

A geometric realisation of affine 0-Schur algebras.

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Chapter 1

Introduction

Chapter 2

The double flag variety approach to q -Schur algebras

Chapter 3

The cyclic flags approach to affine q -Schur algebras

Fix natural numbers n and r .

Definition 3.0.1. A composition of r into n parts is an n -tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ of non-negative integers whose sum equals r . Denote the set of compositions of r into n parts by $\Lambda_0(n, r)$.

Definition 3.0.2. Let $\Lambda_1(n, r)$ be the set of matrices $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ with non-negative integer entries $a_{i,j}$ satisfying the following conditions: each row or column has only finitely many non-zero entries; the sum of the entries in any n consecutive rows or columns equals r ; $a_{i-n,j-n} = a_{i,j}$ for each $i, j \in \mathbb{Z}$. These matrices are referred to as infinite periodic matrices.

Definition 3.0.3. Given $A \in \Lambda_1(n, r)$, let $\text{ro } A$ and $\text{co } A$ be the compositions of r into n parts given by

$$\text{ro } A = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\text{co } A = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right).$$

$A \in \Lambda_1(n, r)$ is said to go from $\text{co } A$ to $\text{ro } A$.

3.1 Cyclic flags

Fix $n, r \in \mathbb{N}$ and let \mathbf{k} be a field. Let \mathcal{S} be the \mathbf{k} -algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let \mathcal{R} be the subalgebra generated by ε , so $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free \mathcal{S} -module of rank r . Let G be the automorphism group of the \mathcal{S} -module V , so G is isomorphic to $\text{GL}_r(\mathcal{S})$. A (\mathcal{R}) -lattice in V is a \mathcal{R} -submodule L of V with $\mathcal{S} \otimes_{\mathcal{R}} L = V$. In particular, a lattice is an \mathcal{R} -submodule of V which is a free \mathcal{R} -module of rank r . Let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of collections $(L_i)_{i \in \mathbb{Z}}$ of lattices in V with $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. These collections of lattices in V are referred to as cyclic flags in V .

G acts on \mathcal{F} by $(g \cdot L)_i = g(L_i)$ for each $i \in \mathbb{Z}$, given $g \in G$ and $L \in \mathcal{F}$. The G -orbits in \mathcal{F} are indexed by the set $\Lambda_0(n, r)$ of compositions of r into n parts: the G -orbit in \mathcal{F} corresponding to $\lambda \in \Lambda_0(n, r)$ is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim \left(\frac{L_i}{L_{i-1}} \right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

Definition 3.1.1. The periodic characteristic matrix of a pair of cyclic flags $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the matrix $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$ with entries

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j} \right)$$

for each $i, j \in \mathbb{Z}$.

The diagonal action of G on $\mathcal{F} \times \mathcal{F}$ has orbits indexed by the set $\Lambda_1(n, r)$ of infinite periodic matrices (see definition 3.0.2). The G -orbit corresponding to $A \in \Lambda_1(n, r)$ is denoted \mathcal{O}_A and consists of those pairs $(L, L') \in \mathcal{F} \times \mathcal{F}$ with periodic characteristic matrix $A(L, L')$ equal to A .

define $Y_{A,B}$, $X_{A,B}$ in terms of the maps π, δ and the product of orbits. Given $A, B \in \Lambda_1(n, r)$ define

$$X_{A,B} = \{(L, L'') : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

Observation 1. There are only finitely many G -orbits in $X_{A,B}$.

3.2 Convolution algebras

Suppose \mathbf{k} is a finite field and let q denote the number of elements of \mathbf{k} . Consider the set S of G -invariant functions $\mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Z}$ with constructible support. S is a free \mathbb{Z} -module with a basis consisting of the indicator functions of the G -orbits in $\mathcal{F} \times \mathcal{F}$. Define an operation \star on S as follows: for each $f, g \in S$, $f \star g \in S$ is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for $(L, L'') \in \mathcal{F} \times \mathcal{F}$.

$f \star g$ is well defined since the supports of f and g consist of finitely many G -orbits, so there are only finitely many $L' \in \mathcal{F}$ such that $f(L, L')g(L', L'') \neq 0$, given $(L, L'') \in \mathcal{F} \times \mathcal{F}$. $f \star g$ is constant on G -orbits and is supported on finitely many G -orbits, so $f \star g \in S$.

Lemma 3.2.1. The set S together with the operation \star is an associative \mathbb{Z} -algebra with identity element ι given by $\iota(L, L) = 1$ and $\iota(L, L') = 0$ for $L' \neq L$.

Proof. Given $f, g, h \in S$ and $(L, L''') \in \mathcal{F} \times \mathcal{F}$,

$$\begin{aligned} ((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''), \end{aligned}$$

thus \star is associative. ι is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L')\iota(L', L'') = f(L, L''),$$

for each $f \in S$ and $(L, L'') \in \mathcal{F} \times \mathcal{F}$. □

Given $A \in \Lambda_1(n, r)$, let $e_A \in S$ denote the indicator function of the orbit \mathcal{O}_A . S is a free \mathbb{Z} -module with basis $\{e_A : A \in \Lambda_1(n, r)\}$. There exist $\gamma_{A,B,C;q} \in \mathbb{Z}$ for $A, B, C \in \Lambda_1(n, r)$ such that

$$e_A \star e_B = \sum_{C \in \Lambda_1(n, r)} \gamma_{A,B,C;q} e_C$$

for each $A, B \in \Lambda_1(n, r)$. Then

$$\begin{aligned} \gamma_{A,B,C;q} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L') e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}, \end{aligned}$$

for any $(L, L'') \in \mathcal{O}_C$.

3.3 Affine q -Schur algebras

There exists a polynomial $\gamma_{A,B,C} \in \mathbb{Z}[q]$ for each $A, B, C \in \Lambda_1(n, r)$ such that $\gamma_{A,B,C}(q) = \gamma_{A,B,C;q}$ for any prime power q , following [1, section 4]. The affine q -Schur algebra $\hat{S}_q(n, r)$ (defined in [\[ADD A REFERENCE\]](#)) is a free $\mathbb{Z}[q]$ -module with basis $\{e_A : A \in \Lambda_1(n, r)\}$ and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

$(L, L') \in \mathcal{O}_A$ if and only if $(L', L) \in \mathcal{O}_{A^\top}$. In fact, the operation of transposition on $\mathcal{F} \times \mathcal{F}$ (or on $\Lambda_1(n, r)$) induces an anti-automorphism of $\hat{S}_q(n, r)$.

Lemma 3.3.1. *Transposition gives a homomorphism of $\mathbb{Z}[q]$ -modules $\top : \hat{S}_q(n, r) \rightarrow \hat{S}_q(n, r)$ with $\top(e_A) = e_{A^\top}$, $\top \circ \top = 1$ and $\top(e_A e_B) = \top(e_B) \top(e_A)$.*

Proof. Let $A, B, C \in \Lambda_1(n, r)$ and let \mathbf{k} be a finite field with $q = \#\mathbf{k}$ elements. If $(L, L'') \in \mathcal{O}_C$ then $(L'', L) \in \mathcal{O}_{C^\top}$ and

$$\begin{aligned} \gamma_{A,B,C;q} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;q} \end{aligned}$$

It then follows that $\top(e_A e_B) = \top(e_B) \top(e_A)$. □

Chapter 4

Quivers with relations for affine q-Schur algebras

4.1 Basic results: TO BE REPLACED WITH A MORE INFORMATIVE NAME.

If $i, j \in \mathbb{Z}$, let $\mathcal{E}_{i,j}$ denote the ‘elementary matrix’ with entries given by $(\mathcal{E}_{i,j})_{s,t} = 1$, for $s, t \in \mathbb{Z}$, whenever $(i, j) \sim (s, t)$ modulo (n, n) and all other entries are zero.

