
CHAPTER 1

PRELIMINARIES

1 Sets, functions and all that

Definition 1.1: Preliminary definitions

1. (Cartesian Product): if A and B are non empty sets, the *Cartesian Product* $A \times B$ is defined as the set of ordered pairs a, b wherein $a \in A, b \in B$. i.e, $A \times B := \{(a, b) : a \in A, b \in B\}$
2. (Function): A function from A to B is a set $f \subseteq A \times B$ such that, $a, b \in f$ and $a, c \in f \implies b = c$. A is called the **Domain of f** . $Range(f) := f(A)$ (see next definition)
3. (Direct Image): Direct image $f(A) := \{y \in B : \exists x \in A \text{ such that } f(x) = y\}$
4. (Inverse Image): $f^{-1}(S \subseteq B) := \{x \in A : f(x) \in S\}$
5. (Relation): Any subset $R \subseteq A \times B$ is a relation from A to B .
We say $x \in X$ is "related to" $y \in Y$ under the relation R , or simply xRy or $R(x) = y$ if $(x, y) \in R \subseteq X \times Y$.
6. (Injection): $f : A \rightarrow B$ is injective if $\forall x_1, x_2 \in A, (x_1, b) \in f \text{ and } (x_2, b) \in f \iff x_1 = x_2$
7. (Surjection): $f : A \rightarrow B$ is surjective if $\forall b \in B, \exists a \in A \text{ such that } f(a) = b$
8. (Bijection): $f : A \rightarrow B$ is bijective if its both surjective and injective.
9. (Identity function on a set): $I_A : A \rightarrow A$ defined by $\forall x \in A, I_A(x) = x$
10. (Permutation): Simply a bijection from A to itself is called a permutation.

Definition 1.2: (Left Inverse)

We say $f : A \rightarrow B$ has a left inverse if there is a function $g : B \rightarrow A$ such that $g \circ f = I_A$

Theorem 1.3

$f : A \rightarrow B$ has a left inverse if and only if it is injective.

Proof for Theorem.

\Rightarrow) If f has a left inverse g , Consider $x, y \in A$ such that $f(x) = f(y) = p$.
We have $g \circ f(x) = g(p) = x = g \circ f(y) = y$. Hence, $x = y$, Injective.
 \Leftarrow) Given that $f : A \rightarrow B$ is injective, define $g : B \rightarrow A$ as:

$$g(z \in B) = \begin{cases} a, & \text{where } f(a) = z, \text{ if } z \in f(A) \\ \text{whatever,} & \text{if } z \notin f(A) \end{cases}$$

consider $g \circ f(x \in A) = g \circ (f(x))$.

Obviously, $f(x) \in f(A)$, therefore, $g(f(x)) =$ that a such that $f(a) = f(x)$.

That a is x . Hence, $g(f(x)) = x$ ■

Definition 1.4: (Right Inverse)

$f : A \rightarrow B$ is said to have a right inverse if there is a function $g : B \rightarrow A$ such that $f \circ g = I_B$

Theorem 1.5

$f : A \rightarrow B$ has a right inverse if and only if f is Surjective.

Proof for Theorem.

\Rightarrow) If f has a right inverse g , such that $f \circ g = I_B : B \rightarrow B$, then it is evident that the range of f is B , for if not, range of $f \circ g$ wouldn't be B either.

\Leftarrow) If f is surjective, then for all $b \in B$, there exists atleast one $a \in A$ such that $f(a) = b$ define g as:

$$g(x \in B) = \text{one of those } a \in A \text{ such that } f(a) = b$$

Consider $f \circ g(x \in B) = f(\text{one of the } a \text{ such that } f(a) = b) = b, \forall b \in B$

Hence, $f \circ g = I_B$ ■

Theorem 1.6

If f has left inverse g_1 and right inverse g_2 , then $g_1 = g_2$. *(True for anything that is Associative, and function composition is associative.)*

Proof for Theorem.

$$\begin{aligned}
 g_1 \circ f &= I_A \text{ and } f \circ g_2 = I_B \\
 g_1 \circ (f \circ g_2) &= g_1 \circ I_B = g \\
 &= (g_1 \circ f) \circ g_2 = I_A \circ g_2 = g_2 \\
 \text{Hence } g_1 &= g_2
 \end{aligned}$$

Corollary 1.7

f is invertible (i.e, both left and right inverse exist) if and only if it is bijective.

Proof for Corollary.

Obvious

1.1 Operations on Relations

If R and S are binary relations over $X \times Y$:

1. $R \cup S := \{(x, y) | xRy \text{ or } xSy\}$
2. $R \cap S := \{(x, y) | xRy \text{ and } xSy\}$
3. Given $S : Y \rightarrow Z$ and $R : X \rightarrow Y$, $S \circ R := \{(x, z) | \exists y \text{ such that } ySz \text{ \& } xRy\}$
4. If R is binary over $X \times Y$, $\bar{R} := \{(x, y) | \neg(xRy)\}$

1.2 Homogeneous Relations

If R is a binary relation over $X \times X$, it is Homogeneous.

Definition 1.8: Definitions Regarding Relations

1. (Reflexive): $\forall x \in X, xRx$
2. (Symmetric): $\forall x, y \in X, xRy \implies yRx$
3. (Transitive): $\forall x, y, z \in X, \text{ if } xRy \ \& \ yRz \implies xRz$
4. (Dense): $\forall x, y \in X, \text{ if } xRy, \text{ then there is some } z \in X \text{ such that } xRz \ \& \ zRy$
5. (**Equivalence Relation**): R is an equivalence relation if it is Reflexive, Symmetric and Transitive.
6. (Equivalence class of $a \in A$ (where there is an equivalence relation defined)): Set of all $b \in A$ such that bRa .
7. (Partition of A): Any collection of sets $\{A_i : i \in I\}$ (where I is some indexing set) such that:

$$A = \bigcup_{i \in I} A_i$$

$$A_i \cap A_j = \phi \text{ if } \forall i, j \in I, i \neq j$$

Theorem 1.9

Let A be a non-empty set. If R defines an equivalence Relation on A , then the set of all equivalence classes of R form a partition of A

Proof for Theorem.

