CHAPTER 1

ASSIGNMENT 1

Lemma 0.1

If α is the supremum of a set S, then $-\alpha$ is the infimum of the set $-S$ defined as

$$-S := \{-x : x \in S\}$$

Proof for Lemma

If α is the supremum, then $\alpha \geq x, \forall x \in S$ and $\alpha \leq M, \forall M$ such that $M \geq x, \forall x \in S$. This means that $-\alpha \leq -x, \forall x \in S$ and $-\alpha \geq -M, \forall - M$ such that $-M \leq -x, \forall - x \in S$ If we re-notate the whole thing we have:

$-\alpha \leq z, \forall z \in -S$ and $-\alpha \geq L, \forall L$ such that $L \leq z \forall z \in -S$. This is precisely the definition for infimum of $-S$, whence we see, we are done.

■

Recall the definitions:

$$LimSup(x_n) := inf(U_n : U_n := sup(\{x_n, x_{n+1} \cdots\}))$$

$$LimInf(x_n) := sup(L_n : L_n := inf(\{x_n, x_{n+1} \cdots\}))$$

1 Problem 2

Let x be $limsup(-x_n)$.

$$x = inf(U_n : U_n = sup\{-x_n, -x_{n+1} \cdots\})$$

$$\implies x = inf(U_n : U_n = -inf\{x_n, x_{n+1}, \cdots\})$$

$$\implies x = inf(-inf\{x_n, x_{n+1} \cdots\}, -inf\{x_{n+1}, x_{n+2} \cdots\}, \cdots)$$

$$\implies x = -sup(inf\{x_n, x_{n+1} \cdots\}, inf\{x_{n+1}, x_{n+2} \cdots\}, \cdots)$$

Whence, we are done.

2 Problem 3

$$Liminf(x_n) \leq (\text{every subsequential limit of} x_n) \leq Limsup(x_n)$$

Where $Liminf(x_n)$ is the infimum, and $Limsup(x_n)$ the supremum of the set of all subsequential limits of x_n

$$Liminf(y_n) \leq (\text{every subsequential limit of} y_n) \leq Limsup(y_n)$$

Where $Liminf(y_n)$ is the infimum, and $Limsup(y_n)$ the supremum of the set of all subsequential limits of y_n

Adding these two inequalities we get:

$Liminf(x_n)+Liminf(y_n) \leq$ (every subsequential limit of x_n+every subsequential limit of y_n) $\leq Limsup(x_n) + Limsup(y_n)$

Since the set of all subsequential limits of $x_n + y_n$ falls as a subset of the sum of the set of all subsequential limits of x_n and y_n respectively, we have:

$Liminf(x_n) + Liminf(y_n) \leq$ (every subsequential limit of $x_n + y_n$) $\leq Limsup(x_n) + Limsup(y_n)$ We see now that $Liminf(x_n) + Liminf(y_n)$ is a lowerbound for the set of all subsequential limits of $x_n + y_n$ which gives us

$$Liminf(x_n) + Liminf(y_n) \leq liminf(x_n + y_n)$$

Similarly we see that $Limsup(x_n) + Limsup(y_n)$ is an upperbound for the set of all subsequential limits of $x_n + y_n$ which gives us

$$Limsup(x_n + y_n) \leq Limsup(x_n) + Limsup(y_n)$$

Equality of (I) holds when the smallest subsequential limit of $x_n + y_n$ is the sum of the smallest possible subsequential limits of x_n and y_n respectively.

Similarly, (II) equailty holds when the largest subsequential limit of $x_n + y_n$ is equal to the sum of the smallest subsequential limits of x_n and y_n respectively.

3 Problem 4

Suppose $\forall n \geq N$, we have $x_n \leq y_n$. It is clear to see that for every $n \geq N$, $U_n^x = sup(x_n, x_{n+1}, \cdots) \leq U_n^y = sup(y_n, y_{n+1}, \cdots)$. We have, in other words: $U_n^x \leq U_n^y$ for all $n \geq N$. Hence, $Limsup(x_n) \leq limsup(y_n)$.

Again consider $\forall n \geq N$, $x_n \leq y_n$ This means that $\forall n \geq N$, $L_n^x = inf(x_n, x_{n+1} \cdots) \leq y_n$ Since for a given n, we have $L_n^x \leq y_n$, and owing to the fact that L_n^x is a monotone increasing sequence, we see:

$$L_n^x \leq y_n$$

$$L_n^x \leq L_{n+1}^x \leq y_{n+1}$$

$$\vdots$$

Therefore, we see that $L_n^x \leq inf\{y_n, y_{n+1} \cdots\} = L_y^x$. From here we can conclude that $Liminf(x_n) \leq Limsup(y_n)$

4 Problem 5

Consider $(x_n)^{\frac{1}{n}}$ and $\frac{x_{n+1}}{x_n}$. Where x_n is a positive, bounded sequence. Let $V = \{v \in \mathbb{R} : \exists\ n_v \in \mathbb{N} :\ \forall n \geq n_v,\ x_n \leq v^n\}$ and $V* = \{v \in \mathbb{R} : \exists\ n_v \in \mathbb{N} :\ \forall n \geq n_v,\ \frac{x_{n+1}}{x_n} \leq v\}$

Consider arbitrary $v \in V$. We have an n_v so that $\forall n \geq n_v$ we have $x_n \leq v^n \implies x_{n+1} \leq v^{n+1} \implies \frac{x_{n+1}}{x_n} \leq v$. Hence, we see that this $v \in V*$ aswell, which gives $V \subseteq V*$.

Consider arbitrary $v \in V*$. We have n_v so that $\forall n \geq n_v$, we have $\frac{x_{n+1}}{x_n} \leq v$

$$\frac{x_{n_v+1}}{x_{n_v}} \leq v$$

$$\frac{x_{n_v+2}}{x_{n_v+1}} \leq v$$

$$\vdots$$

$$\frac{x_{v_v+(n-n_v)}}{x_{n-1}} \leq v$$

Multiply all these equations. We then get:

$$\frac{x_n}{x_{n_v}} \leq v^{n-n_v}$$

which gives, $\forall n \geq n_v$,

$$x_n \leq x_{n_v} v^n v^{-n_v} \implies (x_n)^{1/n} \leq x_{n_v}^{1/n} v (v^{-n_v})^{1/n}$$

We see that, for large enough n, the RHS can be made to go below v. Whence, we see that beyond a certain n, $(x_n)^{1/n} \leq v$. Hence, v that was initially assumed to be in $V*$, is shown to exist in V. Therefore, $V = V*$. Hence, $inf(V) = inf(V*)$ which means $limsup((x_n)^{1/n}) = limsup(\frac{x_{n+1}}{x_n})$.

The very same argument can be re-run, by interchanging the $\leq$-s with $\geq$-s to conclude that $liminf((x_n)^{1/n}) = liminf(\frac{x_{n+1}}{x_n})$. Therefore, if $\frac{x_{n+1}}{x_n}$ converges, then we see $(x_n)^{1/n}$ also does. (See Notes for more info)