

Affine 0-Schur algebras and affine double flag varieties.

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# Chapter 1

## Introduction

The work of this thesis follows the geometric realisation of 0-Schur and 0-Hecke algebras given by Jensen and Su in [23] and is based on Lusztig's geometric approach to affine  $q$ -Schur algebras in [24].

To include in introduction: quantum groups

The first part of the introduction should discuss the context of the topic and mention references. Then proceed to describe the double flag variety realisations of (affine)  $q$ -Schur algebras in the next two sections.

### 1.1 The double flag variety approach to $q$ -Schur algebras

### 1.2 The cyclic flags approach to affine $q$ -Schur algebras

Affine  $q$ -Schur algebras arose from an affine analogue of quantum Schur-Weyl duality and as such are defined as the endomorphism algebra of a certain module over the affine Hecke algebra. The approach used in this thesis is based on Lusztig's construction using double affine flag varieties, which is itself an extension of a similar construction of finite type  $q$ -Schur algebras given by Beilinson, Lusztig and MacPherson in [1]. A good account of these two different realisations of affine  $q$ -Schur algebras can be found in the book [4] by Deng, Du and Fu.

Let  $\mathbf{k}$  be a finite or algebraically closed field and let  $n, r \geq 1$  be integers. Fix a free  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ -module  $V$  of rank  $r$  and let  $G$  be its automorphism group. A lattice in  $V$  is a  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ -submodule of  $V$  which is a free module of rank  $r$  over  $\mathbf{k}[\varepsilon]$ . The space of  $n$ -periodic flags in  $V$  is denoted by  $\mathcal{F}$  and consists of chains of lattices  $L = (L_i)_{i \in \mathbb{Z}}$  in  $V$  such that  $L_i \subset L_{i+1}$  and  $\varepsilon L_i = L_{i-n}$  for each  $i \in \mathbb{Z}$ . The group  $G$  acts naturally on  $\mathcal{F}$  and the orbits are indexed by the set of compositions of  $r$  into  $n$  parts. Given a composition  $\lambda = (\lambda_1, \dots, \lambda_n)$  of  $r$  into  $n$  parts, the corresponding  $G$ -orbit in  $\mathcal{F}$  is

$$\mathcal{F}_\lambda = \{L \in \mathcal{F} : \dim(L_i/L_{i-1}) = \lambda_i \text{ for all } i \in \mathbb{Z}\}.$$

The diagonal action of  $G$  on  $\mathcal{F} \times \mathcal{F}$  has orbits indexed by the set  $\Lambda_1$  of  $\mathbb{Z} \times \mathbb{Z}$  matrices with non-negative integer entries  $a_{i,j}$  such that  $a_{i,j} = a_{i-n,j-n}$  for all  $i, j \in \mathbb{Z}$  and the sum of the entries in any  $n$  consecutive rows or columns is  $r$ . The row vector of  $A$  is the composition  $\text{ro}(A)$  given by adding up the entries in each row and the column vector  $\text{co}(A)$  is the composition given by adding up the entries in each column. The orbit corresponding to  $A \in \Lambda_1$  is denoted by  $\mathcal{O}_A$  and

is the set of pairs of flags  $(L, L')$  such that

$$\dim \left( \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right) = a_{i,j}$$

for all  $i, j \in \mathbb{Z}$ . There are polynomials  $\gamma_{A,B,C} \in \mathbb{Z}[q]$  for  $A, B, C \in \Lambda_1$  such that

$$\gamma_{A,B,C}(\#\mathbf{k}) = \#\{L' \in \mathcal{F} : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

for any finite field  $\mathbf{k}$  and  $(L, L'') \in \mathcal{O}_C$ . The affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  is an associative algebra over  $\mathbb{Z}[q]$ , with a  $\mathbb{Z}[q]$ -basis

$$\{e_A : A \in \Lambda_1\}$$

and multiplication given by

$$e_A e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C} e_C$$

for  $A, B \in \Lambda_1$ .

