

A geometric realisation of affine 0-Schur algebras.

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January 8, 2020

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## Chapter 1

# Introduction

## Chapter 2

# Background: The double flag variety approach to $q$ -Schur algebras

### 2.1 Flag varieties as projective algebraic varieties

Include a discussion of flag varieties in a finite dimensional vector space. Explain: topology of projective space; Plücker embedding of Grassmannian in a projective space; flag varieties as a closed subset in a product of Grassmannians - show that the inclusion of one subspace into another is a closed condition - given by vanishing of some homogenous polynomials which should appear as minors of a matrix.

References for this material include [1][J. Harris: A First Course in Algebraic Geometry]; [2][D. Hudec: The Grassmannian as a Projective Variety]; [4][P. Morandi: Algebraic Groups, Grassmannians and Flag Varieties].

## Chapter 3

# The cyclic flags approach to affine $q$ -Schur algebras

Fix natural numbers  $n$  and  $r$ .

**Definition 3.0.1** (compositions). *A composition of  $r$  into  $n$  parts is an  $n$ -tuple  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$  of non-negative integers whose sum equals  $r$ . Denote the set of compositions of  $r$  into  $n$  parts by  $\Lambda_0$ .*

**Definition 3.0.2** (infinite periodic matrices). *Let  $\Lambda_1$  be the set of matrices  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$  with integer entries  $a_{i,j}$  satisfying the following conditions:*

- $a_{i,j} \geq 0$  for each  $i, j \in \mathbb{Z}$ ;
- each row or column has only finitely many non-zero entries;
- the sum of the entries in any  $n$  consecutive rows or columns equals  $r$ ;
- $a_{i-n,j-n} = a_{i,j}$  for each  $i, j \in \mathbb{Z}$ .

*These matrices are referred to as infinite periodic matrices.*

**Definition 3.0.3** (source and target). *Given  $A \in \Lambda_1$ , let  $\text{ro } A$  and  $\text{co } A$  be the compositions of  $r$  into  $n$  parts given by*

$$\text{ro } A = \left( \sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

*and*

$$\text{co } A = \left( \sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right).$$

*$A \in \Lambda_1$  is said to go from  $\text{co } A$  to  $\text{ro } A$ .*

**Definition 3.0.4** (diagonal matrices). *Given  $\lambda \in \Lambda_0$ , let  $D_\lambda \in \Lambda_1$  be the matrix given by  $(D_\lambda)_{i,j} = 0$  for  $i, j \in \mathbb{Z}$  with  $i \neq j$  and  $(D_\lambda)_{i,i} = \lambda_i$  for  $i \in \mathbb{Z}$ ; where the indices are taken modulo  $n$ .*

### 3.1 Cyclic flags

Fix  $n, r \in \mathbb{N}$  and let  $\mathbf{k}$  be a field. Let  $\mathcal{S}$  be the  $\mathbf{k}$ -algebra  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$  and let  $\mathcal{R}$  be the subalgebra generated by  $\varepsilon$ , so  $\mathcal{R} = \mathbf{k}[\varepsilon]$ . Let  $V$  be a free  $\mathcal{S}$ -module of rank  $r$ . Let  $G$  be the automorphism group of the  $\mathcal{S}$ -module  $V$ , so  $G$  is isomorphic to  $\mathrm{GL}_r(\mathcal{S})$ . A lattice in  $V$  is a  $\mathcal{R}$ -submodule  $L$  of  $V$  with  $\mathcal{S} \otimes_{\mathcal{R}} L = V$ . In particular, a lattice is an  $\mathcal{R}$ -submodule of  $V$  which is a free  $\mathcal{R}$ -module of rank  $r$ .

**Lemma 3.1.1.** *Let  $L$  be a lattice in  $V$ .  $L/\varepsilon L$  is a torsion  $\mathcal{R}$ -module, where  $\varepsilon$  acts as zero.  $L/\varepsilon L$  is a free  $\mathcal{R}/\langle \varepsilon \rangle$ -module of rank  $r$ ; that is,  $L/\varepsilon L$  is an  $r$ -dimensional  $\mathbf{k}$ -vector space.*

*Proof.*  $L$  is a free  $\mathcal{R}$ -module of rank  $r$ , with  $L \subset V$ . Given an  $\mathcal{R}$ -basis  $\{x_1, \dots, x_r\}$  of  $L$ ,  $\{\varepsilon x_1, \dots, \varepsilon x_r\}$  is an  $\mathcal{R}$ -basis of  $\varepsilon L$ . Finally, the cosets  $\{x_1 + \varepsilon L, \dots, x_r + \varepsilon L\}$  give a basis for  $L/\varepsilon L$  over  $\mathcal{R}/\langle \varepsilon \rangle \cong \mathbf{k}$ .  $\square$

Let  $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$  be the set of collections  $(L_i)_{i \in \mathbb{Z}}$  of lattices in  $V$  with  $L_i \subset L_{i+1}$  and  $\varepsilon L_i = L_{i-n}$  for each  $i \in \mathbb{Z}$ . These collections of lattices in  $V$  are referred to as cyclic flags in  $V$ .

$G$  acts on  $\mathcal{F}$  by  $(g \cdot L)_i = g(L_i)$  for each  $i \in \mathbb{Z}$ , given  $g \in G$  and  $L \in \mathcal{F}$ . The  $G$ -orbits in  $\mathcal{F}$  are indexed by the set  $\Lambda_0$  of compositions of  $r$  into  $n$  parts: the  $G$ -orbit in  $\mathcal{F}$  corresponding to  $\lambda \in \Lambda_0$  is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim \left( \frac{L_i}{L_{i-1}} \right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

**Definition 3.1.1.** *The periodic characteristic matrix of a pair of cyclic flags  $(L, L') \in \mathcal{F} \times \mathcal{F}$  is the matrix  $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$  with entries*

$$a_{i,j} = \dim_{\mathbf{k}} \left( \frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j} \right)$$

for each  $i, j \in \mathbb{Z}$ .

The diagonal action of  $G$  on  $\mathcal{F} \times \mathcal{F}$  has orbits indexed by the set  $\Lambda_1$  of infinite periodic matrices (see definition 3.0.2). The  $G$ -orbit corresponding to  $A \in \Lambda_1$  is denoted  $\mathcal{O}_A$  and consists of those pairs  $(L, L') \in \mathcal{F} \times \mathcal{F}$  with periodic characteristic matrix  $A(L, L')$  equal to  $A$ .

**Lemma 3.1.2.** *(alternative expression for characteristic matrix) Alternatively,*

$$a_{i,j} = \dim_{\mathbf{k}} \left( \frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} \right)$$

for each  $i, j \in \mathbb{Z}$ .

*Proof.* Set  $U = L_i \cap L'_j$  and  $U' = L_{i-1} + L_i \cap L'_{j-1}$ . Then  $U + U' = L_{i-1} + L_i \cap L'_j$  and  $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$ . Applying the isomorphism theorems,  $U + U'/U'$  is naturally isomorphic to  $U/U \cap U'$  as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to  $a_{i,j}$ .  $\square$

**Lemma 3.1.3** (transposing characteristic matrix). *Given a pair of flags  $(L, L') \in \mathcal{F}^2$ , the matrices  $A(L, L')$  and  $A(L', L)$  are related by the transpose. In particular,  $A(L, L')_{i,j} = A(L', L)_{j,i}$  for each  $i, j \in \mathbb{Z}$ .*

*Proof.* By swapping the roles of  $i$  and  $j$  and swapping  $L$  and  $L'$  it is clear that  $A(L, L')_{i,j}$  and  $A(L', L)_{j,i}$  are both given by the dimension of the  $\mathbf{k}$ -vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each  $i, j \in \mathbb{Z}$ . □

**Lemma 3.1.4** (a codimension formula). *Given  $(L, L') \in \mathcal{F}^2$  and  $i, j \in \mathbb{Z}$ ,*

$$\dim_{\mathbf{k}} \left( \frac{L_i}{L_i \cap L'_j} \right) = \sum_{s \leq i, t > j} a_{s,t},$$

where  $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$ .

*Proof.* **COMPLETE THIS PROOF** □

**Lemma 3.1.5** (nested flags). *Given  $(L, L') \in \mathcal{F}^2$ ,  $L' \subset L$  if and only if  $A(L, L')_{i,j} = 0$  for  $i, j \in \mathbb{Z}$  with  $i > j$ .*

*Proof.* Suppose  $L, L' \in \mathcal{F}$  with  $L' \subset L$ , meaning  $L'_j \subset L_j$  for each  $j \in \mathbb{Z}$ . Then for  $i > j$ ,  $L_i \cap L'_j = L'_j$ ,  $L_{i-1} \cap L'_j = L'_j$  and  $L_i \cap L'_{j-1}$ , which shows

$$A(L, L')_{i,j} = \dim_{\mathbf{k}} \left( \frac{L'_j}{L'_{j-1} + L'_j} \right) = 0$$

as required. Conversely, suppose  $A(L, L')$  is upper triangular, meaning  $A(L, L')_{i,j} = 0$  when  $i > j$ . Using Lemma 3.1.4,

$$\dim_{\mathbf{k}} \left( \frac{L'_i}{L'_i \cap L_i} \right) = \sum_{s > i, t \leq i} a_{s,t} = 0,$$

so  $L_i \cap L'_i = L'_i$  and thus  $L'_i \subset L_i$  for each  $i \in \mathbb{Z}$ , as required. □

**Corollary 3.1.6** (diagonal orbits). *Given  $L, L' \in \mathcal{F}$ ,  $L = L'$  if and only if  $A(L, L')_{i,j} = 0$  whenever  $i \neq j$ . In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L, L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each  $\lambda \in \Lambda_0$ .

### 3.1.1 A product on orbits

Given  $A, B \in \Lambda_1$  with  $\text{co } A = \text{ro } B$ , define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also  $L \in \mathcal{F}_{\text{ro } A}$ , define the  $L$ -slices of  $Y_{A,B}$  and  $X_{A,B}$  respectively as

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X_{A,B}^L = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$



**Observation 1.** *There are only finitely many  $G$ -orbits in  $X_{A,B}$ .*

**Lemma 3.1.7.** *Given  $A \in \Lambda_1$ ,  $X_{D_\lambda, A} = \mathcal{O}_A$  if  $\lambda = \text{ro } A$  and  $X_{A, D_\lambda} = \mathcal{O}_A$  if  $\lambda = \text{co } A$ .*

*Proof.* Let  $A \in \Lambda_1$  and set  $\lambda = \text{ro } A$ .  $Y_{D_\lambda, A}$  is the set of triples  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_{D_\lambda}$ , thus  $L = L'$  by Corollary 3.1.6, and  $(L', L'') \in \mathcal{O}_A$ .  $X_{D_\lambda, A}$  is the projection of  $Y_{D_\lambda, A}$ , which equals  $\mathcal{O}_A$ .

Similarly, if  $\lambda = \text{co } A$ ,  $Y_{A, D_\lambda}$  is the set of triples  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_A$  and  $L'' = L'$ , so  $X_{A, D_\lambda}$  is exactly the orbit  $\mathcal{O}_B$ .  $\square$

### 3.1.2 Triple products

Given  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ , there are spaces  $X_{A,B,C}$ ,  $Y_{A,B,C}$  and their respective  $L$ -slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y_{A,B,C}^L = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X_{A,B,C}^L = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

## 3.2 Convolution algebras

Suppose  $\mathbf{k}$  is a finite field and let  $q$  denote the number of elements of  $\mathbf{k}$ . Consider the set  $S$  of  $G$ -invariant functions  $\mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Z}$  with constructible support.  $S$  is a free  $\mathbb{Z}$ -module with a basis consisting of the indicator functions of the  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$ . Define an operation  $\star$  on  $S$  as follows: for each  $f, g \in S$ ,  $f \star g \in S$  is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ .

