

A geometric realisation of affine 0-Schur algebras.

Tom Crawley

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Contents

1	Introduction	3
2	Background: The double flag variety approach to q-Schur algebras	4
2.1	Flag varieties as projective algebraic varieties	4
3	The cyclic flags approach to affine q-Schur algebras	5
3.1	Cyclic flags	6
3.1.1	A product on orbits	7
3.1.2	Triple products	8
3.2	Convolution algebras	8
3.3	Affine q -Schur algebras	9
4	Quivers with relations for affine q-Schur algebras	10
4.1	Basic results and notation	10
4.1.1	Elementary matrices	10
4.1.2	Transpose involution	10
4.1.3	A multiplication rule	11
4.2	Relations	12
4.3	quivers with relations	13
4.3.1	Exceptional case $n = 2$	13
4.3.2	Typical case $n > 2$	14
5	A generic affine algebra	15
5.1	Introducing the generic affine algebra	15
5.2	A partial order	16
5.3	Preliminary results	17
5.3.1	Covering with projective varieties	19
5.3.2	Geometry of orbits	20
5.3.3	Geometry of orbit products	22
5.4	Existence of a maximum	23
5.5	Associativity	23
5.6	The generic algebra	25
6	A realisation of affine zero Schur algebras	26
6.1	Preliminary results	26
6.1.1	Elementary basis elements	26
6.1.2	Transpose involution	27
6.1.3	Multiplication rules	27

6.2	Presentation of the generic algebra.	27
6.2.1	The case $n \geq 3$	28
6.2.2	The case $n = 2$	29
7	Further directions	30
7.1	Further results on affine zero Schur algebras	30
7.2	Deformed group algebras of symmetric groups	30

Chapter 1

Introduction

Chapter 2

Background: The double flag variety approach to q -Schur algebras

2.1 Flag varieties as projective algebraic varieties

Include a discussion of flag varieties in a finite dimensional vector space. Explain: topology of projective space; Plücker embedding of Grassmannian in a projective space; flag varieties as a closed subset in a product of Grassmannians - show that the inclusion of one subspace into another is a closed condition - given by vanishing of some homogenous polynomials which should appear as minors of a matrix.

Chapter 3

The cyclic flags approach to affine q -Schur algebras

Fix natural numbers n and r .

Definition 3.0.1 (compositions). *A composition of r into n parts is an n -tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ of non-negative integers whose sum equals r . Denote the set of compositions of r into n parts by Λ_0 .*

Definition 3.0.2 (infinite periodic matrices). *Let Λ_1 be the set of matrices $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ with integer entries $a_{i,j}$ satisfying the following conditions:*

- $a_{i,j} \geq 0$ for each $i, j \in \mathbb{Z}$;
- each row or column has only finitely many non-zero entries;
- the sum of the entries in any n consecutive rows or columns equals r ;
- $a_{i-n,j-n} = a_{i,j}$ for each $i, j \in \mathbb{Z}$.

These matrices are referred to as infinite periodic matrices.

Definition 3.0.3 (source and target). *Given $A \in \Lambda_1$, let $\text{ro } A$ and $\text{co } A$ be the compositions of r into n parts given by*

$$\text{ro } A = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\text{co } A = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right).$$

$A \in \Lambda_1$ is said to go from $\text{co } A$ to $\text{ro } A$.

Definition 3.0.4 (diagonal matrices). *Given $\lambda \in \Lambda_0$, let $D_\lambda \in \Lambda_1$ be the matrix given by $(D_\lambda)_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i \neq j$ and $(D_\lambda)_{i,i} = \lambda_i$ for $i \in \mathbb{Z}$; where the indices are taken modulo n .*

3.1 Cyclic flags

Fix $n, r \in \mathbb{N}$ and let \mathbf{k} be a field. Let \mathcal{S} be the \mathbf{k} -algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let \mathcal{R} be the subalgebra generated by ε , so $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free \mathcal{S} -module of rank r . Let G be the automorphism group of the \mathcal{S} -module V , so G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. A lattice in V is a \mathcal{R} -submodule L of V with $\mathcal{S} \otimes_{\mathcal{R}} L = V$. In particular, a lattice is an \mathcal{R} -submodule of V which is a free \mathcal{R} -module of rank r .

Lemma 3.1.1. *Let L be a lattice in V . $L/\varepsilon L$ is a torsion \mathcal{R} -module, where ε acts as zero. $L/\varepsilon L$ is a free $\mathcal{R}/\langle \varepsilon \rangle$ -module of rank r ; that is, $L/\varepsilon L$ is an r -dimensional \mathbf{k} -vector space.*

Proof. L is a free \mathcal{R} -module of rank r , with $L \subset V$. Given an \mathcal{R} -basis $\{x_1, \dots, x_r\}$ of L , $\{\varepsilon x_1, \dots, \varepsilon x_r\}$ is an \mathcal{R} -basis of εL . Finally, the cosets $\{x_1 + \varepsilon L, \dots, x_r + \varepsilon L\}$ give a basis for $L/\varepsilon L$ over $\mathcal{R}/\langle \varepsilon \rangle \cong \mathbf{k}$. \square

Let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of collections $(L_i)_{i \in \mathbb{Z}}$ of lattices in V with $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. These collections of lattices in V are referred to as cyclic flags in V .

G acts on \mathcal{F} by $(g \cdot L)_i = g(L_i)$ for each $i \in \mathbb{Z}$, given $g \in G$ and $L \in \mathcal{F}$. The G -orbits in \mathcal{F} are indexed by the set Λ_0 of compositions of r into n parts: the G -orbit in \mathcal{F} corresponding to $\lambda \in \Lambda_0$ is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim \left(\frac{L_i}{L_{i-1}} \right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

Definition 3.1.1. *The periodic characteristic matrix of a pair of cyclic flags $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the matrix $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$ with entries*

$$a_{i,j} = \dim_{\mathbf{k}} \left(\frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j} \right)$$

for each $i, j \in \mathbb{Z}$.

The diagonal action of G on $\mathcal{F} \times \mathcal{F}$ has orbits indexed by the set Λ_1 of infinite periodic matrices (see definition 3.0.2). The G -orbit corresponding to $A \in \Lambda_1$ is denoted \mathcal{O}_A and consists of those pairs $(L, L') \in \mathcal{F} \times \mathcal{F}$ with periodic characteristic matrix $A(L, L')$ equal to A .

Lemma 3.1.2. *(alternative expression for characteristic matrix) Alternatively,*

$$a_{i,j} = \dim_{\mathbf{k}} \left(\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} \right)$$

for each $i, j \in \mathbb{Z}$.

Proof. Set $U = L_i \cap L'_j$ and $U' = L_{i-1} + L_i \cap L'_{j-1}$. Then $U + U' = L_{i-1} + L_i \cap L'_j$ and $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$. Applying the isomorphism theorems, $U + U'/U'$ is naturally isomorphic to $U/U \cap U'$ as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to $a_{i,j}$. \square

Lemma 3.1.3 (transposing characteristic matrix). *Given a pair of flags $(L, L') \in \mathcal{F}^2$, the matrices $A(L, L')$ and $A(L', L)$ are related by the transpose. In particular, $A(L, L')_{i,j} = A(L', L)_{j,i}$ for each $i, j \in \mathbb{Z}$.*

Proof. By swapping the roles of i and j and swapping L and L' it is clear that $A(L, L')_{i,j}$ and $A(L', L)_{j,i}$ are both given by the dimension of the \mathbf{k} -vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each $i, j \in \mathbb{Z}$. □

Lemma 3.1.4 (a codimension formula). *Given $(L, L') \in \mathcal{F}^2$ and $i, j \in \mathbb{Z}$,*

$$\dim_{\mathbf{k}} \left(\frac{L_i}{L_i \cap L'_j} \right) = \sum_{s \leq i, t > j} a_{s,t},$$

where $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$.

Proof. **COMPLETE THIS PROOF** □

Lemma 3.1.5 (nested flags). *Given $(L, L') \in \mathcal{F}^2$, $L' \subset L$ if and only if $A(L, L')_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i > j$.*

Proof. Suppose $L, L' \in \mathcal{F}$ with $L' \subset L$, meaning $L'_j \subset L_j$ for each $j \in \mathbb{Z}$. Then for $i > j$, $L_i \cap L'_j = L'_j$, $L_{i-1} \cap L'_j = L'_j$ and $L_i \cap L'_{j-1}$, which shows

$$A(L, L')_{i,j} = \dim_{\mathbf{k}} \left(\frac{L'_j}{L'_{j-1} + L'_j} \right) = 0$$

as required. Conversely, suppose $A(L, L')$ is upper triangular, meaning $A(L, L')_{i,j} = 0$ when $i > j$. Using Lemma 3.1.4,

$$\dim_{\mathbf{k}} \left(\frac{L'_i}{L'_i \cap L_i} \right) = \sum_{s > i, t \leq i} a_{s,t} = 0,$$

so $L_i \cap L'_i = L'_i$ and thus $L'_i \subset L_i$ for each $i \in \mathbb{Z}$, as required. □

Corollary 3.1.6 (diagonal orbits). *Given $L, L' \in \mathcal{F}$, $L = L'$ if and only if $A(L, L')_{i,j} = 0$ whenever $i \neq j$. In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L, L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each $\lambda \in \Lambda_0$.

