

A geometric realisation of affine 0-Schur algebras.

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Contents

1	Introduction	3
2	Background: The double flag variety approach to q-Schur algebras	4
2.1	Flag varieties as projective algebraic varieties	4
3	The cyclic flags approach to affine q-Schur algebras	5
3.1	Cyclic flags	6
3.1.1	A product on orbits	7
3.1.2	Triple products	8
3.2	Convolution algebras	8
3.3	Affine q -Schur algebras	9
4	Quivers with relations for affine q-Schur algebras	10
4.1	Basic results and notation	10
4.1.1	Elementary matrices	10
4.1.2	Transpose involution	10
4.1.3	A multiplication rule	11
4.2	Relations	12
4.3	quivers with relations	13
4.3.1	Exceptional case $n=2$	13
4.3.2	Typical case.	14
5	A generic affine algebra	15
5.1	Introducing the generic affine algebra	15
5.2	A combinatorial partial order	16
5.3	Grassmannians and related varieties	17
5.4	Geometry of affine flag varieties	18
5.4.1	Action through an algebraic group	20
5.4.2	Incidence in affine flag varieties	21
5.5	Geometry of orbits	22
5.6	Geometry of orbit products	23
5.7	Degenerations of orbits and the combinatorial partial order	25
5.8	Associativity of the generic product	25
5.9	The generic algebra	29

6	A realisation of affine zero Schur algebras	30
6.1	Preliminary results	30
6.1.1	Elementary basis elements	30
6.1.2	Transpose involution	31
6.1.3	Multiplication rules	31
6.2	Presentation of the generic algebra.	31
6.2.1	The typical case.	32
6.2.2	Exceptional case.	33
7	Further directions	34
7.1	Further results on affine zero Schur algebras	34
7.2	Deformed group algebras of symmetric groups	34
7.3	back matter	34

Chapter 1

Introduction

Chapter 2

Background: The double flag variety approach to q -Schur algebras

2.1 Flag varieties as projective algebraic varieties

Chapter 3

The cyclic flags approach to affine q -Schur algebras

Fix natural numbers n and r .

Definition 3.0.1 (compositions). *A composition of r into n parts is an n -tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ of non-negative integers whose sum equals r . Denote the set of compositions of r into n parts by Λ_0 .*

Definition 3.0.2 (infinite periodic matrices). *Let Λ_1 be the set of matrices $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ with integer entries $a_{i,j}$ satisfying the following conditions:*

- $a_{i,j} \geq 0$ for each $i, j \in \mathbb{Z}$;
- each row or column has only finitely many non-zero entries;
- the sum of the entries in any n consecutive rows or columns equals r ;
- $a_{i-n,j-n} = a_{i,j}$ for each $i, j \in \mathbb{Z}$.

These matrices are referred to as infinite periodic matrices.

Definition 3.0.3 (source and target). *Given $A \in \Lambda_1$, let $\text{ro}(A)$ and $\text{co}(A)$ be the compositions of r into n parts given by*

$$\text{ro}(A) = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\text{co}(A) = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right).$$

$A \in \Lambda_1$ is said to go from $\text{co}(A)$ to $\text{ro}(A)$.

Definition 3.0.4 (diagonal matrices). *Given $\lambda \in \Lambda_0$, let $D_\lambda \in \Lambda_1$ be the matrix given by $(D_\lambda)_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i \neq j$ and $(D_\lambda)_{i,i} = \lambda_i$ for $i \in \mathbb{Z}$; where the indices are taken modulo n .*

3.1 Cyclic flags

Fix $n, r \in \mathbb{N}$ and let \mathbf{k} be a field. Let \mathcal{S} be the \mathbf{k} -algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let \mathcal{R} be the subalgebra generated by ε , so $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free \mathcal{S} -module of rank r . Let G be the automorphism group of the \mathcal{S} -module V , so G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. A lattice in V is a \mathcal{R} -submodule L of V with $\mathcal{S} \otimes_{\mathcal{R}} L = V$. In particular, a lattice is an \mathcal{R} -submodule of V which is a free \mathcal{R} -module of rank r .

Lemma 3.1.1. *Let L be a lattice in V . $L/\varepsilon L$ is a torsion \mathcal{R} -module, where ε acts as zero. $L/\varepsilon L$ is a free $\mathcal{R}/\langle \varepsilon \rangle$ -module of rank r ; that is, $L/\varepsilon L$ is an r -dimensional \mathbf{k} -vector space.*

Proof. L is a free \mathcal{R} -module of rank r , with $L \subset V$. Given an \mathcal{R} -basis $\{x_1, \dots, x_r\}$ of L , $\{\varepsilon x_1, \dots, \varepsilon x_r\}$ is an \mathcal{R} -basis of εL . Finally, the cosets $\{x_1 + \varepsilon L, \dots, x_r + \varepsilon L\}$ give a basis for $L/\varepsilon L$ over $\mathcal{R}/\langle \varepsilon \rangle \cong \mathbf{k}$. \square

Let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of collections $(L_i)_{i \in \mathbb{Z}}$ of lattices in V with $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. These collections of lattices in V are referred to as cyclic flags in V .

G acts on \mathcal{F} by $(g \cdot L)_i = g(L_i)$ for each $i \in \mathbb{Z}$, given $g \in G$ and $L \in \mathcal{F}$. The G -orbits in \mathcal{F} are indexed by the set Λ_0 of compositions of r into n parts: the G -orbit in \mathcal{F} corresponding to $\lambda \in \Lambda_0$ is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim \left(\frac{L_i}{L_{i-1}} \right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

Definition 3.1.1. *The periodic characteristic matrix of a pair of cyclic flags $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the matrix $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$ with entries*

$$a_{i,j} = \dim_{\mathbf{k}} \left(\frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j} \right)$$

for each $i, j \in \mathbb{Z}$.

The diagonal action of G on $\mathcal{F} \times \mathcal{F}$ has orbits indexed by the set Λ_1 of infinite periodic matrices (see definition 3.0.2). The G -orbit corresponding to $A \in \Lambda_1$ is denoted \mathcal{O}_A and consists of those pairs $(L, L') \in \mathcal{F} \times \mathcal{F}$ with periodic characteristic matrix $A(L, L')$ equal to A .

Lemma 3.1.2. *(alternative expression for characteristic matrix) Alternatively,*

$$a_{i,j} = \dim_{\mathbf{k}} \left(\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} \right)$$

for each $i, j \in \mathbb{Z}$.

Proof. Set $U = L_i \cap L'_j$ and $U' = L_{i-1} + L_i \cap L'_{j-1}$. Then $U + U' = L_{i-1} + L_i \cap L'_j$ and $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$. Applying the isomorphism theorems, $U + U'/U'$ is naturally isomorphic to $U/U \cap U'$ as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to $a_{i,j}$. \square

Lemma 3.1.3 (transposing characteristic matrix). *Given a pair of flags $(L, L') \in \mathcal{F}^2$, the matrices $A(L, L')$ and $A(L', L)$ are related by the transpose. In particular, $A(L, L')_{i,j} = A(L', L)_{j,i}$ for each $i, j \in \mathbb{Z}$.*

Proof. By swapping the roles of i and j and swapping L and L' it is clear that $A(L, L')_{i,j}$ and $A(L', L)_{j,i}$ are both given by the dimension of the \mathbf{k} -vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each $i, j \in \mathbb{Z}$. □

Lemma 3.1.4 (a codimension formula). *Given $(L, L') \in \mathcal{F}^2$ and $i, j \in \mathbb{Z}$,*

$$\dim_{\mathbf{k}} \left(\frac{L_i}{L_i \cap L'_j} \right) = \sum_{s \leq i, t > j} a_{s,t},$$

where $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$.

Proof. **COMPLETE THIS PROOF** □

Lemma 3.1.5 (nested flags). *Given $(L, L') \in \mathcal{F}^2$, $L' \subset L$ if and only if $A(L, L')_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i > j$.*

Proof. Suppose $L, L' \in \mathcal{F}$ with $L' \subset L$, meaning $L'_j \subset L_j$ for each $j \in \mathbb{Z}$. Then for $i > j$, $L_i \cap L'_j = L'_j$, $L_{i-1} \cap L'_j = L'_j$ and $L_i \cap L'_{j-1}$, which shows

$$A(L, L')_{i,j} = \dim_{\mathbf{k}} \left(\frac{L'_j}{L'_{j-1} + L'_j} \right) = 0$$

as required. Conversely, suppose $A(L, L')$ is upper triangular, meaning $A(L, L')_{i,j} = 0$ when $i > j$. Using Lemma 3.1.4,

$$\dim_{\mathbf{k}} \left(\frac{L'_i}{L'_i \cap L_i} \right) = \sum_{s > i, t \leq i} a_{s,t} = 0,$$

so $L_i \cap L'_i = L'_i$ and thus $L'_i \subset L_i$ for each $i \in \mathbb{Z}$, as required. □

Corollary 3.1.6 (diagonal orbits). *Given $L, L' \in \mathcal{F}$, $L = L'$ if and only if $A(L, L')_{i,j} = 0$ whenever $i \neq j$. In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L, L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each $\lambda \in \Lambda_0$.

3.1.1 A product on orbits

Given $A, B \in \Lambda_1$ with $\text{co}(A) = \text{ro}(B)$, define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also $L \in \mathcal{F}_{\text{ro}(A)}$, define the L -slices of $Y_{A,B}$ and $X_{A,B}$ respectively as

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X_{A,B}^L = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

Observation 1. *There are only finitely many G -orbits in $X_{A,B}$.*

Lemma 3.1.7. *Given $A \in \Lambda_1$, $X_{D_\lambda, A} = \mathcal{O}_A$ if $\lambda = \text{ro}(A)$ and $X_{A, D_\lambda} = \mathcal{O}_A$ if $\lambda = \text{co}(A)$.*

Proof. Let $A \in \Lambda_1$ and set $\lambda = \text{ro}(A)$. $Y_{D_\lambda, A}$ is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_{D_\lambda}$, thus $L = L'$ by Corollary 3.1.6, and $(L', L'') \in \mathcal{O}_A$. $X_{D_\lambda, A}$ is the projection of $Y_{D_\lambda, A}$, which equals \mathcal{O}_A .

Similarly, if $\lambda = \text{co}(A)$, Y_{A, D_λ} is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_A$ and $L'' = L'$, so X_{A, D_λ} is exactly the orbit \mathcal{O}_B . \square

3.1.2 Triple products

Given $A, B, C \in \Lambda_1$ with $\text{co}(A) = \text{ro}(B)$ and $\text{co}(B) = \text{ro}(C)$ and $L \in \mathcal{F}_{\text{ro}(A)}$, there are spaces $X_{A,B,C}$, $Y_{A,B,C}$ and their respective L -slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y_{A,B,C}^L = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X_{A,B,C}^L = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

3.2 Convolution algebras

Suppose \mathbf{k} is a finite field and let q denote the number of elements of \mathbf{k} . Consider the set S of G -invariant functions $\mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Z}$ with constructible support. S is a free \mathbb{Z} -module with a basis consisting of the indicator functions of the G -orbits in $\mathcal{F} \times \mathcal{F}$. Define an operation \star on S as follows: for each $f, g \in S$, $f \star g \in S$ is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for $(L, L'') \in \mathcal{F} \times \mathcal{F}$.

$f \star g$ is well defined since the supports of f and g consist of finitely many G -orbits, so there are only finitely many $L' \in \mathcal{F}$ such that $f(L, L')g(L', L'') \neq 0$, given $(L, L'') \in \mathcal{F} \times \mathcal{F}$. $f \star g$ is constant on G -orbits and is supported on finitely many G -orbits, so $f \star g \in S$.

