \cup K^c$. Since E is compact, there would exist a finite subcover such that $E \subset \cup_{j=1}^n F_{\alpha_j} \cup K^c$. We then have $K \subset \cup_{j=1}^n F_{\alpha_j} \cup K^c$, which would mean $K \subset \cup_{j=1}^n F_{\alpha_j}$, whence, we see that K is compact. ■

Corollary 2.10

If F is closed, and K is compact, then $F \cap K$ is compact

Fact 2.11

A compact set is bounded

Proof. Consider (WLOG, a non empty compact set E) and an arbitrary point q in X. $B_\varepsilon(x)$ for every $\varepsilon > 0$ forms an open cover for E (since it is basically X). Which means there is a finite subcover, i.e, a number $\varepsilon_0 > 0$ such that $E \subset B_{\varepsilon_0}(p)$ which makes E bounded. ☐

Theorem 2.12

Finite union of compact sets is compact

Proof for Theorem.

Let $K_1, K_2, \cdots K_r$ be r compact sets. Let $K = \cup_{i=1}^{r} K_i$. Consider an open cover F_α whose union subsumes K. We have that, for every $i \leq r$, $K_i \subset \cup_\alpha F_\alpha$. Since each K_i is compact, there exists a finite of F_α whose union subsumes K_i. For each $i = 1$, to r, we have a finite subcollection, therefore, taking the union of all these finite subcollections gives us a finite subcollection which subsumes whole of K. Hence K is compact.

Remark.

Note that finiteness in the above theorem is important. This is because, each compact may have finite subcollection, but at the end, the union of all these finite collections will be countable, not finite.

Theorem 2.13

If $\{K_\alpha\}$ is a collection of compact sets such that for every finite subcollection $\{K_{\alpha_j} : 1 \leq j \leq k\}$ we have that $\cap_{j=1}^{k} K_{\alpha_j} \neq \phi$. Then $\cap_\alpha K_\alpha \neq \phi$. In pithy words:

"If you have a collection of compact sets for which every finite subcollection's intersection is non-empty, the intersection of the whole collection is non empty"
-Krishna, to Arjuna

Proof for Theorem.

Suppose, on the contrary, let $\cap_\alpha K_\alpha = \phi$ which means $\cup_\alpha K_\alpha^c = X$ which means for every for some α_0, we have $K_{\alpha_0} \subset \cup_\alpha K_\alpha^c$, where $\cup_\alpha K_\alpha^c$ is an open cover of K_{α_0}. This implies that there exists a finite subcollection $\{K_{\alpha_1}^C, K_{\alpha_2}^C \cdots K_{\alpha_r}^C\}$ such that $K_{\alpha_0} \subset \cup_{j=1}^{r} K_{\alpha_j}^C \implies \cap_{j=1}^{r} K_{\alpha_j} \subset K_{\alpha_0}^C$. But this means $\cap_{j=1}^{r} K_{\alpha_j} \cap K\alpha_0 = \phi$, which is absurd since all finite intersection is non empty. ∎

Corollary 2.14

If $K_1, K_2, \cdots$ is a sequence of non-empty compact sets such that $\cdots K_n \subset K_{n-1} \cdots K_3 \subset K_2 \subset K_1$, then $\cap_{i=1}^{\infty} K_i$ is non empty.

Theorem 2.15: Compactness $\implies$ Limit point Compact

If K is a compact set and E is an infinite subset of K, then E has a limit point in K

Proof for Theorem.

Suppose that E has no limit point in K. Since K is closed, E must have no limit points. Hence, E is closed. Since closed subsets of compact sets are compact, E is compact. If no point of E is a limit point of E, then $\forall x \in E$, $\exists \varepsilon_x > 0$ such that no point of E apart form x itself falls into the ε_x-ball of x. Consider the open cover $\{B_{\varepsilon_x}(x) : x \in E\}$ of E. This has a finite subcover $\{B_{\varepsilon_{x_1}}(x_1), B_{\varepsilon_{x_2}}(x_2), \cdots B_{\varepsilon_{x_l}}(x_l)\}$. We see that $E \subseteq \cup_{j=1}^{l} B_{\varepsilon_{x_j}}(x_j)$. But since for every ε_{x_j}-ball around x_j, no point in E except x_j resides, $\cup_{j=1}^{l} B_{\varepsilon_{x_j}}(x_j)$ will have utmost finite points. Since an infinite set E cannot be the subset of a finite set, we have a contradiction. ∎

Definition 2.16: k-cell

a k-cell, E is a set in $\mathbb{R}^k$ such that $E := \{\vec{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^n : a_j \leq x_j \leq b_j$ for given a_j and b_j for every $1 \leq j \leq k\}$

A k-cell is basically a k dimensional cuboid.

Theorem 2.17

k-cells are closed

Proof for Theorem.

Consider a k-cell E. Consider a point z not in E, i.e, $\exists j_0$ such that either $z_j < a_j$ or $z_j > b_j$. WLOG, take the case of $z_j < a_j$. Let $0 < \delta < (a_j - z_j)$. Consider a point q in the δ-ball around z. i.e, $d(z, q) < \delta \implies \sqrt{(z_1 - q_1)^2 + (z_2 - q_2)^2 + \cdots + (z_k - q_k)^2} < \delta \implies (z_1 - q_1)^2 + (z_2 - q_2)^2 + \cdots + (z_k - q_k)^2 < \delta^2 < (a_j - z_j)^2 \implies 0 < (q_j - z_j)^2 < (a_j - z_j)^2 \implies q_j < a_j$. Hence $q \notin E$, which implies there exists, for every x in E^c, a δ for which the δ-ball around x is fully contained in E^c which means E^c is open. This implies E is closed. Same argument applies for the case where $z_j > b_j$. ∎

Theorem 2.18

Closed intervals in $\mathbb{R}$ are compact

Proof for Theorem.

