A geometric realisation of affine 0-Schur algebras.

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Background: The double flag variety approach to q-Schur algebras

2.1 Flag varieties as projective algebraic varieties

Include a discussion of flag varieties in a finite dimensional vector space. Explain: topology of projective space; Plücker embedding of Grassmannian in a projective space; flag varieties as a closed subset in a product of Grassmannians - show that the inclusion of one subspace into another is a closed condition - given by vanishing of some homogenous polynomials which should appear as minors of a matrix.

Chapter 3

The cyclic flags approach to affine q-Schur algebras

Fix natural numbers n and r.

Definition 3.0.1 (compositions). *A composition of r into n parts is an n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ of non-negative integers whose sum equals r. Denote the set of compositions of r into n parts by Λ_0.*

Definition 3.0.2 (infinite periodic matrices). *Let Λ_1 be the set of matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with integer entries $a_{i,j}$ satisfying the following conditions:*

- *$a_{i,j} \geq 0$ for each $i, j \in \mathbb{Z}$;*
- *each row or column has only finitely many non-zero entries;*
- *the sum of the entries in any n consecutive rows or columns equals r;*
- *$a_{i-n,j-n} = a_{i,j}$ for each $i, j \in \mathbb{Z}$.*

These matrices are referred to as infinite periodic matrices.

Definition 3.0.3 (source and target). *Given $A \in \Lambda_1$, let $\operatorname{ro} A$ and $\operatorname{co} A$ be the compositions of r into n parts given by*

$$\operatorname{ro} A = \left(\sum_{j\in\mathbb{Z}} a_{1,j}, \dots, \sum_{j\in\mathbb{Z}} a_{n,j} \right)$$

and

$$\operatorname{co} A = \left(\sum_{i\in\mathbb{Z}} a_{i,1}, \dots, \sum_{i\in\mathbb{Z}} a_{i,n} \right).$$

$A \in \Lambda_1$ is said to go from $\operatorname{co} A$ to $\operatorname{ro} A$.

Definition 3.0.4 (diagonal matrices). *Given $\lambda \in \Lambda_0$, let $D_\lambda \in \Lambda_1$ be the matrix given by $(D_\lambda)_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i \neq j$ and $(D_\lambda)_{i,i} = \lambda_i$ for $i \in \mathbb{Z}$; where the indices are taken modulo n.*

3.1 Cyclic flags

Fix $n, r \in \mathbb{N}$ and let $\mathbf{k}$ be a field. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by ε, so $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free $\mathcal{S}$-module of rank r. Let G be the automorphism group of the $\mathcal{S}$-module V, so G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. A lattice in V is a $\mathcal{R}$-submodule L of V with $\mathcal{S} \otimes_{\mathcal{R}} L = V$. In particular, a lattice is an $\mathcal{R}$-submodule of V which is a free $\mathcal{R}$-module of rank r.

Lemma 3.1.1. *Let L be a lattice in V. $L/\varepsilon L$ is a torsion $\mathcal{R}$-module, where ε acts as zero. $L/\varepsilon L$ is a free $\mathcal{R}/\langle\varepsilon\rangle$-module of rank r; that is, $L/\varepsilon L$ is an r-dimensional $\mathbf{k}$-vector space.*

Proof. L is a free $\mathcal{R}$-module of rank r, with $L \subset V$. Given an $\mathcal{R}$-basis $\{x_1, \ldots, x_r\}$ of L, $\{\varepsilon x_1, \ldots, \varepsilon x_r\}$ is an $\mathcal{R}$-basis of εL. Finally, the cosets $\{x_1 + \varepsilon L, \ldots, x_r + \varepsilon L\}$ give a basis for $L/\varepsilon L$ over $\mathcal{R}/\langle\varepsilon\rangle \cong \mathbf{k}$. $\square$

Let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of collections $(L_i)_{i\in\mathbb{Z}}$ of lattices in V with $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. These collections of lattices in V are referred to as cyclic flags in V.

G acts on $\mathcal{F}$ by $(g \cdot L)_i = g(L_i)$ for each $i \in \mathbb{Z}$, given $g \in G$ and $L \in \mathcal{F}$. The G-orbits in $\mathcal{F}$ are indexed by the set Λ_0 of compositions of r into n parts: the G-orbit in $\mathcal{F}$ corresponding to $\lambda \in \Lambda_0$ is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim\left(\frac{L_i}{L_{i-1}}\right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

Definition 3.1.1. *The periodic characteristic matrix of a pair of cyclic flags $(L, L') \in \mathcal{F}\times\mathcal{F}$ is the matrix $\mathrm{A}(L, L') = (a_{i,j})_{i,j\in\mathbb{Z}}$ with entries*

$$a_{i,j} = \dim_{\mathbf{k}}\left(\frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j}\right)$$

for each $i, j \in \mathbb{Z}$.

The diagonal action of G on $\mathcal{F}\times\mathcal{F}$ has orbits indexed by the set Λ_1 of infinite periodic matrices (see definition 3.0.2). The G-orbit corresponding to $A \in \Lambda_1$ is denoted $\mathcal{O}_A$ and consists of those pairs $(L, L') \in \mathcal{F}\times\mathcal{F}$ with periodic characteristic matrix $\mathrm{A}(L, L')$ equal to A.

Lemma 3.1.2. *(alternative expression for characteristic matrix) Alternatively,*

$$a_{i,j} = \dim_{\mathbf{k}}\left(\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}}\right)$$

for each $i, j \in \mathbb{Z}$.

Proof. Set $U = L_i \cap L'_j$ and $U' = L_{i-1} + L_i \cap L'_{j-1}$. Then $U + U' = L_{i-1} + L_i \cap L'_j$ and $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$. Applying the isomorphism theorems, $U + U'/U'$ is naturally isomorphic to $U/U \cap U'$ as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to $a_{i,j}$. $\square$

Lemma 3.1.3 (transposing characteristic matrix)**.** *Given a pair of flags $(L, L') \in \mathcal{F}^2$, the matrices $\mathrm{A}(L, L')$ and $\mathrm{A}(L', L)$ are related by the transpose. In particular, $\mathrm{A}(L, L')_{i,j} = \mathrm{A}(L', L)_{j,i}$ for each $i, j \in \mathbb{Z}$.*

Proof. By swapping the roles of i and j and swapping L and L' it is clear that $\mathrm{A}(L, L')_{i,j}$ and $\mathrm{A}(L', L)_{j,i}$ are both given by the dimension of the $\mathbf{k}$-vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each $i, j \in \mathbb{Z}$. □

Lemma 3.1.4 (a codimension formula)**.** *Given $(L, L') \in \mathcal{F}^2$ and $i, j \in \mathbb{Z}$,*

$$\dim_{\mathbf{k}} \left(\frac{L_i}{L_i \cap L'_j} \right) = \sum_{s \leq i, t > j} a_{s,t},$$

where $\mathrm{A}(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$.

Proof. COMPLETE THIS PROOF □

Lemma 3.1.5 (nested flags)**.** *Given $(L, L') \in \mathcal{F}^2$, $L' \subset L$ if and only if $\mathrm{A}(L, L')_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i > j$.*

Proof. Suppose $L, L' \in \mathcal{F}$ with $L' \subset L$, meaning $L'_j \subset L_j$ for each $j \in \mathbb{Z}$. Then for $i > j$, $L_i \cap L'_j = L'_j$, $L_{i-1} \cap L'_j = L'_j$ and $L_i \cap L'_{j-1}$, which shows

$$\mathrm{A}(L, L')_{i,j} = \dim_{\mathbf{k}} \left(\frac{L'_j}{L'_{j-1} + L'_j} \right) = 0$$

as required. Conversely, suppose $\mathrm{A}(L, L')$ is upper triangular, meaning $\mathrm{A}(L, L')_{i,j} = 0$ when $i > j$. Using Lemma 3.1.4,

$$\dim_{\mathbf{k}} \left(\frac{L'_i}{L'_i \cap L_i} \right) = \sum_{s > i, t \leq i} a_{s,t} = 0,$$

so $L_i \cap L'_i = L'_i$ and thus $L'_i \subset L_i$ for each $i \in \mathbb{Z}$, as required. □

Corollary 3.1.6 (diagonal orbits)**.** *Given $L, L' \in \mathcal{F}$, $L = L'$ if and only if $\mathrm{A}(L, L')_{i,j} = 0$ whenever $i \neq j$. In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L, L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each $\lambda \in \Lambda_0$.

3.1.1 A product on orbits

Given $A, B \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$, define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also $L \in \mathcal{F}_{\operatorname{ro} A}$, define the L-slices of $Y_{A,B}$ and $X_{A,B}$ respectively as

$$Y^L_{A,B} = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X^L_{A,B} = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

Observation 1. *There are only finitely many G-orbits in $X_{A,B}$.*

Lemma 3.1.7. *Given $A \in \Lambda_1$, $X_{D_\lambda,A} = \mathcal{O}_A$ if $\lambda = \mathrm{ro}\, A$ and $X_{A,D_\lambda} = \mathcal{O}_A$ if $\lambda = \mathrm{co}\, A$.*

Proof. Let $A \in \Lambda_1$ and set $\lambda = \mathrm{ro}\, A$. $Y_{D_\lambda,A}$ is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_{D_\lambda}$, thus $L = L'$ by Corollary 3.1.6, and $(L', L'') \in \mathcal{O}_A$. $X_{D_\lambda,A}$ is the projection of $Y_{D_\lambda,A}$, which equals $\mathcal{O}_A$.