Lemma 3.2.1. *The set S together with the operation \star is an associative \mathbb{Z} -algebra with identity element ι given by $\iota(L, L) = 1$ and $\iota(L, L') = 0$ for $L' \neq L$.*

Proof. Given $f, g, h \in S$ and $(L, L''') \in \mathcal{F} \times \mathcal{F}$,

$$\begin{aligned} ((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''), \end{aligned}$$

thus \star is associative. ι is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L') \iota(L', L'') = f(L, L''),$$

for each $f \in S$ and $(L, L'') \in \mathcal{F} \times \mathcal{F}$. □

Given $A \in \Lambda_1$, let $e_A \in S$ denote the indicator function of the orbit \mathcal{O}_A . S is a free \mathbb{Z} -module with basis $\{e_A : A \in \Lambda_1\}$. There exist $\gamma_{A,B,C;q} \in \mathbb{Z}$ for $A, B, C \in \Lambda_1$ such that

$$e_A \star e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C;q} e_C$$

for each $A, B \in \Lambda_1$. Then

$$\begin{aligned} \gamma_{A,B,C;q} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L') e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}, \end{aligned}$$

for any $(L, L'') \in \mathcal{O}_C$.

3.3 Affine q-Schur algebras

There exist polynomials $\gamma_{A,B,C} \in \mathbb{Z}[q]$ for $A, B, C \in \Lambda_1$ such that $\gamma_{A,B,C}(q) = \gamma_{A,B,C;q}$ for any prime power q , following [32, section 4]. The affine q -Schur algebra $\hat{S}_q(n, r)$ is a $\mathbb{Z}[q]$ -algebra which is a free $\mathbb{Z}[q]$ -module with basis $\{e_A : A \in \Lambda_1\}$ and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these ‘universal polynomials’ $\gamma_{A,B,C} \in \mathbb{Z}[q]$, it follows from Lemma 3.2.1 that $\hat{S}_q(n, r)$ is an associative $\mathbb{Z}[q]$ -algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

Chapter 4

Quivers with relations for affine q-Schur algebras

4.1 Basic results and notation

4.1.1 Elementary matrices

For each $i, j \in \mathbb{Z}$, let $\mathcal{E}_{i,j}$ be the $\mathbb{Z} \times \mathbb{Z}$ ‘elementary periodic matrix’ with entries given by $(\mathcal{E}_{i,j})_{s,t} = 1$ if $(s, t) = (i + cn, j + cn)$ for some $c \in \mathbb{Z}$ and $(\mathcal{E}_{i,j})_{s,t} = 0$ otherwise. Clearly $\mathcal{E}_{i,j} = \mathcal{E}_{i+n,j+n}$ for each $i, j \in \mathbb{Z}$. Recall from Definition 3.0.4 that the diagonal matrix associated to a composition $\lambda \in \Lambda_0$ is

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}.$$

$\{e_{D_\lambda} : \lambda \in \Lambda_0\}$ is a set of pairwise orthogonal idempotents in $\hat{S}_q(n, r)$ with $\sum_{\lambda \in \Lambda_0} e_{D_\lambda} = 1$, as a result of Lemma 3.1.7.

Given $i \in [1, n]$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and define

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}.$$

Given $i \in [1, n]$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and define

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}$$

4.1.2 Transpose involution

Lemma 4.1.1. *Transposition gives a homomorphism of $\mathbb{Z}[q]$ -modules $\top : \hat{S}_q(n, r) \rightarrow \hat{S}_q(n, r)$ with $\top(e_A) = e_{A^\top}$, $\top \circ \top = 1$ and $\top(e_A e_B) = \top(e_B) \top(e_A)$.*

Proof. Let $A, B, C \in \Lambda_1$ and let \mathbf{k} be a finite field with $q = \#\mathbf{k}$ elements. If $(L, L'') \in \mathcal{O}_C$ then $(L'', L) \in \mathcal{O}_{C^\top}$ and

$$\begin{aligned}\gamma_{A,B,C;q} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;q}\end{aligned}$$

It then follows that $\top(e_A e_B) = \top(e_B) \top(e_A)$. \square

The transpose relates the E_i , F_i and 1_λ in the following way: $\top(E_{i,\lambda}) = F_{i,\lambda}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\varepsilon_i+\varepsilon_{i+1}}$ and $\top(1_\lambda) = 1_\lambda$. In particular, $\top(E_i) = F_i$ and $\top(F_i) = E_i$.

4.1.3 A multiplication rule

Lemma 4.1.2. *Given $A \in \Lambda_1$ and $i \in [1, n]$ with $\text{ro}(A)_{i+1} > 0$,*

$$E_i e_A = \sum_{p \in \mathbb{Z}: a_{i+1,p} > 0} q^{\sum_{j>p} a_{i,j}} [[a_{i,p} + 1]] e_{A+\varepsilon_{i,p}-\varepsilon_{i+1,p}}.$$

Given $A \in \Lambda_1$ and $i \in [1, n]$ with $\text{ro}(A)_i > 0$,

$$F_i e_A = \sum_{p \in \mathbb{Z}: a_{i,p} > 0} q^{\sum_{j<p} a_{i+1,j}} [[a_{i+1,p} + 1]] e_{A+\varepsilon_{i+1,p}-\varepsilon_{i,p}}.$$

Note that these formulas are still valid in the cases $E_i e_A = 0$ and $F_i e_A = 0$, provided it is understood that $e_B = 0$ whenever $B \notin \Lambda_1$. There are similar formulas for right multiplication by E_i and F_i , obtained by applying the transpose involution to the above.

Corollary 4.1.3. *Given $A \in \Lambda_1$ and $j \in [1, n]$ with $\text{co}(A)_{j+1} > 0$,*

$$e_A F_j = \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\varepsilon_{p,j}-\varepsilon_{p,j+1}}.$$

Given $A \in \Lambda_1$ and $j \in [1, n]$ with $\text{co}(A)_j > 0$,

$$e_A E_j = \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A+\varepsilon_{p,j+1}-\varepsilon_{p,j}}.$$

Proof.

$$\begin{aligned}e_A F_j &= \top(E_j e_{A^\top}) \\ &= \top \left(\sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A^\top+\varepsilon_{j,p}-\varepsilon_{j+1,p}} \right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\varepsilon_{p,j}-\varepsilon_{p,j+1}},\end{aligned}$$

where the second equality comes from Lemma 4.1.2. Similarly,

$$\begin{aligned}e_A E_j &= \top(F_j e_{A^\top}) \\ &= \top \left(\sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A^\top+\varepsilon_{j+1,p}-\varepsilon_{j,p}} \right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A+\varepsilon_{p,j+1}-\varepsilon_{p,j}}.\end{aligned}$$

□

4.2 Relations

Note that $E_i^{r+1} = F_i^{r+1} = 0$ while

$$E_i^r = [r]! e_r \varepsilon_{i,i+1}$$

and

$$F_i^r = [r]! e_r \varepsilon_{i+1,i}.$$

Lemma 4.2.1 (quantum Serre relations: $n \geq 3$). *Suppose $n \geq 3$. The following relations hold in $\hat{S}_q(n, r)$:*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless $j = i \pm 1$;

$$E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i = 0$$

$$E_i^2 E_{i+1} - (1+q) E_i E_{i+1} E_i + q E_{i+1} E_i^2 = 0$$

and

$$F_{i+1} F_i^2 - (1+q) F_i F_{i+1} F_i + q F_i^2 F_{i+1} = 0$$

$$F_{i+1}^2 F_i - (1+q) F_{i+1} F_i F_{i+1} + q F_i F_{i+1}^2 = 0.$$

Proof. Here we introduce temporary notation for the basis elements: Write $[A] = e_A$.

Take $\lambda \in \Lambda_0$.

$$E_i E_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1} E_i E_{i+1} 1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2 E_i 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then

$$(E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i) 1_\lambda = 0,$$

for each $\lambda \in \Lambda_0$. The relation $E_i E_{i+1}^2 - (1+q) E_{i+1} E_i E_{i+1} + q E_{i+1}^2 E_i = 0$ then follows.

The relations between F_i and F_{i+1} may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping E_i and F_i and reversing the order of multiplication. □

Lemma 4.2.2 (quantum Serre relations: $n = 2$). *In the case $n = 2$, the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

Lemma 4.2.3. $[E_i, F_j] = 0$ unless $j = i$.

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}]) 1_\lambda.$$

For $\lambda \in \Lambda_0$, let $R_\lambda = e_{\lambda_1 \varepsilon_{0,1} + \dots + \lambda_n \varepsilon_{n-1,n}}$. Write $R = \sum_{\lambda \in \Lambda_0} R_\lambda$. Note $R_\lambda = R 1_\lambda$. Given $A \in \Lambda_1$ and $m \in \mathbb{Z}$, let $A[m] \in \Lambda_1$ be given by $A[m]_{i,j} = a_{i,j+m}$ and let $A^{[m]}$ be given by $A^{[m]}_{i,j} = a_{i+m,j}$ for each $i \in \mathbb{Z}$.

Lemma 4.2.4 (Shifting). *If $A \in \Lambda_1$ then*

$$Re_A = e_{A[\pm 1]}$$

and

$$e_A R = e_{A[\pm 1]}.$$

Conjugation by R gives an automorphism ρ of $\hat{S}_q(n, r)$ satisfying $\rho^n = 1$.

4.3 quivers with relations

Denote by Λ_0 the set of compositions of r into n parts. That is, Λ_0 is the set of $\alpha \in \mathbb{Z}^n$ with non-negative entries which sum to r . Let $\varepsilon_i \in \mathbb{Z}^n$ be the i th elementary vector and write $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for each $i \in [1, n]$. Then $\lambda + \alpha_i \in \Lambda_0$ if $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ if $\lambda_i > 0$.

Let $\Gamma = \Gamma(n, r)$ be the quiver with set of vertices Λ_0 , with the following arrows:

For $\lambda \in \Lambda_0$ and $i \in [1, n]$, there is an arrow $e_{i,\lambda} : \lambda \rightarrow \lambda + \alpha_i$ if $\lambda_{i+1} > 0$ and there is an arrow $f_{i,\lambda} : \lambda \rightarrow \lambda - \alpha_i$ if $\lambda_i > 0$.

Denote by $\mathbb{Z}[q]\Gamma$ the path $\mathbb{Z}[q]$ -algebra of Γ . Thus $\mathbb{Z}[q]\Gamma$ is a free $\mathbb{Z}[q]$ -module with a basis given by the set of paths in Γ , with multiplication given by the concatenation of paths. If p starts where q ends, the product pq is the path q followed by p . Write $e_{i,\lambda} = 0$ unless $\lambda, \lambda + \alpha_i \in \Lambda_0$ and write $f_{i,\lambda} = 0$ unless $\lambda, \lambda - \alpha_i \in \Lambda_0$.

By construction, there is a homomorphism of $\mathbb{Z}[q]$ -algebras

$$\phi : \mathbb{Z}[q]\Gamma \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda, \end{aligned}$$

for $i \in [1, n]$ and $\lambda \in \Lambda_0$.

The image of ϕ is the subalgebra of $\hat{S}_q(n, r)$ generated by E_i, F_i for $i \in [1, n]$ and 1_λ for $\lambda \in \Lambda_0$, since $E_{i,\lambda} = E_i 1_\lambda$ and $F_{i,\lambda} = F_i 1_\lambda$, while $E_i = \sum_\lambda E_{i,\lambda}$ and $F_i = \sum_\lambda F_{i,\lambda}$. In general ϕ is not surjective, so this does not always lead to a presentation of $\hat{S}_q(n, r)$.

4.3.1 Exceptional case n=2.

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the q -Schur algebra.

4.3.2 Typical case.

Suppose $n \geq 3$. Then $\Gamma = \Gamma(n, r)$ has vertex set Λ_0 .

Define $e_i, f_i \in \mathbb{Z}[q]\Gamma(n, r)$ by

$$e_i = \sum_{\lambda \in \Lambda_0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0} f_{i,\lambda},$$

with the convention $e_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $f_{i,\lambda} = 0$ unless $\lambda_i > 0$. Let k_λ denote the constant path at vertex λ . $\{k_\lambda : \lambda \in \Lambda_0\}$ is a set of pairwise orthogonal idempotents in $\mathbb{Z}[q]\Gamma(n, r)$.

