CHAPTER 1

INTRODUCTION TO SEQUENCES AND SERIES OF REAL NUMBERS

Covers some of the elementary results regarding sequences and series, more of which will be explored after the section on metric spaces.

1 On Sequences (Introduction)

Definition 1.1: (Sequence of Real numbers)

$X := (x_n : n \in \mathbb{N})$ is a function $x : \mathbb{N} \to \mathbb{R}$. The mapping from N allows us natural ordering.

Definition 1.2: (Limit of a sequence)

A sequence (x_n) in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$ if:

$$(\forall \varepsilon > 0)(\exists\ n_0 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_0)(|x_n - x| < \varepsilon)$$

Whose negation reads: A sequence is *not convergent* to $x \in \mathbb{R}$ if:

$$(\exists \varepsilon_0 > 0)(\forall\ k \in \mathbb{N})(\exists n_k \in \mathbb{N}, n_k \geq k)(|x_n - x| \geq \varepsilon_0)$$

Theorem 1.3: Uniqueness of Limits

If $x_n \to x$, then its limit is unique.

Proof for Theorem.

Suppose two limits exist, x and x'.

$$(\forall \varepsilon > 0)(\exists\ n_1 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_1)(|x_n - x| < \frac{\varepsilon}{2})$$

$$(\forall \varepsilon > 0)(\exists\, n_2 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_2)(|x_n - x'| < \frac{\varepsilon}{2})$$

Choosing $j = max n_1, n_2$ we have:

$$(\forall \varepsilon > 0)(\exists\, j \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_0)(|x_n - x| < \frac{\varepsilon}{2} \& |x_n - x'| < \frac{\varepsilon}{2}) \implies$$

$$(\forall \varepsilon > 0)(\exists\, j \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_0)(|x - x'| < \varepsilon)$$

From "The Lemma", $x = x'$

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Theorem 1.4

Convergence $\implies$ Boundedness

Proof for Theorem.

$$(\forall \varepsilon > 0)(\exists\, n_1 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_1)(-\varepsilon < x_n - x < \varepsilon)$$

Fix $\varepsilon = 1$, and let the corresponding n we get, be n_1. We see that for $n \geq n_1$, the set is bounded. For the numbers x_1 to $x_{n_1 - 1}$, by virtue of being a finite set, it is readily bounded. Hence, the whole sequence is bounded. ■

Fact 1.5

Elementary Results:

1. If $\{x_n\}$ such that $x_n \geq 0 \forall n \in \mathbb{N}$, then, if limit exists, limit(x_n)≥ 0.
2. Given $\{x_n\}$ and $\{y_n\}$ such that $y_n > x_n \forall n \in \mathbb{N}$, then $lim(y_n) \geq lim(x_n)$
3. If $x_n \to x$ and $a \leq x_n \leq b$, then $a \leq x \leq b$.

Theorem 1.6: Squeeze Play

Given sequences x_n, y_n, z_n such that $\forall n \geq n_l, y_n \leq x_n \leq z_n$, and $z_n \to a, y_n \to a$, then $x_n \to a$.

Proof for Theorem.

$$(\forall \varepsilon > 0)(\exists\, n_y \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_y)(a - \varepsilon < y_n \leq x_n)$$

$$(\forall \varepsilon > 0)(\exists\, n_z \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_z)(x_n \geq a + \varepsilon)$$

Combining the two and setting $j = max\{n_y, n_z, n_l\}$ we get

$$(\forall \varepsilon > 0)(\exists\, j \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq j)(a - \varepsilon < x_n < a + \varepsilon)$$

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Definition 1.7: (Unbounded sequence)

A sequence x_n is unbounded if it is neither bounded below, nor bounded above. It is not bounded above if:

$$\forall \xi \in \mathbb{R}, \exists n_k, x_{n_k} > \xi$$

It is not bounded below if:

$$\forall \xi \in \mathbb{R}, \exists n_k, x_{n_k} < \xi$$

Definition 1.8: (Divergence to infinity)

A sequence is said to diverge to +infinity if $\forall \xi \in \mathbb{R}, \exists n_0$ such that $x_n > \xi \forall n \geq n_0$. It is divergent to -infinity if $\forall \xi \in \mathbb{R}, \exists n_0$ such that $x_n < \xi \forall n \geq n_0$.

Theorem 1.9: Multiplication and division of sequences

Multiplication of sequences $x_n \to x$ and $y_n \to y$ gives a sequence $x_n y_n$ that converges to xy.
If $x_n \to x$ and $y_n \to y$ with $y_n \neq 0 \forall n \in \mathbb{N}$ and $y \neq 0$, we have: $\frac{x_n}{y_n} \to \frac{x}{y}$

Proof for Theorem.

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0$ we have $|x_n - x| < \varepsilon$. Similarly, $\forall \varepsilon > 0$, $\exists n_1 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_1$ we have $|y_n - y| < \varepsilon$. Consider $|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy| \leq |y_n||(x_n - x)| + |(x)||(y_n - y)|$. For $\varepsilon = 1$ we have n_{y1} such that $\forall n \geq n_{y1}$, $y - 1 < y_n < y + 1$. And obviously, for $n < n_{y1}$, there exists maxima M. Since y_n is bounded, forall $n \geq n_{y1}$, we have $|y_n||x_n - x| + |x||y_n - y| \leq (Max)|x_n - x| + |x||y_n - y|$. This means that, $\forall \varepsilon > 0, \exists n_0$ such that $|x_n y_n - xy| \leq |y_n||x_n - x| + |x||y_n - y| \leq (Max)|x_n - x| + |x||y_n - y| \leq Max(\varepsilon) + |x|(\varepsilon)$. Hence, we are done.

$$(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_\varepsilon)(|x_n - x| < \varepsilon)$$

$$(\forall \varepsilon > 0)(\exists n'_\varepsilon \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n'_\varepsilon)(|y_n - y| < \varepsilon)$$

Choose $\varepsilon = 1$. After some $k_1 \in \mathbb{N}$, we have $y - 1 < y_n$, which implies $\{|y_n|\}$ has a lower bound. Call this L. Cosider the difference $|\frac{x_n}{y_n} - \frac{x}{y}| = |\frac{x_n y - x y_n}{y y_n}| \leq \frac{|x||y_n - y| + |y||x_n - x|}{|y_n||y|} \implies$

$$\forall n \geq k_1, |\frac{x_n}{y_n} - \frac{x}{y}| \leq \frac{|x||y_n - y| + |y||x_n - x|}{|L||y|}$$

$$\forall \varepsilon, \exists n_i \geq n_0, n_1, k_1, \forall n \in \mathbb{N}, n \geq n_i, |\frac{x_n}{y_n} - \frac{x}{y}| < \frac{1}{|L||y|}|x|\varepsilon + |y|\varepsilon$$

Whence, we are done. ■

Theorem 1.10: Some Results

1. if $a > 1$, then $a^n \to \infty$
2. if $a > 0$, then $a^{\frac{1}{n}} \to 1$

Proof for Theorem.

$a = 1 + \delta \implies a^n = 1 + n\delta + \frac{n(n-1)}{2}\delta^2 \cdots > 1 + n\delta$. This implies a^n diverges to infinity. Given $a > 0$, if $a > 1$, $a^{1/n} > 1$. We have $a^{1/n} = 1 + \delta_n$. $a = (1+\delta_n)^n = 1 + n\delta_n + \frac{n(n-1)}{2}\delta_n^2 \cdots \implies \frac{a}{n} \leq \delta_n$. This means that δ_n converges to 0, which means that $a^{1/n}$ converges to 1.
Suppose $0 < a < 1$, then $\frac{1}{a} > 1$. Therefore $(\frac{1}{a})^{\frac{1}{n}} = (a^{\frac{1}{n}})^{-1}$ converges to 1. This implies $a^{\frac{1}{n}}$ converges to 1 aswell (from the previous theorem on division) ■

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Theorem 1.11: Slick Theorem

Let $\{x_n\}$ be a given sequence and $\{a_n\}; a_n \geq 0$ be a sequence converging to 0. Suppose, also, that for some $C > 0$, we have

$$|x_n - x| \leq Ca_n \forall n \geq n_0$$

, then the sequence x_n converges to x.

Proof for Theorem.

