

# A geometric realisation of affine 0-Schur algebras.

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October 1, 2019

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## Chapter 1

# Introduction

## Chapter 2

# The double flag variety approach to $q$ -Schur algebras

## Chapter 3

# The cyclic flags approach to affine q-Schur algebras

Fix natural numbers  $n$  and  $r$ .

**Definition 3.0.1** (compositions). *A composition of  $r$  into  $n$  parts is an  $n$ -tuple  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$  of non-negative integers whose sum equals  $r$ . Denote the set of compositions of  $r$  into  $n$  parts by  $\Lambda_0(n, r)$ .*

**Definition 3.0.2** (infinite periodic matrices). *Let  $\Lambda_1(n, r)$  be the set of matrices  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$  with integer entries  $a_{i,j}$  satisfying the following conditions:*

- $a_{i,j} \geq 0$  for each  $i, j \in \mathbb{Z}$ ;
- each row or column has only finitely many non-zero entries;
- the sum of the entries in any  $n$  consecutive rows or columns equals  $r$ ;
- $a_{i-n,j-n} = a_{i,j}$  for each  $i, j \in \mathbb{Z}$ .

*These matrices are referred to as infinite periodic matrices.*

**Definition 3.0.3** (source and target). *Given  $A \in \Lambda_1(n, r)$ , let  $\text{ro } A$  and  $\text{co } A$  be the compositions of  $r$  into  $n$  parts given by*

$$\text{ro } A = \left( \sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

*and*

$$\text{co } A = \left( \sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right).$$

*$A \in \Lambda_1(n, r)$  is said to go from  $\text{co } A$  to  $\text{ro } A$ .*

**Definition 3.0.4** (diagonal matrices). *Given  $\lambda \in \Lambda_0(n, r)$ , let  $D_\lambda \in \Lambda_1(n, r)$  be the matrix given by  $(D_\lambda)_{i,j} = 0$  for  $i, j \in \mathbb{Z}$  with  $i \neq j$  and  $(D_\lambda)_{i,i} = \lambda_i$  for  $i \in \mathbb{Z}$ ; where the indices are taken modulo  $n$ .*

### 3.1 Cyclic flags

Fix  $n, r \in \mathbb{N}$  and let  $\mathbf{k}$  be a field. Let  $\mathcal{S}$  be the  $\mathbf{k}$ -algebra  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$  and let  $\mathcal{R}$  be the subalgebra generated by  $\varepsilon$ , so  $\mathcal{R} = \mathbf{k}[\varepsilon]$ . Let  $V$  be a free  $\mathcal{S}$ -module of rank  $r$ . Let  $G$  be the automorphism group of the  $\mathcal{S}$ -module  $V$ , so  $G$  is isomorphic to  $\mathrm{GL}_r(\mathcal{S})$ . A lattice in  $V$  is a  $\mathcal{R}$ -submodule  $L$  of  $V$  with  $\mathcal{S} \otimes_{\mathcal{R}} L = V$ . In particular, a lattice is an  $\mathcal{R}$ -submodule of  $V$  which is a free  $\mathcal{R}$ -module of rank  $r$ .

**Lemma 3.1.1.** *Let  $L$  be a lattice in  $V$ .  $L/\varepsilon L$  is a torsion  $\mathcal{R}$ -module, where  $\varepsilon$  acts as zero.  $L/\varepsilon L$  is a free  $\mathcal{R}/\langle \varepsilon \rangle$ -module of rank  $r$ ; that is,  $L/\varepsilon L$  is an  $r$ -dimensional  $\mathbf{k}$ -vector space.*

*Proof.*  $L$  is a free  $\mathcal{R}$ -module of rank  $r$ , with  $L \subset V$ . Given an  $\mathcal{R}$ -basis  $\{x_1, \dots, x_r\}$  of  $L$ ,  $\{\varepsilon x_1, \dots, \varepsilon x_r\}$  is an  $\mathcal{R}$ -basis of  $\varepsilon L$ . Finally, the cosets  $\{x_1 + \varepsilon L, \dots, x_r + \varepsilon L\}$  give a basis for  $L/\varepsilon L$  over  $\mathcal{R}/\langle \varepsilon \rangle \cong \mathbf{k}$ .  $\square$

Let  $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$  be the set of collections  $(L_i)_{i \in \mathbb{Z}}$  of lattices in  $V$  with  $L_i \subset L_{i+1}$  and  $\varepsilon L_i = L_{i-n}$  for each  $i \in \mathbb{Z}$ . These collections of lattices in  $V$  are referred to as cyclic flags in  $V$ .

$G$  acts on  $\mathcal{F}$  by  $(g \cdot L)_i = g(L_i)$  for each  $i \in \mathbb{Z}$ , given  $g \in G$  and  $L \in \mathcal{F}$ . The  $G$ -orbits in  $\mathcal{F}$  are indexed by the set  $\Lambda_0(n, r)$  of compositions of  $r$  into  $n$  parts: the  $G$ -orbit in  $\mathcal{F}$  corresponding to  $\lambda \in \Lambda_0(n, r)$  is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim \left( \frac{L_i}{L_{i-1}} \right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

**Definition 3.1.1.** *The periodic characteristic matrix of a pair of cyclic flags  $(L, L') \in \mathcal{F} \times \mathcal{F}$  is the matrix  $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$  with entries*

$$a_{i,j} = \dim_{\mathbf{k}} \left( \frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j} \right)$$

for each  $i, j \in \mathbb{Z}$ .

The diagonal action of  $G$  on  $\mathcal{F} \times \mathcal{F}$  has orbits indexed by the set  $\Lambda_1(n, r)$  of infinite periodic matrices (see definition 3.0.2). The  $G$ -orbit corresponding to  $A \in \Lambda_1(n, r)$  is denoted  $\mathcal{O}_A$  and consists of those pairs  $(L, L') \in \mathcal{F} \times \mathcal{F}$  with periodic characteristic matrix  $A(L, L')$  equal to  $A$ .

**Lemma 3.1.2.** *(alternative expression for characteristic matrix) Alternatively,*

$$a_{i,j} = \dim_{\mathbf{k}} \left( \frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} \right)$$

for each  $i, j \in \mathbb{Z}$ .

*Proof.* Set  $U = L_i \cap L'_j$  and  $U' = L_{i-1} + L_i \cap L'_{j-1}$ . Then  $U + U' = L_{i-1} + L_i \cap L'_j$  and  $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$ . Applying the isomorphism theorems,  $U + U'/U'$  is naturally isomorphic to  $U/U \cap U'$  as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to  $a_{i,j}$ .  $\square$

**Lemma 3.1.3** (transposing characteristic matrix). *Given a pair of flags  $(L, L') \in \mathcal{F}^2$ , the matrices  $A(L, L')$  and  $A(L', L)$  are related by the transpose. In particular,  $A(L, L')_{i,j} = A(L', L)_{j,i}$  for each  $i, j \in \mathbb{Z}$ .*

*Proof.* By swapping the roles of  $i$  and  $j$  and swapping  $L$  and  $L'$  it is clear that  $A(L, L')_{i,j}$  and  $A(L', L)_{j,i}$  are both given by the dimension of the  $\mathbf{k}$ -vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each  $i, j \in \mathbb{Z}$ . □

**Lemma 3.1.4** (a codimension formula). *Given  $(L, L') \in \mathcal{F}^2$  and  $i, j \in \mathbb{Z}$ ,*

$$\dim_{\mathbf{k}} \left( \frac{L_i}{L_i \cap L'_j} \right) = \sum_{s \leq i, t > j} a_{s,t},$$

where  $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$ .

*Proof.* **COMPLETE THIS PROOF** □

**Lemma 3.1.5** (nested flags). *Given  $(L, L') \in \mathcal{F}^2$ ,  $L' \subset L$  if and only if  $A(L, L')_{i,j} = 0$  for  $i, j \in \mathbb{Z}$  with  $i > j$ .*

*Proof.* Suppose  $L, L' \in \mathcal{F}$  with  $L' \subset L$ , meaning  $L'_j \subset L_j$  for each  $j \in \mathbb{Z}$ . Then for  $i > j$ ,  $L_i \cap L'_j = L'_j$ ,  $L_{i-1} \cap L'_j = L'_j$  and  $L_i \cap L'_{j-1}$ , which shows

$$A(L, L')_{i,j} = \dim_{\mathbf{k}} \left( \frac{L'_j}{L'_{j-1} + L'_j} \right) = 0$$

as required. Conversely, suppose  $A(L, L')$  is upper triangular, meaning  $A(L, L')_{i,j} = 0$  when  $i > j$ . Using Lemma 3.1.4,

$$\dim_{\mathbf{k}} \left( \frac{L'_i}{L'_i \cap L_i} \right) = \sum_{s > i, t \leq i} a_{s,t} = 0,$$

so  $L_i \cap L'_i = L'_i$  and thus  $L'_i \subset L_i$  for each  $i \in \mathbb{Z}$ , as required. □

**Corollary 3.1.6** (diagonal orbits). *Given  $L, L' \in \mathcal{F}$ ,  $L = L'$  if and only if  $A(L, L')_{i,j} = 0$  whenever  $i \neq j$ . In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L, L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each  $\lambda \in \Lambda_0(n, r)$ .

