A geometric realisation of affine 0-Schur algebras.

Tom Crawley

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The cyclic flags approach to affine q-Schur algebras

Fix natural numbers n and r.

Definition 3.0.1 (compositions). *A composition of r into n parts is an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of non-negative integers whose sum equals r. Denote the set of compositions of r into n parts by Λ_0.*

Definition 3.0.2 (infinite periodic matrices). *Let Λ_1 be the set of matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with integer entries $a_{i,j}$ satisfying the following conditions:*

- *$a_{i,j} \geq 0$ for each $i, j \in \mathbb{Z}$;*
- *each row or column has only finitely many non-zero entries;*
- *the sum of the entries in any n consecutive rows or columns equals r;*
- *$a_{i-n,j-n} = a_{i,j}$ for each $i, j \in \mathbb{Z}$.*

These matrices are referred to as infinite periodic matrices.

Definition 3.0.3 (source and target). *Given $A \in \Lambda_1$, let $\mathrm{ro}(A)$ and $\mathrm{co}(A)$ be the compositions of r into n parts given by*

$$\mathrm{ro}(A) = \left(\sum_{j\in\mathbb{Z}} a_{1,j}, \ldots, \sum_{j\in\mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co}(A) = \left(\sum_{i\in\mathbb{Z}} a_{i,1}, \ldots, \sum_{i\in\mathbb{Z}} a_{i,n} \right).$$

$A \in \Lambda_1$ is said to go from $\mathrm{co}(A)$ to $\mathrm{ro}(A)$.

Definition 3.0.4 (diagonal matrices). *Given $\lambda \in \Lambda_0$, let $D_\lambda \in \Lambda_1$ be the matrix given by $(D_\lambda)_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i \neq j$ and $(D_\lambda)_{i,i} = \lambda_i$ for $i \in \mathbb{Z}$; where the indices are taken modulo n.*

3.1 Cyclic flags

Fix $n, r \in \mathbb{N}$ and let $\mathbf{k}$ be a field. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by ε, so $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free $\mathcal{S}$-module of rank r. Let G be the automorphism group of the $\mathcal{S}$-module V, so G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. A lattice in V is a $\mathcal{R}$-submodule L of V with $\mathcal{S} \otimes_{\mathcal{R}} L = V$. In particular, a lattice is an $\mathcal{R}$-submodule of V which is a free $\mathcal{R}$-module of rank r.

Lemma 3.1.1. *Let L be a lattice in V. $L/\varepsilon L$ is a torsion $\mathcal{R}$-module, where ε acts as zero. $L/\varepsilon L$ is a free $\mathcal{R}/\langle\varepsilon\rangle$-module of rank r; that is, $L/\varepsilon L$ is an r-dimensional $\mathbf{k}$-vector space.*

Proof. L is a free $\mathcal{R}$-module of rank r, with $L \subset V$. Given an $\mathcal{R}$-basis $\{x_1, \ldots, x_r\}$ of L, $\{\varepsilon x_1, \ldots, \varepsilon x_r\}$ is an $\mathcal{R}$-basis of εL. Finally, the cosets $\{x_1 + \varepsilon L, \ldots, x_r + \varepsilon L\}$ give a basis for $L/\varepsilon L$ over $\mathcal{R}/\langle\varepsilon\rangle \cong \mathbf{k}$. □

Let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of collections $(L_i)_{i\in\mathbb{Z}}$ of lattices in V with $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. These collections of lattices in V are referred to as cyclic flags in V.

G acts on $\mathcal{F}$ by $(g \cdot L)_i = g(L_i)$ for each $i \in \mathbb{Z}$, given $g \in G$ and $L \in \mathcal{F}$. The G-orbits in $\mathcal{F}$ are indexed by the set Λ_0 of compositions of r into n parts: the G-orbit in $\mathcal{F}$ corresponding to $\lambda \in \Lambda_0$ is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim\left(\frac{L_i}{L_{i-1}}\right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

Definition 3.1.1. *The periodic characteristic matrix of a pair of cyclic flags $(L, L') \in \mathcal{F}\times\mathcal{F}$ is the matrix $\mathrm{A}(L, L') = (a_{i,j})_{i,j\in\mathbb{Z}}$ with entries*

$$a_{i,j} = \dim_{\mathbf{k}}\left(\frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j}\right)$$

for each $i, j \in \mathbb{Z}$.

The diagonal action of G on $\mathcal{F}\times\mathcal{F}$ has orbits indexed by the set Λ_1 of infinite periodic matrices (see definition 3.0.2). The G-orbit corresponding to $A \in \Lambda_1$ is denoted $\mathcal{O}_A$ and consists of those pairs $(L, L') \in \mathcal{F}\times\mathcal{F}$ with periodic characteristic matrix $\mathrm{A}(L, L')$ equal to A.

Lemma 3.1.2. *(alternative expression for characteristic matrix) Alternatively,*

$$a_{i,j} = \dim_{\mathbf{k}}\left(\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}}\right)$$

for each $i, j \in \mathbb{Z}$.

Proof. Set $U = L_i \cap L'_j$ and $U' = L_{i-1} + L_i \cap L'_{j-1}$. Then $U + U' = L_{i-1} + L_i \cap L'_j$ and $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$. Applying the isomorphism theorems, $U + U'/U'$ is naturally isomorphic to $U/U \cap U'$ as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to $a_{i,j}$. □

Lemma 3.1.3 (transposing characteristic matrix). *Given a pair of flags $(L, L') \in \mathcal{F}^2$, the matrices $\mathrm{A}(L, L')$ and $\mathrm{A}(L', L)$ are related by the transpose. In particular, $\mathrm{A}(L, L')_{i,j} = \mathrm{A}(L', L)_{j,i}$ for each $i, j \in \mathbb{Z}$.*

Proof. By swapping the roles of i and j and swapping L and L' it is clear that $\mathrm{A}(L, L')_{i,j}$ and $\mathrm{A}(L', L)_{j,i}$ are both given by the dimension of the $\mathbf{k}$-vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each $i, j \in \mathbb{Z}$. □

Lemma 3.1.4 (a codimension formula). *Given $(L, L') \in \mathcal{F}^2$ and $i, j \in \mathbb{Z}$,*

$$\dim_{\mathbf{k}} \left(\frac{L_i}{L_i \cap L'_j} \right) = \sum_{s \leq i, t > j} a_{s,t},$$

where $\mathrm{A}(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$.

Proof. COMPLETE THIS PROOF □

Lemma 3.1.5 (nested flags). *Given $(L, L') \in \mathcal{F}^2$, $L' \subset L$ if and only if $\mathrm{A}(L, L')_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i > j$.*

Proof. Suppose $L, L' \in \mathcal{F}$ with $L' \subset L$, meaning $L'_j \subset L_j$ for each $j \in \mathbb{Z}$. Then for $i > j$, $L_i \cap L'_j = L'_j$, $L_{i-1} \cap L'_j = L'_j$ and $L_i \cap L'_{j-1}$, which shows

$$\mathrm{A}(L, L')_{i,j} = \dim_{\mathbf{k}} \left(\frac{L'_j}{L'_{j-1} + L'_j} \right) = 0$$

as required. Conversely, suppose $\mathrm{A}(L, L')$ is upper triangular, meaning $\mathrm{A}(L, L')_{i,j} = 0$ when $i > j$. Using Lemma 3.1.4,

$$\dim_{\mathbf{k}} \left(\frac{L'_i}{L'_i \cap L_i} \right) = \sum_{s > i, t \leq i} a_{s,t} = 0,$$

so $L_i \cap L'_i = L'_i$ and thus $L'_i \subset L_i$ for each $i \in \mathbb{Z}$, as required. □

Corollary 3.1.6 (diagonal orbits). *Given $L, L' \in \mathcal{F}$, $L = L'$ if and only if $\mathrm{A}(L, L')_{i,j} = 0$ whenever $i \neq j$. In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L, L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each $\lambda \in \Lambda_0$.

3.1.1 A product on orbits

Given $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also $L \in \mathcal{F}_{\mathrm{ro}(A)}$, define the L-slices of $Y_{A,B}$ and $X_{A,B}$ respectively as

$$Y^L_{A,B} = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X^L_{A,B} = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

Observation 1. *There are only finitely many G-orbits in $X_{A,B}$.*

Lemma 3.1.7. *Given $A \in \Lambda_1$, $X_{D_\lambda,A} = \mathcal{O}_A$ if $\lambda = \mathrm{ro}(A)$ and $X_{A,D_\lambda} = \mathcal{O}_A$ if $\lambda = \mathrm{co}(A)$.*

Proof. Let $A \in \Lambda_1$ and set $\lambda = \mathrm{ro}(A)$. $Y_{D_\lambda,A}$ is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_{D_\lambda}$, thus $L = L'$ by Corollary 3.1.6, and $(L', L'') \in \mathcal{O}_A$. $X_{D_\lambda,A}$ is the projection of $Y_{D_\lambda,A}$, which equals $\mathcal{O}_A$.

Similarly, if $\lambda = \mathrm{co}(A)$, Y_{A,D_λ} is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_A$ and $L'' = L'$, so X_{A,D_λ} is exactly the orbit $\mathcal{O}_B$. □

3.1.2 Triple products

Given $A, B, C \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(B) = \mathrm{ro}(C)$ and $L \in \mathcal{F}_{\mathrm{ro}(A)}$, there are spaces $X_{A,B,C}$, $Y_{A,B,C}$ and their respective L-slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y^L_{A,B,C} = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X^L_{A,B,C} = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

3.2 Convolution algebras

Suppose $\mathbf{k}$ is a finite field and let q denote the number of elements of $\mathbf{k}$. Consider the set S of G-invariant functions $\mathcal{F}\times\mathcal{F} \to \mathbb{Z}$ with constructible support. S is a free $\mathbb{Z}$-module with a basis consisting of the indicator functions of the G-orbits in $\mathcal{F}\times\mathcal{F}$. Define an operation $\star$ on S as follows: for each $f, g \in S$, $f \star g \in S$ is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for $(L, L'') \in \mathcal{F}\times\mathcal{F}$.

$f \star g$ is well defined since the supports of f and g consist of finitely many G-orbits, so there are only finitely many $L' \in \mathcal{F}$ such that $f(L, L')g(L', L'') \neq 0$, given $(L, L'') \in \mathcal{F}\times\mathcal{F}$. $f \star g$ is constant on G-orbits and is supported on finitely many G-orbits, so $f \star g \in S$.

Lemma 3.2.1. *The set S together with the operation $\star$ is an associative $\mathbb{Z}$-algebra with identity element ι given by $\iota(L, L) = 1$ and $\iota(L, L') = 0$ for $L' \neq L$.*

Proof. Given $f, g, h \in S$ and $(L, L''') \in \mathcal{F}\times\mathcal{F}$,

$$\begin{aligned}((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''}\sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''),\end{aligned}$$

thus $\star$ is associative. ι is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L')\iota(L', L'') = f(L, L''),$$

for each $f \in S$ and $(L, L'') \in \mathcal{F} \times \mathcal{F}$. $\square$

Given $A \in \Lambda_1$, let $e_A \in S$ denote the indicator function of the orbit $\mathcal{O}_A$. S is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1\}$. There exist $\gamma_{A,B,C;\mathrm{q}} \in \mathbb{Z}$ for $A, B, C \in \Lambda_1$ such that

$$e_A \star e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C;\mathrm{q}} e_C$$

for each $A, B \in \Lambda_1$. Then

$$\begin{aligned}
\gamma_{A,B,C;\mathrm{q}} &= (e_A \star e_B)(L, L'') \\
&= \sum_{L'} e_A(L, L') e_B(L', L'') \\
&= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},
\end{aligned}$$

for any $(L, L'') \in \mathcal{O}_C$.

