A geometric realisation of affine 0-Schur algebras.

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Contents

Chapter 1

Introduction

The introduction should contain a summary of the results of the thesis: It should give clear statements of the main results. It is not really clear to me what an introduction should contain. Maybe a summary of the results in the finite case?

Chapter 2

Representations and Hall algebras of cyclic quivers

In this chapter we will review some of the relevant background on quiver representations and establish notation and terminology. The cases of linear quivers of type A and cyclic quivers (type $\tilde{A}$) will be considered in detail. In particular, the aim is to describe the classification of finite dimensional representations of a cyclic quiver over $\mathbb{C}$; or over an 'arbitrary' field. In the case of an arbitrary field, there is a combinatorial characterisation of the isomorphism classes of nilpotent representations, which does not depend on the ground field. This fact is used in defining the Ringel-Hall algebra of a cyclic quiver. (!!) The subalgebra generated by the isoclasses of simple representations, called the composition subalgebra, gives a realisation of the positive part of the quantised enveloping algebra $U_q(\tilde{\mathfrak{gl}}_n)$.

References for this chapter include: [3], [2], [1].

Give the basic definitions and results for an arbitrary connected quiver:

define representations of a quiver, morphisms of quiver representations, simple representations. Define the path algebra of the quiver and give notation for orthogonal idempotents corresponding to the vertices. Emphasise the equivalence between representations of the quiver and modules over the path algebra.

The category of finite dimensional representations of a quiver is Krull-Schmidt, abelian, and so on.

Give Gabriel's theorem and the generalisation due to Kac-(?).

Illustrative example(s): Ringel-Hall algebras for type A_2 and A_3 of different orientations.

2.1 Cyclic quivers

Let n be a positive integer. Let $\Delta = (\Delta_0, \Delta_1)$ denote the (cyclic) quiver with vertex set $\Delta_0 = \mathbb{Z}/n\mathbb{Z}$ and arrows $\Delta_1 = \{i \to i+1 : i \in \Delta_0\}$.

Let $\mathbf{k}$ be a field. A representation $V = (V_i, \rho_i)_{i \in \Delta_0}$ of Δ over $\mathbf{k}$ is a collection of vector spaces $V_i : i \in \Delta_0$ and linear maps $\rho_i \colon V_i \to V_{i+1}$ for $i \in \Delta_0$. A morphism $f \colon V \to W$ is a collection of linear maps $f_i \colon V_i \to W_i$ $(i \in \Delta_0)$ such that $\rho_i^W f_i = f_{i+1} \rho_i^V$ for $i \in \Delta_0$.

The category of finite dimensional representations of Δ over $\mathbf{k}$, denoted $rep_{\mathbf{k}}(\Delta)$, is abelian and Krull-Schmidt. $rep_{\mathbf{k}}(\Delta)$ is equivalent to the category of finite dimensional left modules over the path algebra $\mathbf{k}\Delta$, denoted $mod(\mathbf{k}\Delta)$.

The dimension vector of a finite dimensional representation M is $\underline{\dim}(M) = (\dim(M_1), \ldots, \dim(M_n)) \in \mathbb{Z}^n$. Give consequence of Gabriel's theorem in this case. Discussion of real and imaginary roots.

roughly – real roots ($q(\alpha) = 1$): there is a unique indecomposable with dimension vector α, up to isomorphism; imaginary roots ($q(\alpha) = 0$): a family of isoclasses of indecomposables indexed by $\mathbb{P}^1_k$?.

$M \in mod(\mathbf{k}\Delta)$ is nilpotent if there is $a > 0$ such that $(\mathbf{k}\Delta)^a \cdot M = 0$. The nilpotent modules constitute a full abelian subcategory of $mod(\mathbf{k}\Delta)$, which we denote by $mod^0(\mathbf{k}\Delta)$. The subcategory of nilpotent representations corresponds to the inhomogeneous tube of rank n, with quasi-simples corresponding to the 1-dimensional simple representations at each vertex. In particular, the set of isoclasses in $mod^0(\mathbf{k}\Delta)$ has a combinatorial description which does not depend on the underlying field.

For $i \in \Delta_0$, let S_i denote the simple module $S_i = \mathbf{k}e_i$, where e_i acts as 1 and all other paths act as 0. The indecomposable modules are uniserial and admit a composition series with composition factors (amongst) $S_1, \dots, S_n$ – see this by taking the radical filtration of an indecomposable nilpotent module.

Up to isomorphism, there is a unique module with top S_i and length $l \geq 1$, which we denote by $S_i(l)$. By convention, set $S_i(0) = 0$. Then we have non-split short exact sequences

$$S_{i+1}(l-1) \to S_i(l) \to S_i$$

$$S_{i+l-1} \to S_i(l) \to S_i(l-1),$$

given by embedding of the radical and the quotient by the socle, respectively. In light of these, the convention $S_i(0) = 0$ reflects that simple modules have length 1, so $S_i(1) = S_i$.

Lemma 2.1.1 (extensions of strings)**.** *Let $i_1, i_2 \in \Delta_0$ and $l_1, l_2 \geq 1$. $Ext^1(S_{i_2}(l_2), S_{i_1}(l_1)) = 0$ unless $i_2 = i_1 + j_1$. In the case $i_2 = i_1 + l_1$, we have*

$$\dim Ext^1(S_{i_2}(l_2), S_{i_1}(l_1)) = 1.$$

The class of the non-split extension is given by the short exact sequence

$$S_{i_1+l_1}(l_2) \to S_{i_1}(l_1 + l_2) \to S_{i_1}(l_1).$$

Lemma 2.1.2 (structure theorem)**.** *Any $M \in \mod^0(\mathbf{k}\Delta)$ decomposes uniquely as*

$$M \cong \bigoplus_{i \in \Delta_0; l \geq 1} m_{i,l} S_i(l)$$

2.2 Hall algebras: finite fields

We now define the Hall algebra of $mod(\mathbf{k}\Delta)$, where $\mathbf{k}$ is a finite field with $\mathrm{q} = \#\mathbf{k}$ elements: Let $\mathcal{H}(\mathbf{k}\Delta)$ be a free $\mathbb{Z}$-module with basis $Iso(\mathbf{k}\Delta)$ with a $\mathbb{Z}$-bilinear pairing given by

$$[M][N] = \sum_{[L] \in Iso(\mathbf{k}\Delta)} \phi^{[L]}_{[M],[N]} [L],$$

where

$$\phi^{[L]}_{[M],[N];\mathrm{q}} = \#\{X \leq L : X \cong N, L/X \cong M\}.$$

This is well defined, since the cardinality of the set on the right hand side does not depend on the choice of representatives M, N, L of the isomorphism classes. These cardinalities are finite since L, M, N are finite dimensional and $\mathbf{k}$ is a finite field. With this choice of bilinear pairing, $\mathcal{H}(\mathbf{k}\Delta)$ is a ring with $1 = [0]$, which is known as the Hall algebra of $\mathbf{k}\Delta$. The isomorphism classes of nilpotent representations span a subalgebra of $\mathcal{H}(\mathbf{k}\Delta)$, which may be seen as the Hall algebra of the category $mod^0(\mathbf{k}\Delta)$ of finite dimensional nilpotent $\mathbf{k}\Delta$-modules.

2.3 The Ringel-Hall algebra of a cyclic quiver

In order to define the Ringel-Hall algebra, we first give a combinatorial description of the set of isomorphism classes of finite-dimensional nilpotent $\mathbf{k}\Delta$-modules, where $\mathbf{k}$ is any field. If $M \in mod^0(\mathbf{k}\Delta)$, then

$$M \cong \bigoplus_{i \in \Delta_0, l \geq 1} m_{i,l} S_i(l),$$

for some $m_{i,l} \in \mathbb{N}$. Associate to M the collection $((l_{1,1}, \ldots, l_{1,m_1}), \ldots, (l_{n,1}, \ldots, l_{n,m_n}))$, which may be arranged as an upper triangular tableaux – for now I want to avoid confusion with the matrices indexing orbits in $\mathcal{F} \times \mathcal{F}$.