Let $\mathbb{I}$ be, WLOG, $[-a,a]$. Suppose it is not compact. i.e, There is an open cover G_α of $\mathbb{I}$ such that there exists no finite subcover. $\forall x \in \mathbb{I}$, $\exists \alpha_x$ such that $x \in G_{\alpha_x}$ and $\exists \varepsilon_x$ such that $B_{\varepsilon_x}(x) \subset G_{\alpha_x}$. $\cup_x B_{\varepsilon_x}(x) \subset \cup_\alpha G_\alpha$ is an open cover for $\mathbb{I}$. note that, if no finite subcover for G_α exists, then no finite subcover for $B_{\varepsilon_x}(x)$ exists either. So we can safely work with $B_{\varepsilon_x}(x)$. Split the interval into two halves, $[-a,0]$ and $[0,a]$. One of these intervals is not finitely covered by $B_{\varepsilon_x}(x)$, for if not, the whole thing would be finitely covered. let that interval which is not finitely covered be $\mathbb{I}_1$. This interval's size is a. Split this interval into two again. Yet again, one of the halves must not be finitely covered, for if not, $\mathbb{I}_1$ would be finitely covered, which is contradictory. Let this interval be $\mathbb{I}_2$. This is of size $\frac{a}{2}$. Yet again, keep doing this process to obtain a sequence of intervals $\mathbb{I}_j$, sized $\frac{a}{j}$, which are not finitely covered. These are nested intervals, non empty, and closed. From nested intervals theorem, we see that a point ξ exists in $\cap_{j=1}^{\infty} \mathbb{I}_j$. ξ is a point in $\mathbb{I}$, and there is a corresponding ε_ξ. Consider that j_0 for which $\frac{a}{j_0} < \frac{\varepsilon_\xi}{2}$. We know from Archimedean such a j_0 exists. This means that the interval $\mathbb{I}_{j_0}$ containing ξ, sized $\frac{a}{j_0}$, is completely inside the ε_ξ-ball around ξ, which means it is finitely covered. Contradiction. Hence, $\mathbb{I}$ is compact. ■

Corollary 2.19

Since intervals of the form $[-a,a]$ are compact, every closed interval of the form $[a,b]$ is compact since it would be a closed subset of an interval of the form $[-x,x]$.

Generalisation:

Theorem 2.20

n-cells are compact

Proof for Theorem.

Consider $K := \{\vec{x} \in \mathbb{R}^n : -a \leq x_j \leq a; \forall j \leq n\}$ to be non-compact. There is an open cover G_α of K such that there exists no finite subcover. $\forall x \in K$, $\exists \alpha_x$ such that $x \in G_{\alpha_x}$ and $\exists \varepsilon_x$ such that $B_{\varepsilon_x}(x) \subset G_{\alpha_x}$. $\cup_x B_{\varepsilon_x}(x) \subset \cup_\alpha G_\alpha$ is an open cover for K. note that, if no finite subcover for G_α exists, then no finite subcover for $B_{\varepsilon_x}(x)$ exists either. So we can safely work with $B_{\varepsilon_x}(x)$. Till here, everything is the same as the 1-d case. Note that here, the n-cell is constructed by taking the cartesian product of n- intervals in $\mathbb{R}$ of the kind $[-a,a]$. Construct 2^n subdivisions of K by halving each interval $[-a,a]$ in the construction of K. The total number of subdivisions we make would be $2 \times 2 \times \cdots \times 2$, n times (simple combinatorial argument: for each i, there exists 2 choices, the two half intervals, for crossing. From $i = 1$, you have 2 choices, likewise, $j = 2, 3, \cdots n$). We assume that atleast one of these 2^n subdivisions are not finitely covered by $\{B_{\varepsilon_x}(x)\}$. We let this one be K_1, whose each interval size is now a. Subdivide this yet again into 2^n subsets, and assert that one of these subdivisions is not finitely covered. Call this K_2, whose each interval is of size $\frac{a}{2}$. Construct a sequence of sets K_j, each of whose intervals are sized $\frac{a}{j}$. Each K_j is closed and non empty, hence compact, and are nested. Therefore, $\cap_{j=1}^{\infty} K_j \neq \phi$. Let $p \in \cap_{j=1}^{\infty} K_j$. For this p, there would exist a ε_p and the corresponding ball $B_{\varepsilon_p}(p)$. We require one of our K_j n-cell to fall into this ε_p ball. Let δ be smaller than $\frac{\varepsilon_p}{2}$. Let p be the centre of the $\delta-$ball. Let p be the centre of the n-cube H in the following construction: Consider w to be the side length of H. We require the diagonal length $w\sqrt{n} = \delta$, which gives us $w = \frac{\delta}{\sqrt{n}}$. Consider H such that each side is the interval $[p_j - \frac{w}{2}, p_j + \frac{w}{2}]$. This would force p to fall in the centre of the n-cube H. This cube is fully contained in the δ ball of p which is contained in the ε_p ball of p. We consider that n cell K_j for which each side $\frac{a}{j} < w$. This can be found, and hence, the this K_j cell is finitely covered by $B_{\varepsilon_p}(p)$, which is absurd. Hence, K is compact. ■