3.3 Affine q-Schur algebras

There exist polynomials $\gamma_{A,B,C} \in \mathbb{Z}[q]$ for $A, B, C \in \Lambda_1$ such that $\gamma_{A,B,C}(\mathrm{q}) = \gamma_{A,B,C;\mathrm{q}}$ for any prime power q, following [32, section 4]. The affine q-Schur algebra $\hat{S}_q(n, r)$ is a $\mathbb{Z}[q]$-algebra which is a free $\mathbb{Z}[q]$-module with basis $\{e_A : A \in \Lambda_1\}$ and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these 'universal polynomials' $\gamma_{A,B,C} \in \mathbb{Z}[q]$, it follows from Lemma 3.2.1 that $\hat{S}_q(n, r)$ is an associative $\mathbb{Z}[q]$-algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

Chapter 4

Quivers with relations for affine q-Schur algebras

4.1 Basic results and notation

4.1.1 Elementary matrices

For each $i, j \in \mathbb{Z}$, let $\mathcal{E}_{i,j}$ be the $\mathbb{Z}\times\mathbb{Z}$ 'elementary periodic matrix' with entries given by $(\mathcal{E}_{i,j})_{s,t} = 1$ if $(s,t) = (i+cn, j+cn)$ for some $c \in \mathbb{Z}$ and $(\mathcal{E}_{i,j})_{s,t} = 0$ otherwise. Clearly $\mathcal{E}_{i,j} = \mathcal{E}_{i+n,j+n}$ for each $i, j \in \mathbb{Z}$. Recall from Definition 3.0.4 that the diagonal matrix associated to a composition $\lambda \in \Lambda_0$ is

$$D_\lambda = \lambda_1\mathcal{E}_{1,1} + \cdots + \lambda_n\mathcal{E}_{n,n}.$$

$\{e_{D_\lambda} : \lambda \in \Lambda_0\}$ is a set of pairwise orthogonal idempotents in $\hat{S}_q(n,r)$ with $\sum_{\lambda\in\Lambda_0} e_{D_\lambda} = 1$, as a result of Lemma 3.1.7.

Given $i \in [1,n]$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and define

$$E_i = \sum_{\lambda\in\Lambda_0:\lambda_{i+1}>0} E_{i,\lambda}.$$

Given $i \in [1,n]$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and define

$$F_i = \sum_{\lambda\in\Lambda_0:\lambda_i>0} F_{i,\lambda}$$

4.1.2 Transpose involution

Lemma 4.1.1. *Transposition gives a homomorphism of $\mathbb{Z}[q]$-modules $\top\colon \hat{S}_q(n,r) \to \hat{S}_q(n,r)$ with $\top(e_A) = e_{A^\top}$, $\top\circ\top = 1$ and $\top(e_Ae_B) = \top(e_B)\top(e_A)$.*

Proof. Let $A, B, C \in \Lambda_1$ and let $\mathbf{k}$ be a finite field with $\mathrm{q} = \#\mathbf{k}$ elements. If $(L, L'') \in \mathcal{O}_C$ then $(L'', L) \in \mathcal{O}_{C^\top}$ and

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top,A^\top,C^\top;\mathrm{q}}\end{aligned}$$

It then follows that $\top(e_A e_B) = \top(e_B)\top(e_A)$. $\square$

The transpose relates the E_i, F_i and 1_λ in the following way: $\top(E_{i,\lambda}) = F_{i,\lambda}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\varepsilon_i+\varepsilon_{i+1}}$ and $\top(1_\lambda) = 1_\lambda$. In particular, $\top(E_i) = F_i$ and $\top(F_i) = E_i$.

4.1.3 A multiplication rule

Lemma 4.1.2. *Given $A \in \Lambda_1$ and $i \in [1, n]$ with $\mathrm{ro}(A)_{i+1} > 0$,*

$$E_i e_A = \sum_{p\in\mathbb{Z}:a_{i+1,p}>0} q^{\sum_{j>p} a_{i,j}} [[a_{i,p} + 1]] e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}}.$$

Given $A \in \Lambda_1$ and $i \in [1, n]$ with $\mathrm{ro}(A)_i > 0$,

$$F_i e_A = \sum_{p\in\mathbb{Z}:a_{i,p}>0} q^{\sum_{j<p} a_{i+1,j}} [[a_{i+1,p} + 1]] e_{A+\mathcal{E}_{i+1,p}-\mathcal{E}_{i,p}}.$$

Note that these formulas are still valid in the cases $E_i e_A = 0$ and $F_i e_A = 0$, provided it is understood that $e_B = 0$ whenever $B \notin \Lambda_1$. There are similar formulas for right multiplication by E_i and F_i, obtained by applying the transpose involution to the above.

Corollary 4.1.3. *Given $A \in \Lambda_1$ and $j \in [1, n]$ with $\mathrm{co}(A)_{j+1} > 0$,*

$$e_A F_j = \sum_{p\in\mathbb{Z}:a_{p,j+1}>0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\mathcal{E}_{p,j}-\mathcal{E}_{p,j+1}}.$$

Given $A \in \Lambda_1$ and $j \in [1, n]$ with $\mathrm{co}(A)_j > 0$,

$$e_A E_j = \sum_{p\in\mathbb{Z}:a_{p,j}>0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A+\mathcal{E}_{p,j+1}-\mathcal{E}_{p,j}}.$$

Proof.

$$\begin{aligned}e_A F_j &= \top\left(E_j e_{A^\top}\right) \\ &= \top\left(\sum_{p\in\mathbb{Z}:a_{p,j+1}>0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A^\top+\mathcal{E}_{j,p}-\mathcal{E}_{j+1,p}}\right) \\ &= \sum_{p\in\mathbb{Z}:a_{p,j+1}>0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\mathcal{E}_{p,j}-\mathcal{E}_{p,j+1}},\end{aligned}$$

where the second equality comes from Lemma 4.1.2. Similarly,

$$\begin{aligned}e_A E_j &= \top\left(F_j e_{A^\top}\right) \\ &= \top\left(\sum_{p\in\mathbb{Z}:a_{p,j}>0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A^\top+\mathcal{E}_{j+1,p}-\mathcal{E}_{j,p}}\right) \\ &= \sum_{p\in\mathbb{Z}:a_{p,j}>0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A+\mathcal{E}_{p,j+1}-\mathcal{E}_{p,j}}.\end{aligned}$$

$\square$

4.2 Relations

Note that $E_i^{r+1} = F_i^{r+1} = 0$ while

$$E_i^r = [r]_! e_{r\mathcal{E}_{i,i+1}}$$

and

$$F_i^r = [r]_! e_{r\mathcal{E}_{i+1,i}}.$$

Lemma 4.2.1 (quantum Serre relations: $n \geq 3$)**.** *Suppose $n \geq 3$. The following relations hold in $\hat{S}_q(n,r)$:*

$$E_iE_j - E_jE_i = 0$$

$$F_iF_j - F_jF_i = 0$$

unless $j = i \pm 1$;

$$E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i = 0$$
$$E_i^2E_{i+1} - (1+q)E_iE_{i+1}E_i + qE_{i+1}E_i^2 = 0$$

and

$$F_{i+1}F_i^2 - (1+q)F_iF_{i+1}F_i + qF_i^2F_{i+1} = 0$$
$$F_{i+1}^2F_i - (1+q)F_{i+1}F_iF_{i+1} + qF_iF_{i+1}^2 = 0.$$

Proof. Here we introduce temporary notation for the basis elements: Write $[A] = e_A$.

Take $\lambda \in \Lambda_0$.

$$E_iE_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1}E_iE_{i+1}1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2E_i1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then

$$(E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i)1_\lambda = 0,$$

for each $\lambda \in \Lambda_0$. The relation $E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i = 0$ then follows.

The relations between F_i and F_{i+1} may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping E_i and F_i and reversing the order of multiplication. $\square$

Lemma 4.2.2 (quantum Serre relations: $n = 2$)**.** *In the case $n = 2$, the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

Lemma 4.2.3. $[E_i, F_j] = 0$ *unless $j = i$.*

$$E_iF_i - F_iE_i = \sum_{\lambda\in\Lambda_0} ([\lambda_i] - [\lambda_{i+1}])1_\lambda.$$

For $\lambda \in \Lambda_0$, let $R_\lambda = e_{\lambda_1 \mathcal{E}_{0,1} + \cdots + \lambda_n \mathcal{E}_{n-1,n}}$. Write $R = \sum_{\lambda \in \Lambda_0} R_\lambda$. Note $R_\lambda = R 1_\lambda$. Given $A \in \Lambda_1$ and $m \in \mathbb{Z}$, let $A[m] \in \Lambda_1$ be given by $A[m]_{i,j} = a_{i,j+m}$ and let $A^{[m]}$ be given by $A^{[m]}_{i,j} = a_{i+m,j}$ for each $i \in \mathbb{Z}$.

Lemma 4.2.4 (Shifting). *If $A \in \Lambda_1$ then*

$$Re_A = e_{A[\pm 1]}$$

and

$$e_A R = e_{A_{[\pm 1]}}.$$

Conjugation by R gives an automorphism ρ of $\hat{S}_q(n,r)$ satisfying $\rho^n = 1$.

4.3 quivers with relations

Denote by Λ_0 the set of compositions of r into n parts. That is, Λ_0 is the set of $\alpha \in \mathbb{Z}^n$ with non-negative entries which sum to r. Let $\varepsilon_i \in \mathbb{Z}^n$ be the ith elementary vector and write $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for each $i \in [1,n]$. Then $\lambda + \alpha_i \in \Lambda_0$ if $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ if $\lambda_i > 0$.

Let $\Gamma = \Gamma(n,r)$ be the quiver with set of vertices Λ_0, with the following arrows:

For $\lambda \in \Lambda_0$ and $i \in [1,n]$, there is an arrow $e_{i,\lambda} : \lambda \to \lambda + \alpha_i$ if $\lambda_{i+1} > 0$ and there is an arrow $f_{i,\lambda} : \lambda \to \lambda - \alpha_i$ if $\lambda_i > 0$.

Denote by $\mathbb{Z}[q]\Gamma$ the path $\mathbb{Z}[q]$-algebra of Γ. Thus $\mathbb{Z}[q]\Gamma$ is a free $\mathbb{Z}[q]$-module with a basis given by the set of paths in Γ, with multiplication given by the concatenation of paths. If p starts where q ends, the product pq is the path q followed by p. Write $e_{i,\lambda} = 0$ unless $\lambda, \lambda + \alpha_i \in \Lambda_0$ and write $f_{i,\lambda} = 0$ unless $\lambda, \lambda - \alpha_i \in \Lambda_0$.

By construction, there is a homomorphism of $\mathbb{Z}[q]$-algebras

$$\phi \colon \mathbb{Z}[q]\Gamma \to \hat{S}_q(n,r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda, \end{aligned}$$

for $i \in [1,n]$ and $\lambda \in \Lambda_0$.

The image of ϕ is the subalgebra of $\hat{S}_q(n,r)$ generated by E_i, F_i for $i \in [1,n]$ and 1_λ for $\lambda \in \Lambda_0$, since $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$, while $E_i = \sum_\lambda E_{i,\lambda}$ and $F_i = \sum_\lambda F_{i,\lambda}$. In general ϕ is not surjective, so this does not always lead to a presentation of $\hat{S}_q(n,r)$.

4.3.1 Exceptional case n=2.

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the q-Schur algebra.

4.3.2 Typical case.

Suppose $n \geq 3$. Then $\Gamma = \Gamma(n,r)$ has vertex set Λ_0.

Define $e_i, f_i \in \mathbb{Z}[q]\Gamma(n,r)$ by

$$e_i = \sum_{\lambda \in \Lambda_0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0} f_{i,\lambda},$$

with the convention $e_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $f_{i,\lambda} = 0$ unless $\lambda_i > 0$. Let k_λ denote the constant path at vertex λ. $\{k_\lambda : \lambda \in \Lambda_0\}$ is a set of pairwise orthogonal idempotents in $\mathbb{Z}[q]\Gamma(n,r)$.