$\forall \varepsilon \exists n_1 : \forall n \in \mathbb{N}, n \geq n_1, Ca_n < \varepsilon \implies |x_n - x| < \varepsilon \forall n \geq max\{n_0, n_1\}$ ■

Definition 1.12: (Monotone Sequence)

A sequence is said to be *monotone increasing* if $\forall n \in \mathbb{N}, x_n \geq x_{n-1}$. It would be "ultimately" monotone increasing if $\exists n_0 \in \mathbb{N} \forall n \in \mathbb{N}, n \geq n_0, x_n \geq x_{n-1}$

Likewise, a sequence is said to be *monotone decreasing* if $\forall n \in \mathbb{N}, x_n \leq x_{n-1}$. It would be "ultimately" monotone decreasing if $\exists n_0 \in \mathbb{N} \forall n \in \mathbb{N}, n \geq n_0, x_n \leq x_{n-1}$

A sequence is "monotone" if its either monotone increasing or decreasing.

Theorem 1.13: Monotone Convergence Theorem

A monotone sequence is convergent $\iff$ it is bounded

Proof for Theorem.

$\implies$) Every convergent sequence is bounded.
$\impliedby$) We take the case of monotone increasing sequence that is bounded above. $\exists M \in \mathbb{R}$ such that $x_n \leq M \forall n \in \mathbb{N}$. Consider the set $\{x_n : n \in \mathbb{N}\}$, which is bounded and non empty. Let z be the supremum of this set. Consider an arbitrary $\varepsilon > 0$. $\exists x_{n_0}$ such that $z - \varepsilon \leq x_{n_0} \leq x_n \forall n \geq n_0$. This means: $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, n \in \mathbb{N}$ we have $z - \varepsilon < x_n < z + \varepsilon$. Hence, $x_n \to sup(\{x_n\})$ ■

Euler's Number

Theorem 1.14

Consider the sequence

$$e_n := (1 + \frac{1}{n})^n$$

This sequences is convergent, and $\lim(e_n)$:=e is called the Napier's Constant or Euler's Number.

Proof for Theorem.

$$\begin{aligned} e_n &= (1+\frac{1}{n})^n = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2}(\frac{1}{n^2}) + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}\cdots \\ &= 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \frac{1}{4!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\cdots \end{aligned}$$

And e_n has $n+1$ terms from the binomial theorem.
Consider

$$e_{n+1} = (1+\frac{1}{n+1})^{n+1} = 1 + 1 + \frac{(n+1)(n)}{2}\frac{1}{(n+1)^2} + \frac{(n+1)(n)(n-1)}{3!}\frac{1}{(n+1)^3}\cdots =$$

$$2 + \frac{1}{2!}(1-\frac{1}{n+1}) + \frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1}) + \frac{1}{4!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})(1-\frac{3}{n+1}) + \cdots$$

and e_{n+1} has $n+2$ terms from Binomial. Notice that every term in e_{n+1} is greater than (or equal to) every term of e_n, with there being more terms in e_{n+1}. Therefore, e_n is monotone increasing.

$$\begin{aligned} e_n &= 2 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \frac{1}{4!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\cdots \\ &\leq 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\cdots \end{aligned}$$

Since for $n \geq 3$ we have $2^n \leq n!$, this means $\frac{1}{2^n} \geq \frac{1}{n!}$ and hence,

$$e_n \leq 2 + \frac{1}{2} + \frac{1}{2^2}\cdots \leq 2 + \frac{1}{1-\frac{1}{2}} \leq 4$$

e_n is bounded, hence convergent. ■

Theorem 1.15: Three Beauties

1. (Ratio Test) Let $\{a_n\}$ such that $a_n > 0 \forall n \in \mathbb{N}$. Let $lim(\frac{a_{n+1}}{a_n}) = L$. If $L > 1$, then $lim(a_n) = \infty$. If $L < 1$, $lim(a_n) = 0$ (Test fails if $L = 1$, with the example of $a_n = n$)
2. (Average Convergence Theorem) If $\{a_n\} \to L$, then $\frac{a_1+a_2+\cdots+a_n}{n} \to L$
3. (Cauchy's 2nd) $a_n > 0$, then $lim(a_n)^{\frac{1}{n}} = lim(\frac{a_{n+1}}{a_n})$ provided either $\frac{a_{n+1}}{a_n}$ converges or properly diverges.

Proof for Theorem.

1) Let $\frac{a_{n+1}}{a_n}$ converge to L.

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N}, n \geq n_0, L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon$$

This means that:

$$L - \varepsilon < \frac{a_{n_0+1}}{a_{n_0}} < L + \varepsilon$$
$$L - \varepsilon < \frac{a_{n_0+2}}{a_{n_0+1}} < L + \varepsilon$$
$$\vdots$$
$$L - \varepsilon < \frac{a_m}{a_{m-1}} < L + \varepsilon$$

Multiplying throughout we have:

$$a_{n_0}(L-\varepsilon)^{m-n_0} < a_m < a_{n_0}(L+\varepsilon)^{m-n_0} \qquad ((\text{x}))$$

If $L < 1$ choose a number ε such that $L+\varepsilon < 1$. Therefore, there exists a corresponding n_k such that

$$a_{n_k}(L-\varepsilon)^{m-n_k} < a_m < a_{n_k}(L+\varepsilon)^{m-n_k} < a_{n_k}(z)^{m-n_k} : \forall m \geq n_k, n_0$$

where $z < 1$. Therefore, $a_{n_k}(z)^{m-n_k}$ converges to 0. From squeeze play $a_m \to 0$.
If $L > 1$, choose a number ε such that $L - \varepsilon > 1$. Therefore, there exists a corresponding n_l and a number v such that

$$a_{n_l}(v)^{m-n_l} < a_{n_l}(L-\varepsilon)^{m-n_l} < a_m < a_{n_l}(L+\varepsilon)^{m-n_l} : \forall m \geq n_k, n_0$$

where $v > 1$. Hence, we see that a_m is properly divergent.

2)$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0, L - \varepsilon < a_n < L + \varepsilon \implies$

$$L - \varepsilon < a_{n_0+1} < L + \varepsilon$$

$$L - \varepsilon < a_{n_0+2} < L + \varepsilon$$

$$\vdots$$

$$L - \varepsilon < a_n < L + \varepsilon$$

Adding all these we get:

$$(n - n_0)(L - \varepsilon) < a_{n_0+1} + a_{n_0+2} + \cdots + a_n < (n - n_0)(L + \varepsilon)$$

Consider the set $\{a_1, a_2, \ldots, a_{n_0}\}$, this is a finite set, hence, it has a maximum and a minimum M and m respectively, which means $\forall n \leq n_0, m \leq a_n \leq M$. Therefore, for every $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ and a maxima and minima m and M so that

$$(n_0)m + (n - n_0)(L - \varepsilon) < a_1 + a_2 + \cdots + a_n < (n - n_0)(L + \varepsilon) + (n_0)M \implies$$

$$\mathfrak{L}(\varepsilon, n) = \frac{n_0 m}{n} + \frac{(n - n_0)}{n}(L - \varepsilon) < \frac{a_1 + a_2 \cdots + a_n}{n} < \frac{n_0 M}{n} + \frac{(n - n_0)}{n}(L + \varepsilon) = \mathfrak{U}(\varepsilon, n)$$

It is clear that $\mathfrak{L}(\varepsilon, n)$ and $\mathfrak{U}(\varepsilon, n)$ converge to $L - \varepsilon$ and $L + \varepsilon$ respectively. Therefore:

$$(\forall \varepsilon')(\exists j_l \in \mathbb{N})(\forall n \in \mathbb{N} : n \geq j_l)(L - \varepsilon - \varepsilon' < \mathfrak{L}(\varepsilon, n) \leq \frac{a_1 + a_2 + \cdots_a + n}{n})$$

and

$$(\forall \varepsilon')(\exists j_u \in \mathbb{N})(\forall n \in \mathbb{N} : n \geq j_u)(\frac{a_1 + a_2 + \cdots_a + n}{n} \leq \mathfrak{U}(\varepsilon, n) < L + \varepsilon + \varepsilon')$$

Whence we see that $\forall \varepsilon \forall \varepsilon$ there is some $n_p \geq j_l, j_u, n_0$ such that $\forall n \in \mathbb{N} : n \geq n_p$,

$$L - \varepsilon - \varepsilon' < \frac{a_1 + a_2 + \cdots_a + n}{n} < L + \varepsilon + \varepsilon'$$

Therefore, $\frac{a_1 + a_2 \cdots + a_n}{n}$ converges to L

3)From equation (x), we have that $\forall \varepsilon, \exists n_0(\varepsilon)$ such that

$$a_{n_0}(L - \varepsilon)^{m - n_0} < a_m < a_{n_0}(L + \varepsilon)^{m - n_0} : \forall m \geq n_0(\varepsilon)$$

Either $(L - \varepsilon)$ is positive, whence it is possible to take $m - th$ root which gives:

$$a_{n_0}^{\frac{1}{m}}(L - \varepsilon)^{1 - \frac{n_0}{m}} < a_m^{\frac{1}{m}} < a_{n_0}^{\frac{1}{m}}(L + \varepsilon)^{1 - \frac{n_0}{m}}$$