$f \star g$  is well defined since the supports of  $f$  and  $g$  consist of finitely many  $G$ -orbits, so there are only finitely many  $L' \in \mathcal{F}$  such that  $f(L, L')g(L', L'') \neq 0$ , given  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ .  $f \star g$  is constant on  $G$ -orbits and is supported on finitely many  $G$ -orbits, so  $f \star g \in S$ .

**Lemma 3.2.1.** *The set  $S$  together with the operation  $\star$  is an associative  $\mathbb{Z}$ -algebra with identity element  $\iota$  given by  $\iota(L, L) = 1$  and  $\iota(L, L') = 0$  for  $L' \neq L$ .*

*Proof.* Given  $f, g, h \in S$  and  $(L, L''') \in \mathcal{F} \times \mathcal{F}$ ,

$$\begin{aligned} ((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''), \end{aligned}$$

thus  $\star$  is associative.  $\iota$  is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L') \iota(L', L'') = f(L, L''),$$

for each  $f \in S$  and  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ . □

Given  $A \in \Lambda_1$ , let  $e_A \in S$  denote the indicator function of the orbit  $\mathcal{O}_A$ .  $S$  is a free  $\mathbb{Z}$ -module with basis  $\{e_A : A \in \Lambda_1\}$ . There exist  $\gamma_{A,B,C;q} \in \mathbb{Z}$  for  $A, B, C \in \Lambda_1$  such that

$$e_A \star e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C;q} e_C$$

for each  $A, B \in \Lambda_1$ . Then

$$\begin{aligned} \gamma_{A,B,C;q} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L') e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}, \end{aligned}$$

for any  $(L, L'') \in \mathcal{O}_C$ .

### 3.3 Affine $q$ -Schur algebras

There exist polynomials  $\gamma_{A,B,C} \in \mathbb{Z}[q]$  for  $A, B, C \in \Lambda_1$  such that  $\gamma_{A,B,C}(q) = \gamma_{A,B,C;q}$  for any prime power  $q$ , following [3, section 4]. The affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  (defined in [\[ADD A REFERENCE\]](#)) is a  $\mathbb{Z}[q]$ -algebra which is a free  $\mathbb{Z}[q]$ -module with basis  $\{e_A : A \in \Lambda_1\}$  and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these ‘universal polynomials’  $\gamma_{A,B,C} \in \mathbb{Z}[q]$ , it follows from Lemma 3.2.1 that  $\hat{S}_q(n, r)$  is an associative  $\mathbb{Z}[q]$ -algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

## Chapter 4

# Quivers with relations for affine q-Schur algebras

### 4.1 Basic results and notation

#### 4.1.1 Elementary matrices

For each  $i, j \in \mathbb{Z}$ , let  $\mathcal{E}_{i,j}$  be the  $\mathbb{Z} \times \mathbb{Z}$  ‘elementary periodic matrix’ with entries given by  $(\mathcal{E}_{i,j})_{s,t} = 1$  if  $(s, t) = (i + cn, j + cn)$  for some  $c \in \mathbb{Z}$  and  $(\mathcal{E}_{i,j})_{s,t} = 0$  otherwise. Clearly  $\mathcal{E}_{i,j} = \mathcal{E}_{i+n,j+n}$  for each  $i, j \in \mathbb{Z}$ . Recall from Definition 3.0.4 that the diagonal matrix associated to a composition  $\lambda \in \Lambda_0$  is

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}.$$

$\{e_{D_\lambda} : \lambda \in \Lambda_0\}$  is a set of pairwise orthogonal idempotents in  $\hat{S}_q(n, r)$  with  $\sum_{\lambda \in \Lambda_0} e_{D_\lambda} = 1$ , as a result of Lemma 3.1.7.

Given  $i \in [1, n]$  and  $\lambda \in \Lambda_0$  with  $\lambda_{i+1} > 0$ , define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and define

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}.$$

Given  $i \in [1, n]$  and  $\lambda \in \Lambda_0$  with  $\lambda_i > 0$ , define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and define

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}$$

#### 4.1.2 Transpose involution

**Lemma 4.1.1.** *Transposition gives a homomorphism of  $\mathbb{Z}[q]$ -modules  $\top : \hat{S}_q(n, r) \rightarrow \hat{S}_q(n, r)$  with  $\top(e_A) = e_{A^\top}$ ,  $\top \circ \top = 1$  and  $\top(e_A e_B) = \top(e_B) \top(e_A)$ .*

*Proof.* Let  $A, B, C \in \Lambda_1$  and let  $\mathbf{k}$  be a finite field with  $q = \#\mathbf{k}$  elements. If  $(L, L'') \in \mathcal{O}_C$  then  $(L'', L) \in \mathcal{O}_{C^\top}$  and

$$\begin{aligned}\gamma_{A,B,C;q} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;q}\end{aligned}$$

It then follows that  $\top(e_A e_B) = \top(e_B) \top(e_A)$ .  $\square$

The transpose relates the  $E_i$ ,  $F_i$  and  $1_\lambda$  in the following way:  $\top(E_{i,\lambda}) = F_{i,\lambda}$ ,  $\top(F_{i,\lambda}) = E_{i,\lambda - \varepsilon_i + \varepsilon_{i+1}}$  and  $\top(1_\lambda) = 1_\lambda$ . In particular,  $\top(E_i) = F_i$  and  $\top(F_i) = E_i$ .

### 4.1.3 A multiplication rule

**Lemma 4.1.2.** *Given  $A \in \Lambda_1$  and  $i \in [1, n]$  with  $\text{ro } A_{i+1} > 0$ ,*

$$E_i e_A = \sum_{p \in \mathbb{Z}: a_{i+1,p} > 0} q^{\sum_{j > p} a_{i,j}} [[a_{i,p} + 1]] e_{A + \varepsilon_{i,p} - \varepsilon_{i+1,p}}.$$

*Given  $A \in \Lambda_1$  and  $i \in [1, n]$  with  $\text{ro } A_i > 0$ ,*

$$F_i e_A = \sum_{p \in \mathbb{Z}: a_{i,p} > 0} q^{\sum_{j < p} a_{i+1,j}} [[a_{i+1,p} + 1]] e_{A + \varepsilon_{i+1,p} - \varepsilon_{i,p}}.$$

Note that these formulas are still valid in the cases  $E_i e_A = 0$  and  $F_i e_A = 0$ , provided it is understood that  $e_B = 0$  whenever  $B \notin \Lambda_1$ . There are similar formulas for right multiplication by  $E_i$  and  $F_i$ , obtained by applying the transpose involution to the above.

**Corollary 4.1.3.** *Given  $A \in \Lambda_1$  and  $j \in [1, n]$  with  $\text{co } A_{j+1} > 0$ ,*

$$e_A F_j = \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i > p} a_{i,j}} [[a_{p,j} + 1]] e_{A + \varepsilon_{p,j} - \varepsilon_{p,j+1}}.$$

*Given  $A \in \Lambda_1$  and  $j \in [1, n]$  with  $\text{co } A_j > 0$ ,*

$$e_A E_j = \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i < p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A + \varepsilon_{p,j+1} - \varepsilon_{p,j}}.$$

*Proof.*

$$\begin{aligned}e_A F_j &= \top(E_j e_{A^\top}) \\ &= \top \left( \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i > p} a_{i,j}} [[a_{p,j} + 1]] e_{A^\top + \varepsilon_{j,p} - \varepsilon_{j+1,p}} \right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i > p} a_{i,j}} [[a_{p,j} + 1]] e_{A + \varepsilon_{p,j} - \varepsilon_{p,j+1}},\end{aligned}$$

where the second equality comes from Lemma 4.1.2. Similarly,

$$\begin{aligned}e_A E_j &= \top(F_j e_{A^\top}) \\ &= \top \left( \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i < p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A^\top + \varepsilon_{j+1,p} - \varepsilon_{j,p}} \right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i < p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A + \varepsilon_{p,j+1} - \varepsilon_{p,j}}.\end{aligned}$$

□

## 4.2 Relations

Note that  $E_i^{r+1} = F_i^{r+1} = 0$  while

$$E_i^r = [r]! e_r \mathcal{E}_{i,i+1}$$

and

$$F_i^r = [r]! e_r \mathcal{E}_{i+1,i}.$$

**Lemma 4.2.1** (quantum Serre relations:  $n \geq 3$ ). *Suppose  $n \geq 3$ . The following relations hold in  $\hat{S}_q(n, r)$ :*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless  $j = i \pm 1$ ;

$$E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i = 0$$

$$E_i^2 E_{i+1} - (1+q) E_i E_{i+1} E_i + q E_{i+1} E_i^2 = 0$$

and

$$F_{i+1} F_i^2 - (1+q) F_i F_{i+1} F_i + q F_i^2 F_{i+1} = 0$$

$$F_{i+1}^2 F_i - (1+q) F_{i+1} F_i F_{i+1} + q F_i F_{i+1}^2 = 0.$$

*Proof.* Here we introduce temporary notation for the basis elements: Write  $[A] = e_A$ .

Take  $\lambda \in \Lambda_0$ .

$$E_i E_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1} E_i E_{i+1} 1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2 E_i 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then

$$(E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i) 1_\lambda = 0,$$

for each  $\lambda \in \Lambda_0$ . The relation  $E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i = 0$  then follows.

The relations between  $F_i$  and  $F_{i+1}$  may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping  $E_i$  and  $F_i$  and reversing the order of multiplication. □

**Lemma 4.2.2** (quantum Serre relations:  $n = 2$ ). *In the case  $n = 2$ , the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

**Lemma 4.2.3.**  $[E_i, F_j] = 0$  unless  $j = i$ .

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}]) 1_\lambda.$$

For  $\lambda \in \Lambda_0$ , let  $R_\lambda = e_{\lambda_1 \varepsilon_{0,1} + \dots + \lambda_n \varepsilon_{n-1,n}}$ . Write  $R = \sum_{\lambda \in \Lambda_0} R_\lambda$ . Note  $R_\lambda = R 1_\lambda$ . Given  $A \in \Lambda_1$  and  $m \in \mathbb{Z}$ , let  $A[m] \in \Lambda_1$  be given by  $A[m]_{i,j} = a_{i,j+m}$  and let  $A^{[m]}$  be given by  $A^{[m]}_{i,j} = a_{i+m,j}$  for each  $i \in \mathbb{Z}$ .

**Lemma 4.2.4** (Shifting). *If  $A \in \Lambda_1$  then*

$$Re_A = e_{A[\pm 1]}$$

and

$$e_A R = e_{A[\pm 1]}.$$

Conjugation by  $R$  gives an automorphism  $\rho$  of  $\hat{S}_q(n, r)$  satisfying  $\rho^n = 1$ .

### 4.3 quivers with relations

Denote by  $\Lambda_0$  the set of compositions of  $r$  into  $n$  parts. That is,  $\Lambda_0$  is the set of  $\alpha \in \mathbb{Z}^n$  with non-negative entries which sum to  $r$ . Let  $\varepsilon_i \in \mathbb{Z}^n$  be the  $i$ th elementary vector and write  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for each  $i \in [1, n]$ . Then  $\lambda + \alpha_i \in \Lambda_0$  if  $\lambda_{i+1} > 0$  and  $\lambda - \alpha_i \in \Lambda_0$  if  $\lambda_i > 0$ .

Let  $\Gamma = \Gamma(n, r)$  be the quiver with set of vertices  $\Lambda_0$ , with the following arrows:

For  $\lambda \in \Lambda_0$  and  $i \in [1, n]$ , there is an arrow  $e_{i,\lambda} : \lambda \rightarrow \lambda + \alpha_i$  if  $\lambda_{i+1} > 0$  and there is an arrow  $f_{i,\lambda} : \lambda \rightarrow \lambda - \alpha_i$  if  $\lambda_i > 0$ .