### 3.1.1 A product on orbits

Given  $A, B \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$ , define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also  $L \in \mathcal{F}_{\text{ro } A}$ , define the  $L$ -slices of  $Y_{A,B}$  and  $X_{A,B}$  respectively as

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X_{A,B}^L = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$



**Observation 1.** *There are only finitely many  $G$ -orbits in  $X_{A,B}$ .*

**Lemma 3.1.7.** *Given  $A \in \Lambda_1(n, r)$ ,  $X_{D_\lambda, A} = \mathcal{O}_A$  if  $\lambda = \text{ro } A$  and  $X_{A, D_\lambda} = \mathcal{O}_A$  if  $\lambda = \text{co } A$ .*

*Proof.* Let  $A \in \Lambda_1(n, r)$  and set  $\lambda = \text{ro } A$ .  $Y_{D_\lambda, A}$  is the set of triples  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_{D_\lambda}$ , thus  $L = L'$  by Corollary 3.1.6, and  $(L', L'') \in \mathcal{O}_A$ .  $X_{D_\lambda, A}$  is the projection of  $Y_{D_\lambda, A}$ , which equals  $\mathcal{O}_A$ .

Similarly, if  $\lambda = \text{co } A$ ,  $Y_{A, D_\lambda}$  is the set of triples  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_A$  and  $L'' = L'$ , so  $X_{A, D_\lambda}$  is exactly the orbit  $\mathcal{O}_B$ .  $\square$

### 3.1.2 Triple products

Given  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ , there are spaces  $X_{A,B,C}$ ,  $Y_{A,B,C}$  and their respective  $L$ -slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y_{A,B,C}^L = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X_{A,B,C}^L = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

## 3.2 Convolution algebras

Suppose  $\mathbf{k}$  is a finite field and let  $q$  denote the number of elements of  $\mathbf{k}$ . Consider the set  $S$  of  $G$ -invariant functions  $\mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Z}$  with constructible support.  $S$  is a free  $\mathbb{Z}$ -module with a basis consisting of the indicator functions of the  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$ . Define an operation  $\star$  on  $S$  as follows: for each  $f, g \in S$ ,  $f \star g \in S$  is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ .

$f \star g$  is well defined since the supports of  $f$  and  $g$  consist of finitely many  $G$ -orbits, so there are only finitely many  $L' \in \mathcal{F}$  such that  $f(L, L')g(L', L'') \neq 0$ , given  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ .  $f \star g$  is constant on  $G$ -orbits and is supported on finitely many  $G$ -orbits, so  $f \star g \in S$ .

**Lemma 3.2.1.** *The set  $S$  together with the operation  $\star$  is an associative  $\mathbb{Z}$ -algebra with identity element  $\iota$  given by  $\iota(L, L) = 1$  and  $\iota(L, L') = 0$  for  $L' \neq L$ .*

*Proof.* Given  $f, g, h \in S$  and  $(L, L''') \in \mathcal{F} \times \mathcal{F}$ ,

$$\begin{aligned} ((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''), \end{aligned}$$

thus  $\star$  is associative.  $\iota$  is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L') \iota(L', L'') = f(L, L''),$$

for each  $f \in S$  and  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ . □

Given  $A \in \Lambda_1(n, r)$ , let  $e_A \in S$  denote the indicator function of the orbit  $\mathcal{O}_A$ .  $S$  is a free  $\mathbb{Z}$ -module with basis  $\{e_A : A \in \Lambda_1(n, r)\}$ . There exist  $\gamma_{A,B,C;q} \in \mathbb{Z}$  for  $A, B, C \in \Lambda_1(n, r)$  such that

$$e_A \star e_B = \sum_{C \in \Lambda_1(n, r)} \gamma_{A,B,C;q} e_C$$

for each  $A, B \in \Lambda_1(n, r)$ . Then

$$\begin{aligned} \gamma_{A,B,C;q} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L') e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}, \end{aligned}$$

for any  $(L, L'') \in \mathcal{O}_C$ .

### 3.3 Affine $q$ -Schur algebras

There exist polynomials  $\gamma_{A,B,C} \in \mathbb{Z}[q]$  for  $A, B, C \in \Lambda_1(n, r)$  such that  $\gamma_{A,B,C}(q) = \gamma_{A,B,C;q}$  for any prime power  $q$ , following [2, section 4]. The affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  (defined in [\[ADD A REFERENCE\]](#)) is a  $\mathbb{Z}[q]$ -algebra which is a free  $\mathbb{Z}[q]$ -module with basis  $\{e_A : A \in \Lambda_1(n, r)\}$  and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these ‘universal polynomials’  $\gamma_{A,B,C} \in \mathbb{Z}[q]$ , it follows from Lemma 3.2.1 that  $\hat{S}_q(n, r)$  is an associative  $\mathbb{Z}[q]$ -algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0(n, r)} e_{D_\lambda}.$$

## Chapter 4

# Quivers with relations for affine q-Schur algebras

### 4.1 Basic results and notation

#### 4.1.1 Elementary matrices

If  $i, j \in \mathbb{Z}$ , let  $\mathcal{E}_{i,j}$  denote the ‘elementary matrix’ with entries given by  $(\mathcal{E}_{i,j})_{s,t} = 1$ , for  $s, t \in \mathbb{Z}$ , whenever  $(i, j) \sim (s, t)$  modulo  $(n, n)$  and all other entries are zero.

Given  $\lambda \in \Lambda_0(n, r)$ , let  $D_\lambda \in \Lambda_1(n, r)$  denote the diagonal matrix with  $r(D_\lambda) = c(D_\lambda) = \lambda$ , as in Definition 3.0.4. That is,

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}$$

For  $\lambda \in \Lambda_0(n, r)$ , write  $1_\lambda = e_{D_\lambda}$ . The  $1_\lambda$  are pairwise orthogonal idempotents in  $\hat{S}_q(n, r)$  with  $1 = \sum_{\lambda \in \Lambda_0(n, r)} 1_\lambda$ , as a result of Lemma 3.1.7.

Given  $i, j \in \mathbb{Z}$ , write  $X_{i,j} = \mathcal{E}_{i,j} - \mathcal{E}_{i+1,j}$ . By convention,  $e_A = 0$  unless  $A \in \Lambda_1(n, r)$ .

For  $i \in [1, n]$  and  $\lambda \in \Lambda_0(n, r)$ , write

$$E_{i,\lambda} = e_{D_\lambda + X_{i,i+1}},$$

$$F_{i,\lambda} = e_{D_\lambda - X_{i,i}}.$$

Define

$$E_i = \sum_{\lambda \in \Lambda_0(n, r): \lambda_{i+1} > 0} E_{i,\lambda}$$

$$F_i = \sum_{\lambda \in \Lambda_0(n, r): \lambda_i > 0} F_{i,\lambda}.$$

Observe that  $E_{i,\lambda} = 0$  unless  $\lambda_{i+1} > 0$  and  $F_{i,\lambda} = 0$  unless  $\lambda_i > 0$ . Also,  $E_{i,\lambda} = E_i 1_\lambda$  and  $F_{i,\lambda} = F_i 1_\lambda$ .

#### 4.1.2 Transpose involution

**Lemma 4.1.1.** *Transposition gives a homomorphism of  $\mathbb{Z}[q]$ -modules  $\top: \hat{S}_q(n, r) \rightarrow \hat{S}_q(n, r)$  with  $\top(e_A) = e_{A^\top}$ ,  $\top \circ \top = 1$  and  $\top(e_A e_B) = \top(e_B) \top(e_A)$ .*

*Proof.* Let  $A, B, C \in \Lambda_1(n, r)$  and let  $\mathbf{k}$  be a finite field with  $q = \#\mathbf{k}$  elements. If  $(L, L'') \in \mathcal{O}_C$  then  $(L'', L) \in \mathcal{O}_{C^\top}$  and

$$\begin{aligned}\gamma_{A,B,C;q} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;q}\end{aligned}$$

It then follows that  $\top(e_A e_B) = \top(e_B) \top(e_A)$ . □

### 4.1.3 A multiplication rule

**Lemma 4.1.2.** *Let  $i \in [1, n]$  and  $A \in \Lambda_1(n, r)$ .*

$$E_i e_A = \sum_{p \in \mathbb{Z}} q^{\sum_{j > p} a_{i,j}} [a_{i,p} + 1] e_{A+X_{i,p}}$$

and

$$F_i e_A = \sum_{p \in \mathbb{Z}} q^{\sum_{j < p} a_{i+1,j}} [a_{i+1,p} + 1] e_{A-X_{i,p}}.$$

Note that these formulas are still valid in the cases  $E_i e_A = 0$  and  $F_i e_A = 0$ . There are similar formulas for right multiplication by  $E_i$  and  $F_i$ , which can be obtained by applying the transpose involution to the above formulas. The transpose relates the  $E_i$ ,  $F_i$  and  $1_\lambda$  in the following way:  $\top(E_{i,\lambda}) = F_{i,\lambda}$ ,  $\top(F_{i,\lambda}) = E_{i,\lambda-\varepsilon_i+\varepsilon_{i+1}}$  and  $\top(1_\lambda) = 1_\lambda$ . In particular,  $\top(E_i) = F_i$  and  $\top(F_i) = E_i$ .