There exist polynomials $h^L_{M,N} \in \mathbb{Z}[q]$, for each $L, M, N \in Iso(\mathbb{C}\Delta)$, such that

$$h^L_{M,N}(\mathrm{q}) = \phi^{[L]}_{[M],[N];\mathrm{q}}$$

The generic Ringel Hall algebra of Δ is defined as follows: Let $\mathcal{H}(\Delta)$ be a free $\mathbb{Z}[q]$-module with basis $Iso^0(\mathbb{C}\Delta)$, consisting of the isomorphism classes of finite dimensional nilpotent $\mathbb{C}\Delta$-modules, with $\mathbb{Z}[q]$ bilinear pairing given by

$$[M][N] = \sum_{[L] \in Iso(\mathbb{C}\Delta)} h^L_{M,N}[L]$$

Then $\mathcal{H}(\Delta)$ is an associative $\mathbb{Z}[q]$-algebra with multiplicative identity $1 = [0]$. A short note on this construction: Proving existence of these polynomial structure constants is hard, however their uniqueness is clear and associativity of the multiplication follows from associativity of the multiplication in $\mathcal{H}(\mathbf{k}\Delta)$.

If $\mathbf{k}$ is a finite field with q elements, then the specialisation of $\mathcal{H}(\Delta)$ at $q = \mathrm{q}$ is isomorphic to the Hall algebra of nilpotent $\mathbf{k}\Delta$-modules:

$$\mathbb{Z}[q]/(q - \mathrm{q}) \otimes_{\mathbb{Z}[q]} \mathcal{H}(\Delta) \cong \mathcal{H}(mod^0(\mathbf{k}\Delta))$$

The 0-Hall algebra of Δ is the specialisation of $\mathcal{H}(\Delta)$ at $q = 0$:

$$\mathcal{H}_0(\Delta) \coloneqq \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}[q]} \mathcal{H}(\Delta)$$

Chapter 3

The cyclic flags approach to affine q-Schur algebras.

3.1 Cyclic flags

Fix $n, r \in \mathbb{N}$. Let $\mathbf{k}$ be a field and write $\mathcal{R} = \mathbf{k}[\varepsilon]$ and $\mathcal{S} = \mathbf{k}[\varepsilon, \varepsilon^{-1}]$. Let V be a free $\mathcal{S}$-module of rank r and let G denote the automorphism group of V.

A lattice in V is a $\mathcal{R}$-submodule L of V with $\mathcal{S} \otimes_{\mathcal{R}} L = V$. In particular, a lattice is a rank r free $\mathcal{R}$-module with $L \subset V$. The space of cyclic flags in V is

$$\mathcal{F} = \{L = (L_i)_{i\in\mathbb{Z}} : L_i \subset L_{i+1}, \varepsilon L_i = L_{i-n} \text{ for } i \in \mathbb{Z}\}$$

G acts on $\mathcal{F}$ by the natural G-action on V: $(g.L)_i = g(L_i)$ for each $i \in \mathbb{Z}$. There is a bijection between the set of G-orbits in $\mathcal{F}$ and the set of compositions of r into n parts. More precisely, the map

$$\mathcal{F} \to \Lambda(n, r) : L \mapsto |L| = (\dim(L_1/L_0), \dots, \dim(L_n/L_{n-1}))$$

is constant on G-orbits and gives a bijection $\mathcal{F}/G \cong \Lambda(n, r)$, where $\Lambda(n, r) = \{\lambda \in \mathbb{N}^n : \lambda_1 + \cdots + \lambda_n = r\}$ is the set of compositions of r into n parts. Given $\lambda \in \Lambda(n, r)$, the corresponding G-orbit in $\mathcal{F}$ is $\mathcal{F}_\lambda = \{L \in \mathcal{F} : |L| = \lambda\}$.

Similarly, G acts on $\mathcal{F}\times\mathcal{F}$ by $g \cdot (L, L') = (g \cdot L, g \cdot L')$, for $g \in G$ and $(L, L') \in \mathcal{F}\times\mathcal{F}$. The G-orbits in $\mathcal{F}\times\mathcal{F}$ admit a similar combinatorial description. Given $i, j \in \mathbb{Z}$, define a function

$$a_{i,j} : \mathcal{F}\times\mathcal{F} \to \mathbb{Z} : (L, L') \mapsto \dim\left(\frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j}\right).$$

Note that $a_{i,j}$ is constant on G-orbits and $a_{i,j} = a_{i-n,j-n}$. Let $\Xi(n, r)$ be the set of matrices $A = (a_{i,j})_{i,j\mathbb{Z}}$ with entries $a_{i,j} \in \mathbb{N}$ satisfying the following conditions: $a_{i-n,j-n} = a_{i,j}$ for each $i, j \in \mathbb{Z}$; there are finitely many non-zero entries in each row or column; the sum of the entries in any n consecutive rows (or columns) equals r. The map

$$\mathcal{F}\times\mathcal{F} \to \Xi(n, r) : (L, L') \mapsto (a_{i,j}(L, L'))_{i,j\in\mathbb{Z}}$$

is constant on G-orbits and gives a bijection $\mathcal{F}\times\mathcal{F}/G \cong \Xi(n, r)$.

Given $A \in \Xi(n, r)$, define

$$ro(A) = \left(\sum_{j\in\mathbb{Z}} a_{1,j}, \dots, \sum_{j\in\mathbb{Z}} a_{n,j}\right)$$

and

$$co(A) = \left(\sum_{i\in\mathbb{Z}} a_{i,1}, \ldots, \sum_{i\in\mathbb{Z}} a_{i,n} \right).$$

$ro(A)$ and $co(A)$ are compositions of r into n parts, by definition of $\Xi(n,r)$. If $(L,L') \in \mathcal{O}_A$, then $ro(A) = |L|$ and $co(A) = |L'|$.

3.2 Convolution algebras

Let $\mathbf{k}$ be a finite field. Let V be a free $\mathcal{S}$-module of rank r ($\mathcal{S} = \mathbf{k}[\varepsilon, \varepsilon^{-1}]$) and let G denote the automorphism group of V. $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n,r)$ is the space of n-periodic cyclic flags in V; $\mathcal{F}$ is the set of chains $L = (L_i)_{i\in\mathbb{Z}}$ of lattices in V such that $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. Recall that the G-orbits in $\mathcal{F}$ are indexed by compositions of r into n parts and the G-orbits in $\mathcal{F}\times\mathcal{F}$ are indexed by $\Xi(n,r)$: Write $\mathcal{O}_A$ for the G-orbit in $\mathcal{F}\times\mathcal{F}$ corresponding to A.

Let S be the set of G-invariant functions $\mathcal{F}\times\mathcal{F} \to \mathbb{Z}$ with constructible support. Define a product on S as follows: Given $f, g \in S$ define $f * g \colon \mathcal{F}\times\mathcal{F} \to \mathbb{Z}$ by

$$(f * g)(L, L'') = \sum_{L'} f(L,L')g(L',L'')$$

for $(L, L'') \in \mathcal{F}\times\mathcal{F}$. We will see that this operation defines a $\mathbb{Z}$-algebra structure on S.

$f*g$ is well defined as f and g are supported on finitely many G-orbits, so there are only finitely many $L' \in \mathcal{F}$ such that $f(L,L') \neq 0$ and $g(L',L'') \neq 0$. $f * g$ is G-invariant and is supported on finitely many orbits, so $f * g \in S$. The operation $* : S \times S \to S$ is associative: Given $f, g, h \in S$ and $(L, L''') \in \mathcal{F}\times\mathcal{F}$ we have

$$\begin{aligned}((f * g) * h)(L, L''') &= \sum_{L''} (f * g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L,L')g(L',L'')h(L'',L''') \\ &= (f * (g * h))(L, L''').\end{aligned}$$

The multiplicative identity in S is $\iota \in S$ given by $\iota(L, L') = 1$ if $L = L'$ and $\iota(L,L') = 0$ otherwise. Indeed

$$(\iota * f)(L, L'') = \sum_{L'} \iota(L,L')f(L',L'') = f(L,L'')$$

and

$$(f * \iota)(L, L'') = \sum_{L'} f(L,L')\iota(L',L'') = f(L,L'').$$

It follows $(S, *)$ is a ring with 1.