Denote by  $\mathbb{Z}[q]\Gamma$  the path  $\mathbb{Z}[q]$ -algebra of  $\Gamma$ . Thus  $\mathbb{Z}[q]\Gamma$  is a free  $\mathbb{Z}[q]$ -module with a basis given by the set of paths in  $\Gamma$ , with multiplication given by the concatenation of paths. If  $p$  starts where  $q$  ends, the product  $pq$  is the path  $q$  followed by  $p$ . Write  $e_{i,\lambda} = 0$  unless  $\lambda, \lambda + \alpha_i \in \Lambda_0$  and write  $f_{i,\lambda} = 0$  unless  $\lambda, \lambda - \alpha_i \in \Lambda_0$ .

By construction, there is a homomorphism of  $\mathbb{Z}[q]$ -algebras

$$\phi : \mathbb{Z}[q]\Gamma \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda, \end{aligned}$$

for  $i \in [1, n]$  and  $\lambda \in \Lambda_0$ .

The image of  $\phi$  is the subalgebra of  $\hat{S}_q(n, r)$  generated by  $E_i, F_i$  for  $i \in [1, n]$  and  $1_\lambda$  for  $\lambda \in \Lambda_0$ , since  $E_{i,\lambda} = E_i 1_\lambda$  and  $F_{i,\lambda} = F_i 1_\lambda$ , while  $E_i = \sum_\lambda E_{i,\lambda}$  and  $F_i = \sum_\lambda F_{i,\lambda}$ . In general  $\phi$  is not surjective, so this does not always lead to a presentation of  $\hat{S}_q(n, r)$ .

#### 4.3.1 Exceptional case $n=2$ .

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the  $q$ -Schur algebra.

### 4.3.2 Typical case.

Suppose  $n \geq 3$ . Then  $\Gamma = \Gamma(n, r)$  has vertex set  $\Lambda_0$ . **RESUME HERE...**

Define  $e_i, f_i \in \mathbb{Z}[q]\Gamma(n, r)$  by

$$e_i = \sum_{\lambda \in \Lambda_0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0} f_{i,\lambda},$$

with the convention  $e_{i,\lambda} = 0$  unless  $\lambda_{i+1} > 0$  and  $f_{i,\lambda} = 0$  unless  $\lambda_i > 0$ . Let  $k_\lambda$  denote the constant path at vertex  $\lambda$ .  $\{k_\lambda : \lambda \in \Lambda_0\}$  is a set of pairwise orthogonal idempotents in  $\mathbb{Z}[q]\Gamma(n, r)$ .

Let  $I(n, r) \subset \mathbb{Z}[q]\Gamma(n, r)$  be the ideal generated by the expressions

$$\begin{aligned} & e_i e_{i+1}^2 - (1+q)e_{i+1}e_i e_{i+1} + qe_{i+1}^2 e_i \\ & e_i^2 e_{i+1} - (1+q)e_i e_{i+1} e_i + qe_{i+1} e_i^2 \\ & f_{i+1} f_i^2 - (1+q)f_i f_{i+1} f_i + qf_i^2 f_{i+1} \\ & f_{i+1}^2 f_i - (1+q)f_{i+1} f_i f_{i+1} + qf_i f_{i+1}^2 \\ & e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}]) k_\lambda \end{aligned}$$

Recall that a relation is a  $\mathbb{Z}[q]$ -linear combination of paths with common start and end vertices. The relations involving paths  $\lambda \rightarrow \mu$  are given by  $1_\mu \text{expr} 1_\lambda$ , for each of the above expressions.

**Lemma 4.3.1.** *There is a homomorphism of  $\mathbb{Z}[q]$ -algebras*

$$\phi: \mathbb{Z}[q]\Gamma(n, r)/I(n, r) \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda. \end{aligned}$$

## Chapter 5

# A generic affine algebra

### 5.1 Introducing the generic affine algebra

Assume  $\mathbf{k} = \mathbb{C}$  and fix  $n, r \geq 1$ . Let  $\mathcal{S}$  be the  $\mathbf{k}$ -algebra  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$  and let  $\mathcal{R}$  be the subalgebra generated by  $\varepsilon$ , namely  $\mathcal{R} = \mathbf{k}[\varepsilon]$ . Let  $V$  be a free  $\mathcal{S}$ -module of rank  $r$  and let  $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$  be the set of  $n$ -periodic cyclic flags in  $V$ ; so  $\mathcal{F}$  consists of collections  $L = (L_i)_{i \in \mathbb{Z}}$  of  $\mathcal{R}$ -lattices in  $V$  with  $L_i \subset L_{i+1}$  for  $i \in \mathbb{Z}$  and  $\varepsilon L_i = L_{i-n}$  for  $i \in \mathbb{Z}$ .

Let  $G$  be the group of  $\mathcal{S}$ -module automorphisms of  $V$ . Thus  $G$  is isomorphic to  $\mathrm{GL}_r(\mathcal{S})$ .  $G$  acts on  $\mathcal{F}$  with orbits  $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0\}$ , where  $\Lambda_0$  is the set of compositions of  $r$  into  $n$  parts, as in Definition 3.0.1.

The diagonal action of  $G$  on  $\mathcal{F} \times \mathcal{F}$  has orbits  $\{\mathcal{O}_A : A \in \Lambda_1\}$ , where  $\mathcal{O}_A$  consists of those pairs of flags with periodic characteristic matrix equal to  $A$ . Definitions of the periodic characteristic matrix and the set  $\Lambda_1$  are given in Definition 3.1.1 and Definition 3.0.2 respectively. In particular, the periodic characteristic matrix of a pair  $(L, L') \in \mathcal{F} \times \mathcal{F}$  is the  $\mathbb{Z} \times \mathbb{Z}$  matrix  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ , with

$$a_{i,j} = \dim \left( \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right)$$

for each  $i, j \in \mathbb{Z}$ .

Recall that  $\mathrm{ro}, \mathrm{co}: \Lambda_1 \rightarrow \Lambda_0$  are the maps given by

$$\mathrm{ro} A = \left( \sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co} A = \left( \sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right)$$

for each  $A \in \Lambda_1$ . Given  $A \in \Lambda_1$ , write  $A: \mathrm{co} A \rightarrow \mathrm{ro} A$ .

The purpose of this chapter is to define a category with objects  $\Lambda_0$  and morphisms  $\Lambda_1$ ; where  $\mathrm{Hom}(\lambda, \mu) = \{A \in \Lambda_1 : \mathrm{ro} A = \mu, \mathrm{co} A = \lambda\}$ . Given  $A, B \in \Lambda_1$  let  $\Lambda_{1A,B}$  be the set of  $C \in \Lambda_1$  such that there exist  $L, L', L'' \in \mathcal{F}$  with  $(L, L') \in \mathcal{O}_A$ ,  $(L', L'') \in \mathcal{O}_B$  and  $(L, L'') \in \mathcal{O}_C$ . It will be shown that  $\Lambda_1$  admits a partial order  $\leq$  such that, given  $A, B \in \Lambda_1$  with  $\mathrm{ro} B = \mathrm{co} A$ ,  $\Lambda_{1A,B}$  has a maximum element  $A * B$ . It will be shown that  $*$  is associative, leading to the construction of a category with the described properties.



The generic affine algebra  $\hat{G}(n, r)$  is then defined to be the  $\mathbb{Z}$ -algebra of this category. It will be shown that  $\hat{G}(n, r)$  gives a realisation of the affine 0-Schur algebra  $\hat{S}_0(n, r)$  when  $r < n$ . It is expected that a more refined presentation of the generic algebra and the 0-Schur algebra will allow the conditions on the parameters to be relaxed slightly: the  $r = n$  case is approachable, which may extend to the case  $r < 2n$ .

## 5.2 A partial order

Given  $i, j \in \mathbb{Z}$ , define a map  $d_{i,j}$  on  $\Lambda_1$  by setting

$$d_{i,j}A = \sum_{s \leq i, t > j} a_{s,t}$$

for each  $A \in \Lambda_1$ .

**Lemma 5.2.1.** *Let  $A \in \Lambda_1$ , with  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$  and write  $d_{i,j} = d_{i,j}A$  for  $i, j \in \mathbb{Z}$ . Then*

$$d_{i,j} - d_{i-1,j} = \sum_{t > j} a_{i,t}$$

and

$$d_{i,j} - d_{i,j-1} = - \sum_{s \leq i} a_{s,j}.$$

*Proof.* Let  $i, j \in \mathbb{Z}$ . Then

$$d_{i,j} - d_{i-1,j} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i-1, t > j} a_{s,t} = \sum_{t > j} a_{i,t}.$$

Similarly,

$$d_{i,j} - d_{i,j-1} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i, t > j-1} a_{s,t} = - \sum_{s \leq i} a_{s,j}.$$

□

**Lemma 5.2.2.** *Let  $A \in \Lambda_1$ , with  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$  and write  $d_{i,j} = d_{i,j}A$  for each  $i, j \in \mathbb{Z}$ . Then*

$$a_{i,j} = d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j}$$

for each  $i, j \in \mathbb{Z}$ .

*Proof.* Using Lemma 5.2.1,

$$\begin{aligned} a_{i,j} &= \sum_{t > j-1} a_{i,t} - \sum_{t > j} a_{i,t} \\ &= (d_{i,j-1} - d_{i-1,j-1}) - (d_{i,j} - d_{i-1,j}). \end{aligned}$$

Alternatively,

$$\begin{aligned} a_{i,j} &= \sum_{s \leq i} a_{s,j} - \sum_{s \leq i-1} a_{s,j} \\ &= -(d_{i,j} - d_{i,j-1}) + (d_{i-1,j} - d_{i-1,j-1}). \end{aligned}$$

□

**Lemma 5.2.3.** *The relation  $\leq$  on  $\Lambda_1$ , defined by  $A \leq B$  if and only if  $d_{i,j}A \leq d_{i,j}B$  for all  $i, j \in \mathbb{Z}$ , is a partial order.*

*Proof.* It is clear that  $\leq$  is reflexive and transitive, so it remains to see  $\leq$  is antisymmetric. Suppose  $A, B \in \Lambda_1$  with  $A \leq B$  and  $B \leq A$ . Then  $d_{i,j}A = d_{i,j}B$  for each  $i, j \in \mathbb{Z}$ , which shows  $A = B$  as a result of Lemma 5.2.2.  $\square$

The partial order on  $\Lambda_1$  induces a partial order on the set of  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$ , such that  $\mathcal{O}_A \leq \mathcal{O}_B$  if and only if  $A \leq B$ . The following lemma is rephrased from Lemma 3.1.4 and gives some geometric significance to the partial order on  $\Lambda_1$ .

**Lemma 5.2.4.** *Let  $A \in \Lambda_1$  and take  $(L, L') \in \mathcal{O}_A$ . Then*

$$d_{i,j}A = \dim \left( \frac{L_i}{L_i \cap L'_j} \right)$$

for each  $i, j \in \mathbb{Z}$ .