Given $\lambda \in \Lambda_0(n, r)$, let $D_\lambda \in \Lambda_1(n, r)$ denote the diagonal matrix with $r(D_\lambda) = c(D_\lambda) = \lambda$. That is,

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}$$

For $\lambda \in \Lambda_0(n, r)$, write $1_\lambda = e_{D_\lambda}$. The 1_λ are pairwise orthogonal idempotents in $\hat{S}_q(n, r)$ with $1 = \sum_{\lambda \in \Lambda_0(n, r)} 1_\lambda$.

Given $i, j \in \mathbb{Z}$, write $X_{i,j} = \mathcal{E}_{i,j} - \mathcal{E}_{i+1,j}$. By convention, $e_A = 0$ unless $A \in \Lambda_1(n, r)$.

For $i \in [1, n]$ and $\lambda \in \Lambda_0(n, r)$, write

$$E_{i,\lambda} = e_{D_\lambda + X_{i,i+1}},$$

$$F_{i,\lambda} = e_{D_\lambda - X_{i,i}}.$$

Define

$$E_i = \sum_{\lambda \in \Lambda_0(n, r)} E_{i,\lambda}$$

$$F_i = \sum_{\lambda \in \Lambda_0(n, r)} F_{i,\lambda}.$$

Observe that $E_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $F_{i,\lambda} = 0$ unless $\lambda_i > 0$. Also, $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$.

Lemma 4.1.1. *Let $i \in [1, n]$ and $A \in \Lambda_1(n, r)$.*

$$E_i e_A = \sum_{p \in \mathbb{Z}} q^{\sum_{j>p} a_{i,j}} [a_{i,p} + 1] e_{A+X_{i,p}}$$

and

$$F_i e_A = \sum_{p \in \mathbb{Z}} q^{\sum_{j<p} a_{i+1,j}} [a_{i+1,p} + 1] e_{A-X_{i,p}}.$$

Note that these formulas are still valid in the cases $E_i e_A = 0$ and $F_i e_A = 0$. There are similar formulas for right multiplication by E_i and F_i , which can be obtained by applying the transpose involution to the above formulas. The transpose relates the E_i , F_i and 1_λ in the following way: $\top(E_{i,\lambda}) = F_{i,\lambda}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\varepsilon_i+\varepsilon_{i+1}}$ and $\top(1_\lambda) = 1_\lambda$. In particular, $\top(E_i) = F_i$ and $\top(F_i) = E_i$.

Corollary 4.1.2. *Let $j \in [1, n]$ and $A \in \Lambda_1(n, r)$. Then*

$$e_A F_j = \sum_{p \in \mathbb{Z}} q^{\sum_{i>p} a_{i,j}} [a_{p,j} + 1] e_{A+X_{j,p}^\top}$$

and

$$e_A E_j = \sum_{p \in \mathbb{Z}} q^{\sum_{i<p} a_{i,j+1}} [a_{p,j+1} + 1] e_{A-X_{j,p}^\top}$$

Proof.

$$\begin{aligned} e_A F_j &= \top(E_j e_{A^\top}) \\ &= \top\left(\sum_p q^{\sum_{i>p} a_{i,j}} [a_{p,j} + 1] e_{A^\top + X_{j,p}}\right) \\ &= \sum_p q^{\sum_{i>p} a_{i,j}} [a_{p,j} + 1] e_{A+X_{j,p}^\top} \end{aligned}$$

$$\begin{aligned} e_A E_j &= \top(F_j e_{A^\top}) \\ &= \top\left(\sum_p q^{\sum_{i<p} a_{i,j+1}} [a_{p,j+1} + 1] e_{A^\top - X_{j,p}}\right) \\ &= \sum_p q^{\sum_{i<p} a_{i,j+1}} [a_{p,j+1} + 1] e_{A-X_{j,p}^\top} \end{aligned}$$

□

Note that $E_i^{r+1} = F_i^{r+1} = 0$ while

$$E_i^r = [r]! e_r \varepsilon_{i,i+1}$$

and

$$F_i^r = [r]! e_r \varepsilon_{i+1,i}.$$

Lemma 4.1.3 (quantum Serre relations: $n \geq 3$). *Suppose $n \geq 3$. The following relations hold in $\hat{S}_q(n, r)$:*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless $j = i \pm 1$;

$$\begin{aligned} E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i &= 0 \\ E_i^2 E_{i+1} - (1+q) E_i E_{i+1} E_i + q E_{i+1} E_i^2 &= 0 \end{aligned}$$

and

$$\begin{aligned} F_{i+1} F_i^2 - (1+q) F_i F_{i+1} F_i + q F_i^2 F_{i+1} &= 0 \\ F_{i+1}^2 F_i - (1+q) F_{i+1} F_i F_{i+1} + q F_i F_{i+1}^2 &= 0. \end{aligned}$$

Proof. Here we introduce temporary notation for the basis elements: Write $[A] = e_A$.

Take $\lambda \in \Lambda_0(n, r)$.

$$E_i E_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1, i+2} + X_{i, i+2}] + [2][D_\lambda + 2X_{i+1, i+2} + X_{i, i+1}]$$

$$E_{i+1} E_i E_{i+1} 1_\lambda = [D_\lambda + 2X_{i+1, i+2} + X_{i, i+1}] + [2][D_\lambda + 2X_{i+1, i+1} + X_{i, i+1}]$$

$$E_{i+1}^2 E_i 1_\lambda = [2][D_\lambda + 2X_{i+1, i+2} + X_{i, i+1}]$$

Then

$$(E_i E_{i+1}^2 - (1+q)E_{i+1} E_i E_{i+1} + qE_{i+1}^2 E_i) 1_\lambda = 0,$$

for each $\lambda \in \Lambda_0(n, r)$. The relation $E_i E_{i+1}^2 - (1+q)E_{i+1} E_i E_{i+1} + qE_{i+1}^2 E_i = 0$ then follows.

The relations between F_i and F_{i+1} may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping E_i and F_i and reversing the order of multiplication. \square

Lemma 4.1.4 (quantum Serre relations: $n = 2$). *In the case $n = 2$, the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

Lemma 4.1.5. $[E_i, F_j] = 0$ unless $j = i$.

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0(n, r)} ([\lambda_i] - [\lambda_{i+1}]) 1_\lambda.$$

For $\lambda \in \Lambda_0(n, r)$, let $R_\lambda = e_{\lambda_1 \varepsilon_{0,1} + \dots + \lambda_n \varepsilon_{n-1,n}}$. Write $R = \sum_{\lambda \in \Lambda_0(n, r)} R_\lambda$. Note $R_\lambda = R 1_\lambda$. Given $A \in \Lambda_1(n, r)$ and $m \in \mathbb{Z}$, let $A[m] \in \Lambda_1(n, r)$ be given by $A[m]_{i,j} = a_{i,j+m}$ and let $A^{[m]}$ be given by $A_{i,j}^{[m]} = a_{i+m,j}$ for each $i \in \mathbb{Z}$.

Lemma 4.1.6 (Shifting). *If $A \in \Lambda_1(n, r)$ then*

$$R e_A = e_{A[\pm 1]}$$

and

$$e_A R = e_{A[\pm 1]}.$$

Conjugation by R gives an automorphism ρ of $\hat{S}_q(n, r)$ satisfying $\rho^n = 1$.

4.2 quivers with relations

Denote by $\Lambda_0(n, r)$ the set of compositions of r into n parts. That is, $\Lambda_0(n, r)$ is the set of $\alpha \in \mathbb{Z}^n$ with non-negative entries which sum to r . Let $\varepsilon_i \in \mathbb{Z}^n$ be the i th elementary vector and write $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for each $i \in [1, n]$. Then $\lambda + \alpha_i \in \Lambda_0(n, r)$ if $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0(n, r)$ if $\lambda_i > 0$.