Define our collection $\{A_\alpha\}$ as the set of all equivalence classes of A . Clearly, $\bigcup_{\alpha \in I} A_\alpha = A$. If A only has one element, obviously, that singleton set makes up the partition. Let A_α and $A_{\alpha'}$ be equivalence classes of two elements a and a' in A . If aRa' , then $A_\alpha = A_{\alpha'}$ since every element in the equivalence class of a will, from the transitive property, be in the equivalence class of a' . Suppose $\neg(aRa')$. If, then, $\exists x \in A_\alpha$ such that $x \in A_{\alpha'}$, this means that $xR\alpha$ and $xR\alpha'$, but from transitive property, this means $\alpha R\alpha'$, which is a contradiction. Therefore, the pairwise intersection is disjoint. ■

Theorem 1.10

If $\{A_i : i \in I\}$ is a partition of A , then there exists an equivalence relation R on A whose equivalence classes are $\{A_i : i \in I\}$.

Proof for Theorem.

Define $R(x, y)$ if and only if \exists unique $m \in I$ such that $x \in A_m$ and $y \in A_m$.
 $R(x, x)$ is obvious if non empty, hence R is reflexive.

Suppose $R(x, y)$ and $R(y, z)$. Then, there exists a unique $m \in I$ such that x, y are in A_m . Similarly, there exists a unique $n \in I$ such that y, z are in A_n . Obviously, if $n \neq m$, intersection of A_n and A_m would be non empty, hence, $n = m$. Hence, R is transitive.

Consider $R(x, y)$, which means \exists unique $n \in I$ such that $x, y \in A_n \implies R(y, x)$. Hence, R is an equivalence relation. ■

2 Induction, Naturals, Rationals and the Axiom of Choice

Axiom 2.1: Peano Axioms, characterisation of \mathbb{N}

1. $1 \in \mathbb{N}$
2. every $n \in \mathbb{N}$ has a predecessor $n - 1 \in \mathbb{N}$ except 1
3. if $n \in \mathbb{N} \implies n + 1 \in \mathbb{N}$

Definition 2.2: (Sequence of something)

A sequence of some object is simply a collection of objects $\{O_l : l \in \mathbb{N}\}$ which can be counted.

Axiom 2.3: Well Ordering Property of \mathbb{N}

Every non empty subset of \mathbb{N} has a least element.

Axiom 2.4: Weak Induction

For all subsets $S \subseteq \mathbb{N}$, $((1 \in S) \& ((\forall k \in \mathbb{N})(k \in S \implies k + 1 \in S))) \iff S = \mathbb{N}$

Weak Induction's Negation:(One direction)

There exists subset $S_0 \subseteq \mathbb{N}$, $((1 \in S_0) \& ((\forall k \in \mathbb{N})(k \in S_0 \implies k + 1 \in S_0)))$ but $S_0 \neq \mathbb{N}$

Axiom 2.5: Strong Induction

For all subsets $S \subseteq \mathbb{N}$, $((1 \in S) \& ((\forall k \in \mathbb{N})(1, 2, \dots, k \in S' \implies k + 1 \in S'))) \iff S = \mathbb{N}$

Strong Induction's Negation:(One direction)

There exists subset $S' \subseteq \mathbb{N}$, $((1 \in S') \& ((\forall k \in \mathbb{N})(1, 2, \dots, k \in S' \implies k + 1 \in S')))$ but $S' \neq \mathbb{N}$

Theorem 2.6

Weak Induction \iff Strong Induction.

Proof for Theorem.

\implies) Suppose Weak induction is true, but not strong induction. Take our set to be that S' in the negation of the Strong Induction Statement. $S' \neq \mathbb{N}$ implies that, either $1 \notin S'$ or $\exists k \in \mathbb{N}$ such that $k \in S'$ but $k + 1 \notin S'$. We know that $1 \in S'$, so it must be that $\exists k \in \mathbb{N}$ such that $k \in S'$ but $k + 1 \notin S'$. $\{1\} \in S' \implies \{1, 2\} \in S'$. Assume that for n , $\{1, 2, \dots, n\} \in S'$. This means that $\{1, 2, \dots, n + 1\} \in S'$. This means that for every $n \in \mathbb{N}$, $\{1, 2, \dots, n\} \in S' \implies n \in S'$. Contradiction.

\impliedby) Suppose Strong Induction is true, but not weak induction. Take the set S_0 from the negation of Weak Induction. $S_0 \neq \mathbb{N}$. This means, from strong induction, either $1 \notin S_0$ or $\exists k \in \mathbb{N}$ such that $1, 2, \dots, k \in S_0$ but $k + 1 \notin S_0$. $1 \in S_0$, hence, $2 \in S_0$ and $\{1, 2\} \in S_0$. assume that $\{1, 2, \dots, n\} \in S_0$. This means, $n \in S_0 \implies n + 1 \in S_0$, which means that $\forall k \in \mathbb{N}, \{1, 2, \dots, k\} \in S_0 \implies k + 1 \in S_0$. Therefore, S_0 is \mathbb{N} . ■

Theorem 2.7

Weak Induction \iff Strong Induction \iff Well ordering.

