# A geometric realisation of affine 0-Schur algebras.

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## Chapter 1

# Introduction

# Chapter 2

# Representations and Hall algebras of cyclic quivers

In this chapter we will review some of the relevant background on quiver representations and establish notation and terminology. The cases of linear quivers of type $A$ and cyclic quivers (type $\tilde{A}$) will be considered in detail. In particular, the aim is to describe the classification of finite dimensional representations of a cyclic quiver over $\mathbb{C}$; or over an arbitrary field. In the case of an arbitrary field, there is a combinatorial characterisation of the isomorphism classes of nilpotent representations, which does not depend on the ground field. This fact is used in defining the Ringel-Hall algebra of a cyclic quiver. (!!) The subalgebra generated by the isoclasses of simple representations, called the composition subalgebra, gives a realisation of the positive part of the quantised enveloping algebra $U_q(\tilde{\mathfrak{gl}}_n)$.

References for this chapter include: [3], [2], [1].

Give the basic definitions and results for an arbitrary connected quiver:

define representations of a quiver, morphisms of quiver representations, simple representations. Define the path algebra of the quiver and give notation for orthogonal idempotents corresponding to the vertices. Emphasise the equivalence between representations of the quiver and modules over the path algebra.

The category of finite dimensional representations of a quiver is Krull-Schmidt, abelian, and so on.

Give Gabriel's theorem and the generalisation due to Kac-(?).

Illustrative example(s): Ringel-Hall algebras for type $A_2$ and $A_3$ of different orientations.

## 2.1 Cyclic quivers

Let $n$ be a positive integer. Let $\Delta = (\Delta_0, \Delta_1)$ denote the (cyclic) quiver with vertex set $\Delta_0 = \mathbb{Z}/n\mathbb{Z}$ and arrows $\Delta_1 = \{i \to i+1 : i \in \Delta_0\}$.

Let $\mathbf{k}$ be a field. A representation $V = (V_i, \rho_i)_{i \in \Delta_0}$ of $\Delta$ over $\mathbf{k}$ is a collection of vector spaces $V_i : i \in \Delta_0$ and linear maps $\rho_i \colon V_i \to V_{i+1}$ for $i \in \Delta_0$. A morphism $f \colon V \to W$ is a collection of linear maps $f_i \colon V_i \to W_i$ ($i \in \Delta_0$) such that $\rho_i^W f_i = f_{i+1} \rho_i^V$ for $i \in \Delta_0$.

The category of finite dimensional representations of $\Delta$ over $\mathbf{k}$, denoted $rep_{\mathbf{k}}(\Delta)$, is abelian and Krull-Schmidt. $rep_{\mathbf{k}}(\Delta)$ is equivalent to the category of finite dimensional left modules over the path algebra $\mathbf{k}\Delta$, denoted $mod(\mathbf{k}\Delta)$.

The dimension vector of a finite dimensional representation $M$ is $\underline{\dim}(M) = (\dim(M_1), \ldots, \dim(M_n)) \in \mathbb{Z}^n$. Give consequence of Gabriel's theorem in this case. Discussion of real and imaginary roots.

roughly – real roots ($q(\alpha) = 1$): there is a unique indecomposable with dimension vector $\alpha$, up to isomorphism; imaginary roots ($q(\alpha) = 0$): a family of isoclasses of indecomposables indexed by $\mathbb{P}^1_k$?.

$M \in mod(\mathbf{k}\Delta)$ is nilpotent if there is $a > 0$ such that $(\mathbf{k}\Delta)^a \cdot M = 0$. The nilpotent modules constitute a full abelian subcategory of $mod(\mathbf{k}\Delta)$, which we denote by $mod^0(\mathbf{k}\Delta)$. The subcategory of nilpotent representations corresponds to the inhomogeneous tube of rank $n$, with quasi-simples corresponding to the 1-dimensional simple representations at each vertex. In particular, the set of isoclasses in $mod^0(\mathbf{k}\Delta)$ has a combinatorial description which does not depend on the underlying field.

For $i \in \Delta_0$, let $S_i$ denote the simple module $S_i = \mathbf{k}e_i$, where $e_i$ acts as 1 and all other paths act as 0. The indecomposable modules are uniserial and admit a composition series with composition factors (amongst) $S_1, \dots, S_n$ – see this by taking the radical filtration of an indecomposable nilpotent module.

Up to isomorphism, there is a unique module with top $S_i$ and length $l \geq 1$, which we denote by $S_i(l)$. By convention, set $S_i(0) = 0$. Then we have non-split short exact sequences

$$S_{i+1}(l-1) \to S_i(l) \to S_i$$

$$S_{i+l-1} \to S_i(l) \to S_i(l-1),$$

given by embedding of the radical and the quotient by the socle, respectively. In light of these, the convention $S_i(0) = 0$ reflects that simple modules have length 1, so $S_i(1) = S_i$.

**Lemma 2.1.1** (extensions of strings). *Let $i_1, i_2 \in \Delta_0$ and $l_1, l_2 \geq 1$. $Ext^1(S_{i_2}(l_2), S_{i_1}(l_1)) = 0$ unless $i_2 = i_1 + j_1$. In the case $i_2 = i_1 + l_1$, we have*

$$\dim Ext^1(S_{i_2}(l_2), S_{i_1}(l_1)) = 1.$$

*The class of the non-split extension is given by the short exact sequence*

$$S_{i_1+l_1}(l_2) \to S_{i_1}(l_1 + l_2) \to S_{i_1}(l_1).$$

**Lemma 2.1.2** (structure theorem). *Any $M \in \text{mod}^0(\mathbf{k}\Delta)$ decomposes uniquely as*

$$M \cong \bigoplus_{i \in \Delta_0; l \geq 1} m_{i,l} S_i(l)$$

## 2.2 Hall algebras: finite fields

We now define the Hall algebra of $mod(\mathbf{k}\Delta)$, where $\mathbf{k}$ is a finite field with $\mathrm{q} = \#\mathbf{k}$ elements: Let $\mathcal{H}(\mathbf{k}\Delta)$ be a free $\mathbb{Z}$-module with basis $Iso(\mathbf{k}\Delta)$ with a $\mathbb{Z}$-bilinear pairing given by

$$[M][N] = \sum_{[L] \in Iso(\mathbf{k}\Delta)} \phi^{[L]}_{[M],[N]} [L],$$

where

$$\phi^{[L]}_{[M],[N];\mathrm{q}} = \#\{X \leq L : X \cong N, L/X \cong M\}.$$

This is well defined, since the cardinality of the set on the right hand side does not depend on the choice of representatives $M, N, L$ of the isomorphism classes. These cardinalities are finite since $L, M, N$ are finite dimensional and $\mathbf{k}$ is a finite field. With this choice of bilinear pairing, $\mathcal{H}(\mathbf{k}\Delta)$ is a ring with $1 = [0]$, which is known as the Hall algebra of $\mathbf{k}\Delta$. The isomorphism classes of nilpotent representations span a subalgebra of $\mathcal{H}(\mathbf{k}\Delta)$, which may be seen as the Hall algebra of the category $mod^0(\mathbf{k}\Delta)$ of finite dimensional nilpotent $\mathbf{k}\Delta$-modules.

## 2.3 The Ringel-Hall algebra of a cyclic quiver

In order to define the Ringel-Hall algebra, we first give a combinatorial description of the set of isomorphism classes of finite-dimensional nilpotent $\mathbf{k}\Delta$-modules, where $\mathbf{k}$ is any field. If $M \in mod^0(\mathbf{k}\Delta)$, then

$$M \cong \bigoplus_{i\in\Delta_0, l\geq 1} m_{i,l} S_i(l),$$

for some $m_{i,l} \in \mathbb{N}$. Associate to $M$ the collection $((l_{1,1},\dots,l_{1,m_1}),\dots,(l_{n,1},\dots,l_{n,m_n}))$, which may be arranged as an upper triangular tableaux – for now I want to avoid confusion with the matrices indexing orbits in $\mathcal{F}\times\mathcal{F}$.

There exist polynomials $h^L_{M,N} \in \mathbb{Z}[q]$, for each $L, M, N \in Iso(\mathbb{C}\Delta)$, such that

$$h^L_{M,N}(\mathrm{q}) = \phi^{[L]}_{[M],[N];\mathrm{q}}$$

The generic Ringel Hall algebra of $\Delta$ is defined as follows: Let $\mathcal{H}(\Delta)$ be a free $\mathbb{Z}[q]$-module with basis $Iso^0(\mathbb{C}\Delta)$, consisting of the isomorphism classes of finite dimensional nilpotent $\mathbb{C}\Delta$-modules, with $\mathbb{Z}[q]$ bilinear pairing given by

$$[M][N] = \sum_{[L]\in Iso(\mathbb{C}\Delta)} h^L_{M,N}[L]$$

Then $\mathcal{H}(\Delta)$ is an associative $\mathbb{Z}[q]$-algebra with multiplicative identity $1 = [0]$. A short note on this construction: Proving existence of these polynomial structure constants is hard, however their uniqueness is clear and associativity of the multiplication follows from associativity of the multiplication in $\mathcal{H}(\mathbf{k}\Delta)$.