Let $I(n, r) \subset \mathbb{Z}[q]\Gamma(n, r)$ be the ideal generated by the expressions

$$\begin{aligned} & e_i e_{i+1}^2 - (1+q)e_{i+1}e_i e_{i+1} + qe_{i+1}^2 e_i \\ & e_i^2 e_{i+1} - (1+q)e_i e_{i+1} e_i + qe_{i+1} e_i^2 \\ & f_{i+1} f_i^2 - (1+q)f_i f_{i+1} f_i + qf_i^2 f_{i+1} \\ & f_{i+1}^2 f_i - (1+q)f_{i+1} f_i f_{i+1} + qf_i f_{i+1}^2 \\ & e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}]) k_\lambda \end{aligned}$$

Recall that a relation is a $\mathbb{Z}[q]$ -linear combination of paths with common start and end vertices. The relations involving paths $\lambda \rightarrow \mu$ are given by $1_\mu \text{expr} 1_\lambda$, for each of the above expressions.

Lemma 4.3.1. *There is a homomorphism of $\mathbb{Z}[q]$ -algebras*

$$\phi: \mathbb{Z}[q]\Gamma(n, r)/I(n, r) \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda. \end{aligned}$$

Chapter 5

A generic affine algebra

5.1 Introducing the generic affine algebra

Assume $\mathbf{k} = \mathbb{C}$ and fix $n, r \geq 1$. Let \mathcal{S} be the \mathbf{k} -algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let \mathcal{R} be the subalgebra generated by ε , namely $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free \mathcal{S} -module of rank r and let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of n -periodic cyclic flags in V ; so \mathcal{F} consists of collections $L = (L_i)_{i \in \mathbb{Z}}$ of \mathcal{R} -lattices in V with $L_i \subset L_{i+1}$ for $i \in \mathbb{Z}$ and $\varepsilon L_i = L_{i-n}$ for $i \in \mathbb{Z}$.

Let G be the group of \mathcal{S} -module automorphisms of V . Thus G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. G acts on \mathcal{F} with orbits $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0\}$, where Λ_0 is the set of compositions of r into n parts, as in Definition 3.0.1.

The diagonal action of G on $\mathcal{F} \times \mathcal{F}$ has orbits $\{\mathcal{O}_A : A \in \Lambda_1\}$, where \mathcal{O}_A consists of those pairs of flags with periodic characteristic matrix equal to A . Definitions of the periodic characteristic matrix and the set Λ_1 are given in Definition 3.1.1 and Definition 3.0.2 respectively.

Recall that the periodic characteristic matrix of a pair $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix $A = (a_{i,j})_{i,j \in \mathbb{Z}}$, with

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right)$$

for each $i, j \in \mathbb{Z}$.

Recall that ro and co are the maps $\Lambda_1 \rightarrow \Lambda_0$ given by

$$\mathrm{ro}(A) = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co}(A) = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right)$$

for each $A \in \Lambda_1$. Given $A \in \Lambda_1$, write $A: \mathrm{co}(A) \rightarrow \mathrm{ro}(A)$.

The purpose of this chapter is to define a category with objects Λ_0 and morphisms Λ_1 ; where $\mathrm{Hom}(\lambda, \mu) = \{A \in \Lambda_1 : \mathrm{ro}(A) = \mu, \mathrm{co}(A) = \lambda\}$. Given $A, B \in \Lambda_1$ let $\Lambda_1^{A,B}$ be the set of $C \in \Lambda_1$ such that there exist $L, L', L'' \in \mathcal{F}$ with $(L, L') \in \mathcal{O}_A$, $(L', L'') \in \mathcal{O}_B$ and $(L, L'') \in \mathcal{O}_C$. It will be shown that Λ_1 admits a partial order \leq such that, given $A, B \in \Lambda_1$ with $\mathrm{ro}(B) = \mathrm{co}(A)$, $\Lambda_1^{A,B}$ has a maximum element $A * B$. It will be shown that $*$ is associative, leading to the construction of a category with the described properties.

The generic affine algebra $\hat{G}(n, r)$ is then defined to be the \mathbb{Z} -algebra of this category.

5.2 A combinatorial partial order

Given $i, j \in \mathbb{Z}$, define maps $d_{i,j}$ and $\bar{d}_{i,j}$ on Λ_1 by setting

$$d_{i,j}A = \sum_{s \leq i, t > j} a_{s,t}$$

and

$$\bar{d}_{i,j}A = \sum_{s > i, t \leq j} a_{s,t}$$

for each $A \in \Lambda_1$.

Lemma 5.2.1. *Let $A \in \Lambda_1$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for $i, j \in \mathbb{Z}$. Then*

$$d_{i,j} - d_{i-1,j} = \sum_{t > j} a_{i,t}$$

and

$$d_{i,j} - d_{i,j-1} = - \sum_{s \leq i} a_{s,j}.$$

Proof. Let $i, j \in \mathbb{Z}$. Then

$$d_{i,j} - d_{i-1,j} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i-1, t > j} a_{s,t} = \sum_{t > j} a_{i,t}.$$

Similarly,

$$d_{i,j} - d_{i,j-1} = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i, t > j-1} a_{s,t} = - \sum_{s \leq i} a_{s,j}.$$

□

Lemma 5.2.2. *Let $A \in \Lambda_1$, with $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and write $d_{i,j} = d_{i,j}A$ for each $i, j \in \mathbb{Z}$. Then*

$$a_{i,j} = d_{i,j-1} - d_{i-1,j-1} + d_{i-1,j} - d_{i,j}$$

for each $i, j \in \mathbb{Z}$.

Proof. Using Lemma 5.2.1,

$$\begin{aligned} a_{i,j} &= \sum_{t > j-1} a_{i,t} - \sum_{t > j} a_{i,t} \\ &= (d_{i,j-1} - d_{i-1,j-1}) - (d_{i,j} - d_{i-1,j}). \end{aligned}$$

Alternatively,

$$\begin{aligned} a_{i,j} &= \sum_{s \leq i} a_{s,j} - \sum_{s \leq i-1} a_{s,j} \\ &= -(d_{i,j} - d_{i,j-1}) + (d_{i-1,j} - d_{i-1,j-1}). \end{aligned}$$

□

Lemma 5.2.3. *The relation \leq on Λ_1 , defined by $A \leq B$ if and only if $d_{i,j}A \leq d_{i,j}B$ for all $i, j \in \mathbb{Z}$, is a partial order.*

Proof. It is clear that \leq is reflexive and transitive, so it remains to see \leq is antisymmetric. Suppose $A, B \in \Lambda_1$ with $A \leq B$ and $B \leq A$. Then $d_{i,j}A = d_{i,j}B$ for each $i, j \in \mathbb{Z}$, which shows $A = B$ as a result of Lemma 5.2.2. \square

The partial order on Λ_1 induces a partial order on the set of G -orbits in $\mathcal{F} \times \mathcal{F}$, such that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $A \leq B$. The following lemma is rephrased from Lemma 3.1.4 and gives some geometric significance to the partial order on Λ_1 .

Lemma 5.2.4. *Let $A \in \Lambda_1$ and take $(L, L') \in \mathcal{O}_A$. Then*

$$\dim \left(\frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}A$$

and

$$\dim \left(\frac{L'_j}{L_i \cap L'_j} \right) = \bar{d}_{i,j}A,$$

for each $i, j \in \mathbb{Z}$.

Proof. This is a rephrasing of Lemma 3.1.4. \square

5.3 Grassmannians and related varieties

Here we collect a few elementary results on Grassmannians and some related varieties. In this section, let V be an n -dimensional \mathbf{k} -vector space and let $0 \leq d \leq n$ be an integer. There is a linear map

$$\phi^{(d)}: \Lambda^d(V) \rightarrow \text{Hom}(V, \Lambda^{d+1}(V))$$

given by

$$\phi^{(d)}(\alpha)(v) = \alpha \wedge v$$

for $\alpha \in \Lambda^d(V)$ and $v \in V$. The kernel of $\phi^{(d)}(\alpha)$ is the space of divisors of α ,

$$D_\alpha = \{v \in V : \alpha \wedge v = 0\}.$$

An element $\alpha \in \Lambda^d(V)$ is said to be totally decomposable if $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_d$, where $\alpha_1, \dots, \alpha_d \in V$ are linearly independent. The dimension of D_α is at most d and $\dim(D_\alpha) = d$ precisely when α is totally decomposable. Consequently, the rank of $\phi^{(d)}(\alpha)$ is at least $n - d$ and α is totally decomposable if and only if $\text{rank } \phi^{(d)}(\alpha) \leq n - d$, which holds if and only if the $(n - d + 1) \times (n - d + 1)$ -minors of a matrix of $\phi^{(d)}(\alpha)$ are all zero.

Lemma 5.3.1. *$\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a projective variety, for each $d_1, d_2, a \in \mathbb{N}$ with $d_1, d_2, a \leq n$.*

Proof. As above, there is a linear map $\Psi: \Lambda^{d_1}V \oplus \Lambda^{d_2}V \rightarrow \text{Hom}(V, \Lambda^{d_1+1}(V) \oplus \Lambda^{d_2+1}(V))$ given by $\Psi(\alpha, \beta)(v) = (\alpha \wedge v, \beta \wedge v)$. Given $\alpha \in \Lambda^{d_1}(V)$ and $\beta \in \Lambda^{d_2}(V)$, the kernel of $\Psi(\alpha, \beta)$ is $D_\alpha \cap D_\beta$ and so the rank of $\Psi(\alpha, \beta)$ is $n - \dim(D_\alpha \cap D_\beta)$.

Let $U_i \in \text{Gr}_{d_i}(V)$ and suppose $p_i(U_i) = [\alpha_i]$, where p_i is the Plücker embedding of $\text{Gr}_{d_i}(V)$ in $\mathbb{P}(\Lambda^{d_i}(V))$, so $U_i = D_{\alpha_i} = \ker \phi^{(d_i)}(\alpha_i)$. Therefore the kernel of $\Psi(\alpha_1, \alpha_2)$ is $U_1 \cap U_2$, so the condition that $\dim(U_1 \cap U_2) \geq a$ is equivalent to the condition that $\Psi(\alpha_1, \alpha_2)$ has rank at most $n - a$. After fixing a basis of V , this condition is given by the vanishing of the $(n - a + 1) \times (n - a + 1)$ minors

of the matrix of $\Psi(\alpha_1, \alpha_2)$ with respect to this basis. Therefore $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a closed subset of the product of Grassmannians $\text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V)$, so is a projective variety.

More precisely, the entries of a matrix of $\Psi(\alpha_1, \alpha_2)$ are homogeneous polynomials of degree 1 in the Plücker coordinates on $\text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V)$ since Ψ is linear and so the minors of $\Psi(\alpha_1, \alpha_2)$ are also homogeneous polynomials in the Plücker coordinates. \square

Lemma 5.3.2. *Let V be an n -dimensional vector space over \mathbf{k} and let $d_1, d_2, a \in \mathbb{N}$ with $d_1, d_2, a \leq n$. The following hold:*

1. $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) = a\}$ is a quasiprojective variety;
2. $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : U_1 \subset U_2\}$ is a projective variety;
3. Given $U_2 \in \text{Gr}_{d_2}(V)$, $\{U_1 \in \text{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a projective variety;
4. Given $U_2 \in \text{Gr}_{d_2}(V)$, $\{U_1 \in \text{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) = a\}$ is a quasiprojective variety;
5. Given $U_2 \in \text{Gr}_{d_2}(V)$, $\{U_1 \in \text{Gr}_{d_1}(V) : U_1 \subset U_2\}$ is a projective variety;
6. Given $U_2 \in \text{Gr}_{d_2}(V)$, $\{U_1 \in \text{Gr}_{d_1}(V) : U_2 \subset U_1\}$ is a projective variety.