**Corollary 4.1.3.** *Let  $j \in [1, n]$  and  $A \in \Lambda_1(n, r)$ . Then*

$$e_A F_j = \sum_{p \in \mathbb{Z}} q^{\sum_{i > p} a_{i,j}} [a_{p,j} + 1] e_{A+X_{j,p}^\top}$$

and

$$e_A E_j = \sum_{p \in \mathbb{Z}} q^{\sum_{i < p} a_{i,j+1}} [a_{p,j+1} + 1] e_{A-X_{j,p}^\top}$$

*Proof.*

$$\begin{aligned}e_A F_j &= \top(E_j e_{A^\top}) \\ &= \top\left(\sum_p q^{\sum_{i > p} a_{i,j}} [a_{p,j} + 1] e_{A^\top + X_{j,p}}\right) \\ &= \sum_p q^{\sum_{i > p} a_{i,j}} [a_{p,j} + 1] e_{A+X_{j,p}^\top}\end{aligned}$$

$$\begin{aligned}e_A E_j &= \top(F_j e_{A^\top}) \\ &= \top\left(\sum_p q^{\sum_{i < p} a_{i,j+1}} [a_{p,j+1} + 1] e_{A^\top - X_{j,p}}\right) \\ &= \sum_p q^{\sum_{i < p} a_{i,j+1}} [a_{p,j+1} + 1] e_{A-X_{j,p}^\top}\end{aligned}$$

□

## 4.2 Relations

Note that  $E_i^{r+1} = F_i^{r+1} = 0$  while

$$E_i^r = [r]! e_{r\mathcal{E}_{i,i+1}}$$

and

$$F_i^r = [r]! e_{r\mathcal{E}_{i+1,i}}.$$

**Lemma 4.2.1** (quantum Serre relations:  $n \geq 3$ ). *Suppose  $n \geq 3$ . The following relations hold in  $\hat{S}_q(n, r)$ :*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless  $j = i \pm 1$ ;

$$E_i E_{i+1}^2 - (1+q)E_{i+1}E_i E_{i+1} + qE_{i+1}^2 E_i = 0$$

$$E_i^2 E_{i+1} - (1+q)E_i E_{i+1} E_i + qE_{i+1} E_i^2 = 0$$

and

$$F_{i+1} F_i^2 - (1+q)F_i F_{i+1} F_i + qF_i^2 F_{i+1} = 0$$

$$F_{i+1}^2 F_i - (1+q)F_{i+1} F_i F_{i+1} + qF_i F_{i+1}^2 = 0.$$

*Proof.* Here we introduce temporary notation for the basis elements: Write  $[A] = e_A$ .

Take  $\lambda \in \Lambda_0(n, r)$ .

$$E_i E_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1} E_i E_{i+1} 1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2 E_i 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then

$$(E_i E_{i+1}^2 - (1+q)E_{i+1}E_i E_{i+1} + qE_{i+1}^2 E_i) 1_\lambda = 0,$$

for each  $\lambda \in \Lambda_0(n, r)$ . The relation  $E_i E_{i+1}^2 - (1+q)E_{i+1}E_i E_{i+1} + qE_{i+1}^2 E_i = 0$  then follows.

The relations between  $F_i$  and  $F_{i+1}$  may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping  $E_i$  and  $F_i$  and reversing the order of multiplication.  $\square$

**Lemma 4.2.2** (quantum Serre relations:  $n = 2$ ). *In the case  $n = 2$ , the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

**Lemma 4.2.3.**  $[E_i, F_j] = 0$  unless  $j = i$ .

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0(n, r)} ([\lambda_i] - [\lambda_{i+1}]) 1_\lambda.$$

For  $\lambda \in \Lambda_0(n, r)$ , let  $R_\lambda = e_{\lambda_1 \varepsilon_{0,1} + \dots + \lambda_n \varepsilon_{n-1,n}}$ . Write  $R = \sum_{\lambda \in \Lambda_0(n, r)} R_\lambda$ . Note  $R_\lambda = R 1_\lambda$ . Given  $A \in \Lambda_1(n, r)$  and  $m \in \mathbb{Z}$ , let  $A[m] \in \Lambda_1(n, r)$  be given by  $A[m]_{i,j} = a_{i,j+m}$  and let  $A^{[m]}$  be given by  $A^{[m]}_{i,j} = a_{i+m,j}$  for each  $i \in \mathbb{Z}$ .

**Lemma 4.2.4** (Shifting). *If  $A \in \Lambda_1(n, r)$  then*

$$R e_A = e_{A[\pm 1]}$$

and

$$e_A R = e_{A[\pm 1]}.$$

Conjugation by  $R$  gives an automorphism  $\rho$  of  $\hat{S}_q(n, r)$  satisfying  $\rho^n = 1$ .

### 4.3 quivers with relations

Denote by  $\Lambda_0(n, r)$  the set of compositions of  $r$  into  $n$  parts. That is,  $\Lambda_0(n, r)$  is the set of  $\alpha \in \mathbb{Z}^n$  with non-negative entries which sum to  $r$ . Let  $\varepsilon_i \in \mathbb{Z}^n$  be the  $i$ th elementary vector and write  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for each  $i \in [1, n]$ . Then  $\lambda + \alpha_i \in \Lambda_0(n, r)$  if  $\lambda_{i+1} > 0$  and  $\lambda - \alpha_i \in \Lambda_0(n, r)$  if  $\lambda_i > 0$ .

Let  $\Gamma = \Gamma(n, r)$  be the quiver with set of vertices  $\Lambda_0(n, r)$ , with the following arrows:

For  $\lambda \in \Lambda_0(n, r)$  and  $i \in [1, n]$ , there is an arrow  $e_{i,\lambda} : \lambda \rightarrow \lambda + \alpha_i$  if  $\lambda_{i+1} > 0$  and there is an arrow  $f_{i,\lambda} : \lambda \rightarrow \lambda - \alpha_i$  if  $\lambda_i > 0$ .

Denote by  $\mathbb{Z}[q]\Gamma$  the path  $\mathbb{Z}[q]$ -algebra of  $\Gamma$ . Thus  $\mathbb{Z}[q]\Gamma$  is a free  $\mathbb{Z}[q]$ -module with a basis given by the set of paths in  $\Gamma$ , with multiplication given by the concatenation of paths. If  $p$  starts where  $q$  ends, the product  $pq$  is the path  $q$  followed by  $p$ . Write  $e_{i,\lambda} = 0$  unless  $\lambda, \lambda + \alpha_i \in \Lambda_0(n, r)$  and write  $f_{i,\lambda} = 0$  unless  $\lambda, \lambda - \alpha_i \in \Lambda_0(n, r)$ .

By construction, there is a homomorphism of  $\mathbb{Z}[q]$ -algebras

$$\phi : \mathbb{Z}[q]\Gamma \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda, \end{aligned}$$

for  $i \in [1, n]$  and  $\lambda \in \Lambda_0(n, r)$ .

The image of  $\phi$  is the subalgebra of  $\hat{S}_q(n, r)$  generated by  $E_i, F_i$  for  $i \in [1, n]$  and  $1_\lambda$  for  $\lambda \in \Lambda_0(n, r)$ , since  $E_{i,\lambda} = E_i 1_\lambda$  and  $F_{i,\lambda} = F_i 1_\lambda$ , while  $E_i = \sum_\lambda E_{i,\lambda}$  and  $F_i = \sum_\lambda F_{i,\lambda}$ . In general  $\phi$  is not surjective, so this does not always lead to a presentation of  $\hat{S}_q(n, r)$ .

#### 4.3.1 Exceptional case $n = 2$ .

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the  $q$ -Schur algebra.

### 4.3.2 Typical case $n > 2$ .

Suppose  $n \geq 3$ . Then  $\Gamma = \Gamma(n, r)$  has vertex set  $\Lambda_0(n, r)$ . **RESUME HERE...**

Define  $e_i, f_i \in \mathbb{Z}[q]\Gamma(n, r)$  by

$$e_i = \sum_{\lambda \in \Lambda_0(n, r)} e_{i, \lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0(n, r)} f_{i, \lambda},$$

with the convention  $e_{i, \lambda} = 0$  unless  $\lambda_{i+1} > 0$  and  $f_{i, \lambda} = 0$  unless  $\lambda_i > 0$ . Let  $k_\lambda$  denote the constant path at vertex  $\lambda$ .  $\{k_\lambda : \lambda \in \Lambda_0(n, r)\}$  is a set of pairwise orthogonal idempotents in  $\mathbb{Z}[q]\Gamma(n, r)$ .

Let  $I(n, r) \subset \mathbb{Z}[q]\Gamma(n, r)$  be the ideal generated by the expressions

$$\begin{aligned} & e_i e_{i+1}^2 - (1+q)e_{i+1}e_i e_{i+1} + qe_{i+1}^2 e_i \\ & e_i^2 e_{i+1} - (1+q)e_i e_{i+1} e_i + qe_{i+1} e_i^2 \\ & f_{i+1} f_i^2 - (1+q)f_i f_{i+1} f_i + qf_i^2 f_{i+1} \\ & f_{i+1}^2 f_i - (1+q)f_{i+1} f_i f_{i+1} + qf_i f_{i+1}^2 \\ & e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0(n, r)} ([\lambda_i] - [\lambda_{i+1}]) k_\lambda \end{aligned}$$

Recall that a relation is a  $\mathbb{Z}[q]$ -linear combination of paths with common start and end vertices. The relations involving paths  $\lambda \rightarrow \mu$  are given by  $1_\mu \text{expr} 1_\lambda$ , for each of the above expressions.

**Lemma 4.3.1.** *There is a homomorphism of  $\mathbb{Z}[q]$ -algebras*

$$\phi: \mathbb{Z}[q]\Gamma(n, r)/I(n, r) \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i, \lambda}) &= E_{i, \lambda} \\ \phi(f_{i, \lambda}) &= F_{i, \lambda} \\ \phi(k_\lambda) &= 1_\lambda. \end{aligned}$$

## Chapter 5

# A generic affine algebra

### 5.1 Introducing the generic affine algebra

Assume  $\mathbf{k} = \mathbb{C}$  and fix  $n, r \geq 1$ . Let  $\mathcal{S}$  be the  $\mathbf{k}$ -algebra  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$  and let  $\mathcal{R}$  be the subalgebra generated by  $\varepsilon$ , namely  $\mathcal{R} = \mathbf{k}[\varepsilon]$ . Let  $V$  be a free  $\mathcal{S}$ -module of rank  $r$  and let  $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$  be the set of  $n$ -periodic cyclic flags in  $V$ ; so  $\mathcal{F}$  consists of collections  $L = (L_i)_{i \in \mathbb{Z}}$  of  $\mathcal{R}$ -lattices in  $V$  with  $L_i \subset L_{i+1}$  for  $i \in \mathbb{Z}$  and  $\varepsilon L_i = L_{i-n}$  for  $i \in \mathbb{Z}$ .