If $\mathbf{k}$ is a finite field with q elements, then the specialisation of $\mathcal{H}(\Delta)$ at $q = \mathrm{q}$ is isomorphic to the Hall algebra of nilpotent $\mathbf{k}\Delta$-modules:

$$\mathbb{Z}[q]/(q-\mathrm{q}) \otimes_{\mathbb{Z}[q]} \mathcal{H}(\Delta) \cong \mathcal{H}(mod^0(\mathbf{k}\Delta))$$

The 0-Hall algebra of $\Delta$ is the specialisation of $\mathcal{H}(\Delta)$ at $q = 0$:

$$\mathcal{H}_0(\Delta) \coloneqq \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}[q]} \mathcal{H}(\Delta)$$

# Chapter 3

# The cyclic flags approach to affine q-Schur algebras.

Fix natural numbers $n$ and $r$.

**Definition 3.0.1.** *A composition of $r$ into $n$ parts is an $n$-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of non-negative integers whose sum equals $r$. Denote the set of compositions of $r$ into $n$ parts by $\Lambda_0(n,r)$.*

**Definition 3.0.2.** *Let $\Lambda_1(n,r)$ be the set of matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with non-negative integer entries $a_{i,j}$ satisfying the following conditions: each row or column has only finitely many non-zero entries; the sum of the entries in any $n$ consecutive rows or columns equals $r$; $a_{i-n,j-n} = a_{i,j}$ for each $i,j \in \mathbb{Z}$. These matrices are referred to as infinite periodic matrices.*

**Definition 3.0.3.** *Given $A \in \Lambda_1(n,r)$, let $\operatorname{ro} A$ and $\operatorname{co} A$ be the compositions of $r$ into $n$ parts given by*

$$\operatorname{ro} A = \left( \sum_{j\in\mathbb{Z}} a_{1,j}, \ldots, \sum_{j\in\mathbb{Z}} a_{n,j} \right)$$

*and*

$$\operatorname{co} A = \left( \sum_{i\in\mathbb{Z}} a_{i,1}, \ldots, \sum_{i\in\mathbb{Z}} a_{i,n} \right).$$

*$A \in \Lambda_1(n,r)$ is said to go from $\operatorname{co} A$ to $\operatorname{ro} A$.*

## 3.1 Cyclic flags

Fix $n, r \in \mathbb{N}$ and let $\mathbf{k}$ be a field. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by $\varepsilon$, so $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let $V$ be a free $\mathcal{S}$-module of rank $r$. Let $G$ be the automorphism group of the $\mathcal{S}$-module $V$, so $G$ is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. A ($\mathcal{R}$-)lattice in $V$ is a $\mathcal{R}$-submodule $L$ of $V$ with $\mathcal{S} \otimes_{\mathcal{R}} L = V$. In particular, a lattice is an $\mathcal{R}$-submodule of $V$ which is a free $\mathcal{R}$-module of rank $r$. Let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n,r)$ be the set of collections $(L_i)_{i\in\mathbb{Z}}$ of lattices in $V$ with $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. These collections of lattices in $V$ are referred to as cyclic flags in $V$.

$G$ acts on $\mathcal{F}$ by $(g \cdot L)_i = g(L_i)$ for each $i \in \mathbb{Z}$, given $g \in G$ and $L \in \mathcal{F}$. The $G$-orbits in $\mathcal{F}$ are indexed by the set $\Lambda_0(n,r)$ of compositions of $r$ into $n$ parts: the $G$-orbit in $\mathcal{F}$ corresponding to $\lambda \in \Lambda_0(n,r)$ is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim\left(\frac{L_i}{L_{i-1}}\right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

**Definition 3.1.1.** *The periodic characteristic matrix of a pair of cyclic flags* $(L, L') \in \mathcal{F} \times \mathcal{F}$ *is the matrix* $\mathrm{A}(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$ *with entries*

$$a_{i,j} = \dim \left( \frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j} \right)$$

*for each* $i, j \in \mathbb{Z}$.

The diagonal action of $G$ on $\mathcal{F} \times \mathcal{F}$ has orbits indexed by the set $\Lambda_1(n, r)$ of infinite periodic matrices (see definition 3.0.2). The $G$-orbit corresponding to $A \in \Lambda_1(n, r)$ is denoted $\mathcal{O}_A$ and consists of those pairs $(L, L') \in \mathcal{F} \times \mathcal{F}$ with periodic characteristic matrix $\mathrm{A}(L, L')$ equal to $A$.

define $Y_{A,B}$, $X_{A,B}$ in terms of the maps $\pi, \delta$ and the product of orbits. Given $A, B \in \Lambda_1(n, r)$ define

$$X_{A,B} = \{(L, L'') : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

**Observation 1.** *There are only finitely many* $G$*-orbits in* $X_{A,B}$.

## 3.2 Convolution algebras

Suppose $\mathbf{k}$ is a finite field and let q denote the number of elements of $\mathbf{k}$. Consider the set $S$ of $G$-invariant functions $\mathcal{F} \times \mathcal{F} \to \mathbb{Z}$ with constructible support. $S$ is a free $\mathbb{Z}$-module with a basis consisting of the indicator functions of the $G$-orbits in $\mathcal{F} \times \mathcal{F}$. Define an operation $\star$ on $S$ as follows: for each $f, g \in S$, $f \star g \in S$ is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for $(L, L'') \in \mathcal{F} \times \mathcal{F}$.

$f \star g$ is well defined since the supports of $f$ and $g$ consist of finitely many $G$-orbits, so there are only finitely many $L' \in \mathcal{F}$ such that $f(L, L')g(L', L'') \neq 0$, given $(L, L'') \in \mathcal{F} \times \mathcal{F}$. $f \star g$ is constant on $G$-orbits and is supported on finitely many $G$-orbits, so $f \star g \in S$.

**Lemma 3.2.1.** *The set* $S$ *together with the operation* $\star$ *is an associative* $\mathbb{Z}$*-algebra with identity element* $\iota$ *given by* $\iota(L, L) = 1$ *and* $\iota(L, L') = 0$ *for* $L' \neq L$.

*Proof.* Given $f, g, h \in S$ and $(L, L''') \in \mathcal{F} \times \mathcal{F}$,

$$\begin{aligned} ((f * g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''), \end{aligned}$$

thus $\star$ is associative. $\iota$ is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L')\iota(L', L'') = f(L, L''),$$

for each $f \in S$ and $(L, L'') \in \mathcal{F} \times \mathcal{F}$. $\square$

Given $A \in \Lambda_1(n,r)$, let $e_A \in S$ denote the indicator function of the orbit $\mathcal{O}_A$. $S$ is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1(n,r)\}$. There exist $\gamma_{A,B,C;\mathrm{q}} \in \mathbb{Z}$ for $A, B, C \in \Lambda_1(n,r)$ such that

$$e_A \star e_B = \sum_{C \in \Lambda_1(n,r)} \gamma_{A,B,C;\mathrm{q}} e_C$$

for each $A, B \in \Lambda_1(n,r)$. Then

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L') e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},\end{aligned}$$

for any $(L, L'') \in \mathcal{O}_C$.

## 3.3 Affine q-Schur algebras

There exist a polynomial $\gamma_{A,B,C} \in \mathbb{Z}[q]$ for each $A, B, C \in \Lambda_1(n,r)$ such that $\gamma_{A,B,C}(\mathrm{q}) = \gamma_{A,B,C;\mathrm{q}}$ for any prime power q, following [ADD A REFERENCE][4]. The affine $q$-Schur algebra $\hat{S}_q(n,r)$ (defined in [ADD A REFERENCE]) is a free $\mathbb{Z}[q]$-module with basis $\{e_A : A \in \Lambda_1(n,r)\}$ and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

$(L, L') \in \mathcal{O}_A$ if and only if $(L', L) \in \mathcal{O}_{A^\top}$. In fact, the operation of transposition on $\mathcal{F} \times \mathcal{F}$ (or on $\Lambda_1(n,r)$) induces an anti-automorphism of $\hat{S}_q(n,r)$.

**Lemma 3.3.1.** *Transposition gives a homomorphism of $\mathbb{Z}[q]$-modules $\top \colon \hat{S}_q(n,r) \to \hat{S}_q(n,r)$ with $\top(e_A) = e_{A^\top}$, $\top \circ \top = 1$ and $\top(e_A e_B) = \top(e_B)\top(e_A)$.*

*Proof.* Let $A, B, C \in \Lambda_1(n,r)$ and let $\mathbf{k}$ be a finite field with $\mathrm{q} = \#\mathbf{k}$ elements. If $(L, L'') \in \mathcal{O}_C$ then $(L'', L) \in \mathcal{O}_{C^\top}$ and

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;\mathrm{q}}\end{aligned}$$

It then follows that $\top(e_A e_B) = \top(e_B)\top(e_A)$. ∎

# Chapter 4

# Quivers with relations for affine q-Schur algebras.

## 4.1 Basic results: TO BE REPLACED WITH A MORE INFORMATIVE NAME.

If $i, j \in \mathbb{Z}$, let $\mathcal{E}_{i,j}$ denote the 'elementary matrix' with entries given by $(\mathcal{E}_{i,j})_{s,t} = 1$, for $s, t \in \mathbb{Z}$, whenever $(i, j) \sim (s, t)$ modulo $(n, n)$ and all other entries are zero.

Given $\lambda \in \Lambda_0(n, r)$, let $D_\lambda \in \Lambda_1(n, r)$ denote the diagonal matrix with $r(D_\lambda) = c(D_\lambda) = \lambda$. That is,

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}$$

For $\lambda \in \Lambda_0(n, r)$, write $1_\lambda = e_{D_\lambda}$. The $1_\lambda$ are pairwise orthogonal idempotents in $\hat{S}_q(n, r)$ with $1 = \sum_{\lambda \in \Lambda_0(n,r)} 1_\lambda$.