Proof. Let X_i denote the space in statement i of the lemma. To emphasise the dependence of X_i on a , write $X_{i,a}$.

X_1 is a quasiprojective variety since it is equal to the intersection of the projective variety $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ with the open set $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \leq a\}$.

Given $(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V)$, $U_1 \subset U_2$ if and only if $\dim(U_1 \cap U_2) \geq d_1$, so Lemma 5.3.1 shows X_2 is a projective variety.

Let $\pi_i : \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) \rightarrow \text{Gr}_{d_i}(V)$ be the projection map onto the i -th factor, for $i = 1, 2$. The completeness property of projective varieties ensures that π_i is a closed morphism. Observe that

$$\begin{aligned} X_3 &= \{U_1 \in \text{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\} \\ &= \pi_1(\{(U_1, W) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap W) \geq a\} \cap \pi_2^{-1}\{U_2\}). \end{aligned}$$

The fibre of π_2 over U_2 is closed, so the intersection of the fibre with the variety from Lemma 5.3.1 is closed and then the image of this intersection under π_1 is closed. This shows X_3 is a projective variety.

X_4 is a quasiprojective variety since it is the complement of the subvariety $X_{3,a+1}$ in $X_{3,a}$. Finally, 5-6 follow as special cases of 3 since $X_5 = X_{3,d_1}$ and $X_6 = X_{3,d_2}$. \square

5.4 Geometry of affine flag varieties

Given $L \in \mathcal{F}$, $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0$ define

$$\Pi_{N,\lambda}(L) = \{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi_{N,\lambda}^a(L) = \left\{ L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0, \dim \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right) = a \right\}.$$

Lemma 5.4.1. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$ and $\lambda \in \Lambda_0$,*

$$\Pi_{N,\lambda}(L) = \bigcup_{a:0 \leq a \leq 2Nr} \Pi_{N,\lambda}^a(L).$$

Proof. If $L' \in \Pi_{N,\lambda}(L)$ then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and the $\mathbf{k}[\varepsilon]$ -module $\varepsilon^{-N} L_0 / L'_0$ is naturally isomorphic to $(\varepsilon^{-N} L_0 / \varepsilon^N L_0) / (L'_0 / \varepsilon^N L_0)$, so

$$\dim_{\mathbf{k}} \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right) \leq \dim_{\mathbf{k}} \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0} \right) = 2Nr.$$

□

Lemma 5.4.2. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \leq a \leq 2Nr$, $\Pi_{N,\lambda}^a(L)$ is a projective algebraic variety.*

Proof. Let W be the $\mathbf{k}[\varepsilon]$ -module $\varepsilon^{-1-N} L_0 / \varepsilon^N L_0$, which has dimension $(2N+1)r$ over \mathbf{k} . Let $d_i = 2Nr - a + \lambda_1 + \dots + \lambda_i$ for each $i = 1, \dots, n$. The correspondence between submodules of $\varepsilon^{-1-N} L_0$ which contain $\varepsilon^N L_0$ and submodules of $\varepsilon^{-1-N} L_0 / \varepsilon^N L_0$ determines a map

$$\rho: \Pi_{N,\lambda}^a(L) \rightarrow \text{Gr}_{d_1}(W) \times \dots \times \text{Gr}_{d_n}(W),$$

with $\rho(L') = (L'_1 / \varepsilon^N L_0, \dots, L'_n / \varepsilon^N L_0)$.

Let \mathcal{X} be the space of $(U_1, \dots, U_n) \in \text{Gr}_{d_1}(W) \times \dots \times \text{Gr}_{d_n}(W)$ with $U_i \subset U_{i+1}$ for $i = 1, \dots, n-1$ and $\varepsilon U_n \subset U_1$. Lemma 5.3.2 shows that each of these conditions is closed, so \mathcal{X} is a closed subset of $\text{Gr}_{d_1}(W) \times \dots \times \text{Gr}_{d_n}(W)$, therefore \mathcal{X} is a projective algebraic variety.

The image of ρ is contained in \mathcal{X} since

$$\varepsilon L'_n / \varepsilon^N L_0 = L'_0 / \varepsilon^N L_0 \subset L'_1 / \varepsilon^N L_0 \subset \dots \subset L'_n / \varepsilon^N L_0.$$

Suppose $(U_1, \dots, U_n) \in \mathcal{X}$. Then U_i is a $\mathbf{k}[\varepsilon]$ -module, since $\varepsilon U_i \subset \varepsilon U_n \subset U_1 \subset U_i$, for each $i = 1, \dots, n$, so U_i lifts uniquely to a $\mathbf{k}[\varepsilon]$ -module L'_i with $\varepsilon^N L_0 \subset L'_i \subset \varepsilon^{-1-N} L_0$. Therefore L'_1, \dots, L'_n are $\mathbf{k}[\varepsilon]$ -lattices with $L_i \subset L_{i+1}$ for $i = 1, \dots, n-1$ and $\varepsilon L'_n \subset L'_1$, with

$$\dim(\varepsilon^{-1-N} L_0 / L'_n) = \dim(W / W_n) = (2N+1)r - d_n = a$$

and

$$\dim(L'_i / L'_{i-1}) = \dim(W_i / W_{i-1}) = d_i - d_{i-1} = \lambda_i,$$

for each $i = 2, \dots, n$. Therefore there is a unique $L' \in \Pi_{N,\lambda}^a(L)$ such that $\rho(L') = (W_1, \dots, W_n)$, where L' is given by $L'_{i+cn} = \varepsilon^{-c} L'_i$ for $i = 1, \dots, n$ and $c \in \mathbb{Z}$. It follows ρ is injective and $\text{im } \rho = \mathcal{X}$, which is a projective variety, so $\Pi_{N,\lambda}^a(L)$ is a projective variety. □

Lemma 5.4.3. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \leq a \leq 2Nr$, $\Pi_{N,\lambda}^a(L)$ is closed in $\Pi_{N+1,\lambda}^{a+r}(L)$.*

Proof. If $L' \in \Pi_{N,\lambda}^a(L)$, then $\varepsilon^{N+1} L_0 \subset \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0 \subset \varepsilon^{-(N+1)} L_0$ and

$$\dim \left(\frac{\varepsilon^{-(1+n)} L_0}{L'_0} \right) = \dim \left(\frac{L_0}{\varepsilon L_0} \right) + \dim \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right) = r + a,$$

which shows that $\Pi_{N,\lambda}^a(L) \subset \Pi_{N+1,\lambda}^{a+r}(L)$. For $L' \in \Pi_{N+1,\lambda}^{a+r}(L)$, if additionally $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, then

$$\dim \left(\frac{\varepsilon^{-(N+1)} L_0}{L'_0} \right) = r + \dim \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right),$$

which shows $L' \in \Pi_{N,\lambda}^a(L)$. Therefore $\Pi_{N,\lambda}^a(L)$ is the subspace of $\Pi_{N+1,\lambda}^{a+r}(L)$ defined by the two closed conditions $\varepsilon^N L_0 \subset L'_0$ and $L'_0 \subset \varepsilon^{-N} L_0$, using Lemma 5.3.2. \square

Lemma 5.4.4. *Let $\lambda \in \Lambda_0$, $M, N \in \mathbb{N}$, $L, \tilde{L} \in \mathcal{F}$, $0 \leq a \leq 2Nr$, $0 \leq b \leq 2Mr$. $\Pi_{N,\lambda}^a(L) \cap \Pi_{M,\lambda}^b(\tilde{L})$ is a closed set in $\Pi_{N,\lambda}^a(L)$. In particular, if the intersection is nonempty it is a projective algebraic variety.*

Proof. Observe that $\Pi_{N,\lambda}^a(L) \cap \Pi_{M,\lambda}^b(\tilde{L})$ is the subset of $\Pi_{N,\lambda}^a(L)$ defined by the additional conditions that $\varepsilon^M \tilde{L}_0 \subset L'_0$ and $L'_0 \subset \varepsilon^{-M} \tilde{L}_0$, so is a closed subset of $\Pi_{N,\lambda}^a(L)$, using 5.3.2. \square

Lemma 5.4.5. *Suppose $L \in \mathcal{F}$, $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0$ with $a \leq 2Nr$. For each $g \in G$, the natural map (restriction of the action map) $\Pi_{N,\lambda}^a(L) \rightarrow \Pi_{N,\lambda}^a(gL)$ is an isomorphism of projective varieties.*

Proof. If $L' \in \Pi_{N,\lambda}^a(L)$, then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and so $\varepsilon^N g(L_0) \subset g(L'_0) \subset \varepsilon^{-N} g(L_0)$, so $gL' \in \Pi_{N,\lambda}^a(gL)$. Thus g and g^{-1} induce mutually inverse morphisms of varieties $g: \Pi_{N,\lambda}^a(L) \rightarrow \Pi_{N,\lambda}^a(gL)$ and $g^{-1}: \Pi_{N,\lambda}^a(gL) \rightarrow \Pi_{N,\lambda}^a(L)$. \square

5.4.1 Action through an algebraic group

Given $N \in \mathbb{N}$, define

$$H_N = \left\{ h \in G_L : h = 1 \text{ on } \frac{\varepsilon^{-(1+N)} L_0}{\varepsilon^N L_0} \right\}.$$

Explicitly, the condition $h = 1$ on $\varepsilon^{-(1+N)} L_0 / \varepsilon^N L_0$ means: $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ for $x \in \varepsilon^{-(1+N)} L_0$. Observe that $H_{N+1} \subset H_N$ for $N \in \mathbb{N}$ since $h(x) + \varepsilon^N L_0 = x + \varepsilon^N L_0$ whenever $x \in \varepsilon^{-(1+N)} L_0$.

EDITORIAL REMARK:

Maybe the cleanest way to write this is to describe the natural group homomorphism $G_L \rightarrow \text{GL}(W)$ and state that $H_{N,L}$ is the kernel of this group homomorphism. The next lemma should describe the image and deduce $G_L/H_{N,L}$ is a connected algebraic group, possibly with the last result relegated to a corollary.

Lemma 5.4.6. *Given $L \in \mathcal{F}$ and $N \in \mathbb{N}$, $G_L/H_{N,L}$ is a connected algebraic group.*

Proof. Let W be the $\mathbb{C}[\varepsilon]$ -module $\varepsilon^{-(1+N)} L_0 / \varepsilon^N L_0$. ε^{2N+1} acts as zero on W and $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle \otimes_{\mathbb{C}[\varepsilon]} W$ is a free $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle$ -module of rank r . Given $g \in G_{L_0}$, g is a $\mathbb{C}[\varepsilon]$ -module automorphism of $\varepsilon^{-(1+N)} L_0$ and $\varepsilon^N L_0$ is a g -invariant submodule, so there is an automorphism $\bar{g}: W \rightarrow W$ fitting into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varepsilon^N L_0 & \longrightarrow & \varepsilon^{-1-N} L_0 & \longrightarrow & W \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \varepsilon^N L_0 & \longrightarrow & \varepsilon^{-1-N} L_0 & \longrightarrow & W \longrightarrow 0 \end{array}$$

The natural map $\bar{\cdot} : G_{L_0} \rightarrow \mathrm{GL}(W)$ is a group homomorphism with kernel consisting of those $g \in G_{L_0}$ such that $\bar{g} = 1$: that is, $g(x) \in x + \varepsilon^{2N+1}L_0$ for each $x \in L_0$.