*Proof.* This is a rephrasing of Lemma 3.1.4.  $\square$

**Conjecture 1.** *Suppose  $\mathbf{k} = \mathbb{C}$ . The partial order on  $\Lambda_1$  is compatible with the closure order on  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$ . In particular,  $A \leq B$  if and only if  $\mathcal{O}_A \subset \overline{\mathcal{O}_B}$ .*

### 5.3 Geometry of affine flag varieties

Given  $L \in \mathcal{F}$ ,  $N, a \in \mathbb{N}$  and  $\lambda \in \Lambda_0$  define

$$\Pi_{N,\lambda}(L) = \{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi_{N,\lambda}^a(L) = \left\{ L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0, \dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) = a \right\}.$$

**Lemma 5.3.1.** *Given  $L \in \mathcal{F}$ ,  $N \in \mathbb{N}$  and  $\lambda \in \Lambda_0$ ,*

$$\Pi_{N,\lambda}(L) = \bigcup_{a: 0 \leq a \leq 2Nr} \Pi_{N,\lambda}^a(L).$$

*Proof.* If  $L' \in \Pi_{N,\lambda}(L)$  then  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ , so the  $\mathbf{k}[\varepsilon]$ -module  $\varepsilon^{-N} L_0 / L'_0$  is naturally isomorphic to  $(\varepsilon^{-N} L_0 / \varepsilon^N L_0) / (L'_0 / \varepsilon^N L_0)$ , which is a  $\mathbf{k}$ -vector space of dimension at most  $2Nr = \dim(\varepsilon^{-N} L_0 / \varepsilon^N L_0)$ . Thus  $\Pi_{N,\lambda}^a(L)$  is nonempty only if  $0 \leq a \leq 2Nr$ .  $\square$

**Lemma 5.3.2.** *Given  $L \in \mathcal{F}$ ,  $N \in \mathbb{N}$ ,  $\lambda \in \Lambda_0$  and  $a \in \mathbb{N}$  with  $0 \leq a \leq 2Nr$ ,  $\Pi_{N,\lambda}^a(L)$  is a projective algebraic variety.*

*Proof.* Let  $W$  be the  $\mathbf{k}$ -vector space  $\varepsilon^{-(1+N)} L_0 / \varepsilon^N L_0$ . Thus  $W$  is a  $(2N+1)r$ -dimensional vector space over  $\mathbf{k}$  and the action of  $\varepsilon$  on  $W$  is a nilpotent linear operator with  $\varepsilon^{2N+1} W = 0$ . Then  $\Pi_{N,\lambda}^a(L)$  is in natural bijection with the space of flags  $(0 \leq W_1 \leq \dots \leq W_n \leq W)$  of subspaces of  $W$  with type  $(\lambda, a)$  satisfying the closed condition  $\varepsilon W_n \subset W_1$ . The condition  $\varepsilon W_n \subset W_1$  ensures that each subspace  $W_i \subset W$  is a  $\mathbf{k}[\varepsilon]$ -submodule of  $W$ , since  $\varepsilon W_i \subset \varepsilon W_n \subset W_1 \subset W_i$ . Thus each  $W_i$  lifts to give a  $\mathbf{k}[\varepsilon]$ -submodule  $L_i$  of  $V$  with  $\varepsilon^N L_0 \subset L_i \subset \varepsilon^{-N} L_0$ , thus  $L_i$  is a lattice in  $V$ .  $\square$

**Lemma 5.3.3.** *Given  $L \in \mathcal{F}$ ,  $N \in \mathbb{N}$ ,  $\lambda \in \Lambda_0$  and  $a \in \mathbb{N}$  with  $0 \leq a \leq 2Nr$ ,  $\Pi_{N,\lambda}^a(L)$  is a subvariety of  $\Pi_{N+1,\lambda}^{a+r}(L)$ . In particular,  $\Pi_{N,\lambda}^a(L)$  is a subvariety of  $\Pi_{N+1,\lambda}^{a+r}(L)$ .*

*Proof.* If  $L' \in \Pi_{N+1,\lambda}^{a+r}(L)$ , then  $\varepsilon^{N+1}L_0 \subset \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0 \subset \varepsilon^{-(N+1)}L_0$  and

$$\dim \left( \frac{\varepsilon^{-(1+n)}L_0}{L'_0} \right) = \dim \left( \frac{L_0}{\varepsilon L_0} \right) + \dim \left( \frac{\varepsilon^{-N}L_0}{L'_0} \right) = r + a,$$

which shows that  $\Pi_{N,\lambda}^a(L) \subset \Pi_{N+1,\lambda}^{a+r}(L)$ . For  $L' \in \Pi_{N+1,\lambda}^{a+r}(L)$ , if additionally  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$ , then

$$\dim \left( \frac{\varepsilon^{-(N+1)}L_0}{L'_0} \right) = r + \dim \left( \frac{\varepsilon^{-N}L_0}{L'_0} \right),$$

which shows  $L' \in \Pi_{N,\lambda}^a(L)$ . Thus  $\Pi_{N,\lambda}^a(L)$  is the subspace of  $\Pi_{N+1,\lambda}^{a+r}(L)$  defined by the closed condition  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$ .  $\square$

**Lemma 5.3.4.** *Let  $\lambda \in \Lambda_0$ ,  $N, \tilde{N} \in \mathbb{N}$ ,  $L, \tilde{L} \in \mathcal{F}_\lambda$ ,  $0 \leq a \leq 2Nr$ ,  $0 \leq \tilde{a} \leq 2\tilde{N}r$ .  $\Pi_{N,\lambda}^a(L) \cap \Pi_{\tilde{N},\lambda}^{\tilde{a}}(\tilde{L})$  is a closed set in  $\Pi_{N,\lambda}^a(L)$ . In particular, if the intersection is nonempty it is a projective algebraic variety.*

*Proof.*  $\Pi_{N,\lambda}^a(L)$  is naturally identified with the space of flags of subspaces

$$0 \subset U_1 \subset \cdots \subset U_n \subset \frac{\varepsilon^{-N-1}L_0}{\varepsilon^N L_0}$$

with  $\dim(U_i/U_{i-1}) = \lambda_i$  for  $i = 2, \dots, n$  and  $\dim(W/U_n) = a$ , satisfying the closed condition  $\varepsilon U_n \subset U_1$ . The image of  $\Pi_{N,\lambda}^a(L) \cap \Pi_{\tilde{N},\lambda}^{\tilde{a}}(\tilde{L})$  under this identification is the subspace of flags  $(U_1, \dots, U_n)$  defined by the additional closed conditions  $(\varepsilon^N L_0 + \varepsilon^{\tilde{N}} \tilde{L}_0)/\varepsilon^N L_0 \subset \varepsilon U_n$  and  $U_n \subset (\varepsilon^{-N-1}L_0 \cap \varepsilon^{-\tilde{N}-1}\tilde{L}_0)/\varepsilon^N L_0$ . Thus  $\Pi_{N,\lambda}^a(L) \cap \Pi_{\tilde{N},\lambda}^{\tilde{a}}(\tilde{L})$  is a closed subset of  $\Pi_{N,\lambda}^a(L)$  as claimed.  $\square$

**Remark 1.** *The subsets  $\Pi_{N,\lambda}^a(L)$  of  $\mathcal{F}$  are partially ordered by inclusion, the inclusions are closed and the set theoretic direct limit is the set  $\mathcal{F}$ , so  $\mathcal{F}$  may be endowed with the direct limit topology of a system of projective algebraic varieties.*

**Lemma 5.3.5.** *Suppose  $L \in \mathcal{F}$ ,  $N, a \in \mathbb{N}$  and  $\lambda \in \Lambda_0$  with  $a \leq 2Nr$ . For each  $g \in G$ , the natural map (restriction of the action map)  $\Pi_{N,\lambda}^a(L) \rightarrow \Pi_{N,\lambda}^a(gL)$  is an isomorphism of projective varieties.*

*Proof.* If  $L' \in \Pi_{N,\lambda}^a(L)$ , then  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$  and so  $\varepsilon^N g(L_0) \subset g(L'_0) \subset \varepsilon^{-N}g(L_0)$ , so  $gL' \in \Pi_{N,\lambda}^a(gL)$ . Thus  $g$  and  $g^{-1}$  induce mutually inverse morphisms of varieties  $g: \Pi_{N,\lambda}^a(L) \rightarrow \Pi_{N,\lambda}^a(gL)$  and  $g^{-1}: \Pi_{N,\lambda}^a(gL) \rightarrow \Pi_{N,\lambda}^a(L)$ .  $\square$

### 5.3.1 Action through an algebraic group

Given  $N \in \mathbb{N}$ , define

$$H_N = \left\{ h \in G_L : h = 1 \text{ on } \frac{\varepsilon^{-(1+N)}L_0}{\varepsilon^N L_0} \right\}.$$

Explicitly, the condition  $h = 1$  on  $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$  means:  $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$  for  $x \in \varepsilon^{-(1+N)}L_0$ . Observe that  $H_{N+1} \subset H_N$  for  $N \in \mathbb{N}$  since  $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$  whenever  $x \in \varepsilon^{-(1+N)}L_0$ .

**Lemma 5.3.6.**  $H_N$  is a normal subgroup in  $G_L$ , for any  $N \in \mathbb{N}$ .

*Proof.*  $H_N \subset G_L$  by definition. Suppose  $h, h' \in H_N$  and let  $x \in \varepsilon^{-(1+N)}L_0$ .  $h'(x) \in \varepsilon^{-(1+N)}L_0$  as  $h' \in G_L$ , so  $hh'(x) + \varepsilon^N L_0 = h'(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ , which shows  $hh' \in H_N$ .  $h(x) - x \in \varepsilon^N L_0$ , so  $h^{-1}(x) - x = -h^{-1}(h(x) - x) \in \varepsilon^N L_0$ .  $h^{-1} \in H_N$ , so  $H_N$  is a subgroup of  $G_L$ .

Let  $g \in G_L$ .  $hg^{-1}(x) + \varepsilon^N L_0 = g^{-1}(x) + \varepsilon^N L_0$  as  $g^{-1}(x) \in \varepsilon^{-(1+N)}L_0$ , so  $ghg^{-1}(x) + \varepsilon^N L_0 = gg^{-1}(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ . Thus  $ghg^{-1} \in H_N$ , which proves  $H_N$  is a normal subgroup in  $G_L$ .  $\square$

*Short proof.*  $H_N$  is the kernel of the group homomorphism  $\bar{\cdot}: G_L \rightarrow \text{GL}(W)$ .  $\square$

**Lemma 5.3.7.** Given  $L \in \mathcal{F}$  and  $N \in \mathbb{N}$ ,  $G_L/H_{N,L}$  is a connected algebraic group.

*Proof.* Let  $W$  be the  $\mathbb{C}[\varepsilon]$ -module  $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$ .  $\varepsilon^{2N+1}$  acts as zero on  $W$  and  $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle \otimes_{\mathbb{C}[\varepsilon]} W$  is a free  $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle$ -module of rank  $r$ . Given  $g \in G_{L_0}$ ,  $g$  is a  $\mathbb{C}[\varepsilon]$ -module automorphism of  $\varepsilon^{-(1+N)}L_0$  and  $\varepsilon^N L_0$  is a  $g$ -invariant submodule, so there is an automorphism  $\bar{g}: W \rightarrow W$  fitting into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varepsilon^N L_0 & \longrightarrow & \varepsilon^{-1-N} L_0 & \longrightarrow & W \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \varepsilon^N L_0 & \longrightarrow & \varepsilon^{-1-N} L_0 & \longrightarrow & W \longrightarrow 0 \end{array}$$

The natural map  $\bar{\cdot}: G_{L_0} \rightarrow \text{GL}(W)$  is a group homomorphism with kernel consisting of those  $g \in G_{L_0}$  such that  $\bar{g} = 1$ : that is,  $g(x) \in x + \varepsilon^{2N+1}L_0$  for each  $x \in L_0$ .

The image of  $G_{L_0}$  in  $\text{GL}(W)$  may be described by equations in the coordinates on  $\text{GL}(W)$  with respect to a  $\mathbb{C}$ -basis of  $W$ .  $W$  has a basis  $\{x_1, \dots, x_r\}$  over  $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle$ , therefore the complex vector space  $W$  has a basis  $\{y_j : j \in \mathbb{Z}, 1 - 2Nr \leq j \leq r\}$  given by

$$y_{i-cr} = \varepsilon^c x_i$$

for  $1 \leq i \leq r$  and  $0 \leq c \leq 2N$ . There are coordinate functions  $\gamma_{i,j}: \text{GL}(W) \rightarrow \mathbb{C}$  with respect to this basis, given by

$$g(y_j) = \sum_i \gamma_{ij}(g)y_i.$$

If  $g \in \text{GL}(W)$  is  $\varepsilon$ -linear, then  $g(y_{i-r}) = g(\varepsilon y_i) = \varepsilon g(y_i)$  and therefore  $\gamma_{i-r,j-r}(g) = \gamma_{i,j}(g)$  for all  $i, j$ . Thus the image of  $G_{L_0}$  in  $\text{GL}(W)$  is the parabolic subgroup consisting of elements of the form

$$\begin{array}{c} A_0 A_1 A_2 \cdots A_{2N} \\ 0 A_0 A_1 \cdots A_{2N-1} \\ \dots\dots\dots \\ 0 \cdots 0 A_0 A_1 \\ 0 \cdots 0 A_0, \end{array}$$

which is a closed subgroup of  $\text{GL}(W)$ . The image of  $G_{L_0}$  in  $\text{GL}(W)$  is identified with the (nonempty) open set  $\text{GL}_r(\mathbb{C}) \times M_r(\mathbb{C})^{2N}$  in the affine space  $M_r(\mathbb{C})^{2N+1}$ , so the image of  $G_{L_0}$  is irreducible. This shows that  $G_{L_0}/H_{N,L_0}$  is a connected algebraic group.