Let  $G$  be the group of  $\mathcal{S}$ -module automorphisms of  $V$ . Thus  $G$  is isomorphic to  $\mathrm{GL}_r(\mathcal{S})$ .  $G$  acts on  $\mathcal{F}$  with orbits  $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0(n, r)\}$ , where  $\Lambda_0(n, r)$  is the set of compositions of  $r$  into  $n$  parts, as in Definition 3.0.1.

The diagonal action of  $G$  on  $\mathcal{F} \times \mathcal{F}$  has orbits  $\{\mathcal{O}_A : A \in \Lambda_1(n, r)\}$ , where  $\mathcal{O}_A$  consists of those pairs of flags with periodic characteristic matrix equal to  $A$ . Definitions of the periodic characteristic matrix and the set  $\Lambda_1(n, r)$  are given in Definition 3.1.1 and Definition 3.0.2 respectively. In particular, the periodic characteristic matrix of a pair  $(L, L') \in \mathcal{F} \times \mathcal{F}$  is the  $\mathbb{Z} \times \mathbb{Z}$  matrix  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ , with

$$a_{i,j} = \dim \left( \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right)$$

for each  $i, j \in \mathbb{Z}$ .

#### 5.1.1 De-linearisation

Recall that  $\mathrm{ro}, \mathrm{co}: \Lambda_1(n, r) \rightarrow \Lambda_0(n, r)$  are given by

$$\mathrm{ro} A = \left( \sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co} A = \left( \sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right)$$

for each  $A \in \Lambda_1(n, r)$ .

There has to be a better way to phrase this. Logically, the main result of this chapter will be ‘there exists a category with objects  $\Lambda_0(n, r)$ ; morphisms  $\Lambda_1(n, r)$ ; composition  $*$ ’, then finally the generic affine algebra is defined as the  $\mathbb{Z}$ -algebra of this category.



Given  $A \in \Lambda_1(n, r)$ , write  $A: \text{co } A \rightarrow \text{ro } A$ . The purpose of this chapter is to define a category with objects  $\Lambda_0(n, r)$  and morphisms  $\Lambda_1(n, r)$ ; where  $\text{Hom}(\lambda, \mu) = \{A \in \Lambda_1(n, r) : \text{ro } A = \mu, \text{co } A = \lambda\}$ . Given  $A, B \in \Lambda_1(n, r)$  let  $\Lambda_1(n, r)_{A, B}$  be the set of  $C \in \Lambda_1(n, r)$  such that there exist  $L, L', L'' \in \mathcal{F}$  with  $(L, L') \in \mathcal{O}_A$ ,  $(L', L'') \in \mathcal{O}_B$  and  $(L, L'') \in \mathcal{O}_C$ . It will be shown that  $\Lambda_1(n, r)$  admits a partial order  $\leq$  such that, given  $A, B \in \Lambda_1(n, r)$  with  $\text{ro } B = \text{co } A$ ,  $\Lambda_1(n, r)_{A, B}$  has a maximum element  $A * B$ . It will be shown that  $*$  is associative, leading to the construction of a category with the described properties.

The generic affine Schur algebra  $\hat{G}(n, r)$  will then be a  $\mathbb{Z}$ -algebra defined as a linearisation of this category. It will be shown that  $\hat{G}(n, r)$  gives a realisation of the affine 0-Schur algebra  $\hat{S}_0(n, r)$  when  $r < n$ . It is expected that a more refined presentation of the generic algebra and the 0-Schur algebra will allow the conditions on the parameters to be relaxed slightly: the  $r = n$  case is approachable, which may extend to the case  $r < 2n$ .

## 5.2 A partial order

Given  $i, j \in \mathbb{Z}$ , define a map  $d_{i, j}$  on  $\Lambda_1(n, r)$  by setting

$$d_{i, j}A = \sum_{s \leq i, t > j} a_{s, t}$$

for each  $A \in \Lambda_1(n, r)$ .

**Lemma 5.2.1.** *Let  $A \in \Lambda_1(n, r)$ , with  $A = (a_{i, j})_{i, j \in \mathbb{Z}}$  and write  $d_{i, j} = d_{i, j}A$  for  $i, j \in \mathbb{Z}$ . Then*

$$d_{i, j} - d_{i-1, j} = \sum_{t > j} a_{i, t}$$

and

$$d_{i, j} - d_{i, j-1} = - \sum_{s \leq i} a_{s, j}.$$

*Proof.* Let  $i, j \in \mathbb{Z}$ . Then

$$d_{i, j} - d_{i-1, j} = \sum_{s \leq i, t > j} a_{s, t} - \sum_{s \leq i-1, t > j} a_{s, t} = \sum_{t > j} a_{i, t}.$$

Similarly,

$$d_{i, j} - d_{i, j-1} = \sum_{s \leq i, t > j} a_{s, t} - \sum_{s \leq i, t > j-1} a_{s, t} = - \sum_{s \leq i} a_{s, j}.$$

□

**Lemma 5.2.2.** *Let  $A \in \Lambda_1(n, r)$ , with  $A = (a_{i, j})_{i, j \in \mathbb{Z}}$  and write  $d_{i, j} = d_{i, j}A$  for each  $i, j \in \mathbb{Z}$ . Then*

$$a_{i, j} = d_{i, j-1} - d_{i-1, j-1} + d_{i-1, j} - d_{i, j}$$

for each  $i, j \in \mathbb{Z}$ .

*Proof.* Using Lemma 5.2.1,

$$\begin{aligned} a_{i, j} &= \sum_{t > j-1} a_{i, t} - \sum_{t > j} a_{i, t} \\ &= (d_{i, j-1} - d_{i-1, j-1}) - (d_{i, j} - d_{i-1, j}). \end{aligned}$$

Alternatively,

$$\begin{aligned} a_{i,j} &= \sum_{s \leq i} a_{s,j} - \sum_{s \leq i-1} a_{s,j} \\ &= -(d_{i,j} - d_{i,j-1}) + (d_{i-1,j} - d_{i-1,j-1}). \end{aligned}$$

□

**Lemma 5.2.3.** *The relation  $\leq$  on  $\Lambda_1(n, r)$ , defined by  $A \leq B$  if and only if  $d_{i,j}A \leq d_{i,j}B$  for all  $i, j \in \mathbb{Z}$ , is a partial order.*

*Proof.* It is clear that  $\leq$  is reflexive and transitive, so it remains to see  $\leq$  is antisymmetric. Suppose  $A, B \in \Lambda_1(n, r)$  with  $A \leq B$  and  $B \leq A$ . Then  $d_{i,j}A = d_{i,j}B$  for each  $i, j \in \mathbb{Z}$ , which shows  $A = B$  as a result of Lemma 5.2.2. □

The partial order on  $\Lambda_1(n, r)$  induces a partial order on the set of  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$ , such that  $\mathcal{O}_A \leq \mathcal{O}_B$  if and only if  $A \leq B$ . The following lemma is rephrased from Lemma 3.1.4 and gives some geometric significance to the partial order on  $\Lambda_1(n, r)$ .

**Lemma 5.2.4.** *Let  $A \in \Lambda_1(n, r)$  and take  $(L, L') \in \mathcal{O}_A$ . Then*

$$d_{i,j}A = \dim \left( \frac{L_i}{L_i \cap L'_j} \right)$$

for each  $i, j \in \mathbb{Z}$ .

*Proof.* This is a rephrasing of Lemma 3.1.4. □

**Remark 1.** *It is thought\* that the partial order on  $\Lambda_1(n, r)$  is compatible with the degeneration order (or closure order) on  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$  when  $\mathbf{k} = \mathbb{C}$ . In particular, it is hoped that  $A \leq B$  if and only if  $\mathcal{O}_A \subset \overline{\mathcal{O}_B}$ .*

### 5.3 Preliminary results

Suppose  $A, B \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$ . Recall the notation

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

and

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L', L'') \in Y_{A,B}\}.$$

$X_{A,B}$  is the image of  $Y_{A,B}$  under the projection onto the first and last components.

**Lemma 5.3.1.** *There is  $N \in \mathbb{N}$  such that*

$$\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$$

whenever  $(L, L'') \in X_{A,B}$ .