The image of G_{L_0} in $\mathrm{GL}(W)$ may be described by equations in the coordinates on $\mathrm{GL}(W)$ with respect to a \mathbb{C} -basis of W . W has a basis $\{x_1, \dots, x_r\}$ over $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle$, therefore the complex vector space W has a basis $\{y_j : j \in \mathbb{Z}, 1 - 2Nr \leq j \leq r\}$ given by

$$y_{i-cr} = \varepsilon^c x_i$$

for $1 \leq i \leq r$ and $0 \leq c \leq 2N$. There are coordinate functions $\gamma_{i,j} : \mathrm{GL}(W) \rightarrow \mathbb{C}$ with respect to this basis, given by

$$g(y_j) = \sum_i \gamma_{ij}(g) y_i.$$

If $g \in \mathrm{GL}(W)$ is ε -linear, then $g(y_{i-r}) = g(\varepsilon y_i) = \varepsilon g(y_i)$ and therefore $\gamma_{i-r,j-r}(g) = \gamma_{i,j}(g)$ for all i, j . This shows that the image of G_{L_0} in $\mathrm{GL}(W)$ is the parabolic subgroup consisting of elements of the form

$$\begin{aligned} & A_0 A_1 A_2 \cdots A_{2N} \\ & 0 A_0 A_1 \cdots A_{2N-1} \\ & \dots\dots\dots \\ & 00 \cdots A_0 A_1 \\ & 00 \cdots 0 A_0, \end{aligned}$$

where $A_0 \in \mathrm{GL}_r(\mathbb{C})$ and $A_1, \dots, A_{2N} \in M_r(\mathbb{C})$, which is a closed subgroup of $\mathrm{GL}(W)$. The image of G_{L_0} in $\mathrm{GL}(W)$ is identified with the (nonempty) open set $\mathrm{GL}_r(\mathbb{C}) \times M_r(\mathbb{C})^{2N}$ in the affine space $M_r(\mathbb{C})^{2N+1}$, so the image of G_{L_0} is irreducible. This shows that $G_{L_0}/H_{N,L_0}$ is a connected algebraic group.

Moreover, $G_L = G_{L_1} \cap \cdots \cap G_{L_n}$, so the image of G_L in $\mathrm{GL}(W)$ is a closed subgroup. $G_L/H_{N,L}$ is naturally isomorphic to the subgroup of $\mathrm{GL}(W)$ defined by the equations $\gamma_{i-r,j-r} = \gamma_{i,j}$ and for $j = 1, \dots, r$ the equations $\gamma_{i,j} = 0$ for $i > \lambda_1 + \cdots + \lambda_s$, where s is given by $\lambda_1 + \cdots + \lambda_{s-1} < j \leq \lambda_1 + \cdots + \lambda_s$. Therefore $G_L/H_{N,L}$ is isomorphic to the product $\mathcal{P}_\lambda \times M_r(\mathbb{C}) \times \cdots \times M_r(\mathbb{C})$, where \mathcal{P}_λ is a parabolic subgroup of $\mathrm{GL}(W)$, so is irreducible. \square

Given $g \in G$, the map $G_L \rightarrow G_{gL}$ sending h to ghg^{-1} is a group isomorphism which descends to an isomorphism of algebraic groups $G_L/H_{N,L} \rightarrow G_{gL}/H_{N,gL}$. Thus we have a commuting diagram of morphisms of varieties, where the vertical arrows are isomorphisms:

$$\begin{array}{ccc} G_L/H_{N,L} \times \Pi_{N,\lambda}^a(L) & \longrightarrow & \Pi_{N,\lambda}^a(L) \\ \downarrow & & \downarrow \\ G_{gL}/H_{N,gL} \times \Pi_{N,\lambda}^a(gL) & \longrightarrow & \Pi_{N,\lambda}^a(gL) \end{array}$$

5.4.2 Incidence in affine flag varieties

Lemma 5.4.7. *Given $N, a, b, c \in \mathbb{N}$, $\lambda, \mu \in \Lambda_0$, $L \in \mathcal{F}$ and $i, j \in \mathbb{Z}$,*

$$\left\{ (L', L'') \in \Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L) : \dim \left(\frac{L'_i}{L'_i \cap L''_j} \right) \leq c \right\}$$

is a closed set in the projective variety $\Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L)$.

Proof. There is $M \geq N$ so that $\varepsilon^M L_0 \subset L'_i \subset \varepsilon^{-M} L_0$ and $\varepsilon^M L_0 \subset L''_j \subset \varepsilon^{-M} L_0$. Let $a' = a + (M - N)r$ and $b' = b + (M - N)r$. Lemma 5.4.3 shows that $\Pi_{N,\lambda}^a(L)$ is a subvariety of $\Pi_{M,\lambda}^{a'}(L)$, so $\Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L)$ is a subvariety of $\Pi_{M,\lambda}^{a'}(L) \times \Pi_{M,\mu}^{b'}(L)$.

The fact that

$$\dim \left(\frac{L'_i}{L'_i \cap L''_j} \right) = \dim \left(\frac{L'_i / \varepsilon^M L_0}{L'_i / \varepsilon^M L_0 \cap L''_j / \varepsilon^M L_0} \right),$$

together with Lemma 5.4.2 and Lemma 5.3.1, shows that

$$\left\{ (L', L'') \in \Pi_{M,\lambda}^{a'}(L) \times \Pi_{M,\mu}^{b'}(L) : \dim \left(\frac{L'_i}{L'_i \cap L''_j} \right) \leq c \right\}$$

is closed, so the intersection with $\Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L)$ is closed. \square

Lemma 5.4.8. *Given $N, a, c \in \mathbb{N}$, $\lambda \in \Lambda_0$, $L \in \mathcal{F}$ and $i, j \in \mathbb{Z}$,*

$$\left\{ L' \in \Pi_{N,\lambda}^a(L) : \dim \left(\frac{L_i}{L_i \cap L'_j} \right) \leq c \right\}$$

and

$$\left\{ L' \in \Pi_{N,\lambda}^a(L) : \dim \left(\frac{L'_j}{L_i \cap L'_j} \right) \leq c \right\}$$

are closed sets in $\Pi_{N,\lambda}^a(L)$.

Proof. This is a result of Lemma 5.3.2, since

$$\dim \left(\frac{L_i}{L_i \cap L'_j} \right) = \dim \left(\frac{L_i / \varepsilon^M L_0}{L_i / \varepsilon^M L_0 \cap L'_j / \varepsilon^M L_0} \right),$$

where $M \geq N$ is chosen so that $\varepsilon^M L_0 \subset L_i \subset \varepsilon^{-M} L_0$ and $\varepsilon^M L_0 \subset L'_j \subset \varepsilon^{-M} L_0$ for each $L' \in \Pi_{N,\lambda}^a(L)$. \square

5.5 Geometry of orbits

Let $A \in \Lambda_1$ and $L \in \mathcal{F}_{\text{ro}(A)}$ and write $\lambda = \text{co}(A)$.

Lemma 5.5.1. *There is $N \in \mathbb{N}$ such that $X_A^L \subset \Pi_{N,\lambda}^a(L)$, where $a = d_{nN,0}A$.*

Proof. There is $N \in \mathbb{N}$ so that $a_{i,j} = 0$ whenever $|j - i| > nN$. If $(L, L') \in \mathcal{O}_A$ then

$$\dim \left(\frac{L'_0}{L'_0 \cap \varepsilon^{-N} L_0} \right) = \dim \left(\frac{L'_0}{L'_0 \cap L_{nN}} \right) = \sum_{s > nN, t \leq 0} a_{s,t} = 0,$$

so it follows $L'_0 \subset \varepsilon^{-N} L_0$. Similarly,

$$\dim \left(\frac{\varepsilon^N L_0}{\varepsilon^N L_0 \cap L'_0} \right) = \dim \left(\frac{L_{-nN}}{L_{-nN} \cap L'_0} \right) = \sum_{s \leq -nN, t > 0} a_{s,t} = 0,$$

which shows $\varepsilon^N L_0 \subset L'_0$. Moreover,

$$\dim \left(\frac{\varepsilon^{-N} L_0}{L'_0} \right) = \dim \left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N} L_0 \cap L'_0} \right) = \sum_{s \leq nN, t > 0} a_{s,t} = d_{nN,0}(A),$$

as a result of Lemma 5.2.4. \square

Assume $N \in \mathbb{N}$ is chosen so that $X_A^L \subset \Pi_{N,\lambda}^a(L)$, where $a = d_{nN,0}A$, as in Lemma 5.5.1.

Lemma 5.5.2. X_A^L is a locally closed subset of $\Pi_{N,\lambda}^a(L)$. In particular, X_A^L is a quasiprojective variety.

Proof. If $L' \in \Pi_{N,\lambda}^a(L)$ then

$$L_{-Nn} = \varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset L'_n \subset \varepsilon^{-1-N} L_0 = L_{(N+1)n}.$$

Therefore X_A^L is the subset of $\Pi_{N,\lambda}^a(L)$ defined by the conditions $\dim(L_i/L_i \cap L'_j) = d_{i,j}A$ for $i : -Nn \leq i < j$ and $\dim(L'_j/L_i \cap L'_j) = \bar{d}_{i,j}A$ for $i : j < i \leq (N+1)n$, for $j = 1, \dots, n$.

The set of $L' \in \Pi_{N,\lambda}^a(L)$ with $\dim(L_i/L_i \cap L'_j) \leq d_{i,j}A$ for $j = 1, \dots, n$ and $i : -Nn \leq i < j$ and $\dim(L'_j/L_i \cap L'_j) \leq \bar{d}_{i,j}A$ for $j = 1, \dots, n$ and $i : j < i \leq (N+1)n$ is a closed subset of $\Pi_{N,\lambda}^a(L)$, as a result of Lemma 5.4.8.

On the other hand, the set of $L' \in \Pi_{N,\lambda}^a(L)$ satisfying the conditions $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}A$ (for $i < j$) and $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}A$ (for $i > j$) is open in $\Pi_{N,\lambda}^a(L)$ since the complement is closed, as a result of Lemma 5.4.8.

Therefore X_A^L is the intersection of an open set and a closed set in $\Pi_{N,\lambda}^a(L)$, so X_A^L is locally closed. It follows that X_A^L is an open subset of the projective variety $\overline{X_A^L}$, so is a quasiprojective variety as claimed. \square

Lemma 5.5.3. X_A^L is irreducible.

Proof. For any $L' \in X_A^L$, $X_A^L = G_L/H_{N,L} \cdot L'$. Lemma 5.4.6 shows that $G_L/H_{N,L}$ is a connected algebraic group which acts algebraically on $\Pi_{N,\lambda}^a(L)$. The image of $G_L/H_{N,L}$ under the morphism $g \mapsto gL'$ equals X_A^L , which shows X_A^L is irreducible since $G_L/H_{N,L}$ is irreducible. \square

Consequently, $\overline{X_A^L}$ is an irreducible projective variety and the action of $G_L/H_{N,L}$ on $\Pi_{N,\lambda}^a(L)$ restricts to an algebraic group action on $\overline{X_A^L}$ for which there are finitely many orbits. In particular, $\overline{X_A^L} \setminus X_A^L$ is a union of finitely many orbits which are so-called degenerations of the orbit X_A^L .

5.6 Geometry of orbit products

Let $A, B \in \Lambda_1$ with $\text{co}(A) = \text{ro}(B)$ and write $\lambda = \text{co}(A)$ and $\mu = \text{co}(B)$. Fix $L \in \mathcal{F}_{\text{ro}(A)}$. Recall

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}_\lambda \times \mathcal{F}_\mu : L' \in X_A^L, L'' \in X_B^{L'}\}$$

and

$$X_{A,B}^L = \{L'' \in \mathcal{F}_\mu : \exists L' \in X_A^L \text{ with } L'' \in X_B^{L'}\}$$

Lemma 5.6.1. There is $N \in \mathbb{N}$ such that

$$Y_{A,B}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L),$$

where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$.

Proof. There is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$ for each $(L', L'') \in Y_{A,B}^L$, using Lemma 5.5.1. Set $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$.