S has a $\mathbb{Z}$-basis consisting of the indicator functions of the G-orbits in $\mathcal{F}\times\mathcal{F}$. Given $A \in \Xi(n,r)$, let $e_A \in S$ be the indicator function of the orbit $\mathcal{O}_A$. Write $\gamma_{A,B,C;\mathrm{q}}$ for the structure constants with respect to this basis, where q is the number of elements of $\mathbf{k}$. Then for any $(L, L'') \in \mathcal{O}_C$ we have

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= (e_A * e_B)(L, L'') \\ &= \sum_{L'} e_A(L,L')e_B(L',L'') \\ &= \#\{L' : (L,L') \in \mathcal{O}_A \text{ and } (L',L'') \in \mathcal{O}_B\}.\end{aligned}$$

3.3 Affine q-Schur algebras

Given $A, B, C \in \Xi(n,r)$, there exists a polynomial $\gamma_{A,B,C} \in \mathbb{Z}[q]$ such that $\gamma_{A,B,C}(\mathrm{q}) = \gamma_{A,B,C;\mathrm{q}}$ for any prime power q, following [4]. The affine q-Schur algebra $\hat{S}_q(n,r)$ is a free $\mathbb{Z}[q]$ module with basis $\{e_A : A \in \Xi(n,r)\}$ and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given $A, B \in \Xi(n,r)$ define

$$X_{A,B} = \{(L, L'') : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

Observation 1. *There are only finitely many G-orbits in $X_{A,B}$.*

Proof. There are $a, b \in \mathbb{Z}$ (depending only on A and B) such that

$$\varepsilon^a L \subset L' \subset \varepsilon^{-a} L$$

and

$$\varepsilon^b L' \subset L'' \subset \varepsilon^{-b} L',$$

whenever $(L, L') \in \mathcal{O}_A$ and $(L', L'') \in \mathcal{O}_B$. In this case,

$$\varepsilon^c L \subset L'' \subset \varepsilon^{-c} L$$

where $c = a + b$. The G-orbits in $X_{A,B}$ are indexed by the finite set of $A \in \Xi(n,r)$ with $a_{i,j} = 0$ whenever $|j - i| > cn$. □

Given $A \in \Xi(n,r)$ and $L \in \mathcal{F}$, define

$$X_A^L = \{L' \in \mathcal{F} : (L, L') \in \mathcal{O}_A\}$$

Write $a = ro(A)$, $a' = co(A)$. Then $X_A^L \subset \mathcal{F}_{a'}$ and X_A^L is non-empty if and only if $L \in \mathcal{F}_a$. Let $A, B \in \Xi(n,r)$ and $(L, L'') \in \mathcal{F} \times \mathcal{F}$. Consider the set

$$X_A^L \cap X_{B^T}^{L''} = \{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}$$

Observe that $X_A^L \cap X_{B^\top}^{L''}$ is non-empty precisely when $(L, L'') \in X_{A,B}$. Let $(L, L'') \in X_{A,B}$ and take $g \in G$. The natural map

$$X_A^L \cap X_{B^\top}^{L''} \to X_A^{gL} \cap X_{B^\top}^{gL''} : L' \mapsto gL'$$

is a bijection. In the case $\mathbf{k}$ is a finite field with $\mathrm{q} = \#\mathbf{k}$ elements, the number of elements in $X_A^L \cap X_{B^\top}^{L''}$ depends only on q, A, B and the orbit of (L, L''). Thus there are integers $\gamma_{A,B,C;\mathrm{q}}$ such that, for any finite field $\mathbf{k}$ with q elements and any $(L, L'') \in \mathcal{O}_C$,

$$\gamma_{A,B,C;\mathrm{q}} = \#X_A^L \cap X_{B^\top}^{L''}.$$

3.4 Relation to the Ringel-Hall algebra of a cyclic quiver.

Let $\mathbf{k}$ be a field. $L \in \mathcal{F}$ determines an infinite dimensional $\mathbf{k}\Delta$-module; corresponding to the representation

$$\begin{array}{ccc} L_1 \hookrightarrow L_2 \cdots \hookrightarrow L_n & \qquad\qquad & \to L_1 \\ x \mapsto x \cdots \mapsto & & x \mapsto \varepsilon x. \end{array}$$

If $(L, L') \in \mathcal{F} \times \mathcal{F}$ with $L' \subset L$, we have a short exact sequence in $Mod(\mathbf{k}\Delta)$:

$$L' \to L \to L/L',$$

where L/L' is a finite dimensional nilpotent $\mathbf{k}\Delta$-module.

Chapter 4

Quivers with relations for affine q-Schur algebras.

4.1 Basic results: TO BE REPLACED WITH A MORE INFORMATIVE NAME.

$(L, L') \in \mathcal{O}_A$ if and only if $(L', L) \in \mathcal{O}_{A^\top}$. In fact, the operation of transposition on $\mathcal{F} \times \mathcal{F}$ (or on $\Xi(n, r)$) induces an anti-automorphism of $\hat{S}_q(n, r)$.

Lemma 4.1.1. *Transposition gives a homomorphism of $\mathbb{Z}[q]$-modules $\top\colon \hat{S}_q(n, r) \to \hat{S}_q(n, r)$ with $\top(e_A) = e_{A^\top}$, $\top \circ \top = 1$ and $\top(e_A e_B) = \top(e_B)\top(e_A)$.*

Proof. Let $A, B, C \in \Xi(n, r)$ and let $\mathbf{k}$ be a finite field with $\mathrm{q} = \#\mathbf{k}$ elements. If $(L, L'') \in \mathcal{O}_C$ then $(L'', L) \in \mathcal{O}_{C^\top}$ and

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}\\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\}\\ &= \gamma_{B^\top,A^\top,C^\top;\mathrm{q}}\end{aligned}$$

It then follows that $\top(e_A e_B) = \top(e_B)\top(e_A)$. The other parts of the statements are clear. □

If $i, j \in \mathbb{Z}$, let $\mathcal{E}_{i,j}$ denote the 'elementary matrix' with entries given by $(\mathcal{E}_{i,j})_{s,t} = 1$, for $s, t \in \mathbb{Z}$, whenever $(i, j) \sim (s, t)$ modulo (n, n) and all other entries are zero.

Given $\lambda \in \Lambda(n, r)$, let $D_\lambda \in \Xi(n, r)$ denote the diagonal matrix with $r(D_\lambda) = c(D_\lambda) = \lambda$. That is,

$$D_\lambda = \lambda_1\mathcal{E}_{1,1} + \cdots + \lambda_n\mathcal{E}_{n,n}$$

For $\lambda \in \Lambda(n, r)$, write $1_\lambda = e_{D_\lambda}$. The 1_λ are pairwise orthogonal idempotents in $\hat{S}_q(n, r)$ with $1 = \sum_{\lambda\in\Lambda(n,r)} 1_\lambda$.

Given $i, j \in \mathbb{Z}$, write $X_{i,j} = \mathcal{E}_{i,j} - \mathcal{E}_{i+1,j}$. By convention, $e_A = 0$ unless $A \in \Xi(n, r)$.

For $i \in [1, n]$ and $\lambda \in \Lambda(n, r)$, write

$$E_{i,\lambda} = e_{D_\lambda + X_{i,i+1}},$$

$$F_{i,\lambda} = e_{D_\lambda - X_{i,i}}.$$

Define

$$E_i = \sum_{\lambda\in\Lambda(n,r)} E_{i,\lambda}$$

$$F_i = \sum_{\lambda \in \Lambda(n,r)} F_{i,\lambda}.$$

Observe that $E_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $F_{i,\lambda} = 0$ unless $\lambda_i > 0$. Also, $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$.