Similarly, if $\lambda = \mathrm{co}\, A$, Y_{A,D_λ} is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_A$ and $L'' = L'$, so X_{A,D_λ} is exactly the orbit $\mathcal{O}_B$. □

3.1.2 Triple products

Given $A, B, C \in \Lambda_1$ with $\mathrm{co}\, A = \mathrm{ro}\, B$ and $\mathrm{co}\, B = \mathrm{ro}\, C$ and $L \in \mathcal{F}_{\mathrm{ro}\, A}$, there are spaces $X_{A,B,C}$, $Y_{A,B,C}$ and their respective L-slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y^L_{A,B,C} = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X^L_{A,B,C} = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

3.2 Convolution algebras

Suppose $\mathbf{k}$ is a finite field and let q denote the number of elements of $\mathbf{k}$. Consider the set S of G-invariant functions $\mathcal{F} \times \mathcal{F} \to \mathbb{Z}$ with constructible support. S is a free $\mathbb{Z}$-module with a basis consisting of the indicator functions of the G-orbits in $\mathcal{F} \times \mathcal{F}$. Define an operation $\star$ on S as follows: for each $f, g \in S$, $f \star g \in S$ is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for $(L, L'') \in \mathcal{F} \times \mathcal{F}$.

$f \star g$ is well defined since the supports of f and g consist of finitely many G-orbits, so there are only finitely many $L' \in \mathcal{F}$ such that $f(L, L')g(L', L'') \neq 0$, given $(L, L'') \in \mathcal{F} \times \mathcal{F}$. $f \star g$ is constant on G-orbits and is supported on finitely many G-orbits, so $f \star g \in S$.

Lemma 3.2.1. *The set S together with the operation $\star$ is an associative $\mathbb{Z}$-algebra with identity element ι given by $\iota(L, L) = 1$ and $\iota(L, L') = 0$ for $L' \neq L$.*

Proof. Given $f, g, h \in S$ and $(L, L''') \in \mathcal{F} \times \mathcal{F}$,

$$\begin{aligned}((f * g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''),\end{aligned}$$

thus $\star$ is associative. ι is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L')\iota(L', L'') = f(L, L''),$$

for each $f \in S$ and $(L, L'') \in \mathcal{F} \times \mathcal{F}$. □

Given $A \in \Lambda_1$, let $e_A \in S$ denote the indicator function of the orbit $\mathcal{O}_A$. S is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1\}$. There exist $\gamma_{A,B,C;\mathrm{q}} \in \mathbb{Z}$ for $A, B, C \in \Lambda_1$ such that

$$e_A \star e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C;\mathrm{q}} e_C$$

for each $A, B \in \Lambda_1$. Then

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L')e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},\end{aligned}$$

for any $(L, L'') \in \mathcal{O}_C$.

3.3 Affine q-Schur algebras

There exist polynomials $\gamma_{A,B,C} \in \mathbb{Z}[q]$ for $A, B, C \in \Lambda_1$ such that $\gamma_{A,B,C}(\mathrm{q}) = \gamma_{A,B,C;\mathrm{q}}$ for any prime power q, following [1, section 4]. The affine q-Schur algebra $\hat{S}_q(n, r)$ (defined in [ADD A REFERENCE]) is a $\mathbb{Z}[q]$-algebra which is a free $\mathbb{Z}[q]$-module with basis $\{e_A : A \in \Lambda_1\}$ and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these 'universal polynomials' $\gamma_{A,B,C} \in \mathbb{Z}[q]$, it follows from Lemma 3.2.1 that $\hat{S}_q(n, r)$ is an associative $\mathbb{Z}[q]$-algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

Chapter 4

Quivers with relations for affine q-Schur algebras

4.1 Basic results and notation

4.1.1 Elementary matrices

For each $i, j \in \mathbb{Z}$, let $\mathcal{E}_{i,j}$ be the $\mathbb{Z}\times\mathbb{Z}$ 'elementary periodic matrix' with entries given by $(\mathcal{E}_{i,j})_{s,t} = 1$ if $(s,t) = (i+cn, j+cn)$ for some $c \in \mathbb{Z}$ and $(\mathcal{E}_{i,j})_{s,t} = 0$ otherwise. Clearly $\mathcal{E}_{i,j} = \mathcal{E}_{i+n,j+n}$ for each $i, j \in \mathbb{Z}$. Recall from Definition 3.0.4 that the diagonal matrix associated to a composition $\lambda \in \Lambda_0$ is

$$D_\lambda = \lambda_1\mathcal{E}_{1,1} + \cdots + \lambda_n\mathcal{E}_{n,n}.$$

$\{e_{D_\lambda} : \lambda \in \Lambda_0\}$ is a set of pairwise orthogonal idempotents in $\hat{S}_q(n,r)$ with $\sum_{\lambda\in\Lambda_0} e_{D_\lambda} = 1$, as a result of Lemma 3.1.7.

Given $i \in [1,n]$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and define

$$E_i = \sum_{\lambda\in\Lambda_0:\lambda_{i+1}>0} E_{i,\lambda}.$$

Given $i \in [1,n]$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and define

$$F_i = \sum_{\lambda\in\Lambda_0:\lambda_i>0} F_{i,\lambda}$$

4.1.2 Transpose involution

Lemma 4.1.1. *Transposition gives a homomorphism of $\mathbb{Z}[q]$-modules $\top\colon \hat{S}_q(n,r) \to \hat{S}_q(n,r)$ with $\top(e_A) = e_{A^\top}$, $\top\circ\top = 1$ and $\top(e_Ae_B) = \top(e_B)\top(e_A)$.*

Proof. Let $A, B, C \in \Lambda_1$ and let $\mathbf{k}$ be a finite field with $\mathrm{q} = \#\mathbf{k}$ elements. If $(L, L'') \in \mathcal{O}_C$ then $(L'', L) \in \mathcal{O}_{C^\top}$ and

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;\mathrm{q}}\end{aligned}$$

It then follows that $\top(e_A e_B) = \top(e_B)\top(e_A)$. □

The transpose relates the E_i, F_i and 1_λ in the following way: $\top(E_{i,\lambda}) = F_{i,\lambda}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\varepsilon_i+\varepsilon_{i+1}}$ and $\top(1_\lambda) = 1_\lambda$. In particular, $\top(E_i) = F_i$ and $\top(F_i) = E_i$.

4.1.3 A multiplication rule

Lemma 4.1.2. *Given $A \in \Lambda_1$ and $i \in [1, n]$ with $\operatorname{ro} A_{i+1} > 0$,*

$$E_i e_A = \sum_{p \in \mathbb{Z}: a_{i+1,p} > 0} q^{\sum_{j>p} a_{i,j}} [[a_{i,p} + 1]] e_{A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}}.$$

Given $A \in \Lambda_1$ and $i \in [1, n]$ with $\operatorname{ro} A_i > 0$,

$$F_i e_A = \sum_{p \in \mathbb{Z}: a_{i,p} > 0} q^{\sum_{j<p} a_{i+1,j}} [[a_{i+1,p} + 1]] e_{A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}}.$$

Note that these formulas are still valid in the cases $E_i e_A = 0$ and $F_i e_A = 0$, provided it is understood that $e_B = 0$ whenever $B \notin \Lambda_1$. There are similar formulas for right multiplication by E_i and F_i, obtained by applying the transpose involution to the above.

Corollary 4.1.3. *Given $A \in \Lambda_1$ and $j \in [1, n]$ with $\operatorname{co} A_{j+1} > 0$,*

$$e_A F_j = \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A + \mathcal{E}_{p,j} - \mathcal{E}_{p,j+1}}.$$

Given $A \in \Lambda_1$ and $j \in [1, n]$ with $\operatorname{co} A_j > 0$,

$$e_A E_j = \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A + \mathcal{E}_{p,j+1} - \mathcal{E}_{p,j}}.$$

Proof.

$$\begin{aligned}e_A F_j &= \top\left(E_j e_{A^\top}\right) \\ &= \top\left(\sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A^\top + \mathcal{E}_{j,p} - \mathcal{E}_{j+1,p}}\right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A + \mathcal{E}_{p,j} - \mathcal{E}_{p,j+1}},\end{aligned}$$

where the second equality comes from Lemma 4.1.2. Similarly,

$$\begin{aligned}e_A E_j &= \top\left(F_j e_{A^\top}\right) \\ &= \top\left(\sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A^\top + \mathcal{E}_{j+1,p} - \mathcal{E}_{j,p}}\right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A + \mathcal{E}_{p,j+1} - \mathcal{E}_{p,j}}.\end{aligned}$$

$\square$

4.2 Relations

Note that $E_i^{r+1} = F_i^{r+1} = 0$ while
$$E_i^r = [r]_! e_{r\mathcal{E}_{i,i+1}}$$
and
$$F_i^r = [r]_! e_{r\mathcal{E}_{i+1,i}}.$$

Lemma 4.2.1 (quantum Serre relations: $n \geq 3$)**.** *Suppose $n \geq 3$. The following relations hold in $\hat{S}_q(n,r)$:*
$$E_iE_j - E_jE_i = 0$$
$$F_iF_j - F_jF_i = 0$$
unless $j = i \pm 1$;
$$E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i = 0$$
$$E_i^2E_{i+1} - (1+q)E_iE_{i+1}E_i + qE_{i+1}E_i^2 = 0$$
and
$$F_{i+1}F_i^2 - (1+q)F_iF_{i+1}F_i + qF_i^2F_{i+1} = 0$$
$$F_{i+1}^2F_i - (1+q)F_{i+1}F_iF_{i+1} + qF_iF_{i+1}^2 = 0.$$

Proof. Here we introduce temporary notation for the basis elements: Write $[A] = e_A$.