Moreover,  $G_L = G_{L_1} \cap \cdots \cap G_{L_n}$ , so the image of  $G_L$  in  $\text{GL}(W)$  is a closed subgroup.  $G_L/H_{N,L}$  is naturally isomorphic to the subgroup of  $\text{GL}(W)$  defined by the equations  $\gamma_{i-r,j-r} = \gamma_{i,j}$  and

for  $j = 1, \dots, r$  the equations  $\gamma_{i,j} = 0$  for  $i > \lambda_1 + \dots + \lambda_s$ , where  $s$  is given by  $\lambda_1 + \dots + \lambda_{s-1} < j \leq \lambda_1 + \dots + \lambda_s$ . Therefore  $G_L/H_{N,L}$  is isomorphic to the product  $\mathcal{P}_\lambda \times M_r(\mathbb{C}) \times \dots \times M_r(\mathbb{C})$ , where  $\mathcal{P}_\lambda$  is a parabolic subgroup of  $\mathrm{GL}(W)$ , so is irreducible.  $\square$

Given  $g \in G$ , the map  $G_L \rightarrow G_{gL}$  sending  $h$  to  $ghg^{-1}$  is a group isomorphism which descends to an isomorphism of algebraic groups  $G_L/H_{N,L} \rightarrow G_{gL}/H_{N,gL}$ . Thus we have a commuting diagram of morphisms of varieties, where the vertical arrows are isomorphisms:

$$\begin{array}{ccc} G_L/H_{N,L} \times \Pi_{N,\lambda}^a(L) & \longrightarrow & \Pi_{N,\lambda}^a(L) \\ \downarrow & & \downarrow \\ G_{gL}/H_{N,gL} \times \Pi_{N,\lambda}^a(gL) & \longrightarrow & \Pi_{N,\lambda}^a(gL) \end{array}$$

### 5.3.2 Geometry of orbits

**Lemma 5.3.8.** *Given  $A \in \Lambda_1$ , there is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$  for each  $(L, L') \in \mathcal{O}_A$ . In this case*

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) = d_{nN,0}(A).$$

*Proof.* Given  $A \in \Lambda_1$ , there is  $N \in \mathbb{N}$  so that  $a_{i,j} = 0$  whenever  $|j - i| > nN$ . If  $(L, L') \in \mathcal{O}_A$  then

$$\dim \left( \frac{L'_0}{L'_0 \cap \varepsilon^{-N} L_0} \right) = \dim \left( \frac{L'_0}{L'_0 \cap L_{nN}} \right) = \sum_{s > nN, t \leq 0} a_{s,t} = 0,$$

so it follows  $L'_0 \subset \varepsilon^{-N} L_0$ . Similarly,

$$\dim \left( \frac{\varepsilon^N L_0}{\varepsilon^N L_0 \cap L'_0} \right) = \dim \left( \frac{L_{-nN}}{L_{-nN} \cap L'_0} \right) = \sum_{s \leq -nN, t > 0} a_{s,t} = 0,$$

which shows  $\varepsilon^N L_0 \subset L'_0$ . Moreover,

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) = \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^{-N} L_0 \cap L'_0} \right) = \sum_{s \leq nN, t > 0} a_{s,t} = d_{nN,0}(A),$$

as a result of Lemma 5.2.4.  $\square$

**Lemma 5.3.9.** *Given  $A \in \Lambda_1$  and  $L \in \mathcal{F}_{\mathrm{ro} A}$ ,  $X_A^L$  is a quasiprojective variety. In particular,  $X_A^L$  is a locally closed subset of  $\Pi_{N,\mathrm{co} A}^a(L)$  for some  $N \in \mathbb{N}$  and  $a = d_{nN,0}(A)$ .*

*Proof.* Lemma 5.3.8 shows that there is  $N \in \mathbb{N}$  such that  $X_A^L \subset \Pi_{N,\mathrm{co} A}^a(L)$ , where  $a = d_{nN,0}(A)$ . Then  $X_A^L$  is the subset of  $\Pi_{N,\mathrm{co} A}^a(L)$  given by the conditions  $\dim(L_i/L_i \cap L'_j) = d_{i,j}(A)$  for  $i, j \in \mathbb{Z}$ . For each  $i, j \in \mathbb{Z}$ , the condition  $\dim(L_i/L_i \cap L'_j)$  determines a locally closed subset of  $\Pi_{N,\mathrm{co} A}^a(L)$  since the function  $L' \mapsto \dim(L_i/L_i \cap L'_j)$  is lower semicontinuous. Both  $d_{i,j}(A)$  and  $\dim(L_i/L_i \cap L'_j)$  are invariant under the map  $(i, j) \mapsto (i + n, j + n)$ , so  $X_A^L$  is determined by these conditions for  $(i, j) \in [1, n] \times \mathbb{Z}$ .

There are only finitely many  $(i, j) \in [1, n] \times \mathbb{Z}$  such that  $a_{i,j} > 0$  and

$$a_{i,j} = d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j},$$

which shows that finitely many of these conditions determine  $X_A^L$ , so  $X_A^L$  is the intersection of finitely many locally-closed sets in  $\Pi_{N,\mathrm{co} A}^a(L)$  and is therefore locally closed.  $\square$

**Lemma 5.3.10.** *Given  $A \in \Lambda_1$  and  $L \in \mathcal{F}_{\text{ro}A}$ ,  $X_A^L$  is an irreducible quasi-projective variety.*

*Proof.* There is  $N \in \mathbb{N}$  such that  $X_A^L \subset \Pi_{N, \text{co}A}^a(L)$ , where  $a = d_{nN,0}(A)$ . Lemma 5.3.9 shows that  $X_A^L$  is a quasiprojective variety, so it remains to show  $X_A^L$  is irreducible. Lemma 5.3.7 shows  $G_L/H_{N,L}$  is a connected algebraic group acting algebraically on  $\Pi_{N, \text{co}A}^a(L)$ , so each orbit is irreducible. In particular, if  $L' \in X_A^L$ , then  $X_A^L = G_L \cdot L' = G_L/H_{N,L} \cdot L'$  is irreducible.  $\square$

**Remark 2.** *More is true: the closure of  $X_A^L$  is an irreducible projective variety and  $X_A^L$  is an irreducible quasi-projective variety.*

**Remark 3.** *A question of general topology: is it true that a subspace of a topological space is irreducible if and only if its closure is irreducible?*

### 5.3.3 Geometry of orbit products

Suppose  $A, B \in \Lambda_1$  with  $\text{co} A = \text{ro} B$ . Recall the notation

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

and

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L', L'') \in Y_{A,B}\}.$$

$X_{A,B}$  is the image of  $Y_{A,B}$  under the forgetful map  $(L, L', L'') \mapsto (L, L'')$ .

**Lemma 5.3.11.** *There is  $N \in \mathbb{N}$  such that*

$$\varepsilon^N L_0 \subset L_0'' \subset \varepsilon^{-N} L_0$$

for each  $(L, L'') \in X_{A,B}$ .

*Proof.* There exist  $N_1, N_2 \in \mathbb{N}$  such that

$$\varepsilon^{N_1} L_0 \subset L_0' \subset \varepsilon^{-N_1} L_0$$

and

$$\varepsilon^{N_2} L_0' \subset L_0'' \subset \varepsilon^{-N_2} L_0',$$

for each  $(L, L', L'') \in Y_{A,B}$ . Then, for  $(L, L', L'') \in Y_{A,B}$ ,

$$L_0'' \subset \varepsilon^{-N_2} L_0' \subset \varepsilon^{-(N_1+N_2)} L_0$$

and

$$\varepsilon^{N_1+N_2} L_0 \subset \varepsilon^{N_2} L_0' \subset L_0''.$$

In particular, taking  $N = N_1 + N_2$ , we have

$$\varepsilon^N L_0 \subset L_0'' \subset \varepsilon^{-N} L_0$$

for each  $(L, L'') \in X_{A,B}$ .  $\square$

**Lemma 5.3.12.** Suppose  $N_1, N_2 \in \mathbb{N}$  with  $\varepsilon^{N_1} L_0 \subset L_0 \subset \varepsilon^{-N_1} L_0$  and  $\varepsilon^{N_2} L'_0 \subset L'_0 \subset \varepsilon^{-N_2} L'_0$  for each  $(L, L', L'') \in Y_{A,B}$  and let  $N = N_1 + N_2$ . Then

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right) = d_{nN_1,0}(A) + d_{nN_2,0}(B)$$

and

$$\dim \left( \frac{L''_0}{\varepsilon^N L_0} \right) = 2Nr - d_{nN_1,0}(A) + d_{nN_2,0}(B),$$

for each  $(L, L'') \in X_{A,B}$ .

*Proof.* Suppose  $(L, L'') \in X_{A,B}$  and  $L' \in \mathcal{F}$  so that  $(L, L', L'') \in Y_{A,B}$ . As in lemma 5.3.11,  $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$ , so

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right) + \dim \left( \frac{L''_0}{\varepsilon^N L_0} \right) = \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0} \right).$$

As a  $\mathbf{k}$ -vector space,  $\varepsilon^{-N} L_0 / \varepsilon^N L_0$  is isomorphic to  $(L_0 / \varepsilon L_0)^{2N}$ , which has dimension  $2Nr$ , so

$$\dim \left( \frac{L''_0}{\varepsilon^N L_0} \right) = 2Nr - \dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right).$$

It remains to compute the codimension of  $L''_0$  in  $\varepsilon^{-N} L_0$ . Note  $L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-N} L_0$ , so

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right) = \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0} \right) + \dim \left( \frac{\varepsilon^{-N_2} L'_0}{L''_0} \right).$$

$$\begin{aligned} \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0} \right) &= \dim \left( \frac{\varepsilon^{-N_1} L_0}{L'_0} \right) \\ &= \dim \left( \frac{L_{nN_1}}{L_{nN_1} \cap L'_0} \right) \\ &= \sum_{s \leq nN_1, t > 0} A_{s,t} \\ &= d_{nN_1,0}(A). \end{aligned}$$

$$\begin{aligned} \dim \left( \frac{\varepsilon^{-N_2} L'_0}{L''_0} \right) &= \dim \left( \frac{L'_{nN_2}}{L'_{nN_2} \cap L''_0} \right) \\ &= \sum_{s \leq nN_2, t > 0} B_{s,t} \\ &= d_{nN_2,0}(B). \end{aligned}$$

□

Refer to Section 3.1.1 for definitions of the spaces  $X_{A,B}^L$  and  $Y_{A,B}^L$ .

**Lemma 5.3.13.** *Given  $A, B \in \Lambda_1$  with  $\text{ro } B = \text{co } A$  and  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_A$  and  $(L', L'') \in \mathcal{O}_B$ ,*

$$\begin{aligned} X_{A,B}^L &= G_L G_{L'} L'' \\ &= G_L X_B^{L'}. \end{aligned}$$

*Proof.*  $X_{A,B}^L$  is the image of  $Y_{A,B}^L$  under the forgetful map  $(N', N'') \mapsto N''$ . If  $\alpha \in G_L$  and  $\beta \in G_{L'}$  then  $(L, \alpha L, \alpha \beta L'') \in Y_{A,B}$  since  $(L, \alpha L') = \alpha(L, L') \in \mathcal{O}_A$  and  $(\alpha L', \alpha \beta L'') = \alpha \beta (\beta^{-1} L', L'') = \alpha \beta (L', L'') \in \mathcal{O}_B$ . Consequently,  $G_L G_{L'} L'' \in X_{A,B}^L$ .