3.1.1 A product on orbits

Given $A, B \in \Lambda_1$ with $\text{co } A = \text{ro } B$, define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also $L \in \mathcal{F}_{\text{ro } A}$, define the L -slices of $Y_{A,B}$ and $X_{A,B}$ respectively as

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X_{A,B}^L = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

Observation 1. *There are only finitely many G -orbits in $X_{A,B}$.*

Lemma 3.1.7. *Given $A \in \Lambda_1$, $X_{D_\lambda, A} = \mathcal{O}_A$ if $\lambda = \text{ro } A$ and $X_{A, D_\lambda} = \mathcal{O}_A$ if $\lambda = \text{co } A$.*

Proof. Let $A \in \Lambda_1$ and set $\lambda = \text{ro } A$. $Y_{D_\lambda, A}$ is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_{D_\lambda}$, thus $L = L'$ by Corollary 3.1.6, and $(L', L'') \in \mathcal{O}_A$. $X_{D_\lambda, A}$ is the projection of $Y_{D_\lambda, A}$, which equals \mathcal{O}_A .

Similarly, if $\lambda = \text{co } A$, Y_{A, D_λ} is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_A$ and $L'' = L'$, so X_{A, D_λ} is exactly the orbit \mathcal{O}_B . \square

3.1.2 Triple products

Given $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$ and $L \in \mathcal{F}_{\text{ro } A}$, there are spaces $X_{A,B,C}$, $Y_{A,B,C}$ and their respective L -slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y_{A,B,C}^L = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X_{A,B,C}^L = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

3.2 Convolution algebras

Suppose \mathbf{k} is a finite field and let q denote the number of elements of \mathbf{k} . Consider the set S of G -invariant functions $\mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Z}$ with constructible support. S is a free \mathbb{Z} -module with a basis consisting of the indicator functions of the G -orbits in $\mathcal{F} \times \mathcal{F}$. Define an operation \star on S as follows: for each $f, g \in S$, $f \star g \in S$ is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for $(L, L'') \in \mathcal{F} \times \mathcal{F}$.

$f \star g$ is well defined since the supports of f and g consist of finitely many G -orbits, so there are only finitely many $L' \in \mathcal{F}$ such that $f(L, L')g(L', L'') \neq 0$, given $(L, L'') \in \mathcal{F} \times \mathcal{F}$. $f \star g$ is constant on G -orbits and is supported on finitely many G -orbits, so $f \star g \in S$.

Lemma 3.2.1. *The set S together with the operation \star is an associative \mathbb{Z} -algebra with identity element ι given by $\iota(L, L) = 1$ and $\iota(L, L') = 0$ for $L' \neq L$.*

Proof. Given $f, g, h \in S$ and $(L, L''') \in \mathcal{F} \times \mathcal{F}$,

$$\begin{aligned} ((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''), \end{aligned}$$

thus \star is associative. ι is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L') \iota(L', L'') = f(L, L''),$$

for each $f \in S$ and $(L, L'') \in \mathcal{F} \times \mathcal{F}$. □

Given $A \in \Lambda_1$, let $e_A \in S$ denote the indicator function of the orbit \mathcal{O}_A . S is a free \mathbb{Z} -module with basis $\{e_A : A \in \Lambda_1\}$. There exist $\gamma_{A,B,C;q} \in \mathbb{Z}$ for $A, B, C \in \Lambda_1$ such that

$$e_A \star e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C;q} e_C$$

for each $A, B \in \Lambda_1$. Then

$$\begin{aligned} \gamma_{A,B,C;q} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L') e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}, \end{aligned}$$

for any $(L, L'') \in \mathcal{O}_C$.

3.3 Affine q-Schur algebras

There exist polynomials $\gamma_{A,B,C} \in \mathbb{Z}[q]$ for $A, B, C \in \Lambda_1$ such that $\gamma_{A,B,C}(q) = \gamma_{A,B,C;q}$ for any prime power q , following [1, section 4]. The affine q -Schur algebra $\hat{S}_q(n, r)$ (defined in [\[ADD A REFERENCE\]](#)) is a $\mathbb{Z}[q]$ -algebra which is a free $\mathbb{Z}[q]$ -module with basis $\{e_A : A \in \Lambda_1\}$ and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these ‘universal polynomials’ $\gamma_{A,B,C} \in \mathbb{Z}[q]$, it follows from Lemma 3.2.1 that $\hat{S}_q(n, r)$ is an associative $\mathbb{Z}[q]$ -algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

Chapter 4

Quivers with relations for affine q-Schur algebras

4.1 Basic results and notation

4.1.1 Elementary matrices

For each $i, j \in \mathbb{Z}$, let $\mathcal{E}_{i,j}$ be the $\mathbb{Z} \times \mathbb{Z}$ ‘elementary periodic matrix’ with entries given by $(\mathcal{E}_{i,j})_{s,t} = 1$ if $(s, t) = (i + cn, j + cn)$ for some $c \in \mathbb{Z}$ and $(\mathcal{E}_{i,j})_{s,t} = 0$ otherwise. Clearly $\mathcal{E}_{i,j} = \mathcal{E}_{i+n,j+n}$ for each $i, j \in \mathbb{Z}$. Recall from Definition 3.0.4 that the diagonal matrix associated to a composition $\lambda \in \Lambda_0$ is

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}.$$

$\{e_{D_\lambda} : \lambda \in \Lambda_0\}$ is a set of pairwise orthogonal idempotents in $\hat{S}_q(n, r)$ with $\sum_{\lambda \in \Lambda_0} e_{D_\lambda} = 1$, as a result of Lemma 3.1.7.

Given $i \in [1, n]$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and define

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}.$$

Given $i \in [1, n]$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and define

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}$$

4.1.2 Transpose involution

Lemma 4.1.1. *Transposition gives a homomorphism of $\mathbb{Z}[q]$ -modules $\top : \hat{S}_q(n, r) \rightarrow \hat{S}_q(n, r)$ with $\top(e_A) = e_{A^\top}$, $\top \circ \top = 1$ and $\top(e_A e_B) = \top(e_B) \top(e_A)$.*

Proof. Let $A, B, C \in \Lambda_1$ and let \mathbf{k} be a finite field with $q = \#\mathbf{k}$ elements. If $(L, L'') \in \mathcal{O}_C$ then $(L'', L) \in \mathcal{O}_{C^\top}$ and

$$\begin{aligned}\gamma_{A,B,C;q} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;q}\end{aligned}$$

It then follows that $\top(e_A e_B) = \top(e_B) \top(e_A)$. \square

The transpose relates the E_i , F_i and 1_λ in the following way: $\top(E_{i,\lambda}) = F_{i,\lambda}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\varepsilon_i+\varepsilon_{i+1}}$ and $\top(1_\lambda) = 1_\lambda$. In particular, $\top(E_i) = F_i$ and $\top(F_i) = E_i$.

4.1.3 A multiplication rule

Lemma 4.1.2. *Given $A \in \Lambda_1$ and $i \in [1, n]$ with $\text{ro } A_{i+1} > 0$,*

$$E_i e_A = \sum_{p \in \mathbb{Z}: a_{i+1,p} > 0} q^{\sum_{j>p} a_{i,j}} [[a_{i,p} + 1]] e_{A+\varepsilon_{i,p}-\varepsilon_{i+1,p}}.$$

Given $A \in \Lambda_1$ and $i \in [1, n]$ with $\text{ro } A_i > 0$,

$$F_i e_A = \sum_{p \in \mathbb{Z}: a_{i,p} > 0} q^{\sum_{j<p} a_{i+1,j}} [[a_{i+1,p} + 1]] e_{A+\varepsilon_{i+1,p}-\varepsilon_{i,p}}.$$

Note that these formulas are still valid in the cases $E_i e_A = 0$ and $F_i e_A = 0$, provided it is understood that $e_B = 0$ whenever $B \notin \Lambda_1$. There are similar formulas for right multiplication by E_i and F_i , obtained by applying the transpose involution to the above.

Corollary 4.1.3. *Given $A \in \Lambda_1$ and $j \in [1, n]$ with $\text{co } A_{j+1} > 0$,*

$$e_A F_j = \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\varepsilon_{p,j}-\varepsilon_{p,j+1}}.$$

Given $A \in \Lambda_1$ and $j \in [1, n]$ with $\text{co } A_j > 0$,

$$e_A E_j = \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A+\varepsilon_{p,j+1}-\varepsilon_{p,j}}.$$

Proof.

$$\begin{aligned}e_A F_j &= \top(E_j e_{A^\top}) \\ &= \top\left(\sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A^\top+\varepsilon_{j,p}-\varepsilon_{j+1,p}}\right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\varepsilon_{p,j}-\varepsilon_{p,j+1}},\end{aligned}$$

where the second equality comes from Lemma 4.1.2. Similarly,

$$\begin{aligned}e_A E_j &= \top(F_j e_{A^\top}) \\ &= \top\left(\sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A^\top+\varepsilon_{j+1,p}-\varepsilon_{j,p}}\right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A+\varepsilon_{p,j+1}-\varepsilon_{p,j}}.\end{aligned}$$

□

4.2 Relations

Note that $E_i^{r+1} = F_i^{r+1} = 0$ while

$$E_i^r = [r]! e_r \mathcal{E}_{i,i+1}$$

and

$$F_i^r = [r]! e_r \mathcal{E}_{i+1,i}.$$

Lemma 4.2.1 (quantum Serre relations: $n \geq 3$). *Suppose $n \geq 3$. The following relations hold in $\hat{S}_q(n, r)$:*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless $j = i \pm 1$;

$$E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i = 0$$

$$E_i^2 E_{i+1} - (1+q) E_i E_{i+1} E_i + q E_{i+1} E_i^2 = 0$$

and

$$F_{i+1} F_i^2 - (1+q) F_i F_{i+1} F_i + q F_i^2 F_{i+1} = 0$$

$$F_{i+1}^2 F_i - (1+q) F_{i+1} F_i F_{i+1} + q F_i F_{i+1}^2 = 0.$$

Proof. Here we introduce temporary notation for the basis elements: Write $[A] = e_A$.