Lemma 4.1.2. *Let $i \in [1, n]$ and $A \in \Xi(n, r)$.*

$$E_i e_A = \sum_{p \in \mathbb{Z}} q^{\sum_{j>p} a_{i,j}} [a_{i,p} + 1] e_{A+X_{i,p}}$$

and

$$F_i e_A = \sum_{p \in \mathbb{Z}} q^{\sum_{j<p} a_{i+1,j}} [a_{i+1,p} + 1] e_{A-X_{i,p}}.$$

Note that these formulas are still valid in the cases $E_i e_A = 0$ and $F_i e_A = 0$. There are similar formulas for right multiplication by E_i and F_i, which can be obtained by applying the transpose involution to the above formulas. The transpose relates the E_i, F_i and 1_λ in the following way: $\top(E_{i,\lambda}) = F_{i,\lambda}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\varepsilon_i+\varepsilon_{i+1}}$ and $\top(1_\lambda) = 1_\lambda$. In particular, $\top(E_i) = F_i$ and $\top(F_i) = E_i$.

Corollary 4.1.3. *Let $j \in [1, n]$ and $A \in \Xi(n, r)$. Then*

$$e_A F_j = \sum_{p \in \mathbb{Z}} q^{\sum_{i>p} a_{i,j}} [a_{p,j} + 1] e_{A+X_{j,p}^\top}$$

and

$$e_A E_j = \sum_{p \in \mathbb{Z}} q^{\sum_{i<p} a_{i,j+1}} [a_{p,j+1} + 1] e_{A-X_{j,p}^\top}$$

Proof.

$$\begin{aligned} e_A F_j &= \top(E_j e_{A^\top}) \\ &= \top(\sum_p q^{\sum_{i>p} a_{i,j}} [a_{p,j} + 1] e_{A^\top + X_{j,p}}) \\ &= \sum_p q^{\sum_{i>p} a_{i,j}} [a_{p,j} + 1] e_{A+X_{j,p}^\top} \end{aligned}$$

$$\begin{aligned} e_A E_j &= \top(F_j e_{A^\top}) \\ &= \top(\sum_p q^{\sum_{i<p} a_{i,j+1}} [a_{p,j+1} + 1] e_{A^\top - X_{j,p}}) \\ &= \sum_p q^{\sum_{i<p} a_{i,j+1}} [a_{p,j+1} + 1] e_{A-X_{j,p}^\top} \end{aligned}$$

$\square$

Note that $E_i^{r+1} = F_i^{r+1} = 0$ while

$$E_i^r = [r]_! e_{r\mathcal{E}_{i,i+1}}$$

and

$$F_i^r = [r]_! e_{r\mathcal{E}_{i+1,i}}.$$

Lemma 4.1.4 (quantum Serre relations: $n \geq 3$)**.** *Suppose $n \geq 3$. The following relations hold in $\hat{S}_q(n,r)$:*

$$E_iE_j - E_jE_i = 0$$
$$F_iF_j - F_jF_i = 0$$

unless $j = i \pm 1$;

$$E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i = 0$$
$$E_i^2E_{i+1} - (1+q)E_iE_{i+1}E_i + qE_{i+1}E_i^2 = 0$$

and

$$F_{i+1}F_i^2 - (1+q)F_iF_{i+1}F_i + qF_i^2F_{i+1} = 0$$
$$F_{i+1}^2F_i - (1+q)F_{i+1}F_iF_{i+1} + qF_iF_{i+1}^2 = 0.$$

Proof. Here we introduce temporary notation for the basis elements: Write $[A] = e_A$.

Take $\lambda \in \Lambda(n,r)$.

$$E_iE_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1}E_iE_{i+1}1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2E_i1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then

$$(E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i)1_\lambda = 0,$$

for each $\lambda \in \Lambda(n,r)$. The relation $E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i = 0$ then follows.

The relations between F_i and F_{i+1} may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping E_i and F_i and reversing the order of multiplication. □

Lemma 4.1.5 (quantum Serre relations: $n = 2$)**.** *In the case $n = 2$, the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

Lemma 4.1.6. $[E_i, F_j] = 0$ *unless $j = i$.*

$$E_iF_i - F_iE_i = \sum_{\lambda \in \Lambda(n,r)} ([\lambda_i] - [\lambda_{i+1}])1_\lambda.$$

For $\lambda \in \Lambda(n,r)$, let $R_\lambda = e_{\lambda_1\mathcal{E}_{0,1}+\cdots+\lambda_n\mathcal{E}_{n-1,n}}$. Write $R = \sum_{\lambda\in\Lambda(n,r)} R_\lambda$. Note $R_\lambda = R1_\lambda$. Given $A \in \Xi(n,r)$ and $m \in \mathbb{Z}$, let $A[m] \in \Xi(n,r)$ be given by $A[m]_{i,j} = a_{i,j+m}$ and let $A^{[m]}$ be given by $A^{[m]}_{i,j} = a_{i+m,j}$ for each $i \in \mathbb{Z}$.

Lemma 4.1.7 (Shifting)**.** *If $A \in \Xi(n,r)$ then*

$$Re_A = e_{A[\pm 1]}$$

and

$$e_AR = e_{A_{[\pm 1]}}.$$

Conjugation by R gives an automorphism ρ of $\hat{S}_q(n,r)$ satisfying $\rho^n = 1$.

4.2 quivers with relations

Denote by $\Lambda(n,r)$ the set of compositions of r into n parts. That is, $\Lambda(n,r)$ is the set of $\alpha \in \mathbb{Z}^n$ with non-negative entries which sum to r. Let $\varepsilon_i \in \mathbb{Z}^n$ be the ith elementary vector and write $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for each $i \in [1,n]$. Then $\lambda + \alpha_i \in \Lambda(n,r)$ if $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda(n,r)$ if $\lambda_i > 0$.

Let $\Gamma = \Gamma(n,r)$ be the quiver with set of vertices $\Lambda(n,r)$, with the following arrows:

For $\lambda \in \Lambda(n,r)$ and $i \in [1,n]$, there is an arrow $e_{i,\lambda} : \lambda \to \lambda + \alpha_i$ if $\lambda_{i+1} > 0$ and there is an arrow $f_{i,\lambda} : \lambda \to \lambda - \alpha_i$ if $\lambda_i > 0$.

Denote by $\mathbb{Z}[q]\Gamma$ the path $\mathbb{Z}[q]$-algebra of Γ. Thus $\mathbb{Z}[q]\Gamma$ is a free $\mathbb{Z}[q]$-module with a basis given by the set of paths in Γ, with multiplication given by the concatenation of paths. If p starts where q ends, the product pq is the path q followed by p. Write $e_{i,\lambda} = 0$ unless $\lambda, \lambda + \alpha_i \in \Lambda(n,r)$ and write $f_{i,\lambda} = 0$ unless $\lambda, \lambda - \alpha_i \in \Lambda(n,r)$.

By construction, there is a homomorphism of $\mathbb{Z}[q]$-algebras

$$\phi\colon \mathbb{Z}[q]\Gamma \to \hat{S}_q(n,r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda, \end{aligned}$$

for $i \in [1,n]$ and $\lambda \in \Lambda(n,r)$.

The image of ϕ is the subalgebra of $\hat{S}_q(n,r)$ generated by E_i, F_i for $i \in [1,n]$ and 1_λ for $\lambda \in \Lambda(n,r)$, since $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$, while $E_i = \sum_\lambda E_{i,\lambda}$ and $F_i = \sum_\lambda F_{i,\lambda}$. In general ϕ is not surjective, so this does not always lead to a presentation of $\hat{S}_q(n,r)$.

4.2.1 Exceptional case $n = 2$.

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the q-Schur algebra.

4.2.2 Typical case $n > 2$.

Suppose $n \geq 3$. Then $\Gamma = \Gamma(n,r)$ has vertex set $\Lambda(n,r)$. TO BE CONTINUED.