Let $I(n,r) \subset \mathbb{Z}[q]\Gamma(n,r)$ be the ideal generated by the expressions

$$e_i e_{i+1}^2 - (1+q)e_{i+1}e_ie_{i+1} + qe_{i+1}^2 e_i$$

$$e_i^2 e_{i+1} - (1+q)e_ie_{i+1}e_i + qe_{i+1}e_i^2$$

$$f_{i+1}f_i^2 - (1+q)f_if_{i+1}f_i + qf_i^2 f_{i+1}$$

$$f_{i+1}^2 f_i - (1+q)f_{i+1}f_if_{i+1} + qf_if_{i+1}^2$$

$$e_if_j - f_je_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}])k_\lambda$$

Recall that a relation is a $\mathbb{Z}[q]$-linear combination of paths with common start and end vertices. The relations involving paths $\lambda \to \mu$ are given by $1_\mu \mathrm{expr} 1_\lambda$, for each of the above expressions.

Lemma 4.3.1. *There is a homomorphism of* $\mathbb{Z}[q]$*-algebras*

$$\phi\colon \mathbb{Z}[q]\Gamma(n,r)/I(n,r) \to \hat{S}_q(n,r)$$

given by

$$\phi(e_{i,\lambda}) = E_{i,\lambda}$$

$$\phi(f_{i,\lambda}) = F_{i,\lambda}$$

$$\phi(k_\lambda) = 1_\lambda.$$

Chapter 5

A generic affine algebra

5.1 Introducing the generic affine algebra

Assume $\mathbf{k} = \mathbb{C}$ and fix $n, r \geq 1$. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by ε, namely $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free $\mathcal{S}$-module of rank r and let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of n-periodic cyclic flags in V; so $\mathcal{F}$ consists of collections $L = (L_i)_{i \in \mathbb{Z}}$ of $\mathcal{R}$-lattices in V with $L_i \subset L_{i+1}$ for $i \in \mathbb{Z}$ and $\varepsilon L_i = L_{i-n}$ for $i \in \mathbb{Z}$.

Let G be the group of $\mathcal{S}$-module automorphisms of V. Thus G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. G acts on $\mathcal{F}$ with orbits $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0\}$, where Λ_0 is the set of compositions of r into n parts, as in Definition 3.0.1.

The diagonal action of G on $\mathcal{F} \times \mathcal{F}$ has orbits $\{\mathcal{O}_A : A \in \Lambda_1\}$, where $\mathcal{O}_A$ consists of those pairs of flags with periodic characteristic matrix equal to A. Definitions of the periodic characteristic matrix and the set Λ_1 are given in Definition 3.1.1 and Definition 3.0.2 respectively.

Recall that the periodic characteristic matrix of a pair $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix $A = (a_{i,j})_{i,j \in \mathbb{Z}}$, with

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right)$$

for each $i, j \in \mathbb{Z}$.

Recall that ro and co are the maps $\Lambda_1 \to \Lambda_0$ given by

$$\mathrm{ro}(A) = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \ldots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co}(A) = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \ldots, \sum_{i \in \mathbb{Z}} a_{i,n} \right)$$

for each $A \in \Lambda_1$. Given $A \in \Lambda_1$, write $A\colon \mathrm{co}(A) \to \mathrm{ro}(A)$.

The purpose of this chapter is to define a category with objects Λ_0 and morphisms Λ_1; where $\mathrm{Hom}(\lambda, \mu) = \{A \in \Lambda_1 : \mathrm{ro}(A) = \mu, \mathrm{co}(A) = \lambda\}$. Given $A, B \in \Lambda_1$ let $\Lambda_1^{A,B}$ be the set of $C \in \Lambda_1$ such that there exist $L, L', L'' \in \mathcal{F}$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L, L'') \in \mathcal{O}_C$. It will be shown that Λ_1 admits a partial order $\leq$ such that, given $A, B \in \Lambda_1$ with $\mathrm{ro}(B) = \mathrm{co}(A)$, $\Lambda_1^{A,B}$ has a maximum element $A * B$. It will be shown that $*$ is associative, leading to the construction of a category with the described properties.

The generic affine algebra $\hat{G}(n, r)$ is then defined to be the $\mathbb{Z}$-algebra of this category.

5.2 A combinatorial partial order

Given $i, j \in \mathbb{Z}$, define maps $d_{i,j}$ and $\bar{d}_{i,j}$ on Λ_1 by setting

$$d_{i,j}A = \sum_{s \leq i, t > j} a_{s,t}$$

and

$$\bar{d}_{i,j}A = \sum_{s > i, t \leq j} a_{s,t}$$

for each $A \in \Lambda_1$.

Lemma 5.2.1. *Let $A \in \Lambda_1$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for $i, j \in \mathbb{Z}$. Then*

$$d_{i,j} - d_{i-1,j} = \sum_{t > j} a_{i,t}$$

and

$$d_{i,j} - d_{i,j-1} = -\sum_{s \leq i} a_{s,j}.$$

Proof. Let $i, j \in \mathbb{Z}$. Then

$$d_{i,j} - d_{i-1,j} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i-1, t > j} a_{s,t} = \sum_{t > j} a_{i,t}.$$

Similarly,

$$d_{i,j} - d_{i,j-1} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i, t > j-1} a_{s,t} = -\sum_{s \leq i} a_{s,j}.$$

□

Lemma 5.2.2. *Let $A \in \Lambda_1$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for each $i, j \in \mathbb{Z}$. Then*

$$a_{i,j} = d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j}$$

for each $i, j \in \mathbb{Z}$.

Proof. Using Lemma 5.2.1,

$$\begin{aligned} a_{i,j} &= \sum_{t > j-1} a_{i,t} - \sum_{t > j} a_{i,t} \\ &= (d_{i,j-1} - d_{i-1,j-1}) - (d_{i,j} - d_{i-1,j}). \end{aligned}$$

Alternatively,

$$\begin{aligned} a_{i,j} &= \sum_{s \leq i} a_{s,j} - \sum_{s \leq i-1} a_{s,j} \\ &= -(d_{i,j} - d_{i,j-1}) + (d_{i-1,j} - d_{i-1,j-1}). \end{aligned}$$

□

Lemma 5.2.3. *The relation $\leq$ on Λ_1, defined by $A \leq B$ if and only if $d_{i,j}A \leq d_{i,j}B$ for all $i, j \in \mathbb{Z}$, is a partial order.*

Proof. It is clear that $\leq$ is reflexive and transitive, so it remains to see $\leq$ is antisymmetric. Suppose $A, B \in \Lambda_1$ with $A \leq B$ and $B \leq A$. Then $d_{i,j}A = d_{i,j}B$ for each $i, j \in \mathbb{Z}$, which shows $A = B$ as a result of Lemma 5.2.2. $\square$

The partial order on Λ_1 induces a partial order on the set of G-orbits in $\mathcal{F} \times \mathcal{F}$, such that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $A \leq B$. The following lemma is rephrased from Lemma 3.1.4 and gives some geometric significance to the partial order on Λ_1.

Lemma 5.2.4. *Let $A \in \Lambda_1$ and take $(L, L') \in \mathcal{O}_A$. Then*

$$\dim \left(\frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}A$$

and

$$\dim \left(\frac{L'_j}{L_i \cap L'_j} \right) = \bar{d}_{i,j}A,$$

for each $i, j \in \mathbb{Z}$.

Proof. This is a rephrasing of Lemma 3.1.4. $\square$

5.3 Grassmannians and related varieties

Here we collect a few elementary results on Grassmannians and some related varieties. In this section, let V be an n-dimensional $\mathbf{k}$-vector space and let $0 \leq d \leq n$ be an integer. There is a linear map

$$\phi^{(d)} \colon \Lambda^d(V) \to \operatorname{Hom}(V, \Lambda^{d+1}(V))$$

given by

$$\phi^{(d)}(\alpha)(v) = \alpha \wedge v$$

for $\alpha \in \Lambda^d(V)$ and $v \in V$. The kernel of $\phi^{(d)}(\alpha)$ is the space of divisors of α,

$$D_\alpha = \{v \in V : \alpha \wedge v = 0\}.$$

An element $\alpha \in \Lambda^d(V)$ is said to be totally decomposable if $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_d$, where $\alpha_1, \ldots, \alpha_d \in V$ are linearly independent. The dimension of D_α is at most d and $\dim(D_\alpha) = d$ precisely when α is totally decomposable. Consequently, the rank of $\phi^{(d)}(\alpha)$ is at least $n - d$ and α is totally decomposable if and only if $\operatorname{rank} \phi^{(d)}(\alpha) \leq n - d$, which holds if and only if the $(n-d+1) \times (n-d+1)$-minors of a matrix of $\phi^{(d)}(\alpha)$ are all zero.

Lemma 5.3.1. *$\{(U_1, U_2) \in \operatorname{Gr}_{d_1}(V) \times \operatorname{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a projective variety, for each $d_1, d_2, a \in \mathbb{N}$ with $d_1, d_2, a \leq n$.*

Proof. As above, there is a linear map $\Psi \colon \Lambda^{d_1} V \oplus \Lambda^{d_2} V \to \operatorname{Hom}(V, \Lambda^{d_1+1}(V) \oplus \Lambda^{d_2+1}(V))$ given by $\Psi(\alpha, \beta)(v) = (\alpha \wedge v, \beta \wedge v)$. Given $\alpha \in \Lambda^{d_1}(V)$ and $\beta \in \Lambda^{d_2}(V)$, the kernel of $\Psi(\alpha, \beta)$ is $D_\alpha \cap D_\beta$ and so the rank of $\Psi(\alpha, \beta)$ is $n - \dim(D_\alpha \cap D_\beta)$.

Let $U_i \in \operatorname{Gr}_{d_i}(V)$ and suppose $p_i(U_i) = [\alpha_i]$, where p_i is the Plücker embedding of $\operatorname{Gr}_{d_i}(V)$ in $\mathbb{P}(\Lambda^{d_i}(V))$, so $U_i = D_{\alpha_i} = \ker \phi^{(d_i)}(\alpha)$. Therefore the kernel of $\Psi(\alpha_1, \alpha_2)$ is $U_1 \cap U_2$, so the condition that $\dim(U_1 \cap U_2) \geq a$ is equivalent to the condition that $\Psi(\alpha_1, \alpha_2)$ has rank at most $n - a$. After fixing a basis of V, this condition is given by the vanishing of the $(n - a + 1) \times (n - a + 1)$ minors of the matrix of $\Psi(\alpha_1, \alpha_2)$ with respect to this basis. Therefore $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a closed subset of the product of Grassmannians $\mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V)$, so is a projective variety.

More precisely, the entries of a matrix of $\Psi(\alpha_1, \alpha_2)$ are homogeneous polynomials of degree 1 in the Plücker coordinates on $\mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V)$ since Ψ is linear and so the minors of $\Psi(\alpha_1, \alpha_2)$ are also homogeneous polynomials in the Plücker coordinates. $\square$

Lemma 5.3.2. *Let V be an n-dimensional vector space over $\mathbf{k}$ and let $d_1, d_2, a \in \mathbb{N}$ with $d_1, d_2, a \leq n$. The following hold:*

1. $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) = a\}$ *is a quasiprojective variety;*
2. $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : U_1 \subset U_2\}$ *is a projective variety;*
3. *Given $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a projective variety;*
4. *Given $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) = a\}$ is a quasiprojective variety;*
5. *Given $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : U_1 \subset U_2\}$ is a projective variety;*
6. *Given $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : U_2 \subset U_1\}$ is a projective variety.*

Proof. Let X_i denote the space in statement i of the lemma. To emphasise the dependence of X_i on a, write $X_{i,a}$.

X_1 is a quasiprojective variety since it is equal to the intersection of the projective variety $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ with the open set $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \leq a\}$.

Given $(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V)$, $U_1 \subset U_2$ if and only if $\dim(U_1 \cap U_2) \geq d_1$, so Lemma 5.3.1 shows X_2 is a projective variety.