Then for any $(L', L'') \in Y_{A,B}^L$,

$$\varepsilon^{2N} L_0 \subset \varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0 \subset \varepsilon^{-2N} L_0$$

and

$$\begin{aligned}
\dim \left(\frac{\varepsilon^{-2N} L_0}{L_0''} \right) &= \dim \left(\frac{\varepsilon^{-N} L_0'}{L_0''} \right) + \dim \left(\frac{\varepsilon^{-2N} L_0}{\varepsilon^{-N} L_0'} \right) \\
&= \dim \left(\frac{\varepsilon^{-N} L_0'}{L_0''} \right) + \dim \left(\frac{\varepsilon^{-N} L_0}{L_0'} \right) \\
&= a + b,
\end{aligned}$$

as a result of Lemma 5.2.4, so $(L', L'') \in \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ as required. \square

Now assume $N \in \mathbb{N}$ is chosen so that $Y_{A,B}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$, where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$, using Lemma 5.6.1.

Lemma 5.6.2. $Y_{A,B}^L$ is a locally closed subset of $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$. In particular, $Y_{A,B}^L$ is a quasiprojective variety.

Proof. $Y_{A,B}^L$ is the subset of $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ consisting of those (L', L'') satisfying the following conditions: $\dim(L_i/L_i \cap L_j') = d_{i,j}(A)$ for $i < j$, $\dim(L_j'/L_i \cap L_j') = \bar{d}_{i,j}(A)$ for $i > j$, $\dim(L_i'/L_i' \cap L_j'') = d_{i,j}(B)$ for $i < j$ and $\dim(L_j''/L_i' \cap L_j'') = \bar{d}_{i,j}(B)$. Only finitely many conditions are required to define $Y_{A,B}^L$ since there are only finitely many nonzero entries in A and B modulo the (n, n) -periodicity.

The conditions $\dim(L_i/L_i \cap L_j') \leq d_{i,j}(A)$, $\dim(L_j'/L_i \cap L_j') \leq \bar{d}_{i,j}(A)$ and $\dim(L_i'/L_i' \cap L_j'') \leq d_{i,j}(B)$, $\dim(L_j''/L_i' \cap L_j'') \leq \bar{d}_{i,j}(B)$ define closed subsets of $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ for each $i, j \in \mathbb{Z}$, as a result of Lemma 5.4.7 and Lemma 5.4.8.

On the other hand, the conditions $\dim(L_i/L_i \cap L_j') \geq d_{i,j}(A)$, $\dim(L_j'/L_i \cap L_j') \geq \bar{d}_{i,j}(A)$ and $\dim(L_i'/L_i' \cap L_j'') \geq d_{i,j}(B)$, $\dim(L_j''/L_i' \cap L_j'') \geq \bar{d}_{i,j}(B)$ define open subsets of $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ for each $i, j \in \mathbb{Z}$, using Lemma 5.4.7 and Lemma 5.4.8.

Therefore $Y_{A,B}^L$ is the intersection of finitely many open sets and finitely many closed sets in $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$, so $Y_{A,B}^L$ is locally closed. In particular, $Y_{A,B}^L$ is a quasiprojective variety. \square

Lemma 5.6.3. For any $L' \in X_A^L$, $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$.

Proof. Let $L' \in X_A^L$, then $\{L'\} \times X_B^{L'}$ is contained in $Y_{A,B}^L$ and G_L acts on $Y_{A,B}^L$, so $G_L \cdot (\{L'\} \times X_B^{L'})$ is contained in $Y_{A,B}^L$. If $(N', N'') \in Y_{A,B}^L$, then $N' = \sigma L'$ for some $\sigma \in G_L$, since $N' \in X_A^L$. Then $(N', N'') = \sigma(L', \sigma^{-1} N'')$ and $\sigma^{-1} N'' \in X_B^{\sigma^{-1} N'} = X_B^{L'}$, so $(N', N'') \in \sigma \cdot (\{L'\} \times X_B^{L'})$. Therefore $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$ as claimed. \square

Proposition 5.6.4. $Y_{A,B}^L$ is irreducible.

Proof. Let $L' \in X_A^L$. $G_L/H_{2N,L}$ is a connected algebraic group acting algebraically on $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ by Lemma 5.4.6. $X_B^{L'}$ is an irreducible locally closed subset of $\Pi_{2N,\mu}^{a+b}(L)$, so $\{L'\} \times X_B^{L'}$ is an irreducible locally closed set in $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$. $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'}) = G_L/H_{2N,L} \cdot (\{L'\} \times X_B^{L'})$, by Lemma 5.6.3, so it follows that $Y_{A,B}^L$ is irreducible. \square

Let p_2 be the projection onto the second factor $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \rightarrow \Pi_{2N,\mu}^{a+b}(L)$. p_2 is a closed morphism since $\Pi_{N,\lambda}^a(L)$ is a projective variety and therefore complete, by Lemma 5.4.2. Therefore $p_2(\overline{Y_{A,B}^L}) = \overline{X_B^{L'}}$, since $p_2(Y_{A,B}^L) = X_B^{L'}$.

Lemma 5.6.5. $X_{A,B}^L$ is irreducible and constructible.

Proof. Proposition 5.6.4 shows that $Y_{A,B}^L$ is irreducible and locally closed, so it follows $X_{A,B}^L$ is irreducible and constructible, since $X_{A,B}^L = p_2(Y_{A,B}^L)$. \square

Proposition 5.6.6. There is a unique open G_L -orbit in $X_{A,B}^L$.

Proof. $X_{A,B}^L$ consists of finitely many G_L -orbits and is an irreducible topological space, by Lemma 5.6.5. Consequently, X_C^L is dense in $X_{A,B}^L$ for some $C \in \Lambda_1^{A,B}$. Lemma 5.5.2 shows that X_C^L is locally closed in $X_{A,B}^L$, so X_C^L is open in $\overline{X_C^L} = X_{A,B}^L$. Irreducibility of $X_{A,B}^L$ shows that there is a unique open G_L -orbit, since two nonempty open sets in $X_{A,B}^L$ intersect nontrivially, thus any two open G_L orbits in $X_{A,B}^L$ coincide. \square

Let $A * B \in \Lambda_1$ be the matrix corresponding to the dense open G_L -orbit in $X_{A,B}^L$, so $\overline{X_{A*B}^L} = \overline{X_{A,B}^L}$.

5.7 Degenerations of orbits and the combinatorial partial order

Lemma 5.7.1. Given $A, A' \in \Lambda_1$ with $\text{ro}(A) = \text{ro}(A')$ and $\text{co}(A) = \text{co}(A')$, $A' \leq A$ if and only if $X_{A'}^L \subset \overline{X_A^L}$ for any $L \in \mathcal{F}_{\text{ro}(A)}$.

Proof. Needs a proof. \square

Proposition 5.7.2. Given $A, B \in \Lambda_1$ with $\text{co}(A) = \text{ro}(B)$, $\Lambda_1^{A,B}$ has a maximum element.

Proof. Let $L \in \mathcal{F}_{\text{ro}(A)}$. $X_{A,B}^L$ is irreducible by Lemma 5.6.5 and is the union of finitely many G_L -orbits, namely

$$X_{A,B}^L = \bigcup_{C \in \Lambda_1^{A,B}} X_C^L.$$

This shows that X_C^L is dense in $X_{A,B}^L$ for some $C \in \Lambda_1^{A,B}$. Lemma 5.5.2 shows that the G_L -orbits in $X_{A,B}^L$ are locally closed, so a dense G_L -orbit is open in $X_{A,B}^L$. Lemma 5.7.1 shows that the characteristic matrix of the dense G_L -orbit is a maximum in $\Lambda_1^{A,B}$. \square

5.8 Associativity of the generic product

Let $A, B, C \in \Lambda_1$ with $\text{co}(A) = \text{ro}(B)$ and $\text{co}(B) = \text{ro}(C)$ and fix $L \in \mathcal{F}_{\text{ro}(A)}$. Write $\lambda = \text{co}(A)$, $\mu = \text{co}(B)$ and $\nu = \text{co}(C)$. Define

$$Y_{A,B,C}^L = \left\{ (L', L'', L''') \in \mathcal{F}^3 : L' \in X_A^L, L'' \in X_B^L, L''' \in X_C^L \right\}$$

and

$$X_{A,B,C}^L = \{ L''' \in \mathcal{F} : \exists (L', L'') \in \mathcal{F}^2 \text{ with } (L', L'', L''') \in Y_{A,B,C}^L \} ..$$

Lemma 5.8.1. There is $N \in \mathbb{N}$ such that $Y_{A,B,C}^L$ is contained in $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$, where $a = d_{nN,0}(A)$, $b = d_{nN,0}(B)$ and $c = d_{nN,0}(C)$.

Proof. Lemma 5.5.1 shows that there is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$ and $\varepsilon^N L''_0 \subset L'''_0 \subset \varepsilon^{-N} L''_0$ for each $(L', L'', L''') \in Y_{A,B,C}^L$. Using the proof of Lemma 5.6.1, it follows $L'' \in \Pi_{2N,\mu}^{a+b}(L)$ and $L''' \in \Pi_{2N,\nu}^{b+c}(L') \subset \Pi_{3N,\nu}^{a+b+c}(L)$. \square

Assume $N \in \mathbb{N}$ is chosen so that $Y_{A,B,C}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$, where $a = d_{nN,0}(A)$, $b = d_{nN,0}(B)$ and $c = d_{nN,0}(C)$, as in Lemma 5.8.1.

Lemma 5.8.2. $Y_{A,B,C}^L$ is a locally closed subset of $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$. In particular, $Y_{A,B,C}^L$ is a quasiprojective variety.

Proof. Write $\Pi = \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}(L)$. Then $Y_{A,B,C}^L$ consists of those $(L', L'', L''') \in \Pi$ satisfying the following conditions:

$$\dim \left(\frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}(A), \quad (5.1)$$

$$\dim \left(\frac{L'_i}{L'_i \cap L''_j} \right) = d_{i,j}(B), \quad (5.2)$$

$$\dim \left(\frac{L''_i}{L''_i \cap L'''_j} \right) = d_{i,j}(C), \quad (5.3)$$

for $(i, j) \in [1, n] \times \mathbb{Z}$ with $i < j < (N+1)n$, and

$$\dim \left(\frac{L'_j}{L_i \cap L'_j} \right) = \bar{d}_{i,j}(A), \quad (5.4)$$

$$\dim \left(\frac{L''_j}{L'_i \cap L''_j} \right) = \bar{d}_{i,j}(B), \quad (5.5)$$

$$\dim \left(\frac{L'''_j}{L''_i \cap L'''_j} \right) = \bar{d}_{i,j}(C), \quad (5.6)$$

for $(i, j) \in [1, n] \times \mathbb{Z}$ with $-Nn < j < i$.

For $i < j$, the conditions

$$\begin{aligned} \dim(L_i/L_i \cap L'_j) &\leq d_{i,j}(A), \\ \dim(L'_i/L'_i \cap L''_j) &\leq d_{i,j}(B) \end{aligned}$$

and

$$\dim(L''_i/L''_i \cap L'''_j) \leq d_{i,j}(C)$$

define closed subsets of Π , by Lemma 5.4.7. For $i > j$, the conditions

$$\begin{aligned} \dim(L'_j/L_i \cap L'_j) &\leq \bar{d}_{i,j}(A), \\ \dim(L''_j/L'_i \cap L''_j) &\leq \bar{d}_{i,j}(B) \end{aligned}$$

and

$$\dim(L'''_j/L''_i \cap L'''_j) \leq \bar{d}_{i,j}(C)$$

also define closed subsets of Π .

On the other hand, the conditions $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}(A)$, $\dim(L'_i/L'_i \cap L''_j) \geq d_{i,j}(B)$ and $\dim(L''_i/L''_i \cap L'''_j) \geq d_{i,j}(C)$ for $i < j$ define open subsets of Π . Similarly, the conditions $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}(A)$, $\dim(L''_j/L'_i \cap L''_j) \geq \bar{d}_{i,j}(B)$ and $\dim(L'''_j/L''_i \cap L'''_j) \geq \bar{d}_{i,j}(C)$ for $i > j$ define open subsets of Π .