Proof for Theorem.

\implies) Suppose that, on the contrary, S_0 is a non empty subset of \mathbb{N} , with no least element. Does 1 exist in S_0 ? No, for that will be the least element. Likewise, then, 2 does not belong in S_0 . Assume that $\{1, 2, \dots, n\} \notin S_0$. Does $n + 1$ exist in S_0 ? No, for that will become the least element then. From Strong Induction, $\mathbb{N} - S_0 = \mathbb{N} \implies S_0 = \emptyset$. Contradiction.

\impliedby) Suppose $\exists S_0 \subseteq \mathbb{N}$ such that $1 \in S_0$ and $\forall k \in \mathbb{N}, k \in S_0 \implies k + 1 \in S_0$. Suppose on the contrary, S_0 is not \mathbb{N} . $\mathbb{N} - S_0$ is then, non-empty. From Well Ordering, there is a least element $q \in \mathbb{N} - S_0$. $\implies, q - 1 \in S_0$. But this would imply $q - 1 + 1 \in S_0$. Contradiction. $\mathbb{N} - S_0$ is empty. ■

Definition 2.8: (Finite Sets)

A set X is said to be finite, with n elements in it, if $\exists n \in \mathbb{N}$ such that there exists a bijection $f : \{1, 2, \dots, n\} \rightarrow X$. Set X is *infinite* if it is non-finite.

Theorem 2.9

If A and B are finite sets with m and n elements respectively, and $A \cap B = \emptyset$, then $A \cup B$ is finite, with $m + n$ elements.

Proof for Theorem.

$f : \mathbb{N}_m \rightarrow A$ and $g : \mathbb{N}_n \rightarrow B$.

Define $h : \mathbb{N}_{m+n} \rightarrow A \cup B$ given by:

$$h(i) = \begin{cases} f(i) & \text{if } i = 1, 2, \dots, m \\ g(i - m) & \text{if } i = m + 1, m + 2, \dots, m + n \end{cases}$$

If $i = 1, 2, \dots, m$, $h(i)$ covers all the elements in A through f . If $i = m + 1, \dots, m + n$, $h(i)$ covers all the elements in B through g .

Moreover, $h(i) \neq h(j)$; $i \in [1, m]$, $j \in [m + 1, m + n]$ since $A \cap B = \emptyset$ ■

Theorem 2.10

If C is infinite, and B is finite, then $C - B$ is infinite.

Proof for Theorem.

Suppose $C - B$ is finite. We have $B \cap (C - B) = \emptyset$ and $B \cup (C - B) = C \cup B$

$n(C \cup B) = n(B \cup (C - B)) = n(B) + n(C - B)$ This implies $C \cup B$ is finite. Contradiction. ■

Theorem 2.11

Theorem: Suppose T and S are sets such that $T \subseteq S$. Then:

- a) If S is finite, T is finite.
- b) If T is infinite, S is infinite.

Proof for Theorem.

Given that S is finite, there is a function $f : \mathbb{N}_m \rightarrow S$. Suppose that S has 1 element. Then either T is empty, or S itself, which means T is finite. Suppose that, upto n , it is true that, if S is finite with n elements, all its subsets are finite. Consider S with $n + 1$ elements.

$f : \mathbb{N}_{n+1} \rightarrow S$.

If $f(n + 1) \in T$, consider $T_1 := T - \{f(n + 1)\}$. We have $T_1 \subseteq S - \{f(n + 1)\}$, and since $S - \{f(n + 1)\}$ is a finite set with n elements, from induction hypothesis, T_1 is finite.

Moreover, since $T = T_1 \cup \{f(n + 1)\}$, T is also finite with one more element than T_1 .

If $f(n + 1) \notin T$, then $T \subseteq S - \{f(n + 1)\}$, we are done.

(b) is simply the contrapositive of (a). ■

Definition 2.12: (Countable Sets)

A set S is said to be *countable*, or *denumerable* if, either S is finite, or $\exists f : \mathbb{N} \rightarrow S$ which is a bijection. If S is *not countable*, S is said to be *uncountable*

Theorem 2.13

The set $\mathbb{N} \times \mathbb{N}$ is countable.

Proof for Theorem.

The number of points on diagonals $1, 2, \dots, l$ are: $\psi(k) = 1 + 2 + \dots + k = \frac{k(k+1)}{2}$

The point (m, n) occurs on the $(m + n - 1)$ th diagonal, on which the number $m + n$ is an invariant. The (m, n) point occurs m points down the diagonal. So, to characterise a point, it is enough to specify the diagonal it falls in, and its ordinate (the "rank" of that point on that diagonal). Count the elements till the $m + n - 2$ nd diagonal, then add m , and this would be the position of the point (m, n) .

Define $r : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ given by $r(m, n) = \psi(m + n - 2) + m$. That this is a bijection is pretty clear because we are counting the position of the point (m, n) . For a given point (m, m) , there can only be one unique diagonal on which it exists, and on the diagonal, its rank is unique. Moreover, for every $q \in \mathbb{N}$, there corresponds an (m, n) such that $r(m, n) = q$, for, we simply count along each diagonal in the "zig-zag" manner until we reach that (m, n) for which the position is given by q . Therefore, r is a bijection. (There are other explicit bijections too) ■

Theorem 2.14

The following are equivalent:

1. S is countable
2. \exists a surjective function from $\mathbb{N} \rightarrow S$
3. \exists an injective function from $S \rightarrow \mathbb{N}$

Proof for Theorem.