Take $\lambda \in \Lambda_0$.
$$E_iE_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1}E_iE_{i+1}1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2E_i1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then
$$(E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i)1_\lambda = 0,$$
for each $\lambda \in \Lambda_0$. The relation $E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i = 0$ then follows.

The relations between F_i and F_{i+1} may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping E_i and F_i and reversing the order of multiplication. $\square$

Lemma 4.2.2 (quantum Serre relations: $n = 2$)**.** *In the case $n = 2$, the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

Lemma 4.2.3. $[E_i, F_j] = 0$ *unless* $j = i$.
$$E_iF_i - F_iE_i = \sum_{\lambda\in\Lambda_0} ([\lambda_i] - [\lambda_{i+1}])1_\lambda.$$

For $\lambda \in \Lambda_0$, let $R_\lambda = e_{\lambda_1 \mathcal{E}_{0,1}+\cdots+\lambda_n \mathcal{E}_{n-1,n}}$. Write $R = \sum_{\lambda \in \Lambda_0} R_\lambda$. Note $R_\lambda = R1_\lambda$. Given $A \in \Lambda_1$ and $m \in \mathbb{Z}$, let $A[m] \in \Lambda_1$ be given by $A[m]_{i,j} = a_{i,j+m}$ and let $A^{[m]}$ be given by $A^{[m]}_{i,j} = a_{i+m,j}$ for each $i \in \mathbb{Z}$.

Lemma 4.2.4 (Shifting). *If $A \in \Lambda_1$ then*

$$Re_A = e_{A[\pm 1]}$$

and

$$e_A R = e_{A_{[\pm 1]}}.$$

Conjugation by R gives an automorphism ρ of $\hat{S}_q(n,r)$ satisfying $\rho^n = 1$.

4.3 quivers with relations

Denote by Λ_0 the set of compositions of r into n parts. That is, Λ_0 is the set of $\alpha \in \mathbb{Z}^n$ with non-negative entries which sum to r. Let $\varepsilon_i \in \mathbb{Z}^n$ be the ith elementary vector and write $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for each $i \in [1,n]$. Then $\lambda + \alpha_i \in \Lambda_0$ if $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ if $\lambda_i > 0$.

Let $\Gamma = \Gamma(n,r)$ be the quiver with set of vertices Λ_0, with the following arrows:

For $\lambda \in \Lambda_0$ and $i \in [1,n]$, there is an arrow $e_{i,\lambda} : \lambda \to \lambda + \alpha_i$ if $\lambda_{i+1} > 0$ and there is an arrow $f_{i,\lambda} : \lambda \to \lambda - \alpha_i$ if $\lambda_i > 0$.

Denote by $\mathbb{Z}[q]\Gamma$ the path $\mathbb{Z}[q]$-algebra of Γ. Thus $\mathbb{Z}[q]\Gamma$ is a free $\mathbb{Z}[q]$-module with a basis given by the set of paths in Γ, with multiplication given by the concatenation of paths. If p starts where q ends, the product pq is the path q followed by p. Write $e_{i,\lambda} = 0$ unless $\lambda, \lambda + \alpha_i \in \Lambda_0$ and write $f_{i,\lambda} = 0$ unless $\lambda, \lambda - \alpha_i \in \Lambda_0$.

By construction, there is a homomorphism of $\mathbb{Z}[q]$-algebras

$$\phi\colon \mathbb{Z}[q]\Gamma \to \hat{S}_q(n,r)$$

given by

$$\phi(e_{i,\lambda}) = E_{i,\lambda}$$
$$\phi(f_{i,\lambda}) = F_{i,\lambda}$$
$$\phi(k_\lambda) = 1_\lambda,$$

for $i \in [1,n]$ and $\lambda \in \Lambda_0$.

The image of ϕ is the subalgebra of $\hat{S}_q(n,r)$ generated by E_i, F_i for $i \in [1,n]$ and 1_λ for $\lambda \in \Lambda_0$, since $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$, while $E_i = \sum_\lambda E_{i,\lambda}$ and $F_i = \sum_\lambda F_{i,\lambda}$. In general ϕ is not surjective, so this does not always lead to a presentation of $\hat{S}_q(n,r)$.

4.3.1 Exceptional case $n = 2$.

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the q-Schur algebra.

4.3.2 Typical case $n > 2$.

Suppose $n \geq 3$. Then $\Gamma = \Gamma(n,r)$ has vertex set Λ_0. RESUME HERE...

Define $e_i, f_i \in \mathbb{Z}[q]\Gamma(n,r)$ by

$$e_i = \sum_{\lambda \in \Lambda_0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0} f_{i,\lambda},$$

with the convention $e_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $f_{i,\lambda} = 0$ unless $\lambda_i > 0$. Let k_λ denote the constant path at vertex λ. $\{k_\lambda : \lambda \in \Lambda_0\}$ is a set of pairwise orthogonal idempotents in $\mathbb{Z}[q]\Gamma(n,r)$.

Let $I(n,r) \subset \mathbb{Z}[q]\Gamma(n,r)$ be the ideal generated by the expressions

$$e_i e_{i+1}^2 - (1+q)e_{i+1}e_i e_{i+1} + q e_{i+1}^2 e_i$$

$$e_i^2 e_{i+1} - (1+q)e_i e_{i+1} e_i + q e_{i+1} e_i^2$$

$$f_{i+1} f_i^2 - (1+q) f_i f_{i+1} f_i + q f_i^2 f_{i+1}$$

$$f_{i+1}^2 f_i - (1+q) f_{i+1} f_i f_{i+1} + q f_i f_{i+1}^2$$

$$e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}]) k_\lambda$$

Recall that a relation is a $\mathbb{Z}[q]$-linear combination of paths with common start and end vertices. The relations involving paths $\lambda \to \mu$ are given by $1_\mu \mathrm{expr} 1_\lambda$, for each of the above expressions.

Lemma 4.3.1. *There is a homomorphism of $\mathbb{Z}[q]$-algebras*

$$\phi \colon \mathbb{Z}[q]\Gamma(n,r)/I(n,r) \to \hat{S}_q(n,r)$$

given by

$$\phi(e_{i,\lambda}) = E_{i,\lambda}$$
$$\phi(f_{i,\lambda}) = F_{i,\lambda}$$
$$\phi(k_\lambda) = 1_\lambda.$$

Chapter 5

A generic affine algebra

5.1 Introducing the generic affine algebra

Assume $\mathbf{k} = \mathbb{C}$ and fix $n, r \geq 1$. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by ε, namely $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free $\mathcal{S}$-module of rank r and let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of n-periodic cyclic flags in V; so $\mathcal{F}$ consists of collections $L = (L_i)_{i \in \mathbb{Z}}$ of $\mathcal{R}$-lattices in V with $L_i \subset L_{i+1}$ for $i \in \mathbb{Z}$ and $\varepsilon L_i = L_{i-n}$ for $i \in \mathbb{Z}$.

Let G be the group of $\mathcal{S}$-module automorphisms of V. Thus G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. G acts on $\mathcal{F}$ with orbits $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0\}$, where Λ_0 is the set of compositions of r into n parts, as in Definition 3.0.1.

The diagonal action of G on $\mathcal{F} \times \mathcal{F}$ has orbits $\{\mathcal{O}_A : A \in \Lambda_1\}$, where $\mathcal{O}_A$ consists of those pairs of flags with periodic characteristic matrix equal to A. Definitions of the periodic characteristic matrix and the set Λ_1 are given in Definition 3.1.1 and Definition 3.0.2 respectively. In particular, the periodic characteristic matrix of a pair $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix $A = (a_{i,j})_{i,j \in \mathbb{Z}}$, with

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right)$$

for each $i, j \in \mathbb{Z}$.

Recall that $\mathrm{ro}, \mathrm{co} \colon \Lambda_1 \to \Lambda_0$ are the maps given by

$$\mathrm{ro}\, A = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \ldots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co}\, A = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \ldots, \sum_{i \in \mathbb{Z}} a_{i,n} \right)$$

for each $A \in \Lambda_1$. Given $A \in \Lambda_1$, write $A \colon \mathrm{co}\, A \to \mathrm{ro}\, A$.

The purpose of this chapter is to define a category with objects Λ_0 and morphisms Λ_1; where $\mathrm{Hom}(\lambda, \mu) = \{A \in \Lambda_1 : \mathrm{ro}\, A = \mu, \mathrm{co}\, A = \lambda\}$. Given $A, B \in \Lambda_1$ let $\Lambda_{1A,B}$ be the set of $C \in \Lambda_1$ such that there exist $L, L', L'' \in \mathcal{F}$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L, L'') \in \mathcal{O}_C$. It will be shown that Λ_1 admits a partial order $\leq$ such that, given $A, B \in \Lambda_1$ with $\mathrm{ro}\, B = \mathrm{co}\, A$, $\Lambda_{1A,B}$ has a maximum element $A * B$. It will be shown that $*$ is associative, leading to the construction of a category with the described properties.