Given $i, j \in \mathbb{Z}$, write $X_{i,j} = \mathcal{E}_{i,j} - \mathcal{E}_{i+1,j}$. By convention, $e_A = 0$ unless $A \in \Lambda_1(n, r)$.

For $i \in [1, n]$ and $\lambda \in \Lambda_0(n, r)$, write

$$E_{i,\lambda} = e_{D_\lambda + X_{i,i+1}},$$

$$F_{i,\lambda} = e_{D_\lambda - X_{i,i}}.$$

Define

$$E_i = \sum_{\lambda \in \Lambda_0(n,r)} E_{i,\lambda}$$

$$F_i = \sum_{\lambda \in \Lambda_0(n,r)} F_{i,\lambda}.$$

Observe that $E_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $F_{i,\lambda} = 0$ unless $\lambda_i > 0$. Also, $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$.

**Lemma 4.1.1.** *Let $i \in [1, n]$ and $A \in \Lambda_1(n, r)$.*

$$E_i e_A = \sum_{p \in \mathbb{Z}} q^{\sum_{j>p} a_{i,j}} [a_{i,p} + 1] e_{A + X_{i,p}}$$

*and*

$$F_i e_A = \sum_{p \in \mathbb{Z}} q^{\sum_{j<p} a_{i+1,j}} [a_{i+1,p} + 1] e_{A - X_{i,p}}.$$

Note that these formulas are still valid in the cases $E_i e_A = 0$ and $F_i e_A = 0$. There are similar formulas for right multiplication by $E_i$ and $F_i$, which can be obtained by applying the transpose involution to the above formulas. The transpose relates the $E_i$, $F_i$ and $1_\lambda$ in the following way: $\top(E_{i,\lambda}) = F_{i,\lambda}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\varepsilon_i+\varepsilon_{i+1}}$ and $\top(1_\lambda) = 1_\lambda$. In particular, $\top(E_i) = F_i$ and $\top(F_i) = E_i$.

**Corollary 4.1.2.** *Let $j \in [1,n]$ and $A \in \Lambda_1(n,r)$. Then*

$$e_A F_j = \sum_{p\in\mathbb{Z}} q^{\sum_{i>p} a_{i,j}} [a_{p,j}+1] e_{A+X_{j,p}^\top}$$

*and*

$$e_A E_j = \sum_{p\in\mathbb{Z}} q^{\sum_{i<p} a_{i,j+1}} [a_{p,j+1}+1] e_{A-X_{j,p}^\top}$$

*Proof.*

$$\begin{aligned} e_A F_j &= \top(E_j e_{A^\top}) \\ &= \top(\sum_p q^{\sum_{i>p} a_{i,j}} [a_{p,j}+1] e_{A^\top + X_{j,p}}) \\ &= \sum_p q^{\sum_{i>p} a_{i,j}} [a_{p,j}+1] e_{A+X_{j,p}^\top} \end{aligned}$$

$$\begin{aligned} e_A E_j &= \top(F_j e_{A^\top}) \\ &= \top(\sum_p q^{\sum_{i<p} a_{i,j+1}} [a_{p,j+1}+1] e_{A^\top - X_{j,p}}) \\ &= \sum_p q^{\sum_{i<p} a_{i,j+1}} [a_{p,j+1}+1] e_{A-X_{j,p}^\top} \end{aligned}$$

□

Note that $E_i^{r+1} = F_i^{r+1} = 0$ while

$$E_i^r = [r]_! e_{r\mathcal{E}_{i,i+1}}$$

and

$$F_i^r = [r]_! e_{r\mathcal{E}_{i+1,i}}.$$

**Lemma 4.1.3** (quantum Serre relations: $n \geq 3$)**.** *Suppose $n \geq 3$. The following relations hold in $\hat{S}_q(n,r)$:*

$$E_i E_j - E_j E_i = 0$$
$$F_i F_j - F_j F_i = 0$$

*unless $j = i \pm 1$;*

$$\begin{aligned} E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i &= 0 \\ E_i^2 E_{i+1} - (1+q) E_i E_{i+1} E_i + q E_{i+1} E_i^2 &= 0 \end{aligned}$$

*and*

$$\begin{aligned} F_{i+1} F_i^2 - (1+q) F_i F_{i+1} F_i + q F_i^2 F_{i+1} &= 0 \\ F_{i+1}^2 F_i - (1+q) F_{i+1} F_i F_{i+1} + q F_i F_{i+1}^2 &= 0. \end{aligned}$$

*Proof.* Here we introduce temporary notation for the basis elements: Write $[A] = e_A$.

Take $\lambda \in \Lambda_0(n,r)$.

$$E_iE_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1}E_iE_{i+1}1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2 E_i 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then

$$(E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i)1_\lambda = 0,$$

for each $\lambda \in \Lambda_0(n,r)$. The relation $E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i = 0$ then follows.

The relations between $F_i$ and $F_{i+1}$ may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping $E_i$ and $F_i$ and reversing the order of multiplication. □

**Lemma 4.1.4** (quantum Serre relations: $n = 2$)**.** *In the case $n = 2$, the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

**Lemma 4.1.5.** $[E_i, F_j] = 0$ *unless* $j = i$.

$$E_iF_i - F_iE_i = \sum_{\lambda\in\Lambda_0(n,r)} ([\lambda_i] - [\lambda_{i+1}])1_\lambda.$$

For $\lambda \in \Lambda_0(n,r)$, let $R_\lambda = e_{\lambda_1\mathcal{E}_{0,1}+\cdots+\lambda_n\mathcal{E}_{n-1,n}}$. Write $R = \sum_{\lambda\in\Lambda_0(n,r)} R_\lambda$. Note $R_\lambda = R1_\lambda$. Given $A \in \Lambda_1(n,r)$ and $m \in \mathbb{Z}$, let $A[m] \in \Lambda_1(n,r)$ be given by $A[m]_{i,j} = a_{i,j+m}$ and let $A^{[m]}$ be given by $A^{[m]}_{i,j} = a_{i+m,j}$ for each $i \in \mathbb{Z}$.

**Lemma 4.1.6** (Shifting)**.** *If $A \in \Lambda_1(n,r)$ then*

$$Re_A = e_{A^{[\pm 1]}}$$

*and*

$$e_AR = e_{A_{[\pm 1]}}.$$

Conjugation by $R$ gives an automorphism $\rho$ of $\hat{S}_q(n,r)$ satisfying $\rho^n = 1$.

## 4.2 quivers with relations

Denote by $\Lambda_0(n,r)$ the set of compositions of $r$ into $n$ parts. That is, $\Lambda_0(n,r)$ is the set of $\alpha \in \mathbb{Z}^n$ with non-negative entries which sum to $r$. Let $\varepsilon_i \in \mathbb{Z}^n$ be the $i$th elementary vector and write $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for each $i \in [1,n]$. Then $\lambda + \alpha_i \in \Lambda_0(n,r)$ if $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0(n,r)$ if $\lambda_i > 0$.

Let $\Gamma = \Gamma(n,r)$ be the quiver with set of vertices $\Lambda_0(n,r)$, with the following arrows:

For $\lambda \in \Lambda_0(n,r)$ and $i \in [1,n]$, there is an arrow $e_{i,\lambda} : \lambda \to \lambda + \alpha_i$ if $\lambda_{i+1} > 0$ and there is an arrow $f_{i,\lambda} : \lambda \to \lambda - \alpha_i$ if $\lambda_i > 0$.

Denote by $\mathbb{Z}[q]\Gamma$ the path $\mathbb{Z}[q]$-algebra of $\Gamma$. Thus $\mathbb{Z}[q]\Gamma$ is a free $\mathbb{Z}[q]$-module with a basis given by the set of paths in $\Gamma$, with multiplication given by the concatenation of paths. If $p$ starts where $q$ ends, the product $pq$ is the path $q$ followed by $p$. Write $e_{i,\lambda} = 0$ unless $\lambda, \lambda+\alpha_i \in \Lambda_0(n,r)$ and write $f_{i,\lambda} = 0$ unless $\lambda, \lambda - \alpha_i \in \Lambda_0(n,r)$.

By construction, there is a homomorphism of $\mathbb{Z}[q]$-algebras

$$\phi \colon \mathbb{Z}[q]\Gamma \to \hat{S}_q(n,r)$$

given by

$$\begin{aligned}\phi(e_{i,\lambda}) &= E_{i,\lambda}\\ \phi(f_{i,\lambda}) &= F_{i,\lambda}\\ \phi(k_\lambda) &= 1_\lambda,\end{aligned}$$

for $i \in [1,n]$ and $\lambda \in \Lambda_0(n,r)$.

The image of $\phi$ is the subalgebra of $\hat{S}_q(n,r)$ generated by $E_i$, $F_i$ for $i \in [1,n]$ and $1_\lambda$ for $\lambda \in \Lambda_0(n,r)$, since $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$, while $E_i = \sum_\lambda E_{i,\lambda}$ and $F_i = \sum_\lambda F_{i,\lambda}$. In general $\phi$ is not surjective, so this does not always lead to a presentation of $\hat{S}_q(n,r)$.

### 4.2.1 Exceptional case $n = 2$.

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the q-Schur algebra.

### 4.2.2 Typical case $n > 2$.

Suppose $n \geq 3$. Then $\Gamma = \Gamma(n,r)$ has vertex set $\Lambda_0(n,r)$. RESUME HERE...