Let $\Gamma = \Gamma(n, r)$ be the quiver with set of vertices $\Lambda_0(n, r)$, with the following arrows:

For $\lambda \in \Lambda_0(n, r)$ and $i \in [1, n]$, there is an arrow $e_{i,\lambda} : \lambda \rightarrow \lambda + \alpha_i$ if $\lambda_{i+1} > 0$ and there is an arrow $f_{i,\lambda} : \lambda \rightarrow \lambda - \alpha_i$ if $\lambda_i > 0$.

Denote by $\mathbb{Z}[q]\Gamma$ the path $\mathbb{Z}[q]$ -algebra of Γ . Thus $\mathbb{Z}[q]\Gamma$ is a free $\mathbb{Z}[q]$ -module with a basis given by the set of paths in Γ , with multiplication given by the concatenation of paths. If p starts

where q ends, the product pq is the path q followed by p . Write $e_{i,\lambda} = 0$ unless $\lambda, \lambda + \alpha_i \in \Lambda_0(n, r)$ and write $f_{i,\lambda} = 0$ unless $\lambda, \lambda - \alpha_i \in \Lambda_0(n, r)$.

By construction, there is a homomorphism of $\mathbb{Z}[q]$ -algebras

$$\phi: \mathbb{Z}[q]\Gamma \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned}\phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda,\end{aligned}$$

for $i \in [1, n]$ and $\lambda \in \Lambda_0(n, r)$.

The image of ϕ is the subalgebra of $\hat{S}_q(n, r)$ generated by E_i, F_i for $i \in [1, n]$ and 1_λ for $\lambda \in \Lambda_0(n, r)$, since $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$, while $E_i = \sum_\lambda E_{i,\lambda}$ and $F_i = \sum_\lambda F_{i,\lambda}$. In general ϕ is not surjective, so this does not always lead to a presentation of $\hat{S}_q(n, r)$.

4.2.1 Exceptional case $n = 2$.

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the q -Schur algebra.

4.2.2 Typical case $n > 2$.

Suppose $n \geq 3$. Then $\Gamma = \Gamma(n, r)$ has vertex set $\Lambda_0(n, r)$. **RESUME HERE...**

Define $e_i, f_i \in \mathbb{Z}[q]\Gamma(n, r)$ by

$$e_i = \sum_{\lambda \in \Lambda_0(n, r)} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0(n, r)} f_{i,\lambda},$$

with the convention $e_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $f_{i,\lambda} = 0$ unless $\lambda_i > 0$. Let $k_\lambda \in \mathbf{k}\Delta$ denote the constant path at vertex λ . $\{k_\lambda : \lambda \in \Lambda_0(n, r)\}$ is a set of pairwise orthogonal idempotents in $\mathbb{Z}[q]\Gamma(n, r)$.

Let $I(n, r) \subset \mathbb{Z}[q]\Gamma(n, r)$ be the ideal generated by the expressions

$$\begin{aligned}& e_i e_{i+1}^2 - (1+q)e_{i+1}e_i e_{i+1} + qe_{i+1}^2 e_i \\& e_i^2 e_{i+1} - (1+q)e_i e_{i+1} e_i + qe_{i+1} e_i^2 \\& f_{i+1} f_i^2 - (1+q)f_i f_{i+1} f_i + qf_i^2 f_{i+1} \\& f_{i+1}^2 f_i - (1+q)f_{i+1} f_i f_{i+1} + qf_i f_{i+1}^2 \\& e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0(n, r)} ([\lambda_i] - [\lambda_{i+1}]) k_\lambda\end{aligned}$$

Recall that a relation is a $\mathbb{Z}[q]$ -linear combination of paths with common start and end vertices. The relations involving paths $\lambda \rightarrow \mu$ are given by $1_\mu \text{expr} 1_\lambda$, for each of the above expressions.

Lemma 4.2.1. *There is a homomorphism of $\mathbb{Z}[q]$ -algebras*

$$\phi: \mathbb{Z}[q]\Gamma(n, r)/I(n, r) \rightarrow \hat{S}_q(n, r)$$

given by

$$\phi(e_{i,\lambda}) = E_{i,\lambda}$$

$$\phi(f_{i,\lambda}) = F_{i,\lambda}$$

$$\phi(k_\lambda) = 1_\lambda.$$

Chapter 5

A generic affine Schur algebra

5.1 Introducing the affine generic algebra

Assume $\mathbf{k} = \mathbb{C}$ and fix $n, r \geq 1$. Let \mathcal{S} be the \mathbf{k} -algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let \mathcal{R} be the subalgebra generated by ε , namely $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free \mathcal{S} -module of rank r and let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of n -periodic cyclic flags in V ; so \mathcal{F} consists of collections $L = (L_i)_{i \in \mathbb{Z}}$ of \mathcal{R} -lattices in V with $L_i \subset L_{i+1}$ for $i \in \mathbb{Z}$ and $\varepsilon L_i = L_{i-n}$ for $i \in \mathbb{Z}$.

Let G be the group of \mathcal{S} -module automorphisms of V . Thus G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. G acts on \mathcal{F} with orbits $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0(n, r)\}$, where $\Lambda_0(n, r)$ is the set of compositions of r into n parts.

The diagonal action of G on $\mathcal{F} \times \mathcal{F}$ has orbits $\{\mathcal{O}_A : A \in \Lambda_1(n, r)\}$, where \mathcal{O}_A consists of those pairs of flags with periodic characteristic matrix equal to A . In particular, the periodic characteristic matrix of a pair $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix $A = (a_{i,j})_{i,j \in \mathbb{Z}}$, with

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right),$$

for each $i, j \in \mathbb{Z}$.

5.1.1 Not quite a category

There are maps $\mathrm{ro}, \mathrm{co} : \Lambda_1(n, r) \rightarrow \Lambda_0(n, r)$ given by

$$\mathrm{ro} A = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co} A = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right).$$

Given $A \in \Lambda_1(n, r)$, write $\mathrm{co} A \xrightarrow{A} \mathrm{ro} A$. The purpose of this chapter is to define a category with objects $\Lambda_0(n, r)$ and morphisms $\Lambda_1(n, r)$; where $\mathrm{Hom}(\lambda, \mu) = \{A \in \Lambda_1(n, r) : \mathrm{ro} A = \mu, \mathrm{co} A = \lambda\}$. Given $A, B \in \Lambda_1(n, r)$ let $\Lambda_1(n, r)_{A,B}$ be the set of $C \in \Lambda_1(n, r)$ such that there exist $L, L', L'' \in \mathcal{F}$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L'', L''') \in \mathcal{O}_C$. It will be shown that $\Lambda_1(n, r)$ admits a partial order \leq such that $\Lambda_1(n, r)_{A,B}$ has a maximum element $A * B$, whenever $\mathrm{co} A = \mathrm{ro} B$. It

will be shown that $*$ is associative, so defining the composition of morphisms in the category formed by $\Lambda_0(n, r)$ and $\Lambda_1(n, r)$.

The generic affine Schur algebra $\hat{G}(n, r)$ will then be a \mathbb{Z} -algebra defined as a linearisation of this category. It will be shown that $\hat{G}(n, r)$ gives a realisation of the affine 0-Schur algebra $\hat{S}_0(n, r)$ when $r < n$. It is expected that a more refined presentation of the generic algebra and the 0-Schur algebra will allow the conditions on the parameters to be relaxed slightly: the $r = n$ case is approachable, which may extend to the case $r < 2n$.

5.2 A partial order

Given $i, j \in \mathbb{Z}$, define a map $d_{i,j}$ on $\Lambda_1(n, r)$ by setting

$$d_{i,j}A = \sum_{s \leq i, t > j} a_{s,t}$$

for each $A \in \Lambda_1(n, r)$.