Or it is negative, where it is pretty obvious that

$$a_{n_0}^{\frac{1}{m}}(L - \varepsilon) < a_m^{\frac{1}{m}} < a_{n_0}^{\frac{1}{m}}(L + \varepsilon)^{1 - \frac{n_0}{m}}$$

Hence, $\forall \varepsilon > 0$, $\exists n_0(\varepsilon) \in \mathbb{N}$ such that $\forall m \geq n_0(\varepsilon)$

$$\begin{cases} a_{n_0}^{\frac{1}{m}}(L - \varepsilon)^{1 - \frac{n_0}{m}} < a_m^{\frac{1}{m}} < a_{n_0}^{\frac{1}{m}}(L + \varepsilon)^{1 - \frac{n_0}{m}} & \text{if } L - \varepsilon > 0 \\ a_{n_0}^{\frac{1}{m}}(L - \varepsilon) < a_m^{\frac{1}{m}} < a_{n_0}^{\frac{1}{m}}(L + \varepsilon)^{1 - \frac{n_0}{m}} & \text{if } L - \varepsilon < 0 \end{cases}$$

Call $a_{n_0}^{\frac{1}{m}}(L+\varepsilon)^{1-\frac{n_0}{m}} = \mathfrak{U}(\varepsilon, m)$
Call $a_{n_0}^{\frac{1}{m}}(L-\varepsilon)^{1-\frac{n_0}{m}} = \mathfrak{L}(\varepsilon, m)$

It is clear to see that both $\mathfrak{U}(\varepsilon, m)$ and $\mathfrak{L}(\varepsilon, m)$ converge, and do so to $(L+\varepsilon)$ and $(L-\varepsilon)$ respectively. Therefore: $\forall \varepsilon' > 0\ \exists j \in \mathbb{N}$ such that

$$L - \varepsilon - \varepsilon' < \mathfrak{L}(\varepsilon, n) \forall n \geq j$$

and

$$\mathfrak{U}(\varepsilon, n) < L + \varepsilon + \varepsilon' \forall n \geq j \implies$$
$$L - \varepsilon - \varepsilon' < a_m^{\frac{1}{m}} < L + \varepsilon + \varepsilon' : \forall m \geq t = \max\{j, n_0\}$$

This means that $\forall \varepsilon \forall \varepsilon' > 0$, $\exists t \in \mathbb{N}$ such that $\forall m \in \mathbb{N}, m \geq t$ we have:

$$L - \varepsilon - \varepsilon' < a_m^{\frac{1}{m}} < L + \varepsilon + \varepsilon'$$

The same argument can be applied for $\varepsilon' s$ where $(L-\varepsilon)$ is negative. Hence, $a_m^{\frac{1}{m}} \to L$
Suppose that $\frac{a_{m+1}}{a_m} \to \infty$, then $\frac{a_m}{a_{m+1}} \to 0$. Therefore, applying the previous result to this, we have $(\frac{1}{a_m})^{\frac{1}{m}} \to 0 \implies (a_m)^{\frac{1}{m}} \to \infty$ (Proof left as an exercise to the reader) ■

Definition 1.16: (Subsequences)

Given a sequence $\{x_n\}$, we define the subsequence $\{x_{n_k}\}$ as the sequence within $\{x_n\}$ generated through the increasing sequence of natural numbers $n_1 < n_2 < n_3 < \ldots$ with $k \geq n_k$

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Theorem 1.17

Given $x_n \to x$, then all subsequences of x_n converge to x.

Proof for Theorem.

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N} : n \geq n_0)(|x_n - x| < \varepsilon) \implies$$
$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n_k \in \mathbb{N} : n_k \geq k \geq n_0)(|x_{n_k} - x| < \varepsilon)$$

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Theorem 1.18

Let $A \subseteq \mathbb{R}$ be an infinite subset that is bounded, and non empty, with supremum S. Then, $\exists \{x_n\}$ in A such that $x_n \to S$, with x_n being monotone increasing.

Proof for Theorem.

Choose $\delta_1 = 1$. There would exist some $x_1 \in A$ such that $S - 1 \leq x_1 < S$. Choose $\delta_2 = \frac{d(x_1,S)}{2}$ where $d(x_1, S)$ is the Euclidean distance from S to x_1. There would exist, again, some x_2 such that $S - \delta_2 = S - \frac{d(x_1,S)}{2} \leq x_2 < S$. It is easy to see here that $x_1 \leq x_2$. Having found x_n using $\delta_n = \frac{d(x_{n-1},S)}{n}$, now choose x_{n+1} using $\delta_{n+1} = \frac{d(x_n,S)}{n+1}$. Via this compression, we see that the sequence converges to S through squeeze play. Moreover, by construction this sequence is increasing. ■

Theorem 1.19: Equivalent statements pertaining to Divergence

1. $X_n \not\to x$
2. $\exists \varepsilon > 0$ such that $\forall k \in \mathbb{N}\ \exists n_k \geq k\ |x_{n_k} - x| \geq \varepsilon$
3. $\exists \varepsilon > 0$ and a subsequence x_{n_k} such that $\forall k \in \mathbb{N}\ |x_{n_k} - x| \geq \varepsilon$

Proof for Theorem.

(1) $\implies$ (2)) Comes directly from the negation of convergence, with tweaks to the notation.
(2) $\implies$ (3)) For $k = 1$, $\exists n_1$ such that $|x_{n_1} - x| \geq \varepsilon$, likewise, $fork = j$, $\exists n_j$ such that $|x_{n_j} - x| \geq \varepsilon$. Note that $n_k \geq k$ which means that x_{n_k} would form a subsequence of x_n.
(3) $\implies$ (1)) If x_n on the contrary, converged to x, then all its subsequences converge to x as well, but for the subsequence x_{n_k} from (2), we have that $\exists \varepsilon > 0$ such that $\forall k \in \mathbb{N}$ $\exists n_k \in \mathbb{N}, n_k \geq k$ such that $|x_{n_k} - x| \geq \varepsilon$ which is the definition of divergence. Absurd. ■

Theorem 1.20: Monotone Subsequence Theorem

Every sequence x_n has a monotone subsequence

Proof for Theorem.

First Proof:

Call a point x_n a "peak" if $x_n \geq x_m \forall m \geq n$. i.e, it is larger than all the terms that come after it. Consider the case where there are finite peak points. List them as $x_{n_1}, x_{n_2}, x_{n_3} \ldots . x_{n_k}$. The term x_{n_k+1} is not a peak, which means after $n_k + 1$-th term, there exists another term such that it is larger than this one. Since that term is not a peak either, there must exist another term with an index larger than both the previous terms, such that it is larger than both. As such, keep choosing such non peak terms to generate a monotone increasing sequence. If there are infinite peak points, $x_{n_1}, x_{n_2} \cdots x_{n_k} \cdots$, simply set the sequence to be these peak points. This would be a monotone decreasing sequence

Alternate Proof:

Case 1: The sequence is unbounded:
Call $S_0 := \{x_1, x_2, \cdots x_n \cdots\}$. For $\varepsilon = 1, \exists x_{n_0}$ such that $x_{n_0} \geq 1$. Consider the set $S_1 :=$