Define $e_i, f_i \in \mathbb{Z}[q]\Gamma(n,r)$ by

$$e_i = \sum_{\lambda \in \Lambda_0(n,r)} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0(n,r)} f_{i,\lambda},$$

with the convention $e_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $f_{i,\lambda} = 0$ unless $\lambda_i > 0$. Let $k_\lambda \in \mathbf{k}\Delta$ denote the constant path at vertex $\lambda$. $\{k_\lambda : \lambda \in \Lambda_0(n,r)\}$ is a set of pairwise orthogonal idempotents in $\mathbb{Z}[q]\Gamma(n,r)$.

Let $I(n,r) \subset \mathbb{Z}[q]\Gamma(n,r)$ be the ideal generated by the expressions

$$\begin{gathered}e_i e_{i+1}^2 - (1+q)e_{i+1}e_i e_{i+1} + q e_{i+1}^2 e_i\\ e_i^2 e_{i+1} - (1+q)e_i e_{i+1} e_i + q e_{i+1} e_i^2\\ f_{i+1} f_i^2 - (1+q) f_i f_{i+1} f_i + q f_i^2 f_{i+1}\\ f_{i+1}^2 f_i - (1+q) f_{i+1} f_i f_{i+1} + q f_i f_{i+1}^2\\ e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0(n,r)} ([\lambda_i] - [\lambda_{i+1}]) k_\lambda\end{gathered}$$

Recall that a relation is a $\mathbb{Z}[q]$-linear combination of paths with common start and end vertices. The relations involving paths $\lambda \to \mu$ are given by $1_\mu \mathrm{expr} 1_\lambda$, for each of the above expressions.

**Lemma 4.2.1.** *There is a homomorphism of $\mathbb{Z}[q]$-algebras*

$$\phi\colon \mathbb{Z}[q]\Gamma(n,r)/I(n,r) \to \hat{S}_q(n,r)$$

*given by*

$$\phi(e_{i,\lambda}) = E_{i,\lambda}$$
$$\phi(f_{i,\lambda}) = F_{i,\lambda}$$
$$\phi(k_\lambda) = 1_\lambda.$$

# Chapter 5

# A generic affine Schur algebra.

## 5.1 Introducing the affine generic algebra

Assume $\mathbf{k} = \mathbb{C}$ and fix $n, r \geq 1$. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by $\varepsilon$, namely $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let $V$ be a free $\mathcal{S}$-module of rank $r$ and let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of $n$-periodic cyclic flags in $V$; so $\mathcal{F}$ consists of collections $L = (L_i)_{i \in \mathbb{Z}}$ of $\mathcal{R}$-lattices in $V$ with $L_i \subset L_{i+1}$ for $i \in \mathbb{Z}$ and $\varepsilon L_i = L_{i-n}$ for $i \in \mathbb{Z}$.

Let $G$ be the group of $\mathcal{S}$-module automorphisms of $V$. Thus $G$ is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. $G$ acts on $\mathcal{F}$ with orbits $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0(n, r)\}$, where $\Lambda_0(n, r)$ is the set of compositions of $r$ into $n$ parts.

The diagonal action of $G$ on $\mathcal{F} \times \mathcal{F}$ has orbits $\{\mathcal{O}_A : A \in \Lambda_1(n, r)\}$, where $\mathcal{O}_A$ consists of those pairs of flags with periodic characteristic matrix equal to $A$. In particular, the periodic characteristic matrix of a pair $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix $A = (a_{i,j})_{i,j \in \mathbb{Z}}$, with

$$a_{i,j} = \dim \left( \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right),$$

for each $i, j \in \mathbb{Z}$.

### 5.1.1 Not quite a category

There are maps $\mathrm{ro}, \mathrm{co} \colon \Lambda_1(n, r) \to \Lambda_0(n, r)$ given by

$$\mathrm{ro}\, A = \left( \sum_j a_{1,j}, \ldots, \sum_j a_{n,j} \right)$$

and

$$\mathrm{co}\, A = \left( \sum_i a_{i,1}, \ldots, \sum_i a_{i,n} \right).$$

Given $A \in \Lambda_1(n, r)$, write $\mathrm{co}\, A \xrightarrow{A} \mathrm{ro}\, A$. The purpose of this chapter is to define a category with objects $\Lambda_0(n, r)$ and morphisms $\Lambda_1(n, r)$; where $\mathrm{Hom}(\lambda, \mu) = \{A \in \Lambda_1(n, r) : \mathrm{ro}\, A = \mu, \mathrm{co}\, A = \lambda\}$. Given $A, B \in \Lambda_1(n, r)$ let $\Lambda_1(n, r)_{A,B}$ be the set of $C \in \Lambda_1(n, r)$ such that there exist $L, L', L'' \in \mathcal{F}$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L'', L''') \in \mathcal{O}_C$. It will be shown that $\Lambda_1(n, r)$ admits a partial order $\leq$ such that $\Lambda_1(n, r)_{A,B}$ has a maximum element $A * B$, whenever $\mathrm{co}\, A = \mathrm{ro}\, B$. It will be shown that $*$ is associative, so defining the composition of morphisms in the category formed by $\Lambda_0(n,r)$ and $\Lambda_1(n,r)$.

The generic affine Schur algebra $\hat{G}(n,r)$ will then be a $\mathbb{Z}$-algebra defined as a linearisation of this category. It will be shown that $\hat{G}(n,r)$ gives a realisation of the affine 0-Schur algebra $\hat{S}_0(n,r)$ when $r < n$. It is expected that a more refined presentation of the generic algebra and the 0-Schur algebra will allow the conditions on the parameters to be relaxed slightly: the $r = n$ case is approachable, which may extend to the case $r < 2n$.

## 5.2 A partial order

Given $i, j \in \mathbb{Z}$, define a map $d_{i,j}$ on $\Lambda_1(n,r)$ by setting

$$d_{i,j}A = \sum_{s \leq i, t > j} a_{s,t}$$

for each $A \in \Lambda_1(n,r)$.

**Lemma 5.2.1.** *Let $A \in \Lambda_1(n,r)$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for $i, j \in \mathbb{Z}$. Then*

$$d_{i+1,j} - d_{i,j} = \sum_{t>j} a_{i+1,t}$$

*and*

$$d_{i,j+1} - d_{i,j} = -\sum_{s \leq i} a_{s,j}.$$

*Proof.* Let $i, j \in \mathbb{Z}$. Then

$$d_{i+1,j} - d_{i,j} = \sum_{s \leq i+1, t > j} a_{s,t} - \sum_{s \leq i, t > j} a_{s,t} = \sum_{t > j} a_{i+1,t}.$$

Similarly,

$$d_{i,j+1} - d_{i,j} = \sum_{s \leq i, t > j+1} a_{s,t} - \sum_{s \leq i, t > j} a_{s,t} = -\sum_{s \leq i} a_{s,j}.$$

□

**Lemma 5.2.2.** *Let $A \in \Lambda_1(n,r)$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for each $i, j \in \mathbb{Z}$. Then*

$$a_{i,j} = d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j}$$

*for each $i, j \in \mathbb{Z}$.*

*Proof.* Using lemma 5.2.1,

$$\begin{aligned} a_{i,j} &= \sum_{t > j-1} a_{i,t} - \sum_{t>j} a_{i,t} \\ &= (d_{i,j-1} - d_{i-1,j-1}) - (d_{i,j} - d_{i-1,j}). \end{aligned}$$

Alternatively,

$$\begin{aligned} a_{i,j} &= \sum_{s \leq i} a_{s,j} - \sum_{s \leq i-1} a_{s,j} \\ &= (d_{i,j-1} - d_{i,j}) - (d_{i-1,j-1} - d_{i-1,j}). \end{aligned}$$

□

**Lemma 5.2.3.** *The relation $\leq$ on $\Lambda_1(n,r)$, defined by $A \leq B$ if and only if $d_{i,j}A \leq d_{i,j}B$ for all $i,j \in \mathbb{Z}$, is a partial order.*

*Proof.* It is clear that $\leq$ is reflexive and transitive, so it remains to see $\leq$ is antisymmetric. Suppose $A, B \in \Lambda_1(n,r)$ with $A \leq B$ and $B \leq A$. Then $d_{i,j}A = d_{i,j}B$ for each $i,j \in \mathbb{Z}$, which shows $A = B$ as a result of lemma 5.2.2. $\square$

The partial order on $\Lambda_1(n,r)$ induces a partial order on the set of $G$-orbits in $\mathcal{F} \times \mathcal{F}$, such that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $A \leq B$. The next lemma gives some geometric significance to the partial order on $\Lambda_1(n,r)$.

**Lemma 5.2.4.** *Let $A \in \Lambda_1(n,r)$ and take $(L, L') \in \mathcal{O}_A$. Then*

$$d_{i,j}A = \dim\left(\frac{L_i}{L_i \cap L'_j}\right)$$

*for each $i,j \in \mathbb{Z}$.*

It is thought* that the partial order on $\Lambda_1(n,r)$ is compatible with the degeneration order (or closure order) on $G$-orbits in $\mathcal{F} \times \mathcal{F}$ when $\mathbf{k} = \mathbb{C}$. In particular, it is hoped that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $\mathcal{O}_A \subset \overline{\mathcal{O}_B}$.

## 5.3 Preliminary results

Fix $L \in \mathcal{F}$.