Let $\pi_i \colon \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) \to \mathrm{Gr}_{d_i}(V)$ be the projection map onto the i-th factor, for $i = 1, 2$. The completeness property of projective varieties ensures that π_i is a closed morphism. Observe that

$$\begin{aligned} X_3 &= \{U_1 \in \mathrm{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\} \\ &= \pi_1(\{(U_1, W) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap W) \geq a\} \cap \pi_2^{-1}\{U_2\}). \end{aligned}$$

The fibre of π_2 over U_2 is closed, so the intersection of the fibre with the variety from Lemma 5.3.1 is closed and then the image of this intersection under π_1 is closed. This shows X_3 is a projective variety.

X_4 is a quasiprojective variety since it is the complement of the subvariety $X_{3,a+1}$ in $X_{3,a}$. Finally, *5-6* follow as special cases of *3* since $X_5 = X_{3,d_1}$ and $X_6 = X_{3,d_2}$. $\square$

5.4 Geometry of affine flag varieties

Given $L \in \mathcal{F}$, $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0$ define

$$\Pi_{N,\lambda}(L) = \{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L_0' \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi_{N,\lambda}^a(L) = \left\{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L_0' \subset \varepsilon^{-N} L_0, \dim\left(\frac{\varepsilon^{-N} L_0}{L_0'}\right) = a\right\}.$$

Lemma 5.4.1. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$ and $\lambda \in \Lambda_0$,*

$$\Pi_{N,\lambda}(L) = \bigcup_{a:0 \le a \le 2Nr} \Pi^a_{N,\lambda}(L).$$

Proof. If $L' \in \Pi_{N,\lambda}(L)$ then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and the $\mathbf{k}[\varepsilon]$-module $\varepsilon^{-N} L_0 / L'_0$ is naturally isomorphic to $(\varepsilon^{-N} L_0/\varepsilon^N L_0)/(L'_0/\varepsilon^N L_0)$, so

$$\dim_{\mathbf{k}} \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right) \le \dim_{\mathbf{k}} \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0} \right) = 2Nr.$$

□

Lemma 5.4.2. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \le a \le 2Nr$, $\Pi^a_{N,\lambda}(L)$ is a projective algebraic variety.*

Proof. Let W be the $\mathbf{k}[\varepsilon]$-module $\varepsilon^{-1-N} L_0/\varepsilon^N L_0$, which has dimension $(2N+1)r$ over $\mathbf{k}$. Let $d_i = 2Nr - a + \lambda_1 + \cdots + \lambda_i$ for each $i = 1, \ldots, n$. The correspondence between submodules of $\varepsilon^{-1-N} L_0$ which contain $\varepsilon^N L_0$ and submodules of $\varepsilon^{-1-N} L_0/\varepsilon^N L_0$ determines a map

$$\rho \colon \Pi^a_{N,\lambda}(L) \to \mathrm{Gr}_{d_1}(W) \times \cdots \times \mathrm{Gr}_{d_n}(W),$$

with $\rho(L') = (L'_1/\varepsilon^N L_0, \ldots, L'_n/\varepsilon^N L_0)$.

Let $\mathcal{X}$ be the space of $(U_1, \ldots, U_n) \in \mathrm{Gr}_{d_1}(W) \times \cdots \times \mathrm{Gr}_{d_n}(W)$ with $U_i \subset U_{i+1}$ for $i = 1, \ldots, n-1$ and $\varepsilon U_n \subset U_1$. Lemma 5.3.2 shows that each of these conditions is closed, so $\mathcal{X}$ is a closed subset of $\mathrm{Gr}_{d_1}(W) \times \cdots \times \mathrm{Gr}_{d_n}(W)$, therefore $\mathcal{X}$ is a projective algebraic variety.

The image of ρ is contained in $\mathcal{X}$ since

$$\varepsilon L'_n/\varepsilon^N L_0 = L'_0/\varepsilon^N L_0 \subset L'_1/\varepsilon^N L_0 \subset \cdots \subset L'_n/\varepsilon^N L_0.$$

Suppose $(U_1, \ldots, U_n) \in \mathcal{X}$. Then U_i is a $\mathbf{k}[\varepsilon]$-module, since $\varepsilon U_i \subset \varepsilon U_n \subset U_1 \subset U_i$, for each $i = 1, \ldots, n$, so U_i lifts uniquely to a $\mathbf{k}[\varepsilon]$-module L'_i with $\varepsilon^N L_0 \subset L'_i \subset \varepsilon^{-1-N} L_0$. Therefore $L'_1, \ldots, L'_n$ are $\mathbf{k}[\varepsilon]$-lattices with $L_i \subset L_{i+1}$ for $i = 1, \ldots, n-1$ and $\varepsilon L'_n \subset L'_1$, with

$$\dim\left(\varepsilon^{-1-N} L_0/L'_n\right) = \dim\left(W/W_n\right) = (2N+1)r - d_n = a$$

and

$$\dim\left(L'_i/L'_{i-1}\right) = \dim\left(W_i/W_{i-1}\right) = d_i - d_{i-1} = \lambda_i,$$

for each $i = 2, \ldots, n$. Therefore there is a unique $L' \in \Pi^a_{N,\lambda}(L)$ such that $\rho(L') = (W_1, \ldots, W_n)$, where L' is given by $L'_{i+cn} = \varepsilon^{-c} L'_i$ for $i = 1, \ldots, n$ and $c \in \mathbb{Z}$. It follows ρ is injective and $\operatorname{im} \rho = \mathcal{X}$, which is a projective variety, so $\Pi^a_{N,\lambda}(L)$ is a projective variety. □

Lemma 5.4.3. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \le a \le 2Nr$, $\Pi^a_{N,\lambda}(L)$ is closed in $\Pi^{a+r}_{N+1,\lambda}(L)$.*

Proof. If $L' \in \Pi^a_{N,\lambda}(L)$, then $\varepsilon^{N+1} L_0 \subset \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0 \subset \varepsilon^{-(N+1)} L_0$ and

$$\dim\left(\frac{\varepsilon^{-(1+n)} L_0}{L'_0}\right) = \dim\left(\frac{L_0}{\varepsilon L_0}\right) + \dim\left(\frac{\varepsilon^{-N} L_0}{L'_0}\right) = r + a,$$

which shows that $\Pi^a_{N,\lambda}(L) \subset \Pi^{a+r}_{N+1,\lambda}(L)$. For $L' \in \Pi^{a+r}_{N+1,\lambda}(L)$, if additionally $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, then

$$\dim\left(\frac{\varepsilon^{-(N+1)}L_0}{L'_0}\right) = r + \dim\left(\frac{\varepsilon^{-N}L_0}{L'_0}\right),$$

which shows $L' \in \Pi^a_{N,\lambda}(L)$. Therefore $\Pi^a_{N,\lambda}(L)$ is the subspace of $\Pi^{a+r}_{N+1,\lambda}(L)$ defined by the two closed conditions $\varepsilon^N L_0 \subset L'_0$ and $L'_0 \subset \varepsilon^{-N} L_0$, using Lemma 5.3.2. □

Lemma 5.4.4. *Let $\lambda \in \Lambda_0$, $M, N \in \mathbb{N}$, $L, \tilde{L} \in \mathcal{F}$, $0 \le a \le 2Nr$, $0 \le b \le 2Mr$. $\Pi^a_{N,\lambda}(L) \cap \Pi^b_{M,\lambda}(\tilde{L})$ is a closed set in $\Pi^a_{N,\lambda}(L)$. In particular, if the intersection is nonempty it is a projective algebraic variety.*

Proof. Observe that $\Pi^a_{N,\lambda}(L) \cap \Pi^b_{M,\lambda}(\tilde{L})$ is the subset of $\Pi^a_{N,\lambda}(L)$ defined by the additional conditions that $\varepsilon^M \tilde{L}_0 \subset L'_0$ and $L'_0 \subset \varepsilon^{-M}\tilde{L}_0$, so is a closed subset of $\Pi^a_{N,\lambda}(L)$, using 5.3.2. □

Lemma 5.4.5. *Suppose $L \in \mathcal{F}$, $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0$ with $a \le 2Nr$. For each $g \in G$, the natural map (restriction of the action map) $\Pi^a_{N,\lambda}(L) \to \Pi^a_{N,\lambda}(gL)$ is an isomorphism of projective varieties.*

Proof. If $L' \in \Pi^a_{N,\lambda}(L)$, then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and so $\varepsilon^N g(L_0) \subset g(L'_0) \subset \varepsilon^{-N} g(L_0)$, so $gL' \in \Pi^a_{N,\lambda}(L)$. Thus g and g^{-1} induce mutually inverse morphisms of varieties $g\colon \Pi^a_{N,\lambda}(L) \to \Pi^a_{N,\lambda}(gL)$ and $g^{-1}\colon \Pi^a_{N,\lambda}(gL) \to \Pi^a_{N,\lambda}(L)$. □

5.4.1 Action through an algebraic group

Given $N \in \mathbb{N}$, define

$$H_N = \left\{ h \in G_L : h = 1 \text{ on } \frac{\varepsilon^{-(1+N)}L_0}{\varepsilon^N L_0} \right\}.$$

Explicitly, the condition $h = 1$ on $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$ means: $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)}L_0$. Observe that $H_{N+1} \subset H_N$ for $N \in \mathbb{N}$ since $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ whenever $x \in \varepsilon^{-(1+N)}L_0$.

EDITORIAL REMARK:

Maybe the cleanest way to write this is to describe the natural group homomorphism $G_L \to \mathrm{GL}(W)$ and state that $H_{N,L}$ is the kernel of this group homomorphism. The next lemma should describe the image and deduce $G_L/H_{N,L}$ is a connected algebraic group, possibly with the last result relegated to a corollary.

Lemma 5.4.6. *Given $L \in \mathcal{F}$ and $N \in \mathbb{N}$, $G_L/H_{N,L}$ is a connected algebraic group.*

Proof. Let W be the $\mathbb{C}[\varepsilon]$-module $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$. ε^{2N+1} acts as zero on W and $\mathbb{C}[\varepsilon]/\langle\varepsilon^{2N+1}\rangle \otimes_{\mathbb{C}[\varepsilon]} W$ is a free $\mathbb{C}[\varepsilon]/\langle\varepsilon^{2N+1}\rangle$-module of rank r. Given $g \in G_{L_0}$, g is a $\mathbb{C}[\varepsilon]$-module automorphism of $\varepsilon^{-(1+N)}L_0$ and $\varepsilon^N L_0$ is a g-invariant submodule, so there is an automorphism $\bar{g}\colon W \to W$ fitting into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varepsilon^N L_0 \arrow[r] \arrow[d] & \varepsilon^{-1-N} L_0 \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varepsilon^N L_0 \arrow[r] & \varepsilon^{-1-N} L_0 \arrow[r] & W \arrow[r] & 0
\end{tikzcd}$$

The natural map $\bar{\ }: G_{L_0} \to \mathrm{GL}(W)$ is a group homomorphism with kernel consisting of those $g \in G_{L_0}$ such that $\bar{g} = 1$: that is, $g(x) \in x + \varepsilon^{2N+1} L_0$ for each $x \in L_0$.

The image of G_{L_0} in $\mathrm{GL}(W)$ may be described by equations in the coordinates on $\mathrm{GL}(W)$ with respect to a $\mathbb{C}$-basis of W. W has a basis $\{x_1, \ldots, x_r\}$ over $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle$, therefore the complex vector space W has a basis $\{y_j : j \in \mathbb{Z}, 1 - 2Nr \leq j \leq r\}$ given by

$$y_{i-cr} = \varepsilon^c x_i$$

for $1 \leq i \leq r$ and $0 \leq c \leq 2N$. There are coordinate functions $\gamma_{i,j} \colon \mathrm{GL}(W) \to \mathbb{C}$ with respect to this basis, given by

$$g(y_j) = \sum_i \gamma_{ij}(g) y_i.$$

If $g \in \mathrm{GL}(W)$ is ε-linear, then $g(y_{i-r}) = g(\varepsilon y_i) = \varepsilon g(y_i)$ and therefore $\gamma_{i-r,j-r}(g) = \gamma_{i,j}(g)$ for all i, j. This shows that the image of G_{L_0} in $\mathrm{GL}(W)$ is the parabolic subgroup consisting of elements of the form

$$\begin{array}{r} A_0 A_1 A_2 \cdots A_{2N} \\ 0 A_0 A_1 \cdots A_{2N-1} \\ \cdots\cdots\cdots \\ 00 \cdots A_0 A_1 \\ 00 \cdots 0 A_0, \end{array}$$

where $A_0 \in \mathrm{GL}_r(\mathbb{C})$ and $A_1, \ldots, A_{2N} \in \mathrm{M}_r(\mathbb{C})$, which is a closed subgroup of $\mathrm{GL}(W)$. The image of G_{L_0} in $\mathrm{GL}(W)$ is identified with the (nonempty) open set $\mathrm{GL}_r(\mathbb{C}) \times M_r(\mathbb{C})^{2N}$ in the affine space $M_r(\mathbb{C})^{2N+1}$, so the image of G_{L_0} is irreducible. This shows that $G_{L_0}/H_{N,L_0}$ is a connected algebraic group.