$S_0 - \{x_1, x_2, \cdots, x_{n_0}\}$. This is still unbounded, hence, choose $\varepsilon = x_{n_0}$ $\exists x_{n_1} \geq x_{n_0}$ with $n_1 > n_2$. Having chosen x_{n_k}, choose $x_{n_{k+1}}$ by taking the set $S_{k+1} := S_0 - \{x_1, x_2, \cdots x_{n_k}\}$. This set is unbounded. Therefore, for $\varepsilon_{k+1} = x_{n_k}$, $\exists x_{n_{k+1}} \geq x_{n_k}$ with $n_{k+1} > n_k$. This forms a subsequence that is monotonic increasing.
Case 2: The sequence is bounded:
Consider the set S_k to be defined as $S_k := \{x_n : n \geq m\} = \{x_m, x_{m+1}, x_{m+2}, \cdots, x_n \cdots\}$ and define the supremum's of each of these sets as $U_k = sup(S_k)$ (easy to see that they exist) Notice that $U_{k+1} \leq U_k$.
If only finite sets $S_{n_1}, S_{n_2} \cdots S_{n_j}$ has its own supremum, then for the sets $\{S_{n_j+1}, S_{n_j+2} \cdots\}$, the supremum of these sets are not within themselves. This would mean that all the sets $\{S_{n_j+1}, S_{n_j+2} \cdots\}$ contain the same supremum. To see this, suppose that S_{n_j+1} and S_{n_j+2} have different supremums. The fact is that these sets differ by just one element x_{n_j}. If, due to removing this one element, the supremum changes, that would imply that element is the supremum. But since supremums don't exist in these sets, we conclude they share the same supremum U. We deal with the set $S_{n_j+1} = \{x_{n_j+1}, x_{n_j+2}, \cdots\}$ whose supremum is U, lying outside the set. For $\varepsilon = 1$, $\exists x_{a_1} \in S_{n_j+1}$ such that $U - 1 \leq x_{a_1} < U$. Take the set $S_{a_1+1} := \{x_{a_1+1}, x_{a_1+2} \cdots\}$ Whose supremum is also U. Choose $\varepsilon = U - x_{a_1}$. We have then $U - \varepsilon = U - (U - x_{a_1} \leq x_{a_2} < U$. Having found x_{a_k}, consider the set $S_{a_k+1} = \{x_{a_k+1}, x_{a_k+2} \cdots\}$ whose supremum is U. Now we take ε to be $u - x_{a_k}$ which would imply $\exists x_{a_{k+1}} \in S_{a_k+1}$ such that $U - \varepsilon = U - (U - x_{a_k}) \leq x_{a_{k+1}} < U$. Hence we see that $x_{a_k} \leq x_{a_{k+1}} \forall k \in \mathbb{N}$ and by construction, $a_k < a_{k+1}$. This is then, a monotone increasing subsequence.
If there are infinitely many sets S_{k_n} that contain their own supremum, then simply create a sequence of these supremums $U_{k_1}, U_{k_2}, \cdots$. This is obviously monotone decreasing, and it is a subsequence since, by construction, $k_n \leq k_{n+1}$. ■

Theorem 1.21: Bolzano Weierstrass

Every bounded sequence has a convergent subsequence.

Proof for Theorem.

From monotone subsequence theorem, every sequence has a monotone subsequence. If the main sequence is bounded, every subsequence is bounded. Hence, this monotone sequence is bounded, hence, convergent. ■

Theorem 1.22

If x_n is a bounded sequence such that every convergent subsequence converges to x, then the main sequence converges to x.

Proof for Theorem.

Suppose that the main sequence *does not* converge to x, which means that there exists $\varepsilon > 0 \in \mathbb{R}$ and a subsequce x_{n_k} such that $\forall k \in \mathbb{N}$, $|x_{n_k} - x| \geq \varepsilon$. This subsequence is bounded, hence it has a sub-subsequence $x_{n_{k_j}}$ that is convergent. This sub-subsequence converges to x. But this raises a contradiction since for a particular ε, every term in this subsequence, and by extention, the sub-subsequence, falls outside the ε neighbourhood of x. ■

Definition 1.23: (LimSup and LimInf)

Given a sequence that is bounded (hence forth, all theorems involving limsup and liminf assumes a bounded sequence as given):

1. $Limsup(x_n) := inf(V := \{v \in \mathbb{R} : \exists n_v \in \mathbb{N} \text{ such that } \forall n \geq n_v, x_n \leq v\})$
2. $Liminf(x_n) := sup(U = \{u \in \mathbb{R} : \exists n_u \in \mathbb{N} \text{ such that } \forall n \geq n_u, x_n \geq v\})$

Theorem 1.24

The following are equivalent:

1. x is the LimSup(x_n)
2. if $\varepsilon > 0$, then $\exists$ utmost finite $n \in \mathbb{N}$ such that $x + \varepsilon < x_n$ but infinite $n \in \mathbb{N}$ such that $x - \varepsilon < x_n$. This implies $x + \varepsilon \in V$ but $x - \varepsilon \notin V$
3. If $S_m : \{x_m, x_{m+1}, \cdots\}$ and $U_m = sup(S_m)$, ,then $lim(U_m) = inf(U_m) = x$
4. If S is the set of all subsequential limits of x_n, then $sup(S) = x$

Proof for Theorem.

(1) $\implies$ (2)) Since x is the infimum of V, $\forall \varepsilon > 0$, $\exists z \in V$ such that $z \leq x + \varepsilon$. We see that, $\exists n_z$ such that $\forall n \geq n_z, x_n < z \leq x + \varepsilon$. Hence, $x + \varepsilon \in V$, or rather, there exists utmost finite n such that $x_n > x + \varepsilon$. $x - \varepsilon$ cannot belong in V since x is the infimum, therefore, $\forall k \in \mathbb{N}, \exists n_k \geq k$ such that $x_{n_k} > x - \varepsilon$, or rather, there would exist infinite n such that $x - \varepsilon < x_n$.

(2) $\implies$ (3)) We know that $U_m \geq U_{m+1}$, is a monotone decreasing sequence that is bounded below. Hence, from monotone convergence theorem, we have $lim(U_m) = inf(\{U_m\})$. From (2), we know that $\forall \varepsilon > 0$, $\exists n_\varepsilon$ such that $\forall n \geq n_z$ we have $x_n \leq x + \varepsilon$. Therefore, $U_{n_\varepsilon} \leq x + \varepsilon$. Hence, $\forall \varepsilon > 0, inf(U_n) = lim(U_n) \leq x + \varepsilon$. There exists infinite x_n such that $x - \varepsilon < x_n$ which means that $x - \varepsilon < U_n \forall n \in \mathbb{N}$. This implies $x - \varepsilon \leq inf(U_n)$. Therefore means $\forall \varepsilon > -, |inf(U_n) - x| \leq \varepsilon$. From the lemma, $inf(U_n) = x$.

(3) $\implies$ (4)) Since $inf(U_n) = x$, $\forall \varepsilon, \exists n_0(\varepsilon) \in \mathbb{N}$ such that $U_{n_0(\varepsilon)} \leq x + \varepsilon \implies \forall n \geq n_0(\varepsilon), x_n \leq x + \varepsilon$ so for every convergent subsequence, x_{n_k}, $lim(x_{n_k}) \leq x + \varepsilon$. Since the set of all subsequential limits is bounded (and non empty from Bolzano Weierstrass Theorem), $sup(S = \text{set of all subsequential limits}) \leq x + \varepsilon$. $\forall \varepsilon > 0, x - \varepsilon < inf(U_n) \implies \forall \varepsilon \forall n \in \mathbb{N}, x - \varepsilon < U_n$.
Choose $\varepsilon = 1$ and the set S_1, for which $\exists x_{n_1} \in S_1$ such that $U_1 - 1 \leq x_{n_1} < U_1$. Choose $\varepsilon = \frac{1}{2}$ and the set S_{n_1} for which $\exists x_{n_2} \in S_{n_1}$ such that $U_{n_1} - \frac{1}{2} \leq x_{n_2} < U_{n_1}$. From

construction, $n_2 > n_1$. Having chosen $\varepsilon = \frac{1}{j}$ and the set $S_{n_{j-1}}$ and obtaining n_j such that $\exists x_{n_j} \in S_{n_{j-1}}$ so that $U_{n_{j-1}} - \frac{1}{j} \leq x_{n_j} < U_{n_{j-1}}$ such that $n_j > n_{j-1}$, we choose $\varepsilon = \frac{1}{j+1}$ and the set S_{n_j}. The construction continues and we create a sequence x_{n_j} which from squeeze play, converges to x.To see this, we have that $\forall j \in \mathbb{N}$

$$U_{n_{j-1}} - \frac{1}{j} \leq x_{n_j} < U_{n_{j-1}}$$

Taking limit on both LHS and RHS we see that x_{n_j} converges to x. Therefore x itself is a subsequential limit, which means $x \leq sup(S)$. We already had $\forall \varepsilon > 0, sup(S) \leq x + \varepsilon$, which gives, $\forall \varepsilon > 0, x \leq sup(S) \leq x + \varepsilon$, which means $sup(S) = x$.

(4) $\implies$ (1)) Consider the set $V := \{v \in \mathbb{R} : \exists n_v \in \mathbb{N}$ such that $\forall n \geq n_v, x_n \leq v\}$. if $z \in V$, it means that every subsequential limit of x_n goes below z. Therefore, $sup(S) = x \leq z \forall z \in V$. This means $x \leq limsup(x_n)$. Suppose $sup(S) = x < limsup(x_n)$. This means $Sup(S) + \delta = limsup(x_n)$ or $x = limsup(x_n) - \delta$. $limsup(x_n)$ is an upper bound to the set of all subsequential limits S. Consider an arbitrary subsequential limit y. $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_\varepsilon$ we have $y - \varepsilon < x_n < y + \varepsilon < limsup(x_n) - \delta + \varepsilon$. Choose an ε slightly larger than δ, which would make $limsup(x_n) - (\delta - \varepsilon)$ slightly smaller than $limsup(x_n)$. This gives: for the chosen $\varepsilon \exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_\varepsilon$ we have $x_n < limsup(x_n) - \delta + \varepsilon < limsup(x_n)$. This means that a number slightly smaller than $inf(V)$ exists in V. This is absurd. Hence, $limsup(x_n) = sup(S =$ set of all subsequential limits $)$.