The generic affine algebra $\hat{G}(n,r)$ is then defined to be the $\mathbb{Z}$-algebra of this category. It will be shown that $\hat{G}(n,r)$ gives a realisation of the affine 0-Schur algebra $\hat{S}_0(n,r)$ when $r < n$. It is expected that a more refined presentation of the generic algebra and the 0-Schur algebra will allow the conditions on the parameters to be relaxed slightly: the $r = n$ case is approachable, which may extend to the case $r < 2n$.

5.2 A partial order

Given $i, j \in \mathbb{Z}$, define a map $d_{i,j}$ on Λ_1 by setting

$$d_{i,j}A = \sum_{s \leq i, t > j} a_{s,t}$$

for each $A \in \Lambda_1$.

Lemma 5.2.1. *Let $A \in \Lambda_1$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for $i, j \in \mathbb{Z}$. Then*

$$d_{i,j} - d_{i-1,j} = \sum_{t>j} a_{i,t}$$

and

$$d_{i,j} - d_{i,j-1} = -\sum_{s \leq i} a_{s,j}.$$

Proof. Let $i, j \in \mathbb{Z}$. Then

$$d_{i,j} - d_{i-1,j} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i-1, t > j} a_{s,t} = \sum_{t>j} a_{i,t}.$$

Similarly,

$$d_{i,j} - d_{i,j-1} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i, t > j-1} a_{s,t} = -\sum_{s \leq i} a_{s,j}.$$

□

Lemma 5.2.2. *Let $A \in \Lambda_1$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for each $i, j \in \mathbb{Z}$. Then*

$$a_{i,j} = d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j}$$

for each $i, j \in \mathbb{Z}$.

Proof. Using Lemma 5.2.1,

$$\begin{aligned} a_{i,j} &= \sum_{t>j-1} a_{i,t} - \sum_{t>j} a_{i,t} \\ &= (d_{i,j-1} - d_{i-1,j-1}) - (d_{i,j} - d_{i-1,j}). \end{aligned}$$

Alternatively,

$$\begin{aligned} a_{i,j} &= \sum_{s \leq i} a_{s,j} - \sum_{s \leq i-1} a_{s,j} \\ &= -(d_{i,j} - d_{i,j-1}) + (d_{i-1,j} - d_{i-1,j-1}). \end{aligned}$$

□

Lemma 5.2.3. *The relation $\leq$ on Λ_1, defined by $A \leq B$ if and only if $d_{i,j}A \leq d_{i,j}B$ for all $i,j \in \mathbb{Z}$, is a partial order.*

Proof. It is clear that $\leq$ is reflexive and transitive, so it remains to see $\leq$ is antisymmetric. Suppose $A, B \in \Lambda_1$ with $A \leq B$ and $B \leq A$. Then $d_{i,j}A = d_{i,j}B$ for each $i,j \in \mathbb{Z}$, which shows $A = B$ as a result of Lemma 5.2.2. $\square$

The partial order on Λ_1 induces a partial order on the set of G-orbits in $\mathcal{F}\times\mathcal{F}$, such that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $A \leq B$. The following lemma is rephrased from Lemma 3.1.4 and gives some geometric significance to the partial order on Λ_1.

Lemma 5.2.4. *Let $A \in \Lambda_1$ and take $(L, L') \in \mathcal{O}_A$. Then*

$$d_{i,j}A = \dim\left(\frac{L_i}{L_i \cap L'_j}\right)$$

for each $i,j \in \mathbb{Z}$.

Proof. This is a rephrasing of Lemma 3.1.4. $\square$

Conjecture 1. *Suppose $\mathbf{k} = \mathbb{C}$. The partial order on Λ_1 is compatible with the closure order on G-orbits in $\mathcal{F}\times\mathcal{F}$. In particular, $A \leq B$ if and only if $\mathcal{O}_A \subset \overline{\mathcal{O}_B}$.*

5.3 Preliminary results

Suppose $A, B \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$. Recall the notation

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

and

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L', L'') \in Y_{A,B}\}.$$

$X_{A,B}$ is the image of $Y_{A,B}$ under the forgetful map $(L, L', L'') \mapsto (L, L'')$.

Lemma 5.3.1. *There is $N \in \mathbb{N}$ such that*

$$\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$$

for each $(L, L'') \in X_{A,B}$.

Proof. There exist $N_1, N_2 \in \mathbb{N}$ such that

$$\varepsilon^{N_1} L_0 \subset L'_0 \subset \varepsilon^{-N_1} L_0$$

and

$$\varepsilon^{N_2} L'_0 \subset L''_0 \subset \varepsilon^{-N_2} L'_0,$$

for each $(L, L', L'') \in Y_{A,B}$. Then, for $(L, L', L'') \in Y_{A,B}$,

$$L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-(N_1+N_2)} L_0$$

and

$$\varepsilon^{N_1+N_2} L_0 \subset \varepsilon^{N_2} L'_0 \subset L''_0.$$

In particular, taking $N = N_1 + N_2$, we have

$$\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$$

for each $(L, L'') \in X_{A,B}$. $\square$

Lemma 5.3.2. *Suppose $N_1, N_2 \in \mathbb{N}$ with $\varepsilon^{N_1} L_0 \subset L_0 \subset \varepsilon^{-N_1} L_0$ and $\varepsilon^{N_2} L_0' \subset L_0'' \subset \varepsilon^{-N_2} L_0'$ for each $(L, L', L'') \in Y_{A,B}$ and let $N = N_1 + N_2$. Then*

$$\dim\left(\frac{\varepsilon^{-N} L_0}{L_0''}\right) = d_{nN_1,0}(A) + d_{nN_2,0}(B)$$

and

$$\dim\left(\frac{L_0''}{\varepsilon^{N} L_0}\right) = 2Nr - d_{nN_1,0}(A) + d_{nN_2,0}(B),$$

for each $(L, L'') \in X_{A,B}$.

Proof. Suppose $(L, L'') \in X_{A,B}$ and $L' \in \mathcal{F}$ so that $(L, L', L'') \in Y_{A,B}$. As in lemma 5.3.1, $\varepsilon^N L_0 \subset L_0'' \subset \varepsilon^{-N} L_0$, so

$$\dim\left(\frac{\varepsilon^{-N} L_0}{L_0''}\right) + \dim\left(\frac{L_0''}{\varepsilon^{N} L_0}\right) = \dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{N} L_0}\right).$$

As a $\mathbf{k}$-vector space, $\varepsilon^{-N} L_0 / \varepsilon^N L_0$ is isomorphic to $(L_0/\varepsilon L_0)^{2N}$, which has dimension $2Nr$, so

$$\dim\left(\frac{L_0''}{\varepsilon^{N} L_0}\right) = 2Nr - \dim\left(\frac{\varepsilon^{-N} L_0}{L_0''}\right).$$

It remains to compute the codimension of L_0'' in $\varepsilon^{-N} L_0$. Note $L_0'' \subset \varepsilon^{-N_2} L_0' \subset \varepsilon^{-N} L_0$, so

$$\dim\left(\frac{\varepsilon{-N} L_0}{L_0''}\right) = \dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L_0'}\right) + \dim\left(\frac{\varepsilon^{-N_2} L_0'}{L_0''}\right).$$

$$\begin{aligned}
\dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L_0'}\right) &= \dim\left(\frac{\varepsilon^{-N_1} L_0}{L_0'}\right) \\
&= \dim\left(\frac{L_{nN_1}}{L_{nN_1} \cap L_0'}\right) \\
&= \sum_{s \le nN_1, t > 0} A_{s,t} \\
&= d_{nN_1,0}(A).
\end{aligned}$$

$$\begin{aligned}
\dim\left(\frac{\varepsilon^{-N_2} L_0'}{L_0''}\right) &= \dim\left(\frac{L'_{nN_2}}{L'_{nN_2} \cap L_0''}\right) \\
&= \sum_{s \le nN_2, t > 0} B_{s,t} \\
&= d_{nN_2,0}(B).
\end{aligned}$$

□

5.3.1 Covering with projective varieties

Given $L \in \mathcal{F}$, $N \in \mathbb{N}$ and $\lambda \in \Lambda_0$ define

$$\Pi_{N,\lambda}(L) = \{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi^a_{N,\lambda} = \left\{ L'' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L''_0 \subset \varepsilon^N L_0, \dim\left(\frac{\varepsilon^{-N} L_0}{L''_0}\right) = a \right\},$$

for each $a \in \mathbb{N}$.