There is a set of orthogonal idempotents  $\{1_\lambda : \lambda \in \Lambda_0\}$  in  $\hat{S}_q(n, r)$  with  $1 = \sum_{\lambda \in \Lambda_0} 1_\lambda$ , where  $1_\lambda$  is the basis element corresponding to the diagonal matrix with column vector equal to  $\lambda$ . There are distinguished elements  $E_i$  and  $F_i$  for  $i \in \{1, \dots, n\}$ , where  $E_i$  is the sum of those basis elements  $e_A$  such that  $a_{j,j+1} = 1$  when  $j = i$  modulo  $n$  with all other off-diagonal entries are zero, and  $F_i$  is the sum of those  $e_A$  such that  $a_{j+1,j} = 1$  when  $j = i$  modulo  $n$  and all other off-diagonal entries are zero.

There are the following multiplication rules for  $E_i$  and  $F_i$  which allow computations to be done in a clear combinatorial way and are later used to derive a set of relations. For an integer  $m \geq 1$ , the  $q$ -integer  $[[m]]$  is the polynomial

$$\frac{1 - q^m}{1 - q} = 1 + q + \dots + q^{m-1} \in \mathbb{Z}[q]$$

and  $[[0]] = 0$ . For  $i, j \in \mathbb{Z}$ , let  $\mathcal{E}_{i,j}$  be the elementary periodic matrix with entries equal to 1 in positions  $(i + cn, j + cn)$  for  $c \in \mathbb{Z}$  and all other entries equal to zero. Given  $A \in \Lambda_1$ ,

$$\begin{aligned} E_i e_A &= \sum_{p \in \mathbb{Z}: a_{i+1,p} > 0} q^{\sum_{j > p} a_{i,j}} [[a_{i,p} + 1]] e_{A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}} \\ F_i e_A &= \sum_{p \in \mathbb{Z}: a_{i,p} > 0} q^{\sum_{j < p} a_{i+1,j}} [[a_{i+1,p} + 1]] e_{A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}} \end{aligned}$$

for each  $i \in \{1, \dots, n\}$ .

### 1.3 Main results

Specialising the affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  at  $q = 0$  gives an associative  $\mathbb{Z}$ -algebra

$$\hat{S}_0(n, r) = \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}[q]} \hat{S}_q(n, r)$$

called the affine 0-Schur algebra. The ultimate goal of the project is to study the structure of the affine  $q$ -Schur algebra and to define a new associative algebra with a modified version of the product in  $\hat{S}_q(n, r)$ , the generic affine algebra, where the standard basis elements  $e_A$  form a

multiplicative basis and to finally investigate the link between the generic affine algebra and the affine 0-Schur algebra. Interestingly, this product may be understood geometrically in terms of degenerations of orbits, by a purely combinatorial approach and from a representation theoretic viewpoint by considering the Hall algebra for a cyclic quiver and the corresponding product given by generic extensions of representations.

[Replace lemma with some sentences.]

**Lemma.** *There is an invertible element  $R$  in  $\hat{S}_q(n, r)$  such that acting on a basis element  $e_A$  on the left corresponds to shifting all entries up by one row and acting on the right by  $R$  corresponds to shifting all entries of  $A$  to the right by one column. Moreover, the map*

$$e_A \mapsto Re_A R^{-1}$$

*is a unipotent automorphism of  $\hat{S}_q(n, r)$  of degree  $n$ , corresponding to a shift along the diagonal of  $A$ .*

We define a quiver  $\Gamma$  for  $\hat{S}_q(n, r)$  and give a set of  $\mathbb{Z}[q]$ -linear relations with the aim of giving a presentation of  $\hat{S}_q(n, r)$  over an extended ground ring and later use the same quiver with the  $q = 0$  form of the relations to study both the affine zero Schur algebra  $\hat{S}_0(n, r)$  and the generic affine algebra  $\hat{G}(n, r)$ . Let  $\Gamma$  be the quiver with set of vertices  $\Lambda_0$  and arrows

$$\begin{aligned} e_{i,\lambda}: \lambda &\rightarrow \lambda + \alpha_i : \lambda_{i+1} > 0 \\ f_{i,\lambda}: \lambda &\rightarrow \lambda - \alpha_i : \lambda_i > 0, \end{aligned}$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$ , where  $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathbb{Z}^n$  is the simple root. Let  $I = I(n, r)$  be the ideal of relations in  $\mathbb{Z}[q]\Gamma$  generated by