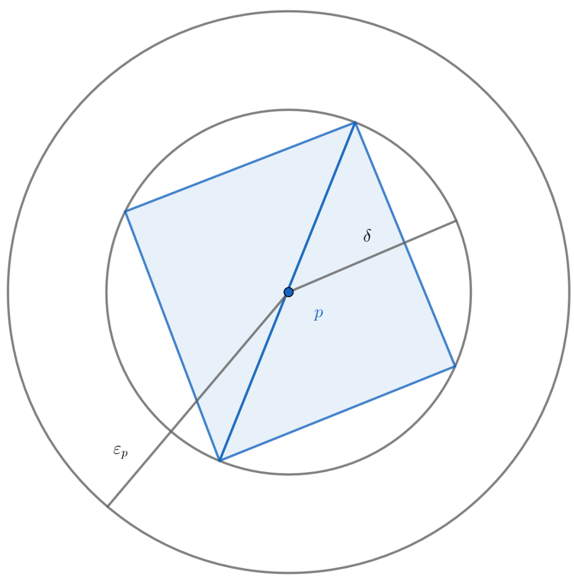

Figure 1.2: Figure for the proof: n-cells are compact.(The ε_p-ball around p, and the n-cell construction)

Remark.

> We proved the result for k-cells of the kind $[-a, a]^k$, but it is easily generalised by noting that arbitrary k-cells are contained in some k-cell of the above kind. By virtue of being closed, they also are compact.

Theorem 2.21: Heine-Borel

Given a set $E \subset \mathbb{R}^n$, the following are equivalent:

1. E is closed and bounded
2. E is compact

Proof for Theorem.

$\Longleftarrow$) We know that all compact sets are closed.

$\Longrightarrow$) If $E \subset \mathbb{R}^n$ is closed and bounded, it is contained in some n-cell, which is compact. By virtue of being a closed subset of a compact set, E is also compact. ■

Theorem 2.22

If $\{x_n\}$ is a sequence in X convergent to $x \in X$, then the set $\{x_n\}$ has only one limit point, which is x.

Proof for Theorem.

That x is a limit point is clear. We note that $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \in \mathbb{N}, n \geq n_0$ we have $d(x_n, x) < \varepsilon$. i.e, beyond a particular n_0, every point of $\{X_n\}$ falls in the ε-ball of x. Therefore, only finite points lie outside this ε-ball of x. Suppose it has another limit point y, other than x. Therefore, there would exist a δ such that the δ-ball around y lies completely outside the δ-ball around x. This means that, only finite points of x_n lie in the δ ball of y, making it unviable to be a limit point. ■

Extending Heine Borel we have:

Theorem 2.23: (Extension)

For a subset $E \subset \mathbb{R}^n$, the following are equivalent:

1. E is closed and bounded
2. E is compact
3. every infinite subset K of E has a limit point in E

Proof for Theorem.

(1) $\Longrightarrow$ (2)) Heine Borel

(2) $\Longrightarrow$ (3)) Already seen

(3) $\Longrightarrow$ (1)) Let us assume that E is either not closed, or not bounded. We start by assuming it is not closed. Which means that $\exists q$ outside E such that there exists a

sequence in E that converges to q. We take this sequence $\{x_n\}$ as our infinite set, and we see that, from the previous theorem, this has only one limit point q, which lies outside E. Hence, there exists an infinite set $\{x_n\}$ which has no limit point in E.

Suppose that E is unbounded. We then have that, $\forall x \in X$, $\forall \varepsilon > 0$, $\exists y \in E$ such that $d(x, y) > \varepsilon$. Fix some x_0 in X. Choose some y_0 that is a distance $z_{00} = d(y_0, x_0)$ away from x_0. Look if there is a point y_1 so that its distance from x_0 is more than z_{00} but less than $2(z_{00})$. If it doesn't exist, check for less than $3(z_{00})$. Find some $k_1(z_{00})$ so that distance of y_1 to x_0 is more than z_{00} but less than $k_1(z_{00})$. Same way, for y_2, find y_2 so that its distance from x_0 is more than $k_1(z_{00})$ but less than some other $k_2(z_{00})$. Inductively, find y_j whose distance is more than $k_{j-1}(z_{00})$ but less than $k_j(z_{00})$. Note that $1 < k_1 < k_2 < \cdots$. Hence, for any ε-ball around x_0, only finite y_j exists in that ball, since there would exist some $k_q(z_{00})$ and $k_{q-1}(z_{00})$ between which ε lies. And inside $k_{q-1}(z_{00})$ ball around x_0, utmost finite points y_j exists. Hence, x_0 is clearly not a limit point for the set of y_j-s. Consider any other point $a \in X$. For some every ε-ball around x_0, only finite points exists. For some, perhaps larger δ-ball around a, a chosen ε-ball around x_0 gets subsumed into the δ-ball around a. This implies that only finite points of y_j-s exists in the δ-ball around a as well, making a a non viable limit point. We see that, for this infinite subset $\{y_j\}$ of E, no limit point exists. ■

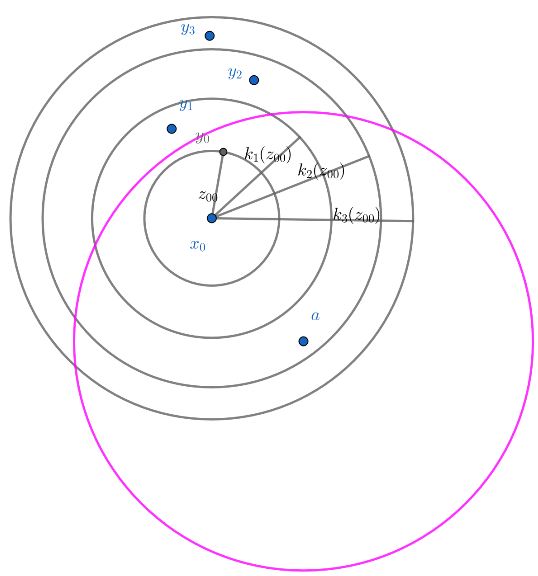

Figure 1.3: Figure for proof: Lim point compact $\implies$ closed+bounded. (The construction of an unbounded sequence)

Remark.