Lemma 5.3.3. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$ and $\lambda \in \Lambda_0$,*

$$\Pi_{N,\lambda}(L) = \bigcup_{a:0\leq a\leq 2Nr} \Pi^a_{N,\lambda}(L)$$

Proof. If $L' \in \Pi_{N,\lambda}(L)$ then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, so the $\mathbf{k}[\varepsilon]$-module $\varepsilon^{-N} L_0 / L'_0$ is isomorphic to $(\varepsilon^{-N} L_0/\varepsilon^N L_0)/(L'_0/\varepsilon^N L_0)$, which is a $\mathbf{k}$-vector space of dimension at most $2Nr = \dim(\varepsilon^{-N} L_0/\varepsilon^N L_0)$. □

Lemma 5.3.4. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \leq a \leq 2Nr$, $\Pi^a_{N,\lambda}(L)$ is a projective algebraic variety.*

Proof. Let W be the $\mathbf{k}$-vector space $\varepsilon^{-(1+N)} L_0/\varepsilon^N L_0$. Thus W is a $(2N+1)r$-dimensional vector space over $\mathbf{k}$ and the action of ε on W is a nilpotent linear operator with $\varepsilon^{2N+1} W = 0$. Then $\Pi^a_{N,\lambda}(L)$ is in natural bijection with the space of flags $(0 \leq W_1 \leq \cdots \leq W_n \leq W)$ of subspaces of W with type (λ, a) satisfying the closed condition $\varepsilon W_n \subset W_1$. The condition $\varepsilon W_n \subset W_1$ ensures that each subspace $W_i \subset W$ is a $\mathbf{k}[\varepsilon]$-submodule of W, since $\varepsilon W_i \subset \varepsilon W_n \subset W_1 \subset W_i$. Thus each W_i lifts to give a $\mathbf{k}[\varepsilon]$-submodule L_i of V with $\varepsilon^N L_0 \subset L_i \subset \varepsilon^{-N} L_0$, thus L_i is a lattice in V. □

Remark 1. *It should be made clear why $\varepsilon W_n \subset W_1$ is a closed condition in the variety of flags in W. That flag varieties are projective algebraic varieties should be considered an 'elementary' result not needing a reference or further explanation here, though a short discussion will be found in the early sections of the thesis.*

Lemma 5.3.5. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \leq a \leq 2Nr$, $\Pi^a_{N,\lambda}(L)$ is a closed subspace of $\Pi^{a+r}_{N+1,\lambda}(L)$.*

Proof. If $L' \in \Pi^{a+r}_{N+1,\lambda}(L)$, then $\varepsilon^{N+1} L_0 \subset \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0 \subset \varepsilon^{-(N+1)} L_0$ and

$$\dim\left(\frac{\varepsilon^{-(1+n)} L_0}{L'_0}\right) = \dim\left(\frac{L_0}{\varepsilon L_0}\right) + \dim\left(\frac{\varepsilon^{-N} L_0}{L'_0}\right) = r + a,$$

which shows that $\Pi^a_{N,\lambda}(L) \subset \Pi^{a+r}_{N+1,\lambda}(L)$. For $L' \in \Pi^{a+r}_{N+1,\lambda}(L)$, if additionally $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, then

$$\dim\left(\frac{\varepsilon^{-(N+1)} L_0}{L'_0}\right) = r + \dim\left(\frac{\varepsilon^{-N} L_0}{L'_0}\right),$$

which shows $L' \in \Pi^a_{N,\lambda}(L)$. Thus $\Pi^a_{N,\lambda}(L)$ is the subspace of $\Pi^{a+r}_{N+1,\lambda}(L)$ defined by the closed condition $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$. □

$$\begin{aligned}\Pi^{a}_{N,\lambda}(L) &= \{L' \in \Pi^{a+r}_{N+1,\lambda}(L) : \varepsilon L' \in \Pi^{a+2r}_{N+1,\lambda}(L), \varepsilon^{-1}L' \in \Pi^{a}_{N+1,\lambda}(L)\}\\ &= \varepsilon \Pi^{a}_{N+1,\lambda}(L) \cap \varepsilon^{-1}\Pi^{a+2r}_{N+1,\lambda}(L)\\ &= \Pi^{a}_{N+1,\lambda}(\varepsilon L) \cap \Pi^{a+2r}_{N+1,\lambda}(\varepsilon^{-1}L)\end{aligned}$$

Conjecture 2. *If $L, \tilde{L} \in \mathcal{F}$, $N_1, N_2 \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a_1, a_2 \in \mathbb{N}$ with $0 \le a_1 \le 2N_1 r$ and $0 \le a_2 \le 2N_2 r$ such that $\Pi^{a_1}_{N_1,\lambda}(L) \subset \Pi^{a_2}_{N_2,\lambda}(\tilde{L})$, then $\Pi^{a_1}_{N_1,\lambda}(L)$ is a closed subspace of $\Pi^{a_2}_{N_2,\lambda}(\tilde{L})$.*

Thus the subsets $\Pi^{a}_{N,\lambda}(L)$ of $\mathcal{F}$ are partially ordered by inclusion, the inclusions are closed and the set theoretic direct limit is the set $\mathcal{F}$, so $\mathcal{F}$ may be endowed with the direct limit topology of a system of projective algebraic varieties. If this is true, then subsets of $\mathcal{F}$ may be given the subspace topology without reference to an explicit embedding in a $\Pi^{a}_{N,\lambda}(L)$.

5.3.2 Geometry of orbits

Lemma 5.3.6. *Given $L \in \mathcal{F}$, the G_L-orbits in $\mathcal{F}$ are locally closed.*

Proof. This seems to require defining a limit topology on $\mathcal{F}$. Given $L \in \mathcal{F}$ and $A \in \Lambda_1$ with $|L| = \operatorname{ro} A$ and $\lambda = \operatorname{co} A$, there is a natural number N so that $X^L_A \subset \Pi^{\lambda}_N(L)$, which is a projective algebraic variety. Moreover, X^L_A is given by the conditions

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) = d_{i,j}A.$$

Lower(?) semicontinuity of the map $L' \mapsto \dim\left(L_i/L_i \cap L'_j\right)$ shows that each condition defines a locally closed subset of $\Pi^{\lambda}_N(L)$. Finally it will be shown that X^L_A is determined by finitely many of these conditions. $\square$

Given $N \in \mathbb{N}$, define

$$H_N = \left\{ h \in G_L : h = 1 \text{ on } \frac{\varepsilon^{-(1+N)}L_0}{\varepsilon^N L_0} \right\}.$$

Explicitly, the condition $h = 1$ on $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$ means: $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)}L_0$. Observe that $H_{N+1} \subset H_N$ for $N \in \mathbb{N}$ since $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ whenever $x \in \varepsilon^{-(1+N)}L_0$.

Lemma 5.3.7. *H_N is a normal subgroup in G_L, for any $N \in \mathbb{N}$.*

Proof. $H_N \subset G_L$ by definition. Suppose $h, h' \in H_N$ and let $x \in \varepsilon^{-(1+N)}L_0$. $h'(x) \in \varepsilon^{-(1+N)}L_0$ as $h' \in G_L$, so $hh'(x) + \varepsilon^N L_0 = h'(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$, which shows $hh' \in H_N$. $h(x) - x \in \varepsilon^N L_0$, so $h^{-1}(x) - x = -h^{-1}(h(x) - x) \in \varepsilon^N L_0$. $h^{-1} \in H_N$, so H_N is a subgroup of G_L.

Let $g \in G_L$. $hg^{-1}(x) + \varepsilon^N L_0 = g^{-1}(x)$ as $g^{-1}(x) \in \varepsilon^{-(1+N)}L_0$, so $ghg^{-1}(x) + \varepsilon^N L_0 = gg^{-1}(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$. Thus $ghg^{-1} \in H_N$, which proves H_N is a normal subgroup in G_L. $\square$

The H_N form a descending chain of normal subgroups in G_L: $\cdots \subset H_1 \subset H_0 \subset G_L \subset G$.

Lemma 5.3.8. *G_L/H_N is an irreducible algebraic group for any $N \in \mathbb{N}$.*

Proof. See the discussion in [1][section 4]. Should be able to give an explicit presentation of G_L/H_N in terms of the block structure.

$\sigma \in G_L$ induces an automorphism $\bar{\sigma}$ of $\varepsilon^{-N}L_0/\varepsilon^N L_0$, with inverse induced by σ^{-1}. Moreover, the natural map

$$G_L/H \to GL(\varepsilon^{-N}L_0/\varepsilon^N L_0)$$

is a group homomorphism. In fact, this homomorphism is injective: if $\sigma = \tau$ on $\varepsilon^{-N}L_0/\varepsilon^N L_0$, then $\sigma\tau^{-1} = 1$ on $\varepsilon^{-N}L_0/\varepsilon^N L_0$ and so $\sigma H = \tau H$. Thus G_L/H is isomorphic to its image in $\mathrm{GL}(\varepsilon^{-(N+1)}L_0/\varepsilon^N L_0)$. $\square$

Lemma 5.3.9. *Given $A \in \Lambda_1$ and $L \in \mathcal{F}_{\mathrm{co}\,A}$, X_A^L is an irreducible subspace of $\Pi^a_{N,\mathrm{co}\,A}(L)$ for some $N, a \in \mathbb{N}$.*

Proof. There exist $N, a \in \mathbb{N}$ so that $X_A^L \subset \Pi^a_{N,\mathrm{co}\,A}(L)$. Then $X_A^L = G_L/H_N L'$ for any $L' \in X_A^L$, where G_L/H_N is an irreducible algebraic group acting naturally on $\Pi^a_{N,\mathrm{co}\,A}(L)$, thus X_A^L is irreducible. $\square$

5.3.3 Geometry of orbit products

Refer to Section 3.1.1 for definitions of the spaces $X^L_{A,B}$ and $Y^L_{A,B}$.