Lemma 5.2.1. *Let $A \in \Lambda_1(n, r)$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for $i, j \in \mathbb{Z}$. Then*

$$d_{i,j} - d_{i-1,j} = \sum_{t > j} a_{i,t}$$

and

$$d_{i,j} - d_{i,j-1} = - \sum_{s \leq i} a_{s,j}.$$

Proof. Let $i, j \in \mathbb{Z}$. Then

$$d_{i,j} - d_{i-1,j} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i-1, t > j} a_{s,t} = \sum_{t > j} a_{i,t}.$$

Similarly,

$$d_{i,j} - d_{i,j-1} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i, t > j-1} a_{s,t} = - \sum_{s \leq i} a_{s,j}.$$

□

Lemma 5.2.2. *Let $A \in \Lambda_1(n, r)$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for each $i, j \in \mathbb{Z}$. Then*

$$a_{i,j} = d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j}$$

for each $i, j \in \mathbb{Z}$.

Proof. Using Lemma 5.2.1,

$$\begin{aligned} a_{i,j} &= \sum_{t > j-1} a_{i,t} - \sum_{t > j} a_{i,t} \\ &= (d_{i,j-1} - d_{i-1,j-1}) - (d_{i,j} - d_{i-1,j}). \end{aligned}$$

Alternatively,

$$\begin{aligned} a_{i,j} &= \sum_{s \leq i} a_{s,j} - \sum_{s \leq i-1} a_{s,j} \\ &= -(d_{i,j} - d_{i,j-1}) + (d_{i-1,j} - d_{i-1,j-1}). \end{aligned}$$

□

Lemma 5.2.3. *The relation \leq on $\Lambda_1(n, r)$, defined by $A \leq B$ if and only if $d_{i,j}A \leq d_{i,j}B$ for all $i, j \in \mathbb{Z}$, is a partial order.*

Proof. It is clear that \leq is reflexive and transitive, so it remains to see \leq is antisymmetric. Suppose $A, B \in \Lambda_1(n, r)$ with $A \leq B$ and $B \leq A$. Then $d_{i,j}A = d_{i,j}B$ for each $i, j \in \mathbb{Z}$, which shows $A = B$ as a result of Lemma 5.2.2. \square

The partial order on $\Lambda_1(n, r)$ induces a partial order on the set of G -orbits in $\mathcal{F} \times \mathcal{F}$, such that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $A \leq B$. The next lemma gives some geometric significance to the partial order on $\Lambda_1(n, r)$.

Lemma 5.2.4. *Let $A \in \Lambda_1(n, r)$ and take $(L, L') \in \mathcal{O}_A$. Then*

$$d_{i,j}A = \dim \left(\frac{L_i}{L_i \cap L'_j} \right)$$

for each $i, j \in \mathbb{Z}$.

It is thought* that the partial order on $\Lambda_1(n, r)$ is compatible with the degeneration order (or closure order) on G -orbits in $\mathcal{F} \times \mathcal{F}$ when $\mathbf{k} = \mathbb{C}$. In particular, it is hoped that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $\mathcal{O}_A \subset \overline{\mathcal{O}_B}$.

5.3 Preliminary results

Fix $L \in \mathcal{F}$.

Lemma 5.3.1. *$L_0/\varepsilon L_0$ is a torsion $\mathbf{k}[\varepsilon]$ -module, where ε acts as zero, with dimension r as a \mathbf{k} -vector space.*

Proof. Let $V = \mathbf{k}[\varepsilon, \varepsilon^{-1}]^r$. L_0 is a free $\mathbf{k}[\varepsilon]$ -module of rank r , with $L_0 \subset V$. So we may take a $\mathbf{k}[\varepsilon]$ -basis $x_1, \dots, x_r \in V$ for L_0 . The action of ε gives an automorphism of V mapping L_0 to εL_0 , so $\varepsilon x_1, \dots, \varepsilon x_r$ give a basis for εL_0 over $\mathbf{k}[\varepsilon]$. Therefore, the cosets $x_1 + \varepsilon L_0, \dots, x_r + \varepsilon L_0$ give a basis for $L_0/\varepsilon L_0$ over \mathbf{k} . \square

Suppose $A, B \in \Lambda_1(n, r)$ with $\text{co } A = \text{ro } B$. Recall the notation

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

and

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L', L'') \in Y_{A,B}\}.$$

$X_{A,B}$ is the image of $Y_{A,B}$ under the projection onto the first and last components.

Lemma 5.3.2. *There is $N \in \mathbb{N}$ such that*

$$\varepsilon^N L_0 \subset L_0'' \subset \varepsilon^{-N} L_0$$

whenever $(L, L'') \in X_{A,B}$.

Proof. There exist $N_1, N_2 \in \mathbb{N}$ such that

$$\varepsilon^{N_1} L_0 \subset L'_0 \subset \varepsilon^{-N_1} L_0$$

and

$$\varepsilon^{N_2} L'_0 \subset L''_0 \subset \varepsilon^{-N_2} L'_0,$$

whenever $(L, L', L'') \in Y_{A,B}$. Then, for $(L, L', L'') \in Y_{A,B}$,

$$L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-(N_1+N_2)} L_0$$

and

$$\varepsilon^{N_1+N_2} L_0 \subset \varepsilon^{N_2} L'_0 \subset L''_0.$$

In particular, taking $N = N_1 + N_2$, we have

$$\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$$

whenever $(L, L'') \in X_{A,B}$. □

Lemma 5.3.3. Suppose $N_1, N_2 \in \mathbb{N}$ with $\varepsilon^{N_1} L_0 \subset L_0 \subset \varepsilon^{-N_1} L_0$ and $\varepsilon^{N_2} L'_0 \subset L''_0 \subset \varepsilon^{-N_2} L'_0$ whenever $(L, L', L'') \in Y_{A,B}$ and let $N = N_1 + N_2$. Then

$$\dim \left(\frac{\varepsilon^{-N} L_0}{L''_0} \right) = d_{nN_1,0}(A) + d_{nN_2,0}(B)$$

and

$$\dim \left(\frac{L''_0}{\varepsilon^N L_0} \right) = 2Nr - d_{nN_1,0}(A) + d_{nN_2,0}(B),$$

whenever $(L, L'') \in X_{A,B}$.

Proof. Suppose $(L, L'') \in X_{A,B}$ and $L' \in \mathcal{F}$ so that $(L, L', L'') \in Y_{A,B}$. As in lemma 5.3.2, $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$, so

$$\dim \left(\frac{\varepsilon^{-N} L_0}{L''_0} \right) + \dim \left(\frac{L''_0}{\varepsilon^N L_0} \right) = \dim \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0} \right).$$

As a \mathbf{k} -vector space, $\varepsilon^{-N} L_0 / \varepsilon^N L_0$ is isomorphic to $(L_0 / \varepsilon L_0)^{2N}$, which has dimension $2Nr$, so

$$\dim \left(\frac{L''_0}{\varepsilon^N L_0} \right) = 2Nr - \dim \left(\frac{\varepsilon^{-N} L_0}{L''_0} \right).$$

It remains to compute the codimension of L''_0 in $\varepsilon^{-N} L_0$. Note $L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-N} L_0$, so

$$\dim \left(\frac{\varepsilon^{-N} L_0}{L''_0} \right) = \dim \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0} \right) + \dim \left(\frac{\varepsilon^{-N_2} L'_0}{L''_0} \right).$$

$$\begin{aligned} \dim \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0} \right) &= \dim \left(\frac{\varepsilon^{-N_1} L_0}{L'_0} \right) \\ &= \dim \left(\frac{L_{nN_1}}{L_{nN_1} \cap L'_0} \right) \\ &= \sum_{s \leq nN_1, t > 0} A_{s,t} \\ &= d_{nN_1,0}(A). \end{aligned}$$

$$\begin{aligned}
\dim\left(\frac{\varepsilon^{-N_2}L'_0}{L''_0}\right) &= \dim\left(\frac{L'_{nN_2}}{L'_{nN_2} \cap L''_0}\right) \\
&= \sum_{s \leq nN_2, t > 0} B_{s,t} \\
&= d_{nN_2,0}(B).
\end{aligned}$$

□

Fix $L \in \mathcal{F}$. Given $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0(n, r)$, define

$$\Pi_{N,\lambda} = \{L'' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'' \subset \varepsilon^{-N} L_0\}$$

and

$$\Pi_{N,\lambda}^a = \left\{ L'' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'' \subset \varepsilon^N L_0, \dim\left(\frac{\varepsilon^{-N} L_0}{L''_0}\right) = a \right\}.$$

$\Pi_{N,\lambda}$ is the (disjoint) union of the $\Pi_{N,\lambda}^a$ for $a \in \mathbb{N}$. In fact, we will see $\Pi_{N,\lambda}^a$ is empty whenever $a > 2Nr$.