$$\begin{aligned} e_i e_j - e_j e_i \\ f_i f_j - f_j f_i \end{aligned}$$

for  $i, j \in \{1, \dots, n\}$  such that  $j \neq i \pm 1$ ,

$$\begin{aligned} e_i e_{i+1}^2 - [[2]] e_{i+1} e_i e_{i+1} + q e_{i+1}^2 e_i \\ e_i^2 e_{i+1} - [[2]] e_i e_{i+1} e_i + q e_{i+1} e_i^2 \\ f_{i+1}^2 f_i - [[2]] f_{i+1} f_i f_{i+1} + q f_i f_{i+1}^2 \\ f_{i+1} f_i^2 - [[2]] f_i f_{i+1} f_i + q f_i^2 f_{i+1} \end{aligned}$$

for  $i \in \{1, \dots, n\}$ ,

$$e_i f_j - f_j e_i$$

for  $i, j \in \{1, \dots, n\}$  such that  $i \neq j$ ,

$$e_i f_i - f_i e_i - \sum_{\lambda \in \Lambda_0} ([[\lambda_i]] - [[\lambda_{i+1}]]) k_\lambda$$

for  $i \in \{1, \dots, n\}$ ; where

$$\begin{aligned} e_i &= \sum_{\lambda \in \Lambda_0: \lambda_{i+1} > 0} e_{i,\lambda} \\ f_i &= \sum_{\lambda \in \Lambda_0: \lambda_i > 0} f_{i,\lambda}. \end{aligned}$$

There is a unique homomorphism of  $\mathbb{Z}[q]$ -algebras

$$\phi: \mathbb{Z}[q]\Gamma/I(n, r) \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned}\phi(k_\lambda + I(n, r)) &= 1_\lambda \\ \phi(e_{i,\lambda} + I(n, r)) &= E_i 1_\lambda \\ \phi(f_{i,\lambda} + I(n, r)) &= F_i 1_\lambda\end{aligned}$$

and it is **conjectured** that if  $\mathcal{Q}$  is a subring of  $\mathbb{Q}(q)$  such that the  $q$ -integers are invertible and  $q$  is not invertible, then the induced  $\mathcal{Q}$ -algebra homomorphism  $\mathcal{Q} \otimes \phi$  is an isomorphism. Surjectivity is proven in Proposition 3.2.17 and I believe injectivity will follow once the presentation of  $\hat{G}(n, r)$  is proven to be an isomorphism, which depends on the technical lemma on transforming paths in  $\Gamma$  to standard paths.

In order to gain geometric insight on the product of  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$  we consider finite dimensional slices of the orbits and orbit products together with the action of a finite dimensional quotient of a stabiliser in  $G$ . For  $A, B \in \Lambda_1$  and  $L \in \mathcal{F}_{\text{ro}(A)}$  we consider the spaces

$$X_A^L = \{L' \in \mathcal{F} : (L, L') \in \mathcal{O}_A\},$$

$$X_{A,B}^L = \{L'' \in \mathcal{F} : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

and

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F} \times \mathcal{F} : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

together with the action of the stabiliser  $G_L$ . It is proven that the  $G_L$ -orbit  $X_A^L$  is an irreducible quasiprojective algebraic variety in Lemmas 4.4.3 and 4.4.2. It is shown that  $Y_{A,B}^L$  is an irreducible quasiprojective variety in Lemma 4.5.2 and then  $X_{A,B}^L$  is irreducible and constructible, thus establishing existence and uniqueness generic orbits.

**Proposition.** *Given  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$ , there is a unique element  $A * B \in \Lambda_1$  such that  $X_{A*B}^L$  is open in  $X_{A,B}^L$  for any  $L \in \mathcal{F}_{\text{ro}(A)}$ .*

This proposition leads to the definition of the generic affine algebra  $\hat{G}(n, r)$ , which is a free  $\mathbb{Z}$ -module with basis  $\{e_A : A \in \Lambda_1\}$  together with the generic product  $*$  given by

$$e_A * e_B = \begin{cases} e_{A*B} & : \text{ if } \text{co}(A) = \text{ro}(B), \\ 0 & : \text{ if } \text{co}(A) \neq \text{ro}(B). \end{cases}$$