For the reverse inclusion, if  $(N', N'') \in Y_{A,B}^L$  then  $(L, N') \in \mathcal{O}_A$  and  $(N', N'') \in \mathcal{O}_B$ , so there exist  $\sigma_1, \sigma_2 \in G$  such that  $(L, N') = \sigma_1(L, L')$  and  $(N', N'') \in \sigma_2(N', N'')$ . Then  $(L, N', N'') = (L, \sigma_1 L', \sigma_1 (\sigma_1^{-1} \sigma_2) L'')$  with  $\sigma_1 \in G_L$  and  $\sigma_1^{-1} \sigma_2 \in G_{L'}$ . Thus  $X_{A,B}^L = G_L G_{L'} L''$ .

The second equality follows since  $X_B^{L'} = G_{L'} L''$ .  $\square$

**Lemma 5.3.14.** *There is  $N \in \mathbb{N}$  such that  $H_N \subset G_{L'}$ . Consequently,  $H_{N'} \subset G_{L'}$  whenever  $N' \geq N$ .*

*Proof.* Choose  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ . Then

$$\varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset \cdots \subset L'_n \subset \varepsilon^{-(1+N)} L_0.$$

Let  $h \in H_N$ .  $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$  for  $x \in \varepsilon^{-(1+N)} L_0$ , so  $h(L'_i) \subset L'_i$  for  $i = 0, 1, \dots, n$ . Moreover,  $h^{-1}$  stabilises  $L'_i$ , so  $h(L'_i) = L'_i$  for  $i = 0, 1, \dots, n$  and therefore for  $i \in \mathbb{Z}$ . This shows  $h \in G_{L'}$  as required, so  $H_N \subset G_{L'}$ .  $\square$

$H_N$  is generally not normal in  $G_{L'}$ , though the space of (right) cosets of  $H_N$  in  $G_{L'}$  will still be irreducible.

**Lemma 5.3.15.**  *$G_{L'}/H_N$  is irreducible, provided  $H_N \subset G_{L'}$ .*

*Proof.* Needs a proof.  $\square$

**Lemma 5.3.16.** *Given  $A, B \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $Y_{A,B}^L$  is a quasiprojective variety.*

*Proof.* There is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$  and  $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$  for each  $(L, L', L'')$  with  $(L, L') \in \mathcal{O}_A$  and  $(L', L'') \in \mathcal{O}_B$ . Let  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ . Then  $Y_{A,B}^L$  is the subset of  $\Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L)$  defined by the conditions

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}(A)$$

and

$$\dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) = d_{i,j}(B)$$

for each  $i, j \in \mathbb{Z}$ .

it remains to justify why these are locally closed conditions on the product and why there are effectively only finitely many conditions. Thus  $Y_{A,B}^L$  is a locally closed subset of  $\Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L)$  so is a quasiprojective variety.  $\square$



**Proposition 5.3.17.** *Given  $A, B \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $Y_{A,B}^L$  is an irreducible quasi-projective variety.*

*Proof.* There is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$  and  $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$  for each  $(L', L'') \in Y_{A,B}^L$ . Write  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ , so  $Y_{A,B}^L \subset \Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L)$ . Lemma 5.3.16 shows that  $Y_{A,B}^L$  is a locally closed subset of  $\Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L)$  and thus a quasiprojective variety, so it remains to show  $Y_{A,B}^L$  is irreducible.

For any  $L' \in X_A^L$ ,  $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'}) = G_L/H_{2N,L} \cdot (\{L'\} \times X_B^{L'})$ .  $\overline{X_B^{L'}}$  is an irreducible subvariety of  $\Pi_{N,\text{co } B}^b(L')$ , which is a subvariety of  $\Pi_{2N,\text{co } B}^{a+b}(L)$ , so  $\{L'\} \times \overline{X_B^{L'}}$  is an irreducible subvariety of  $\Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L)$ . In particular,  $\{L'\} \times X_B^{L'}$  is an irreducible (locally closed) subspace of  $\Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L)$ .  $G_L/H_{2N,L}$  is a connected algebraic group acting algebraically on  $\Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L)$  and  $Y_{A,B}^L$  is the image of  $G_L/H_{2N,L} \times \{L'\} \times X_B^{L'}$  under the action map, so  $Y_{A,B}^L$  is irreducible.  $\square$

**Lemma 5.3.18.** *Given  $A, B \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $X_{A,B}^L$  is an irreducible topological space.*

*Proof.*  $X_{A,B}^L$  is the image of  $Y_{A,B}^L$  under the projection  $\mathcal{F}_{\text{co } A} \times \mathcal{F}_{\text{co } B} \rightarrow \mathcal{F}_{\text{co } B}$  and  $Y_{A,B}^L$  is irreducible, by Proposition 5.3.17, so  $X_{A,B}^L$  is irreducible.  $\square$

*Proof.* There is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$  and  $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$  for each  $(L, L') \in \mathcal{O}_A$  and  $(L', L'') \in \mathcal{O}_B$ . Then  $\overline{X_B^{L'}}$  is an irreducible subvariety of  $\Pi_{N,\text{co } B}^b(L')$ , which is in turn a subvariety of  $\Pi_{2N,\text{co } B}^{a+b}(L)$ . Thus  $X_B^{L'}$  is an irreducible subspace of  $\Pi_{2N,\text{co } B}^{a+b}(L)$ .  $G_L/H_{2N,L}$  is a connected algebraic group acting morphically on  $\Pi_{2N,\text{co } B}^{a+b}(L)$ , therefore  $X_{A,B}^L = G_L/H_{2N,L} \cdot X_B^{L'}$  is irreducible.  $\square$

**Remark 4.**  $X_{A,B}^L$  is a union of finitely many  $G_L$ -orbits, each of which is locally closed, so  $X_{A,B}^L$  is constructible. Lemma 5.3.18 shows that  $X_{A,B}^L$  is irreducible. Investigate whether  $X_{A,B}^L$  is actually locally closed and therefore an irreducible quasiprojective variety.

**Proposition 5.3.19.** *Given  $A, B \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $L \in \mathcal{F}_{\text{ro } A}$ , there is a unique open  $G_L$ -orbit in  $X_{A,B}^L$ .*

*Proof.*  $X_{A,B}^L$  consists of finitely many  $G_L$ -orbits and is an irreducible topological space, by Lemma 5.3.18. Consequently,  $X_C^L$  is dense in  $X_{A,B}^L$  for some  $C \in \Lambda_{1A,B}$ . Lemma 5.3.9 shows that  $X_C^L$  is locally closed in  $X_{A,B}^L$ , so  $X_C^L$  is open in  $\overline{X_C^L} = X_{A,B}^L$ . Irreducibility of  $X_{A,B}^L$  shows that there is a unique open  $G_L$ -orbit, since two non-empty open sets in  $X_{A,B}^L$  intersect non-trivially, thus any two open  $G_L$  orbits in  $X_{A,B}^L$  coincide.  $\square$

## 5.4 Existence of a maximum

**Lemma 5.4.1.** *Given  $A, A' \in \Lambda_1$  with  $\text{ro } A = \text{ro } A'$  and  $\text{co } A = \text{co } A'$ ,  $A' \leq A$  if and only if  $X_{A'}^L \subset X_A^L$  for any  $L \in \mathcal{F}_{\text{ro } A}$ .*

*Proof.* Needs a proof.  $\square$

**Proposition 5.4.2.** *Given  $A, B \in \Lambda_1$  with  $\text{co } A = \text{ro } B$ ,  $\Lambda_{1A,B}$  has a maximum element.*

*Proof.* Let  $L \in \mathcal{F}_{\text{ro } A}$ .  $X_{A,B}^L$  is irreducible by Lemma 5.3.18 and is the union of finitely many  $G_L$ -orbits, namely

$$X_{A,B}^L = \bigcup_{C \in \Lambda_{1A,B}} X_C^L.$$

This shows that  $X_C^L$  is dense in  $X_{A,B}^L$  for some  $C \in \Lambda_{1A,B}$ . Lemma 5.3.9 shows that the  $G_L$ -orbits in  $X_{A,B}^L$  are locally closed, so a dense  $G_L$ -orbit is open in  $X_{A,B}^L$ . Lemma 5.4.1 shows that the characteristic matrix of the dense  $G_L$ -orbit is a maximum in  $\Lambda_{1A,B}$ .  $\square$

## 5.5 Associativity

Refer to Section 3.1.2 for definitions of the spaces  $X_{A,B,C}^L$  and  $Y_{A,B,C}^L$ . Recall that  $X_{A,B,C}^L$  is the image of  $Y_{A,B,C}^L$  under the forgetful map  $f$ , given by  $f(L', L'', L''') = L'''$  for each  $(L', L'', L''') \in Y_{A,B,C}^L$ .

**Lemma 5.5.1.** *Given  $A, B, C \in \Lambda_1$  with  $\text{ro } C = \text{co } B$ ,  $\text{ro } B = \text{co } A$  and a tuple of flags  $(L, L', L'', L''') \in \mathcal{F}^4$  with  $(L, L') \in \mathcal{O}_A$ ,  $(L', L'') \in \mathcal{O}_B$  and  $(L'', L''') \in \mathcal{O}_C$ ,*

$$X_{A,B,C}^L = G_L G_{L'} G_{L''} L''''.$$

*Proof.* Given  $\alpha \in G_L$ ,  $\beta \in G_{L'}$  and  $\gamma \in G_{L''}$ ,  $(L, \alpha L', \alpha \beta L'', \alpha \beta \gamma L''') \in Y_{A,B,C}^L$  since  $(L, \alpha L') = \alpha(L, L') \in \mathcal{O}_A$ ,  $(\alpha L', \alpha \beta L'') = \alpha \beta(L', L'') \in \mathcal{O}_B$  and  $(\alpha \beta L'', \alpha \beta \gamma L''') = \alpha \beta \gamma(L'', L''') \in \mathcal{O}_C$ . This shows  $G_L G_{L'} G_{L''} L'''' \in X_{A,B,C}^L$ .

Given  $(N', N'', N''') \in Y_{A,B,C}^L$ , there exist  $\sigma_1, \sigma_2, \sigma_3 \in G$  such that  $(L, N') = \sigma_1(L, L')$ ,  $(N', N'') = \sigma_2(L', L'')$  and  $(N'', N''') = \sigma_3(L'', L''')$ ; then  $N' = \sigma_1 L' = \sigma_2 L'$ ,  $N'' = \sigma_2 L'' = \sigma_3 L''$  and  $N''' = \sigma_3 L'''$ . Thus

$$(L, N', N'', N''') = (L, \sigma_1 L', \sigma_1(\sigma_1^{-1} \sigma_2) L'', \sigma_1(\sigma_1^{-1} \sigma_2)(\sigma_2^{-1} \sigma_3) L''')$$

where  $\sigma_1 \in G_L$ ,  $\sigma_1^{-1} \sigma_2 \in G_{L'}$  and  $\sigma_2^{-1} \sigma_3 \in G_{L''}$ .  $\square$

**Lemma 5.5.2.** *Given  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $Y_{A,B,C}^L$  is a quasiprojective variety.*

*Proof.* There is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ ,  $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$  and  $\varepsilon^N L''_0 \subset L'''_0 \subset \varepsilon^{-N} L''_0$  for each  $(L', L'', L''') \in Y_{A,B,C}^L$ . Let  $a = d_{nN,0}(A)$ ,  $b = d_{nN,0}(B)$  and  $c = d_{nN,0}(C)$ . Then  $Y_{A,B,C}^L$  is the subset of  $\Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L) \times \Pi_{3N,\text{co } C}^{a+b+c}(L)$  defined by the conditions

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}(A),$$

$$\dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) = d_{i,j}(B),$$

and

$$\dim \left( \frac{L''_i}{L''_i \cap L'''_j} \right) = d_{i,j}(C)$$

for each  $i, j \in \mathbb{Z}$ .

it remains to show these conditions define locally closed subsets of the triple product and that there are effectively finitely many conditions.