> In the previous proof, we note that (3), which is called Limit Point Compactness, implies (1) Closed and Bounded, in any metric space, not just $\mathbb{R}^n$, as we see in the proof, no property of $\mathbb{R}^n$ was used.
>
> **Spoiler Alert:** In any metric space, Limit Point Compact $\iff$ Compact

Theorem 2.24: Weierstrass Theorem

Every Bounded, infinite set in $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.

Proof for Theorem.

> If a set is bounded in $\mathbb{R}^n$, it is the subset of a compact set (i.e, a closed and bounded set). From the previous equivalence, an infinite subset E of a compact set has a limit point in the compact set, which means the bounded, infinite set we have has a limit point in the compact set that contains it, hence, it has a limit point in $\mathbb{R}^n$. ■

Remark.

> The above "Weierstrass Theorem" is just the "Bolzano-Weierstrass" Theorem we saw in sequences. Actually, the "Bolzano-Weierstrass" Theorem is a direct corollary of the more general "Weierstrass Theorem". Let $\{x_n\}$ be any sequence in $\mathbb{R}$ that is bounded. This means that this sequence is the subset of a compact set, hence, has a limit point in $\mathbb{R}$. This implies, a subsequence of $\{x_n\}$ converges in $\mathbb{R}$. Hence, every bounded sequence has a convergent subsequence.

Fact 2.25

Let $X = \mathbb{R}^n$. The closure of any open ball is the corresponding closed ball.

Proof. Consider $B := B_\delta(x_0) := \{y \in \mathbb{R}^n : ||y - x_0|| < \delta\}$. Let z_0 be a point on the rim of B, i.e $d(x_0, z_0) = \delta$. Such a point obviously exists. Consider $\vec{\gamma}(t) = t\vec{z}_0 + (1-t)\vec{x}_0$ with $t \in (0,1)$. For every $t \in (0,1)$, $\vec{\gamma}(t)$ belongs in B. To see this, consider $||\vec{\gamma}(t) - \vec{x}_0|| = ||t\vec{z}_0 + (1-t)\vec{x}_0 - \vec{x}_0|| = ||t(\vec{z}_0 - \vec{x}_0)|| = t||\vec{z}_0 - \vec{x}_0|| < t(\delta) < \delta$. Suppose we are given an arbitrary $\eta > 0$. Does there exist a $t \in (0,1)$ so that $\vec{\gamma}(t)$ belongs in the η-ball of $\vec{z}_0$? i.e, we need a t so that $||\vec{\gamma}(t) - \vec{z}_0|| = ||t\vec{z}_0 + (1-t)\vec{x}_0 - \vec{z}_0|| = ||(1-t)(\vec{z}_0 - \vec{x}_0)|| < \eta \implies (1-t)||\vec{z}_0 - \vec{x}_0|| < \eta \implies 1 - t < \frac{\eta}{\delta} \implies 1 - \frac{\eta}{\delta} < t < 1$. Such a t exists for every η. Hence, $\vec{z}_0$ is a limit point of B (by virtue of there existing a sequence of $\vec{\gamma}(t_j)$ that converges to $\vec{z}_0$). Hence, every point on the rim is a limit point. Moreover, no point w so that $d(w, x_0) > \delta$ is a limit point of B, since there would exist an ε-ball around w so that no point of B falls into it (from openness). Hence, closure of B is the corresponding closed ball, in $\mathbb{R}^n$.

□

3 Perfect Sets

Definition 3.1: Perfect Set

A set $E \subset X$ is perfect if every point of E is a limit point of E, and E is closed

Theorem 3.2

Perfect subsets in $\mathbb{R}^n$ are uncountable.

Proof for Theorem.

Suppose E is a perfect set in $\mathbb{R}^n$ but is countable. i.e, it can be enumerated as $E = \{x_1, x_2, \cdots\}$.

Choose x_1, and $\varepsilon_0 = 1$. Let V_0 denote the ε_0-ball around x_1. This ball is non empty, moreover, $\bar{V_0} \cap E$(which is the corresponding closed ball of V_0) is non empty, and is compact by virtue of being closed and bounded. Inside, $V_0 \cap E$, there exists infinite points of E, since x_1 is a limit point of E.

Choose an arbitrary point z_1 in V_0 that is not x_1. Now let $\varepsilon_1 < d(x_1, z_1)$. Let V_1 be the ε_1-ball around z_1. Notice the following: z_1 is a limit point of E, hence, there are infinite points of E in V_1. x_1 is not in $\bar{V_1}$. $\bar{V_1} \cap E$ is closed, bounded and non empty, hence Compact.

Choose a point z_2 in V_1 that is not x_2, and let $\varepsilon_2 < min\{\varepsilon_1, d(x_2, z_2)\}$. Let V_2 be the ε_2-ball around z_2. Note that, x_2 is not in $\bar{V_2}$. Also note yet again that there are infinitely many points of E in V_2. It is crucial to note now that $\bar{V_2} \cap E \subset \bar{V_1} \cap E \subset \bar{V_0} \cap E$.