Take $\lambda \in \Lambda_0$.

$$E_i E_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1} E_i E_{i+1} 1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2 E_i 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then

$$(E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i) 1_\lambda = 0,$$

for each $\lambda \in \Lambda_0$. The relation $E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i = 0$ then follows.

The relations between F_i and F_{i+1} may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping E_i and F_i and reversing the order of multiplication. □

Lemma 4.2.2 (quantum Serre relations: $n = 2$). *In the case $n = 2$, the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

Lemma 4.2.3. $[E_i, F_j] = 0$ unless $j = i$.

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}]) 1_\lambda.$$

For $\lambda \in \Lambda_0$, let $R_\lambda = e_{\lambda_1 \varepsilon_{0,1} + \dots + \lambda_n \varepsilon_{n-1,n}}$. Write $R = \sum_{\lambda \in \Lambda_0} R_\lambda$. Note $R_\lambda = R 1_\lambda$. Given $A \in \Lambda_1$ and $m \in \mathbb{Z}$, let $A[m] \in \Lambda_1$ be given by $A[m]_{i,j} = a_{i,j+m}$ and let $A^{[m]}$ be given by $A^{[m]}_{i,j} = a_{i+m,j}$ for each $i \in \mathbb{Z}$.

Lemma 4.2.4 (Shifting). *If $A \in \Lambda_1$ then*

$$Re_A = e_{A[\pm 1]}$$

and

$$e_A R = e_{A[\pm 1]}.$$

Conjugation by R gives an automorphism ρ of $\hat{S}_q(n, r)$ satisfying $\rho^n = 1$.

4.3 quivers with relations

Denote by Λ_0 the set of compositions of r into n parts. That is, Λ_0 is the set of $\alpha \in \mathbb{Z}^n$ with non-negative entries which sum to r . Let $\varepsilon_i \in \mathbb{Z}^n$ be the i th elementary vector and write $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for each $i \in [1, n]$. Then $\lambda + \alpha_i \in \Lambda_0$ if $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ if $\lambda_i > 0$.

Let $\Gamma = \Gamma(n, r)$ be the quiver with set of vertices Λ_0 , with the following arrows:

For $\lambda \in \Lambda_0$ and $i \in [1, n]$, there is an arrow $e_{i,\lambda} : \lambda \rightarrow \lambda + \alpha_i$ if $\lambda_{i+1} > 0$ and there is an arrow $f_{i,\lambda} : \lambda \rightarrow \lambda - \alpha_i$ if $\lambda_i > 0$.

Denote by $\mathbb{Z}[q]\Gamma$ the path $\mathbb{Z}[q]$ -algebra of Γ . Thus $\mathbb{Z}[q]\Gamma$ is a free $\mathbb{Z}[q]$ -module with a basis given by the set of paths in Γ , with multiplication given by the concatenation of paths. If p starts where q ends, the product pq is the path q followed by p . Write $e_{i,\lambda} = 0$ unless $\lambda, \lambda + \alpha_i \in \Lambda_0$ and write $f_{i,\lambda} = 0$ unless $\lambda, \lambda - \alpha_i \in \Lambda_0$.

By construction, there is a homomorphism of $\mathbb{Z}[q]$ -algebras

$$\phi : \mathbb{Z}[q]\Gamma \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda, \end{aligned}$$

for $i \in [1, n]$ and $\lambda \in \Lambda_0$.

The image of ϕ is the subalgebra of $\hat{S}_q(n, r)$ generated by E_i, F_i for $i \in [1, n]$ and 1_λ for $\lambda \in \Lambda_0$, since $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$, while $E_i = \sum_\lambda E_{i,\lambda}$ and $F_i = \sum_\lambda F_{i,\lambda}$. In general ϕ is not surjective, so this does not always lead to a presentation of $\hat{S}_q(n, r)$.

4.3.1 Exceptional case $n = 2$.

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the q -Schur algebra.

4.3.2 Typical case $n > 2$.

Suppose $n \geq 3$. Then $\Gamma = \Gamma(n, r)$ has vertex set Λ_0 . **RESUME HERE...**

Define $e_i, f_i \in \mathbb{Z}[q]\Gamma(n, r)$ by

$$e_i = \sum_{\lambda \in \Lambda_0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0} f_{i,\lambda},$$

with the convention $e_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $f_{i,\lambda} = 0$ unless $\lambda_i > 0$. Let k_λ denote the constant path at vertex λ . $\{k_\lambda : \lambda \in \Lambda_0\}$ is a set of pairwise orthogonal idempotents in $\mathbb{Z}[q]\Gamma(n, r)$.

Let $I(n, r) \subset \mathbb{Z}[q]\Gamma(n, r)$ be the ideal generated by the expressions

$$\begin{aligned} & e_i e_{i+1}^2 - (1+q)e_{i+1}e_i e_{i+1} + qe_{i+1}^2 e_i \\ & e_i^2 e_{i+1} - (1+q)e_i e_{i+1} e_i + qe_{i+1} e_i^2 \\ & f_{i+1} f_i^2 - (1+q)f_i f_{i+1} f_i + qf_i^2 f_{i+1} \\ & f_{i+1}^2 f_i - (1+q)f_{i+1} f_i f_{i+1} + qf_i f_{i+1}^2 \\ & e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}]) k_\lambda \end{aligned}$$

Recall that a relation is a $\mathbb{Z}[q]$ -linear combination of paths with common start and end vertices. The relations involving paths $\lambda \rightarrow \mu$ are given by $1_\mu \text{expr} 1_\lambda$, for each of the above expressions.

Lemma 4.3.1. *There is a homomorphism of $\mathbb{Z}[q]$ -algebras*

$$\phi: \mathbb{Z}[q]\Gamma(n, r)/I(n, r) \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda. \end{aligned}$$

Chapter 5

A generic affine algebra

5.1 Introducing the generic affine algebra

Assume $\mathbf{k} = \mathbb{C}$ and fix $n, r \geq 1$. Let \mathcal{S} be the \mathbf{k} -algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let \mathcal{R} be the subalgebra generated by ε , namely $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free \mathcal{S} -module of rank r and let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of n -periodic cyclic flags in V ; so \mathcal{F} consists of collections $L = (L_i)_{i \in \mathbb{Z}}$ of \mathcal{R} -lattices in V with $L_i \subset L_{i+1}$ for $i \in \mathbb{Z}$ and $\varepsilon L_i = L_{i-n}$ for $i \in \mathbb{Z}$.

Let G be the group of \mathcal{S} -module automorphisms of V . Thus G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. G acts on \mathcal{F} with orbits $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0\}$, where Λ_0 is the set of compositions of r into n parts, as in Definition 3.0.1.

The diagonal action of G on $\mathcal{F} \times \mathcal{F}$ has orbits $\{\mathcal{O}_A : A \in \Lambda_1\}$, where \mathcal{O}_A consists of those pairs of flags with periodic characteristic matrix equal to A . Definitions of the periodic characteristic matrix and the set Λ_1 are given in Definition 3.1.1 and Definition 3.0.2 respectively. In particular, the periodic characteristic matrix of a pair $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix $A = (a_{i,j})_{i,j \in \mathbb{Z}}$, with

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right)$$

for each $i, j \in \mathbb{Z}$.

Recall that $\mathrm{ro}, \mathrm{co}: \Lambda_1 \rightarrow \Lambda_0$ are the maps given by

$$\mathrm{ro} A = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co} A = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right)$$

for each $A \in \Lambda_1$. Given $A \in \Lambda_1$, write $A: \mathrm{co} A \rightarrow \mathrm{ro} A$.

The purpose of this chapter is to define a category with objects Λ_0 and morphisms Λ_1 ; where $\mathrm{Hom}(\lambda, \mu) = \{A \in \Lambda_1 : \mathrm{ro} A = \mu, \mathrm{co} A = \lambda\}$. Given $A, B \in \Lambda_1$ let $\Lambda_{1A,B}$ be the set of $C \in \Lambda_1$ such that there exist $L, L', L'' \in \mathcal{F}$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L, L'') \in \mathcal{O}_C$. It will be shown that Λ_1 admits a partial order \leq such that, given $A, B \in \Lambda_1$ with $\mathrm{ro} B = \mathrm{co} A$, $\Lambda_{1A,B}$ has a maximum element $A * B$. It will be shown that $*$ is associative, leading to the construction of a category with the described properties.

The generic affine algebra $\hat{G}(n, r)$ is then defined to be the \mathbb{Z} -algebra of this category. It will be shown that $\hat{G}(n, r)$ gives a realisation of the affine 0-Schur algebra $\hat{S}_0(n, r)$ when $r < n$. It is expected that a more refined presentation of the generic algebra and the 0-Schur algebra will allow the conditions on the parameters to be relaxed slightly: the $r = n$ case is approachable, which may extend to the case $r < 2n$.