Lemma 5.3.4. *Let $N, a \in \mathbb{N}$, $\lambda \in \Lambda_0(n, r)$. Then $\Pi_{N,\lambda}^a$ is nonempty exactly when $0 \leq a \leq 2Nr$.*

Proof. Suppose $L'' \in \Pi_{N,\lambda}$. By definition, $\varepsilon^{-N} L_0 \subset L'' \subset \varepsilon^N L_0$, which shows

$$\dim\left(\frac{\varepsilon^{-N} L_0}{L''_0}\right) \leq \dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0}\right) = 2Nr.$$

Therefore, $\Pi_{N,\lambda}^a$ is empty unless $a \leq 2Nr$.

Now assume $0 \leq a \leq 2Nr$. We may choose an ε -invariant subspace W' of $W = \varepsilon^{-N} L_0 / \varepsilon^N L_0$ of codimension a . W' lifts to give a \mathcal{R} -module, say L''_0 , with $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$ and with $\dim(\varepsilon^{-N} L_0 / L''_0) = \dim(W/W') = a$. Similarly, a flag of type λ in $L''_0 / \varepsilon L''_0$ lifts to give \mathcal{R} -modules $(L''_{-n+1}, \dots, L''_0)$ with

$$\varepsilon L''_0 \subset L''_{-n+1} \subset \dots \subset L''_{-1} \subset L''_0 \subset \varepsilon^{-N} L_0$$

and such that the dimensions of successive quotients are given by $\lambda_1, \dots, \lambda_n, a$, from left to right. Thus, $(L''_{-n+1}, \dots, L''_0)$ extends by periodicity to give an element of $\Pi_{N,\lambda}^a$, as desired. □

Lemma 5.3.5. *$\Pi_{N,\lambda}^a$ is a (quasi)projective variety, provided $0 \leq a \leq 2Nr$.*

Proof. Let $W = \varepsilon^{-(1+N)} L_0 / \varepsilon^N L_0$ and let

$$X = \left\{ W_1 \leq \dots \leq W_n \leq W : \dim\left(\frac{W}{W_n}\right) = a, \dim\left(\frac{W_i}{W_{i-1}}\right) = \lambda_i \text{ for } i = 2, \dots, n \right\}.$$

X is known to be a projective variety [CITATION NEEDED]

Let X' be the subset of X consisting of those (W_1, \dots, W_n) , where each W_i is ε -invariant and $\varepsilon W_n \leq W_1$. X' is a closed subset of X , though is not necessarily irreducible.

The correspondence between the set of \mathcal{R} -submodules of $\varepsilon^{-(1+N)} L_0$ which contain $\varepsilon^N L_0$ and the set of \mathcal{R} -submodules of $\varepsilon^{-(1+N)} L_0 / \varepsilon^N L_0$ gives a pair of mutually inverse maps $\Pi_{N,\lambda}^a \leftrightarrow X'$.

– the idea that is relevant to the proof is that inclusion relations $L_i \subset L_{i+1}$ describe a closed set in a product of grassmanians. Unsure here – Is it true that irreducible components of X' are projective varieties. In this case, should the statement be that $\Pi_{N,\lambda}^a$ is a projective algebraic set, rather than a quasi projective variety? □

Lemma 5.3.6. Suppose $(L', L'') \in \mathcal{O}_B$ with $(L, L') \in \mathcal{O}_A$. Then $X_{A,B}^L$ is the image of the map

$$G_L \times G_{L'} \rightarrow \mathcal{F} : (\alpha, \beta) \mapsto \alpha\beta L''.$$

Proof. Suppose $\alpha \in G_L$ and $\beta \in G_{L'}$. $(L, \alpha L', \alpha\beta L'') \in Y_{A,B}$ since $(L, \alpha L') \sim (L, L') \in \mathcal{O}_A$ and $(\alpha L', \alpha\beta L'') \sim (L', L'') \in \mathcal{O}_B$. This shows $(L, \alpha\beta L'') \in X_{A,B}$ and thus $\alpha\beta L'' \in X_{A,B}^L$.

Conversely, suppose $N'' \in X_{A,B}^L$. $(L, N'') \in X_{A,B}$, so there is N' such that $(L, N') \in \mathcal{O}_A$ and $(N', N'') \in \mathcal{O}_B$. There exist $\gamma, \delta \in G$ such that $\gamma(L, L') = (N, N')$ and $\delta(L', L'') = (N', N'')$. Then $(L, N', N'') = (L, \gamma L', \delta L'') = (L, \gamma L', \gamma(\gamma^{-1}\delta)L'')$, where $\gamma \in G_L$ and $\gamma^{-1}\delta \in G_{L'}$. This shows $N'' \in G_L G_{L'} L''$ as required. \square

Given $N \in \mathbb{N}$, define

$$H_N = \left\{ h \in G_L : h = 1 \text{ on } \frac{\varepsilon^{-(1+N)} L_0}{\varepsilon^N L_0} \right\}.$$

Explicitly, the condition $h = 1$ on $\varepsilon^{-(1+N)} L_0 / \varepsilon^N L_0$ means: $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)} L_0$. Observe that $H_{N+1} \subset H_N$ for $N \in \mathbb{N}$ since $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ whenever $x \in \varepsilon^{-(1+N)} L_0$.

Lemma 5.3.7. H_N is a normal subgroup in G_L , for any $N \in \mathbb{N}$.

Proof. $H_N \subset G_L$ by definition. Suppose $h, h' \in H_N$ and let $x \in \varepsilon^{-(1+N)} L_0$. $h'(x) \in \varepsilon^{-(1+N)} L_0$ as $h' \in G_L$, so $hh'(x) + \varepsilon^N L_0 = h'(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$, which shows $hh' \in H_N$. $h(x) - x \in \varepsilon^N L_0$, so $h^{-1}(x) - x = -h^{-1}(h(x) - x) \in \varepsilon^N L_0$. $h^{-1} \in H_N$, so H_N is a subgroup of G_L .

Let $g \in G_L$. $hg^{-1}(x) + \varepsilon^N L_0 = g^{-1}(x)$ as $g^{-1}(x) \in \varepsilon^{-(1+N)} L_0$, so $ghg^{-1}(x) + \varepsilon^N L_0 = gg^{-1}(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$. Thus $ghg^{-1} \in H_N$, which proves H_N is a normal subgroup in G_L . \square

The H_N form a descending chain of normal subgroups in G_L : $\cdots \subset H_1 \subset H_0 \subset G_L \subset G$.

Lemma 5.3.8. G_L/H_N is an irreducible algebraic group for any $N \in \mathbb{N}$.

Proof. See the discussion in [1][section 4]. Should be able to give an explicit presentation of G_L/H_N in terms of the block structure. \square

Lemma 5.3.9. There is $N \in \mathbb{N}$ such that $H_N \subset G_{L'}$. Consequently, $H_{N'} \subset G_{L'}$ whenever $N' \geq N$.