Lemma 5.3.10. *Given $A, B \in \Lambda_1$ with $\mathrm{ro}\,B = \mathrm{co}\,A$ and $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_A$ and $(L', L'') \in \mathcal{O}_B$,*

$$X^L_{A,B} = G_L G_{L'} L''.$$

Proof. $X^L_{A,B}$ is the image of $Y^L_{A,B}$ under the forgetful map $(N', N'') \mapsto N''$. If $\alpha \in G_L$ and $\beta \in G_{L'}$ then $(L, \alpha L, \alpha\beta L'') \in Y_{A,B}$ since $(L, \alpha L') = \alpha(L, L') \in \mathcal{O}_A$ and $(\alpha L', \alpha\beta L'') = \alpha\beta(\beta^{-1}L', L'') = \alpha\beta(L', L'') \in \mathcal{O}_B$. Consequently, $G_L G_{L'} L'' \in X^L_{A,B}$.

For the reverse inclusion, if $(N', N'') \in Y^L_{A,B}$ then $(L, N') \in \mathcal{O}_A$ and $(N', N'') \in \mathcal{O}_B$, so there exist $\sigma_1, \sigma_2 \in G$ such that $(L, N') = \sigma_1(L, L')$ and $(N', N'') \in \sigma_2(N', N'')$. Then $(L, N', N'') = (L, \sigma_1 L', \sigma_1(\sigma_1^{-1}\sigma_2)L'')$ with $\sigma_1 \in G_L$ and $\sigma_1^{-1}\sigma_2 \in G_{L'}$. Thus $X^L_{A,B} = G_L G_{L'} L''$. $\square$

Lemma 5.3.11. *There is $N \in \mathbb{N}$ such that $H_N \subset G_{L'}$. Consequently, $H_{N'} \subset G_{L'}$ whenever $N' \geq N$.*

Proof. Choose $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$. Then

$$\varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset \cdots \subset L'_n \subset \varepsilon^{-(1+N)}L_0.$$

Let $h \in H_N$. $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)}L_0$, so $h(L'_i) \subset L'_i$ for $i = 0, 1, \ldots, n$. Moreover, h^{-1} stabilises L'_i, so $h(L'_i) = L'_i$ for $i = 0, 1, \ldots, n$ and therefore for $i \in \mathbb{Z}$. This shows $h \in G_{L'}$ as required, so $H_N \subset G_{L'}$. $\square$

Note that H_N is generally not a normal subgroup of $G_{L'}$, though the space of (right) cosets of H_N in $G_{L'}$ will still be irreducible.

Example 1. *Give an example here to illustrate that H^L_N need not be a normal subgroup in $G_{L'}$ for $L' \neq L$.*

Lemma 5.3.12. *$G_{L'}/H_N$ is irreducible, provided $H_N \subset G_{L'}$.*

Proof. Needs a proof. □

Lemma 5.3.13. *Given $A, B \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $L \in \mathcal{F}_{\operatorname{ro} A}$, $X^L_{A,B}$ is an irreducible topological space.*

Proof. Write up this proof properly - this is only a sketch. There is $N \in \mathbb{N}$ sufficiently large that $X^L_{A,B}$ is contained in $\Pi_{N,\operatorname{co} B}$, using Lemma 5.3.1. Suppose $(L, L') \in \mathcal{O}_A$, then $X^L_{A,B} = G_L X^{L'}_B$. G_L acts on $\Pi_{N,\lambda}$ through a quotient G_L/H which is an irreducible algebraic group, as a result of Lemma 5.3.8. $X^{L'}_B$ is an irreducible subspace of $\Pi_{N,\lambda}$. $X^L_{A,B}$ is the image of an irreducible subspace of $\Pi_{N,\lambda}$ under the action of a connected algebraic group, so $X^L_{A,B}$ is irreducible. □

Proposition 5.3.14. *Given $A, B \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $L \in \mathcal{F}_{\operatorname{ro} A}$, there is a unique open G_L-orbit in $X^L_{A,B}$.*

Proof. $X^L_{A,B}$ consists of finitely many G_L-orbits and is an irreducible topological space, by Lemma 5.3.13. Consequently, X^L_C is dense in $X^L_{A,B}$ for some $C \in \Lambda_{1A,B}$. Lemma 5.3.6 shows that X^L_C is locally closed in $X^L_{A,B}$, so X^L_C is open in $\overline{X^L_C} = X^L_{A,B}$. Irreducibility of $X^L_{A,B}$ shows that there is a unique open G_L-orbit, since two non-empty open sets in $X^L_{A,B}$ intersect non-trivially, thus any two open G_L orbits in $X^L_{A,B}$ coincide. □

5.4 Existence of a maximum

Lemma 5.4.1. *Given $A, A' \in \Lambda_1$ with $\operatorname{ro} A = \operatorname{ro} A'$ and $\operatorname{co} A = \operatorname{co} A'$, $A' \leq A$ if and only if $X^L_{A'} \subset \overline{X^L_A}$ for any $L \in \mathcal{F}_{\operatorname{ro} A}$.*

Proof. Needs a proof. □

Proposition 5.4.2. *Given $A, B \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$, $\Lambda_{1A,B}$ has a maximum element.*

Proof. Let $L \in \mathcal{F}_{\operatorname{ro} A}$. $X^L_{A,B}$ is irreducible by Lemma 5.3.13 and is the union of finitely many G_L-orbits, namely

$$X^L_{A,B} = \bigcup_{C \in \Lambda_{1A,B}} X^L_C.$$

This shows that X^L_C is dense in $X^L_{A,B}$ for some $C \in \Lambda_{1A,B}$. Lemma 5.3.6 shows that the G_L-orbits in $X^L_{A,B}$ are locally closed, so a dense G_L-orbit is open in $X^L_{A,B}$. Lemma 5.4.1 shows that the characteristic matrix of the dense G_L-orbit is a maximum in $\Lambda_{1A,B}$. □

5.5 Associativity

Refer to Section 3.1.2 for definitions of the spaces $X^L_{A,B,C}$ and $Y^L_{A,B,C}$. Recall that $X^L_{A,B,C}$ is the image of $Y^L_{A,B,C}$ under the forgetful map f, given by $f(L', L'', L''') = L'''$ for each $(L', L'', L''') \in Y^L_{A,B,C}$.

Lemma 5.5.1. *Given $A, B, C \in \Lambda_1$ with $\operatorname{ro} C = \operatorname{co} B$, $\operatorname{ro} B = \operatorname{co} A$ and a tuple of flags $(L, L', L'', L''') \in \mathcal{F}^4$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L'', L''') \in \mathcal{O}_C$,*

$$X^L_{A,B,C} = G_L G_{L'} G_{L''} L'''.$$

Proof. Given $\alpha \in G_L$, $\beta \in G_{L'}$ and $\gamma \in G_{L''}$, $(L, \alpha L', \alpha\beta L'', \alpha\beta\gamma L''') \in Y_{A,B,C}$ since $(L, \alpha L') = \alpha(L, L') \in \mathcal{O}_A$, $(\alpha L', \alpha\beta L'') = \alpha\beta(L', L'') \in \mathcal{O}_B$ and $(\alpha\beta L'', \alpha\beta\gamma L''') = \alpha\beta\gamma(L'', L''') \in \mathcal{O}_C$. This shows $G_L G_{L'} G_{L''} L''' \in X^L_{A,B,C}$.

Given $(N', N'', N''') \in Y^L_{A,B,C}$, there exist $\sigma_1, \sigma_2, \sigma_3 \in G$ such that $(L, N') = \sigma_1(L, L')$, $(N', N'') = \sigma_2(L', L'')$ and $(N'', N''') = \sigma_3(L'', L''')$; then $N' = \sigma_1 L' = \sigma_2 L'$, $N'' = \sigma_2 L'' = \sigma_3 L''$ and $N''' = \sigma_3 L'''$. Thus

$$(L, N', N'', N''') = (L, \sigma_1 L', \sigma_1(\sigma_1^{-1}\sigma_2) L'', \sigma_1(\sigma_1^{-1}\sigma_2)(\sigma_2^{-1}\sigma_3) L''')$$

where $\sigma_1 \in G_L$, $\sigma_1^{-1}\sigma_2 \in G_{L'}$ and $\sigma_2^{-1}\sigma_3 \in G_{L''}$. □

Lemma 5.5.2. *Given $A, B, C \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$ and $L \in \mathcal{F}_{\operatorname{ro} A}$, $Y^L_{A,B,C}$ is an irreducible topological space.*

Proof. Needs a proof. This is still a sticking point I think. □

Corollary 5.5.3. *Given $A, B, C \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$ and $L \in \mathcal{F}_{\operatorname{ro} A}$, $X^L_{A,B,C}$ is an irreducible topological space.*

Proof. $X^L_{A,B,C}$ is the image of $Y^L_{A,B,C}$ under the forgetful map f and $Y^L_{A,B,C}$ is irreducible, by Lemma 5.5.2, so $X^L_{A,B,C}$ is irreducible. □

Lemma 5.5.4. *Given matrices $A, B, C \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$ and $L \in \mathcal{F}_{\operatorname{ro} A}$, there is a unique open G_L-orbit in $X^L_{A,B,C}$.*