5.2 A partial order

Given $i, j \in \mathbb{Z}$, define a map $d_{i,j}$ on Λ_1 by setting

$$d_{i,j}A = \sum_{s \leq i, t > j} a_{s,t}$$

for each $A \in \Lambda_1$.

Lemma 5.2.1. *Let $A \in \Lambda_1$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for $i, j \in \mathbb{Z}$. Then*

$$d_{i,j} - d_{i-1,j} = \sum_{t > j} a_{i,t}$$

and

$$d_{i,j} - d_{i,j-1} = - \sum_{s \leq i} a_{s,j}.$$

Proof. Let $i, j \in \mathbb{Z}$. Then

$$d_{i,j} - d_{i-1,j} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i-1, t > j} a_{s,t} = \sum_{t > j} a_{i,t}.$$

Similarly,

$$d_{i,j} - d_{i,j-1} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i, t > j-1} a_{s,t} = - \sum_{s \leq i} a_{s,j}.$$

□

Lemma 5.2.2. *Let $A \in \Lambda_1$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for each $i, j \in \mathbb{Z}$. Then*

$$a_{i,j} = d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j}$$

for each $i, j \in \mathbb{Z}$.

Proof. Using Lemma 5.2.1,

$$\begin{aligned} a_{i,j} &= \sum_{t > j-1} a_{i,t} - \sum_{t > j} a_{i,t} \\ &= (d_{i,j-1} - d_{i-1,j-1}) - (d_{i,j} - d_{i-1,j}). \end{aligned}$$

Alternatively,

$$\begin{aligned} a_{i,j} &= \sum_{s \leq i} a_{s,j} - \sum_{s \leq i-1} a_{s,j} \\ &= -(d_{i,j} - d_{i,j-1}) + (d_{i-1,j} - d_{i-1,j-1}). \end{aligned}$$

□

Lemma 5.2.3. *The relation \leq on Λ_1 , defined by $A \leq B$ if and only if $d_{i,j}A \leq d_{i,j}B$ for all $i, j \in \mathbb{Z}$, is a partial order.*

Proof. It is clear that \leq is reflexive and transitive, so it remains to see \leq is antisymmetric. Suppose $A, B \in \Lambda_1$ with $A \leq B$ and $B \leq A$. Then $d_{i,j}A = d_{i,j}B$ for each $i, j \in \mathbb{Z}$, which shows $A = B$ as a result of Lemma 5.2.2. \square

The partial order on Λ_1 induces a partial order on the set of G -orbits in $\mathcal{F} \times \mathcal{F}$, such that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $A \leq B$. The following lemma is rephrased from Lemma 3.1.4 and gives some geometric significance to the partial order on Λ_1 .

Lemma 5.2.4. *Let $A \in \Lambda_1$ and take $(L, L') \in \mathcal{O}_A$. Then*

$$d_{i,j}A = \dim \left(\frac{L_i}{L_i \cap L'_j} \right)$$

for each $i, j \in \mathbb{Z}$.

Proof. This is a rephrasing of Lemma 3.1.4. \square

Conjecture 1. *Suppose $\mathbf{k} = \mathbb{C}$. The partial order on Λ_1 is compatible with the closure order on G -orbits in $\mathcal{F} \times \mathcal{F}$. In particular, $A \leq B$ if and only if $\mathcal{O}_A \subset \overline{\mathcal{O}_B}$.*

5.3 Preliminary results

Suppose $A, B \in \Lambda_1$ with $\text{co } A = \text{ro } B$. Recall the notation

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

and

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L', L'') \in Y_{A,B}\}.$$

$X_{A,B}$ is the image of $Y_{A,B}$ under the forgetful map $(L, L', L'') \mapsto (L, L'')$.

Lemma 5.3.1. *There is $N \in \mathbb{N}$ such that*

$$\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$$

for each $(L, L'') \in X_{A,B}$.

Proof. There exist $N_1, N_2 \in \mathbb{N}$ such that

$$\varepsilon^{N_1} L_0 \subset L'_0 \subset \varepsilon^{-N_1} L_0$$

and

$$\varepsilon^{N_2} L'_0 \subset L''_0 \subset \varepsilon^{-N_2} L'_0,$$

for each $(L, L', L'') \in Y_{A,B}$. Then, for $(L, L', L'') \in Y_{A,B}$,

$$L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-(N_1+N_2)} L_0$$

and

$$\varepsilon^{N_1+N_2} L_0 \subset \varepsilon^{N_2} L'_0 \subset L''_0.$$

In particular, taking $N = N_1 + N_2$, we have

$$\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$$

for each $(L, L'') \in X_{A,B}$. \square

Lemma 5.3.2. Suppose $N_1, N_2 \in \mathbb{N}$ with $\varepsilon^{N_1} L_0 \subset L_0 \subset \varepsilon^{-N_1} L_0$ and $\varepsilon^{N_2} L'_0 \subset L''_0 \subset \varepsilon^{-N_2} L'_0$ for each $(L, L', L'') \in Y_{A,B}$ and let $N = N_1 + N_2$. Then

$$\dim \left(\frac{\varepsilon^{-N} L_0}{L''_0} \right) = d_{nN_1,0}(A) + d_{nN_2,0}(B)$$

and

$$\dim \left(\frac{L''_0}{\varepsilon^N L_0} \right) = 2Nr - d_{nN_1,0}(A) + d_{nN_2,0}(B),$$

for each $(L, L'') \in X_{A,B}$.

Proof. Suppose $(L, L'') \in X_{A,B}$ and $L' \in \mathcal{F}$ so that $(L, L', L'') \in Y_{A,B}$. As in lemma 5.3.1, $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$, so

$$\dim \left(\frac{\varepsilon^{-N} L_0}{L''_0} \right) + \dim \left(\frac{L''_0}{\varepsilon^N L_0} \right) = \dim \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0} \right).$$

As a \mathbf{k} -vector space, $\varepsilon^{-N} L_0 / \varepsilon^N L_0$ is isomorphic to $(L_0 / \varepsilon L_0)^{2N}$, which has dimension $2Nr$, so

$$\dim \left(\frac{L''_0}{\varepsilon^N L_0} \right) = 2Nr - \dim \left(\frac{\varepsilon^{-N} L_0}{L''_0} \right).$$

It remains to compute the codimension of L''_0 in $\varepsilon^{-N} L_0$. Note $L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-N} L_0$, so

$$\dim \left(\frac{\varepsilon^{-N} L_0}{L''_0} \right) = \dim \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0} \right) + \dim \left(\frac{\varepsilon^{-N_2} L'_0}{L''_0} \right).$$

$$\begin{aligned} \dim \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0} \right) &= \dim \left(\frac{\varepsilon^{-N_1} L_0}{L'_0} \right) \\ &= \dim \left(\frac{L_{nN_1}}{L_{nN_1} \cap L'_0} \right) \\ &= \sum_{s \leq nN_1, t > 0} A_{s,t} \\ &= d_{nN_1,0}(A). \end{aligned}$$

$$\begin{aligned} \dim \left(\frac{\varepsilon^{-N_2} L'_0}{L''_0} \right) &= \dim \left(\frac{L'_{nN_2}}{L'_{nN_2} \cap L''_0} \right) \\ &= \sum_{s \leq nN_2, t > 0} B_{s,t} \\ &= d_{nN_2,0}(B). \end{aligned}$$

□

5.3.1 Covering with projective varieties

Given $L \in \mathcal{F}$, $N \in \mathbb{N}$ and $\lambda \in \Lambda_0$ define

$$\Pi_{N,\lambda}(L) = \{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi_{N,\lambda}^a = \left\{ L'' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L''_0 \subset \varepsilon^N L_0, \dim \left(\frac{\varepsilon^{-N} L_0}{L''_0} \right) = a \right\},$$

for each $a \in \mathbb{N}$.

Lemma 5.3.3. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$ and $\lambda \in \Lambda_0$,*

$$\Pi_{N,\lambda}(L) = \bigcup_{a: 0 \leq a \leq 2Nr} \Pi_{N,\lambda}^a(L)$$

Proof. If $L' \in \Pi_{N,\lambda}(L)$ then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, so the $\mathbf{k}[\varepsilon]$ -module $\varepsilon^{-N} L_0 / L'_0$ is isomorphic to $(\varepsilon^{-N} L_0 / \varepsilon^N L_0) / (L'_0 / \varepsilon^N L_0)$, which is a \mathbf{k} -vector space of dimension at most $2Nr = \dim(\varepsilon^{-N} L_0 / \varepsilon^N L_0)$. \square

Lemma 5.3.4. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \leq a \leq 2Nr$, $\Pi_{N,\lambda}^a(L)$ is a projective algebraic variety.*

Proof. Let W be the \mathbf{k} -vector space $\varepsilon^{-(1+N)} L_0 / \varepsilon^N L_0$. Thus W is a $(2N+1)r$ -dimensional vector space over \mathbf{k} and the action of ε on W is a nilpotent linear operator with $\varepsilon^{2N+1} W = 0$. Then $\Pi_{N,\lambda}^a(L)$ is in natural bijection with the space of flags $(0 \leq W_1 \leq \dots \leq W_n \leq W)$ of subspaces of W with type (λ, a) satisfying the closed condition $\varepsilon W_n \subset W_1$. The condition $\varepsilon W_n \subset W_1$ ensures that each subspace $W_i \subset W$ is a $\mathbf{k}[\varepsilon]$ -submodule of W , since $\varepsilon W_i \subset \varepsilon W_n \subset W_1 \subset W_i$. Thus each W_i lifts to give a $\mathbf{k}[\varepsilon]$ -submodule L_i of V with $\varepsilon^N L_0 \subset L_i \subset \varepsilon^{-N} L_0$, thus L_i is a lattice in V . \square