Suppose you have already constructed V_k by finding z_k in V_{k-1} that is not x_k and an $\varepsilon_k < mind(z_k, x_k), \varepsilon_{k-1}$ such that $x_k \notin \bar{V_k}$, $\bar{V_k} \cap E$ is compact, non empty and $\bar{V_k} \cap E \subset \bar{V_{k-1}} \cap E \cdots$.

Now, choose $z_{k+1} \neq x_{k+1}$, inside V_k. Choose $\varepsilon_{k+1} < min\{d(z_{k+1}, x_{k+1}), \varepsilon_k\}$. Let V_{k+1} be the ε_{k+1}-ball around z_{k+1}. Yet again, we see that $\bar{V_{k+1}} \cap E$ is non empty, x_{k+1} is not in $\bar{V_{k+1}}$, and $\bar{V_{k+1}} \cap E \subset \bar{V_k} \cap E$. Hence, we have a sequence of non empty, nested compact sets. This implies that $\exists \xi \in E \subset \mathbb{R}^n$ such that $\xi \in \cap_{i=1}^{\infty}(\bar{V_i} \cap E)$. Is ξ any one of x_j enumerated? No, because if it was, from the construction, x_j would not belong in V_j. Hence, ξ is not in the enumeration of E. Contradiction. ∎

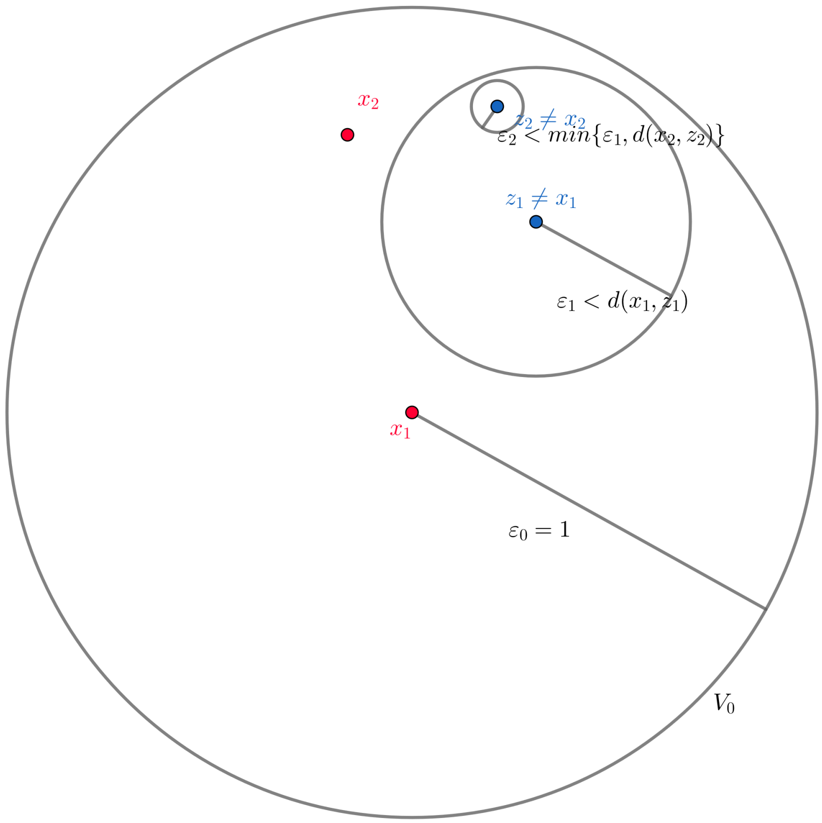

Figure 1.4: Figure: Perfect sets are uncountable. Construction of the nested sequence of compact sets by choosing $z_k \neq x_k \in V_{k-1}$.

Remark.

> It is easily seen that, closed intervals in $\mathbb{R}$ are perfect: From density theorem, for every point in I, there would exist a sequence of rationals converging to that point. Moreover, closed intervals in $\mathbb{R}$ are closed since closed balls in metric spaces are closed. Therefore, we see that intervals are uncountable.

3.1 The Cantor Set

The following is the construction of an uncountable, perfect set that contains no intervals: The Cantor Set.

Let $I_0 = [0, 1]$. size of the interval(s) in I_0 is 1, and there are $2^0 = 1$ intervals.

Let $I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ be constructed by trisecting I_0 and tossing the middle one. Here, we have each interval sized $\frac{1}{3^1}$, and there are $2^1 = 2$ intervals total.

Let $I_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ be generated by taking each of the two sub intervals in I_1, trisecting them, and tossing the middle one, and joining them finally. We have each interval sized $\frac{1}{3^2}$ and there are $2^2 = 4$ intervals total.

Inductively keep making these trisections+tossings to make a sequence of closed, nested intervals (Compact, too) I_k, each containing 2^k intervals each of size $\frac{1}{3^k}$.

Finally, define the Cantor set P as

$$P := \cap_{i=1}^{\infty} I_i$$

Note that P is compact since it is the closed subset of a compact set. It is also non empty by virtue of being the intersection of a sequence of nested, non empty, compact sets.

Note that, no interval of the kind $[a, b]$ exists in the Cantor Set. The size of each interval in I_j is $\frac{1}{3^j}$. We can find j so that $\frac{1}{3^j} < b - a \implies \frac{1}{b-a} < 3^j \implies \log_3(\frac{1}{b-a}) < j$. For such I_j, we notice that $[a, b]$ has "inbetween" points that doesn't exist in any of I_j's intervals. Hence, taking the intersection, these "inbetween" terms don't survive. Hence, no intervals exist.

Theorem 3.3

The Cantor set P is perfect.

Proof for Theorem.