**Lemma 5.3.1.** *$L_0/\varepsilon L_0$ is a torsion $\mathbf{k}[\varepsilon]$-module, where $\varepsilon$ acts as zero, with dimension $r$ as a $\mathbf{k}$-vector space.*

*Proof.* Let $V = \mathbf{k}[\varepsilon, \varepsilon^{-1}]^r$. $L_0$ is a free $\mathbf{k}[\varepsilon]$-module of rank $r$, with $L_0 \subset V$. So we may take a $\mathbf{k}[\varepsilon]$-basis $x_1, \ldots, x_r \in V$ for $L_0$. The action of $\varepsilon$ gives an automorphism of $V$ mapping $L_0$ to $\varepsilon L_0$, so $\varepsilon x_1, \ldots, \varepsilon x_r$ give a basis for $\varepsilon L_0$ over $\mathbf{k}[\varepsilon]$. Therefore, the cosets $x_1 + \varepsilon L_0, \ldots x_r + \varepsilon L_0$ give a basis for $L_0/\varepsilon L_0$ over $\mathbf{k}$. $\square$

Suppose $A, B \in \Lambda_1(n,r)$ with $\operatorname{co} A = \operatorname{ro} B$. Recall the notation

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

and

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L', L'') \in Y_{A,B}\}.$$

$X_{A,B}$ is the image of $Y_{A,B}$ under the projection onto the first and last components.

**Lemma 5.3.2.** *There is $N \in \mathbb{N}$ such that*

$$\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$$

*whenever $(L, L'') \in X_{A,B}$.*

*Proof.* There exist $N_1, N_2 \in \mathbb{N}$ such that

$$\varepsilon^{N_1} L_0 \subset L'_0 \subset \varepsilon^{-N_1} L_0$$

and

$$\varepsilon^{N_2} L'_0 \subset L''_0 \subset \varepsilon^{-N_2} L'_0,$$

whenever $(L, L', L'') \in Y_{A,B}$. Then, for $(L, L', L'') \in Y_{A,B}$,

$$L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-(N_1+N_2)} L_0$$

and

$$\varepsilon^{N_1+N_2} L_0 \subset \varepsilon^{N_2} L'_0 \subset L''_0.$$

In particular, taking $N = N_1 + N_2$, we have

$$\varepsilon^{N} L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$$

whenever $(L, L'') \in X_{A,B}$. □

**Lemma 5.3.3.** *Suppose $N_1, N_2 \in \mathbb{N}$ with $\varepsilon^{N_1} L_0 \subset L_0 \subset \varepsilon^{-N_1} L_0$ and $\varepsilon^{N_2} L'_0 \subset L''_0 \subset \varepsilon^{-N_2} L'_0$ whenever $(L, L', L'') \in Y_{A,B}$ and let $N = N_1 + N_2$. Then*

$$\dim\left(\frac{\varepsilon^{-N} L_0}{L''_0}\right) = d_{nN_1,0}(A) + d_{nN_2,0}(B)$$

*and*

$$\dim\left(\frac{L''_0}{\varepsilon^{N} L_0}\right) = 2Nr - d_{nN_1,0}(A) + d_{nN_2,0}(B),$$

*whenever $(L, L'') \in X_{A,B}$.*

*Proof.* Suppose $(L, L'') \in X_{A,B}$ and $L' \in \mathcal{F}$ so that $(L, L', L'') \in Y_{A,B}$. As in lemma 5.3.2, $\varepsilon^{N} L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$, so

$$\dim\left(\frac{\varepsilon^{-N} L_0}{L''_0}\right) + \dim\left(\frac{L''_0}{\varepsilon^{N} L_0}\right) = \dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{N} L_0}\right).$$

As a $\mathbf{k}$-vector space, $\varepsilon^{-N} L_0 / \varepsilon^{N} L_0$ is isomorphic to $(L_0/\varepsilon L_0)^{2N}$, which has dimension $2Nr$, so

$$\dim\left(\frac{L''_0}{\varepsilon^{N} L_0}\right) = 2Nr - \dim\left(\frac{\varepsilon^{-N} L_0}{L''_0}\right).$$

It remains to compute the codimension of $L''_0$ in $\varepsilon^{-N} L_0$. Note $L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-N} L_0$, so

$$\dim\left(\frac{\varepsilon - N L_0}{L''_0}\right) = \dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0}\right) + \dim\left(\frac{\varepsilon^{-N_2} L'_0}{L''_0}\right).$$

$$\begin{aligned}
\dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0}\right) &= \dim\left(\frac{\varepsilon^{-N_1} L_0}{L'_0}\right) \\
&= \dim\left(\frac{L_{nN_1}}{L_{nN_1} \cap L'_0}\right) \\
&= \sum_{s \leq nN_1, t > 0} A_{s,t} \\
&= d_{nN_1,0}(A).
\end{aligned}$$

$$\begin{aligned}\dim\left(\frac{\varepsilon^{-N_2}L'_0}{L''_0}\right) &= \dim\left(\frac{L'_{nN_2}}{L'_{nN_2}\cap L''_0}\right)\\ &= \sum_{s\le nN_2, t>0} B_{s,t}\\ &= d_{nN_2,0}(B).\end{aligned}$$

□

Fix $L \in \mathcal{F}$. Given $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0(n,r)$, define

$$\Pi_{N,\lambda} = \{L'' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0\}$$

and

$$\Pi^a_{N,\lambda} = \left\{L'' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L''_0 \subset \varepsilon^N L_0, \dim\left(\frac{\varepsilon^{-N}L_0}{L''_0}\right) = a\right\}.$$

$\Pi_{N,\lambda}$ is the (disjoint) union of the $\Pi^a_{N,\lambda}$ for $a \in \mathbb{N}$. In fact, we will see $\Pi^a_{N,\lambda}$ is empty whenever $a > 2Nr$.

**Lemma 5.3.4.** *Let $N, a \in \mathbb{N}$, $\lambda \in \Lambda_0(n,r)$. Then $\Pi^a_{N,\lambda}$ is nonempty exactly when $0 \le a \le 2Nr$.*

*Proof.* Suppose $L'' \in \Pi_{N,\lambda}$. By definition, $\varepsilon^{-N}L_0 \subset L''_0 \subset \varepsilon^{-N}L_0$, which shows

$$\dim\left(\frac{\varepsilon^{-N}L_0}{L''_0}\right) \le \dim\left(\frac{\varepsilon^{-N}L_0}{\varepsilon^N L_0}\right) = 2Nr.$$

Therefore, $\Pi^a_{N,\lambda}$ is empty unless $a \le 2Nr$.

Now assume $0 \le a \le 2Nr$. We may choose an $\varepsilon$-invariant subspace $W'$ of $W = \varepsilon^{-N}L_0/\varepsilon^N L_0$ of codimension $a$. $W'$ lifts to give a $\mathcal{R}$-module, say $L''_0$, with $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N}L_0$ and with $\dim(\varepsilon^{-N}L_0/L''_0) = \dim(W/W') = a$. Similarly, a flag of type $\lambda$ in $L''_0/\varepsilon L''_0$ lifts to give $\mathcal{R}$-modules $(L''_{-n+1}, \dots, L''_0)$ with

$$\varepsilon L''_0 \subset L''_{-n+1} \subset \dots \subset L''_{-1} \subset L''_0 \subset \varepsilon^{-N}L_0$$

and such that the dimensions of successive quotients are given by $\lambda_1, \dots, \lambda_n, a$, from left to right. Thus, $(L''_{-n+1}, \dots, L''_0)$ extends by periodicity to give an element of $\Pi^a_{N,\lambda}$, as desired. □

**Lemma 5.3.5.** *$\Pi^a_{N,\lambda}$ is a (quasi)projective variety, provided $0 \le a \le 2Nr$.*

*Proof.* Let $W = \varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$ and let

$$X = \left\{W_1 \le \dots \le W_n \le W : \dim\left(\frac{W}{W_n}\right) = a, \dim\left(\frac{W_i}{W_{i-1}}\right) = \lambda_i \text{ for } i = 2, \dots, n\right\}.$$

$X$ is known to be a projective variety [CITATION NEEDED]

Let $X'$ be the subset of $X$ consisting of those $(W_1, \dots, W_n)$, where each $W_i$ is $\varepsilon$-invariant and $\varepsilon W_n \le W_1$. $X'$ is a closed subset of $X$, though is not necessarily irreducible.

The correspondence between the set of $\mathcal{R}$-submodules of $\varepsilon^{-(1+N)}L_0$ which contain $\varepsilon^N L_0$ and the set of $\mathcal{R}$-submodules of $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$ gives a pair of mutually inverse maps $\Pi^a_{N,\lambda} \leftrightarrow X'$.

– the idea that is relevant to the proof is that inclusion relations $L_i \subset L_{i+1}$ describe a closed set in a product of grassmanians. Unsure here – Is it true that irreducible components of $X'$ are projective varieties. In this case, should the statement be that $\Pi^a_{N,\lambda}$ is a projective algebraic set, rather that a quasi projective variety? □

**Lemma 5.3.6.** *Suppose $(L', L'') \in \mathcal{O}_B$ with $(L, L') \in \mathcal{O}_A$. Then $X^L_{A,B}$ is the image of the map*

$$G_L \times G_{L'} \to \mathcal{F} : (\alpha, \beta) \mapsto \alpha\beta L''.$$

*Proof.* Suppose $\alpha \in G_L$ and $\beta \in G_{L'}$. $(L, \alpha L', \alpha\beta L'') \in Y_{A,B}$ since $(L, \alpha L') \sim (L, L') \in \mathcal{O}_A$ and $(\alpha L', \alpha\beta L'') \sim (L', L'') \in \mathcal{O}_B$. This shows $(L, \alpha\beta L'') \in X_{A,B}$ and thus $\alpha\beta L'' \in X^L_{A,B}$.