■

Theorem 1.25

The following are equivalent:

1. y is the Liminf(x_n)
2. if $\varepsilon > 0$, then $\exists$ utmost finite $n \in \mathbb{N}$ such that $x_n < y - \varepsilon$ but infinite $n \in \mathbb{N}$ such that $x_n < y + \varepsilon$. This implies $y + \varepsilon \notin U$ but $y - \varepsilon \in U$
3. If $S_m : \{x_m, x_{m+1}, \cdots\}$ and $L_m = inf(S_m)$, ,then $lim(L_m) = sup(L_m) = y$
4. If S is the set of all subsequential limits of x_n, then $inf(S) = y$

Proof for Theorem.

(1) $\implies$ (2)) Given $y = Liminf(x_n) := sup(U = \{u \in \mathbb{R} : \exists n_u \in \mathbb{N}$ such that $\forall n \geq n_u, x_n \geq v\})$. If $\varepsilon > 0$ we have a $z \in U$ such that $y - \varepsilon \leq z$. There exists only finite n such that $x_n < z$ which means there exists only finite n such that $x_n < y - \varepsilon$. Therefore, $y - \varepsilon \in U$. Consider $y + \varepsilon$. Since $y + \varepsilon \notin U$, we have that $\forall k \in \mathbb{N}\ \exists n_k \geq k$ such that $x_n < y + \varepsilon$ or infinite n_k such that x_{n_k} such that $x_{n_k} < y + \varepsilon$.

(2) $\implies$ (3)) We can see that, if $S_m := \{x_n : n \geq m\}$, and $L_m := inf(S_m)$, then $L_m \leq L_{m-1}$, which is a monotone increasing sequence, which is bounded, hence is convergent to $sup(\{L_m\}) = lim(L_m)$. Since from (2), $\exists$ infinite n_k such that $x_n < y + \varepsilon$, we see that $\forall m, inf(S_m) = L_m \leq y + \varepsilon \implies lim(L_m) \leq y + \varepsilon \forall \varepsilon > 0$. Since there exists only finite n such that $x_n < y - \varepsilon \implies \forall n \geq n_y(\varepsilon), x_n \geq y - \varepsilon$. This means that $y - \varepsilon \leq L_{n_y} \implies y - \varepsilon \leq lim(L_m) = sup(L_m)$. Hence $\forall \varepsilon > 0$, $y - \varepsilon \leq lim(L_m) \leq y + \varepsilon$, hence $y = lim(L_m)$.

(3) $\implies$ (4)) Sunce $y = sup(L_m) = lim(L_m)$. For an $\varepsilon > 0$, we have an L_{n_1} such that $y - \varepsilon \leq L_{n_l}$. Since L_{n_1} is the infimum of S_{n_1}, we have $y - \varepsilon < x_n, \forall n \geq n_l$. This would mean that every subsequence converges to a point larger than $y - \varepsilon$. Therefore $y - \varepsilon < t\ \forall t \in S$ where S is the set of all subsequential limits (This set is non empty from Bolzano Weierstrass, and is bounded, hence has a supremum and infinmum). Hence $\forall \varepsilon > 0$, $y - \varepsilon \leq inf(S)$. $y + \varepsilon$ is an upper bound for $\{L_m : m \in \mathbb{N}\}$. Choose $\varepsilon = 1$ and the set S_1. $L_1 + 1 \geq L_1$. Since L_1 is infimum of S_1, then $\exists x_{n_1} \in S_1$ such that $L_1 \leq x_{n_1} \leq L_1 + 1$. Choose $\varepsilon = \frac{1}{2}$, and the set S_{n_1}. $\exists x_{n_2} \in S_{n_1}$ such that $L_{n_1} \leq x_{n_2} \leq L_{n_1} + \frac{1}{2}$. Having chosen $\varepsilon = \frac{1}{j}$ and the set $S_{n_{j-1}}$, we have an $x_{n_j} \in S_{n_{j-1}}$ such that

$$L_{n_{j-1}} \leq x_{n_j} \leq L_{n_{j-1}} + \frac{1}{j}$$

Notice that, by construction of our sets, $n_j > n_{j-1}$. Hence, we have a subsequence of x_n which is x_{n_j} which, from squeeze play theorem, converges to y. Therefore, $y \in S$, which means $inf(S) \leq y$. We therefore have $\forall \varepsilon > 0$, $y - \varepsilon \leq inf(S) \leq y$. This means $inf(S) = y$.

(4) $\implies$ (1)) Given y is the infimum of the set of all subsequential limits. Say $\alpha =$

$Liminf(x_n) := sup(U = \{u \in \mathbb{R} : \exists n_u \in \mathbb{N}$ such that $\forall n \geq n_u, x_n \geq v\})$. If $z \in U$, it means that after some n_z, every $x_n \geq z$ which means every subsequence converges above z. Therefore, z is a lowerbound for the set of all subsequential limits S. $z \leq inf(S) = y, \forall z \in U$. We can see that $sup(U) = \alpha \leq y$ from this. Suppose $Sup(U) = \alpha < y \implies \alpha = y - \delta$ for some δ. Consider an arbitrary subsequential limit q. $\forall \varepsilon, \exists n_q(\varepsilon) \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_q(\varepsilon)$ we have $y - \varepsilon + \delta - \delta \leq q - \varepsilon < x_n < q + \varepsilon$. $(y - \delta) - (\varepsilon - \delta) = \alpha + (\delta - \varepsilon) \leq q - \varepsilon < x_n < q + \varepsilon$. This means that $\exists n_0$ such that $\forall n \geq n_0$, $\alpha + (\delta - \varepsilon) < x_n$. This means that, if we choose ε smaller than δ, we would have a number larger than $sup(U) = \alpha$ being inside U. Absurd. Hence $\alpha = y$. ■

Theorem 1.26

A bounded sequence is convergent if and only if $limsup(x_n) = liminf(x_n)$

Proof for Theorem.

$\implies$) If a bounded sequence is convergent, all its subsequences converge to the same limit x. Therefore x is both the supremum and the infimum of the set of all subsequential limits, which is also the limsup and liminf.

$\impliedby$) If limsup=liminf, then the set of all subsequential limits has infimum and the supremum equal. This means that the set of all subsequential limits is singleton, with $x \in S$. If x_n is bounded and all its convergent subsequences converge to x, then x_n converges to x. ■

Theorem 1.27: Shuffle Lemma

If x_n and y_n are sequences in $\mathbb{R}$, let the shuffled sequence z_n be defined as $z_{2n} = y_n$ and $z_{2n-1} = x_n$. Then, z_n is convergent $\iff$ x_n and y_n are convergent, and $lim(x_n) = lim(y_n)$.

Proof for Theorem.

$\implies$) If z_n converges, then for all $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that after n_0 every term of z_n lies in the ε neighbourhood of some limit z. This means that beyond some $n_{\text{something else}}$, every term of x_n and y_n - the sequences that make up z_n - falls into the ε neighbourhood of z. Therefore, both x_n and y_n are convergent, and to z.

$\impliedby$) If x_n and y_n are convergent to z, then $\forall \varepsilon > 0$,$\exists n_x, n_y \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_x$, $z - \varepsilon < x_n < z + \varepsilon$ and $z - \varepsilon < y_n < z + \varepsilon \implies z - \varepsilon < z_{2n} < z + \varepsilon$ and $z - \varepsilon < z_{2n-1} < z + \varepsilon$. Hence z_n is also convergent, and converges to z. ■

Definition 1.28: Cauchy Sequences

A sequence is said to be **Cauchy** if $\forall \varepsilon > 0$, $\exists n_0(\varepsilon) \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_0$ we have $|x_n - x_m| < \varepsilon$

Theorem 1.29

Cauchy Sequences are bounded

Proof for Theorem.

Suppose x_n is cauchy. In the definition, fix $\varepsilon = 1$ and fix one element $x_j \geq n_0$. We then have $\forall n \in N, n \geq n_0$ $|x_n - x_j < 1 \implies x_j - 1 < x_n < x_j + 1$. We can see that $\forall n \geq n_0$, it is bounded, Since $x_1, x_2, \cdots x_{n_0}$ is a finite set, it too is bounded. Therefore, Cauchy sequences are bounded. ■

Definition 1.30: Contractive sequence

A sequence x_n is contractive if $\exists C > 0$ and $n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0$ we have $|x_{n+2} - x_{n+1}| < C|x_{n+1} - x_n|$

Theorem 1.31

A sequence is cauchy in $\mathbb{R}$ if and only if it is convergent in $\mathbb{R}$

Proof for Theorem.