Thus  $Y_{A,B,C}^L$  is a locally closed subset of the projective variety  $\Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L) \times \Pi_{3N,\text{co } C}^{a+b+c}(L)$ , so  $Y_{A,B,C}^L$  is a quasiprojective variety.  $\square$

**Lemma 5.5.3.** *Given  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $Y_{A,B,C}^L$  is an irreducible quasiprojective variety.*

*Proof.* There is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ ,  $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$  and  $\varepsilon^N L''_0 \subset L'''_0 \subset \varepsilon^{-N} L''_0$  for each  $(L', L'', L''') \in Y_{A,B,C}^L$ . Lemma 5.3.7 shows that  $G_L/H_{3N,L}$  is a connected algebraic group acting algebraically on  $\Pi = \Pi_{N,\text{co } A}^a(L) \times \Pi_{2N,\text{co } B}^{a+b}(L) \times \Pi_{3N,\text{co } C}^{a+b+c}(L)$ .

Let  $L' \in X_A^L$ .  $Y_{A,B,C}^L = G_L \cdot (\{L'\} \times Y_{B,C}^{L'})$ .  $Y_{B,C}^{L'}$  is an irreducible quasiprojective variety;  $\overline{Y_{B,C}^{L'}}$  is an irreducible subvariety of  $\Pi_{N,\text{co } B}^b(L') \times \Pi_{2N,\text{co } C}^{b+c}(L')$ , which is a subvariety of  $\Pi_{2N,\text{co } B}^{a+b}(L) \times \Pi_{3N,\text{co } C}^{a+b+c}(L)$ . Thus  $\{L'\} \times \overline{Y_{B,C}^{L'}}$  is an irreducible subvariety of  $\Pi$ . Therefore  $Y_{A,B,C}^L$  is the image of the irreducible space  $G_L/H_{3N,L} \times \{L'\} \times Y_{B,C}^{L'}$  under the action map, so  $Y_{A,B,C}^L$  is irreducible. Lemma 5.5.2 shows that  $Y_{A,B,C}^L$  is quasiprojective, so  $Y_{A,B,C}^L$  is an irreducible quasiprojective variety.  $\square$

**Corollary 5.5.4.** *Given  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $X_{A,B,C}^L$  is an irreducible topological space.*

*Proof.*  $X_{A,B,C}^L$  is the image of  $Y_{A,B,C}^L$  under the forgetful map  $f$  and  $Y_{A,B,C}^L$  is irreducible, by Lemma 5.5.3, so  $X_{A,B,C}^L$  is irreducible.  $\square$

**Lemma 5.5.5.** *Given matrices  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ , there is a unique open  $G_L$ -orbit in  $X_{A,B,C}^L$ .*

*Proof.*  $X_{A,B,C}^L$  is irreducible, by Corollary 5.5.4, and consists of finitely many  $G_L$ -orbits, so contains a dense  $G_L$ -orbit. In particular, there is  $D \in \Lambda_1$  such that  $\overline{X_D^L} = X_{A,B,C}^L$ . Lemma 5.3.9 shows that the  $G_L$ -orbits are locally closed in  $X_{A,B,C}^L$ . In particular,  $X_D^L$  is open in  $\overline{X_D^L} = X_{A,B,C}^L$ . Therefore, there is an open  $G_L$ -orbit in  $X_{A,B,C}^L$ . There is a unique open  $G_L$ -orbit since  $X_{A,B,C}^L$  is irreducible.  $\square$

**Definition 5.5.1.** *Given  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ , define spaces*

$$\begin{aligned}\tilde{Y}_{(AB)C}^L &= f^{-1} X_{(A*B)*C}^L \\ \tilde{Y}_{A(BC)}^L &= f^{-1} X_{A*(B*C)}^L.\end{aligned}$$

**Lemma 5.5.6.** *Given  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $\tilde{Y}_{(AB)C}^L$  is open in  $Y_{A,B,C}^L$ .*

*Proof.*

$$f^{-1} X_{A*B,C}^L = \left\{ (L', L'', L''') \in Y_{A,B,C}^L : \dim \left( \frac{L_i}{L_i \cap L_j''} \right) \text{ is maximal, for each } i, j \in \mathbb{Z} \right\}$$

is open in  $Y_{A,B,C}^L$  since  $f^{-1}X_{A*B,C}^L$  is given by **finitely many** open conditions; the function on  $X_{A,B}^L$  given by  $L'' \mapsto \dim\left(\frac{L_i}{L_i \cap L_j''}\right)$  is lower semicontinuous, so maximising such a function is an open condition in  $X_{A,B}^L$ .  $\square$

**Lemma 5.5.7.** *Given  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $\tilde{Y}_{A(BC)}^L$  is open in  $Y_{A,B,C}^L$ .*

*Proof.*

$$f^{-1}X_{A,B*C}^L = \left\{ (L', L'', L''') \in Y_{A,B,C}^L : \dim\left(\frac{L'_i}{L'_i \cap L''_j}\right) \text{ is maximal, for each } i, j \in \mathbb{Z} \right\}$$

is open in  $Y_{A,B,C}^L$ , as it is determined by **finitely many** open conditions; the function on  $X_A^L \times X_{A,B*C}^L$  given by  $(L', L''') \mapsto \dim\left(\frac{L'_i}{L'_i \cap L''_j}\right)$  is lower semicontinuous, so maximising such a function is an open condition in  $X_A^L \times X_{A,B*C}^L$ .  $\square$

**Conjecture 2.** *Given  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $X_{A*B,C}^L$  and  $X_{A,B*C}^L$  are open and dense in  $X_{A,B,C}^L$ .*

**Remark 5.** *If  $f$  is shown to be an open map then this result follows from Lemma 5.5.6 and Lemma 5.5.7.*

**Proposition 5.5.8.** *Given  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$ ,  $(A * B) * C = A * (B * C)$ .*

*Proof.* Take  $A, B, C \in \Lambda_1$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and fix  $L \in \mathcal{F}_{\text{ro } A}$ .

$X_{(A*B)*C}^L$  is open in  $X_{A*B,C}^L$ , so  $f^{-1}X_{(A*B)*C}^L$  is open in  $f^{-1}X_{A*B,C}^L$ . Lemma 5.5.6 shows that  $f^{-1}X_{A*B,C}^L$  is open in  $Y_{A,B,C}^L$ , so  $f^{-1}X_{(A*B)*C}^L$  is open in  $Y_{A,B,C}^L$ . Similarly,  $X_{A*(B*C)}^L$  is open in  $X_{A,B*C}^L$  and  $f^{-1}X_{A,B*C}^L$  is open in  $Y_{A,B,C}^L$ , by Lemma 5.5.7, so  $f^{-1}X_{A*(B*C)}^L$  is open in  $Y_{A,B,C}^L$ .

Lemma 5.5.3 shows that  $Y_{A,B,C}^L$  is irreducible, so  $f^{-1}X_{(A*B)*C}^L$  and  $f^{-1}X_{A*(B*C)}^L$  have nonempty intersection. Therefore the  $G_L$ -orbits  $X_{(A*B)*C}^L$  and  $X_{A*(B*C)}^L$  intersect nontrivially, so are the same  $G_L$ -orbit. Thus  $(A * B) * C = A * (B * C)$ .  $\square$

## 5.6 The generic algebra

**Lemma 5.6.1.** *Given  $\lambda \in \Lambda_0$  and  $A \in \Lambda_1$ ,  $D_\lambda * A = A$  if  $\text{ro } A = \lambda$  and  $A * D_\lambda = A$  if  $\text{co } A = \lambda$ .*

*Proof.* Lemma 3.1.7 shows that  $\Lambda_{1D_\lambda,A} = \{A\}$  if  $\lambda = \text{ro } A$  and  $\Lambda_{1A,D_\lambda} = \{A\}$  if  $\lambda = \text{co } A$ , which proves the result.  $\square$

**Theorem 5.6.2.** *The following constitutes a small category: the set of objects is  $\Lambda_0$  and the set of morphisms is  $\Lambda_1$ . Given compositions  $\lambda, \mu \in \Lambda_0$ , the morphisms with source  $\lambda$  and target  $\mu$  are those matrices  $A \in \Lambda_1$  with  $\text{co } A = \lambda$  and  $\text{ro } A = \mu$ . Given  $\lambda, \mu, \nu \in \Lambda_0$  and  $A, B \in \Lambda_1$  with  $B: \lambda \rightarrow \mu$  and  $A: \mu \rightarrow \nu$  the composition is  $A * B: \lambda \rightarrow \nu$ .*

*Proof.* Proposition 5.4.2 shows that the composition is well defined while Proposition 5.5.8 establishes associativity of the composition. Lemma 5.6.1 shows that  $D_\lambda: \lambda \rightarrow \lambda$  is the identity morphism for each  $\lambda \in \Lambda_0$ . Thus  $(\Lambda_0, \Lambda_1, \text{co}, \text{ro}, *)$  is a category.  $\square$

Write  $\mathcal{G}(n, r)$  to denote this so-called ‘generic category’.

**Example 1.** *The objects in  $\mathcal{G}(2, 2)$  are compositions of 2 into 2 parts, namely  $(0, 2)$ ,  $(1, 1)$  and  $(2, 0)$ . The set of morphisms from  $\lambda$  to  $\mu$  is the set of infinite periodic matrices  $A \in \Lambda_1[2, 2]$  with  $\text{co } A = \lambda$  and  $\text{ro } A = \mu$ , which is a countably infinite set for any pair of compositions  $\lambda, \mu \in \Lambda_0[2, 2]$ .*

**Definition 5.6.1** (Generic algebra). *The generic affine algebra  $\hat{G}(n, r)$  is the category  $\mathbb{Z}$ -algebra of  $\mathcal{G}(n, r)$ . In particular,  $\hat{G}(n, r)$  is a free  $\mathbb{Z}$ -module with basis  $\{e_A : A \in \Lambda_1\}$  and with associative multiplication given by*

$$e_A * e_B = \begin{cases} e_{A*B} & \text{if } \text{co } A = \text{ro } B \\ 0 & \text{if } \text{co } A \neq \text{ro } B. \end{cases}$$

*The multiplicative identity in  $\hat{G}(n, r)$  is*

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

## Chapter 6

# A realisation of affine zero Schur algebras

We aim to prove the isomorphism theorem in the cases  $r < n$  and  $n \leq r < 2n$  separately. Below are crude versions of the statements we want to prove.

**Theorem 6.0.1.** *Assume  $r < n$ . The map  $\psi: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$ , given by  $\psi(E_i) = E_i$ ,  $\psi(F_i) = F_i$  and  $\psi(1_\lambda) = 1_\lambda$ , is an isomorphism of  $\mathbb{Z}$ -algebras.*

**Theorem 6.0.2.** *Assume  $n \leq r < 2n$ . There is a unique homomorphism of  $\mathbb{Z}$ -algebras  $\hat{\psi}: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$  such that  $\hat{\psi}(R) = R$  and  $\hat{\psi} = \psi$  on the subalgebra of  $\hat{G}(n, r)$  generated by the  $E_i$ ,  $F_i$  and  $1_\lambda$ .  $\hat{\psi}$  is an isomorphism of  $\mathbb{Z}$ -algebras.*

### 6.1 Preliminary results

Recall from Definition 5.6.1 that the generic algebra  $\hat{G}(n, r)$  is an associative  $\mathbb{Z}$ -algebra which is a free  $\mathbb{Z}$ -module with an atomic basis  $\{e_A : A \in \Lambda_1\}$ : given  $A, B \in \Lambda_1$  with  $\text{co } A = \text{ro } B$ ,  $e_A e_B = e_{A*B}$ .