Moreover, $G_L = G_{L_1} \cap \cdots \cap G_{L_n}$, so the image of G_L in $\mathrm{GL}(W)$ is a closed subgroup. $G_L/H_{N,L}$ is naturally isomorphic to the subgroup of $\mathrm{GL}(W)$ defined by the equations $\gamma_{i-r,j-r} = \gamma_{i,j}$ and for $j = 1, \ldots, r$ the equations $\gamma_{i,j} = 0$ for $i > \lambda_1 + \cdots + \lambda_s$, where s is given by $\lambda_1 + \cdots + \lambda_{s-1} < j \leq \lambda_1 + \cdots + \lambda_s$. Therefore $G_L/H_{N,L}$ is isomorphic to the product $\mathcal{P}_\lambda \times M_r(\mathbb{C}) \times \cdots \times M_r(\mathbb{C})$, where $\mathcal{P}_\lambda$ is a parabolic subgroup of $\mathrm{GL}(W)$, so is irreducible. □

Given $g \in G$, the map $G_L \to G_{gL}$ sending h to ghg^{-1} is a group isomorphism which descends to an isomorphism of algebraic groups $G_L/H_{N,L} \to G_{gL}/H_{N,gL}$. Thus we have a commuting diagram of morphisms of varieties, where the vertical arrows are isomorphisms:

$$\begin{array}{ccc} G_L/H_{N,L} \times \Pi^a_{N,\lambda}(L) & \longrightarrow & \Pi^a_{N,\lambda}(L) \\ \downarrow & & \downarrow \\ G_{gL}/H_{N,gL} \times \Pi^a_{N,\lambda}(gL) & \longrightarrow & \Pi^a_{N,\lambda}(gL) \end{array}$$

5.4.2 Incidence in affine flag varieties

Lemma 5.4.7. *Given $N, a, b, c \in \mathbb{N}$, $\lambda, \mu \in \Lambda_0$, $L \in \mathcal{F}$ and $i, j \in \mathbb{Z}$,*

$$\left\{ (L', L'') \in \Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L) : \dim \left(\frac{L'_i}{L'_i \cap L''_j} \right) \leq c \right\}$$

is a closed set in the projective variety $\Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L)$.

Proof. There is $M \geq N$ so that $\varepsilon^M L_0 \subset L'_i \subset \varepsilon^{-M} L_0$ and $\varepsilon^M L_0 \subset L''_j \subset \varepsilon^{-M} L_0$. Let $a' = a + (M-N)r$ and $b' = b + (M-N)r$. Lemma 5.4.3 shows that $\Pi^a_{N,\lambda}(L)$ is a subvariety of $\Pi^{a'}_{M,\lambda}(L)$, so $\Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L)$ is a subvariety of $\Pi^{a'}_{M,\lambda}(L) \times \Pi^{b'}_{M,\mu}(L)$.

The fact that

$$\dim\left(\frac{L'_i}{L'_i \cap L''_j}\right) = \dim\left(\frac{L'_i/\varepsilon^M L_0}{L'_i/\varepsilon^M L_0 \cap L''_j/\varepsilon^M L_0}\right),$$

together with Lemma 5.4.2 and Lemma 5.3.1, shows that

$$\left\{(L', L'') \in \Pi^{a'}_{M,\lambda}(L) \times \Pi^{b'}_{M,\mu}(L) : \dim\left(\frac{L'_i}{L'_i \cap L''_j}\right) \leq c\right\}$$

is closed, so the intersection with $\Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L)$ is closed. □

Lemma 5.4.8. *Given $N, a, c \in \mathbb{N}$, $\lambda \in \Lambda_0$, $L \in \mathcal{F}$ and $i, j \in \mathbb{Z}$,*

$$\left\{L' \in \Pi^a_{N,\lambda}(L) : \dim\left(\frac{L_i}{L_i \cap L'_j}\right) \leq c\right\}$$

and

$$\left\{L' \in \Pi^a_{N,\lambda}(L) : \dim\left(\frac{L'_j}{L_i \cap L'_j}\right) \leq c\right\}$$

are closed sets in $\Pi^a_{N,\lambda}(L)$.

Proof. This is a result of Lemma 5.3.2, since

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) = \dim\left(\frac{L_i/\varepsilon^M L_0}{L_i/\varepsilon^M L_0 \cap L'_j/\varepsilon^M L_0}\right),$$

where $M \geq N$ is chosen so that $\varepsilon^M L_0 \subset L_i \subset \varepsilon^{-M} L_0$ and $\varepsilon^M L_0 \subset L'_j \subset \varepsilon^{-M} L_0$ for each $L' \in \Pi^a_{N,\lambda}(L)$. □

5.5 Geometry of orbits

Let $A \in \Lambda_1$ and $L \in \mathcal{F}_{\mathrm{ro}(A)}$ and write $\lambda = \mathrm{co}(A)$.

Lemma 5.5.1. *There is $N \in \mathbb{N}$ such that $X^L_A \subset \Pi^a_{N,\lambda}(L)$, where $a = d_{nN,0}A$.*

Proof. There is $N \in \mathbb{N}$ so that $a_{i,j} = 0$ whenever $|j - i| > nN$. If $(L, L') \in \mathcal{O}_A$ then

$$\dim\left(\frac{L'_0}{L'_0 \cap \varepsilon^{-N} L_0}\right) = \dim\left(\frac{L'_0}{L'_0 \cap L_{nN}}\right) = \sum_{s > nN, t \leq 0} a_{s,t} = 0,$$

so it follows $L'_0 \subset \varepsilon^{-N} L_0$. Similarly,

$$\dim\left(\frac{\varepsilon^N L_0}{\varepsilon^N L_0 \cap L'_0}\right) = \dim\left(\frac{L_{-nN}}{L_{-nN} \cap L'_0}\right) = \sum_{s \leq -nN, t > 0} a_{s,t} = 0,$$

which shows $\varepsilon^N L_0 \subset L'_0$. Moreover,

$$\dim\left(\frac{\varepsilon^{-N} L_0}{L'_0}\right) = \dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N} L_0 \cap L'_0}\right) = \sum_{s \leq nN, t > 0} a_{s,t} = d_{nN,0}(A),$$

as a result of Lemma 5.2.4. □

Assume $N \in \mathbb{N}$ is chosen so that $X_A^L \subset \Pi_{N,\lambda}^a(L)$, where $a = d_{nN,0}A$, as in Lemma 5.5.1.

Lemma 5.5.2. *X_A^L is a locally closed subset of $\Pi_{N,\lambda}^a(L)$. In particular, X_A^L is a quasiprojective variety.*

Proof. If $L' \in \Pi_{N,\lambda}^a(L)$ then

$$L_{-Nn} = \varepsilon^N L_0 \subset L_0' \subset L_1' \subset L_n' \subset \varepsilon^{-1-N} L_0 = L_{(N+1)n}.$$

Therefore X_A^L is the subset of $\Pi_{N,\lambda}^a(L)$ defined by the conditions $\dim(L_i/L_i \cap L_j') = d_{i,j}A$ for $i : -Nn \le i < j$ and $\dim(L_j'/L_i \cap L_j') = \bar{d}_{i,j}A$ for $i : j < i \le (N+1)n$, for $j = 1, \dots, n$.

The set of $L' \in \Pi_{N,\lambda}^a(L)$ with $\dim(L_i/L_i \cap L_j') \le d_{i,j}A$ for $j = 1, \dots, n$ and $i : -Nn \le i < j$ and $\dim(L_j'/L_i \cap L_j') \le \bar{d}_{i,j}A$ for $j = 1, \dots, n$ and $i : j < i \le (N+1)n$ is a closed subset of $\Pi_{N,\lambda}^a(L)$, as a result of Lemma 5.4.8.

On the other hand, the set of $L' \in \Pi_{N,\lambda}^a(L)$ satisfying the conditions $\dim(L_i/L_i \cap L_j') \ge d_{i,j}A$ (for $i < j$) and $\dim(L_j'/L_i \cap L_j') \ge \bar{d}_{i,j}A$ (for $i > j$) is open in $\Pi_{N,\lambda}^a(L)$ since the complement is closed, as a result of Lemma 5.4.8.

Therefore X_A^L is the intersection of an open set and a closed set in $\Pi_{N,\lambda}^a(L)$, so X_A^L is locally closed. It follows that X_A^L is an open subset of the projective variety $\overline{X_A^L}$, so is a quasiprojective variety as claimed. □

Lemma 5.5.3. *X_A^L is irreducible.*

Proof. For any $L' \in X_A^L$, $X_A^L = G_L/H_{N,L} \cdot L'$. Lemma 5.4.6 shows that $G_L/H_{N,L}$ is a connected algebraic group which acts algebraically on $\Pi_{N,\lambda}^a(L)$. The image of $G_L/H_{N,L}$ under the morphism $g \mapsto gL'$ equals X_A^L, which shows X_A^L is irreducible since $G_L/H_{N,L}$ is irreducible. □

Consequently, $\overline{X_A^L}$ is an irreducible projective variety and the action of $G_L/H_{N,L}$ on $\Pi_{N,\lambda}^a(L)$ restricts to an algebraic group action on $\overline{X_A^L}$ for which there are finitely many orbits. In particular, $\overline{X_A^L} \setminus X_A^L$ is a union of finitely many orbits which are so-called degenerations of the orbit X_A^L.

5.6 Geometry of orbit products

Let $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$ and write $\lambda = \mathrm{co}(A)$ and $\mu = \mathrm{co}(B)$. Fix $L \in \mathcal{F}_{\mathrm{ro}(A)}$. Recall

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}_\lambda \times \mathcal{F}_\mu : L' \in X_A^L, L'' \in X_B^{L'}\}$$

and

$$X_{A,B}^L = \{L'' \in \mathcal{F}_\mu : \exists L' \in X_A^L \text{ with } L'' \in X_B^{L'}\}$$

Lemma 5.6.1. *There is $N \in \mathbb{N}$ such that*

$$Y_{A,B}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L),$$

where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$.

Proof. There is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L_0' \subset \varepsilon^{-N} L_0$ and $\varepsilon^N L_0' \subset L_0'' \subset \varepsilon^{-N} L_0'$ for each $(L', L'') \in Y_{A,B}^L$, using Lemma 5.5.1. Set $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$.

Then for any $(L', L'') \in Y_{A,B}^L$,

$$\varepsilon^{2N} L_0 \subset \varepsilon^N L_0' \subset L_0'' \subset \varepsilon^{-N} L_0' \subset \varepsilon^{-2N} L_0$$

and

$$\begin{aligned}\dim\left(\frac{\varepsilon^{-2N}L_0}{L_0''}\right) &= \dim\left(\frac{\varepsilon^{-N}L_0'}{L_0''}\right) + \dim\left(\frac{\varepsilon^{-2N}L_0}{\varepsilon^{-N}L_0'}\right)\\ &= \dim\left(\frac{\varepsilon^{-N}L_0'}{L_0''}\right) + \dim\left(\frac{\varepsilon^{-N}L_0}{L_0'}\right)\\ &= a+b,\end{aligned}$$

as a result of Lemma 5.2.4, so $(L', L'') \in \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ as required. □

Now assume $N \in \mathbb{N}$ is chosen so that $Y^L_{A,B} \subset \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$, where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$, using Lemma 5.6.1.

