CHAPTER 1

ASSIGNMENT 1

Lemma 0.1

If α is the supremum of a set S, then $-\alpha$ is the infimum of the set $-S$ defined as

$$-S := \{-x : x \in S\}$$

Proof for Lemma

If α is the supremum, then $\alpha \geq x, \forall x \in S$ and $\alpha \leq M, \forall M$ such that $M \geq x, \forall x \in S$. This means that $-\alpha \leq -x, \forall x \in S$ and $-\alpha \geq -M, \forall -M$ such that $-M \leq -x, \forall -x \in S$ If we re-notate the whole thing we have:

$-\alpha \leq z, \forall z \in -S$ and $-\alpha \geq L, \forall L$ such that $L \leq z \forall z \in -S$. This is precisely the definition for infimum of $-S$, whence we see, we are done.

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Recall the definitions:

$$LimSup(x_n) := inf(U_n : U_n := sup(\{x_n, x_{n+1} \cdots\}))$$

$$LimInf(x_n) := sup(L_n : L_n := inf(\{x_n, x_{n+1} \cdots\}))$$

1 Problem 1

Let x be $limsup(-x_n)$.

$$x = inf(U_n : U_n = sup\{-x_n, -x_{n+1} \cdots\})$$

$$\implies x = inf(U_n : U_n = -inf\{x_n, x_{n+1}, \cdots\})$$

$$\implies x = inf(-inf\{x_n, x_{n+1} \cdots\}, -inf\{x_{n+1}, x_{n+2} \cdots\}, \cdots)$$

$$\implies x = -sup(inf\{x_n, x_{n+1} \cdots\}, inf\{x_{n+1}, x_{n+2} \cdots\}, \cdots)$$

Whence, we are done.

2 Problem 2

$$Liminf(x_n) \leq (\text{every subsequential limit of} x_n) \leq Limsup(x_n)$$

Where $Liminf(x_n)$ is the infimum, and $Limsup(x_n)$ the supremum of the set of all subsequential limits of x_n

$$Liminf(y_n) \leq (\text{every subsequential limit of} y_n) \leq Limsup(y_n)$$

Where $Liminf(y_n)$ is the infimum, and $Limsup(y_n)$ the supremum of the set of all subsequential limits of y_n

Adding these two inequalities we get:

$Liminf(x_n)+Liminf(y_n) \leq (\text{every subsequential limit of } x_n + \text{every subsequential limit of } y_n)$ $\leq Limsup(x_n) + Limsup(y_n)$

Since the set of all subsequential limits of $x_n + y_n$ falls as a subset of the sum of the set of all subsequential limits of x_n and y_n respectively, we have:

$Liminf(x_n) + Liminf(y_n) \leq (\text{every subsequential limit of } x_n + y_n) \leq Limsup(x_n) + Limsup(y_n)$ We see now that $Liminf(x_n) + Liminf(y_n)$ is a lowerbound for the set of all subsequential limits of $x_n + y_n$ which gives us

$$Liminf(x_n) + Liminf(y_n) \leq liminf(x_n + y_n)$$

Similarly we see that $Limsup(x_n) + Limsup(y_n)$ is an upperbound for the set of all subsequential limits of $x_n + y_n$ which gives us

$$Limsup(x_n + y_n) \leq Limsup(x_n) + Limsup(y_n)$$

Equality of (I) holds when the smallest subsequential limit of $x_n + y_n$ is the sum of the smallest possible subsequential limits of x_n and y_n respectively.

Similarly, (II) equailty holds when the largest subsequential limit of $x_n + y_n$ is equal to the sum of the smallest subsequential limits of x_n and y_n respectively.

3 Problem 3