Conversely, suppose $N'' \in X^L_{A,B}$. $(L, N'') \in X_{A,B}$, so there is $N'$ such that $(L, N') \in \mathcal{O}_A$ and $(N', N'') \in \mathcal{O}_B$. There exist $\gamma, \delta \in G$ such that $\gamma(L, L') = (N, N')$ and $\delta(L', L'') = (N', N'')$. Then $(L, N', N'') = (L, \gamma L', \delta L'') = (L, \gamma L', \gamma(\gamma^{-1}\delta)L'')$, where $\gamma \in G_L$ and $\gamma^{-1}\delta \in G_{L'}$. This shows $N'' \in G_L G_{L'} L''$ as required. □

Given $N \in \mathbb{N}$, define

$$H_N = \left\{ h \in G_L : h = 1 \text{ on } \frac{\varepsilon^{-(1+N)} L_0}{\varepsilon^N L_0} \right\}.$$

Explicitly, the condition $h = 1$ on $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$ means: $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)}L_0$. Observe that $H_{N+1} \subset H_N$ for $N \in \mathbb{N}$ since $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ whenever $x \in \varepsilon^{-(1+N)}L_0$.

**Lemma 5.3.7.** *$H_N$ is a normal subgroup in $G_L$, for any $N \in \mathbb{N}$.*

*Proof.* $H_N \subset G_L$ by definition. Suppose $h, h' \in H_N$ and let $x \in \varepsilon^{-(1+N)}L_0$. $h'(x) \in \varepsilon^{-(1+N)}L_0$ as $h' \in G_L$, so $hh'(x) + \varepsilon^N L_0 = h'(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$, which shows $hh' \in H_N$. $h(x) - x \in \varepsilon^N L_0$, so $h^{-1}(x) - x = -h^{-1}(h(x) - x) \in \varepsilon^N L_0$. $h^{-1} \in H_N$, so $H_N$ is a subgroup of $G_L$.

Let $g \in G_L$. $hg^{-1}(x) + \varepsilon^N L_0 = g^{-1}(x)$ as $g^{-1}(x) \in \varepsilon^{-(1+N)}L_0$, so $ghg^{-1}(x) + \varepsilon^N L_0 = gg^{-1}(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$. Thus $ghg^{-1} \in H_N$, which proves $H_N$ is a normal subgroup in $G_L$. □

**Lemma 5.3.8.** *$G_L/H_N$ is an irreducible algebraic group, for any $N \in \mathbb{N}$.*

*Proof.* COMPLETE THIS PROOF. □

The $H_N$ form a descending chain of normal subgroups in $G_L$: $\cdots \subset H_1 \subset H_0 \subset G_L \subset G$.

**Lemma 5.3.9.** *There is $N \in \mathbb{N}$ such that $H_N \subset G_{L'}$. Consequently, $H_{N'} \subset G_{L'}$ whenever $N' \geq N$.*

*Proof.* Choose $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$. Then

$$\varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset \cdots \subset L'_n \subset \varepsilon^{-(1+N)} L_0.$$

Let $h \in H_N$. $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)}L_0$, so $h(L'_i) \subset L'_i$ for $i = 0, 1, \ldots, n$. Moreover, $h^{-1}$ stabilises $L'_i$, so $h(L'_i) = L'_i$ for $i = 0, 1, \ldots, n$ and therefore for $i \in \mathbb{Z}$. This shows $h \in G_{L'}$ as required, so $H_N \subset G_{L'}$. □

Note that $H_N$ is generally not a normal subgroup of $G_{L'}$, though the space of (right) cosets of $H_N$ in $G_{L'}$ will still be irreducible.

**Lemma 5.3.10.** *$G_{L'}/H_N$ is irreducible, provided $H_N \subset G_{L'}$.*

*Proof.* COMPLETE THIS PROOF. □

## 5.4 Existence of a maximum

**Proposition 5.4.1.** *Given $A, B \in \Lambda_1(n,r)$ with* $\operatorname{co} A = \operatorname{ro} B$*, $\Lambda_1(n,r)_{A,B}$ has a maximum element.*

*Draft 1.* $\Lambda_1(n,r)_{A,B}$ is non-empty since $\operatorname{co} A = \operatorname{ro} B$. The partial order on $\Lambda_1(n,r)_{A,B}$ is given by the partial order on $\Lambda_1(n,r)$; where $C' \leq C$ if and only if $d_{i,j}C' \leq d_{i,j}C$ for all $i,j \in \mathbb{Z}$.

To prove existence of a maximum element in $\Lambda_1(n,r)_{A,B}$ we will consider the poset of $G$-orbits in $\mathcal{F}\times\mathcal{F}$ and prove existence of a maximum orbit in $X_{A,B}$ using an open orbits argument. Recall $X_{A,B}$ consists of $(L, L'') \in \mathcal{F}\times\mathcal{F}$ such that there exists $L' \in \mathcal{F}$ with $(L, L') \in \mathcal{O}_A$ and $(L', L'') \in \mathcal{O}_B$.

There is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$ whenever $(L, L'') \in X_{A,B}$. Fix $L \in \mathcal{F}_{\operatorname{ro} A}$ and write

$$X^L_{A,B} = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

With the above choice of $N$, write

$$\Pi = \{L'' \in \mathcal{F}_{\operatorname{co} B} : \varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0\}.$$

$\Pi$ is a complex projective variety (not generally irreducible), closed under the action of $G_L$. [ADD A REFERENCE]The closure $\overline{X^L_{A,B}}$ of $X^L_{A,B}$ in $\Pi$ is an irreducible complex projective variety.

Proposition [ADD A REFERENCE]shows there is a unique $G_L$-orbit in $X^L_{A,B}$ which is open in $\overline{X^L_{A,B}}$, say $\mathcal{O}^L_C$ for some $C \in \Lambda_1(n,r)_{A,B}$. It will be shown that $C$ is the maximum element of $\Lambda_1(n,r)_{A,B}$. Given $i,j \in \mathbb{Z}$, let $m_{i,j}$ denote the maximum of $\{d_{i,j}C : C \in \Lambda_1(n,r)_{A,B}\}$ and define

$$\mathcal{M}_{i,j} = \{L'' \in \overline{X^L_{A,B}} : d_{i,j}(L, L'') = m_{i,j}\}.$$

$\mathcal{M}_{i,j}$ is non-empty by definition of the $m_{i,j}$ and is closed under the action of $G_L$. $\mathcal{M}_{i,j}$ is open in $\overline{X^L_{A,B}}$ since the function

$$d^L_{i,j} : \Pi \to \mathbb{Z} : L'' \mapsto \dim\left(\frac{L_i}{L_i \cap L''_j}\right)$$

is lower semi-continuous [ADD A REFERENCE]and

$$\mathcal{M}_{i,j} = \overline{X^L_{A,B}} \setminus \{L'' \in \overline{X^L_{A,B}} : d^L_{i,j}(L'') \leq m_{i,j} - 1\}.$$

It follows that $\mathcal{O}^L_C$ and $\mathcal{M}_{i,j}$ intersect non-trivially, since $\overline{X^L_{A,B}}$ is irreducible and therefore $\mathcal{O}^L_C \subset \mathcal{M}_{i,j}$ as both are closed under the action of $G_L$. This proves $C$ is a maximum element of $\Lambda_1(n,r)_{A,B}$, since

$$d_{i,j}C = d_{i,j}(L, L'') = m_{i,j}$$

for any $L'' \in \mathcal{O}^L_C$. □

*Draft 2.* $\Lambda_1(n,r)_{A,B}$ is non-empty since $\operatorname{co} A = \operatorname{ro} B$. For each $i,j \in \mathbb{Z}$, define

$$m_{i,j} = \max_{C \in \Lambda_1(n,r)_{A,B}} d_{i,j}C.$$

It will be shown that there is a unique element $A * B \in \Lambda_1(n,r)_{A,B}$ with $d_{i,j}(A * B) = m_{i,j}$: such an element is necessarily a maximum in $\Lambda_1(n,r)_{A,B}$. Fix $L \in \mathcal{F}_{\operatorname{ro} A}$ and assume $N \in \mathbb{N}$ is sufficiently large that $X^L_{A,B} \subset \Pi_N$; where

$$\Pi_N = \{L'' \in \mathcal{F}_{\operatorname{co} B} : \varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0\}.$$

Lusztig notes [4] that $\Pi_N$ is a projective algebraic variety, closed under the action of $G_L$. Lemma [ADD A REFERENCE]shows that the closure of $X^L_{A,B}$ in $\Pi_N$, denoted $\overline{X^L_{A,B}}$, is an irreducible complex projective variety.

For each $i, j \in \mathbb{Z}$, write

$$\mathcal{M}_{i,j} = \{L'' \in \overline{X^L_{A,B}} : d_{i,j}(L, L'') = m_{i,j}\}.$$

$\mathcal{M}_{i,j}$ is non-empty since $d_{i,j}(L, -)$ attains a maximum on $X^L_{A,B}$. $\mathcal{M}_{i,j}$ is open in $\overline{X^L_{A,B}}$ since

$$\overline{X^L_{A,B}} \setminus \mathcal{M}_{i,j} = \{L'' \in \overline{X^L_{A,B}} : d_{i,j}(L, L'') \leq m_{i,j} - 1\}$$

and the function

$$d_{i,j}(L, -) \colon \Pi_N \to \mathbb{Z} : L'' \mapsto \dim\left(\frac{L_i}{L_i \cap L''_j}\right)$$

is lower semi-continuous, by lemma [[ADD A REFERENCE]: lower semi-continuity].