We already know that the Cantor set is closed. We need to show that every point in the cantor set is a limit point. First, observe that, for any I_k, if z is the end point of any of the sub interval of I_k, it survives the ∞-intersection. This is because, after I_k-s trisection, the end points still stay endpoints. Let ξ be any point in the cantor set, which means it is a point in every I_k. Let $\delta > 0$ be given. Consider the interval $(\xi - \delta, \xi + \delta)$. This interval is sized 2δ. ξ exists in one of the sub intervals of I_k for all $K \geq k_0$ for some k_0. Choose j so that $\frac{1}{3^j} < \delta$. Then, the interval in I_j containing ξ would fall completely inside $(\xi - \delta, \xi + \delta)$. Choose q as one of the end points of this sub interval of I_j. Therefore, $\forall \xi \in P$, $\forall \delta > 0$, $\exists q \in P, q \neq \xi$ so that $q \in (\xi - \delta, \xi + \delta)$. Therefore, every $\xi \in P$ is a limit point of P. Hence, P is perfect. ∎

3.2 Connected Sets

Definition 3.4: Separated Sets

$A \subset X$ and $B \subset X$ are said to be *separated* if $\bar{A} \cap B = \bar{B} \cap A = \phi$, i.e, they are disjoint and no point of one, is the limit point of the other.

Definition 3.5: Connected Set

A set $E \subset X$ is said to be *connected* if it is *not* the union of two non-empty separated sets. In other words, for every "split" of E into two non empty sets, none of them are separated. Even if one split of E is separated, then E is *not connected*.

Example : Separated $\implies$ Disjoint, but Disjoint $\not\Longrightarrow$ Separated.
$[0,1]$and $(1,2)$ are disjoint, but are not connected since a sequence in $(1,2)$ converges to 1 in $[0,1]$.

Theorem 3.6

$E \subset \mathbb{R}$ is connected $\iff \forall x, y \in E,\ x < z < y \implies z \in E$.

Proof for Theorem.

$\implies$) Suppose $\exists x_0, y_0 \in E$ so that $\exists z, x_0 < z < y_0$, but $z \notin E$. Consider $A := (-\infty, z)$ and $B := (z, \infty)$. A and B are seen to be separated, and E is a subset of $A \cup B$, which makes it disconnected.

$\impliedby$) Suppose that we have E disconnected, which means it is the union of two separated sets A and B that are non-empty. $x_0 \in A$ and $y_0 \in B$. Consider $z(t) = x_0 + t(y_0 - x_0)$ for $t \in [0,1]$. Note that $z(0) = x_0$ and $z(1) = y_0$.

Conjecture: There exists a $t_B \in (0,1)$ so that for every $t < t_B$, $z(t)$ does not belong in B. If it is not true, then for every $t \in (0,1)$, there exists a point $t_B < t$ so that $z(t_B)$ is in B. Choose $t = 1$ to get $z(t_1)$ in B. Chooe $t = \frac{t_1}{2}$ to get $z(t_2)$ in B with $t_2 < t_1$ and $t_2 < \frac{1}{2}$. Keep going with $t = \frac{t_{n-1}}{2^{n-1}}$ to get $z(t_n)$ in B with $t_n < t_{n-1}$ and $t_n < \frac{1}{2^{n-1}}$. This gives us a sequence $z(t_k)$ which we can see is monotone decreasing assuming $x_0 < y_0$. This sequence converges to $z(0)$ which is in A which means that there exists a sequence in B, $z(t_n)$ that converges to A. Absurd.

In a similar vein, we can show that there exists $t_A \in (0,1)$ so that for every $t > t_A$, $z(t)$ is not in A. Consider

$$S_A := \{t \in [0,1] : z(t) \in A \cap [x_0, y_0]\}$$

and

$$S_B := \{t \in [0,1] : z(t) \in B \cap [x_0, y_0]\}$$

It is easy to see that S_A and S_B are disjoint. If a sequence in one converges in another, say $t_n \in S_B$ converges to $t_0 \in S_A$. Then $z(t_n) \in B$ by definition, for every n. But then by definition, $z(t_n) = x_0 + t_n(y_0 - x_0) \in B$ such that $lim(z(t_n)) = x_0 + t_0(y_0 - x_0) \in A$, which means a sequence in B converges in A. Absurd. So S_A and S_B are separated.

Note that, for $t > t_A$, no $z(t)$ is in A. Hence, we see that for every t so that $z(t)$ falls in A, there is an upperbound. Likewise, for every t such that $z(t)$ falls in B, there is a lowerbound. Hence, S_A has a supremum $sup(S_A)$ and S_B has an infimum $inf(S_B)$.

At this point, we may as well assume that for every $t < inf(S_B)$, $t \in S_A$ for if not, what we wanted to prove would get proved. Suppose then, for argument sake, that for every $t > Sup(S_A)$, $t \in S_B$, and likewise, for every $t < Inf(S_B)$. $t \in S_A$. Now then, does $Sup(S_A)$ belong in S_A? we see that for every $t > sup(S_A)$, $t \in S_B$ which means we can construct a sequence in S_B using those ts, which converge to $Sup(S_A)$ in S_A. So that is ruled out. So is $Sup(S_A)$ in S_B? That is not possible either, since S_A is a bounded, infinite set (mainly because supremum isn't in the set), we know that there is a monotone subsequence in S_A converging to $Sup(S_A)$ which is in S_B. We therefore conclude that, there exists a point $t \in (0,1)$ so that t is neither in S_A, nor in S_B. This translates to there being a point $z = z(t)$, between x_0 and y_0 so that $z(t) \notin A \cup B \implies \notin E$. We are, therefore, done.