Define $e_i, f_i \in \mathbb{Z}[q]\Gamma(n,r)$ by

$$e_i = \sum_{\lambda \in \Lambda(n,r)} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda(n,r)} f_{i,\lambda},$$

with the convention $e_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $f_{i,\lambda} = 0$ unless $\lambda_i > 0$. Let $k_\lambda \in \mathbf{k}\Delta$ denote the constant path at vertex λ. $\{k_\lambda : \lambda \in \Lambda(n,r)\}$ is a set of pairwise orthogonal idempotents in $\mathbb{Z}[q]\Gamma(n,r)$.

Let $I(n,r) \subset \mathbb{Z}[q]\Gamma(n,r)$ be the ideal generated by the expressions

$$e_i e_{i+1}^2 - (1+q) e_{i+1} e_i e_{i+1} + q e_{i+1}^2 e_i$$

$$e_i^2 e_{i+1} - (1+q)e_i e_{i+1} e_i + q e_{i+1} e_i^2$$
$$f_{i+1} f_i^2 - (1+q) f_i f_{i+1} f_i + q f_i^2 f_{i+1}$$
$$f_{i+1}^2 f_i - (1+q) f_{i+1} f_i f_{i+1} + q f_i f_{i+1}^2$$
$$e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda(n,r)} ([\lambda_i] - [\lambda_{i+1}]) k_\lambda$$

Recall that a relation is a $\mathbb{Z}[q]$-linear combination of paths with common start and end vertices. The relations involving paths $\lambda \to \mu$ are given by $1_\mu \mathrm{expr} 1_\lambda$, for each of the above expressions.

Lemma 4.2.1. *There is a homomorphism of $\mathbb{Z}[q]$-algebras*

$$\phi \colon \mathbb{Z}[q]\Gamma(n,r)/I(n,r) \to \hat{S}_q(n,r)$$

given by

$$\phi(e_{i,\lambda}) = E_{i,\lambda}$$
$$\phi(f_{i,\lambda}) = F_{i,\lambda}$$
$$\phi(k_\lambda) = 1_\lambda.$$

Chapter 5

The poset of orbits and a generic multiplication.

5.1 Poset of orbits

First we recall some notation. Let $\mathbf{k}$ be a field and let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n,r)$ be the set cyclic flags of period n in $V = \mathcal{S}^r$. That is, $\mathcal{F}$ is the set of collections $L = (L_i)_{i\in\mathbb{Z}}$, where each L_i is a lattice in V (which means $\mathcal{S} \otimes_{\mathcal{R}} L_i = V$), $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$.

Given $i, j \in \mathbb{Z}$, define functions

$$a_{i,j} \colon \mathcal{F}\times\mathcal{F} \to \mathbb{Z} : (L, L') \mapsto \dim\left(\frac{L_i \cap L'_j}{L_{i-1}\cap L'_j + L_i \cap L'_{j-1}}\right)$$

and

$$d_{i,j} \colon \mathcal{F}\times\mathcal{F} \to \mathbb{Z} : (L, L') \mapsto \dim\left(\frac{L_i}{L_i \cap L'_j}\right).$$

We have seen that the functions $\{a_{i,j} : i, j \in \mathbb{Z}\}$ give a labelling of the orbits in $\mathcal{F}\times\mathcal{F}$ by integer matrices satisfying some combinatorial conditions. In particular, the map

$$\mathcal{F}\times\mathcal{F}/G \to \Xi(n,r) \colon G(L, L') \mapsto (a_{i,j}(L, L'))_{i,j\in\mathbb{Z}}$$

is a bijection. The orbit in $\mathcal{F}\times\mathcal{F}$ corresponding to $A \in \Xi(n,r)$ is denoted $\mathcal{O}_A$.

$$d_{i,j}(L, L') = \sum_{s\leq i, t>j} a_{s,t}(L, L')$$

$$a_{i,j}(L, L') = (d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j})\,(L, L')$$

There is a partial order $\leq$ on $\mathcal{F}\times\mathcal{F}/G$, given by $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $d_{i,j}(A) \leq d_{i,j}(B)$ for all $i, j \in \mathbb{Z}$. Thus $\mathcal{F}\times\mathcal{F}/G$ is regarded as a poset where the ordering is independent of the underlying field. Indeed, we may identify $\mathcal{F}\times\mathcal{F}/G$ with $\Xi(n,r)$ with the compatible partial ordering.

Given $A, B \in \Xi(n,r)$, write $\Xi_{A,B} = \{C \in \Xi(n,r) : g_{A,B,C} \neq 0\}$ where $g_{A,B,C} \in \mathbb{Z}[q]$ is the structure polynomial giving the coefficient of e_C in the product $e_A e_B$ in $\hat{S}_q(n,r)$. The set of G-orbits in $X_{A,B}$ may be identified with $\Xi_{A,B}$, with partial ordering $A' \leq A$ if and only if

$$\sum_{s\leq i, t>j} a'_{s,t} \leq \sum_{s\leq i, t>j} a_{s,t}$$

for all $i, j \in \mathbb{Z}$.

5.2 Preliminary results

Given $A, B \in \Xi(n,r)$, write $Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$.

Lemma 5.2.1. *Assume $N \in \mathbb{N}$ is chosen sufficiently large that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$ whenever $(L, L', L'') \in Y_{A,B}$. Then*

$$\dim\left(\frac{\varepsilon^{-2N} L_0}{L''_0}\right) = d_{nN,0}(A) + d_{nN,0}(B)$$

and

$$\dim\left(\frac{L''_0}{\varepsilon^{2N} L_0}\right) = 4Nr - d_{nN,0}(A) - d_{nN,0}(B)$$

for $(L, L'') \in X_{A,B}$

Proof. Given $(L, L'') \in X_{A,B}$, fix $L \in \mathcal{F}$ such that $(L, L', L'') \in Y_{A,B}$. By choice of N,

$$\varepsilon^{2N} L_0 \subset \varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0 \subset \varepsilon^{-2N} L_0.$$

Then

$$\begin{aligned}
\dim\left(\frac{L''_0}{\varepsilon^{2N} L_0}\right) &= \dim\left(\frac{L''_0}{\varepsilon^{N} L'_0}\right) + \dim\left(\frac{\varepsilon^N L'_0}{\varepsilon^{2N} L_0}\right) \\
&= \dim\left(\frac{L''_0}{\varepsilon^{N} L'_0}\right) + \dim\left(\frac{L'_0}{\varepsilon^{N} L_0}\right) \\
&= d_{nN,0}(A) + d_{nN,0}(B).
\end{aligned}$$

$$\dim\left(\frac{\varepsilon^{-2N} L_0}{L''_0}\right) + \dim\left(\frac{L''_0}{\varepsilon^{2N} L_0}\right) = \dim\left(\frac{\varepsilon^{-2N} L_0}{\varepsilon^{2N} L_0}\right) = 4Nr.$$

Combining these expressions gives

$$\dim\left(\frac{L''_0}{\varepsilon^{2N} L_0}\right) = 4Nr - d_{nN,0}(A) - d_{nN,0}(B).$$

□

Lemma 5.2.2.

$$\Pi = \{L'' \in \mathcal{F}_{co(B)} : \varepsilon^{2N} \subset L''_0 \subset \varepsilon^{-2N} L_0, \dim\left(\frac{\varepsilon^{-2N} L_0}{L''_0}\right) = d_{nN,0}(A) + d_{nN,0}(B)\}$$

is a complex projective variety, closed under the action of G_L.

Proof. □

5.3 Existence of a maximum

Proposition 5.3.1. *Given $A, B \in \Xi(n,r)$ with $co(A) = ro(B)$, $\Xi_{A,B}$ has a maximum element.*

Proof. The assumption $co(A) = ro(B)$ ensures $\Xi_{A,B}$ is non-empty. The partial order on $\Xi_{A,B}$ is given by the partial order on $\Xi(n,r)$; where $A' \leq A$ if and only if $d_{i,j}(A') \leq d_{i,j}(A)$ for all $i, j \in \mathbb{Z}$.