*Proof.* There exist  $N_1, N_2 \in \mathbb{N}$  such that

$$\varepsilon^{N_1} L_0 \subset L'_0 \subset \varepsilon^{-N_1} L_0$$

and

$$\varepsilon^{N_2} L'_0 \subset L''_0 \subset \varepsilon^{-N_2} L'_0,$$

whenever  $(L, L', L'') \in Y_{A,B}$ . Then, for  $(L, L', L'') \in Y_{A,B}$ ,

$$L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-(N_1+N_2)} L_0$$

and

$$\varepsilon^{N_1+N_2} L_0 \subset \varepsilon^{N_2} L'_0 \subset L''_0.$$

In particular, taking  $N = N_1 + N_2$ , we have

$$\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$$

whenever  $(L, L'') \in X_{A,B}$ . □

**Lemma 5.3.2.** Suppose  $N_1, N_2 \in \mathbb{N}$  with  $\varepsilon^{N_1} L_0 \subset L_0 \subset \varepsilon^{-N_1} L_0$  and  $\varepsilon^{N_2} L'_0 \subset L''_0 \subset \varepsilon^{-N_2} L'_0$  whenever  $(L, L', L'') \in Y_{A,B}$  and let  $N = N_1 + N_2$ . Then

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right) = d_{nN_1,0}(A) + d_{nN_2,0}(B)$$

and

$$\dim \left( \frac{L''_0}{\varepsilon^N L_0} \right) = 2Nr - d_{nN_1,0}(A) + d_{nN_2,0}(B),$$

whenever  $(L, L'') \in X_{A,B}$ .

*Proof.* Suppose  $(L, L'') \in X_{A,B}$  and  $L' \in \mathcal{F}$  so that  $(L, L', L'') \in Y_{A,B}$ . As in lemma 5.3.1,  $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$ , so

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right) + \dim \left( \frac{L''_0}{\varepsilon^N L_0} \right) = \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0} \right).$$

As a  $\mathbf{k}$ -vector space,  $\varepsilon^{-N} L_0 / \varepsilon^N L_0$  is isomorphic to  $(L_0 / \varepsilon L_0)^{2N}$ , which has dimension  $2Nr$ , so

$$\dim \left( \frac{L''_0}{\varepsilon^N L_0} \right) = 2Nr - \dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right).$$

It remains to compute the codimension of  $L''_0$  in  $\varepsilon^{-N} L_0$ . Note  $L''_0 \subset \varepsilon^{-N_2} L'_0 \subset \varepsilon^{-N} L_0$ , so

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right) = \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0} \right) + \dim \left( \frac{\varepsilon^{-N_2} L'_0}{L''_0} \right).$$

$$\begin{aligned} \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^{-N_2} L'_0} \right) &= \dim \left( \frac{\varepsilon^{-N_1} L_0}{L'_0} \right) \\ &= \dim \left( \frac{L_{nN_1}}{L_{nN_1} \cap L'_0} \right) \\ &= \sum_{s \leq nN_1, t > 0} A_{s,t} \\ &= d_{nN_1,0}(A). \end{aligned}$$

$$\begin{aligned}
\dim \left( \frac{\varepsilon^{-N_2} L'_0}{L''_0} \right) &= \dim \left( \frac{L'_{nN_2}}{L'_{nN_2} \cap L''_0} \right) \\
&= \sum_{s \leq nN_2, t > 0} B_{s,t} \\
&= d_{nN_2,0}(B).
\end{aligned}$$

□

### 5.3.1 A quasi-projective variety

Fix  $L \in \mathcal{F}$ . Given  $N \in \mathbb{N}$  and  $\lambda \in \Lambda_0(n, r)$ , define

$$\Pi_{N,\lambda} = \{L'' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'' \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi_{N,\lambda}^a = \left\{ L'' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'' \subset \varepsilon^{-N} L_0, \dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right) = a \right\}.$$

$\Pi_{N,\lambda}$  is the (disjoint) union of the  $\Pi_{N,\lambda}^a$  for  $a \in \mathbb{N}$ . In fact, we will see  $\Pi_{N,\lambda}^a$  is empty whenever  $a > 2Nr$ .

**Lemma 5.3.3.** *Let  $N, a \in \mathbb{N}$ ,  $\lambda \in \Lambda_0(n, r)$ . Then  $\Pi_{N,\lambda}^a$  is nonempty exactly when  $0 \leq a \leq 2Nr$ .*

*Proof.* Suppose  $L'' \in \Pi_{N,\lambda}$ . By definition,  $\varepsilon^{-N} L_0 \subset L'' \subset \varepsilon^{-N} L_0$ , which shows

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L''_0} \right) \leq \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0} \right) = 2Nr.$$

Therefore,  $\Pi_{N,\lambda}^a$  is empty unless  $a \leq 2Nr$ .

Now assume  $0 \leq a \leq 2Nr$ . We may choose an  $\varepsilon$ -invariant subspace  $W'$  of  $W = \varepsilon^{-N} L_0 / \varepsilon^N L_0$  of codimension  $a$ .  $W'$  lifts to give a  $\mathcal{R}$ -module, say  $L''_0$ , with  $\varepsilon^N L_0 \subset L''_0 \subset \varepsilon^{-N} L_0$  and with  $\dim(\varepsilon^{-N} L_0 / L''_0) = \dim(W/W') = a$ . Similarly, a flag of type  $\lambda$  in  $L''_0 / \varepsilon L''_0$  lifts to give  $\mathcal{R}$ -modules  $(L''_{-n+1}, \dots, L''_0)$  with

$$\varepsilon L''_0 \subset L''_{-n+1} \subset \dots \subset L''_{-1} \subset L''_0 \subset \varepsilon^{-N} L_0$$

and such that the dimensions of successive quotients are given by  $\lambda_1, \dots, \lambda_n, a$ , from left to right. Thus,  $(L''_{-n+1}, \dots, L''_0)$  extends by periodicity to give an element of  $\Pi_{N,\lambda}^a$ , as desired. □

**Lemma 5.3.4.** *Given  $\lambda \in \Lambda_0(n, r)$ ,  $N \in \mathbb{N}$  and  $a \in \mathbb{Z}$  with  $0 \leq a \leq 2Nr$ ,  $\Pi_{N,\lambda}^a$  is a quasi-projective variety.*

*Proof.* Let  $W = \varepsilon^{-(1+N)} L_0 / \varepsilon^N L_0$  and let

$$X = \left\{ W_1 \leq \dots \leq W_n \leq W : \dim \left( \frac{W}{W_n} \right) = a, \dim \left( \frac{W_i}{W_{i-1}} \right) = \lambda_i \text{ for } i = 2, \dots, n \right\}.$$

$X$  is known to be a projective variety [CITATION NEEDED]

Let  $X'$  be the subset of  $X$  consisting of those  $(W_1, \dots, W_n)$ , where each  $W_i$  is  $\varepsilon$ -invariant and  $\varepsilon W_n \leq W_1$ .  $X'$  is a closed subset of  $X$ , though is not necessarily irreducible.

The correspondence between the set of  $\mathcal{R}$ -submodules of  $\varepsilon^{-(1+N)}L_0$  which contain  $\varepsilon^N L_0$  and the set of  $\mathcal{R}$ -submodules of  $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$  gives a pair of mutually inverse maps  $\Pi_{N,\lambda}^a \leftrightarrow X'$ .

– the idea that is relevant to the proof is that inclusion relations  $L_i \subset L_{i+1}$  describe a closed set in a product of grassmanians. Unsure here – Is it true that irreducible components of  $X'$  are projective varieties. In this case, should the statement be that  $\Pi_{N,\lambda}^a$  is a projective algebraic set, rather than a quasi projective variety?  $\square$

### 5.3.2 Geometry of orbit products

Refer to Section 3.1.1 for definitions of the spaces  $X_{A,B}^L$  and  $Y_{A,B}^L$ .

**Lemma 5.3.5.** *Given  $A, B \in \Lambda_1(n, r)$  with  $\text{ro } B = \text{co } A$  and  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_A$  and  $(L', L'') \in \mathcal{O}_B$ ,*

$$X_{A,B}^L = G_L G_{L'} L''.$$

*Proof.*  $X_{A,B}^L$  is the image of  $Y_{A,B}^L$  under the forgetful map  $(N', N'') \mapsto N''$ . If  $\alpha \in G_L$  and  $\beta \in G_{L'}$  then  $(L, \alpha L, \alpha \beta L'') \in Y_{A,B}$  since  $(L, \alpha L') = \alpha(L, L') \in \mathcal{O}_A$  and  $(\alpha L', \alpha \beta L'') = \alpha \beta (\beta^{-1} L', L'') = \alpha \beta (L', L'') \in \mathcal{O}_B$ . Consequently,  $G_L G_{L'} L'' \in X_{A,B}^L$ .

For the reverse inclusion, if  $(N', N'') \in Y_{A,B}^L$  then  $(L, N') \in \mathcal{O}_A$  and  $(N', N'') \in \mathcal{O}_B$ , so there exist  $\sigma_1, \sigma_2 \in G$  such that  $(L, N') = \sigma_1(L, L')$  and  $(N', N'') \in \sigma_2(N', N'')$ . Then  $(L, N', N'') = (L, \sigma_1 L', \sigma_1(\sigma_1^{-1} \sigma_2) L'')$  with  $\sigma_1 \in G_L$  and  $\sigma_1^{-1} \sigma_2 \in G_{L'}$ . Thus  $X_{A,B}^L = G_L G_{L'} L''$ .  $\square$

Given  $N \in \mathbb{N}$ , define

$$H_N = \left\{ h \in G_L : h = 1 \text{ on } \frac{\varepsilon^{-(1+N)} L_0}{\varepsilon^N L_0} \right\}.$$

Explicitly, the condition  $h = 1$  on  $\varepsilon^{-(1+N)} L_0 / \varepsilon^N L_0$  means:  $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$  for  $x \in \varepsilon^{-(1+N)} L_0$ . Observe that  $H_{N+1} \subset H_N$  for  $N \in \mathbb{N}$  since  $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$  whenever  $x \in \varepsilon^{-(1+N)} L_0$ .