(1 \implies 2) is obvious

(2 \implies 3) $f : \mathbb{N} \rightarrow S$, every element of S has atleast one preimage in \mathbb{N} . Define a function from $S \rightarrow \mathbb{N}$ by taking for each $s \in S$ the least such $n \in \mathbb{N}$ such that $f(n) = s$. This defines an injection.

(3 \implies 1) If there is an injection from $S \rightarrow \mathbb{N}$, then there is a bijection from $S \rightarrow$ a subset of \mathbb{N} , which implies S is countable. ■

Corollary 2.15

The set of Rational Numbers \mathbb{Q} is countable.

Proof for Corollary.

We know that a surjection from $\mathbb{N} \times \mathbb{N}$ to \mathbb{Q} exists (where $f(0, 0) = 0$, and $f(m, n) = \frac{m}{n}$). We know that $\mathbb{N} \times \mathbb{N}$ is bijective to \mathbb{N} . This means \mathbb{N} is surjective to \mathbb{Q} . We are Done. ■

Theorem 2.16

Every infinite subset of a countable set is countable.

Proof for Theorem.

Consider $N_s \subseteq \mathbb{N}$ which is infinite.

Define $g(1) = \text{least number in } N_s$

Having defined $g(n)$, define $g(n+1) = \text{least number in } N_s \text{ which is larger than } g(n)$.

That it is an injection is obvious, for $g(m) > g(n)$ if $m > n$.

Suppose it is not a surjection, i.e, $g(\mathbb{N}) \neq N_s \implies g(\mathbb{N}) \subset N_s \implies N_s - g(\mathbb{N}) \neq \emptyset$. Therefore, $N_s - g(\mathbb{N})$ has a least element, k . This means that $k-1$ is in $g(\mathbb{N})$. Therefore, there exists q in \mathbb{N} such that $g(q) = k-1$. But then, $g(q+1) = \text{least number in } N_s \text{ such that it is bigger than } g(q)$. This would, ofcourse be, k , which means $k = g(q+1)$, which puts k in $g(\mathbb{N})$. Contradiction. Hence, $g(\mathbb{N}) = N_s$, therefore, g is a bijection from $\mathbb{N} \rightarrow N_s$. Since every countable set is bijective to \mathbb{N} , and every infinite subset of a countable set is bijective to an infinite subset of \mathbb{N} , the theorem holds generally for countable sets. ■

Theorem 2.17

$\mathbb{N} \times \mathbb{N} \cdots \mathbb{N}$ is bijective to \mathbb{N}

Proof for Theorem.

$\mathbb{N} \times \mathbb{N}$ is bijective to \mathbb{N} obviously. Assume that $f : \mathbb{N} \rightarrow \mathbb{N} \cdots \mathbb{N}$ (n times) is bijective.

Consider $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \cdots \mathbb{N}$ ($n+1$ times) given by $g(m, n) = (f(m), n)$. Clearly, this is bijective. ■

2.1 Axiom of Choice**Axiom 2.18: Axiom of Choice (AC)**

For any collection of non empty sets $C = \{A_l : l \in L\}$, there exists a function f called the "counting function" which maps each set A_l to an element in A_l .

Formally: $f : C \rightarrow \bigcup_l A_l$ such that $\forall l \in L, f(A_l) \in A_l$

Theorem 2.19

Countable union of Countable sets is countable (This theorem is an example of a theorem that requires Axiom of Choice)

Proof for Theorem.

Suppose we are given a sequence of countable sets $\{S_n : n \in \mathbb{N}\}$. Since each S_j is countable, we have for each j , at least one bijective map $f_j : \mathbb{N} \rightarrow S_j$. Define $k : \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_j S_j$ given by: $k(m, n) = f_m(n)$. Suppose $x \in \bigcup_j S_j$, i.e. $x \in S_j$ for some j . This means that, $f(n) = x$ for some n . Therefore, $k(j, n) = x$. Hence, k is surjective. From theorem 2.14, we are done.

(Remark: Keep in mind, for each S_j , there are a myriad of functions $(f_j)_k : \mathbb{N} \rightarrow S_j$. For each S_j , which is countably infinite, we have to choose one of the many functions that biject \mathbb{N} to S_j . So we have a countable collection of sets $C = \{E_j : j \in \mathbb{N}\}$, where E_j denotes the set of all functions that biject \mathbb{N} into S_j . So for every element in C , we need to choose one element in each element of C . This is where the Axiom of Choice comes into play.) ■

Theorem 2.20

If $f : A \rightarrow B$ is a surjection, then B is bijective to a subset of A

Proof for Theorem.

We are told that $f(A) = B$, i.e. for every $b \in B$, $\exists x_b$ (many such x_b -s are possible) such that $f(x_b) = b$. Define a function $g : B \rightarrow A$ as: $g(b) =$ one of those x_b such that $f(x_b) = b$. g is bijective to the set of all the chosen x_b for every b ■

Remark.

We make use of the Axiom of Choice in the previous theorem when we choose an x_b from a set of all possible x_b -s for b . Let A_b be the set of all possible x_b -s. Then the collection $\{A_b : b \in B\}$ is a collection of non-empty sets. And we are to select "one" element from each A_b . This requires AC.

Definition 2.21: (Power Set of a set)

Power set of A , denoted by $P(A)$ is the set of all subsets of A .

Theorem 2.22: Cantor's Theorem

For any set A , there *does not exist* any surjection from A onto $P(A)$

Proof for Theorem.