Proof. $X^L_{A,B,C}$ is irreducible, by Corollary 5.5.3, and consists of finitely many G_L-orbits, so contains a dense G_L-orbit. In particular, there is $D \in \Lambda_1$ such that $\overline{X^L_D} = X^L_{A,B,C}$. Lemma 5.3.6 shows that the G_L-orbits are locally closed in $X^L_{A,B,C}$. In particular, X^L_D is open in $\overline{X^L_D} = X^L_{A,B,C}$. Therefore, there is an open G_L-orbit in $X^L_{A,B,C}$. There is a unique open G_L-orbit since $X^L_{A,B,C}$ is irreducible. □

Definition 5.5.1. *Given $A, B, C \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$ and $L \in \mathcal{F}_{\operatorname{ro} A}$, define spaces*

$$\tilde{Y}^L_{(AB)C} = f^{-1} X^L_{(A*B)*C}$$

$$\tilde{Y}^L_{A(BC)} = f^{-1} X^L_{A*(B*C)}.$$

Lemma 5.5.5. *Given $A, B, C \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$ and $L \in \mathcal{F}_{\operatorname{ro} A}$, $\tilde{Y}^L_{(AB)C}$ is open in $Y^L_{A,B,C}$.*

Proof.

$$f^{-1}X^L_{A*B,C} = \left\{(L',L'',L''') \in Y^L_{A,B,C} : \dim\left(\frac{L_i}{L_i \cap L''_j}\right) \text{ is maximal, for each } i,j \in \mathbb{Z}\right\}$$

is open in $Y^L_{A,B,C}$ since $f^{-1}X^L_{A*B,C}$ is given by finitely many open conditions; the function on $X^L_{A,B}$ given by $L'' \mapsto \dim\left(\frac{L_i}{L_i\cap L''_j}\right)$ is lower semicontinuous, so maximising such a function is an open condition in $X^L_{A,B}$. □

Lemma 5.5.6. *Given $A,B,C \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$ and $L \in \mathcal{F}_{\operatorname{ro} A}$, $\tilde{Y}^L_{A(BC)}$ is open in $Y^L_{A,B,C}$.*

Proof.

$$f^{-1}X^L_{A,B*C} = \left\{(L',L'',L''') \in Y^L_{A,B,C} : \dim\left(\frac{L'_i}{L'_i \cap L'''_j}\right) \text{ is maximal, for each } i,j \in \mathbb{Z}\right\}$$

is open in $Y^L_{A,B,C}$, as it is determined by finitely many open conditions; the function on $X^L_A \times X^L_{A,B*C}$ given by $(L',L''') \mapsto \dim\left(\frac{L'_i}{L'_i\cap L'''_j}\right)$ is lower semicontinuous, so maximising such a function is an open condition in $X^L_A \times X^L_{A,B*C}$. □

Conjecture 3. *Given $A,B,C \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$ and $L \in \mathcal{F}_{\operatorname{ro} A}$, $X^L_{A*B,C}$ and $X^L_{A,B*C}$ are open and dense in $X^L_{A,B,C}$.*

Remark 2. *If f is shown to be an open map then this result follows from Lemma 5.5.5 and Lemma 5.5.6.*

Proposition 5.5.7. *Given $A,B,C \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$, $(A*B)*C = A*(B*C)$.*

Proof. Take $A,B,C \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$ and $\operatorname{co} B = \operatorname{ro} C$ and fix $L \in \mathcal{F}_{\operatorname{ro} A}$.

$X^L_{(A*B)*C}$ is open in $X^L_{A*B,C}$, so $f^{-1}X^L_{(A*B)*C}$ is open in $f^{-1}X^L_{A*B,C}$. Lemma 5.5.5 shows that $f^{-1}X^L_{A*,B,C}$ is open in $Y^L_{A,B,C}$, so $f^{-1}X^L_{(A*B)*C}$ is open in $Y^L_{A,B,C}$. Similarly, $X^L_{A*(B*C)}$ is open in $X^L_{A,B*C}$ and $f^{-1}X^L_{A,B*C}$ is open in $Y^L_{A,B,C}$, by Lemma 5.5.6, so $f^{-1}X^L_{A*(B*C)}$ is open in $Y^L_{A,B,C}$.

Lemma 5.5.2 shows that $Y^L_{A,B,C}$ is irreducible, so $f^{-1}X^L_{(A*B)*C}$ and $f^{-1}X^L_{A*(B*C)}$ have nonempty intersection. Therefore the G_L-orbits $X^L_{(A*B)*C}$ and $X^L_{A*(B*C)}$ intersect nontrivially, so are the same G_L-orbit. Thus $(A*B)*C = A*(B*C)$. □

5.6 The generic algebra

Lemma 5.6.1. *Given $\lambda \in \Lambda_0$ and $A \in \Lambda_1$, $D_\lambda * A = A$ if $\operatorname{ro} A = \lambda$ and $A * D_\lambda = A$ if $\operatorname{co} A = \lambda$.*

Proof. Lemma 3.1.7 shows that ${\Lambda_1}_{D_\lambda,A} = \{A\}$ if $\lambda = \operatorname{ro} A$ and ${\Lambda_1}_{A,D_\lambda} = \{A\}$ if $\lambda = \operatorname{co} A$, which proves the result. □

Theorem 5.6.2. *The following constitutes a small category: the set of objects is Λ_0 and the set of morphisms is Λ_1. Given compositions $\lambda,\mu \in \Lambda_0$, the morphisms with source λ and target μ are those matrices $A \in \Lambda_1$ with $\operatorname{co} A = \lambda$ and $\operatorname{ro} A = \mu$. Given $\lambda,\mu,\nu \in \Lambda_0$ and $A,B \in \Lambda_1$ with $B\colon \lambda \to \mu$ and $A\colon \mu \to \nu$ the composition is $A*B\colon \lambda \to \nu$.*

Proof. Proposition 5.4.2 shows that the composition is well defined while Proposition 5.5.7 establishes associativity of the composition. Lemma 5.6.1 shows that $D_\lambda \colon \lambda \to \lambda$ is the identity morphism for each $\lambda \in \Lambda_0$. Thus $(\Lambda_0, \Lambda_1, \mathrm{co}, \mathrm{ro}, *)$ is a category. $\square$

Write $\mathcal{G}(n, r)$ to denote this so-called 'generic category'.

Example 2. *The objects in $\mathcal{G}(2,2)$ are compositions of 2 into 2 parts, namely $(0,2)$, $(1,1)$ and $(2,0)$. The set of morphisms from λ to μ is the set of infinite periodic matrices $A \in \Lambda_1[2,2]$ with $\mathrm{co}\, A = \lambda$ and $\mathrm{ro}\, A = \mu$, which is a countably infinite set for any pair of compositions $\lambda, \mu \in \Lambda_0[2,2]$.*

Definition 5.6.1 (Generic algebra)**.** *The generic affine algebra $\hat{G}(n,r)$ is the category $\mathbb{Z}$-algebra of $\mathcal{G}(n,r)$. In particular, $\hat{G}(n,r)$ is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1\}$ and with associative multiplication given by*

$$e_A * e_B = \begin{cases} e_{A*B} & \text{if } \mathrm{co}\, A = \mathrm{ro}\, B \\ 0 & \text{if } \mathrm{co}\, A \neq \mathrm{ro}\, B. \end{cases}$$

The multiplicative identity in $\hat{G}(n,r)$ is

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

Chapter 6

A realisation of affine zero Schur algebras

We aim to prove the isomorphism theorem in the cases $r < n$ and $n \leq r < 2n$ separately. Below are crude versions of the statements we want to prove.

Theorem 6.0.1. *Assume $r < n$. The map $\psi \colon \hat{G}(n,r) \to \hat{S}_0(n,r)$, given by $\psi(E_i) = E_i$, $\psi(F_i) = F_i$ and $\psi(1_\lambda) = 1_\lambda$, is an isomorphism of $\mathbb{Z}$-algebras.*

Theorem 6.0.2. *Assume $n \leq r < 2n$. There is a unique homomorphism of $\mathbb{Z}$-algebras $\hat{\psi} \colon \hat{G}(n,r) \to \hat{S}_0(n,r)$ such that $\hat{\psi}(R) = R$ and $\hat{\psi} = \psi$ on the subalgebra of $\hat{G}(n,r)$ generated by the E_i, F_i and 1_λ. $\hat{\psi}$ is an isomorphism of $\mathbb{Z}$-algebras.*

6.1 Preliminary results

Recall from Definition 5.6.1 that the generic algebra $\hat{G}(n,r)$ is an associative $\mathbb{Z}$-algebra which is a free $\mathbb{Z}$-module with an atomic basis $\{e_A : A \in \Lambda_1\}$: given $A, B \in \Lambda_1$ with $\operatorname{co} A = \operatorname{ro} B$, $e_A e_B = e_{A*B}$.

6.1.1 Elementary basis elements

Given $i \in [1,n]$ and $\lambda \in \Lambda_0$ such that $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}.$$

Given $i \in [1,n]$ and $\lambda \in \Lambda_0$ such that $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}.$$

6.1.2 Transpose involution

Lemma 6.1.1. *The $\mathbb{Z}$-module automorphism $\top$ of $\hat{G}(n,r)$ given by $e_A \mapsto e_{A^\top}$ is a $\mathbb{Z}$-algebra antihomomorphism: that is,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A$$

for each $A, B \in \Lambda_1$. Moreover, $\top(E_{i,\lambda}) = F_{i,\lambda+\alpha_i}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}$ and $\top(1_\lambda) = 1_\lambda$, for permissible $(i,\lambda) \in \mathbb{Z} \times \Lambda_0$.