**Lemma 5.3.6.**  *$H_N$  is a normal subgroup in  $G_L$ , for any  $N \in \mathbb{N}$ .*

*Proof.*  $H_N \subset G_L$  by definition. Suppose  $h, h' \in H_N$  and let  $x \in \varepsilon^{-(1+N)} L_0$ .  $h'(x) \in \varepsilon^{-(1+N)} L_0$  as  $h' \in G_L$ , so  $hh'(x) + \varepsilon^N L_0 = h'(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ , which shows  $hh' \in H_N$ .  $h(x) - x \in \varepsilon^N L_0$ , so  $h^{-1}(x) - x = -h^{-1}(h(x) - x) \in \varepsilon^N L_0$ .  $h^{-1} \in H_N$ , so  $H_N$  is a subgroup of  $G_L$ .

Let  $g \in G_L$ .  $hg^{-1}(x) + \varepsilon^N L_0 = g^{-1}(x) + \varepsilon^N L_0$  as  $g^{-1}(x) \in \varepsilon^{-(1+N)} L_0$ , so  $ghg^{-1}(x) + \varepsilon^N L_0 = gg^{-1}(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ . Thus  $ghg^{-1} \in H_N$ , which proves  $H_N$  is a normal subgroup in  $G_L$ .  $\square$

The  $H_N$  form a descending chain of normal subgroups in  $G_L$ :  $\cdots \subset H_1 \subset H_0 \subset G_L \subset G$ .

**Lemma 5.3.7.**  *$G_L/H_N$  is an irreducible algebraic group for any  $N \in \mathbb{N}$ .*

*Proof.* See the discussion in [2][section 4]. Should be able to give an explicit presentation of  $G_L/H_N$  in terms of the block structure.

$\sigma \in G_L$  induces an automorphism  $\bar{\sigma}$  of  $\varepsilon^{-N} L_0 / \varepsilon^N L_0$ , with inverse induced by  $\sigma^{-1}$ . Moreover, the natural map

$$G_L/H \rightarrow GL(\varepsilon^{-N} L_0 / \varepsilon^N L_0)$$

is a group homomorphism. In fact, this homomorphism is injective: if  $\sigma = \tau$  on  $\varepsilon^{-N}L_0/\varepsilon^N L_0$ , then  $\sigma\tau^{-1} = 1$  on  $\varepsilon^{-N}L_0/\varepsilon^N L_0$  and so  $\sigma H = \tau H$ . Thus  $G_L/H$  is isomorphic to its image in  $\text{GL}(\varepsilon)^{-N}L_0/\varepsilon^N L_0$ .  $\square$

**Lemma 5.3.8.** *There is  $N \in \mathbb{N}$  such that  $H_N \subset G_{L'}$ . Consequently,  $H_{N'} \subset G_{L'}$  whenever  $N' \geq N$ .*

*Proof.* Choose  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ . Then

$$\varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset \cdots \subset L'_n \subset \varepsilon^{-(1+N)} L_0.$$

Let  $h \in H_N$ .  $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$  for  $x \in \varepsilon^{-(1+N)} L_0$ , so  $h(L'_i) \subset L'_i$  for  $i = 0, 1, \dots, n$ . Moreover,  $h^{-1}$  stabilises  $L'_i$ , so  $h(L'_i) = L'_i$  for  $i = 0, 1, \dots, n$  and therefore for  $i \in \mathbb{Z}$ . This shows  $h \in G_{L'}$  as required, so  $H_N \subset G_{L'}$ .  $\square$

Note that  $H_N$  is generally not a normal subgroup of  $G_{L'}$ , though the space of (right) cosets of  $H_N$  in  $G_{L'}$  will still be irreducible. [ADD AN EXAMPLE](#)

**Lemma 5.3.9.**  *$G_{L'}/H_N$  is irreducible, provided  $H_N \subset G_{L'}$ .*

*Proof.* Needs a proof.  $\square$

**Lemma 5.3.10.** *Given  $L \in \mathcal{F}$ , the  $G_L$ -orbits in  $\mathcal{F}$  are locally closed.*

*Proof.* Look at proposition 8.3 "Closed Orbits" in [1], which shows that the orbits under an algebraic group action are locally closed.  $\square$

**Lemma 5.3.11.** *Given  $A, B \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $X_{A,B}^L$  is an irreducible topological space.*

*Proof.* Write up this proof properly - this is only a sketch. There is  $N \in \mathbb{N}$  sufficiently large that  $X_{A,B}^L$  is contained in  $\Pi_{N, \text{co } B}$ , using Lemma 5.3.1. Suppose  $(L, L') \in \mathcal{O}_A$ , then  $X_{A,B}^L = G_L X_B^{L'}$ .  $G_L$  acts on  $\Pi_{N, \lambda}$  through a quotient  $G_L/H$  which is an irreducible algebraic group, as a result of Lemma 5.3.7.  $X_B^{L'}$  is an irreducible subspace of  $\Pi_{N, \lambda}$ .  $X_{A,B}^L$  is the image of an irreducible subspace of  $\Pi_{N, \lambda}$  under the action of a connected algebraic group, so  $X_{A,B}^L$  is irreducible.  $\square$

**Proposition 5.3.12.** *Given  $A, B \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $L \in \mathcal{F}_{\text{ro } A}$ , there is a unique open  $G_L$ -orbit in  $X_{A,B}^L$ .*

*Proof.*  $X_{A,B}^L$  consists of finitely many  $G_L$ -orbits and is an irreducible topological space, by Lemma 5.3.11. Consequently,  $X_C^L$  is dense in  $X_{A,B}^L$  for some  $C \in \Lambda_1(n, r)_{A,B}$ . Lemma 5.3.10 shows that  $X_C^L$  is locally closed in  $X_{A,B}^L$ , so  $X_C^L$  is open in  $\overline{X_C^L} = X_{A,B}^L$ . Irreducibility of  $X_{A,B}^L$  shows that there is a unique open  $G_L$ -orbit, since two non-empty open sets in  $X_{A,B}^L$  intersect non-trivially, thus any two open  $G_L$  orbits in  $X_{A,B}^L$  coincide.  $\square$

## 5.4 Existence of a maximum

**Lemma 5.4.1.** *Given  $A, A' \in \Lambda_1(n, r)$  with  $\text{ro } A = \text{ro } A'$  and  $\text{co } A = \text{co } A'$ ,  $A' \leq A$  if and only if  $X_{A'}^L \subset \overline{X_A^L}$  for any  $L \in \mathcal{F}_{\text{ro } A}$ .*

*Proof.* **ADD PROOF** □

**Proposition 5.4.2.** *Given  $A, B \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$ ,  $\Lambda_1(n, r)_{A, B}$  has a maximum element.*

*Proof.* Let  $L \in \mathcal{F}_{\text{ro } A}$ .  $X_{A, B}^L$  is irreducible by Lemma 5.3.11 and is the union of finitely many  $G_L$ -orbits, namely

$$X_{A, B}^L = \bigcup_{C \in \Lambda_1(n, r)_{A, B}} X_C^L.$$

This shows that  $X_C^L$  is dense in  $X_{A, B}^L$  for some  $C \in \Lambda_1(n, r)_{A, B}$ . Lemma 5.3.10 shows that the  $G_L$ -orbits in  $X_{A, B}^L$  are locally closed, so a dense  $G_L$ -orbit is open in  $X_{A, B}^L$ . Lemma 5.4.1 shows that the characteristic matrix of the dense  $G_L$ -orbit is a maximum in  $\Lambda_1(n, r)_{A, B}$ . □

## 5.5 Associativity

Refer to Section 3.1.2 for definitions of the spaces  $X_{A, B, C}^L$  and  $Y_{A, B, C}^L$ .

**Lemma 5.5.1.** *Given  $A, B, C \in \Lambda_1(n, r)$  with  $\text{ro } C = \text{co } B$ ,  $\text{ro } B = \text{co } A$  and a tuple of flags  $(L, L', L'', L''') \in \mathcal{F}^4$  with  $(L, L') \in \mathcal{O}_A$ ,  $(L', L'') \in \mathcal{O}_B$  and  $(L'', L''') \in \mathcal{O}_C$ ,*

$$X_{A, B, C}^L = G_L G_{L'} G_{L''} L''''.$$

*Proof.* Given  $\alpha \in G_L$ ,  $\beta \in G_{L'}$  and  $\gamma \in G_{L''}$ ,  $(L, \alpha L', \alpha \beta L'', \alpha \beta \gamma L''') \in Y_{A, B, C}^L$  since  $(L, \alpha L') = \alpha(L, L') \in \mathcal{O}_A$ ,  $(\alpha L', \alpha \beta L'') = \alpha \beta(L', L'') \in \mathcal{O}_B$  and  $(\alpha \beta L'', \alpha \beta \gamma L''') = \alpha \beta \gamma(L'', L''') \in \mathcal{O}_C$ . This shows  $G_L G_{L'} G_{L''} L'''' \in X_{A, B, C}^L$ .