Suppose, on the contrary, a surjection $\psi : A \rightarrow P(A)$ exists. For every subset A_s of A , there exists an element x of A such that $\psi(x) = A_s$. Either this x exists in A_s , or it doesn't. Consider $D := \{x \in A : x \notin \psi(x)\}$. D is a subset of A , so there must be some element $y \in A$ such that $\psi(y) = D$. Does y belong in D ? If so, $y \notin \psi(y) = D$. Which means $y \notin D$. If, though, $y \notin D$, that implies $y \in \psi(y) \implies y \in D$. Contradictions left and right. ■

3 Elementary Results regarding Integers

Definition 3.1

1. (Divides) We say $a \in \mathbb{Z} \setminus \{0\}$ divides $b \in \mathbb{Z}$ if there exists an integer δ such that $a\delta = b$. We denote it by $a|b$.
2. (GCD) We call a number d the "Greatest Common Divisor" of two integers a and b if $d|a$ and $d|b$, and d is the largest such number that divides both a and b (that the largest such number exists is clear, since divisors are finite).
3. (LCM) We call a number l the "Least Common Multiple" of two integers a and b if $a|l$ and $b|l$ and l is the smallest such integer.

Definition 3.2: Prime Number

A number p in \mathbb{N} is *prime* if it has only itself and 1 as divisors. Non-primes are called composite.

Theorem 3.3

1. The GCD d of $a, b \in \mathbb{Z}$ is unique, and has the property that, if any other integer q is a divisor of a and b , then q divides d .
2. The LCM l of $a, b \in \mathbb{Z}$ is unique, and has the property that, if another integer p is a multiple of a and b , then l divides p .
3. If LCM of a, b and GCD of a, b are l and d respectively, then $dl = ab$.

Proof for Theorem.

(1) This will be proved below with the division algorithm. For now note that, if every divisor divides d , then d is the GCD.

(2) This is also proved using the division algorithm. For now note that if every multiple of a, b is divisible by a multiple l , then it is the least common multiple.

(3) Suppose d is the unique GCD of a, b and l is the unique LCM of a, b .

Note that $d|ab$ which means $dc_0 = ab$ for some c_0 . $c_0 = a(\frac{b}{d})$ and $c_0 = b(\frac{a}{d})$. This means c_0 is a multiple of a and b which means $l|c_0$. We have $lq_0 = c_0$ for some q_0 . This means $dlq_0 = dc_0 = ab$. $(dq_0)(\frac{l}{a}) = b$ and $(dq_0)(\frac{l}{b}) = a$ which makes dq_0 a divisor. This would necessarily mean $q_0 = 1$, whence, we are done. ■

3.1 Euclid's Divison Algorithm

Lemma 3.4: The Lemma

Given integers $a, b \in \mathbb{Z}$ with $b \neq 0$, we get a unique $q \in \mathbb{Z}$ and $r \in \mathbb{Z}$ such that:

$$a = bq + r$$

with $0 \leq r < |b|$.

Proof for Lemma

We Prove for the case that $a, b \in \mathbb{N}$. Assume that for $a_0 \in \mathbb{N}$, the divison lemma works, i.e, $\exists q_0$ and r_0 so that

$$a_0 = bq_0 + r_0$$

with $0 \leq r < |b|$. Look at $a_0 + 1 = bq_0 + r_0 + 1$. We have that, either $r_0 + 1 = b$, or $r_0 + 1 < b$. If its the former, then we see that $a_0 + 1 = bq_0 + b = b(q_0 + 1) + 0$ whence we see that the new quotient is $q_0 + 1$ and the new remainder is 0. Hence, by induction, the lemma is proved.

For the cases where $a < 0$ or $b < 0$, we can simply multiply by -1 to get the result. ■

The Algorithm:

We start with $a, b \in \mathbb{Z} \setminus \{0\}$, and without loss of generality, we assume that $a \geq b$. We then have:

$$a = bq_0 + r_0 \text{ with } 0 \leq r_0 < |b|$$

$$b = r_0q_1 + r_1 \text{ with } 0 \leq r_1 < r_0 < |b|$$

$$r_0 = r_1q_2 + r_2 \text{ with } 0 \leq r_2 < r_1 < r_0 < |b|$$

$$r_1 = r_2q_3 + r_3 \text{ with } 0 \leq r_3 < r_2 < r_1 < r_0 < |b|$$

$$\vdots$$

$$r_{n_0-1} = r_{n_0}q_{n_0+1} + r_{n_0+1} \text{ with } 0 \leq r_{n_0+1} < r_{n_0} \cdots b$$

$$r_{n_0} = r_{n_0+1}q_{n_0+2} + r_{n_0+2} \text{ with } 0 \leq r_{n_0+2} < r_{n_0+1} \cdots < b$$

Since we cannot have a sequence of strictly decreasing positive integers, we note that at some point, $r_n = 0 (= r_{n_0+2}$ for our sake).

We would then have (in the last step),

$$r_{n_0} = r_{n_0+1}q_{n_0+2} + 0$$

and back substituting,

$$r_{n_0-1} = r_{n_0+1}(q_{n_0+2}q_{n_0+1} + 1)$$

$$r_{n_0-2} = (r_{n_0+1}(q_{n_0+2}q_{n_0+1} + 1))q_{n_0} + r_{n_0+1}q_{n_0+2}$$

And finally we would end up with

$$a = r_{n_0+1}(\text{something}_1)$$

and

$$b = r_{n_0+1}(\text{something}_2)$$

with the added fact that r_{n_0+1} divides every remainder r_n in the division algorithm performed with a and b .