Therefore $Y_{A,B,C}^L$ is the intersection of finitely many closed sets in Π with finitely many open subsets of Π , so $Y_{A,B,C}^L$ is locally closed. In particular, $Y_{A,B,C}^L$ is a quasiprojective variety. \square

Lemma 5.8.3. *For any $(L', L'', L''') \in Y_{A,B,C}^L$,*

$$Y_{A,B,C}^L = \{ \alpha \cdot (L', \beta L'', \beta \gamma L''') : \alpha \in G_L, \beta \in G_{L'}, \gamma \in G_{L''} \}.$$

In particular,

$$Y_{A,B,C}^L = G_L \cdot (\{L'\} \times Y_{B,C}^{L'})$$

for each $L' \in X_A^L$.

Proof. Let $(L', L'', L''') \in Y_{A,B,C}^L$. Given $\alpha \in G_L$, $\beta \in G_{L'}$ and $\gamma \in G_{L''}$, $(L, \alpha L', \alpha \beta L'', \alpha \beta \gamma L''')$ is in $Y_{A,B,C}^L$ since

$$\begin{aligned} (L, \alpha L') &= \alpha(L, L') \in \mathcal{O}_A \\ (\alpha L', \alpha \beta L'') &= \alpha \beta(L', L'') \in \mathcal{O}_B \\ (\alpha \beta L'', \alpha \beta \gamma L''') &= \alpha \beta \gamma(L'', L''') \in \mathcal{O}_C \end{aligned}$$

For each $(N', N'', N''') \in Y_{A,B,C}^L$ there exist $\sigma_1, \sigma_2, \sigma_3 \in G$ with

$$\begin{aligned} (L, N') &= \sigma_1(L, L') \\ (N', N'') &= \sigma_2(L', L'') \\ (N'', N''') &= \sigma_3(L'', L'''). \end{aligned}$$

Let $\alpha = \sigma_1$, $\beta = \sigma_1^{-1} \sigma_2$ and $\gamma = \sigma_2^{-1} \sigma_3$, so $\sigma_2 = \alpha \beta$ and $\sigma_3 = \alpha \beta \gamma$. It follows that

$$(N', N'', N''') = (\alpha L', \alpha \beta L'', \alpha \beta \gamma L'''),$$

which proves the first claim. The second claim follows from the first since $(L'', L''') \in Y_{B,C}^{L'}$ and therefore

$$Y_{B,C}^{L'} = \{(\beta L'', \beta \gamma L''') : \beta \in G_{L'}, \gamma \in G_{L''}\},$$

as required. \square

Lemma 5.8.4. *$Y_{A,B,C}^L$ is irreducible.*

Proof. There is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$ and $\varepsilon^N L''_0 \subset L'''_0 \subset \varepsilon^{-N} L''_0$ for each $(L', L'', L''') \in Y_{A,B,C}^L$. Lemma 5.4.6 shows that $G_L/H_{3N,L}$ is a connected algebraic group acting algebraically on $\Pi = \Pi_{N,\text{co}(A)}^a(L) \times \Pi_{2N,\text{co}(B)}^{a+b}(L) \times \Pi_{3N,\text{co}(C)}^{a+b+c}(L)$.

Let $L' \in X_A^L$. $Y_{A,B,C}^L = G_L \cdot (\{L'\} \times Y_{B,C}^{L'})$. $Y_{B,C}^{L'}$ is an irreducible quasiprojective variety; $\overline{Y_{B,C}^{L'}}$ is an irreducible subvariety of $\Pi_{N,\text{co}(B)}^b(L') \times \Pi_{2N,\text{co}(C)}^{b+c}(L')$, which is a subvariety of $\Pi_{2N,\text{co}(B)}^{a+b}(L) \times \Pi_{3N,\text{co}(C)}^{a+b+c}(L)$. Thus $\{L'\} \times \overline{Y_{B,C}^{L'}}$ is an irreducible subvariety of Π . Therefore $Y_{A,B,C}^L$ is the image of the irreducible space $G_L/H_{3N,L} \times \{L'\} \times Y_{B,C}^{L'}$ under the action map, so $Y_{A,B,C}^L$ is irreducible. Lemma 5.8.2 shows that $Y_{A,B,C}^L$ is quasiprojective, so $Y_{A,B,C}^L$ is an irreducible quasiprojective variety. \square

Let p_3 be the projection of $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$ onto the third factor. By the completeness property of projective varieties, p_3 is a closed morphism. The image of $Y_{A,B,C}^L$ under p_3 is $X_{A,B,C}^L$, so $p_3(\overline{Y_{A,B,C}^L}) = \overline{X_{A,B,C}^L}$.

Lemma 5.8.5. $X_{A,B,C}^L$ is irreducible and constructible.

Proof. Lemma 5.8.2 and Lemma 5.8.4 show that $Y_{A,B,C}^L$ is irreducible and locally closed. It follows $X_{A,B}^L$ is irreducible and constructible, since $X_{A,B,C}^L$ is the image of $Y_{A,B,C}^L$ under the morphism p_3 . \square

Lemma 5.8.6. There is a unique open and dense G_L -orbit in $X_{A,B,C}^L$.

Proof. There are only finitely many G_L -orbits in $X_{A,B,C}^L$. In particular,

$$X_{A,B,C}^L = \bigcup_{D \in \Lambda_1^{A,B}} X_{D,C}^L = \bigcup_{D \in \Lambda_1^{A,B}} \bigcup_{D' \in \Lambda_1^{D,C}} X_{D'}^L$$

and

$$\overline{X_{A,B,C}^L} = \bigcup_{D \in \Lambda_1^{A,B}} \bigcup_{D' \in \Lambda_1^{D,C}} \overline{X_{D'}^L}.$$

There is $D \in \Lambda_1$ such that $\overline{X_D^L} = \overline{X_{A,B,C}^L}$, since $X_{A,B,C}^L$ is irreducible, by Lemma 5.8.5. By Lemma 5.5.2, X_D^L is open in $\overline{X_D^L} = \overline{X_{A,B,C}^L}$, so X_D^L is open in $X_{A,B,C}^L$.

If X_D^L and $X_{D'}^L$ are open in $X_{A,B,C}^L$, then X_D^L and $X_{D'}^L$ have nonempty intersection since $X_{A,B,C}^L$ is irreducible, then $X_D^L = X_{D'}^L$. \square

Lemma 5.8.7. $p_3^{-1}(X_{A*B,C}^L)$ is open in $\overline{Y_{A,B,C}^L}$.

Proof. Projection onto the second component is a closed morphism of varieties $p_2: \overline{Y_{A,B,C}^L} \rightarrow \overline{X_{A,B}^L}$ with $p_2(Y_{A,B,C}^L) = X_{A,B}^L$. It follows that $p_3^{-1}(X_{A*B,C}^L)$ is open in $\overline{Y_{A,B,C}^L}$ since $p_3^{-1}(X_{A*B,C}^L) = p_2^{-1}(X_{A*B}^L)$ and X_{A*B}^L is open in $\overline{X_{A,B}^L}$. \square

Lemma 5.8.8. $p_3^{-1}(X_{A,B*C}^L)$ is open in $\overline{Y_{A,B,C}^L}$.

Proof. $p_3^{-1}(X_{A,B*C}^L)$ consists of those $(L', L'', L''') \in \overline{Y_{A,B,C}^L}$ such that $\dim(L'_i/L'_i \cap L''_j) \geq d_{i,j}(B * C)$ for $i < j$ and $\dim(L''_j/L'_i \cap L'''_j) \geq \bar{d}_{i,j}(B * C)$ for $i > j$. Each of these conditions defines an open subset of $\overline{Y_{A,B,C}^L}$ as a result of Lemma 5.4.7 and only finitely many conditions are required to determine $p_3^{-1}(X_{A,B*C}^L)$, as before. Therefore $p_3^{-1}(X_{A,B*C}^L)$ is the intersection of finitely many open sets in $\overline{Y_{A,B,C}^L}$, so is open as claimed. \square

Proposition 5.8.9. $X_{A*(B*C)}^L = X_{(A*B)*C}^L$.

Proof. The unique open G_L -orbit in $X_{A*B,C}^L$ is $X_{(A*B)*C}^L$, so $p_3^{-1}(X_{(A*B)*C}^L)$ is open in $p_3^{-1}(X_{A*B,C}^L)$. Lemma 5.8.7 shows that $p_3^{-1}(X_{A*B,C}^L)$ is open in $\overline{Y_{A,B,C}^L}$, so $p_3^{-1}(X_{(A*B)*C}^L)$ is open in $\overline{Y_{A,B,C}^L}$.

Similarly, $X_{A*(B*C)}^L$ is open in $X_{A,B*C}^L$, so $p_3^{-1}(X_{A*(B*C)}^L)$ is open in $p_3^{-1}(X_{A,B*C}^L)$. Lemma 5.8.8 shows that $p_3^{-1}(X_{A,B*C}^L)$ is open in $\overline{Y_{A,B,C}^L}$, so it follows $p_3^{-1}(X_{A*(B*C)}^L)$ is open in $\overline{Y_{A,B,C}^L}$.

Therefore $f^{-1}(X_{A*(B*C)}^L)$ has nonempty intersection with $f^{-1}(X_{(A*B)*C}^L)$, since $Y_{A,B,C}^L$ is irreducible by Lemma 5.8.4. It follows that the G_L -orbits $X_{A*(B*C)}^L$ and $X_{(A*B)*C}^L$ have nonempty intersection and therefore $X_{A*(B*C)}^L$ equals $X_{(A*B)*C}^L$. \square

5.9 The generic algebra

Lemma 5.9.1. *Given $\lambda \in \Lambda_0$ and $A \in \Lambda_1$, $D_\lambda * A = A$ if $\text{ro}(A) = \lambda$ and $A * D_\lambda = A$ if $\text{co}(A) = \lambda$.*

Proof. Lemma 3.1.7 shows that $\Lambda_1^{D_\lambda, A} = \{A\}$ if $\lambda = \text{ro}(A)$ and $\Lambda_1 A, D_\lambda = \{A\}$ if $\lambda = \text{co}(A)$, which proves the result. \square

Theorem 5.9.2. *The following constitutes a small category: the set of objects is Λ_0 and the set of morphisms is Λ_1 . Given compositions $\lambda, \mu \in \Lambda_0$, the morphisms with source λ and target μ are those matrices $A \in \Lambda_1$ with $\text{co}(A) = \lambda$ and $\text{ro}(A) = \mu$. Given $\lambda, \mu, \nu \in \Lambda_0$ and $A, B \in \Lambda_1$ with $B: \lambda \rightarrow \mu$ and $A: \mu \rightarrow \nu$ the composition is $A * B: \lambda \rightarrow \nu$.*

Proof. Proposition 5.7.2 shows that the composition is well defined while Proposition 5.8.9 establishes associativity of the composition. Lemma 5.9.1 shows that $D_\lambda: \lambda \rightarrow \lambda$ is the identity morphism for each $\lambda \in \Lambda_0$. Thus $(\Lambda_0, \Lambda_1, \text{co}(\cdot), \text{ro}(\cdot), *)$ is a category. \square

Write $\mathcal{G}(n, r)$ to denote this so-called ‘generic category’.