Remark 1. *It should be made clear why $\varepsilon W_n \subset W_1$ is a closed condition in the variety of flags in W . That flag varieties are projective algebraic varieties should be considered an ‘elementary’ result not needing a reference or further explanation here, though a short discussion will be found in the early sections of the thesis.*

Lemma 5.3.5. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \leq a \leq 2Nr$, $\Pi_{N,\lambda}^a(L)$ is a closed subspace of $\Pi_{N+1,\lambda}^{a+r}(L)$.*

Proof. If $L' \in \Pi_{N+1,\lambda}^{a+r}(L)$, then $\varepsilon^{N+1} L_0 \subset \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0 \subset \varepsilon^{-(N+1)} L_0$ and

$$\dim \left(\frac{\varepsilon^{-(1+n)} L_0}{L'_0} \right) = \dim \left(\frac{L_0}{\varepsilon L_0} \right) + \dim \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right) = r + a,$$

which shows that $\Pi_{N,\lambda}^a(L) \subset \Pi_{N+1,\lambda}^{a+r}(L)$. For $L' \in \Pi_{N+1,\lambda}^{a+r}(L)$, if additionally $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, then

$$\dim \left(\frac{\varepsilon^{-(N+1)} L_0}{L'_0} \right) = r + \dim \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right),$$

which shows $L' \in \Pi_{N,\lambda}^a(L)$. Thus $\Pi_{N,\lambda}^a(L)$ is the subspace of $\Pi_{N+1,\lambda}^{a+r}(L)$ defined by the closed condition $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$. \square

$$\begin{aligned}
\Pi_{N,\lambda}^a(L) &= \{L' \in \Pi_{N+1,\lambda}^{a+r}(L) : \varepsilon L' \in \Pi_{N+1,\lambda}^{a+2r}(L), \varepsilon^{-1} L' \in \Pi_{N+1,\lambda}^a(L)\} \\
&= \varepsilon \Pi_{N+1,\lambda}^a(L) \cap \varepsilon^{-1} \Pi_{N+1,\lambda}^{a+2r}(L) \\
&= \Pi_{N+1,\lambda}^a(\varepsilon L) \cap \Pi_{N+1,\lambda}^{a+2r}(\varepsilon^{-1} L)
\end{aligned}$$

Conjecture 2. If $L, \tilde{L} \in \mathcal{F}$, $N_1, N_2 \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a_1, a_2 \in \mathbb{N}$ with $0 \leq a_1 \leq 2N_1r$ and $0 \leq a_2 \leq 2N_2r$ such that $\Pi_{N_1,\lambda}^{a_1}(L) \subset \Pi_{N_2,\lambda}^{a_2}(\tilde{L})$, then $\Pi_{N_1,\lambda}^{a_1}(L)$ is a closed subspace of $\Pi_{N_2,\lambda}^{a_2}(\tilde{L})$.

Thus the subsets $\Pi_{N,\lambda}^a(L)$ of \mathcal{F} are partially ordered by inclusion, the inclusions are closed and the set theoretic direct limit is the set \mathcal{F} , so \mathcal{F} may be endowed with the direct limit topology of a system of projective algebraic varieties. If this conjecture is true, then subsets of \mathcal{F} may be given the subspace topology without reference to an embedding in $\Pi_{N,\lambda}^a(L)$.

5.3.2 Geometry of orbits

Lemma 5.3.6. Given $A \in \Lambda_1$, there is $N \in \mathbb{N}$ such that for each $(L, L') \in \mathcal{O}_A$, $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$. In this case

$$\dim \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right) = d_{nN,0}(A).$$

Proof. Given $A \in \Lambda_1$, there is $N \in \mathbb{N}$ so that $a_{i,j} = 0$ whenever $|j - i| > nN$. If $(L, L') \in \mathcal{O}_A$ then

$$\dim \left(\frac{L'_0}{L'_0 \cap \varepsilon^{-N} L_0} \right) = \dim \left(\frac{L'_0}{L'_0 \cap L_{nN}} \right) = \sum_{s > nN, t \leq 0} a_{s,t} = 0,$$

so it follows $L'_0 \subset \varepsilon^{-N} L_0$. Similarly,

$$\dim \left(\frac{\varepsilon^N L_0}{\varepsilon^N L_0 \cap L'_0} \right) = \dim \left(\frac{L_{-nN}}{L_{-nN} \cap L'_0} \right) = \sum_{s \leq -nN, t > 0} a_{s,t} = 0,$$

which shows $\varepsilon^N L_0 \subset L'_0$. Moreover,

$$\dim \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right) = \dim \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N} L_0 \cap L'_0} \right) = \sum_{s \leq nN, t > 0} a_{s,t} = d_{nN,0}(A),$$

as a result of Lemma 5.2.4. □

Lemma 5.3.7. Given $A \in \Lambda_1$ and $L \in \mathcal{F}_{\text{ro } A}$, suppose $N \in \mathbb{N}$ so that $X_A^L \subset \Pi_{N,\text{co } A}^a(L)$. Then X_A^L is a locally-closed subspace of $\Pi_{N,\text{co } A}^a(L)$, where $a = d_{nN,0}(A)$.

Proof. Lemma 5.3.6 shows that there is $N \in \mathbb{N}$ such that $X_A^L \subset \Pi_{N,\text{co } A}^a(L)$, where $a = d_{nN,0}(A)$. Then X_A^L is the subset of $\Pi_{N,\text{co } A}^a(L)$ given by the conditions $\dim(L_i/L_i \cap L'_j) = d_{i,j}(A)$ for $i, j \in \mathbb{Z}$. For each $i, j \in \mathbb{Z}$, the condition $\dim(L_i/L_i \cap L'_j)$ determines a locally closed subset of $\Pi_{N,\text{co } A}^a(L)$ since the function $L' \mapsto \dim(L_i/L_i \cap L'_j)$ is lower semicontinuous. Both $d_{i,j}(A)$ and $\dim(L_i/L_i \cap L'_j)$ are invariant under the map $(i, j) \mapsto (i + n, j + n)$, so X_A^L is determined by these conditions for $(i, j) \in [1, n] \times \mathbb{Z}$.

There are only finitely many $(i, j) \in [1, n] \times \mathbb{Z}$ such that $a_{i,j} > 0$ and

$$a_{i,j} = d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j},$$

which shows that finitely many of these conditions determine X_A^L , so X_A^L is the intersection of finitely many locally-closed sets in $\Pi_{N,\text{co } A}^a(L)$ and is therefore locally closed. \square

Given $N \in \mathbb{N}$, define

$$H_N = \left\{ h \in G_L : h = 1 \text{ on } \frac{\varepsilon^{-(1+N)}L_0}{\varepsilon^N L_0} \right\}.$$

Explicitly, the condition $h = 1$ on $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$ means: $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)}L_0$. Observe that $H_{N+1} \subset H_N$ for $N \in \mathbb{N}$ since $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ whenever $x \in \varepsilon^{-(1+N)}L_0$.

Lemma 5.3.8. H_N is a normal subgroup in G_L , for any $N \in \mathbb{N}$.

Proof. $H_N \subset G_L$ by definition. Suppose $h, h' \in H_N$ and let $x \in \varepsilon^{-(1+N)}L_0$. $h'(x) \in \varepsilon^{-(1+N)}L_0$ as $h' \in G_L$, so $hh'(x) + \varepsilon^N L_0 = h'(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$, which shows $hh' \in H_N$. $h(x) - x \in \varepsilon^N L_0$, so $h^{-1}(x) - x = -h^{-1}(h(x) - x) \in \varepsilon^N L_0$. $h^{-1} \in H_N$, so H_N is a subgroup of G_L .

Let $g \in G_L$. $hg^{-1}(x) + \varepsilon^N L_0 = g^{-1}(x)$ as $g^{-1}(x) \in \varepsilon^{-(1+N)}L_0$, so $ghg^{-1}(x) + \varepsilon^N L_0 = gg^{-1}(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$. Thus $ghg^{-1} \in H_N$, which proves H_N is a normal subgroup in G_L . \square

The H_N form a descending chain of normal subgroups in G_L : $\cdots \subset H_1 \subset H_0 \subset G_L \subset G$.

Lemma 5.3.9. G_L/H_N is an irreducible algebraic group for any $N \in \mathbb{N}$.

Proof. See the discussion in [1][section 4]. Should be able to give an explicit presentation of G_L/H_N in terms of the block structure.

$\sigma \in G_L$ induces an automorphism $\bar{\sigma}$ of $\varepsilon^{-N}L_0/\varepsilon^N L_0$, with inverse induced by σ^{-1} . Moreover, the natural map

$$G_L/H \rightarrow GL(\varepsilon^{-N}L_0/\varepsilon^N L_0)$$

is a group homomorphism. In fact, this homomorphism is injective: if $\sigma = \tau$ on $\varepsilon^{-N}L_0/\varepsilon^N L_0$, then $\sigma\tau^{-1} = 1$ on $\varepsilon^{-N}L_0/\varepsilon^N L_0$ and so $\sigma H = \tau H$. Thus G_L/H is isomorphic to its image in $GL(\varepsilon^{-(N+1)}L_0/\varepsilon^N L_0)$. \square

Lemma 5.3.10. Given $A \in \Lambda_1$ and $L \in \mathcal{F}_{\text{co } A}$, X_A^L is an irreducible subspace of $\Pi_{N,\text{co } A}^a(L)$ for some $N, a \in \mathbb{N}$.