Proof that any divisor divides the GCD: Suppose that z is a divisor of a and b . From $a = bq_0 + r_0$, and

$$\frac{a}{z} = \frac{b}{z}q_0 + \frac{r_0}{z}$$

we see that z divides r_0 . From $b = r_0q_1 + r_1$ and

$$\frac{b}{z} = \frac{r_0}{z}q_1 + \frac{r_1}{z}$$

we see that z divides r_1 too. Suppose z divides all remainders till r_{n_0} . $r_{n_0-1} = r_{n_0}q_{n_0+1} + r_{n_0+1}$ gives us $\frac{r_{n_0-1}}{z} = \frac{r_{n_0}}{z}q_{n_0+1} + \frac{r_{n_0+1}}{z}$ whence we see that r_{n_0+1} is divisible by z . Therefore, r_{n_0+1} is the GCD.

Proof that any multiple is divisible by LCM: Suppose that l and m are multiples of a, b and l is the least such multiple. Then $l \leq m$ with equality case being trivial. Suppose $l < m$. From Euclid's division lemma, we have $m = lq_0 + r_0$ with $0 < r_0 < l < m$. Both a and b divide m and l , which means

$$\frac{m}{a} = \frac{l}{a}q_0 + \frac{r_0}{a}$$

and

$$\frac{m}{b} = \frac{l}{b}q_0 + \frac{r_0}{b}$$

Which makes r_0 a multiple of a and b , which is absurd.

Theorem 3.5: Bezout's Theorem

If $a, b \in \mathbb{Z} \setminus \{0\}$, then there exists $x, y \in \mathbb{Z}$ so that $\gcd(a, b) = xa + yb$

Proof for Theorem.

From the first equation we see

$$a = bq_0 + r_0 \implies r_0 = a - bq_0$$

putting r_0 in the 2nd equation we see:

$$b = (a - bq_0)q_1 + r_1 \implies r_1 = b - (a - bq_0)q_1$$

Putting r_1 and r_0 into the 3rd equation, we get, likewise, r_2 in terms of a and b (a linear combination of a and b) As such, we can keep doing this to express r_{n_0+1} as a linear combination of a and b , like, $xa + yb$. ■

Theorem 3.6: Fundamental Theorem of Arithmetic

Given an integer $a > 1$, we can decompose a as a product of primes uniquely (upto ordering)

Proof for Theorem.

If $a = 2$, obviously we can. Suppose we can decompose every positive integer q $1 < q < n_0$ as a product of primes. Consider $n_0 + 1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} + 1$. Either $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} + 1$ is a prime or is composite. If it is prime, we are done. If it is a composite number, then $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} + 1 = xy$ where x and y are numbers smaller than $n_0 + 1$. But since x and y can be expressed as a product of primes, we see that $n_0 + 1$ can also be represented as a product of primes.

Suppose $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, but also $a = p_1^{\alpha'_1} p_2^{\alpha'_2} \cdots p_m^{\alpha'_m}$.

We have ($m \geq n$):

$$1 = p_1^{\alpha_1 - \alpha'_1} p_2^{\alpha_2 - \alpha'_2} \cdots p_n^{\alpha_n - \alpha'_n} p_{n+1}^{-\alpha'_{n+1}} \cdots p_m^{-\alpha'_m}$$

$$1 = p_1^{\alpha'_1 - \alpha_1} p_2^{\alpha'_2 - \alpha_2} \cdots p_n^{\alpha'_n - \alpha_n} p_{n+1}^{\alpha'_{n+1}} \cdots p_m^{\alpha'_m}$$

Obviously, not all $\alpha_j - \alpha'_j$ are positive, in the same way they all aren't negative. Suppose some of these powers are positive, while some negative. Consider some $p_j^{\delta_j}$ for which the power is negative, which moves it to the denominator. We have:

$$1 = \frac{\text{some primes raised to negative} \cdot \text{some primes raised to positive}}{p_j^{\delta_j}}$$

If we multiply the negative powers out to both sides, we get to a stage where we see that p_j divides a product of primes (raised to positive powers). But, p_j is different from all the primes, and hence does not divide any of them individually, which means, from the lemma(s) below, that p_j actually does not divide the whole product. A contradiction. Hence, the only case remaining is that of product of primes raised to 0 powers, which makes it unique.

Requisite Lemmas:

Lemma: Suppose m is such that $\gcd(a, m) = 1$, and $|ab|$. Then, we have that $m|b$.

To see this, we use Bezout's Theorem: there exists x and y integers such that $ax + my = 1$.

This means $abx + mby = b \implies \frac{ab}{m}x + by = \frac{b}{m}$ whence it becomes clear.

Lemma: If a prime p divides z^n for some integer z and some natural n , then p divides z .

To see this, Suppose that p does not divide z . Which means, from the lemma above, that p divides z^{n-1} , which likewise means it divides z^{n-2} and finally reaching to a contradiction that it finally divides z . ■

If we arrange the primes p_1, p_2, \dots, p_n in ascending order, we find that this representation becomes the only one (no order permutations).

AN ALTERNATE LOOK AT GCD AND LCM:

Suppose that $a = p_1^{q_1} p_2^{q_2} \dots p_n^{q_n}$ and $b = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n} \dots p_m^{r_m}$. Since all numbers whatsoever are product of primes, we have that every divisor is of the form (for a number $z = p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}$) $p_1^{j_1} p_2^{j_2} \dots p_k^{j_k}$ with $j_1 \leq l_1, j_2 \leq l_2 \dots j_k \leq l_k$. The number of divisors for a given $z = p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}$ would therefore be: $\binom{l_1+1}{1} \binom{l_2+1}{1} \dots \binom{l_k+1}{1}$.