To prove existence of a maximum element in $\Xi_{A,B}$ we will consider the poset of G-orbits in $\mathcal{F} = \mathcal{F}_{\mathbb{C}}(n,r)$ and prove existence of a maximum orbit in $X_{A,B}$ using an open orbits argument. Fix a free $\mathbb{C}[\varepsilon, \varepsilon^{-1}]$-module of rank r and write $G = \mathrm{Aut}(V)$. Then $\mathcal{F} = \mathcal{F}_{\mathbb{C}}(n,r)$ denotes the space of n-cyclic flags of $\mathbb{C}[\varepsilon]$-lattices in V.

Recall $X_{A,B}$ denotes the set of $(L, L'') \in \mathcal{F} \times \mathcal{F}$ where there exists $L' \in \mathcal{F}$ with $(L, L') \in \mathcal{O}_A$ and $(L', L'') \in \mathcal{O}_B$. $X_{A,B}/G$ is identified with $\Xi_{A,B}$ by the map $\mathcal{O}_C \mapsto C$ and $d_{i,j}(C) = \dim(L_i / L_i \cap L'_j)$.

There is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$ whenever $(L, L'') \in X_{A,B}$. Fix $L \in \mathcal{F}_{ro(A)}$ and write

$$X^L_{A,B} = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

With the above choice of N, write

$$\Pi = \{L'' \in \mathcal{F}_{co(B)} : \varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0\}.$$

Π is a (disjoint union of) complex projective varieties (!), closed under the action of G_L. [REF] The closure $\overline{X^L_{A,B}}$ of $X^L_{A,B}$ in Π is irreducible so is a complex projective variety.

Proposition [REF] shows there is a unique G_L-orbit in $X^L_{A,B}$ which is open in $\overline{X^L_{A,B}}$, say $\mathcal{O}^L_C$ for some $C \in \Xi_{A,B}$. It will be shown that C is the maximum element of $\Xi_{A,B}$. Given $i, j \in \mathbb{Z}$, let $m_{i,j}$ denote the maximum of $\{d_{i,j}(C) : C \in \Xi_{A,B}\}$ and define

$$\mathcal{M}_{i,j} = \{L'' \in \overline{X^L_{A,B}} : d_{i,j}(L, L'') = m_{i,j}\}.$$

$\mathcal{M}_{i,j}$ is non-empty by definition of the $m_{i,j}$ and is closed under the action of G_L. $\mathcal{M}_{i,j}$ is open in $\overline{X^L_{A,B}}$ since the function

$$d^L_{i,j} : \Pi \to \mathbb{Z} : L'' \mapsto \dim\left(\frac{L_i}{L_i \cap L''_j}\right)$$

is lower semi-continuous and

$$\mathcal{M}_{i,j} = \overline{X^L_{A,B}} \setminus \{L'' \in \overline{X^L_{A,B}} : d^L_{i,j}(L'') \leq m_{i,j} - 1\}.$$

It follows that $\mathcal{O}^L_C$ and $\mathcal{M}_{i,j}$ intersect non-trivially, since $\overline{X^L_{A,B}}$ is irreducible and therefore $\mathcal{O}^L_C \subset \mathcal{M}_{i,j}$ as both are closed under the action of G_L. This proves C is a maximum element of $\Xi_{A,B}$, since

$$d_{i,j}(C) = d_{i,j}(L, L'') = m_{i,j}$$

for any $L'' \in \mathcal{O}^L_C$.

□

We denote the maximum element of $\Xi_{A,B}$ by $A * B$.

5.4 Associativity

Proposition 5.4.1. *Given $A, B, C \in \Xi(n,r)$ with $co(A) = ro(B)$ and $co(B) = ro(C)$, $(A*B)*C = A*(B*C)$.*

Proof. INCLUDE PROOF. $\square$

5.5 The generic algebra

This now leads to the construction of a so-called generic version of the affine q-Schur algebra. Let $\hat{G}(n,r)$ be a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Xi(n,r)\}$ with $\mathbb{Z}$-bilinear product

$$e_A * e_B = \begin{cases} e_{A*B} \text{ if } co(A) = ro(B) \\ 0 \text{ if } co(A) \neq ro(B). \end{cases}$$

Theorem 5.5.1. *$\hat{G}(n,r)$ is a ring with* 1.

Proof. Given $A, B \in \Xi(n,r)$ with $co(A) = ro(B)$, proposition 5.3.1 shows that there is a maximum element $C \in \Xi(n,r)$ such that $g_{A,B,C} \neq 0$, which we denote by $A*B$. This shows that the product on $\hat{G}(n,r)$ is well-defined. To establish associativity in $\hat{G}(n,r)$ it suffices to show $(e_A * e_B) * e_C = e_A * (e_B * e_C)$ for $A, B, C \in \Xi(n,r)$. If $co(A) \neq ro(B)$ or $co(B) \neq ro(C)$, then $(e_A * e_B) * e_C = 0 = e_A * (e_B * e_C)$. If $co(A) = ro(B)$ and $co(B) = ro(C)$, then proposition 5.4.1 shows that

$$(e_A * e_B) * e_C = e_{(A*B)*C} = e_{A*(B*C)} = e_A * (e_B * e_C).$$

INCLUDE EXISTENCE OF THE MULTIPLICATIVE IDENTITY. $\square$

5.6 Multiplication rules

Write

$$E_i = \sum_{\lambda \in \Lambda(n,r)} E_{i,\lambda}$$

$$F_i = \sum_{\lambda \in \Lambda(n,r)} F_{i,\lambda}.$$

Then $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$.

Lemma 5.6.1. *Let $A \in \Xi(n,r)$, $i \in [1,n]$ and let $\lambda = ro(A)$. The following multiplication rules hold:*

$$E_i e_A = \begin{cases} e_{A+X_{i,p}} & \text{if } \lambda_{i+1} > 0 \\ 0 & \text{if } \lambda_{i+1} = 0; \end{cases}$$

where p is such that $A_{i+1,p} > 0$ and $A_{i+1,j} = 0$ for $j > p$. Also

$$F_i e_A = \begin{cases} e_{A-X_{i,p}} & \text{if } \lambda_i > 0 \\ 0 & \text{if } \lambda_i = 0; \end{cases}$$

where p is such that $A_{i,p} > 0$ and $A_{i,j} = 0$ for $j < p$.

Similar formulas for right multiplication by E_i and F_i are obtained by applying the transpose.

Lemma 5.6.2. *The following relations hold in $\hat{G}(n,r)$ $(n \geq 3)$:*

$$E_iE_j - E_jE_i = 0$$

$$F_iF_j - F_jF_i = 0$$

unless $|j-i| = 1$.

$$E_iE_{i+1}^2 - E_{i+1}E_iE_{i+1} = 0$$
$$E_i^2E_{i+1} - E_iE_{i+1}E_i = 0$$
$$F_{i+1}F_i^2 - F_iF_{i+1}F_i = 0$$
$$F_{i+1}^2F_i - F_{i+1}F_iF_{i+1} = 0$$

$$E_iF_j - F_jE_i = 0$$

unless $j = i$.

$$E_iFi - F_iE_i + \sum_{\lambda:\lambda_i=0,\lambda_{i+1}>0} 1_\lambda - \sum_{\lambda:\lambda_i>0,\lambda_{i+1}=0} 1_\lambda = 0.$$

5.7 — Rough technical results —

Given $A, B \in \Xi(n,r)$ and $L \in \mathcal{F}$, define

$$X_{A,B}^L = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

$X_{A,B}^L \subset \mathcal{F}_b$, where $b = co(B)$. $X_{A,B}^L \neq \emptyset$ if and only if $co(A) = ro(B)$ and $L \in \mathcal{F}_a$, where $a = ro(A)$. Let G_L denote the stabiliser of L in G: $G_L = \{g \in G : gL = L\}$. G_L acts on $X_{A,B}^L$ by restriction of the G-action on $\mathcal{F}\times\mathcal{F}$. The G_L orbits in $X_{A,B}^L$ correspond bijectively to the G orbits in $X_{A,B}$. We seek to prove existence of a unique generic G-orbit in $X_{A,B}$ by considering the degeneration order on G_L orbits in $X_{A,B}^L$.