Proof. There exist $N, a \in \mathbb{N}$ so that $X_A^L \subset \Pi_{N,\text{co } A}^a(L)$. Then $X_A^L = G_L/H_N L'$ for any $L' \in X_A^L$, where G_L/H_N is an irreducible algebraic group acting naturally on $\Pi_{N,\text{co } A}^a(L)$, thus X_A^L is irreducible. \square

5.3.3 Geometry of orbit products

Refer to Section 3.1.1 for definitions of the spaces $X_{A,B}^L$ and $Y_{A,B}^L$.

Lemma 5.3.11. *Given $A, B \in \Lambda_1$ with $\text{ro } B = \text{co } A$ and $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_A$ and $(L', L'') \in \mathcal{O}_B$,*

$$X_{A,B}^L = G_L G_{L'} L''.$$

Proof. $X_{A,B}^L$ is the image of $Y_{A,B}^L$ under the forgetful map $(N', N'') \mapsto N''$. If $\alpha \in G_L$ and $\beta \in G_{L'}$ then $(L, \alpha L, \alpha \beta L'') \in Y_{A,B}^L$ since $(L, \alpha L') = \alpha(L, L') \in \mathcal{O}_A$ and $(\alpha L', \alpha \beta L'') = \alpha \beta (\beta^{-1} L', L'') = \alpha \beta (L', L'') \in \mathcal{O}_B$. Consequently, $G_L G_{L'} L'' \in X_{A,B}^L$.

For the reverse inclusion, if $(N', N'') \in Y_{A,B}^L$ then $(L, N') \in \mathcal{O}_A$ and $(N', N'') \in \mathcal{O}_B$, so there exist $\sigma_1, \sigma_2 \in G$ such that $(L, N') = \sigma_1(L, L')$ and $(N', N'') \in \sigma_2(N', N'')$. Then $(L, N', N'') = (L, \sigma_1 L', \sigma_1 (\sigma_1^{-1} \sigma_2) L'')$ with $\sigma_1 \in G_L$ and $\sigma_1^{-1} \sigma_2 \in G_{L'}$. Thus $X_{A,B}^L = G_L G_{L'} L''$. \square

Lemma 5.3.12. *There is $N \in \mathbb{N}$ such that $H_N \subset G_{L'}$. Consequently, $H_{N'} \subset G_{L'}$ whenever $N' \geq N$.*

Proof. Choose $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$. Then

$$\varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset \cdots \subset L'_n \subset \varepsilon^{-(1+N)} L_0.$$

Let $h \in H_N$. $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)} L_0$, so $h(L'_i) \subset L'_i$ for $i = 0, 1, \dots, n$. Moreover, h^{-1} stabilises L'_i , so $h(L'_i) = L'_i$ for $i = 0, 1, \dots, n$ and therefore for $i \in \mathbb{Z}$. This shows $h \in G_{L'}$ as required, so $H_N \subset G_{L'}$. \square

Note that H_N is generally not a normal subgroup of $G_{L'}$, though the space of (right) cosets of H_N in $G_{L'}$ will still be irreducible.

Example 1. *Give an example here to illustrate that H_N^L need not be a normal subgroup in $G_{L'}$ for $L' \neq L$.*

Lemma 5.3.13. *$G_{L'}/H_N$ is irreducible, provided $H_N \subset G_{L'}$.*

Proof. Needs a proof. \square

Lemma 5.3.14. *Given $A, B \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $L \in \mathcal{F}_{\text{ro } A}$, $X_{A,B}^L$ is an irreducible topological space.*

Proof. Write up this proof properly - this is only a sketch. There is $N \in \mathbb{N}$ sufficiently large that $X_{A,B}^L$ is contained in $\Pi_{N, \text{co } B}$, using Lemma 5.3.1. Suppose $(L, L') \in \mathcal{O}_A$, then $X_{A,B}^L = G_L X_B^{L'}$. G_L acts on $\Pi_{N, \lambda}$ through a quotient G_L/H which is an irreducible algebraic group, as a result of Lemma 5.3.9. $X_B^{L'}$ is an irreducible subspace of $\Pi_{N, \lambda}$. $X_{A,B}^L$ is the image of an irreducible subspace of $\Pi_{N, \lambda}$ under the action of a connected algebraic group, so $X_{A,B}^L$ is irreducible. \square

Proposition 5.3.15. *Given $A, B \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $L \in \mathcal{F}_{\text{ro } A}$, there is a unique open G_L -orbit in $X_{A,B}^L$.*

Proof. $X_{A,B}^L$ consists of finitely many G_L -orbits and is an irreducible topological space, by Lemma 5.3.14. Consequently, X_C^L is dense in $X_{A,B}^L$ for some $C \in \Lambda_{1A,B}$. Lemma 5.3.7 shows that X_C^L is locally closed in $X_{A,B}^L$, so X_C^L is open in $\overline{X_C^L} = X_{A,B}^L$. Irreducibility of $X_{A,B}^L$ shows that there is a unique open G_L -orbit, since two non-empty open sets in $X_{A,B}^L$ intersect non-trivially, thus any two open G_L orbits in $X_{A,B}^L$ coincide. \square

5.4 Existence of a maximum

Lemma 5.4.1. *Given $A, A' \in \Lambda_1$ with $\text{ro } A = \text{ro } A'$ and $\text{co } A = \text{co } A'$, $A' \leq A$ if and only if $X_{A'}^L \subset \overline{X_A^L}$ for any $L \in \mathcal{F}_{\text{ro } A}$.*

Proof. Needs a proof. □

Proposition 5.4.2. *Given $A, B \in \Lambda_1$ with $\text{co } A = \text{ro } B$, $\Lambda_{1A,B}$ has a maximum element.*

Proof. Let $L \in \mathcal{F}_{\text{ro } A}$. $X_{A,B}^L$ is irreducible by Lemma 5.3.14 and is the union of finitely many G_L -orbits, namely

$$X_{A,B}^L = \bigcup_{C \in \Lambda_{1A,B}} X_C^L.$$

This shows that X_C^L is dense in $X_{A,B}^L$ for some $C \in \Lambda_{1A,B}$. Lemma 5.3.7 shows that the G_L -orbits in $X_{A,B}^L$ are locally closed, so a dense G_L -orbit is open in $X_{A,B}^L$. Lemma 5.4.1 shows that the characteristic matrix of the dense G_L -orbit is a maximum in $\Lambda_{1A,B}$. □

5.5 Associativity

Refer to Section 3.1.2 for definitions of the spaces $X_{A,B,C}^L$ and $Y_{A,B,C}^L$. Recall that $X_{A,B,C}^L$ is the image of $Y_{A,B,C}^L$ under the forgetful map f , given by $f(L', L'', L''') = L'''$ for each $(L', L'', L''') \in Y_{A,B,C}^L$.

Lemma 5.5.1. *Given $A, B, C \in \Lambda_1$ with $\text{ro } C = \text{co } B$, $\text{ro } B = \text{co } A$ and a tuple of flags $(L, L', L'', L''') \in \mathcal{F}^4$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L'', L''') \in \mathcal{O}_C$,*

$$X_{A,B,C}^L = G_L G_{L'} G_{L''} L'''.$$

Proof. Given $\alpha \in G_L$, $\beta \in G_{L'}$ and $\gamma \in G_{L''}$, $(L, \alpha L', \alpha \beta L'', \alpha \beta \gamma L''') \in Y_{A,B,C}^L$ since $(L, \alpha L') = \alpha(L, L') \in \mathcal{O}_A$, $(\alpha L', \alpha \beta L'') = \alpha \beta(L', L'') \in \mathcal{O}_B$ and $(\alpha \beta L'', \alpha \beta \gamma L''') = \alpha \beta \gamma(L'', L''') \in \mathcal{O}_C$. This shows $G_L G_{L'} G_{L''} L''' \in X_{A,B,C}^L$.