$X_{A, B, C}^L$  is the image of  $Y_{A, B, C}^L$  under the forgetful map  $(N', N'', N''') \mapsto N''''$ . Given  $(N', N'', N''') \in Y_{A, B, C}^L$ , there exist  $\sigma_1, \sigma_2, \sigma_3 \in G$  such that  $(L, N') = \sigma_1(L, L')$ ,  $(N', N'') = \sigma_2(L', L'')$  and  $(N'', N''') = \sigma_3(L'', L''')$ ; then  $N' = \sigma_1 L' = \sigma_2 L'$ ,  $N'' = \sigma_2 L'' = \sigma_3 L''$  and  $N''' = \sigma_3 L'''$ . Thus

$$(L, N', N'', N''') = (L, \sigma_1 L', \sigma_1(\sigma_1^{-1} \sigma_2) L'', \sigma_1(\sigma_1^{-1} \sigma_2)(\sigma_2^{-1} \sigma_3) L''')$$

where  $\sigma_1 \in G_L$ ,  $\sigma_1^{-1} \sigma_2 \in G_{L'}$  and  $\sigma_2^{-1} \sigma_3 \in G_{L''}$ . □

**Lemma 5.5.2.** *Given  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $Y_{A, B, C}^L$  is an irreducible topological space.*

*Proof – to be written.* □

**Corollary 5.5.3.** *Given  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $X_{A, B, C}^L$  is an irreducible topological space*

*Proof.*  $X_{A, B, C}^L$  is the image of  $Y_{A, B, C}^L$  under the forgetful map  $(L', L'', L''') \mapsto L''''$ , which is continuous. Lemma 5.5.2 shows that  $Y_{A, B, C}^L$  is irreducible, so it follows  $X_{A, B, C}^L$  is irreducible. □

**Definition 5.5.1.** Given  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ , define spaces

$$\tilde{Y}_{(AB)C}^L = f^{-1}X_{(A*B)*C}^L$$

$$\tilde{Y}_{A(BC)}^L = f^{-1}X_{A*(B*C)}^L.$$

**Lemma 5.5.4.** Given  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $\tilde{Y}_{(AB)C}^L$  is open in  $Y_{A,B,C}^L$ .

*Proof – to be written.* □

**Lemma 5.5.5.** Given  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $\tilde{Y}_{A(BC)}^L$  is open in  $Y_{A,B,C}^L$ .

*Proof – to be written.* □

**Lemma 5.5.6** (Conjecture??). Given  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and  $L \in \mathcal{F}_{\text{ro } A}$ ,  $X_{A*B,C}^L$  and  $X_{A,B*C}^L$  are open and dense in  $X_{A,B,C}^L$ .

*Proof – to be written.* Question: should this result be removed or changed into a conjecture, since it is stronger than the results above which will be used to prove associativity? In particular, if  $f$  is shown to be an open map then this result follows from Lemma 5.5.4 and Lemma 5.5.5. □

**Proposition 5.5.7.** Given  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$ ,  $(A * B) * C = A * (B * C)$ .

*Proof - original.* Take  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and fix  $L \in \mathcal{F}_{\text{ro } A}$ .  $X_{A,B,C}^L$  is irreducible, by Lemma 5.5.3, and is the union of finitely many disjoint locally closed subsets, namely

$$X_{A,B,C}^L = \bigcup_{D \in \Lambda_1(n, r)_{A,B,C}} X_D^L.$$

Therefore, exactly one of the  $X_D^L$  is open and dense in  $X_{A,B,C}^L$ .  $X_{A*B,C}^L$  is open and dense in  $X_{A,B,C}^L$ , by Lemma 5.5.6. It then follows that the maximum  $G_L$ -orbit  $X_{(A*B)*C}^L$  in  $X_{A*B,C}^L$  is open and dense in  $X_{A,B,C}^L$ . Similarly,  $X_{A*(B*C)}^L$  is open and dense in  $X_{A,B*C}^L$  which is in turn open and dense in  $X_{A,B,C}^L$ .  $X_{(A*B)*C}^L$  and  $X_{A*(B*C)}^L$  are both a single orbit for the action of  $G_L$  and intersect non-trivially since  $X_{A,B,C}^L$  is irreducible, therefore  $X_{(A*B)*C}^L = X_{A*(B*C)}^L$ , which means  $(A * B) * C = A * (B * C)$ . □

*Proof.* Take  $A, B, C \in \Lambda_1(n, r)$  with  $\text{co } A = \text{ro } B$  and  $\text{co } B = \text{ro } C$  and fix  $L \in \mathcal{F}_{\text{ro } A}$ . Let  $f: Y_{A,B,C}^L \rightarrow X_{A,B,C}^L$  be the restriction of the forgetful map  $(L', L'', L''') \mapsto L'''$ .  $Y_{A,B,C}^L$  is an irreducible topological space, by 5.5.2.

$$f^{-1}X_{A*B,C}^L = \left\{ (L', L'', L''') \in Y_{A,B,C}^L : \dim \left( \frac{L_i}{L_i \cap L_j''} \right) \text{ is maximal, for each } i, j \in \mathbb{Z} \right\}$$

is open in  $Y_{A,B,C}^L$  since  $f^{-1}X_{A*B,C}^L$  is given by **finitely many open conditions**.

$X_{(A*B)*C}^L$  is open in  $X_{A*B,C}^L$ , so  $f^{-1}X_{(A*B)*C}^L$  is open in  $f^{-1}X_{A*B,C}^L$ , which is open in  $Y_{A,B,C}^L$ , so  $f^{-1}X_{(A*B)*C}^L$  is open and dense in  $Y_{A,B,C}^L$ .



$$f^{-1}X_{A,B*C}^L = \left\{ (L', L'', L''') \in Y_{A,B,C}^L : \dim \left( \frac{L'_i}{L'_i \cap L'''_j} \right) \text{ is maximal, for each } i, j \in \mathbb{Z} \right\}$$

is open in  $Y_{A,B,C}^L$ , **being determined by finitely many open conditions.**

$X_{A*(B*C)}^L$  is open in  $X_{A,B*C}^L$ , so  $f^{-1}X_{A*(B*C)}^L$  is open in  $f^{-1}X_{A,B*C}^L$ , which is open in  $Y_{A,B,C}^L$ , so  $f^{-1}X_{A*(B*C)}^L$  is open in  $Y_{A,B,C}^L$ . Then  $f^{-1}X_{(A*B)*C}^L$  and  $f^{-1}X_{A*(B*C)}^L$  are both nonempty and open in the irreducible space  $Y_{A,B,C}^L$ , so  $f^{-1}X_{(A*B)*C}^L \cap f^{-1}X_{A*(B*C)}^L$  is non-empty. It follows that the  $G_L$ -orbits  $X_{(A*B)*C}^L$  and  $X_{A*(B*C)}^L$  coincide and thus  $(A * B) * C = A * (B * C)$ .  $\square$

## 5.6 The generic algebra

**Lemma 5.6.1.** *Given  $\lambda \in \Lambda_0(n, r)$  and  $A \in \Lambda_1(n, r)$ ,  $D_\lambda * A = A$  if  $\text{ro } A = \lambda$  and  $A * D_\lambda = A$  if  $\text{co } A = \lambda$ .*

*Proof.* Lemma 3.1.7 shows that  $\Lambda_1(n, r)_{D_\lambda, A} = \{A\}$  if  $\lambda = \text{ro } A$  and  $\Lambda_1(n, r)_{A, D_\lambda} = \{A\}$  if  $\lambda = \text{co } A$ , which proves the result.  $\square$

**Theorem 5.6.2.** *The following constitutes a small category: the set of objects is  $\Lambda_0(n, r)$  and the set of morphisms is  $\Lambda_1(n, r)$ . Given compositions  $\lambda, \mu \in \Lambda_0(n, r)$ , the morphisms with source  $\lambda$  and target  $\mu$  are those matrices  $A \in \Lambda_1(n, r)$  with  $\text{co } A = \lambda$  and  $\text{ro } A = \mu$ . Given  $\lambda, \mu, \nu \in \Lambda_0(n, r)$  and  $A, B \in \Lambda_1(n, r)$  with  $B: \lambda \rightarrow \mu$  and  $A: \mu \rightarrow \nu$  the composition is  $A * B: \lambda \rightarrow \nu$ .*

*Proof.* Proposition 5.4.2 shows that the composition is well defined while Proposition 5.5.7 establishes associativity of the composition. Lemma 5.6.1 shows that  $D_\lambda: \lambda \rightarrow \lambda$  is the identity morphism for each  $\lambda \in \Lambda_0(n, r)$ . Thus  $(\Lambda_0(n, r), \Lambda_1(n, r), \text{co}, \text{ro}, *)$  is a category.  $\square$

Write  $\mathcal{G}(n, r)$  to denote this so-called ‘generic category’.

**Example 1.** *The objects in  $\mathcal{G}(2, 2)$  are compositions of 2 into 2 parts, namely  $(0, 2)$ ,  $(1, 1)$  and  $(2, 0)$ . The set of morphisms from  $\lambda$  to  $\mu$  is the set of infinite periodic matrices  $A \in \Lambda_1(2, 2)$  with  $\text{co } A = \lambda$  and  $\text{ro } A = \mu$ , which is a countably infinite set for any pair of compositions  $\lambda, \mu \in \Lambda_0(2, 2)$ .*

**Definition 5.6.1** (Generic algebra). *The generic affine algebra  $\hat{G}(n, r)$  is the category  $\mathbb{Z}$ -algebra of  $\mathcal{G}(n, r)$ . In particular,  $\hat{G}(n, r)$  is a free  $\mathbb{Z}$ -module with basis  $\{e_A : A \in \Lambda_1(n, r)\}$  and with associative multiplication given by*

$$e_A * e_B = \begin{cases} e_{A*B} & \text{if } \text{co } A = \text{ro } B \\ 0 & \text{if } \text{co } A \neq \text{ro } B. \end{cases}$$

*The multiplicative identity in  $\hat{G}(n, r)$  is*

$$1 = \sum_{\lambda \in \Lambda_0(n, r)} e_{D_\lambda}.$$

## Chapter 6

# A realisation of affine zero Schur algebras

We aim to prove the isomorphism theorem in the cases  $r < n$  and  $n \leq r < 2n$  separately. Below are crude versions of the statements we want to prove.