Slicker Argument: Suppose $E = A \cup B$ with $\bar{A} \cap B = \bar{B} \cap A = \phi$. Consider $x_0 \in A$ and $y_0 \in B$ and WLOG assume $x_0 < y_0$. Define $z = sup(A \cap [x_0, y_0])$. There would be a sequence in A that converges to z, by virtue of being the supremum. $z \in \bar{A} \implies z \notin B$. This means $x_0 \leq z < y_0$. If $z \notin A$, we would be done. If $z \in A$, then $z \notin \bar{B}$. Therefore, z is in an open set $\bar{B}^C$. There would exist an ε_z-ball around z so that it is fully contained outside $\bar{B}$. Choose $z + \frac{\varepsilon_z}{2}$ as your z'. Note that z' is greater than the supremum of A. We see that z' is not in B, and not in A either. Hence, we are done. ■

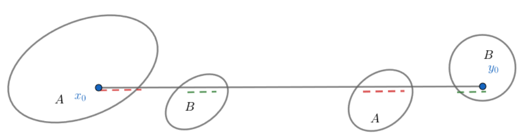

Figure 1.5: Figure: Proof for the equivalence for connectedness for sets in $\mathbb{R}$. A look at $A \cap [x_0, y_0]$ and $B \cap [x_0, y_0]$.

4 Misc Knowledge

Theorem 4.1

Suppose $A_1, A_2 \cdots, A_n \cdots \in X$. Then,

1. If $B_n = \cup_{i=1}^n A_i$, then $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$
2. $B = \cup_{i=1}^\infty A_i$, $\bar{B} \supseteq \cup_{i=1}^\infty \bar{A}_i$ with possibility of strict inequality.

Proof for Theorem.

Suppose $x \in \cup_{i=1}^{n} \bar{A}_i$. Which means $x \in \bar{A}_i$ for some i. It is clear that x is either a point of A_i or a limit point of A_i. We have that, whatever maybe the case either x is a point in B_n or a limit point of B_n. Therfore $\bar{B_n} \supset \cup_{i=1}^{n} \bar{A}_i$.

Suppose x is a point in $\bar{B_n}$. If it is a point of B_n, we are done. Suppose it is the limit point of B_n, but not a point. Also suppose that x is not a limit point of any $A_i : i = 1 \to n$. This means that,

$$\exists \varepsilon_1 \text{ such that } \forall q \in A_1, q \neq x, \text{ we have } d(q,x) \geq \varepsilon_1$$

$$\exists \varepsilon_2 \text{ such that } \forall q \in A_2, q \neq x, \text{ we have } d(q,x) \geq \varepsilon_2$$

$$\vdots$$

$$\exists \varepsilon_n \text{ such that } \forall q \in A_n, q \neq x, \text{ we have } d(q,x) \geq \varepsilon_n$$

If we choose $0 < \varepsilon_0 < min\{\varepsilon_i : i = 1 \to n\}$, we would have that, for every point q in $A_1 \cup A_2 \cdots A_n$, $q \neq x$(which is needless to say), we have $d(q,x) \geq \varepsilon_0$ which makes x a non-limit point of B_n, which is absurd. Hence we see that if x is point of B_n or a limit point of B_n, then it is a point or the limit point of some A_j.

For a good counterexample, we look to $\mathbb{Q} := \{q : \text{ q is rational}\}$. This set is countable. Let $\{q_1, q_2, \cdots\}$ be the enumeration of $\mathbb{Q}$. Consider $\mathbb{Q} := \cup_{j=1}^{n} \{q_j\}$. The closure of $\mathbb{Q}$ is $\mathbb{R}$ but since these singleton sets are by definition closed, the union of them only gives you $\mathbb{Q}$. ■

Definition 4.2: Interior of a Set

Given $S \in X$, the interior $\underline{\mathrm{S}}$ is defined as:

$$\underline{\mathrm{S}} := \{x \in X : \exists \varepsilon_x > 0 \text{ such that } B_{\varepsilon_x}(x) \subset S\}$$

Theorem 4.3

The Interior is an open set

Proof for Theorem.

Consider $(\underline{\mathrm{S}})^C := \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap S^C \neq \phi\}$. It is possible that x is a point of S^C, if it is in $\underline{\mathrm{S}}^C$. Suppose it is a point of $\underline{\mathrm{S}}^C$ but not a point of ^{C}. From the definition, we see that $\forall \varepsilon > 0, \exists q \in S^C, q \neq x$ such that $d(q,x) < \varepsilon$. This makes x a limit point of S^C, which means, for every x in $\underline{\mathrm{S}}^C$, x is either a point of S^C or a limit point of S^C. Hence,

$$\underline{\mathrm{S}}^C \subseteq \bar{S^C}$$

Suppose x is a point of $\bar{S^C}$. Say it is a point of S^C, then obviously, it is a point of $(\underline{\mathrm{S}})^C$. Suppose x is not a point of S^C, but a limit point of S^C. This means $\forall \varepsilon > 0, \exists q \in S^C, q \neq x$ so that $d(q,x) < \varepsilon$. This is precisely the condition for which x is a point of $(\underline{\mathrm{S}})^C$. Hence we see $(\underline{S})^C \supseteq (\bar{S^C})$. Therefore, $(\underline{S})^C = (\bar{S^C})$. From here we see that $\underline{S}$ is an open set.