Lemma [[ADD A REFERENCE]: open orbit] shows that there is a unique $G_L$-orbit in $X^L_{A,B}$ which is open in $\overline{X^L_{A,B}}$, say $\mathcal{O}^L_{A*B}$ for some $A * B \in \Lambda_1(n, r)_{A,B}$. $\mathcal{M}_{i,j}$ intersects the open orbit $\mathcal{O}^L_{A*B}$ non-trivially, since $\mathcal{M}_{i,j}$ and $\mathcal{O}^L_{A*B}$ are both non-empty and open in the irreducible space $\overline{X^L_{A,B}}$. Moreover, $\mathcal{O}^L_{A*B} \subset \mathcal{M}_{i,j}$, since $\mathcal{M}_{i,j}$ is closed under the action of $G_L$. In particular, we have $A*B \in \Lambda_1(n, r)_{A,B}$ with $d_{i,j}(A*B) = m_{i,j}$ for each $i, j \in \mathbb{Z}$, which shows $A*B$ is a maximum in $\Lambda_1(n, r)_{A,B}$.

More specifically, we may compute:

$$a_{i,j}(A * B) = m_{i,j-1} - m_{i-1,j-1} + m_{i-1,j} - m_{i,j}$$

for each $i, j \in \mathbb{Z}$. □

**Lemma 5.4.2.** *For each $\lambda \in \Lambda_0(n, r)$, $D_\lambda * A = A$ whenever $\operatorname{ro} A = \lambda$ and $A * D_\lambda = A$ whenever $\operatorname{co} A = \lambda$.*

## 5.5 Associativity

**Proposition 5.5.1.** *Given $A, B, C \in \Lambda_1(n, r)$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$, $(A * B) * C = A * (B * C)$.*

*Proof.* INCLUDE PROOF. □

## 5.6 The generic algebra

**Definition 5.6.1.** *For each $n, r \geq 1$, the generic category $\mathcal{G}(n, r)$ is the category with set of objects $\Lambda_0(n, r)$ and set of morphisms $\Lambda_1(n, r)$ where; the morphisms from $\lambda$ to $\mu$ are those matrices $A \in \Lambda_1(n, r)$ with $\operatorname{co} A = \lambda$ and $\operatorname{ro} A = \mu$; the composition of morphisms $A \colon \lambda \to \mu$ and $B \colon \mu \to \nu$ is $B * A \colon \lambda \to \nu$, where $B * A$ is the maximum element in $\Lambda_1(n, r)_{A,B}$. For each $\lambda \in \Lambda_0(n, r)$, the identity morphism $D_\lambda \colon \lambda \to \lambda$ is given by $(D_\lambda)_{i,i} = \lambda_i$ and $(D_\lambda)_{i,j} = 0$ whenever $i \neq j$.*

**Example 1.** *The objects in $\mathcal{G}(n, r)2, 2$ are compositions of 2 into 2 parts, namely $(0, 2)$, $(1, 1)$ and $(2, 0)$. The set of morphisms from $\lambda$ to $\mu$ is the set of infinite periodic matrices $A \in \Lambda_1(n, r)2, 2$ with $\operatorname{co} A = \lambda$ and $\operatorname{ro} A = \mu$, which is a countably infinite set for any pair of compositions $\lambda, \mu \in \Lambda_0(n, r)2, 2$.*

**Definition 5.6.2** (Generic algebra)**.** *The affine generic algebra $\hat{G}(n,r)$ is the category $\mathbb{Z}$-algebra of $\mathcal{G}(n,r)$. In particular, $\hat{G}(n,r)$ is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1(n,r)\}$ and with associative multiplication given by*

$$e_A * e_B = \begin{cases} e_{A*B} & \text{if } \operatorname{co} A = \operatorname{ro} B \\ 0 & \text{if } \operatorname{co} A \neq \operatorname{ro} B. \end{cases}$$

Given $\lambda \in \Lambda_0(n,r)$, let $1_\lambda = e_{D_\lambda}$.

**Corollary 5.6.1.** *$\{1_\lambda : \lambda \in \Lambda_0(n,r)\}$ is a set of pairwise orthogonal idempotents in $\hat{G}(n,r)$ with $\sum_{\lambda \in \Lambda_0(n,r)} 1_\lambda = 1$.*

**Theorem 5.6.2.** *$\hat{G}(n,r)$ is an associative $\mathbb{Z}$-algebra with $1$.*

*Proof.* Given $A, B \in \Lambda_1(n,r)$ with $\operatorname{co} A = \operatorname{ro} B$, proposition 5.4.1 shows that there is a maximum element in $\{C \in \Lambda_1(n,r) : g_{A,B,C} \neq 0\}$, which is denoted $A * B$. This shows that the product on $\hat{G}(n,r)$ is well-defined. If $\operatorname{co} A \neq \operatorname{ro} B$ or $\operatorname{co} B \neq \operatorname{ro} C$, then $(e_A * e_B) * e_C = 0 = e_A * (e_B * e_C)$. If $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$, then proposition 5.5.1 shows that

$$(e_A * e_B) * e_C = e_{(A*B)*C} = e_{A*(B*C)} = e_A * (e_B * e_C).$$

Corollary 5.6.1 shows that the sum of the idempotents $1_\lambda$ for $\lambda \in \Lambda_0(n,r)$ is a multiplicative identity. $\square$

## 5.7 Multiplication rules

Write

$$E_i = \sum_{\lambda \in \Lambda_0(n,r)} E_{i,\lambda}$$

$$F_i = \sum_{\lambda \in \Lambda_0(n,r)} F_{i,\lambda}.$$

Then $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$.

**Lemma 5.7.1.** *Let $A \in \Lambda_1(n,r)$, $i \in [1,n]$ and let $\lambda = \operatorname{ro} A$. The following multiplication rules hold:*

$$E_i e_A = \begin{cases} e_{A+X_{i,p}} & \text{if } \lambda_{i+1} > 0 \\ 0 & \text{if } \lambda_{i+1} = 0; \end{cases}$$

*where $p$ is such that $A_{i+1,p} > 0$ and $A_{i+1,j} = 0$ for $j > p$. Also*

$$F_i e_A = \begin{cases} e_{A-X_{i,p}} & \text{if } \lambda_i > 0 \\ 0 & \text{if } \lambda_i = 0; \end{cases}$$

*where $p$ is such that $A_{i,p} > 0$ and $A_{i,j} = 0$ for $j < p$.*

Similar formulas for right multiplication by $E_i$ and $F_i$ are obtained by applying the transpose.

**Lemma 5.7.2.** *The following relations hold in $\hat{G}(n,r)$ $(n \geq 3)$:*

$$E_iE_j - E_jE_i = 0$$

$$F_iF_j - F_jF_i = 0$$

*unless $|j-i| = 1$.*

$$E_iE_{i+1}^2 - E_{i+1}E_iE_{i+1} = 0$$
$$E_i^2E_{i+1} - E_iE_{i+1}E_i = 0$$
$$F_{i+1}F_i^2 - F_iF_{i+1}F_i = 0$$
$$F_{i+1}^2F_i - F_{i+1}F_iF_{i+1} = 0$$

$$E_iF_j - F_jE_i = 0$$

*unless $j = i$.*

$$E_iFi - F_iE_i + \sum_{\lambda:\lambda_i=0,\lambda_{i+1}>0} 1_\lambda - \sum_{\lambda:\lambda_i>0,\lambda_{i+1}=0} 1_\lambda = 0.$$

## 5.8 – Chapter draft bin –

Define

$$\Pi = \left\{ L'' \in \mathcal{F}_{\mathrm{co}\,B} : \varepsilon^N L_0 \subset L_0'' \subset \cdots \subset L_n'' \subset \varepsilon^{-N} L_0 \text{ and } \dim\left(L_0''/\varepsilon^N L_0\right) = -Nr + d^-_{-Nn,0}(A) + d^-_{-Nn,0}(B) \right\}.$$

**Lemma 5.8.1.** $\Pi$ *is a projective algebraic variety, closed under the action of $G_L$.*

By choice of $N$, we have $X^L_{A,B} \subset \Pi$.