Proof. This is a consequence of Lemma 4.1.1. It must also be shown that the transpose operation on Λ_1 is order preserving. □

6.1.3 Multiplication rules

Lemma 6.1.2. *Given $A \in \Lambda_1$ and $i \in [1,n]$ such that $\operatorname{ro} A_{i+1} > 0$,*

$$E_i e_A = e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}},$$

where $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$.

Given $A \in \Lambda_1$ and $i \in [1,n]$ such that $\operatorname{ro} A_i > 0$,

$$F_i e_A = e_{A+\mathcal{E}_{i+1,p}-\mathcal{E}_{i,p}},$$

where $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$.

Similar formulas for right multiplication by E_i and F_i are obtained by applying the transpose.

6.2 Presentation of the generic algebra.

Recall that Λ_0 denotes the set of compositions of r into n parts. That is, Λ_0 is the set of tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with each λ_i non-negative and $\lambda_1 + \cdots + \lambda_n = r$. Given $i \in [1,n]$, let $\varepsilon_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^n$ be the i–th elementary vector and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then given $\lambda \in \Lambda_0$, we have $\lambda + \alpha_i \in \Lambda_0$ provided $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ provided $\lambda_i > 0$.

Let $\Gamma = \Gamma(n,r)$ be the quiver with set of vertices Λ_0 with arrows $e_{i,\lambda} \colon \lambda \to \lambda + \alpha_i$ (if $\lambda_{i+1} > 0$) and $f_{i,\lambda} \colon \lambda \to \lambda - \alpha_i$ (if $\lambda_i > 0$). Thus there are no arrows between λ and μ unless $\lambda = \mu \pm \alpha_i$ for some $i \in [1,n]$.

If $n \geq 3$ then neighbouring vertices are connected by two arrows, one of each direction. In the case $n = 2$, neighbouring vertices are joined by four arrows, two of each direction. The $\mathbb{Z}\Gamma$ denote the path $\mathbb{Z}$ algebra of Γ. By construction of Γ, there is a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}\Gamma \to \hat{G}(n,r)$ with $e_{i,\lambda} \mapsto E_{i,\lambda}$, $f_{i,\lambda} \mapsto F_{i,\lambda}$ and $k_\lambda = 1_\lambda$. We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [1]).

Definition 6.2.1. *(aperiodicity) $A \in \Lambda_1$ is aperiodic if for each $l \in \mathbb{Z} \setminus \{0\}$ there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$. Denote the set of aperiodic elements in Λ_1 by ${\Lambda_1}^{ap}$. Note that ${\Lambda_1}^{ap} = \Lambda_1$ if $r < n$. Linear combinations of the basis elements corresponding to aperiodic matrices are also said to be aperiodic - if A is aperiodic, we say e_A is aperiodic.*

Lemma 6.2.1. *Let $A \in \Lambda_1$ and write $\lambda = \operatorname{ro} A$. If A is aperiodic and $\lambda_{i+1} > 0$, then $E_i * e_A$ is aperiodic. If A is aperiodic and $\lambda_i > 0$, then $F_i * e_A$ is aperiodic.*

Proof. Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_{i+1} > 0$, where $\lambda = \mathrm{ro}\, A$. There is $p \in \mathbb{Z}$ such that $a_{i+1,p} > 0$ and $a_{i+1,p'} = 0$ whenever $p' > p$. Lemma 6.1.2 shows that $E_i * e_A = e_B$, where $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p-i, p-i-1\}$, then $b_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $b_{s,s+l} = a_{s,s+l} = 0$, since A is aperiodic. If $l = p - i$, then $b_{i+1,i+1+l} = b_{i+1,p+1} = a_{i+1,p+1} = 0$, by maximality of p. If $l = p-i-1$, there is $s \neq i+1$ such that $a_{s,s+l} = 0$, since A is aperiodic and $a_{i+1,i+1+l} = a_{i+1,p} > 0$, so $b_{s,s+l} = a_{s,s+l} = 0$. Therefore, $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ is aperiodic.

Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_i > 0$, where $\lambda = \mathrm{ro}\, A$. Lemma 6.1.2 shows that $F_i * e_A = e_C$ where $C = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$ and $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p-i, p-i-1\}$ then $c_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $c_{s,s+p} = a_{s,s+p} = 0$, by aperiodicity of A. If $l = p - i$, then $a_{i,i+l} = a_{i,p} > 0$, so there is $s \neq i$ such that $a_{s,s+l} = 0$. Then $c_{s,s+l} = a_{s,s+l} = 0$. Finally, if $l = p-i-1$, then $c_{i,i+l} = a_{i,p-1} = 0$ by minimality of p. Thus C is aperiodic as required. ☐

Definition 6.2.2. *(Weight function) Define the weight function* $\mathrm{wt}\colon \Lambda_1 \to \mathbb{Z}$ *by*

$$\mathrm{wt}\, A = \sum_{i \in [1,n], j \in \mathbb{Z}} |j-i| a_{i,j}$$

for each $A \in \Lambda_1$*. The sum is taken over a transversal of the set of congruence classes of* (i,j) *modulo* (n,n) *for* $i, j \in \mathbb{Z}$*.*

Lemma 6.2.2. *Let* $A \in \Lambda_1$ *and write* $\lambda = \mathrm{ro}\, A$*. Suppose* $\lambda_{i+1} > 0$ *and set* $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$*. If* $p > i$ *then* $\mathrm{wt}\, e_{i,\lambda} * A = 1 + \mathrm{wt}\, A$*. If* $p \leq i$ *then* $\mathrm{wt}\, e_{i,\lambda} * A = -1 + \mathrm{wt}\, A$*. Suppose* $\lambda_i > 0$ *and set* $q = \min\{q' \in \mathbb{Z} : a_{i,q'} > 0\}$*. If* $q \leq i$ *then* $\mathrm{wt}\, f_{i,\lambda} * A = 1 + \mathrm{wt}\, A$*. If* $q > i$ *then* $\mathrm{wt}\, f_{i,\lambda} * A = -1 + \mathrm{wt}\, A$*.*

Proof. Lemma 6.1.2 shows that $e_i A = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$, so $\mathrm{wt}\, e_i A - \mathrm{wt}\, A = |p-i| - |p-i-1|$, which equals 1 if $p > i$ and equals -1 if $p \leq i$. Similarly, $f_i A = A + \mathcal{E}_{i+1,q} - \mathcal{E}_{i,q}$ by Lemma 6.1.2, so $\mathrm{wt}\, f_i A - \mathrm{wt}\, A = |q-i-1| - |q-i|$, which equals -1 if $q > i$ and equals 1 if $q \leq i$. ☐

Lemma 6.2.3. *If* $A \in \Lambda_1$ *is aperiodic, then* e_A *may be obtained from* $1_{\mathrm{co}\, A}$ *by finitely many applications of* E_i *and* F_i *for* $i \in [1,n]$*.*

Proposition 6.2.4. *The* $\mathbb{Z}$*-subalgebra of* $\hat{G}(n,r)$ *generated by* $E_{i,\lambda}$*,* $F_{i,\lambda}$ *and* 1_λ *has* $\mathbb{Z}$*-basis* $\{e_A : A \in {\Lambda_1}^{ap}\}$*, where* ${\Lambda_1}^{ap} \subset \Lambda_1$ *is the set of aperiodic elements.*

Proof. ☐

6.2.1 The case $n \geq 3$.

Lemma 6.2.5. *The following relations hold in* $\hat{G}(n,r)$ *(*$n \geq 3$*):*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless $|j-i| = 1$*.*

$$E_i E_{i+1}^2 - E_{i+1} E_i E_{i+1} = 0$$

$$E_i^2 E_{i+1} - E_i E_{i+1} E_i = 0$$

$$F_{i+1} F_i^2 - F_i F_{i+1} F_i = 0$$

$$F_{i+1}^2 F_i - F_{i+1} F_i F_{i+1} = 0$$

$$E_i F_j - F_j E_i = 0$$

unless $j = i$.

$$E_i F i - F_i E_i + \sum_{\lambda:\lambda_i=0,\lambda_{i+1}>0} 1_\lambda - \sum_{\lambda:\lambda_i>0,\lambda_{i+1}=0} 1_\lambda = 0.$$

6.2.2 The case $n = 2$.

In this case, the quiver $\Gamma(2, r)$ has vertices $\Lambda_0[2, r] = \{(0, r), (1, r-1), \ldots, (r, 0)\}$; adjacent vertices are connected by two pairs of arrows with opposite orientation: (e_1, f_1) and (e_2, f_2). The relations arising from $\hat{G}(2, r)$ are of a more complicated form - in particular, the serre relations of total degree 3 will not hold in this case - so this case will be treated separately and at a later date.

Chapter 7

Further directions

7.1 Further results on affine zero Schur algebras

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

7.2 Deformed group algebras of symmetric groups

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algebras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for S_3 and S_4. Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: 'these' relations hold; these generators and relations give a presentation of the degenerate group algebras.

Terminology: deformed group algebra.

Bibliography

[1] George Lusztig. “Aperiodicity in quantum affine gln”. In: *Asian Journal of Mathematics* 3.1 (1999), pp. 147–178.