Alternate Argument (similar): Consider $(\underline{S})^C := \{x \in X : \forall\varepsilon > 0, B_\varepsilon(x) \cap S^C \neq \phi\}$ Consider a limit point p of $(\underline{S})^C$. $\forall\varepsilon > 0, \exists q_\varepsilon \in \underline{S}^C$ such that $d(q_\varepsilon, p) < \frac{\varepsilon}{2}$. Since q_ε is in $(\underline{S})^C$, we have that: $\forall\delta > 0, \exists r_\delta \in S^C, r_\delta \neq q_\varepsilon$ such that $d(r_\delta, q_\varepsilon) < \delta$

Combining these we have:

$$(\forall\varepsilon > 0)(\exists q_\varepsilon \in \underline{S}^C)(\exists\delta > 0)(\exists q_\delta \in S^C)$$

$$(d(q_\delta, p) \leq d(q_\delta, q_\varepsilon) + d(q_\varepsilon, p) < \frac{\varepsilon}{2} + \delta < \varepsilon)$$

This means p is a limit point of S^C. Hence, $\underline{S}^C$ is closed. ■

Theorem 4.4

$\underline{S} = S \iff S$ is open

Proof for Theorem.

$\Longrightarrow$) if $\underline{S} = S$, obviously S is open.

$\Longleftarrow$) If S is open, then by definition $\underline{S}$ = set of all points in S so that there's an ε-ball of x in S. But that is every point of S. ■

Theorem 4.5

$\underline{S}$ is the largest open set contained in S

Proof for Theorem.

Consider an open subset of S. These are subsets of S whose each point has an ε-ball around it so that the ball is contained in the subset, which is contained in S. So by definition, these points in these subsets are contained in $\underline{S}$. ■

Theorem 4.6

$$(\underline{S})^C = \overline{(S^C)}$$

"The compliment of the interior is the closure of the compliment"

Proof for Theorem.

Refer to the proof of "Interiors of sets are Open", to see this construction.

Alternate method(slicker):

$\underline{S} \subseteq S \implies S^C \subset (\underline{S})^C$ where $(\underline{S})^C$ is a closed set containing S^C. Since $\overline{S^C}$ is the smallest closed set that contains S^C, we have $\overline{S^C} \subseteq (\underline{S})^C$.

Note that $S^C \subseteq \overline{S^C} \implies (\overline{S^C})^C \subseteq S$ where $(\overline{S^C})^C$ is an open set inside S. Since $\underline{S}$ is the largest open set containing S, we have that $(\overline{S^C})^C \subseteq \underline{S} \implies \underline{S}^C \subseteq \overline{S^C}$. Combining these two set inequalities, we are done. ■

Theorem 4.7

1. If A and B are closed, disjoint subsets of X, then A and B are separated.
2. If A and B are open, disjoint sets, then A and B are separated.

Proof for Theorem.

(1) A B closed implies $A = \overline{A}$ and $B = \overline{B}$ which are disjoint. From here it is obvious.

(2) A and B are open disjoint sets, then we see that $A \subseteq B^C$ and $B \subseteq A^C$ where A^C and B^C are closed by definition. Since closure is the smallest closed set containing A (and B), we see that $\overline{A} \subseteq B^C$ and $\overline{B} \subseteq A^C$. It is now trivial to see that $\overline{A} \cap B \subseteq B^C \cap B = \phi = \overline{B} \cap A$ which is the definition of separated. ■

Corollary 4.8

Let $p \in X$ and $\delta > 0$. Define $A := B_\delta(p)$ and $B = (B_{[\delta]}(p))^C$. A and B are, then, separeted.

Proof for Corollary.

Easy to see that they are both open sets that are disjoint. ■

Theorem 4.9

Every connected metric space with atleast two points is uncountable.

Proof for Theorem.

Let a and b be in X. Let $\xi \leq d(a, b)$. Note that, $P = B_\xi(a)$ and $Q = (B_{[\xi]}(a))^C$ are non empty, separated sets (from the previous corollary). If X is not the union of P and Q, then there is a point z_ξ in X so that it is neither in P not in Q. That means that it is exactly ξ distance away from p. For every $\xi < d(a, b)$, there exists a point z_ξ so that its distance from p is exactly ξ. Therefore, every z_ξ is unique (from positivity property of metric spaces) which means there are uncountable z_ξ-s. ■

Theorem 4.10

If P and Q are connected such that $P \cap Q \neq \phi$, then $P \cup Q$ is also connected.

Proof for Theorem.

Suppose $P \cup Q$ is actually not connected. This means $P \cup Q = A \cup B$ for non empty, separated sets A and B. Suppose P is fully contained in A. This means that Q has intersection with A and intersection with B which are non empty. Obviously $Q \subseteq A \cup B$ which means $Q = (A \cap Q) \cup (B \cap Q)$ where $(A \cap Q)$ and $(B \cap Q)$ are separated and non empty. Since Q is connected, this is absurd. Suppose then that P is not fully contained in A. This means that $P \cap A$ and $P \cap B$ is non empty each. This means $P = (P \cap A) \cup (P \cap B)$. From the same reasoning, this is absurd. ■

Lemma 4.11

Given two balls B_1 and B_2 in $\mathbb{R}^n$ that are closed, with $B_1 \cap B_2 = \{z\}$ with $z \in \mathbb{R}^n$, then the interior of $B_1 \cup B_2$, i.e, $\underline{B_1 \cup B_2}$, is $\underline{B_1} \cup \underline{B_2}$, which are their respective open ball counterparts.

Proof for Lemma

We understand that $\underline{B_1} \cup \underline{B_2} \subseteq \underline{B_1 \cup B_2}$. Note that none of the "rim" points of B_1 or B_2, are in the interior. This would conclude the result. ■