Example 1. *The objects in $\mathcal{G}(2, 2)$ are compositions of 2 into 2 parts, namely $(0, 2)$, $(1, 1)$ and $(2, 0)$. The set of morphisms from λ to μ is the set of infinite periodic matrices $A \in \Lambda_1^{2,2}$ with $\text{co}(A) = \lambda$ and $\text{ro}(A) = \mu$, which is a countably infinite set for any pair of compositions $\lambda, \mu \in \Lambda_0[2, 2]$.*

Definition 5.9.1 (Generic algebra). *The generic affine algebra $\hat{G}(n, r)$ is the category \mathbb{Z} -algebra of $\mathcal{G}(n, r)$. In particular, $\hat{G}(n, r)$ is a free \mathbb{Z} -module with basis $\{e_A : A \in \Lambda_1\}$ and with associative multiplication given by*

$$e_A * e_B = \begin{cases} e_{A*B} & \text{if } \text{co}(A) = \text{ro}(B) \\ 0 & \text{if } \text{co}(A) \neq \text{ro}(B). \end{cases}$$

The multiplicative identity in $\hat{G}(n, r)$ is

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

Chapter 6

A realisation of affine zero Schur algebras

We aim to prove the isomorphism theorem in the cases $r < n$ and $n \leq r < 2n$ separately. Below are crude versions of the statements we want to prove.

Theorem 6.0.1. *Assume $r < n$. The map $\psi: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$, given by $\psi(E_i) = E_i$, $\psi(F_i) = F_i$ and $\psi(1_\lambda) = 1_\lambda$, is an isomorphism of \mathbb{Z} -algebras.*

Theorem 6.0.2. *Assume $n \leq r < 2n$. There is a unique homomorphism of \mathbb{Z} -algebras $\hat{\psi}: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$ such that $\hat{\psi}(R) = R$ and $\hat{\psi} = \psi$ on the subalgebra of $\hat{G}(n, r)$ generated by the E_i , F_i and 1_λ . $\hat{\psi}$ is an isomorphism of \mathbb{Z} -algebras.*

6.1 Preliminary results

Recall from Definition 5.9.1 that the generic algebra $\hat{G}(n, r)$ is an associative \mathbb{Z} -algebra which is a free \mathbb{Z} -module with an atomic basis $\{e_A : A \in \Lambda_1\}$: given $A, B \in \Lambda_1$ with $\text{co}(A) = \text{ro}(B)$, $e_A e_B = e_{A*B}$.

6.1.1 Elementary basis elements

Given $i \in [1, n]$ and $\lambda \in \Lambda_0$ such that $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and

$$E_i = \sum_{\lambda \in \Lambda_0: \lambda_{i+1} > 0} E_{i,\lambda}.$$

Given $i \in [1, n]$ and $\lambda \in \Lambda_0$ such that $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and

$$F_i = \sum_{\lambda \in \Lambda_0: \lambda_i > 0} F_{i,\lambda}.$$

6.1.2 Transpose involution

Lemma 6.1.1. *The \mathbb{Z} -module automorphism \top of $\hat{G}(n, r)$ given by $e_A \mapsto e_{A^\top}$ is a \mathbb{Z} -algebra antihomomorphism: that is,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A$$

for each $A, B \in \Lambda_1$. Moreover, $\top(E_{i,\lambda}) = F_{i,\lambda+\alpha_i}$, $\top(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}$ and $\top(1_\lambda) = 1_\lambda$, for permissible $(i, \lambda) \in \mathbb{Z} \times \Lambda_0$.

Proof. This is a consequence of Lemma 4.1.1. It must also be shown that the transpose operation on Λ_1 is order preserving. \square

6.1.3 Multiplication rules

Lemma 6.1.2. *Given $A \in \Lambda_1$ and $i \in [1, n]$ such that $\text{ro}(A)_{i+1} > 0$,*

$$E_i e_A = e_{A+\varepsilon_{i,p}-\varepsilon_{i+1,p}},$$

where $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$.

Given $A \in \Lambda_1$ and $i \in [1, n]$ such that $\text{ro}(A)_i > 0$,

$$F_i e_A = e_{A+\varepsilon_{i+1,p}-\varepsilon_{i,p}},$$

where $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$.

Similar formulas for right multiplication by E_i and F_i are obtained by applying the transpose.

6.2 Presentation of the generic algebra.

Recall that Λ_0 denotes the set of compositions of r into n parts. That is, Λ_0 is the set of tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with each λ_i non-negative and $\lambda_1 + \dots + \lambda_n = r$. Given $i \in [1, n]$, let $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$ be the i -th elementary vector and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then given $\lambda \in \Lambda_0$, we have $\lambda + \alpha_i \in \Lambda_0$ provided $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ provided $\lambda_i > 0$.

Let $\Gamma = \Gamma(n, r)$ be the quiver with set of vertices Λ_0 with arrows $e_{i,\lambda}: \lambda \rightarrow \lambda + \alpha_i$ (if $\lambda_{i+1} > 0$) and $f_{i,\lambda}: \lambda \rightarrow \lambda - \alpha_i$ (if $\lambda_i > 0$). Thus there are no arrows between λ and μ unless $\lambda = \mu \pm \alpha_i$ for some $i \in [1, n]$.

If $n \geq 3$ then neighbouring vertices are connected by two arrows, one of each direction. In the case $n = 2$, neighbouring vertices are joined by four arrows, two of each direction. The $\mathbb{Z}\Gamma$ denote the path \mathbb{Z} algebra of Γ . By construction of Γ , there is a \mathbb{Z} -algebra homomorphism $\mathbb{Z}\Gamma \rightarrow \hat{G}(n, r)$ with $e_{i,\lambda} \mapsto E_{i,\lambda}$, $f_{i,\lambda} \mapsto F_{i,\lambda}$ and $k_\lambda = 1_\lambda$. We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [32]).

Definition 6.2.1. (*aperiodicity*) $A \in \Lambda_1$ is aperiodic if for each $l \in \mathbb{Z} \setminus \{0\}$ there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$. If $r < n$ then and $A \in \Lambda_1$ is aperiodic. Linear combinations of the basis elements corresponding to aperiodic matrices are also said to be aperiodic - if A is aperiodic, we say e_A is aperiodic.

Lemma 6.2.1. *Let $A \in \Lambda_1$ and write $\lambda = \text{ro}(A)$. If A is aperiodic and $\lambda_{i+1} > 0$, then $E_i * e_A$ is aperiodic. If A is aperiodic and $\lambda_i > 0$, then $F_i * e_A$ is aperiodic.*

Proof. Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_{i+1} > 0$, where $\lambda = \text{ro}(A)$. There is $p \in \mathbb{Z}$ such that $a_{i+1,p} > 0$ and $a_{i+1,p'} = 0$ whenever $p' > p$. Lemma 6.1.2 shows that $E_i * e_A = e_B$, where $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p-i, p-i-1\}$, then $b_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $b_{s,s+l} = a_{s,s+l} = 0$, since A is aperiodic. If $l = p-i$, then $b_{i+1,i+1+l} = b_{i+1,p+1} = a_{i+1,p+1} = 0$, by maximality of p . If $l = p-i-1$, there is $s \neq i+1$ such that $a_{s,s+l} = 0$, since A is aperiodic and $a_{i+1,i+1+l} = a_{i+1,p} > 0$, so $b_{s,s+l} = a_{s,s+l} = 0$. Therefore, $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ is aperiodic.

Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_i > 0$, where $\lambda = \text{ro}(A)$. Lemma 6.1.2 shows that $F_i * e_A = e_C$ where $C = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$ and $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p-i, p-i-1\}$ then $c_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $c_{s,s+l} = a_{s,s+l} = 0$, by aperiodicity of A . If $l = p-i$, then $a_{i,i+l} = a_{i,p} > 0$, so there is $s \neq i$ such that $a_{s,s+l} = 0$. Then $c_{s,s+l} = a_{s,s+l} = 0$. Finally, if $l = p-i-1$, then $c_{i,i+l} = a_{i,p-1} = 0$ by minimality of p . Thus C is aperiodic as required. \square

Definition 6.2.2. (*Weight function*) Define the weight function $\text{wt} : \Lambda_1 \rightarrow \mathbb{Z}$ by

$$\text{wt } A = \sum_{i \in [1,n], j \in \mathbb{Z}} |j-i| a_{i,j}$$

for each $A \in \Lambda_1$. The sum is taken over a transversal of the set of congruence classes of (i, j) modulo (n, n) for $i, j \in \mathbb{Z}$.

Lemma 6.2.2. Let $A \in \Lambda_1$ and write $\lambda = \text{ro}(A)$. Suppose $\lambda_{i+1} > 0$ and set $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$. If $p > i$ then $\text{wt } e_{i,\lambda} * A = 1 + \text{wt } A$. If $p \leq i$ then $\text{wt } e_{i,\lambda} * A = -1 + \text{wt } A$. Suppose $\lambda_i > 0$ and set $q = \min\{q' \in \mathbb{Z} : a_{i,q'} > 0\}$. If $q \leq i$ then $\text{wt } f_{i,\lambda} * A = 1 + \text{wt } A$. If $q > i$ then $\text{wt } f_{i,\lambda} * A = -1 + \text{wt } A$.

Proof. Lemma 6.1.2 shows that $e_i A = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$, so $\text{wt } e_i A - \text{wt } A = |p-i| - |p-i-1|$, which equals 1 if $p > i$ and equals -1 if $p \leq i$. Similarly, $f_i A = A + \mathcal{E}_{i+1,q} - \mathcal{E}_{i,q}$ by Lemma 6.1.2, so $\text{wt } f_i A - \text{wt } A = |q-i-1| - |q-i|$, which equals -1 if $q > i$ and equals 1 if $q \leq i$. \square

Lemma 6.2.3. If $A \in \Lambda_1$ is aperiodic, then e_A may be obtained from $1_{\text{co}(A)}$ by finitely many applications of E_i and F_i for $i \in [1, n]$.

Proposition 6.2.4. The \mathbb{Z} -subalgebra of $\hat{G}(n, r)$ generated by $E_{i,\lambda}$, $F_{i,\lambda}$ and 1_λ has \mathbb{Z} -basis $\{e_A : A \in \Lambda_1 \text{ is aperiodic}\}$.

Proof. \square

6.2.1 The typical case.

Lemma 6.2.5. The following relations hold in $\hat{G}(n, r)$ ($n \geq 3$):

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless $|j-i| = 1$.

$$E_i E_{i+1}^2 - E_{i+1} E_i E_{i+1} = 0$$

$$E_i^2 E_{i+1} - E_i E_{i+1} E_i = 0$$

$$F_{i+1} F_i^2 - F_i F_{i+1} F_i = 0$$

$$F_{i+1}^2 F_i - F_{i+1} F_i F_{i+1} = 0$$

$$E_i F_j - F_j E_i = 0$$

unless $j = i$.

$$E_i F_i - F_i E_i + \sum_{\lambda: \lambda_i=0, \lambda_{i+1}>0} 1_\lambda - \sum_{\lambda: \lambda_i>0, \lambda_{i+1}=0} 1_\lambda = 0.$$

6.2.2 Exceptional case.

In this case, the quiver $\Gamma(2, r)$ has vertices $\Lambda_0[2, r] = \{(0, r), (1, r-1), \dots, (r, 0)\}$; adjacent vertices are connected by two pairs of arrows with opposite orientation: (e_1, f_1) and (e_2, f_2) . The relations arising from $\hat{G}(2, r)$ are of a more complicated form - in particular, the serre relations of total degree 3 will not hold in this case - so this case will be treated separately and at a later date.

Chapter 7

Further directions

7.1 Further results on affine zero Schur algebras

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

7.2 Deformed group algebras of symmetric groups

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algebras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for S_3 and S_4 . Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: ‘these’ relations hold; these generators and relations give a presentation of the degenerate group algebras.

Terminology: deformed group algebra.

7.3 back matter

[1] $Y_{A,B}^L$ is the image of $G_L \times G_{L'}$ under the action map $(\alpha, \beta) \mapsto \alpha\beta \cdot (L', L'')$, for any $(L', L'') \in Y_{A,B}^L$. Lemma 5.4.6 shows that $G_L/H_{N,L}$ is a connected algebraic group. Moreover, $G_{L'}/H_{2N,L}$ is an irreducible affine variety, so $G_L/H_{N,L} \times G_{L'}/H_{2N,L}$ is an irreducible affine variety. It follows that $Y_{A,B}^L$ is irreducible and constructible.

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