We note that, given $a = p_1^{x_1} p_2^{x_2} \dots p_m^{x_m}$ and $b = p_1^{y_1} p_2^{y_2} \dots p_m^{y_m}$ with $x_j, y_j \geq 0$,

$$\gcd(a, b) = p_1^{\min(x_1, y_1)} p_2^{\min(x_2, y_2)} \dots p_m^{\min(x_m, y_m)}$$

and

$$\text{lcm}(a, b) = p_1^{\max(x_1, y_1)} p_2^{\max(x_2, y_2)} \dots p_m^{\max(x_m, y_m)}$$

Euler Totient Function (ϕ):

The Euler Totient function ϕ is defined as follows:

$$\phi(n) = \text{number of positive integers } z \text{ less than or equal to } n \text{ such that } \gcd(n, z) = 1$$

For primes p , every number smaller than p is coprime to p , hence $\phi(p) = p - 1$. More generally we have for primes p and natural q , $\phi(p^q) = p^q - p^{q-1}$.

Proof. Consider $\phi(p^k) = ||\text{all such numbers } x \text{ such that } x \leq p^k \text{ and } \gcd(x, p^k) = 1||$. Note that apart from the multiples of p, p^2, p^3, \dots , all else are coprimes with p^k . The numbers that are not coprimes with p^k can be counted as: $1p, 2p, \dots, p(p), \dots, (p^2)p \dots p^{k-1}p$. Hence, the number of multiples of p in this list are p^{k-1} . Total number of "numbers" smaller or equal to p^k are obviously, p^k . From here, the answer is obvious. □

Just accept the following fact

Fact 3.7

If $(m, n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$

3.2 Modular Arithmetic:

Given $n \in \mathbb{N}$ and the set of all integers \mathbb{Z} , we define the following equivalence class:

$$aRb \iff n|(b-a)$$

We have aRa obviously. If aRb , obviously bRa . If aRb and bRc , then $n|(b-a)$ and $n|(c-b)$ which means $n|(b-a) + (c-b) \implies n|c-a$. Hence, R is an equivalence relation. Consider arbitrary $a \in \mathbb{Z}$. A number x is in the equivalence class of a under the relation defined above if and only if $n|(x-a)$ which means $\exists z \in \mathbb{Z}$ so that $nz = x-a \implies x = a + nz$. A number x is in the equivalence class of a under R (also called as *congruence class of $a \bmod n$*) if and only if x is a plus an integral multiple of n .

It is also fruitful to view this from the following perspective:

Conjecture: zRa (i.e., $n|z-a$) if and only if z , when divided by n , gives the same remainder as a when divided by n .

Proof. (\implies) Consider $n|z-a$ which means $\exists p$ so that $z = a + pn$. If $a < n$, we have the remainder when a is divided by n , as a itself, whence we see that the remainder when z is divided by x is a as well. If $a > n$, then $a = qn + r_0$ with $0 \leq r_0 < n$. We then have $z = pn + a = pn + qn + r_0 = z_0n + r_0$ where $r_0 < n$. Therefore, same remainder.

(\impliedby) Suppose z and a give the same remainder when divided by n . This means $z = q_zn + r$ and $a = q_an + r$ which means $z - a = (q_z - q_a)n$ which means $n|z - a$. \square

Notation: We say zRb in the above context if either $n|z-b$ or z and b share the same remainders when divided by n . The equivalence class of a in this context is sometimes also called as *congruence class of $a \bmod n$* . A symbolic way to say zRa in this context is

$$z \equiv a \bmod(n)$$

We also denote the congruence class (or residue class) of an integer a by \bar{a} .

Theorem 3.8

If z is a given integer, and n is a given natural number, then the congruence class that z falls in would be one of the congruence classes formed by the $n-1$ numbers before n . Therefore, the entire partition created by R can be listed out as the congruence classes of the $n-1$ numbers before n

Proof for Theorem.

Suppose $z < n$. then obviously there is an integer q less than n (which is z itself) so that $q \equiv z \bmod(n)$. If $z > n$, then $z = qn + r$ where $0 \leq r < n$, which also means that, when r is divided by n , the remainder is r , just like z . This means $z \equiv r \bmod(n)$. From here it is clear that every number in \mathbb{Z} would be in the equivalence class of some integer less than n . It is also obvious that for any two integers less than n , they each form unique residue classes. \blacksquare

From the previous lemma, it is clear to see that all the residue classes, or "partitions" of \mathbb{Z} under the congruence mod (n) relation, can be succinctly listed out as

$$\overline{0}, \overline{1}, \overline{2} \dots \overline{n-1}$$

Definition 3.9: $\mathbb{Z}/n\mathbb{Z}$

The set of all equivalence classes (or residue classes) under the relation defined by n on \mathbb{Z} is denoted by

$$\mathbb{Z}/n\mathbb{Z}$$

which is basically $\overline{0}, \overline{1}, \overline{2} \dots \overline{n-1}$

Theorem 3.10

If $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n}$, then $a_1 + a_2 \in \overline{b_1 + b_2}$ and $a_1 a_2 \in \overline{b_1 b_2}$

Proof for Theorem.

(1) We have $a_1 = b_1 + kn$ and $a_2 = b_2 + ln$ which gives $a_1 + a_2 = (k+l)n + b_1 + b_2$ where we see that $a_1 + a_2$ is integer multiple of $n + b_1 + b_2$. Therefore, $a_1 + a_2 \in \overline{b_1 + b_2}$.