5.7.1 reducing to the finite setting

There is $N \in \mathbb{N}$ such that

$$\varepsilon^N L_0 \subset L_0' \subset \cdots \subset L_n' \subset \varepsilon^{-N} L_0$$
$$\varepsilon^N L_0' \subset L_0'' \subset \cdots \subset L_n'' \subset \varepsilon^{-N} L_0',$$
$$\varepsilon^N L_0 \subset L_0'' \subset \cdots \subset L_n'' \subset \varepsilon^{-N} L_0,$$

if $(L, L', L'') \in \mathcal{F}\times\mathcal{F}\times\mathcal{F}$ with $(L, L') \in \mathcal{O}_A$ and $(L', L'') \in \mathcal{O}_B$.

Lemma 5.7.1. *If $L'' \in X_{A,B}^L$, then*

$$\dim\left(L_0''/\varepsilon^N L_0\right) = -Nr + d^-_{-Nn,0}(A) + d^-_{-Nn,0}(B).$$

In particular, $\dim(L_0''/\varepsilon^N L_0)$ does not depend on the choice of $L'' \in X_{A,B}$, depending only on A and B. Also,

$$\dim\left(\varepsilon^{-N} L_0/L_n''\right) + \dim\left(L_0''/\varepsilon^N L_0\right) = (2N-1)r.$$

It seems there should be a formula for the top dimension in terms of the $d^+_{\cdot,\cdot}$.

Proof.

$$\varepsilon^{2N}L_0 \subset \varepsilon^N L'_0 \subset L''_0$$
$$\varepsilon^{2N}L_0 \subset \varepsilon^N L_0 \subset L''_0$$

Comparing the dimensions of the quotients above, we get

$$\dim\left(L''_0/\varepsilon^N L_0\right) + \dim\left(L_0/\varepsilon^N L_0\right) = \dim\left(L'_0/\varepsilon^N L_0\right) + \dim\left(L''_0/\varepsilon^N L'_0\right).$$

$\dim(L_0/\varepsilon^N L_0) = Nr$.

$$\dim(L'_0/\varepsilon^N L_0) = \sum_{s>-Nn,t\leq 0} a_{s,t} = d^-_{-Nn,0}(A)$$

and

$$\dim(L''_0/\varepsilon^N L'_0) = \sum_{s>-Nn,t\leq 0} b_{s,t} = d^-_{-Nn,0}(B)$$

For the final claim, computing the dimensions of quotients in the filtration

$$\varepsilon^N L_0 \subset L''_0 \subset L''_n \subset \varepsilon^{-N}L_0$$

gives

$$\dim(\varepsilon^{-N}L_0/L''_n) + r + \dim(L''_0/\varepsilon^N L_0) = 2Nr.$$

□

Define

$$\Pi = \left\{L'' \in \mathcal{F}_{co(B)} : \varepsilon^N L_0 \subset L''_0 \subset \cdots \subset L''_n \subset \varepsilon^{-N}L_0 \text{ and } \dim\left(L''_0/\varepsilon^N L_0\right) = -Nr + d^-_{-Nn,0}(A) + d^-_{-Nn,0}(B)\right\}.$$

Lemma 5.7.2. Π *is a projective algebraic variety, closed under the action of* G_L.

By choice of N, we have $X^L_{A,B} \subset \Pi$.

Write

$$H = \{g \in G_L : g = 1 \text{ on } \varepsilon^{-N}L_0/\varepsilon^N L_0\}.$$

The condition $g = 1$ on $\varepsilon^{-N}L_0/\varepsilon^N L_0$ means $g(x) - x \in \varepsilon^N L_0$ whenever $x \in \varepsilon^{-N}L_0$. It follows that the action of H fixes each lattice $\mathcal{L}$ with $\varepsilon^N L_0 \subset \mathcal{L} \subset \varepsilon^{-N}L_0$.

Lemma 5.7.3. H *is a normal subgroup in* G_L *and* H *acts trivially on* Π.

Proof. If $g, g' \in H$, then $gg' \in G_L$ and for $x \in \varepsilon^{-N}L_0$ we have

$$gg'(x) - x = g(g'(x) - x) + g(x) - x.$$

$g(x) - x$ and $g'(x) - x$ are contained in $\varepsilon^N L_0$ and g stabilises L, so it follows $gg'(x) - x \in \varepsilon^N L_0$ as required.

$g^{-1}(x) - x = g^{-1}(x - g(x)) \in \varepsilon^N L_0$, since $g^{-1} \in G_L$. Thus $g^{-1} \in H^L_N$. It remains to check normality in G_L.

Now suppose $g \in H^L_N$, $h \in G_L$ and take $x \in \varepsilon^{-N}L_0$. $h^{-1}(x)\varepsilon^{-N}L_0$, since $h^{-1} \in G_L$, so $gh^{-1}(x) - h^{-1}(x) \in \varepsilon^{-N}L_0$ then applying h gives that $hgh^{-1} = 1$ on $\varepsilon^{-N}L_0/\varepsilon^N L_0$.

If $h \in H$ and $L'' \in \Pi$, then

$$\varepsilon^N L_0 \subset L''_0 \subset \cdots \subset L''_n \subset \varepsilon^{-N}L_0$$

, so $h(L''_i) = L''_i$ for $i = 0, 1, \ldots, n$. This proves $hL'' = L''$. □

Lemma 5.7.4. *H is a normal subgroup in $G_{L'}$.*

Proof. H fixes any lattice $\mathcal{L}$ with $\varepsilon^N L_0 \subset \mathcal{L} \subset \varepsilon^{-N} L_0$. In particular,

$$\varepsilon^N L_0 \subset L_0' \subset \cdots \subset L_n' \subset \varepsilon^{-N} L_0,$$

so H fixes $L_0', \dots, L_n'$. It follows H is a subgroup in $G_{L'}$, so it remains to prove normality. □

Lemma 5.7.5. *The groups G_L/H and $G_{L'}/H$ are (finite dimensional/ connected/ ...) algebraic groups.*

Proof. $\sigma \in G_L$ naturally induces an automorphism $\bar{\sigma}$ of $\varepsilon^{-N} L_0/\varepsilon^N L_0$, with inverse induced by σ^{-1}. Moreover, the natural map

$$G_L/H \to GL(\varepsilon^{-N} L_0/\varepsilon^N L_0)$$

is a group homomorphism. In fact, this homomorphism is injective: if $\sigma = \tau$ on $\varepsilon^{-N} L_0/\varepsilon^N L_0$, then $\sigma\tau^{-1} = 1$ on $\varepsilon^{-N} L_0/\varepsilon^N L_0$ and so $\sigma H = \tau H$. Thus G_L/H is isomorphic to its image in $GL(\varepsilon^{-N} L_0/\varepsilon^N L_0)$. this image is an algebraic group, then I need to deduce G_L/H is an algebraic group. □

Lemma 5.7.6. *Suppose $(L, L', L''), (N, N', N'') \in \beta^{-1}(\mathcal{O}_A \times \mathcal{O}_B)$. Then there are $\sigma, \tau \in G$, with $\tau \in G_{L'}$, such that $(N, N', N'') = \sigma(L, L', \tau L'')$.*

Proof. There exist $g, g' \in G$ such that $(N, N') = g(L, L')$ and $(N', N'') = g'(L', L'')$. Then $(N, N', N'') = g(L, L', g^{-1}g'L'')$. Taking $\sigma = g$ and $\tau = g^{-1}g'$ gives the required result. □

Proposition 5.7.7. *Suppose $X_{A,B}^L \neq \emptyset$. Then $X_{A,B}^L \subset \mathcal{F}_{co(B)}$ is finite dimensional and irreducible.*

Proof. The map

$$G_L/H \times G_{L'}/H \to \Pi$$

has image $X_{A,B}^L$, so the closure of $X_{A,B}^L$ in Π is irreducible due to some properties of the above groups. □

5.7.2 locally closed orbits

Proposition 5.7.8. *Suppose $X_{A,B}^L \neq \emptyset$. The G_L-orbits in $X_{A,B}^L$ are locally closed.*

Proof. The G_L orbit of $L'' \in X_{A,B}^L$ is the image of the map

$$G_L/H \to \Pi : g \mapsto gL''.$$

Justify why this image must be locally closed. □

Proposition 5.7.9. *Let $A, B \in \Xi(n, r)$, $L \in \mathcal{F}$ and suppose $X_{A,B}^L \neq \emptyset$. There is a unique open G_L-orbit in $X_{A,B}^L$.*

Proof. Write $X = X^L_{A,B}$. X is irreducible and finite dimensional, using Lemma 5.7.7. We have

$$X = \bigcup_C O_C,$$

where the union is taken over the finite set $\{C \in \Xi(n,r) : \mathcal{O}_C \subset X_{A,B}\}$.