$\Longleftarrow$) $\forall \varepsilon > 0$ $\exists n_1 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_1$

$$|x_n - x| < \frac{\varepsilon}{2}$$

$\forall \varepsilon > 0$ $\exists n_1 \in \mathbb{N}$ such that $\forall m \in \mathbb{N}, m \geq n_1$

$$|x - x_m| < \frac{\varepsilon}{2}$$

Adding these two we get: $\forall \varepsilon > 0$ $\exists n_1 \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_1$

$$|x_n - x_m| \leq |x_n - x| + |x - x_m| < 2\frac{\varepsilon}{2} = \varepsilon$$

$\Longrightarrow$)Say x_n is Cauchy, but not convergent. Since x_n is bounded, it has a convergent subsequence $x_{n_k} \to x$. Suppose x_n doesn't converge to x. This means that $\exists \varepsilon > 0$ and a subsequence x_{n_j} such that $\forall j \in \mathbb{N}$, we have $|x_{n_j} - x| \geq \varepsilon$. Since x_n is Cauchy,$\forall \varepsilon$, $\exists n_0$ such that $\forall j, k \in \mathbb{N}, j, k \geq n_0$

$$|x_{n_j} - x_{n_k}| < \frac{\varepsilon}{2}$$

Since $x_{n_k} \to x$, we have $\forall \varepsilon > 0$ $\exists l \in \mathbb{N}$ such that $\forall n_k \in \mathbb{N}, n_k \geq l$

$$|x_{n_k} - x| < \frac{\varepsilon}{2}$$

Adding those two we have $\forall \varepsilon$ $\exists n_{\text{something}} \in \mathbb{N}$ such that $\forall n_j, n_k \geq n_{\text{something}}$ we have

$$|x_{n_j} - x| \leq |x_{n_j} - x_{n_k}| + |x_{n_k} - x| < 2\frac{\varepsilon}{2} = \varepsilon$$

We see that x_{n_j} actually converges to x, contrary to the divergence criteria. ■

Theorem 1.32

Contractive sequences are Cauchy.

Proof for Theorem.

$\forall n \geq n_0$ we have $\forall n \geq n_0$, $|x_n - x_{n-1}| \leq C|x_{n-1} - x_{n-2}|$ where $C < 1$. $|x_n - x_{n-1}| \leq C|x_{n-1} - x_{n-2}| \leq C^2|x_{n-2} - x_{n-3}| \leq C^3|x_{n-3} - x_{n-4}| \cdots \leq C^{n-n_0-1}|x_{n_0+1} - x_{n_0}|$

Consider $|x_m - x_n|$ where WLOG $m > n$. $|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - x_{m-3} + x_{m-3} - \cdots x_{n+1} - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \leq (C^{m-n_0-1} + C^{m-n_0-2} + C^{m-n_0-3} \cdots + C^{n-n_0})|x_{n_0+1} - x_{n_0}|$. The term in the brackets can be made smaller than any ε for large enough m, n. This means that, x_n is Cauchy, hence convergent. ■

Example : (Applying Contraction) If $x_n = \frac{x_{n-1}+x_{n-2}}{2}$ ***Then*** x_n ***is cauchy.***
We see that $2x_n = x_{n-1} + x_{n-2} \implies 2x_n - 2x_{n-1} = -(x_{n-1} - x_{n-2})$. This implies: $2(|x_n - x_{n-1}|) = |x_{n-1} - x_{n-2}| \implies |x_n - x_{n-1}| \leq \frac{2}{3}|x_{n-1} - x_{n-2}|$. Therefore, x_n is Cauchy by virtue of being contractive. There are two methods to find the limit of this sequence.

Method 1 (Courtesy of TYS Arjun):

$$x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$$
$$x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$$
$$x_{n-1} = \frac{1}{2}(x_{n-2} + x_{n-3})$$
$$\vdots$$
$$x_4 = \frac{1}{2}(x_3 + x_2)$$
$$x_3 = \frac{1}{2}(x_2 + x_1)$$

Add all these to get:

$$x_3 + x_4 + \cdots x_{n-1} + x_n + x_{n+1} = \frac{1}{2}(x_n) + x_{n-1} + x_{n-2} \cdots + x_3 + x_2 + \frac{1}{2}(x_1) \implies$$

$$\frac{1}{2}(x_n) + x_{n+1} = x_2 + \frac{1}{2}(x_1)$$

Passing to the limit which was shown to exist we get:

$$\frac{1}{2}x + x = x_2 + \frac{1}{2}(x_1)$$

Method 2:

$$(x_n - x_{n-1}) = -\frac{1}{2}(x_{n-1} - x_{n-2}) = \frac{1}{2^2}(x_{n-2} - x_{n-3}) = -\frac{1}{2^3}(x_{n-3} - x_{n-4})\cdots$$
$$= (-1)^j\frac{1}{2^j}(x_{n-j} - x_{n-j-1}) = (-1)^{n-n_0+1}\frac{1}{2^{n-n_0+1}}(x_{n-(n-n_0+1)} - x_{n_0})$$

$\implies$

$$x_n - x_1 = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + (x_{n-2} - x_{n-3})\cdots(x_2 - x_1) \implies$$

$$x_n - x_1 = \sum_{k=1}^{n-1}(x_{k+1} - x_k) = \sum_{k=1}^{n}(-1)^{-n_0}(-1)^k((-2)^{n_0-1})(\frac{1}{2^k}(x_{n_0+1} - x_{n_0})) =$$

$$x_n - x_1 = (-1)^{-n_0}(-2)^{n_0-1}(x_{n_0+1} - x_{n_0})\sum_{k=1}^{n}(\frac{-1}{2})^k =$$

$$(-1)^{-n_0}(-2)^{n_0-1}(x_{n_0+1} - x_{n_0})(\frac{1}{3})((\frac{-1}{2})^n - 1)$$

Therefore,

$$lim(x_n - x_1) = (-1)^{-n_0}(-2)^{n_0-1}(x_{n_0+1} - x_{n_0})\sum_{k=1}^{n}(\frac{-1}{2})^k =$$

$$(-1)^{-n_0}(-2)^{n_0-1}(x_{n_0+1} - x_{n_0})(\frac{1}{3})lim((\frac{-1}{2})^n - 1)$$

■

Example : (Fibonacci) $f_1 = 1$***,*** $f_2 = 1$ ***and*** $f_n = f_{n-1} + f_{n-2}$ ***characterises the fibonacci sequence. The sequence*** $x_n = \frac{f_n}{f_{n+1}}$ ***is convergent.***

$x_n = \frac{f_n}{f_{n+1}} = \frac{f_n}{f_n+f_{n-1}} = \frac{1}{1+\frac{f_{n-1}}{f_n}} = \frac{1}{1+x_{n-1}}$. Since $x_{n-1} = \frac{1}{1+x_{n-2}}$, we have $x_n = \frac{1}{1+\frac{1}{1+x_{n-2}}}$.
Notice that $x_1 > x_3 > x_5$ and $x_2 < x_4 < x_6$. Suppose, till $n = n_0$, we have $x_{2n-1} < x_{2n-3}$ and $x_{2n} > x_{2n-2}$. Consider x_{2n_0+1} and x_{2n_0-1}. $x_{2n_0+1} = \frac{1}{1+x_{2n_0}}$ and $x_{2n_0-1} = \frac{1}{1+x_{2n_0-2}}$.
Since $1 + x_{2n_0-2} < 1 + x_{2n_0}$ we have $\frac{1}{1+x_{2n_0-2}} > \frac{1}{1+x_{2n_0}} \implies x_{2n_0-1} > x_{2n_0+1}$. Hence, it is true for all $n \in \mathbb{N}$ from induction.
In a similar fashion, we can show via induction that the even subsequences are monotone increasing. Both the odd subsequences and even subsequences are monotone decreasing and increasing respectively, whilst being bounded. Hence, they are convergent. From the fact that $x_n = \frac{1}{1+x_{n-1}}$, we can see that both of these converge to the same number.

■

Definition 1.33: (Proper Divergence)

A sequence is said to diverge to +infinity if $\forall\xi \in \mathbb{R}, \exists n_0$ such that $x_n > \xi \forall n \geq n_0$.

It is divergent to -infinity if $\forall\xi \in \mathbb{R}, \exists n_0$ such that $x_n < \xi \forall n \geq n_0$.

Theorem 1.34

If x_n is monotone, then it is unbounded $\iff$ it is properly divergent.

Proof for Theorem.