Consider $a_1 a_2 = (kn + b_1)(ln + b_2) = (kln^2 + b_2kn + b_1ln) + b_1b_2$. Obvious from here. ■

Definition 3.11: Modular Arithmetic

Treating the residue classes that form $\mathbb{Z}/n\mathbb{Z}$ as elements with which arithmetic can be done, we define addition and multiplication as follows:

$$\overline{a} + \overline{b} = \overline{a + b}$$

i.e, the sum of two residue classes is the residue class of the sum of an element from the class of a and the class of b . (This sum is well defined from the previous theorem.)

$$\overline{a} \overline{b} = \overline{ab}$$

i.e, the product of residue class of a and b is the residue class of the product of an element in the class of a and an element from the class of b . (Yet again, well defined)

Definition 3.12: $(\mathbb{Z}/n\mathbb{Z})^*$

$$(\mathbb{Z}/n\mathbb{Z})^* := \{\overline{a} \in \mathbb{Z}/n\mathbb{Z} : \exists \overline{c} \in \mathbb{Z}/n\mathbb{Z} \text{ such that } \overline{c} \cdot \overline{a} = \overline{1}\}$$

Lemma 3.13

If $\gcd(a, n) = 1$, i.e, a and n are coprimes, then every element in the residue class of $a \bmod(n)$ is coprimes with n

Proof for Lemma

Consider $z = a + kn$ for some k to be "non coprimes with n ". i.e, $\gcd(z, n) = j \neq 1$. We have $j|a + kn$ and $j|n$. But this means directly that $j|a$ whence we see that $j \neq 1$ is a divisor of a and n . Absurd. ■

Lemma 3.14

If $a \in \bar{a} \in \mathbb{Z}/n\mathbb{Z}$, $a \leq n$ and if $\gcd(a, n) = j$, then for any $z \in \bar{a}$, $\gcd(z, n) = j$.

Proof for Lemma

$\gcd(a, n) = k$. Consider $\gcd(a + gn, n)$. We apply the division algorithm:

$$(a + gn) = g_0(n) + a$$

$$n = g_1a + r_1$$

$$\vdots$$

Notice that after step (1), the procedure is exactly the same as the procedure to find $\gcd(a, n)$. The last living remainder is the GCD, and from here we can clearly see that the gcd are the same. ■

Lemma 3.15

If $1 \leq a \leq n$, with $n \geq 2$, and $(a, n) = j \neq 1 (\geq 2)$, then there exists a number $1 \leq b < n$ so that $\bar{a} \cdot \bar{b} = \bar{0}$. A corollary of this is that there exists *no* $c \in \mathbb{Z}$ so that $\bar{a} \cdot \bar{c} = \bar{1}$

Proof for Lemma

We know $\gcd(a, n) = j \neq 1 (\geq 2)$. This means $j\alpha = a$, $j\mu = n$ where $\alpha < a$ and $\mu < n$. We can see that $a\mu = j\mu\alpha = n\alpha$. So the b in the lemma is the μ here. Therefore, $\bar{a} \cdot \bar{b} = \bar{0}$. Suppose $ac \equiv 1 \bmod(n)$. This means from the multiplicative property of $\bmod(n)$, we have $ac(b) \equiv b \bmod(n)$, but if you commute the multiplication, we see

$$c(ab) = \bar{c} \cdot \overline{ab} = \bar{c} \cdot \bar{0} \equiv 0 \bmod(n)$$

But this would imply $b \equiv 0 \bmod(n)$ which is not true. Hence, there can be no c so that $ac \equiv 1 \bmod(n)$. ■

Proposition 3.16

$\mathbb{Z}/n\mathbb{Z}^*$ is the same as $\{\bar{a} \in \mathbb{Z}/n\mathbb{Z} : (a, n) = 1\}$

Proof. Let

$$A = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} : \exists \bar{z} \in \mathbb{Z}/n\mathbb{Z} \text{ such that } \bar{z} \cdot \bar{a} = \bar{1}\}$$

and

$$B = \{\bar{b} \in \mathbb{Z}/n\mathbb{Z} : (b, n) = 1\}$$

Consider an arbitrary \bar{a} in B such that $\forall z \in \bar{a}$ we have $\gcd(z, n) = 1$. We know that $z = a + k_z n$. From Bezout Identity, we have x_z and y_z so that $x_z(a + k_z n) + y_z(n) = 1$ which gives us $x_z a = (-y_z - k_z x_z)(n) + 1$. We see That $x_z a \in \bar{1}$. Consider arbitrary $t \in \bar{x}_z$. We have $t = x_z + jn$ or $x_z = t - jn$. Plugging this back we have $x_z a = (t - jn)a = (-y_z - k_z(t - jn))n + 1$ whence we can easily see that $ta \in \bar{1}$. Therefore, $\bar{a} \cdot \bar{x} = \bar{1}$. Therefore, if \bar{b} is such that $(b, n) = 1$, then an inverse exists for it. Hence, $B \subseteq A$.

We showed that if x is in B , it must be in A . Consider x not in B . i.e, it is not co-primes with n . From the previous proposition, we see that there would exist no c so that $\bar{x} \cdot \bar{c} = \bar{1}$, i.e, no inverse element for any x not in B . Inexistence in B therefore implies inexistence in A , which gives us $A \subseteq B$. We can conclude that $A = B$ or in pithy words:

"The set of all congruence classes b so that $\gcd(b, n) = 1$ is the same as the set of all congruence classes b so that there exists another congruence class c so that $\bar{b} \cdot \bar{c} = \bar{1}$ "

Symbolically:

$$\{\bar{b} \in \mathbb{Z}/n\mathbb{Z} : \gcd(b, n) = 1\} = \{\bar{b} \in \mathbb{Z}/n\mathbb{Z} : \exists \bar{c} \in \mathbb{Z}/n\mathbb{Z} : \bar{b} \cdot \bar{c} = \bar{1}\} = (\mathbb{Z}/n\mathbb{Z})^*$$

□