Lemma 5.6.2. *$Y^L_{A,B}$ is a locally closed subset of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$. In particular, $Y^L_{A,B}$ is a quasiprojective variety.*

Proof. $Y^L_{A,B}$ is the subset of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ consisting of those (L', L'') satisfying the following conditions: $\dim(L_i/L_i \cap L'_j) = d_{i,j}(A)$ for $i < j$, $\dim(L'_j/L_i \cap L'_j) = \bar{d}_{i,j}(A)$ for $i > j$, $\dim(L'_i/L'_i \cap L''_j) = d_{i,j}(B)$ for $i < j$ and $\dim(L''_j/L'_i \cap L''_j) = \bar{d}_{i,j}(B)$. Only finitely many conditions are required to define $Y^L_{A,B}$ since there are only finitely many nonzero entries in A and B modulo the (n,n)-periodicity.

The conditions $\dim(L_i/L_i \cap L'_j) \leq d_{i,j}(A)$, $\dim(L'_i/L'_i \cap L''_j) \leq d_{i,j}(B)$, $\dim(L'_j/L_i \cap L'_j) \leq \bar{d}_{i,j}(A)$ and $\dim(L''_j/L'_i \cap L''_j) \leq \bar{d}_{i,j}(B)$ define closed subsets of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ for each $i, j \in \mathbb{Z}$, as a result of Lemma 5.4.7 and Lemma 5.4.8.

On the other hand, the conditions $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}(A)$, $\dim(L'_i/L'_i \cap L''_j) \geq d_{i,j}(B)$, $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}(A)$ and $\dim(L''_j/L'_i \cap L''_j) \geq \bar{d}_{i,j}(B)$ define open subsets of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ for each $i, j \in \mathbb{Z}$, using Lemma 5.4.7 and Lemma 5.4.8.

Therefore $Y^L_{A,B}$ is the intersection of finitely many open sets and finitely many closed sets in $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$, so $Y^L_{A,B}$ is locally closed. In particular, $Y^L_{A,B}$ is a quasiprojective variety. □

Lemma 5.6.3. *For any $L' \in X^L_A$, $Y^L_{A,B} = G_L \cdot (\{L'\} \times X^{L'}_B)$.*

Proof. Let $L' \in X^L_A$, then $\{L'\} \times X^{L'}_B$ is contained in $Y^L_{A,B}$ and G_L acts on $Y^L_{A,B}$, so $G_L \cdot (\{L'\} \times X^{L'}_B)$ is contained in $Y^L_{A,B}$. If $(N', N'') \in Y^L_{A,B}$, then $N' = \sigma L'$ for some $\sigma \in G_L$, since $N' \in X^L_A$. Then $(N', N'') = \sigma(L', \sigma^{-1}N'')$ and $\sigma^{-1}N'' \in X^{\sigma^{-1}N'}_B = X^{L'}_B$, so $(N', N'') \in \sigma \cdot (\{L'\} \times X^{L'}_B)$. Therefore $Y^L_{A,B} = G_L \cdot (\{L'\} \times X^{L'}_B)$ as claimed. □

Proposition 5.6.4. *$Y^L_{A,B}$ is irreducible.*

Proof. Let $L' \in X^L_A$. $G_L/H_{2N,L}$ is a connected algebraic group acting algebraically on $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ by Lemma 5.4.6. $X^{L'}_B$ is an irreducible locally closed subset of $\Pi^{a+b}_{2N,\mu}(L)$, so $\{L'\} \times X^{L'}_B$ is an irreducible locally closed set in $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$. $Y^L_{A,B} = G_L \cdot (\{L'\} \times X^{L'}_B) = G_L/H_{2N,L} \cdot (\{L'\} \times X^{L'}_B)$, by Lemma 5.6.3, so it follows that $Y^L_{A,B}$ is irreducible. □

Let p_2 be the projection onto the second factor $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \to \Pi^{a+b}_{2N,\mu}(L)$. p_2 is a closed morphism since $\Pi^a_{N,\lambda}(L)$ is a projective variety and therefore complete, by Lemma 5.4.2. Therefore $p_2(\overline{Y^L_{A,B}}) = \overline{X^L_{A,B}}$, since $p_2(Y^L_{A,B}) = X^L_{A,B}$.

Lemma 5.6.5. *$X^L_{A,B}$ is irreducible and constructible.*

Proof. Proposition 5.6.4 shows that $Y^L_{A,B}$ is irreducible and locally closed, so it follows $X^L_{A,B}$ is irreducible and constructible, since $X^L_{A,B} = p_2(Y^L_{A,B})$. $\square$

Proposition 5.6.6. *There is a unique open G_L-orbit in $X^L_{A,B}$.*

Proof. $X^L_{A,B}$ consists of finitely many G_L-orbits and is an irreducible topological space, by Lemma 5.6.5. Consequently, X^L_C is dense in $X^L_{A,B}$ for some $C \in \Lambda_1^{A,B}$. Lemma 5.5.2 shows that X^L_C is locally closed in $X^L_{A,B}$, so X^L_C is open in $\overline{X^L_C} = X^L_{A,B}$. Irreducibility of $X^L_{A,B}$ shows that there is a unique open G_L-orbit, since two nonempty open sets in $X^L_{A,B}$ intersect nontrivially, thus any two open G_L orbits in $X^L_{A,B}$ coincide. $\square$

Let $A * B \in \Lambda_1$ be the matrix corresponding to the dense open G_L-orbit in $X^L_{A,B}$, so $\overline{X^L_{A*B}} = \overline{X^L_{A,B}}$.

5.7 Degenerations of orbits and the combinatorial partial order

Lemma 5.7.1. *Given $A, A' \in \Lambda_1$ with $\mathrm{ro}(A) = \mathrm{ro}(A')$ and $\mathrm{co}(A) = \mathrm{co}(A')$, $A' \leq A$ if and only if $X^L_{A'} \subset \overline{X^L_A}$ for any $L \in \mathcal{F}_{\mathrm{ro}(A)}$.*

Proof. Needs a proof. $\square$

Proposition 5.7.2. *Given $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, $\Lambda_1^{A,B}$ has a maximum element.*

Proof. Let $L \in \mathcal{F}_{\mathrm{ro}(A)}$. $X^L_{A,B}$ is irreducible by Lemma 5.6.5 and is the union of finitely many G_L-orbits, namely

$$X^L_{A,B} = \bigcup_{C \in \Lambda_1^{A,B}} X^L_C.$$

This shows that X^L_C is dense in $X^L_{A,B}$ for some $C \in \Lambda_1^{A,B}$. Lemma 5.5.2 shows that the G_L-orbits in $X^L_{A,B}$ are locally closed, so a dense G_L-orbit is open in $X^L_{A,B}$. Lemma 5.7.1 shows that the characteristic matrix of the dense G_L-orbit is a maximum in $\Lambda_1^{A,B}$. $\square$

5.8 Associativity of the generic product

Let $A, B, C \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(B) = \mathrm{ro}(C)$ and fix $L \in \mathcal{F}_{\mathrm{ro}(A)}$. Write $\lambda = \mathrm{co}(A)$, $\mu = \mathrm{co}(B)$ and $\nu = \mathrm{co}(C)$. Define

$$Y^L_{A,B,C} = \left\{ (L', L'', L''') \in \mathcal{F}^3 : L' \in X^L_A, L'' \in X^{L'}_B, L''' \in X^{L''}_C \right\}$$

and

$$X^L_{A,B,C} = \left\{ L''' \in \mathcal{F} : \exists (L', L'') \in \mathcal{F}^2 \text{ with } (L', L'', L''') \in Y^L_{A,B,C} \right\}..$$

Lemma 5.8.1. *There is $N \in \mathbb{N}$ such that $Y^L_{A,B,C}$ is contained in $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi^{a+b+c}_{3N,\nu}(L)$, where $a = d_{nN,0}(A)$, $b = d_{nN,0}(B)$ and $c = d_{nN,0}(C)$.*

Proof. Lemma 5.5.1 shows that there is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$ and $\varepsilon^N L''_0 \subset L'''_0 \subset \varepsilon^{-N} L''_0$ for each $(L', L'', L''') \in Y^L_{A,B,C}$. Using the proof of Lemma 5.6.1, it follows $L'' \in \Pi^{a+b}_{2N,\mu}(L)$ and $L''' \in \Pi^{b+c}_{2N,\nu}(L') \subset \Pi^{a+b+c}_{3N,\nu}(L)$. □

Assume $N \in \mathbb{N}$ is chosen so that $Y^L_{A,B,C} \subset \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi^{a+b+c}_{3N,\nu}(L)$, where $a = d_{nN,0}(A)$, $b = d_{nN,0}(B)$ and $c = d_{nN,0}(C)$, as in Lemma 5.8.1.

Lemma 5.8.2. *$Y^L_{A,B,C}$ is a locally closed subset of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi^{a+b+c}_{3N,\nu}(L)$. In particular, $Y^L_{A,B,C}$ is a quasiprojective variety.*

Proof. Write $\Pi = \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi_{3N,\nu}(L)$. Then $Y^L_{A,B,C}$ consists of those $(L', L'', L''') \in \Pi$ satisfying the following conditions:

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) = d_{i,j}(A), \tag{5.1}$$

$$\dim\left(\frac{L'_i}{L'_i \cap L''_j}\right) = d_{i,j}(B), \tag{5.2}$$

$$\dim\left(\frac{L''_i}{L''_i \cap L'''_j}\right) = d_{i,j}(C), \tag{5.3}$$

for $(i,j) \in [1,n] \times \mathbb{Z}$ with $i < j < (N+1)n$, and

$$\dim\left(\frac{L'_j}{L_i \cap L'_j}\right) = \bar{d}_{i,j}(A), \tag{5.4}$$

$$\dim\left(\frac{L''_j}{L'_i \cap L''_j}\right) = \bar{d}_{i,j}(B), \tag{5.5}$$

$$\dim\left(\frac{L'''_j}{L''_i \cap L'''_j}\right) = \bar{d}_{i,j}(C), \tag{5.6}$$

for $(i,j) \in [1,n] \times \mathbb{Z}$ with $-Nn < j < i$.

For $i < j$, the conditions

$$\dim\left(L_i / L_i \cap L'_j\right) \leq d_{i,j}(A),$$
$$\dim\left(L'_i / L'_i \cap L''_j\right) \leq d_{i,j}(B)$$

and

$$\dim\left(L''_i / L''_i \cap L'''_j\right) \leq d_{i,j}(C)$$

define closed subsets of Π, by Lemma 5.4.7. For $i > j$, the conditions

$$\dim\left(L'_j / L_i \cap L'_j\right) \leq \bar{d}_{i,j}(A),$$
$$\dim\left(L''_j / L'_i \cap L''_j\right) \leq \bar{d}_{i,j}(B)$$

and

$$\dim\left(L'''_j / L''_i \cap L'''_j\right) \leq \bar{d}_{i,j}(C)$$

also define closed subsets of Π.

On the other hand, the conditions $\dim\left(L_i/L_i\cap L'_j\right)\geq d_{i,j}(A)$, $\dim\left(L'_i/L'_i\cap L''_j\right)\geq d_{i,j}(B)$ and $\dim\left(L''_i/L''_i\cap L'''_j\right)\geq d_{i,j}(C)$ for $i<j$ define open subsets of Π. Similarly, the conditions $\dim\left(L'_j/L_i\cap L'_j\right)\geq \bar d_{i,j}(A)$, $\dim\left(L''_j/L'_i\cap L''_j\right)\geq \bar d_{i,j}(B)$ and $\dim\left(L'''_j/L''_i\cap L'''_j\right)\geq \bar d_{i,j}(C)$ for $i>j$ define open subsets of Π.