#### 6.1.1 Elementary basis elements

Given  $i \in [1, n]$  and  $\lambda \in \Lambda_0$  such that  $\lambda_{i+1} > 0$ , define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and

$$E_i = \sum_{\lambda \in \Lambda_0: \lambda_{i+1} > 0} E_{i,\lambda}.$$

Given  $i \in [1, n]$  and  $\lambda \in \Lambda_0$  such that  $\lambda_i > 0$ , define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and

$$F_i = \sum_{\lambda \in \Lambda_0: \lambda_i > 0} F_{i,\lambda}.$$

### 6.1.2 Transpose involution

**Lemma 6.1.1.** *The  $\mathbb{Z}$ -module automorphism  $\top$  of  $\hat{G}(n, r)$  given by  $e_A \mapsto e_{A^\top}$  is a  $\mathbb{Z}$ -algebra antihomomorphism: that is,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A$$

*for each  $A, B \in \Lambda_1$ . Moreover,  $\top(E_{i,\lambda}) = F_{i,\lambda+\alpha_i}$ ,  $\top(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}$  and  $\top(1_\lambda) = 1_\lambda$ , for permissible  $(i, \lambda) \in \mathbb{Z} \times \Lambda_0$ .*

*Proof.* This is a consequence of Lemma 4.1.1. It must also be shown that the transpose operation on  $\Lambda_1$  is order preserving.  $\square$

### 6.1.3 Multiplication rules

**Lemma 6.1.2.** *Given  $A \in \Lambda_1$  and  $i \in [1, n]$  such that  $\text{ro } A_{i+1} > 0$ ,*

$$E_i e_A = e_{A+\varepsilon_{i,p}-\varepsilon_{i+1,p}},$$

*where  $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$ .*

*Given  $A \in \Lambda_1$  and  $i \in [1, n]$  such that  $\text{ro } A_i > 0$ ,*

$$F_i e_A = e_{A+\varepsilon_{i+1,p}-\varepsilon_{i,p}},$$

*where  $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$ .*

Similar formulas for right multiplication by  $E_i$  and  $F_i$  are obtained by applying the transpose.

## 6.2 Presentation of the generic algebra.

Recall that  $\Lambda_0$  denotes the set of compositions of  $r$  into  $n$  parts. That is,  $\Lambda_0$  is the set of tuples  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$  with each  $\lambda_i$  non-negative and  $\lambda_1 + \dots + \lambda_n = r$ . Given  $i \in [1, n]$ , let  $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$  be the  $i$ -th elementary vector and let  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ . Then given  $\lambda \in \Lambda_0$ , we have  $\lambda + \alpha_i \in \Lambda_0$  provided  $\lambda_{i+1} > 0$  and  $\lambda - \alpha_i \in \Lambda_0$  provided  $\lambda_i > 0$ .

Let  $\Gamma = \Gamma(n, r)$  be the quiver with set of vertices  $\Lambda_0$  with arrows  $e_{i,\lambda}: \lambda \rightarrow \lambda + \alpha_i$  (if  $\lambda_{i+1} > 0$ ) and  $f_{i,\lambda}: \lambda \rightarrow \lambda - \alpha_i$  (if  $\lambda_i > 0$ ). Thus there are no arrows between  $\lambda$  and  $\mu$  unless  $\lambda = \mu \pm \alpha_i$  for some  $i \in [1, n]$ .

If  $n \geq 3$  then neighbouring vertices are connected by two arrows, one of each direction. In the case  $n = 2$ , neighbouring vertices are joined by four arrows, two of each direction. The  $\mathbb{Z}\Gamma$  denote the path  $\mathbb{Z}$  algebra of  $\Gamma$ . By construction of  $\Gamma$ , there is a  $\mathbb{Z}$ -algebra homomorphism  $\mathbb{Z}\Gamma \rightarrow \hat{G}(n, r)$  with  $e_{i,\lambda} \mapsto E_{i,\lambda}$ ,  $f_{i,\lambda} \mapsto F_{i,\lambda}$  and  $k_\lambda = 1_\lambda$ . We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [3]).

**Definition 6.2.1.** (aperiodicity)  *$A \in \Lambda_1$  is aperiodic if for each  $l \in \mathbb{Z} \setminus \{0\}$  there exists  $i \in \mathbb{Z}$  such that  $a_{i,i+l} = 0$ . Denote the set of aperiodic elements in  $\Lambda_1$  by  $\Lambda_1^{\text{ap}}$ . Note that  $\Lambda_1^{\text{ap}} = \Lambda_1$  if  $r < n$ . Linear combinations of the basis elements corresponding to aperiodic matrices are also said to be aperiodic - if  $A$  is aperiodic, we say  $e_A$  is aperiodic.*

**Lemma 6.2.1.** *Let  $A \in \Lambda_1$  and write  $\lambda = \text{ro } A$ . If  $A$  is aperiodic and  $\lambda_{i+1} > 0$ , then  $E_i * e_A$  is aperiodic. If  $A$  is aperiodic and  $\lambda_i > 0$ , then  $F_i * e_A$  is aperiodic.*

*Proof.* Suppose  $A \in \Lambda_1$  is aperiodic and  $\lambda_{i+1} > 0$ , where  $\lambda = \text{ro } A$ . There is  $p \in \mathbb{Z}$  such that  $a_{i+1,p} > 0$  and  $a_{i+1,p'} = 0$  whenever  $p' > p$ . Lemma 6.1.2 shows that  $E_i * e_A = e_B$ , where  $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ . Let  $l \in \mathbb{Z} \setminus \{0\}$ . If  $l \notin \{p-i, p-i-1\}$ , then  $b_{s,s+l} = a_{s,s+l}$  for each  $s \in \mathbb{Z}$ , so there is  $s \in \mathbb{Z}$  such that  $b_{s,s+l} = a_{s,s+l} = 0$ , since  $A$  is aperiodic. If  $l = p-i$ , then  $b_{i+1,i+1+l} = b_{i+1,p+1} = a_{i+1,p+1} = 0$ , by maximality of  $p$ . If  $l = p-i-1$ , there is  $s \neq i+1$  such that  $a_{s,s+l} = 0$ , since  $A$  is aperiodic and  $a_{i+1,i+1+l} = a_{i+1,p} > 0$ , so  $b_{s,s+l} = a_{s,s+l} = 0$ . Therefore,  $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$  is aperiodic.

Suppose  $A \in \Lambda_1$  is aperiodic and  $\lambda_i > 0$ , where  $\lambda = \text{ro } A$ . Lemma 6.1.2 shows that  $F_i * e_A = e_C$  where  $C = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$  and  $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$ . Let  $l \in \mathbb{Z} \setminus \{0\}$ . If  $l \notin \{p-i, p-i-1\}$  then  $c_{s,s+l} = a_{s,s+l}$  for each  $s \in \mathbb{Z}$ , so there is  $s \in \mathbb{Z}$  such that  $c_{s,s+l} = a_{s,s+l} = 0$ , by aperiodicity of  $A$ . If  $l = p-i$ , then  $a_{i,i+l} = a_{i,p} > 0$ , so there is  $s \neq i$  such that  $a_{s,s+l} = 0$ . Then  $c_{s,s+l} = a_{s,s+l} = 0$ . Finally, if  $l = p-i-1$ , then  $c_{i,i+l} = a_{i,p-1} = 0$  by minimality of  $p$ . Thus  $C$  is aperiodic as required.  $\square$

**Definition 6.2.2.** (*Weight function*) Define the weight function  $\text{wt} : \Lambda_1 \rightarrow \mathbb{Z}$  by

$$\text{wt } A = \sum_{i \in [1,n], j \in \mathbb{Z}} |j-i| a_{i,j}$$

for each  $A \in \Lambda_1$ . The sum is taken over a transversal of the set of congruence classes of  $(i, j)$  modulo  $(n, n)$  for  $i, j \in \mathbb{Z}$ .

**Lemma 6.2.2.** Let  $A \in \Lambda_1$  and write  $\lambda = \text{ro } A$ . Suppose  $\lambda_{i+1} > 0$  and set  $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$ . If  $p > i$  then  $\text{wt } e_{i,\lambda} * A = 1 + \text{wt } A$ . If  $p \leq i$  then  $\text{wt } e_{i,\lambda} * A = -1 + \text{wt } A$ . Suppose  $\lambda_i > 0$  and set  $q = \min\{q' \in \mathbb{Z} : a_{i,q'} > 0\}$ . If  $q \leq i$  then  $\text{wt } f_{i,\lambda} * A = 1 + \text{wt } A$ . If  $q > i$  then  $\text{wt } f_{i,\lambda} * A = -1 + \text{wt } A$ .

*Proof.* Lemma 6.1.2 shows that  $e_i A = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ , so  $\text{wt } e_i A - \text{wt } A = |p-i| - |p-i-1|$ , which equals 1 if  $p > i$  and equals -1 if  $p \leq i$ . Similarly,  $f_i A = A + \mathcal{E}_{i+1,q} - \mathcal{E}_{i,q}$  by Lemma 6.1.2, so  $\text{wt } f_i A - \text{wt } A = |q-i-1| - |q-i|$ , which equals -1 if  $q > i$  and equals 1 if  $q \leq i$ .  $\square$

**Lemma 6.2.3.** If  $A \in \Lambda_1$  is aperiodic, then  $e_A$  may be obtained from  $1_{\text{co } A}$  by finitely many applications of  $E_i$  and  $F_i$  for  $i \in [1, n]$ .

**Proposition 6.2.4.** The  $\mathbb{Z}$ -subalgebra of  $\hat{G}(n, r)$  generated by  $E_{i,\lambda}$ ,  $F_{i,\lambda}$  and  $1_\lambda$  has  $\mathbb{Z}$ -basis  $\{e_A : A \in \Lambda_1^{\text{ap}}\}$ , where  $\Lambda_1^{\text{ap}} \subset \Lambda_1$  is the set of aperiodic elements.

*Proof.*  $\square$

### 6.2.1 The typical case.

**Lemma 6.2.5.** The following relations hold in  $\hat{G}(n, r)$  ( $n \geq 3$ ):

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless  $|j-i| = 1$ .

$$E_i E_{i+1}^2 - E_{i+1} E_i E_{i+1} = 0$$

$$E_i^2 E_{i+1} - E_i E_{i+1} E_i = 0$$

$$F_{i+1} F_i^2 - F_i F_{i+1} F_i = 0$$



$$F_{i+1}^2 F_i - F_{i+1} F_i F_{i+1} = 0$$

$$E_i F_j - F_j E_i = 0$$

unless  $j = i$ .

$$E_i F_i - F_i E_i + \sum_{\lambda: \lambda_i=0, \lambda_{i+1}>0} 1_\lambda - \sum_{\lambda: \lambda_i>0, \lambda_{i+1}=0} 1_\lambda = 0.$$

### 6.2.2 Exceptional case.

In this case, the quiver  $\Gamma(2, r)$  has vertices  $\Lambda_0[2, r] = \{(0, r), (1, r-1), \dots, (r, 0)\}$ ; adjacent vertices are connected by two pairs of arrows with opposite orientation:  $(e_1, f_1)$  and  $(e_2, f_2)$ . The relations arising from  $\hat{G}(2, r)$  are of a more complicated form - in particular, the serre relations of total degree 3 will not hold in this case - so this case will be treated separately and at a later date.

## Chapter 7

# Further directions

### 7.1 Further results on affine zero Schur algebras

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

### 7.2 Deformed group algebras of symmetric groups

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algebras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for  $S_3$  and  $S_4$ . Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: ‘these’ relations hold; these generators and relations give a presentation of the degenerate group algebras.

Terminology: deformed group algebra.

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