$\implies$) $\forall \varepsilon > 0$, $\exists n(\varepsilon) \in \mathbb{N}$ such that $x_{n(\varepsilon)} > \varepsilon \implies x_n > \varepsilon \forall n \geq n(\varepsilon)$.

$\impliedby$)Properly Divergent is stronger than unboundedness. ■

Theorem 1.35: Comparision test #1

If $lim(x_n) = \infty$ and $x_n \leq y_n$, then $y_n \to \infty$. Similarly, if $lim(y_n) \to -\infty$, then $x_n \to -\infty$

Proof for Theorem.

Obvious ■

Theorem 1.36: Comparision test #2

If x_n and y_n and Positive sequences, and if $L > 0$, and if $lim(\frac{x_n}{y_n}) = L$, then $x_n \to \infty \iff y_n \to \infty$.

Proof for Theorem.

$\forall \varepsilon \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0$ we have $y_n(L - \varepsilon) < x_n < (L + \varepsilon)y_n$. Choose $\varepsilon = \frac{L}{2}$ so that we have $\forall n \geq n_0$, $\frac{L}{2}y_n < x_n < \frac{3}{2}y_n$ whence we see from test #1 that $x_n \to \infty \iff y_n \to \infty$. If $L = 0$. ■

Lemma 1.37: Useful Lemma

A monotone sequence is bounded if one of its subsequences is bounded

Proof for Lemma

Say the main sequence is properly divergent (which essentially means unboundede), then after some n_0 dependent on ε, all terms of all the subsequences coming after the index n_0 will be greater than ε. This is true for every ε, which means that all subsequences are unbounded. Therefore, the contrapositive gives that if one subsequence is bounded, the main sequence is bounded. ■

2 On Series (Introduction)

Definition 2.1: Series

Given a sequence x_n, we say the series generated by x_n is s_n if $s_n = \sim_{i=1}^{n} x_i$. (Sequence of partial sums defined inductively)

Lemma 2.2: n-th Term Test

A series $s_n = \sum_{i=1}^{n} x_i$ is convergent $\implies lim(x_n) = 0$

Proof for Lemma

$s_n = x_n + s_{n-1} \implies x_n = s_n - s_{n-1}$, and passing to the limit gives $lim(x_n) = 0$ ■

Theorem 2.3: Cauchy Criterion for Series

A series $s_n = \sum_{i=1}^{n} x_i$ is convergent $\iff \forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}, m, n \geq n_0$ we have $|s_m - s_n| = |x_{n+1} + x_{n+2} \cdots + x_m| < \varepsilon$

Example : The 1 harmonic: $\sum_{i=1}^{n} \frac{1}{i}$ is divergent.
Method 1:

Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$, and consider the subsequence of H_n which is $H_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n-1}} + \cdots \frac{1}{2^n}$

$$H_{2^1} = 1 + \frac{1}{2} \geq 1 + \frac{1}{2}$$

$$H_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) = 1 + \frac{1}{2} + \frac{1}{2}$$
$$= 1 + \frac{2}{2}$$

$$H_{2^n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) \cdots + (\frac{1}{2^{n-1}} + \frac{1}{2^{n-1} + 1} \cdots \frac{1}{2^n}$$
$$\geq 1 + \frac{n}{2}$$

Hence, we see that H_{2^n} is properly divergent, which means that the main sequence is properly divergent.

Method 2:

Consider $|H_m - H_n| = |\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$ with the assumption that $m > n$. Note that $H_m - H_n$ has $m - n$ terms. $|H_m - H_n| > \frac{m-n}{m}$. Suppose $m = 2n$. We then have $|H_m - H_n| > \frac{n}{2n} = \frac{1}{2}$. Choose $\varepsilon = \frac{1}{2}$. We now have: $\exists \varepsilon = \frac{1}{2}$ such that $\forall k \in \mathbb{N}$, $\exists m(k), n(k) \in \mathbb{N}, m(k), n(k) \geq k$ with $m(k) = 2n(k)$ such that $|H_{m(k)} - H_{n(k)}| \geq \varepsilon = \frac{1}{2}$.

Hence, from negation of cauchy criteria, the 1 harmonic properly diverges.

Method 3:

Suppose that H_n is actually convergent. Consider H_{2n}. We have

$$H_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} \cdots \frac{1}{2n}$$

$$H_{2n} > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{6} + \frac{1}{6}) + (\frac{1}{8} + \frac{1}{8}) + \cdots + (\frac{1}{2n} + \frac{1}{2n}) =$$

$$H_{2n} = \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{1}{2} + H_n$$

Passing to the limit we have:

$$H \geq \frac{1}{2} + H$$

which is absurd.

Lemma 2.4

A positive termed series either converges or properly diverges

Proof for Lemma

Obvious

Example : The 2 harmonic $S_n = \sum_{i=1}^{n} \frac{1}{i^2}$ ***is convergent***

Method 1:

$$S_n = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq \frac{1}{1} + \frac{1}{(2)(1)} + \frac{1}{(3)(2)} + \cdots + \frac{1}{(n)(n-1)} \implies$$

$$S_n \leq 1 + \frac{2-1}{(2)(1)} + \frac{3-2}{(3)(2)} + \cdots + \frac{n-(n-1)}{n(n-1)} =$$

$$1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots \frac{1}{n-1} - \frac{1}{n} \implies$$

$1 \leq S_n \leq 2 - \frac{1}{n} \leq 2$ which means this monotone increasing sequence is bounded above.

Method 2:

Consider the subsequence of S_n, S_{2^n-1}.

$$S_{2^1-1} = S_1 = \frac{1}{1}$$

$$S_{2^2-1} = S_3 = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} \leq 1 + \frac{1}{2}$$

$$S_{2^3-1} = S_7 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots \frac{1}{7^2} \leq 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}$$
$$\leq 1 + \frac{1}{2} + \frac{1}{2^2}$$

We can likewise easily see that $S_{2^n-1} \leq 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^{n-1}} \leq 2$. Which means S_n is convergent.

Theorem 2.5: Comparision test

Given X_n and Y_n, and $S_n = \sum X_i$ and $T_n = \sum Y_i$, and $\forall n \geq k_{\text{something}}, 0 \leq x_n \leq y_n$, then, if T_n converges, then S_n converges.

Proof for Theorem.

If T_n converges, we have from cauchy criteria that $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}; m, n \geq n_0$ we have $|y_{n+1} + y_{n+2} \cdots y_m| < \varepsilon$. After $n \geq max\{n_0, k_{\text{something}}\}$ we have $|x_{n+1} + x_{n+2} \cdots + x_m| < \varepsilon$ which fulfils the cauchy criteria for S_n.

Example : The alternating harmonic series: $S_n = \frac{(-1)^{1+1}}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)^{n+1}}{n}$ ***is convergent.***

Consider the odd subsequences:

$$S_{2n+1} = \frac{(-1)^{1+1}}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)^{2n+1}}{2n} + \frac{(-1)^{2n+2}}{2n+1}$$

$$S_{2n+1} = \frac{(-1)^2}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)}{2n} + \frac{(1)}{2n+1} = S_{2n} + \frac{1}{2n+1} = S_{2n-1} - (\frac{1}{2n} - \frac{1}{2n+1})$$

We see that odd subsequences are decreasing.

$$S_{2n} = \frac{(-1)^2}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)}{2n} = S_{2n-1} + \frac{-1}{2n} = S_{2n-2} + (\frac{1}{2n-1} - \frac{1}{2n})$$

We see that even subsequences are increasing. S_{2n+1} has $2n+1$ terms, with

$$S_{2n+1} = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) \cdots + (\frac{1}{2n-1} - \frac{1}{2n}) + \frac{1}{2n+1}$$

Hence, odd subsequences are bounded below by 0. Therefore, $S_{2n+1} > 0, \forall n \in \mathbb{N}$. For even subsequences, S_{2n} has $2n$ terms, and

$$S_{2n} = 1 - (\frac{1}{2} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{5}) \cdots - (\frac{1}{2n-1} - \frac{1}{2n})$$

We see that even terms are bounded above by 1. Hence, both even and odd subsequences converge.

we have $S_{2n+2} - S_{2n} = \frac{1}{2n+1} - \frac{1}{2n+2} \implies S_{2n+2} = S_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2}$. Hence, even limit= odd limit. From shuffle play theorem, the alternating harmonic series converges.

Theorem 2.6: Limit Comparision Test

Given that x_n and y_n are such that $x_n > 0$ and $y_n > 0\ \forall n \in \mathbb{N}$, and $\exists r \in \mathbb{R}^+ \cup \{U\}$ such that

$$r = lim(\frac{x_n}{y_n})$$

Then:

1. if $r \neq 0$, $\sum x_n$ converges iff $\sum y_n$ converges.
2. if $r = 0$, then $\sum y_n$ converges $\implies$ $\sum x_n$ converges.