A proper, non-empty, closed subset of X has strictly smaller dimension than X, so there is C such that $\overline{O_C} = X$. O_C is locally closed, by Lemma 5.7.8, so it follows that O_C is open in $\overline{O_C} = X$.

Now suppose O_C is an open G_L orbit and let $D \in \Xi(n,r)$. $O_D \subset X \setminus O_C$ and thus $\overline{O_D} \subset X \setminus O_C$. This shows O_D is not open in X and thus the claim is proven. □

5.7.3 Associativity of the generic product

Given $A, B, C \in \Xi(n,r)$ and $L \in \mathcal{F}$ let

$$X^L_{A,B,C} = \{L''' \in \mathcal{F} : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A, (L'', L''') \in \mathcal{O}_C\}.$$

Note that $X^L_{A,B,C} \subset$ is contained in $\mathcal{F}_{co(C)}$ and is non-empty only if $L \in \mathcal{F}_{ro(A)}$, $co(A) = ro(B)$ and $co(B) = ro(C)$. $X^L_{A,B,C}$ consists of finitely many G_L-orbits. Using a similar argument to the existence of generic orbits we show that there is a unique generic orbit in $X^L_{A,B,C}$, which will establish associativity of the generic product. We now suppose $X^L_{A,B,C}$ is non-empty and fix $(L, L', L'', L''') \in \mathcal{F}^4$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L'', L''') \in \mathcal{O}_C$.

Lemma 5.7.10. *$X^L_{A,B,C}$ is the image of the map*

$$\phi \colon G_L \times G_{L'} \times G_{L''} \to \mathcal{F} : (\alpha, \beta, \gamma) \mapsto \alpha\beta\gamma L'''.$$

Lemma 5.7.11. *Unsure with terminology here – treat this with suspicion We may choose N sufficiently large that*

$$H = \{g \in G : g = 1 \text{ on } \varepsilon^{-N} L_0 / \varepsilon^N L_0\}$$

is a normal subgroup in G_L, $G_{L'}$ and $G_{L''}$. Moreover, N may be chosen so that the quotients G_L/H, $G_{L'}/H$ and $G_{L''}/H$ are algebraic groups, irreducible as varieties. In this case $G_L \times G_{L'} \times G_{L''}$ is an irreducible algebraic group.

Proposition 5.7.12. *The closure $\overline{X^L_{A,B,C}}$ of $X^L_{A,B,C}$ in $\mathcal{F}$ is irreducible.*

Proof. assume chosen N sufficiently large and $H = H_N$ so that H is normal in each of the three stabilisers. Then the product of the quotient groups is an irreducible projective variety and thus the image under the orbit map has irreducible closure. □

Proposition 5.7.13. *There is a unique generic G_L-orbit in $X^L_{A,B,C}$.*

Chapter 6

A realisation of affine zero Schur algebras

We aim to prove the isomorphism theorem in the cases $r < n$ and $n \le r < 2n$ separately. Below are crude versions of the statements we want to prove.

Theorem 6.0.1. *Assume $r < n$. The map $\psi\colon \hat{G}(n,r) \to \hat{S}_0(n,r)$, given by $\psi(E_i) = E_i$, $\psi(F_i) = F_i$ and $\psi(1_\lambda) = 1_\lambda$, is an isomorphism of $\mathbb{Z}$-algebras.*

Proof. Below are some of the pieces: [1] The elements E_i, F_i, 1_λ generate $\hat{G}(n,r)$.

Provided $r < n$, any $A \in \Xi(n,r)$ may be obtained from the diagonal matrix D_λ with $\lambda = ro(A)$ by a sequence of transitions $A \mapsto A \pm X_{i,p}$.

[2] Give a complete set of generating relations for $\hat{G}(n,r)$. □

Theorem 6.0.2. *Assume $n \le r < 2n$. There is a unique homomorphism of $\mathbb{Z}$-algebras $\hat{\psi}\colon \hat{G}(n,r) \to \hat{S}_0(n,r)$ such that $\hat{\psi}(R) = R$ and $\hat{\psi} = \psi$ on the subalgebra of $\hat{G}(n,r)$ generated by the E_i, F_i and 1_λ. $\hat{\psi}$ is an isomorphism of $\mathbb{Z}$-algebras.*

6.1 Quivers with relations for the generic algebra.

Recall that $\Lambda(n,r)$ denotes the set of compositions of r into n parts. That is, $\Lambda(n,r)$ is the set of tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with each λ_i non-negative and $\lambda_1 + \cdots + \lambda_n = r$. Given $i \in [1,n]$, let $\varepsilon_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^n$ be the i–th elementary vector and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then given $\lambda \in \Lambda(n,r)$, we have $\lambda + \alpha_i \in \Lambda(n,r)$ provided $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda(n,r)$ provided $\lambda_i > 0$.

Let $\Gamma = \Gamma(n,r)$ be the quiver with set of vertices $\Lambda(n,r)$ with arrows $e_{i,\lambda}\colon \lambda \to \lambda + \alpha_i$ (if $\lambda_{i+1} > 0$) and $f_{i,\lambda}\colon \lambda \to \lambda - \alpha_i$ (if $\lambda_i > 0$). Thus there are no arrows between λ and μ unless $\lambda = \mu \pm \alpha_i$ for some $i \in [1,n]$.

If $n \ge 3$ then neighbouring vertices are connected by two arrows, one of each direction. In the case $n = 2$, neighbouring vertices are joined by four arrows, two of each direction. The $\mathbb{Z}\Gamma$ denote the path $\mathbb{Z}$ algebra of Γ. By construction of Γ, there is a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}\Gamma \to \hat{G}(n,r)$ with $e_{i,\lambda} \mapsto E_{i,\lambda}$, $f_{i,\lambda} \mapsto F_{i,\lambda}$ and $k_\lambda = 1_\lambda$. We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [4]).

$A \in \Xi(n,r)$ is said to be aperiodic if for each $l \in \mathbb{Z} \setminus \{0\}$ there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$. Denote the set of aperiodic elements in $\Xi(n,r)$ by $\Xi(n,r)^{ap}$. Note that $\Xi(n,r)^{ap} = \Xi(n,r)$ if $r < n$.

Proposition 6.1.1. *The subalgebra of $\hat{G}(n,r)$ generated by $E_{i,\lambda}$, $F_{i,\lambda}$ and 1_λ has $\mathbb{Z}$-basis $\{e_A : A \in \Xi(n,r)^{ap}\}$, where $\Xi(n,r)^{ap} \subset \Xi(n,r)$ is the set of aperiodic elements.*

Chapter 7

Further directions

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

Bibliography

[1] Deng. B and J. Du. *Hall algebras of cyclic quivers and q-deformed Fock spaces.* 2015. eprint: `arXiv:1507.03064`.

[2] Andrew Hubery. *Hall polynomials for affine quivers.* 2007. eprint: `arXiv:math/0703178`.

[3] Andrew Hubery. "The composition algebra of an affine quiver". In: *arXiv preprint math/0403206* (2004).

[4] George Lusztig. "Aperiodicity in quantum affine gln". In: *Asian Journal of Mathematics* 3.1 (1999), pp. 147–178.