Proof. Choose $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$. Then

$$\varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset \cdots \subset L'_n \subset \varepsilon^{-(1+N)} L_0.$$

Let $h \in H_N$. $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)} L_0$, so $h(L'_i) \subset L'_i$ for $i = 0, 1, \dots, n$. Moreover, h^{-1} stabilises L'_i , so $h(L'_i) = L'_i$ for $i = 0, 1, \dots, n$ and therefore for $i \in \mathbb{Z}$. This shows $h \in G_{L'}$ as required, so $H_N \subset G_{L'}$. \square

Note that H_N is generally not a normal subgroup of $G_{L'}$, though the space of (right) cosets of H_N in $G_{L'}$ will still be irreducible. [ADD AN EXAMPLE](#)

Lemma 5.3.10. $G_{L'}/H_N$ is irreducible, provided $H_N \subset G_{L'}$.

Proof. **COMPLETE THIS PROOF.** \square

5.4 Existence of a maximum

Proposition 5.4.1. *Given $A, B \in \Lambda_1(n, r)$ with $\text{co } A = \text{ro } B$, $\Lambda_1(n, r)_{A, B}$ has a maximum element.*

Draft 1. $\Lambda_1(n, r)_{A, B}$ is non-empty since $\text{co } A = \text{ro } B$. The partial order on $\Lambda_1(n, r)_{A, B}$ is given by the partial order on $\Lambda_1(n, r)$; where $C' \leq C$ if and only if $d_{i, j}C' \leq d_{i, j}C$ for all $i, j \in \mathbb{Z}$.

To prove existence of a maximum element in $\Lambda_1(n, r)_{A, B}$ we will consider the poset of G -orbits in $\mathcal{F} \times \mathcal{F}$ and prove existence of a maximum orbit in $X_{A, B}$ using an open orbits argument. Recall $X_{A, B}$ consists of $(L, L'') \in \mathcal{F} \times \mathcal{F}$ such that there exists $L' \in \mathcal{F}$ with $(L, L') \in \mathcal{O}_A$ and $(L', L'') \in \mathcal{O}_B$.

There is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$ whenever $(L, L'') \in X_{A, B}$. Fix $L \in \mathcal{F}_{\text{ro } A}$ and write

$$X_{A, B}^L = \{L'' \in \mathcal{F} : (L, L'') \in X_{A, B}\}.$$

With the above choice of N , write

$$\Pi = \{L'' \in \mathcal{F}_{\text{co } B} : \varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0\}.$$

Π is a complex projective variety (not generally irreducible), closed under the action of G_L . [\[ADD A REFERENCE\]](#) The closure $\overline{X_{A, B}^L}$ of $X_{A, B}^L$ in Π is an irreducible complex projective variety.

Proposition [\[ADD A REFERENCE\]](#) shows there is a unique G_L -orbit in $X_{A, B}^L$ which is open in $\overline{X_{A, B}^L}$, say \mathcal{O}_C^L for some $C \in \Lambda_1(n, r)_{A, B}$. It will be shown that C is the maximum element of $\Lambda_1(n, r)_{A, B}$. Given $i, j \in \mathbb{Z}$, let $m_{i, j}$ denote the maximum of $\{d_{i, j}C : C \in \Lambda_1(n, r)_{A, B}\}$ and define

$$\mathcal{M}_{i, j} = \{L'' \in \overline{X_{A, B}^L} : d_{i, j}(L, L'') = m_{i, j}\}.$$

$\mathcal{M}_{i, j}$ is non-empty by definition of the $m_{i, j}$ and is closed under the action of G_L . $\mathcal{M}_{i, j}$ is open in $\overline{X_{A, B}^L}$ since the function

$$d_{i, j}^L : \Pi \rightarrow \mathbb{Z} : L'' \mapsto \dim \left(\frac{L_i}{L_i \cap L_j''} \right)$$

is lower semi-continuous [\[ADD A REFERENCE\]](#) and

$$\mathcal{M}_{i, j} = \overline{X_{A, B}^L} \setminus \{L'' \in \overline{X_{A, B}^L} : d_{i, j}^L(L'') \leq m_{i, j} - 1\}.$$

It follows that \mathcal{O}_C^L and $\mathcal{M}_{i, j}$ intersect non-trivially, since $\overline{X_{A, B}^L}$ is irreducible and therefore $\mathcal{O}_C^L \subset \mathcal{M}_{i, j}$ as both are closed under the action of G_L . This proves C is a maximum element of $\Lambda_1(n, r)_{A, B}$, since

$$d_{i, j}C = d_{i, j}(L, L'') = m_{i, j}$$

for any $L'' \in \mathcal{O}_C^L$. □

Draft 2. $\Lambda_1(n, r)_{A, B}$ is non-empty since $\text{co } A = \text{ro } B$. For each $i, j \in \mathbb{Z}$, define

$$m_{i, j} = \max_{C \in \Lambda_1(n, r)_{A, B}} d_{i, j}C.$$

It will be shown that there is a unique element $A * B \in \Lambda_1(n, r)_{A, B}$ with $d_{i, j}(A * B) = m_{i, j}$: such an element is necessarily a maximum in $\Lambda_1(n, r)_{A, B}$. Fix $L \in \mathcal{F}_{\text{ro } A}$ and assume $N \in \mathbb{N}$ is sufficiently large that $X_{A, B}^L \subset \Pi_N$; where

$$\Pi_N = \{L'' \in \mathcal{F}_{\text{co } B} : \varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0\}.$$

Lusztig notes [1] that Π_N is a projective algebraic variety, closed under the action of G_L . Lemma [ADD A REFERENCE] shows that the closure of $X_{A,B}^L$ in Π_N , denoted $\overline{X_{A,B}^L}$, is an irreducible complex projective variety.

For each $i, j \in \mathbb{Z}$, write

$$\mathcal{M}_{i,j} = \{L'' \in \overline{X_{A,B}^L} : d_{i,j}(L, L'') = m_{i,j}\}.$$

$\mathcal{M}_{i,j}$ is non-empty since $d_{i,j}(L, -)$ attains a maximum on $X_{A,B}^L$. $\mathcal{M}_{i,j}$ is open in $\overline{X_{A,B}^L}$ since

$$\overline{X_{A,B}^L} \setminus \mathcal{M}_{i,j} = \{L'' \in \overline{X_{A,B}^L} : d_{i,j}(L, L'') \leq m_{i,j} - 1\}$$

and the function

$$d_{i,j}(L, -) : \Pi_N \rightarrow \mathbb{Z} : L'' \mapsto \dim \left(\frac{L_i}{L_i \cap L_j''} \right)$$

is lower semi-continuous, by lemma [[ADD A REFERENCE]: lower semi-continuity].

Lemma [[ADD A REFERENCE]: open orbit] shows that there is a unique G_L -orbit in $X_{A,B}^L$ which is open in $\overline{X_{A,B}^L}$, say \mathcal{O}_{A*B}^L for some $A * B \in \Lambda_1(n, r)_{A,B}$. $\mathcal{M}_{i,j}$ intersects the open orbit \mathcal{O}_{A*B}^L non-trivially, since $\mathcal{M}_{i,j}$ and \mathcal{O}_{A*B}^L are both non-empty and open in the irreducible space $\overline{X_{A,B}^L}$. Moreover, $\mathcal{O}_{A*B}^L \subset \mathcal{M}_{i,j}$, since $\mathcal{M}_{i,j}$ is closed under the action of G_L . In particular, we have $A * B \in \Lambda_1(n, r)_{A,B}$ with $d_{i,j}(A * B) = m_{i,j}$ for each $i, j \in \mathbb{Z}$, which shows $A * B$ is a maximum in $\Lambda_1(n, r)_{A,B}$.