**Theorem 6.0.1.** *Assume  $r < n$ . The map  $\psi: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$ , given by  $\psi(E_i) = E_i$ ,  $\psi(F_i) = F_i$  and  $\psi(1_\lambda) = 1_\lambda$ , is an isomorphism of  $\mathbb{Z}$ -algebras.*

**Theorem 6.0.2.** *Assume  $n \leq r < 2n$ . There is a unique homomorphism of  $\mathbb{Z}$ -algebras  $\hat{\psi}: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$  such that  $\hat{\psi}(R) = R$  and  $\hat{\psi} = \psi$  on the subalgebra of  $\hat{G}(n, r)$  generated by the  $E_i$ ,  $F_i$  and  $1_\lambda$ .  $\hat{\psi}$  is an isomorphism of  $\mathbb{Z}$ -algebras.*

## 6.1 Preliminary results

### 6.1.1 Elementary basis elements

Give notation for the elementary basis elements  $E_{i,\lambda}$ ,  $F_{i,\lambda}$  and  $1_\lambda$ .

### 6.1.2 Transpose involution

**Lemma 6.1.1.** *The  $\mathbb{Z}$ -module automorphism  $\top$  of  $\hat{G}(n, r)$  given by  $e_A \mapsto e_{A^\top}$  is a  $\mathbb{Z}$ -algebra antihomomorphism: that is,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A$$

for each  $A, B \in \Lambda_1(n, r)$ . Moreover,  $\top(E_{i,\lambda}) = F_{i,\lambda+\alpha_i}$ ,  $\top(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}$  and  $\top(1_\lambda) = 1_\lambda$ , for permissible  $(i, \lambda) \in \mathbb{Z} \times \Lambda_0(n, r)$ .

*Proof.* This is a consequence of Lemma 4.1.1. It must also be shown that the transpose operation on  $\Lambda_1(n, r)$  is order preserving.  $\square$

### 6.1.3 Multiplication rules

Write

$$E_i = \sum_{\lambda \in \Lambda_0(n, r): \lambda_{i+1} > 0} E_{i,\lambda}$$

$$F_i = \sum_{\lambda \in \Lambda_0(n, r): \lambda_i > 0} F_{i,\lambda}.$$

Then  $E_{i,\lambda} = E_i 1_\lambda$  and  $F_{i,\lambda} = F_i 1_\lambda$ .

**Lemma 6.1.2.** *Let  $A \in \Lambda_1(n, r)$ ,  $i \in [1, n]$  and let  $\lambda = \text{ro } A$ . The following multiplication rules hold:*

$$E_i e_A = \begin{cases} e_{A+X_{i,p}} & \text{if } \lambda_{i+1} > 0 \\ 0 & \text{if } \lambda_{i+1} = 0; \end{cases}$$

where  $p$  is such that  $A_{i+1,p} > 0$  and  $A_{i+1,j} = 0$  for  $j > p$ . Also

$$F_i e_A = \begin{cases} e_{A-X_{i,p}} & \text{if } \lambda_i > 0 \\ 0 & \text{if } \lambda_i = 0; \end{cases}$$

where  $p$  is such that  $A_{i,p} > 0$  and  $A_{i,j} = 0$  for  $j < p$ .

Similar formulas for right multiplication by  $E_i$  and  $F_i$  are obtained by applying the transpose.   
 **Add formulas for right multiplication by  $E_i$  and  $F_i$ .**

## 6.2 Presentation of the generic algebra.

Recall that  $\Lambda_0(n, r)$  denotes the set of compositions of  $r$  into  $n$  parts. That is,  $\Lambda_0(n, r)$  is the set of tuples  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$  with each  $\lambda_i$  non-negative and  $\lambda_1 + \dots + \lambda_n = r$ . Given  $i \in [1, n]$ , let  $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$  be the  $i$ -th elementary vector and let  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ . Then given  $\lambda \in \Lambda_0(n, r)$ , we have  $\lambda + \alpha_i \in \Lambda_0(n, r)$  provided  $\lambda_{i+1} > 0$  and  $\lambda - \alpha_i \in \Lambda_0(n, r)$  provided  $\lambda_i > 0$ .

Let  $\Gamma = \Gamma(n, r)$  be the quiver with set of vertices  $\Lambda_0(n, r)$  with arrows  $e_{i,\lambda}: \lambda \rightarrow \lambda + \alpha_i$  (if  $\lambda_{i+1} > 0$ ) and  $f_{i,\lambda}: \lambda \rightarrow \lambda - \alpha_i$  (if  $\lambda_i > 0$ ). Thus there are no arrows between  $\lambda$  and  $\mu$  unless  $\lambda = \mu \pm \alpha_i$  for some  $i \in [1, n]$ .

If  $n \geq 3$  then neighbouring vertices are connected by two arrows, one of each direction. In the case  $n = 2$ , neighbouring vertices are joined by four arrows, two of each direction. The  $\mathbb{Z}\Gamma$  denote the path  $\mathbb{Z}$  algebra of  $\Gamma$ . By construction of  $\Gamma$ , there is a  $\mathbb{Z}$ -algebra homomorphism  $\mathbb{Z}\Gamma \rightarrow \hat{G}(n, r)$  with  $e_{i,\lambda} \mapsto E_{i,\lambda}$ ,  $f_{i,\lambda} \mapsto F_{i,\lambda}$  and  $k_\lambda = 1_\lambda$ . We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [2]).

**Definition 6.2.1.** (aperiodicity)  $A \in \Lambda_1(n, r)$  is aperiodic if for each  $l \in \mathbb{Z} \setminus \{0\}$  there exists  $i \in \mathbb{Z}$  such that  $a_{i,i+l} = 0$ . Denote the set of aperiodic elements in  $\Lambda_1(n, r)$  by  $\Lambda_1(n, r)^{ap}$ . Note that  $\Lambda_1(n, r)^{ap} = \Lambda_1(n, r)$  if  $r < n$ . Linear combinations of the basis elements corresponding to aperiodic matrices are also said to be aperiodic - if  $A$  is aperiodic, we say  $e_A$  is aperiodic.

**Lemma 6.2.1.** *If  $A \in \Lambda_1(n, r)$  is aperiodic, then  $E_i * e_A$  and  $F_i * e_A$  are aperiodic.*

*Proof.* Suppose  $A \in \Lambda_1(n, r)$  is aperiodic and  $E_i * A \neq 0$ . There is  $l \in \mathbb{Z}$  such that  $a_{i+1,l} > 0$  and  $a_{i+1,l'} = 0$  for  $l' > l$ . Then  $E_i * e_A = e_{A+\varepsilon_{i,l}-\varepsilon_{i+1,l}}$ , from Lemma 6.1.2. **FINISH THIS PROOF**  $\square$

**Lemma 6.2.2.** *If  $A \in \Lambda_1(n, r)$  is aperiodic, then  $e_A$  may be obtained from  $1_{\text{co } A}$  by finitely many applications of  $E_i$  and  $F_i$  for  $i \in [1, n]$ .*

**Proposition 6.2.3.** *The subalgebra of  $\hat{G}(n, r)$  generated by  $E_{i,\lambda}$ ,  $F_{i,\lambda}$  and  $1_\lambda$  has  $\mathbb{Z}$ -basis  $\{e_A : A \in \Lambda_1(n, r)^{ap}\}$ , where  $\Lambda_1(n, r)^{ap} \subset \Lambda_1(n, r)$  is the set of aperiodic elements, as in Definition 6.2.1.*

*Proof.* Combining Lemma 6.2.1 and Lemma 6.2.2 proves the result.  $\square$

### 6.2.1 The case $n \geq 3$ .

**Lemma 6.2.4.** *The following relations hold in  $\hat{G}(n, r)$  ( $n \geq 3$ ):*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

*unless  $|j - i| = 1$ .*

$$E_i E_{i+1}^2 - E_{i+1} E_i E_{i+1} = 0$$

$$E_i^2 E_{i+1} - E_i E_{i+1} E_i = 0$$

$$F_{i+1} F_i^2 - F_i F_{i+1} F_i = 0$$

$$F_{i+1}^2 F_i - F_{i+1} F_i F_{i+1} = 0$$

$$E_i F_j - F_j E_i = 0$$

*unless  $j = i$ .*

$$E_i F_i - F_i E_i + \sum_{\lambda: \lambda_i=0, \lambda_{i+1}>0} 1_\lambda - \sum_{\lambda: \lambda_i>0, \lambda_{i+1}=0} 1_\lambda = 0.$$

### 6.2.2 The case $n = 2$ .

In this case, the quiver  $\Gamma(2, r)$  has vertices  $\Lambda_0(2, r) = \{(0, r), (1, r-1), \dots, (r, 0)\}$ ; adjacent vertices are connected by two pairs of arrows with opposite orientation:  $(e_1, f_1)$  and  $(e_2, f_2)$ . The relations arising from  $\hat{G}(2, r)$  are of a more complicated form - in particular, the serre relations of total degree 3 will not hold in this case - so this case will be treated separately and at a later date.

## Chapter 7

# Further directions

### 7.1 Further results on affine zero Schur algebras

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

### 7.2 Deformed group algebras of symmetric groups

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algebras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for  $S_3$  and  $S_4$ . Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: ‘these’ relations hold; these generators and relations give a presentation of the degenerate group algebras.

Terminology: deformed group algebra.

# Bibliography

- [1] J. Humphreys. *Linear Algebraic Groups*. Springer-Verlag, 1981.
- [2] George Lusztig. “Aperiodicity in quantum affine  $\mathfrak{gl}_n$ ”. In: *Asian Journal of Mathematics* 3.1 (1999), pp. 147–178.