Given $(N', N'', N''') \in Y_{A,B,C}^L$, there exist $\sigma_1, \sigma_2, \sigma_3 \in G$ such that $(L, N') = \sigma_1(L, L')$, $(N', N'') = \sigma_2(L', L'')$ and $(N'', N''') = \sigma_3(L'', L''')$; then $N' = \sigma_1 L' = \sigma_2 L'$, $N'' = \sigma_2 L'' = \sigma_3 L''$ and $N''' = \sigma_3 L'''$. Thus

$$(L, N', N'', N''') = (L, \sigma_1 L', \sigma_1(\sigma_1^{-1} \sigma_2) L'', \sigma_1(\sigma_1^{-1} \sigma_2)(\sigma_2^{-1} \sigma_3) L''')$$

where $\sigma_1 \in G_L$, $\sigma_1^{-1} \sigma_2 \in G_{L'}$ and $\sigma_2^{-1} \sigma_3 \in G_{L''}$. □

Lemma 5.5.2. *Given $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$ and $L \in \mathcal{F}_{\text{ro } A}$, $Y_{A,B,C}^L$ is an irreducible topological space.*

Proof. Needs a proof. This is still a sticking point I think. □

Corollary 5.5.3. *Given $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$ and $L \in \mathcal{F}_{\text{ro } A}$, $X_{A,B,C}^L$ is an irreducible topological space.*

Proof. $X_{A,B,C}^L$ is the image of $Y_{A,B,C}^L$ under the forgetful map f and $Y_{A,B,C}^L$ is irreducible, by Lemma 5.5.2, so $X_{A,B,C}^L$ is irreducible. \square

Lemma 5.5.4. *Given matrices $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$ and $L \in \mathcal{F}_{\text{ro } A}$, there is a unique open G_L -orbit in $X_{A,B,C}^L$.*

Proof. $X_{A,B,C}^L$ is irreducible, by Corollary 5.5.3, and consists of finitely many G_L -orbits, so contains a dense G_L -orbit. In particular, there is $D \in \Lambda_1$ such that $\overline{X_D^L} = X_{A,B,C}^L$. Lemma 5.3.7 shows that the G_L -orbits are locally closed in $X_{A,B,C}^L$. In particular, X_D^L is open in $\overline{X_D^L} = X_{A,B,C}^L$. Therefore, there is an open G_L -orbit in $X_{A,B,C}^L$. There is a unique open G_L -orbit since $X_{A,B,C}^L$ is irreducible. \square

Definition 5.5.1. *Given $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$ and $L \in \mathcal{F}_{\text{ro } A}$, define spaces*

$$\begin{aligned}\tilde{Y}_{(AB)C}^L &= f^{-1}X_{(A*B)*C}^L \\ \tilde{Y}_{A(BC)}^L &= f^{-1}X_{A*(B*C)}^L.\end{aligned}$$

Lemma 5.5.5. *Given $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$ and $L \in \mathcal{F}_{\text{ro } A}$, $\tilde{Y}_{(AB)C}^L$ is open in $Y_{A,B,C}^L$.*

Proof.

$$f^{-1}X_{A*B,C}^L = \left\{ (L', L'', L''') \in Y_{A,B,C}^L : \dim \left(\frac{L_i}{L_i \cap L_j''} \right) \text{ is maximal, for each } i, j \in \mathbb{Z} \right\}$$

is open in $Y_{A,B,C}^L$ since $f^{-1}X_{A*B,C}^L$ is given by **finitely many** open conditions; the function on $X_{A,B}^L$ given by $L'' \mapsto \dim \left(\frac{L_i}{L_i \cap L_j''} \right)$ is lower semicontinuous, so maximising such a function is an open condition in $X_{A,B}^L$. \square

Lemma 5.5.6. *Given $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$ and $L \in \mathcal{F}_{\text{ro } A}$, $\tilde{Y}_{A(BC)}^L$ is open in $Y_{A,B,C}^L$.*

Proof.

$$f^{-1}X_{A,B*C}^L = \left\{ (L', L'', L''') \in Y_{A,B,C}^L : \dim \left(\frac{L'_i}{L'_i \cap L_j'''} \right) \text{ is maximal, for each } i, j \in \mathbb{Z} \right\}$$

is open in $Y_{A,B,C}^L$, as it is determined by **finitely many** open conditions; the function on $X_A^L \times X_{A,B*C}^L$ given by $(L', L''') \mapsto \dim \left(\frac{L'_i}{L'_i \cap L_j'''} \right)$ is lower semicontinuous, so maximising such a function is an open condition in $X_A^L \times X_{A,B*C}^L$. \square

Conjecture 3. *Given $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$ and $L \in \mathcal{F}_{\text{ro } A}$, $X_{A*B,C}^L$ and $X_{A,B*C}^L$ are open and dense in $X_{A,B,C}^L$.*

Remark 2. *If f is shown to be an open map then this result follows from Lemma 5.5.5 and Lemma 5.5.6.*

Proposition 5.5.7. *Given $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$, $(A * B) * C = A * (B * C)$.*

Proof. Take $A, B, C \in \Lambda_1$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$ and fix $L \in \mathcal{F}_{\text{ro } A}$.

$X_{(A*B)*C}^L$ is open in $X_{A*B,C}^L$, so $f^{-1}X_{(A*B)*C}^L$ is open in $f^{-1}X_{A*B,C}^L$. Lemma 5.5.5 shows that $f^{-1}X_{A*B,C}^L$ is open in $Y_{A,B,C}^L$, so $f^{-1}X_{(A*B)*C}^L$ is open in $Y_{A,B,C}^L$. Similarly, $X_{A*(B*C)}^L$ is open in $X_{A,B*C}^L$ and $f^{-1}X_{A,B*C}^L$ is open in $Y_{A,B,C}^L$, by Lemma 5.5.6, so $f^{-1}X_{A*(B*C)}^L$ is open in $Y_{A,B,C}^L$.

Lemma 5.5.2 shows that $Y_{A,B,C}^L$ is irreducible, so $f^{-1}X_{(A*B)*C}^L$ and $f^{-1}X_{A*(B*C)}^L$ have nonempty intersection. Therefore the G_L -orbits $X_{(A*B)*C}^L$ and $X_{A*(B*C)}^L$ intersect nontrivially, so are the same G_L -orbit. Thus $(A * B) * C = A * (B * C)$. \square

5.6 The generic algebra

Lemma 5.6.1. *Given $\lambda \in \Lambda_0$ and $A \in \Lambda_1$, $D_\lambda * A = A$ if $\text{ro } A = \lambda$ and $A * D_\lambda = A$ if $\text{co } A = \lambda$.*

Proof. Lemma 3.1.7 shows that $\Lambda_{1D_\lambda,A} = \{A\}$ if $\lambda = \text{ro } A$ and $\Lambda_{1A,D_\lambda} = \{A\}$ if $\lambda = \text{co } A$, which proves the result. \square

Theorem 5.6.2. *The following constitutes a small category: the set of objects is Λ_0 and the set of morphisms is Λ_1 . Given compositions $\lambda, \mu \in \Lambda_0$, the morphisms with source λ and target μ are those matrices $A \in \Lambda_1$ with $\text{co } A = \lambda$ and $\text{ro } A = \mu$. Given $\lambda, \mu, \nu \in \Lambda_0$ and $A, B \in \Lambda_1$ with $B: \lambda \rightarrow \mu$ and $A: \mu \rightarrow \nu$ the composition is $A * B: \lambda \rightarrow \nu$.*

Proof. Proposition 5.4.2 shows that the composition is well defined while Proposition 5.5.7 establishes associativity of the composition. Lemma 5.6.1 shows that $D_\lambda: \lambda \rightarrow \lambda$ is the identity morphism for each $\lambda \in \Lambda_0$. Thus $(\Lambda_0, \Lambda_1, \text{co}, \text{ro}, *)$ is a category. \square

Write $\mathcal{G}(n, r)$ to denote this so-called ‘generic category’.

Example 2. *The objects in $\mathcal{G}(2, 2)$ are compositions of 2 into 2 parts, namely $(0, 2)$, $(1, 1)$ and $(2, 0)$. The set of morphisms from λ to μ is the set of infinite periodic matrices $A \in \Lambda_1[2, 2]$ with $\text{co } A = \lambda$ and $\text{ro } A = \mu$, which is a countably infinite set for any pair of compositions $\lambda, \mu \in \Lambda_0[2, 2]$.*

Definition 5.6.1 (Generic algebra). *The generic affine algebra $\hat{G}(n, r)$ is the category \mathbb{Z} -algebra of $\mathcal{G}(n, r)$. In particular, $\hat{G}(n, r)$ is a free \mathbb{Z} -module with basis $\{e_A : A \in \Lambda_1\}$ and with associative multiplication given by*

$$e_A * e_B = \begin{cases} e_{A*B} & \text{if } \text{co } A = \text{ro } B \\ 0 & \text{if } \text{co } A \neq \text{ro } B. \end{cases}$$

The multiplicative identity in $\hat{G}(n, r)$ is

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

Chapter 6

A realisation of affine zero Schur algebras

We aim to prove the isomorphism theorem in the cases $r < n$ and $n \leq r < 2n$ separately. Below are crude versions of the statements we want to prove.

Theorem 6.0.1. *Assume $r < n$. The map $\psi: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$, given by $\psi(E_i) = E_i$, $\psi(F_i) = F_i$ and $\psi(1_\lambda) = 1_\lambda$, is an isomorphism of \mathbb{Z} -algebras.*

Theorem 6.0.2. *Assume $n \leq r < 2n$. There is a unique homomorphism of \mathbb{Z} -algebras $\hat{\psi}: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$ such that $\hat{\psi}(R) = R$ and $\hat{\psi} = \psi$ on the subalgebra of $\hat{G}(n, r)$ generated by the E_i , F_i and 1_λ . $\hat{\psi}$ is an isomorphism of \mathbb{Z} -algebras.*

6.1 Preliminary results

Recall from Definition 5.6.1 that the generic algebra $\hat{G}(n, r)$ is an associative \mathbb{Z} -algebra which is a free \mathbb{Z} -module with an atomic basis $\{e_A : A \in \Lambda_1\}$: given $A, B \in \Lambda_1$ with $\text{co } A = \text{ro } B$, $e_A e_B = e_{A*B}$.