More specifically, we may compute:

$$a_{i,j}(A * B) = m_{i,j-1} - m_{i-1,j-1} + m_{i-1,j} - m_{i,j}$$

for each $i, j \in \mathbb{Z}$. □

Lemma 5.4.2. *For each $\lambda \in \Lambda_0(n, r)$, $D_\lambda * A = A$ whenever $\text{ro } A = \lambda$ and $A * D_\lambda = A$ whenever $\text{co } A = \lambda$.*

5.5 Associativity

Proposition 5.5.1. *Given $A, B, C \in \Lambda_1(n, r)$ with $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$, $(A * B) * C = A * (B * C)$.*

Proof. INCLUDE PROOF. □

5.6 The generic algebra

Definition 5.6.1. *For each $n, r \geq 1$, the generic category $\mathcal{G}(n, r)$ is the category with set of objects $\Lambda_0(n, r)$ and set of morphisms $\Lambda_1(n, r)$ where; the morphisms from λ to μ are those matrices $A \in \Lambda_1(n, r)$ with $\text{co } A = \lambda$ and $\text{ro } A = \mu$; the composition of morphisms $A : \lambda \rightarrow \mu$ and $B : \mu \rightarrow \nu$ is $B * A : \lambda \rightarrow \nu$, where $B * A$ is the maximum element in $\Lambda_1(n, r)_{A,B}$. For each $\lambda \in \Lambda_0(n, r)$, the identity morphism $D_\lambda : \lambda \rightarrow \lambda$ is given by $(D_\lambda)_{i,i} = \lambda_i$ and $(D_\lambda)_{i,j} = 0$ whenever $i \neq j$.*

Example 1. *The objects in $\mathcal{G}(n, r)_{2,2}$ are compositions of 2 into 2 parts, namely $(0, 2)$, $(1, 1)$ and $(2, 0)$. The set of morphisms from λ to μ is the set of infinite periodic matrices $A \in \Lambda_1(n, r)_{2,2}$ with $\text{co } A = \lambda$ and $\text{ro } A = \mu$, which is a countably infinite set for any pair of compositions $\lambda, \mu \in \Lambda_0(n, r)_{2,2}$.*

Definition 5.6.2 (Generic algebra). *The affine generic algebra $\hat{G}(n, r)$ is the category \mathbb{Z} -algebra of $\mathcal{G}(n, r)$. In particular, $\hat{G}(n, r)$ is a free \mathbb{Z} -module with basis $\{e_A : A \in \Lambda_1(n, r)\}$ and with associative multiplication given by*

$$e_A * e_B = \begin{cases} e_{A*B} & \text{if } \text{co } A = \text{ro } B \\ 0 & \text{if } \text{co } A \neq \text{ro } B. \end{cases}$$

Given $\lambda \in \Lambda_0(n, r)$, let $1_\lambda = e_{D_\lambda}$.

Corollary 5.6.1. $\{1_\lambda : \lambda \in \Lambda_0(n, r)\}$ is a set of pairwise orthogonal idempotents in $\hat{G}(n, r)$ with $\sum_{\lambda \in \Lambda_0(n, r)} 1_\lambda = 1$.

Theorem 5.6.2. $\hat{G}(n, r)$ is an associative \mathbb{Z} -algebra with 1.

Proof. Given $A, B \in \Lambda_1(n, r)$ with $\text{co } A = \text{ro } B$, proposition 5.4.1 shows that there is a maximum element in $\{C \in \Lambda_1(n, r) : g_{A,B,C} \neq 0\}$, which is denoted $A * B$. This shows that the product on $\hat{G}(n, r)$ is well-defined. If $\text{co } A \neq \text{ro } B$ or $\text{co } B \neq \text{ro } C$, then $(e_A * e_B) * e_C = 0 = e_A * (e_B * e_C)$. If $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$, then proposition 5.5.1 shows that

$$(e_A * e_B) * e_C = e_{(A*B)*C} = e_{A*(B*C)} = e_A * (e_B * e_C).$$

Corollary 5.6.1 shows that the sum of the idempotents 1_λ for $\lambda \in \Lambda_0(n, r)$ is a multiplicative identity. \square

5.7 Multiplication rules

Write

$$E_i = \sum_{\lambda \in \Lambda_0(n, r)} E_{i,\lambda}$$

$$F_i = \sum_{\lambda \in \Lambda_0(n, r)} F_{i,\lambda}.$$

Then $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$.

Lemma 5.7.1. *Let $A \in \Lambda_1(n, r)$, $i \in [1, n]$ and let $\lambda = \text{ro } A$. The following multiplication rules hold:*

$$E_i e_A = \begin{cases} e_{A+X_{i,p}} & \text{if } \lambda_{i+1} > 0 \\ 0 & \text{if } \lambda_{i+1} = 0; \end{cases}$$

where p is such that $A_{i+1,p} > 0$ and $A_{i+1,j} = 0$ for $j > p$. Also

$$F_i e_A = \begin{cases} e_{A-X_{i,p}} & \text{if } \lambda_i > 0 \\ 0 & \text{if } \lambda_i = 0; \end{cases}$$

where p is such that $A_{i,p} > 0$ and $A_{i,j} = 0$ for $j < p$.

Similar formulas for right multiplication by E_i and F_i are obtained by applying the transpose.

Lemma 5.7.2. *The following relations hold in $\hat{G}(n, r)$ ($n \geq 3$):*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless $|j - i| = 1$.

$$E_i E_{i+1}^2 - E_{i+1} E_i E_{i+1} = 0$$

$$E_i^2 E_{i+1} - E_i E_{i+1} E_i = 0$$

$$F_{i+1} F_i^2 - F_i F_{i+1} F_i = 0$$

$$F_{i+1}^2 F_i - F_{i+1} F_i F_{i+1} = 0$$

$$E_i F_j - F_j E_i = 0$$

unless $j = i$.

$$E_i F_i - F_i E_i + \sum_{\lambda: \lambda_i=0, \lambda_{i+1}>0} 1_\lambda - \sum_{\lambda: \lambda_i>0, \lambda_{i+1}=0} 1_\lambda = 0.$$

5.8 – Chapter draft bin –

Define

$$\Pi = \left\{ L'' \in \mathcal{F}_{\text{co}B} : \varepsilon^N L_0 \subset L_0'' \subset \dots \subset L_n'' \subset \varepsilon^{-N} L_0 \text{ and } \dim(L_0''/\varepsilon^N L_0) = -Nr + d_{-Nn,0}^-(A) + d_{-Nn,0}^-(B) \right\}.$$

Lemma 5.8.1. *Π is a projective algebraic variety, closed under the action of G_L .*

By choice of N , we have $X_{A,B}^L \subset \Pi$.

Lemma 5.8.2. *The group G_L/H is an irreducible algebraic group.*

Proof. $\sigma \in G_L$ naturally induces an automorphism $\bar{\sigma}$ of $\varepsilon^{-N} L_0/\varepsilon^N L_0$, with inverse induced by σ^{-1} . Moreover, the natural map

$$G_L/H \rightarrow GL(\varepsilon^{-N} L_0/\varepsilon^N L_0)$$

is a group homomorphism. In fact, this homomorphism is injective: if $\sigma = \tau$ on $\varepsilon^{-N} L_0/\varepsilon^N L_0$, then $\sigma\tau^{-1} = 1$ on $\varepsilon^{-N} L_0/\varepsilon^N L_0$ and so $\sigma H = \tau H$. Thus G_L/H is isomorphic to its image in $GL(\varepsilon^{-N} L_0/\varepsilon^N L_0)$. **this image is an algebraic group, then I need to deduce G_L/H is an algebraic group. First isomorphism theorem?** \square

Lemma 5.8.3. *Suppose $(L, L', L''), (N, N', N'') \in \beta^{-1}(\mathcal{O}_A \times \mathcal{O}_B)$. Then there are $\sigma, \tau \in G$, with $\tau \in G_{L'}$, such that $(N, N', N'') = \sigma(L, L', \tau L'')$.*