Proof for Theorem.

$\forall \varepsilon > 0$, $\exists n_0 > 0$ such that $\forall n \geq n_0$, we have

$$y_n(r - \varepsilon) < x_n < y_n(r + \varepsilon)$$

If we choose ε appropriately, we would have $\forall n \geq n_0$

$$y_n(\frac{r}{2}) < x_n < y_n(\frac{3r}{2})$$

Whence we can see that the $\iff$ statement is true from the first comparision test for series'.

If $r = 0$, then we would have $y_n(-\varepsilon) < x_n < y_n(\varepsilon)$, where we see that from the first comparision test, if $\sum y_n$ converges, we have $\sum x_n$ converges. To see that the forward implication does not hold, consider $x_n = \frac{1}{n^2}$ and $y_n = \frac{1}{n}$. $lim(\frac{x_n}{y_n}) = 0$, but $\sum x_n$ converges, whilst $\sum y_n$ diverges. ∎

Theorem 2.7: Addition of Series

If $\sum x_n$ and $\sum y_n$ converge, then the series $\sum x_n + y_n$ also converges.

Proof for Theorem.

$\forall \varepsilon > 0$, $\exists n_x \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_x$ we have

$$|x_{n+1} + x_{n+2} \cdots + x_m| < \frac{\varepsilon}{2}$$

And $\forall \varepsilon > 0$, $\exists n_y \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_y$ we have

$$|y_{n+1} + y_{n+2} \cdots + y_m| < \frac{\varepsilon}{2}$$

Hence, $\forall \varepsilon > 0$, $\exists j = max\{n_x, n_y\} \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq j$ we have

$$|(x_{n+1} + y_{n+1}) + \cdots (x_m + y_m)| \leq |x_{n+1} + x_{n+2} \cdots + x_m| + |y_{n+1} + y_{n+2} \cdots + y_m| < \varepsilon$$

Which is the cauchy criteria for $\sum x_n + y_n$.

∎

Theorem 2.8

Let $S_n = \sum_{j=1}^{n} a_j$ be a given series constructed from $\{a_n\}$, and suppose $T_n := \sum_{i=1}^{n} b_i$ constructed from the non-zero terms of $\{a_n\}$, maintaining order. Then $lim(S_n) = a \iff lim(T_n) = a$

Proof for Theorem.

$\implies$)

$$a_1 + a_2 + \cdots a_n$$

is the same as

$$b_1 + b_2 + \cdots b_k$$

WLOG, assume infinite terms exists (non-zero). This means, $\forall k \in \mathbb{N}$, $\exists n(k) \geq k \in \mathbb{N}$ such that

$$\sum_{j=1}^{k} b_j = \sum_{i=1}^{n(k)} a_i$$

We see that $\sum_{i=1}^{n(k)}$ is a subsequence of $\sum a_j$, hence is convergent to a.

$\impliedby$) Suppose that (and assume WLOG that there are infinite non zero terms) $lim(\sum_{j=1}^{k} b_j) = a$ $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall k \geq n_0, k \in \mathbb{N}$ we have

$$|\sum_{j=1}^{k} b_j - a| < \varepsilon$$

$\forall k \in \mathbb{N}$, $\exists n(k) \in \mathbb{N}, n(k) \geq k$ such that

$$\sum_{i=1}^{n(k)} a_i = \sum_{j=1}^{k} b_j$$

Note that $n(k) \geq k, n(k+1) > n(k)$ and $\forall n \notin \{n_1, n_2, \cdots\}$ we have $a_n = 0$. If our n in consideration falls on some n_k, then for such an n, we already have

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n(k)} a_i = \sum_{j=1}^{k} b_j$$

and as such, for sufficiently large n in this consideration,

$$|\sum_{i=1}^{n} a_i - a| < \varepsilon$$

Suppose our n doesn't fall on some n_k. Then it must belong between some n_{k_0} and n_{k_0+1}. Therefore, $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n_{k_0}} a_i = \sum_{j=1}^{k_0} b_j$ which means, for sufficiently large n (sufficiency governed by the ε), we have

$$|\sum_{i=1}^{n} -a| < \varepsilon$$

Therefore, we have covered all the n-s. We finally have: $\forall\varepsilon\ \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0$

$$|S_n - a| < \varepsilon$$

Theorem 2.9

Convergence of a series is not affected by altering a finite number of terms. The limit, ofcourse, can change.

Proof for Theorem.

Let S_n be the given series, and S'_n be the altered series, altering the terms $\{a_{n_1}, a_{n_2} \cdots a_{n_k}\}$. We have $\forall\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}, n, m \geq n_0$,

$$|S_m - S_n| < \varepsilon$$

Let $j(\varepsilon) = max(\{n_0, k\})$. We are done.

Theorem 2.10: Cauchy Condensation Test

Suppose $a(n)$ is a monotone decreasing, positive termed sequence. Then $\sum_{i=1}^{n} a(i)$ converges $\iff$ $\sum_{j=1}^{n} 2^j a(2^j)$ converges.

Proof for Theorem.

We are told $a_1 \geq a_2 \geq a_3 \cdots$. Consider

$$2S_{2^n} = 2a_1 + 2a_2 + \cdots 2a_{2^{n-1}} + 2a_{2^{n-1}+1} + \cdots + 2a_{2^n}$$

$$2S_{2^n} \geq a_1 + 2a_2 + 2a_4 + 2a_4 + 2a_8 + 2a_8 + 2a_8 + 2a_8 + \cdots + \underbrace{2a_{2^n} + \cdots 2a_{2^n}}_{2^n-1-2^{n-1}-1+1=\text{ terms}} +2a_{2^n}$$

We therefore have:

$$2S_{2^n} \geq a_1 + 2a_2 + 4a_4 + 8a_8 \cdots + 2a_{2^n}(2^n - 1 - \frac{2^n}{2}) + 2a_{2^n} =$$

$$a_1 + 2a_2 + 4a_4 + \cdots 2^n a_{2^n}$$

Or

$$\frac{1}{2}(a(1) + 2a(2) + 4a(4) + \cdots 2^n a(2^n)) \leq S_{2^n}$$

Consider another distribution scheme:

$$2S_{2^n} = 2a_1 + 2a_2 + 2a_3 + 2a_4 \cdots 2a_{2^{n-1}} + 2a_{2^{n-1}+1} + \cdots 2a_{2^n-1} + 2a_{2^n}$$

$$2S_{2^n} \leq 2a_1 + 2a_2 + 2a_2 + 2a_4 + 2a_4 + 2a_4 + 2a_4 + \cdots 2a_{2^{n-1}} + 2(2^n - 1 - 2^{n-1})a_{2^{n-1}} + 2a_{2^n} \implies$$

$$2S_{2^n} \leq 2a_1 + 4a_2 + 8a_4 + \cdots 2^n a_{2^{n-1}} + 2a_{2^n} \implies$$

$$S_{2^n} \leq a_1 + 2a_2 + 4a_4 + \cdots 2^{n-1}a_{2^{n-1}} + a_{2^n}$$

Finally we have

$$\sum_{j=1}^{n} \frac{1}{2} 2^j a(2^j) \leq S_{2^n} \leq \sum_{i=1}^{n-1} 2^j a(2^j) + a_{2^n}$$

And from Limit comparision test, the result is obvious.

Example : The p-harmonic series: $H_n^p = \sum_{i=1}^{n} \frac{1}{i^p}$ diverges if $p \leq 1$ and Converges if $p > 1$

We already know from limit comparision test that the p series, by virtue of the 1 series diverging, diverges for $p \leq 1$. Consider the case of $p > 1$.

Method 1:

Consider the subsequence $H^p(2^n - 1)$.

$$H^p(1) = 1$$

$$H^p(3) = 1 + \frac{1}{2^p} + \frac{1}{3^p} \leq 1 + \frac{1}{2^{p-1}}$$

$$H^p(7) = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \frac{1}{7^p}$$

$$\leq 1 + (\frac{1}{2^p} + \frac{1}{2^p}) + (\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}) \leq 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}}$$

Likewise, we can see that $H^p(2^n - 1) \leq 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \cdots \frac{1}{(2^{p-1})^{n-1}}$. Hence, the sequence $H^p(n)$ converges by virtue of being bounded.

Method 2:

Applying cauchy condensation:

$$\sum_{j=1}^{n} 2^j a(2^j) = \sum_{j=1}^{n} 2^j \frac{1}{(2^j)^p} = \sum_{j=1}^{n} (\frac{1}{2^j})^{p-1}$$

This series converges (geometric series), and hence, from Cauchy Condensation, the main series converges.