Therefore $Y^L_{A,B,C}$ is the intersection of finitely many closed sets in Π with finitely many open subsets of Π, so $Y^L_{A,B,C}$ is locally closed. In particular, $Y^L_{A,B,C}$ is a quasiprojective variety. □

Lemma 5.8.3. *For any $(L',L'',L''')\in Y^L_{A,B,C}$, $Y^L_{A,B,C}$ is the image of the product of stabilisers $G_L\times G_{L'}\times G_{L''}$ under the map*

$$(\alpha,\beta,\gamma)\mapsto(\alpha L',\alpha\beta L'',\alpha\beta\gamma L''').$$

In particular,

$$Y^L_{A,B,C}=G_L\cdot(\{L'\}\times Y^{L'}_{B,C}),$$

for any $L'\in X^L_A$.

Proof. Given $\alpha\in G_L$, $\beta\in G_{L'}$ and $\gamma\in G_{L''}$, $(L,\alpha L',\alpha\beta L'',\alpha\beta\gamma L''')\in Y_{A,B,C}$ since $(L,\alpha L')=\alpha(L,L')\in\mathcal{O}_A$, $(\alpha L',\alpha\beta L'')=\alpha\beta(L',L'')\in\mathcal{O}_B$ and $(\alpha\beta L'',\alpha\beta\gamma L''')=\alpha\beta\gamma(L'',L''')\in\mathcal{O}_C$. This shows $G_LG_{L'}G_{L''}L'''\in X^L_{A,B,C}$.

Given $(N',N'',N''')\in Y^L_{A,B,C}$, there exist $\sigma_1,\sigma_2,\sigma_3\in G$ such that $(L,N')=\sigma_1(L,L')$, $(N',N'')=\sigma_2(L',L'')$ and $(N'',N''')=\sigma_3(L'',L''')$; then $N'=\sigma_1L'=\sigma_2L'$, $N''=\sigma_2L''=\sigma_3L''$ and $N'''=\sigma_3L'''$. Thus

$$(L,N',N'',N''')=(L,\sigma_1L',\sigma_1(\sigma_1^{-1}\sigma_2)L'',\sigma_1(\sigma_1^{-1}\sigma_2)(\sigma_2^{-1}\sigma_3)L''')$$

where $\sigma_1\in G_L$, $\sigma_1^{-1}\sigma_2\in G_{L'}$ and $\sigma_2^{-1}\sigma_3\in G_{L''}$. □

Lemma 5.8.4. *$Y^L_{A,B,C}$ is irreducible.*

Proof. There is $N\in\mathbb{N}$ such that $\varepsilon^NL_0\subset L'_0\subset\varepsilon^{-N}L_0$, $\varepsilon^NL'_0\subset L''_0\subset\varepsilon^{-N}L'_0$ and $\varepsilon^NL''_0\subset L'''_0\subset\varepsilon^{-N}L''_0$ for each $(L',L'',L''')\in Y^L_{A,B,C}$. Lemma 5.4.6 shows that $G_L/H_{3N,L}$ is a connected algebraic group acting algebraically on $\Pi=\Pi^a_{N,\mathrm{co}(A)}(L)\times\Pi^{a+b}_{2N,\mathrm{co}(B)}(L)\times\Pi^{a+b+c}_{3N,\mathrm{co}(C)}(L)$.

Let $L'\in X^L_A$. $Y^L_{A,B,C}=G_L\cdot(\{L'\}\times Y^{L'}_{B,C}$. $Y^{L'}_{B,C}$ is an irreducible quasiprojective variety; $\overline{Y^{L'}_{B,C}}$ is an irreducible subvariety of $\Pi^b_{N,\mathrm{co}(B)}(L')\times\Pi^{b+c}_{2N,\mathrm{co}(C)}(L')$, which is a subvariety of $\Pi^{a+b}_{2N,\mathrm{co}(B)}(L)\times\Pi^{a+b+c}_{3N,\mathrm{co}(C)}(L)$. Thus $\{L'\}\times\overline{Y^{L'}_{B,C}}$ is an irreducible subvariety of Π. Therefore $Y^L_{A,B,C}$ is the image of the irreducible space $G_L/H_{3N,L}\times\{L'\}\times Y^{L'}_{B,C}$ under the action map, so $Y^L_{A,B,C}$ is irreducible. Lemma 5.8.2 shows that $Y^L_{A,B,C}$ is quasiprojective, so $Y^L_{A,B,C}$ is an irreducible quasiprojective variety. □

Let p_3 be the projection of $\Pi^a_{N,\lambda}(L)\times\Pi^{a+b}_{2N,\mu}(L)\times\Pi^{a+b+c}_{3N,\nu}(L)$ onto the third factor. By the completeness property of projective varieties, p_3 is a closed morphism. The image of $Y^L_{A,B,C}$ under p_3 is $X^L_{A,B,C}$, so $p_3(\overline{Y^L_{A,B,C}})=\overline{X^L_{A,B,C}}$.

Lemma 5.8.5. *$X^L_{A,B,C}$ is irreducible and constructible.*

Proof. Lemma 5.8.2 and Lemma 5.8.4 show that $Y^L_{A,B,C}$ is irreducible and locally closed. It follows $X^L_{A,B}$ is irreducible and constructible, since $X^L_{A,B,C}$ is the image of $Y^L_{A,B,C}$ under the morphism p_3. □

Lemma 5.8.6. *There is a unique open and dense G_L-orbit in $X^L_{A,B,C}$.*

Proof. There are only finitely many G_L-orbits in $X^L_{A,B,C}$. In particular,

$$X^L_{A,B,C} = \bigcup_{D\in\Lambda_1^{A,B}} X^L_{D,C} = \bigcup_{D\in\Lambda_1^{A,B}} \bigcup_{D'\in\Lambda_1^{D,C}} X^L_{D'}$$

and

$$\overline{X^L_{A,B,C}} = \bigcup_{D\in\Lambda_1^{A,B}} \bigcup_{D'\in\Lambda_1^{D,C}} \overline{X^L_{D'}}.$$

There is $D \in \Lambda_1$ such that $\overline{X^L_D} = \overline{X^L_{A,B,C}}$, since $X^L_{A,B,C}$ is irreducible, by Lemma 5.8.5. By Lemma 5.5.2, X^L_D is open in $\overline{X^L_D} = \overline{X^L_{A,B,C}}$, so X^L_D is open in $X^L_{A,B,C}$.

If X^L_D and $X^L_{D'}$ are open in $X^L_{A,B,C}$, then X^L_D and $X^L_{D'}$ have nonempty intersection since $X^L_{A,B,C}$ is irreducible, then $X^L_D = X^L_{D'}$. □

Lemma 5.8.7. *$p_3^{-1}(X^L_{A*B,C})$ is open in $\overline{Y^L_{A,B,C}}$.*

Proof. Projection onto the second component is a closed morphism of varieties $p_2 \colon \overline{Y^L_{A,B,C}} \to \overline{X^L_{A,B}}$ with $p_2(Y^L_{A,B,C}) = X^L_{A,B}$. It follows that $p_3^{-1}(X^L_{A*B,C})$ is open in $\overline{Y^L_{A,B,C}}$ since $p_3^{-1}(X^L_{A*B,C}) = p_2^{-1}(X^L_{A*B})$ and X^L_{A*B} is open in $\overline{X^L_{A,B}}$. □

Lemma 5.8.8. *$p_3^{-1}(X^L_{A,B*C})$ is open in $\overline{Y^L_{A,B,C}}$.*

Proof. $p_3^{-1}(X^L_{A,B*C})$ consists of those $(L', L'', L''') \in \overline{Y^L_{A,B,C}}$ such that $\dim\left(L'_i/L'_i \cap L'''_j\right) \geq d_{i,j}(B*C)$ for $i < j$ and $\dim\left(L'''_j/L'_i \cap L'''_j\right) \geq \bar{d}_{i,j}(B*C)$ for $i > j$. Each of these conditions defines an open subset of $\overline{Y^L_{A,B,C}}$ as a result of Lemma 5.4.7 and only finitely many conditions are required to determine $p_3^{-1}(X^L_{A,B*C})$, as before. Therefore $p_3^{-1}(X^L_{A,B*C})$ is the intersection of finitely many open sets in $\overline{Y^L_{A,B,C}}$, so is open as claimed. □

Proposition 5.8.9. $X^L_{A*(B*C)} = X^L_{(A*B)*C}$.

Proof. The unique open G_L-orbit in $X^L_{A*B,C}$ is $X^L_{(A*B)*C}$, so $p_3^{-1}(X^L_{(A*B)*C})$ is open in $p_3^{-1}(X^L_{A*B,C})$. Lemma 5.8.7 shows that $p_3^{-1}(X^L_{A*B,C})$ is open in $\overline{Y^L_{A,B,C}}$, so $p_3^{-1}(X^L_{(A*B)*C})$ is open in $\overline{Y^L_{A,B,C}}$.

Similarly, $X^L_{A*(B*C)}$ is open in $X^L_{A,B*C}$, so $p_3^{-1}(X^L_{A*(B*C)})$ is open in $p_3^{-1}(X^L_{A,B*C})$. Lemma 5.8.8 shows that $p_3^{-1}(X^L_{A,B*C})$ is open in $\overline{Y^L_{A,B,C}}$, so it follows $p_3^{-1}(X^L_{A*(B*C)})$ is open in $\overline{Y^L_{A,B,C}}$.

Therefore $f^{-1}(X^L_{A*(B*C)})$ has nonempty intersection with $f^{-1}(X^L_{(A*B)*C})$, since $Y^L_{A,B,C}$ is irreducible by Lemma 5.8.4. It follows that the G_L-orbits $X^L_{A*(B*C)}$ and $X^L_{(A*B)*C}$ have nonempty intersection and therefore $X^L_{A*(B*C)}$ equals $X^L_{(A*B)*C}$. □

5.9 The generic algebra

Lemma 5.9.1. *Given $\lambda \in \Lambda_0$ and $A \in \Lambda_1$, $D_\lambda * A = A$ if $\mathrm{ro}(A) = \lambda$ and $A * D_\lambda = A$ if $\mathrm{co}(A) = \lambda$.*

Proof. Lemma 3.1.7 shows that $\Lambda_1^{D_\lambda,A} = \{A\}$ if $\lambda = \mathrm{ro}(A)$ and $\Lambda_1 A, D_\lambda = \{A\}$ if $\lambda = \mathrm{co}(A)$, which proves the result. □

Theorem 5.9.2. *The following constitutes a small category: the set of objects is Λ_0 and the set of morphisms is Λ_1. Given compositions $\lambda, \mu \in \Lambda_0$, the morphisms with source λ and target μ are those matrices $A \in \Lambda_1$ with $\mathrm{co}(A) = \lambda$ and $\mathrm{ro}(A) = \mu$. Given $\lambda, \mu, \nu \in \Lambda_0$ and $A, B \in \Lambda_1$ with $B\colon \lambda \to \mu$ and $A\colon \mu \to \nu$ the composition is $A * B\colon \lambda \to \nu$.*

Proof. Proposition 5.7.2 shows that the composition is well defined while Proposition 5.8.9 establishes associativity of the composition. Lemma 5.9.1 shows that $D_\lambda\colon \lambda \to \lambda$ is the identity morphism for each $\lambda \in \Lambda_0$. Thus $(\Lambda_0, \Lambda_1, \mathrm{co}(,)\, \mathrm{ro}(,)*)$ is a category. $\square$

Write $\mathcal{G}(n, r)$ to denote this so-called 'generic category'.