**Lemma 5.8.2.** *The group $G_L/H$ is an irreducible algebraic group.*

*Proof.* $\sigma \in G_L$ naturally induces an automorphism $\bar{\sigma}$ of $\varepsilon^{-N}L_0/\varepsilon^N L_0$, with inverse induced by $\sigma^{-1}$. Moreover, the natural map

$$G_L/H \to GL(\varepsilon^{-N}L_0/\varepsilon^N L_0)$$

is a group homomorphism. In fact, this homomorphism is injective: if $\sigma = \tau$ on $\varepsilon^{-N}L_0/\varepsilon^N L_0$, then $\sigma\tau^{-1} = 1$ on $\varepsilon^{-N}L_0/\varepsilon^N L_0$ and so $\sigma H = \tau H$. Thus $G_L/H$ is isomorphic to its image in $GL(\varepsilon^{-N}L_0/\varepsilon^N L_0)$. this image is an algebraic group, then I need to deduce $G_L/H$ is an algebraic group. First isomorphism theorem? ☐

**Lemma 5.8.3.** *Suppose $(L, L', L''), (N, N', N'') \in \beta^{-1}(\mathcal{O}_A \times \mathcal{O}_B)$. Then there are $\sigma, \tau \in G$, with $\tau \in G_{L'}$, such that $(N, N', N'') = \sigma(L, L', \tau L'')$.*

*Proof.* There exist $g, g' \in G$ such that $(N, N') = g(L, L')$ and $(N', N'') = g'(L', L'')$. Then $(N, N', N'') = g(L, L', g^{-1}g'L'')$. Taking $\sigma = g$ and $\tau = g^{-1}g'$ gives the required result. ☐

**Proposition 5.8.4.** *Suppose $X^L_{A,B} \neq \emptyset$. Then $X^L_{A,B} \subset \mathcal{F}_{\mathrm{co}\,B}$ is finite dimensional and irreducible.*

*Proof.* The map

$$G_L/H \times G_{L'}/H \to \Pi$$

has image $X^L_{A,B}$, so the closure of $X^L_{A,B}$ in $\Pi$ is irreducible due to some properties of the above groups. ☐

### 5.8.1 locally closed orbits

**Proposition 5.8.5.** *Suppose $X^L_{A,B} \neq \emptyset$. The $G_L$-orbits in $X^L_{A,B}$ are locally closed.*

*Proof.* The $G_L$ orbit of $L'' \in X^L_{A,B}$ is the image of the map

$$G_L/H \to \Pi : g \mapsto gL''.$$

Justify why this image must be locally closed. □

**Proposition 5.8.6.** *Let $A, B \in \Lambda_1(n,r)$, $L \in \mathcal{F}$ and suppose $X^L_{A,B} \neq \emptyset$. There is a unique open $G_L$-orbit in $X^L_{A,B}$.*

*Proof.* Write $X = X^L_{A,B}$. $X$ is irreducible and finite dimensional, using Lemma 5.8.4. We have

$$X = \bigcup_C O_C,$$

where the union is taken over the finite set $\{C \in \Lambda_1(n,r) : \mathcal{O}_C \subset X_{A,B}\}$.

A proper, non-empty, closed subset of $X$ has strictly smaller dimension than $X$, so there is $C$ such that $\overline{O_C} = X$. $O_C$ is locally closed, by Lemma 5.8.5, so it follows that $O_C$ is open in $\overline{O_C} = X$.

Now suppose $O_C$ is an open $G_L$ orbit and let $D \in \Lambda_1(n,r)$. $O_D \subset X \setminus O_C$ and thus $\overline{O_D} \subset X \setminus O_C$. This shows $O_D$ is not open in $X$ and thus the claim is proven. □

### 5.8.2 Associativity of the generic product

Given $A, B, C \in \Lambda_1(n,r)$ and $L \in \mathcal{F}$ let

$$X^L_{A,B,C} = \{L''' \in \mathcal{F} : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A, (L'', L''') \in \mathcal{O}_C\}.$$

Note that $X^L_{A,B,C} \subset$ is contained in $\mathcal{F}_{\mathrm{co}\,C}$ and is non-empty only if $L \in \mathcal{F}_{\mathrm{ro}\,A}$, $\mathrm{co}\,A = \mathrm{ro}\,B$ and $\mathrm{co}\,B = \mathrm{ro}\,C$. $X^L_{A,B,C}$ consists of finitely many $G_L$-orbits. Using a similar argument to the existence of generic orbits we show that there is a unique generic orbit in $X^L_{A,B,C}$, which will establish associativity of the generic product. We now suppose $X^L_{A,B,C}$ is non-empty and fix $(L, L', L'', L''') \in \mathcal{F}^4$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L'', L''') \in \mathcal{O}_C$.

**Lemma 5.8.7.** *$X^L_{A,B,C}$ is the image of the map*

$$\phi\colon G_L \times G_{L'} \times G_{L''} \to \mathcal{F} : (\alpha, \beta, \gamma) \mapsto \alpha\beta\gamma L'''.$$

**Proposition 5.8.8.** *The closure $\overline{X^L_{A,B,C}}$ of $X^L_{A,B,C}$ in $\mathcal{F}$ is irreducible.*

**Proposition 5.8.9.** *There is a unique generic $G_L$-orbit in $X^L_{A,B,C}$.*

# Chapter 6

# A realisation of affine zero Schur algebras

We aim to prove the isomorphism theorem in the cases $r < n$ and $n \leq r < 2n$ separately. Below are crude versions of the statements we want to prove.

**Theorem 6.0.1.** *Assume $r < n$. The map $\psi\colon \hat{G}(n,r) \to \hat{S}_0(n,r)$, given by $\psi(E_i) = E_i$, $\psi(F_i) = F_i$ and $\psi(1_\lambda) = 1_\lambda$, is an isomorphism of $\mathbb{Z}$-algebras.*

*Proof.* Below are some of the pieces: [1] The elements $E_i$, $F_i$, $1_\lambda$ generate $\hat{G}(n,r)$.

Provided $r < n$, any $A \in \Lambda_1(n,r)$ may be obtained from the diagonal matrix $D_\lambda$ with $\lambda = \operatorname{ro} A$ by a sequence of transitions $A \mapsto A \pm X_{i,p}$.

[2] Give a complete set of generating relations for $\hat{G}(n,r)$. □

**Theorem 6.0.2.** *Assume $n \leq r < 2n$. There is a unique homomorphism of $\mathbb{Z}$-algebras $\hat{\psi}\colon \hat{G}(n,r) \to \hat{S}_0(n,r)$ such that $\hat{\psi}(R) = R$ and $\hat{\psi} = \psi$ on the subalgebra of $\hat{G}(n,r)$ generated by the $E_i$, $F_i$ and $1_\lambda$. $\hat{\psi}$ is an isomorphism of $\mathbb{Z}$-algebras.*

## 6.1 Quivers with relations for the generic algebra.

Recall that $\Lambda_0(n,r)$ denotes the set of compositions of $r$ into $n$ parts. That is, $\Lambda_0(n,r)$ is the set of tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with each $\lambda_i$ non-negative and $\lambda_1 + \cdots + \lambda_n = r$. Given $i \in [1,n]$, let $\varepsilon_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^n$ be the $i$–th elementary vector and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then given $\lambda \in \Lambda_0(n,r)$, we have $\lambda + \alpha_i \in \Lambda_0(n,r)$ provided $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0(n,r)$ provided $\lambda_i > 0$.

Let $\Gamma = \Gamma(n,r)$ be the quiver with set of vertices $\Lambda_0(n,r)$ with arrows $e_{i,\lambda}\colon \lambda \to \lambda + \alpha_i$ (if $\lambda_{i+1} > 0$) and $f_{i,\lambda}\colon \lambda \to \lambda - \alpha_i$ (if $\lambda_i > 0$). Thus there are no arrows between $\lambda$ and $\mu$ unless $\lambda = \mu \pm \alpha_i$ for some $i \in [1,n]$.

If $n \geq 3$ then neighbouring vertices are connected by two arrows, one of each direction. In the case $n = 2$, neighbouring vertices are joined by four arrows, two of each direction. The $\mathbb{Z}\Gamma$ denote the path $\mathbb{Z}$ algebra of $\Gamma$. By construction of $\Gamma$, there is a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}\Gamma \to \hat{G}(n,r)$ with $e_{i,\lambda} \mapsto E_{i,\lambda}$, $f_{i,\lambda} \mapsto F_{i,\lambda}$ and $k_\lambda = 1_\lambda$. We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [4]).

$A \in \Lambda_1(n,r)$ is said to be aperiodic if for each $l \in \mathbb{Z}\backslash\{0\}$ there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$. Denote the set of aperiodic elements in $\Lambda_1(n,r)$ by $\Lambda_1(n,r)^{ap}$. Note that $\Lambda_1(n,r)^{ap} = \Lambda_1(n,r)$ if $r < n$.

**Proposition 6.1.1.** *The subalgebra of $\hat{G}(n,r)$ generated by $E_{i,\lambda}$, $F_{i,\lambda}$ and $1_\lambda$ has $\mathbb{Z}$-basis $\{e_A : A \in \Lambda_1(n,r)^{ap}\}$, where $\Lambda_1(n,r)^{ap} \subset \Lambda_1(n,r)$ is the set of aperiodic elements.*

# Chapter 7

# Further directions

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algeras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for $S_3$ and $S_4$. Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: 'these' relations hold; these generators and relations give a presentation of the degenerate group algebras.

# Chapter 8

# A brief report of progress

The stated goal for the summer was to have the existence and associativity of generic multiplication written up carefully and to have some new ideas in presenting affine zero schur algebras by the generic algebras.

I think the existence and associativity parts are almost coherent, but still don't think my document is convincing and needs a lot of editing, as I have ended up neglecting typing in favour of trying to make the argument clear to myself.

Progress towards presenting generic algebras: Have a systematic way to express the aperiodic basis elements (any $n$ and $r$) in terms of $E_i$, $F_i$ and $1_\lambda$ for $i \in [1, n]$ and $\lambda \in \Lambda_0(n, r)$. To prove the algorithm works, try induction on the number of nonzero entries - modulo the periodicity condition - off the diagonal.

Work on degenerate group algebras for symmetric groups:

Presentation by generators and relations for the cases $S_3$ and $S_4$. Explicit calculation done on paper. Identified parts that should generalise to $S_n$ for any $n$. This is to be typed in a separate document in only a few pages.

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