Proof. There exist $g, g' \in G$ such that $(N, N') = g(L, L')$ and $(N', N'') = g'(L', L'')$. Then $(N, N', N'') = g(L, L', g^{-1}g'L'')$. Taking $\sigma = g$ and $\tau = g^{-1}g'$ gives the required result. \square

Proposition 5.8.4. *Suppose $X_{A,B}^L \neq \emptyset$. Then $X_{A,B}^L \subset \mathcal{F}_{\text{co}B}$ is finite dimensional and irreducible.*

Proof. The map

$$G_L/H \times G_{L'}/H \rightarrow \Pi$$

has image $X_{A,B}^L$, so the closure of $X_{A,B}^L$ in Π is irreducible **due to some properties of the above groups.** \square

5.8.1 locally closed orbits

Proposition 5.8.5. *Suppose $X_{A,B}^L \neq \emptyset$. The G_L -orbits in $X_{A,B}^L$ are locally closed.*

Proof. The G_L orbit of $L'' \in X_{A,B}^L$ is the image of the map

$$G_L/H \rightarrow \Pi : g \mapsto gL''.$$

Justify why this image must be locally closed. □

Proposition 5.8.6. *Let $A, B \in \Lambda_1(n, r)$, $L \in \mathcal{F}$ and suppose $X_{A,B}^L \neq \emptyset$. There is a unique open G_L -orbit in $X_{A,B}^L$.*

Proof. Write $X = X_{A,B}^L$. X is irreducible and finite dimensional, using Lemma 5.8.4. We have

$$X = \bigcup_C O_C,$$

where the union is taken over the finite set $\{C \in \Lambda_1(n, r) : O_C \subset X_{A,B}\}$.

A proper, non-empty, closed subset of X has strictly smaller dimension than X , so there is C such that $\overline{O_C} = X$. O_C is locally closed, by Lemma 5.8.5, so it follows that O_C is open in $\overline{O_C} = X$.

Now suppose O_C is an open G_L orbit and let $D \in \Lambda_1(n, r)$. $O_D \subset X \setminus O_C$ and thus $\overline{O_D} \subset X \setminus O_C$. This shows O_D is not open in X and thus the claim is proven. □

5.8.2 Associativity of the generic product

Given $A, B, C \in \Lambda_1(n, r)$ and $L \in \mathcal{F}$ let

$$X_{A,B,C}^L = \{L''' \in \mathcal{F} : \exists(L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A, (L'', L''') \in \mathcal{O}_C\}.$$

Note that $X_{A,B,C}^L \subset \mathcal{F}$ is contained in $\mathcal{F}_{\text{co } C}$ and is non-empty only if $L \in \mathcal{F}_{\text{ro } A}$, $\text{co } A = \text{ro } B$ and $\text{co } B = \text{ro } C$. $X_{A,B,C}^L$ consists of finitely many G_L -orbits. Using a similar argument to the existence of generic orbits we show that there is a unique generic orbit in $X_{A,B,C}^L$, which will establish associativity of the generic product. We now suppose $X_{A,B,C}^L$ is non-empty and fix $(L, L', L'', L''') \in \mathcal{F}^4$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L'', L''') \in \mathcal{O}_C$.

Lemma 5.8.7. *$X_{A,B,C}^L$ is the image of the map*

$$\phi : G_L \times G_{L'} \times G_{L''} \rightarrow \mathcal{F} : (\alpha, \beta, \gamma) \mapsto \alpha\beta\gamma L'''.$$

Proposition 5.8.8. *The closure $\overline{X_{A,B,C}^L}$ of $X_{A,B,C}^L$ in \mathcal{F} is irreducible.*

Proposition 5.8.9. *There is a unique generic G_L -orbit in $X_{A,B,C}^L$.*

Chapter 6

A realisation of affine zero Schur algebras

We aim to prove the isomorphism theorem in the cases $r < n$ and $n \leq r < 2n$ separately. Below are crude versions of the statements we want to prove.

Theorem 6.0.1. *Assume $r < n$. The map $\psi: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$, given by $\psi(E_i) = E_i$, $\psi(F_i) = F_i$ and $\psi(1_\lambda) = 1_\lambda$, is an isomorphism of \mathbb{Z} -algebras.*

Proof. Below are some of the pieces: [1] The elements $E_i, F_i, 1_\lambda$ generate $\hat{G}(n, r)$.

Provided $r < n$, any $A \in \Lambda_1(n, r)$ may be obtained from the diagonal matrix D_λ with $\lambda = \text{ro } A$ by a sequence of transitions $A \mapsto A \pm X_{i,p}$.

[2] Give a complete set of generating relations for $\hat{G}(n, r)$. □

Theorem 6.0.2. *Assume $n \leq r < 2n$. There is a unique homomorphism of \mathbb{Z} -algebras $\hat{\psi}: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$ such that $\hat{\psi}(R) = R$ and $\hat{\psi} = \psi$ on the subalgebra of $\hat{G}(n, r)$ generated by the E_i, F_i and 1_λ . $\hat{\psi}$ is an isomorphism of \mathbb{Z} -algebras.*

6.1 Presentation of the generic algebra.

Recall that $\Lambda_0(n, r)$ denotes the set of compositions of r into n parts. That is, $\Lambda_0(n, r)$ is the set of tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with each λ_i non-negative and $\lambda_1 + \dots + \lambda_n = r$. Given $i \in [1, n]$, let $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$ be the i -th elementary vector and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then given $\lambda \in \Lambda_0(n, r)$, we have $\lambda + \alpha_i \in \Lambda_0(n, r)$ provided $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0(n, r)$ provided $\lambda_i > 0$.

Let $\Gamma = \Gamma(n, r)$ be the quiver with set of vertices $\Lambda_0(n, r)$ with arrows $e_{i,\lambda}: \lambda \rightarrow \lambda + \alpha_i$ (if $\lambda_{i+1} > 0$) and $f_{i,\lambda}: \lambda \rightarrow \lambda - \alpha_i$ (if $\lambda_i > 0$). Thus there are no arrows between λ and μ unless $\lambda = \mu \pm \alpha_i$ for some $i \in [1, n]$.

If $n \geq 3$ then neighbouring vertices are connected by two arrows, one of each direction. In the case $n = 2$, neighbouring vertices are joined by four arrows, two of each direction. The $\mathbb{Z}\Gamma$ denote the path \mathbb{Z} algebra of Γ . By construction of Γ , there is a \mathbb{Z} -algebra homomorphism $\mathbb{Z}\Gamma \rightarrow \hat{G}(n, r)$ with $e_{i,\lambda} \mapsto E_{i,\lambda}$, $f_{i,\lambda} \mapsto F_{i,\lambda}$ and $k_\lambda = 1_\lambda$. We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [1]).

$A \in \Lambda_1(n, r)$ is said to be aperiodic if for each $l \in \mathbb{Z} \setminus \{0\}$ there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$. Denote the set of aperiodic elements in $\Lambda_1(n, r)$ by $\Lambda_1(n, r)^{ap}$. Note that $\Lambda_1(n, r)^{ap} = \Lambda_1(n, r)$ if $r < n$.

Proposition 6.1.1. *The subalgebra of $\hat{G}(n, r)$ generated by $E_{i,\lambda}$, $F_{i,\lambda}$ and 1_λ has \mathbb{Z} -basis $\{e_A : A \in \Lambda_1(n, r)^{ap}\}$, where $\Lambda_1(n, r)^{ap} \subset \Lambda_1(n, r)$ is the set of aperiodic elements.*

Chapter 7

Further directions

7.1 Further results on affine zero Schur algebras

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

7.2 Deformed group algebras of symmetric groups

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algebras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for S_3 and S_4 . Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: ‘these’ relations hold; these generators and relations give a presentation of the degenerate group algebras.

Terminology: deformed group algebra.

Bibliography

- [1] George Lusztig. “Aperiodicity in quantum affine \mathfrak{gl}_n ”. In: *Asian Journal of Mathematics* 3.1 (1999), pp. 147–178.