Example 1. *The objects in $\mathcal{G}(2, 2)$ are compositions of 2 into 2 parts, namely $(0, 2)$, $(1, 1)$ and $(2, 0)$. The set of morphisms from λ to μ is the set of infinite periodic matrices $A \in \Lambda_1^{2,2}$ with $\mathrm{co}(A) = \lambda$ and $\mathrm{ro}(A) = \mu$, which is a countably infinite set for any pair of compositions $\lambda, \mu \in \Lambda_0[2, 2]$.*

Definition 5.9.1 (Generic algebra)**.** *The generic affine algebra $\hat{G}(n, r)$ is the category $\mathbb{Z}$-algebra of $\mathcal{G}(n, r)$. In particular, $\hat{G}(n, r)$ is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1\}$ and with associative multiplication given by*

$$e_A * e_B = \begin{cases} e_{A*B} & \text{if } \mathrm{co}(A) = \mathrm{ro}(B) \\ 0 & \text{if } \mathrm{co}(A) \neq \mathrm{ro}(B). \end{cases}$$

The multiplicative identity in $\hat{G}(n, r)$ is

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

Chapter 6

A realisation of affine zero Schur algebras

We aim to prove the isomorphism theorem in the cases $r < n$ and $n \leq r < 2n$ separately. Below are crude versions of the statements we want to prove.

Theorem 6.0.1. *Assume $r < n$. The map $\psi\colon \hat{G}(n,r) \to \hat{S}_0(n,r)$, given by $\psi(E_i) = E_i$, $\psi(F_i) = F_i$ and $\psi(1_\lambda) = 1_\lambda$, is an isomorphism of $\mathbb{Z}$-algebras.*

Theorem 6.0.2. *Assume $n \leq r < 2n$. There is a unique homomorphism of $\mathbb{Z}$-algebras $\hat{\psi}\colon \hat{G}(n,r) \to \hat{S}_0(n,r)$ such that $\hat{\psi}(R) = R$ and $\hat{\psi} = \psi$ on the subalgebra of $\hat{G}(n,r)$ generated by the E_i, F_i and 1_λ. $\hat{\psi}$ is an isomorphism of $\mathbb{Z}$-algebras.*

6.1 Preliminary results

Recall from Definition 5.9.1 that the generic algebra $\hat{G}(n,r)$ is an associative $\mathbb{Z}$-algebra which is a free $\mathbb{Z}$-module with an atomic basis $\{e_A : A \in \Lambda_1\}$: given $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, $e_A e_B = e_{A*B}$.

6.1.1 Elementary basis elements

Given $i \in [1,n]$ and $\lambda \in \Lambda_0$ such that $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}.$$

Given $i \in [1,n]$ and $\lambda \in \Lambda_0$ such that $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}.$$

6.1.2 Transpose involution

Lemma 6.1.1. *The $\mathbb{Z}$-module automorphism $\top$ of $\hat{G}(n,r)$ given by $e_A \mapsto e_{A^\top}$ is a $\mathbb{Z}$-algebra antihomomorphism: that is,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A$$

for each $A, B \in \Lambda_1$. Moreover, $\top(E_{i,\lambda}) = F_{i,\lambda+\alpha_i}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}$ and $\top(1_\lambda) = 1_\lambda$, for permissible $(i,\lambda) \in \mathbb{Z} \times \Lambda_0$.

Proof. This is a consequence of Lemma 4.1.1. It must also be shown that the transpose operation on Λ_1 is order preserving. $\square$

6.1.3 Multiplication rules

Lemma 6.1.2. *Given $A \in \Lambda_1$ and $i \in [1,n]$ such that $\mathrm{ro}(A)_{i+1} > 0$,*

$$E_i e_A = e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}},$$

where $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$.

Given $A \in \Lambda_1$ and $i \in [1,n]$ such that $\mathrm{ro}(A)_i > 0$,

$$F_i e_A = e_{A+\mathcal{E}_{i+1,p}-\mathcal{E}_{i,p}},$$

where $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$.

Similar formulas for right multiplication by E_i and F_i are obtained by applying the transpose.

6.2 Presentation of the generic algebra.

Recall that Λ_0 denotes the set of compositions of r into n parts. That is, Λ_0 is the set of tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with each λ_i non-negative and $\lambda_1 + \cdots + \lambda_n = r$. Given $i \in [1,n]$, let $\varepsilon_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^n$ be the i–th elementary vector and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then given $\lambda \in \Lambda_0$, we have $\lambda + \alpha_i \in \Lambda_0$ provided $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ provided $\lambda_i > 0$.

Let $\Gamma = \Gamma(n,r)$ be the quiver with set of vertices Λ_0 with arrows $e_{i,\lambda} \colon \lambda \to \lambda + \alpha_i$ (if $\lambda_{i+1} > 0$) and $f_{i,\lambda} \colon \lambda \to \lambda - \alpha_i$ (if $\lambda_i > 0$). Thus there are no arrows between λ and μ unless $\lambda = \mu \pm \alpha_i$ for some $i \in [1,n]$.

If $n \geq 3$ then neighbouring vertices are connected by two arrows, one of each direction. In the case $n = 2$, neighbouring vertices are joined by four arrows, two of each direction. The $\mathbb{Z}\Gamma$ denote the path $\mathbb{Z}$ algebra of Γ. By construction of Γ, there is a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}\Gamma \to \hat{G}(n,r)$ with $e_{i,\lambda} \mapsto E_{i,\lambda}$, $f_{i,\lambda} \mapsto F_{i,\lambda}$ and $k_\lambda = 1_\lambda$. We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [32]).

Definition 6.2.1. *(aperiodicity) $A \in \Lambda_1$ is aperiodic if for each $l \in \mathbb{Z} \setminus \{0\}$ there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$. If $r < n$ then and $A \in \Lambda_1$ is aperiodic. Linear combinations of the basis elements corresponding to aperiodic matrices are also said to be aperiodic - if A is aperiodic, we say e_A is aperiodic.*

Lemma 6.2.1. *Let $A \in \Lambda_1$ and write $\lambda = \mathrm{ro}(A)$. If A is aperiodic and $\lambda_{i+1} > 0$, then $E_i * e_A$ is aperiodic. If A is aperiodic and $\lambda_i > 0$, then $F_i * e_A$ is aperiodic.*

Proof. Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_{i+1} > 0$, where $\lambda = \mathrm{ro}(A)$. There is $p \in \mathbb{Z}$ such that $a_{i+1,p} > 0$ and $a_{i+1,p'} = 0$ whenever $p' > p$. Lemma 6.1.2 shows that $E_i * e_A = e_B$, where $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p-i, p-i-1\}$, then $b_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $b_{s,s+l} = a_{s,s+l} = 0$, since A is aperiodic. If $l = p - i$, then $b_{i+1,i+1+l} = b_{i+1,p+1} = a_{i+1,p+1} = 0$, by maximality of p. If $l = p - i - 1$, there is $s \neq i+1$ such that $a_{s,s+l} = 0$, since A is aperiodic and $a_{i+1,i+1+l} = a_{i+1,p} > 0$, so $b_{s,s+l} = a_{s,s+l} = 0$. Therefore, $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ is aperiodic.

Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_i > 0$, where $\lambda = \mathrm{ro}(A)$. Lemma 6.1.2 shows that $F_i * e_A = e_C$ where $C = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$ and $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p-i, p-i-1\}$ then $c_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $c_{s,s+p} = a_{s,s+p} = 0$, by aperiodicity of A. If $l = p - i$, then $a_{i,i+l} = a_{i,p} > 0$, so there is $s \neq i$ such that $a_{s,s+l} = 0$. Then $c_{s,s+l} = a_{s,s+l} = 0$. Finally, if $l = p - i - 1$, then $c_{i,i+l} = a_{i,p-1} = 0$ by minimality of p. Thus C is aperiodic as required. ☐

Definition 6.2.2. *(Weight function) Define the weight function* $\mathrm{wt}\colon \Lambda_1 \to \mathbb{Z}$ *by*

$$\mathrm{wt}\, A = \sum_{i \in [1,n], j \in \mathbb{Z}} |j - i| a_{i,j}$$

for each $A \in \Lambda_1$. *The sum is taken over a transversal of the set of congruence classes of* (i,j) *modulo* (n,n) *for* $i,j \in \mathbb{Z}$.

Lemma 6.2.2. *Let* $A \in \Lambda_1$ *and write* $\lambda = \mathrm{ro}(A)$. *Suppose* $\lambda_{i+1} > 0$ *and set* $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$. *If* $p > i$ *then* $\mathrm{wt}\, e_{i,\lambda} * A = 1 + \mathrm{wt}\, A$. *If* $p \leq i$ *then* $\mathrm{wt}\, e_{i,\lambda} * A = -1 + \mathrm{wt}\, A$. *Suppose* $\lambda_i > 0$ *and set* $q = \min\{q' \in \mathbb{Z} : a_{i,q'} > 0\}$. *If* $q \leq i$ *then* $\mathrm{wt}\, f_{i,\lambda} * A = 1 + \mathrm{wt}\, A$. *If* $q > i$ *then* $\mathrm{wt}\, f_{i,\lambda} * A = -1 + \mathrm{wt}\, A$.

Proof. Lemma 6.1.2 shows that $e_i A = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$, so $\mathrm{wt}\, e_i A - \mathrm{wt}\, A = |p - i| - |p - i - 1|$, which equals 1 if $p > i$ and equals -1 if $p \leq i$. Similarly, $f_i A = A + \mathcal{E}_{i+1,q} - \mathcal{E}_{i,q}$ by Lemma 6.1.2, so $\mathrm{wt}\, f_i A - \mathrm{wt}\, A = |q - i - 1| - |q - i|$, which equals -1 if $q > i$ and equals 1 if $q \leq i$. ☐

Lemma 6.2.3. *If* $A \in \Lambda_1$ *is aperiodic, then* e_A *may be obtained from* $1_{\mathrm{co}(A)}$ *by finitely many applications of* E_i *and* F_i *for* $i \in [1,n]$.

Proposition 6.2.4. *The* $\mathbb{Z}$*-subalgebra of* $\hat{G}(n,r)$ *generated by* $E_{i,\lambda}$, $F_{i,\lambda}$ *and* 1_λ *has* $\mathbb{Z}$*-basis* $\{e_A : A \in \Lambda_1$ *is aperiodic.*$\}$.

Proof. ☐

6.2.1 The typical case.

Lemma 6.2.5. *The following relations hold in* $\hat{G}(n,r)$ $(n \geq 3)$*:*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless $|j - i| = 1$.

$$E_i E_{i+1}^2 - E_{i+1} E_i E_{i+1} = 0$$

$$E_i^2 E_{i+1} - E_i E_{i+1} E_i = 0$$

$$F_{i+1} F_i^2 - F_i F_{i+1} F_i = 0$$

$$F_{i+1}^2 F_i - F_{i+1} F_i F_{i+1} = 0$$

$$E_i F_j - F_j E_i = 0$$

unless $j = i$.

$$E_i F i - F_i E_i + \sum_{\lambda:\lambda_i = 0, \lambda_{i+1} > 0} 1_\lambda - \sum_{\lambda:\lambda_i > 0, \lambda_{i+1} = 0} 1_\lambda = 0.$$

6.2.2 Exceptional case.

In this case, the quiver $\Gamma(2, r)$ has vertices $\Lambda_0[2, r] = \{(0, r), (1, r-1), \ldots, (r, 0)\}$; adjacent vertices are connected by two pairs of arrows with opposite orientation: (e_1, f_1) and (e_2, f_2). The relations arising from $\hat{G}(2, r)$ are of a more complicated form - in particular, the serre relations of total degree 3 will not hold in this case - so this case will be treated separately and at a later date.

Chapter 7

Further directions

7.1 Further results on affine zero Schur algebras

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

7.2 Deformed group algebras of symmetric groups

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algebras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for S_3 and S_4. Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: 'these' relations hold; these generators and relations give a presentation of the degenerate group algebras.

Terminology: deformed group algebra.

7.3 back matter

[1] $Y^L_{A,B}$ is the image of $G_L \times G_{L'}$ under the action map $(\alpha, \beta) \mapsto \alpha\beta \cdot (L', L'')$, for any $(L', L'') \in Y^L_{A,B}$. Lemma 5.4.6 shows that $G_L/H_{N,L}$ is a connected algebraic group. Moreover, $G_{L'}/H_{2N,L}$ is an irreducible affine variety, so $G_L/H_{N,L} \times G_{L'}/H_{2N,L}$ is an irreducible affine variety. It follows that $Y^L_{A,B}$ is irreducible and constructible.

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