6.1.1 Elementary basis elements

Given $i \in [1, n]$ and $\lambda \in \Lambda_0$ such that $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and

$$E_i = \sum_{\lambda \in \Lambda_0: \lambda_{i+1} > 0} E_{i,\lambda}.$$

Given $i \in [1, n]$ and $\lambda \in \Lambda_0$ such that $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and

$$F_i = \sum_{\lambda \in \Lambda_0: \lambda_i > 0} F_{i,\lambda}.$$

6.1.2 Transpose involution

Lemma 6.1.1. *The \mathbb{Z} -module automorphism \top of $\hat{G}(n, r)$ given by $e_A \mapsto e_{A^\top}$ is a \mathbb{Z} -algebra antihomomorphism: that is,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A$$

for each $A, B \in \Lambda_1$. Moreover, $\top(E_{i,\lambda}) = F_{i,\lambda+\alpha_i}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}$ and $\top(1_\lambda) = 1_\lambda$, for permissible $(i, \lambda) \in \mathbb{Z} \times \Lambda_0$.

Proof. This is a consequence of Lemma 4.1.1. [It must also be shown that the transpose operation on \$\Lambda_1\$ is order preserving.](#) \square

6.1.3 Multiplication rules

Lemma 6.1.2. *Given $A \in \Lambda_1$ and $i \in [1, n]$ such that $\text{ro } A_{i+1} > 0$,*

$$E_i e_A = e_{A+\varepsilon_{i,p}-\varepsilon_{i+1,p}},$$

where $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$.

Given $A \in \Lambda_1$ and $i \in [1, n]$ such that $\text{ro } A_i > 0$,

$$F_i e_A = e_{A+\varepsilon_{i+1,p}-\varepsilon_{i,p}},$$

where $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$.

Similar formulas for right multiplication by E_i and F_i are obtained by applying the transpose.

6.2 Presentation of the generic algebra.

Recall that Λ_0 denotes the set of compositions of r into n parts. That is, Λ_0 is the set of tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with each λ_i non-negative and $\lambda_1 + \dots + \lambda_n = r$. Given $i \in [1, n]$, let $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$ be the i -th elementary vector and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then given $\lambda \in \Lambda_0$, we have $\lambda + \alpha_i \in \Lambda_0$ provided $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ provided $\lambda_i > 0$.

Let $\Gamma = \Gamma(n, r)$ be the quiver with set of vertices Λ_0 with arrows $e_{i,\lambda}: \lambda \rightarrow \lambda + \alpha_i$ (if $\lambda_{i+1} > 0$) and $f_{i,\lambda}: \lambda \rightarrow \lambda - \alpha_i$ (if $\lambda_i > 0$). Thus there are no arrows between λ and μ unless $\lambda = \mu \pm \alpha_i$ for some $i \in [1, n]$.

If $n \geq 3$ then neighbouring vertices are connected by two arrows, one of each direction. In the case $n = 2$, neighbouring vertices are joined by four arrows, two of each direction. The $\mathbb{Z}\Gamma$ denote the path \mathbb{Z} algebra of Γ . By construction of Γ , there is a \mathbb{Z} -algebra homomorphism $\mathbb{Z}\Gamma \rightarrow \hat{G}(n, r)$ with $e_{i,\lambda} \mapsto E_{i,\lambda}$, $f_{i,\lambda} \mapsto F_{i,\lambda}$ and $k_\lambda = 1_\lambda$. We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [1]).

Definition 6.2.1. *(aperiodicity) $A \in \Lambda_1$ is aperiodic if for each $l \in \mathbb{Z} \setminus \{0\}$ there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$. Denote the set of aperiodic elements in Λ_1 by Λ_1^{ap} . Note that $\Lambda_1^{\text{ap}} = \Lambda_1$ if $r < n$. Linear combinations of the basis elements corresponding to aperiodic matrices are also said to be aperiodic - if A is aperiodic, we say e_A is aperiodic.*

Lemma 6.2.1. *Let $A \in \Lambda_1$ and write $\lambda = \text{ro } A$. If A is aperiodic and $\lambda_{i+1} > 0$, then $E_i * e_A$ is aperiodic. If A is aperiodic and $\lambda_i > 0$, then $F_i * e_A$ is aperiodic.*

Proof. Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_{i+1} > 0$, where $\lambda = \text{ro } A$. There is $p \in \mathbb{Z}$ such that $a_{i+1,p} > 0$ and $a_{i+1,p'} = 0$ whenever $p' > p$. Lemma 6.1.2 shows that $E_i * e_A = e_B$, where $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p-i, p-i-1\}$, then $b_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $b_{s,s+l} = a_{s,s+l} = 0$, since A is aperiodic. If $l = p-i$, then $b_{i+1,i+1+l} = b_{i+1,p+1} = a_{i+1,p+1} = 0$, by maximality of p . If $l = p-i-1$, there is $s \neq i+1$ such that $a_{s,s+l} = 0$, since A is aperiodic and $a_{i+1,i+1+l} = a_{i+1,p} > 0$, so $b_{s,s+l} = a_{s,s+l} = 0$. Therefore, $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ is aperiodic.

Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_i > 0$, where $\lambda = \text{ro } A$. Lemma 6.1.2 shows that $F_i * e_A = e_C$ where $C = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$ and $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p-i, p-i-1\}$ then $c_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $c_{s,s+l} = a_{s,s+l} = 0$, by aperiodicity of A . If $l = p-i$, then $a_{i,i+l} = a_{i,p} > 0$, so there is $s \neq i$ such that $a_{s,s+l} = 0$. Then $c_{s,s+l} = a_{s,s+l} = 0$. Finally, if $l = p-i-1$, then $c_{i,i+l} = a_{i,p-1} = 0$ by minimality of p . Thus C is aperiodic as required. \square

Definition 6.2.2. (*Weight function*) Define the weight function $\text{wt} : \Lambda_1 \rightarrow \mathbb{Z}$ by

$$\text{wt } A = \sum_{i \in [1,n], j \in \mathbb{Z}} |j-i| a_{i,j}$$

for each $A \in \Lambda_1$. The sum is taken over a transversal of the set of congruence classes of (i, j) modulo (n, n) for $i, j \in \mathbb{Z}$.

Lemma 6.2.2. Let $A \in \Lambda_1$ and write $\lambda = \text{ro } A$. Suppose $\lambda_{i+1} > 0$ and set $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$. If $p > i$ then $\text{wt } e_{i,\lambda} * A = 1 + \text{wt } A$. If $p \leq i$ then $\text{wt } e_{i,\lambda} * A = -1 + \text{wt } A$. Suppose $\lambda_i > 0$ and set $q = \min\{q' \in \mathbb{Z} : a_{i,q'} > 0\}$. If $q \leq i$ then $\text{wt } f_{i,\lambda} * A = 1 + \text{wt } A$. If $q > i$ then $\text{wt } f_{i,\lambda} * A = -1 + \text{wt } A$.

Proof. Lemma 6.1.2 shows that $e_i A = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$, so $\text{wt } e_i A - \text{wt } A = |p-i| - |p-i-1|$, which equals 1 if $p > i$ and equals -1 if $p \leq i$. Similarly, $f_i A = A + \mathcal{E}_{i+1,q} - \mathcal{E}_{i,q}$ by Lemma 6.1.2, so $\text{wt } f_i A - \text{wt } A = |q-i-1| - |q-i|$, which equals -1 if $q > i$ and equals 1 if $q \leq i$. \square

Lemma 6.2.3. If $A \in \Lambda_1$ is aperiodic, then e_A may be obtained from $1_{\text{co } A}$ by finitely many applications of E_i and F_i for $i \in [1, n]$.

Proposition 6.2.4. The \mathbb{Z} -subalgebra of $\hat{G}(n, r)$ generated by $E_{i,\lambda}$, $F_{i,\lambda}$ and 1_λ has \mathbb{Z} -basis $\{e_A : A \in \Lambda_1^{\text{ap}}\}$, where $\Lambda_1^{\text{ap}} \subset \Lambda_1$ is the set of aperiodic elements.

Proof. \square

6.2.1 The case $n \geq 3$.

Lemma 6.2.5. The following relations hold in $\hat{G}(n, r)$ ($n \geq 3$):

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless $|j-i| = 1$.

$$E_i E_{i+1}^2 - E_{i+1} E_i E_{i+1} = 0$$

$$E_i^2 E_{i+1} - E_i E_{i+1} E_i = 0$$

$$F_{i+1} F_i^2 - F_i F_{i+1} F_i = 0$$

$$F_{i+1}^2 F_i - F_{i+1} F_i F_{i+1} = 0$$

$$E_i F_j - F_j E_i = 0$$

unless $j = i$.

$$E_i F_i - F_i E_i + \sum_{\lambda: \lambda_i=0, \lambda_{i+1}>0} 1_\lambda - \sum_{\lambda: \lambda_i>0, \lambda_{i+1}=0} 1_\lambda = 0.$$

6.2.2 The case $n = 2$.

In this case, the quiver $\Gamma(2, r)$ has vertices $\Lambda_0[2, r] = \{(0, r), (1, r-1), \dots, (r, 0)\}$; adjacent vertices are connected by two pairs of arrows with opposite orientation: (e_1, f_1) and (e_2, f_2) . The relations arising from $\hat{G}(2, r)$ are of a more complicated form - in particular, the serre relations of total degree 3 will not hold in this case - so this case will be treated separately and at a later date.

Chapter 7

Further directions

7.1 Further results on affine zero Schur algebras

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

7.2 Deformed group algebras of symmetric groups

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algebras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for S_3 and S_4 . Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: ‘these’ relations hold; these generators and relations give a presentation of the degenerate group algebras.

Terminology: deformed group algebra.

Bibliography

- [1] George Lusztig. “Aperiodicity in quantum affine \mathfrak{gl}_n ”. In: *Asian Journal of Mathematics* 3.1 (1999), pp. 147–178.