**Theorem.** *The generic affine algebra  $\hat{G}(n, r)$  is an associative  $\mathbb{Z}$ -algebra with 1.*

Let  $\mathcal{J}$  be the ideal of relations in  $\mathbb{Z}\Gamma$  given by specialising the relations in  $I(n, r)$  at  $q = 0$ , so  $\mathcal{J}$  is generated by

$$\begin{aligned}e_i e_j - e_j e_i &: |j - i| \neq 1 \\ f_i f_j - f_j f_i &: |j - i| \neq 1 \\ e_i^2 e_{i+1} - e_i e_{i+1} e_i \\ e_i e_{i+1}^2 - e_{i+1} e_i e_{i+1} \\ f_{i+1}^2 f_i - f_{i+1} f_i f_{i+1} \\ f_{i+1} f_i^2 - f_i f_{i+1} f_i\end{aligned}$$

and

$$\begin{aligned} e_{i,\lambda-\alpha_i}f_{i,\lambda} - f_{i,\lambda+\alpha_i}e_{i,\lambda} : \lambda_i > 0, \lambda_{i+1} > 0 \\ e_{i,\lambda-\alpha_i}f_{i,\lambda} - k_\lambda : \lambda_i > 0, \lambda_{i+1} = 0 \\ f_{i,\lambda+\alpha_i}e_{i,\lambda} - k_\lambda : \lambda_i = 0, \lambda_{i+1} > 0 \end{aligned}$$

for  $i, j \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$ .

An element  $A \in \Lambda_1$  is said to be aperiodic if for every  $s \neq 0$  there is  $i \in \{1, \dots, n\}$  such that  $a_{i,i+s} = 0$ .

**Proposition.** *There is a  $\mathbb{Z}$ -algebra homomorphism*

$$\rho: \mathbb{Z}\Gamma/\mathcal{J} \rightarrow \hat{G}(n, r)$$

defined by

$$\begin{aligned} \rho(k_\lambda + \mathcal{J}) &= 1_\lambda \\ \rho(e_{i,\lambda} + \mathcal{J}) &= E_i 1_\lambda \\ \rho(f_{i,\lambda} + \mathcal{J}) &= F_i 1_\lambda. \end{aligned}$$

The image of  $\rho$  is spanned by the aperiodic basis elements. If  $r < n$  then  $\rho$  is surjective.

It is strongly believed that  $\rho$  is an isomorphism of  $\mathbb{Z}$ -algebras when  $r < n$  and the proof depends on a single technical lemma for which we seem to be close to a proof. Once this result is proven we have the following conjecture.

**Conjecture.** Suppose  $r < n$ . There is an isomorphism of  $\mathbb{Z}$ -algebras

$$\Psi: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$$

such that

$$\begin{aligned} \Psi(1_\lambda) &= 1_\lambda \\ \Psi(E_i) &= E_i \\ \Psi(F_i) &= F_i. \end{aligned}$$

The first chapter introduces the construction of affine  $q$ -Schur algebras via a convolution product of orbits in double affine flag varieties, which is due to Lusztig. The second chapter begins to study the structure of affine  $q$ -Schur algebras, including fundamental multiplication rules, some symmetries and a notion of periodic shifting as well as giving a quiver and a set of relations for the  $q$ -Schur algebra. In the third chapter, the geometry of affine flag varieties is studied further and is used to establish existence of a *generic product* of orbits, as has been shown in the finite case by Jensen and Su in [23]. Remarkably, the generic product of orbits is shown to be associative, which leads to the construction of an associative  $\mathbb{Z}$ -algebra called the *generic affine algebra*. The fourth chapter is dedicated to studying the relationship of the generic affine algebra to the affine 0-Schur algebra.



## Chapter 2

# Geometric approach to affine q-Schur algebras

### 2.1 The cyclic flags realisation of affine q-Schur algebras

Fix integers  $n, r \geq 1$ .

**Definition 2.1.1.** A *composition of  $r$  into  $n$  parts* is an  $n$ -tuple  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$  of non-negative integers whose sum equals  $r$ . Denote the set of compositions of  $r$  into  $n$  parts by  $\Lambda_0$ .

A composition  $\lambda \in \Lambda_0$  is said to be *sincere* if  $\lambda_i > 0$  for each  $i \in \{1, \dots, n\}$  and otherwise  $\lambda$  is said to be *insincere*.

For each  $i \in \{1, \dots, n\}$ , let

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1},$$

where  $\varepsilon_{n+1} = \varepsilon_1$ .

**Definition 2.1.2.** Let  $\Lambda_1$  be the set of matrices  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$  with integer entries  $a_{i,j}$  satisfying the following conditions:

- i.  $a_{i,j} \geq 0$  for each  $i, j \in \mathbb{Z}$ ;
- ii. each row or column has only finitely many non-zero entries;
- iii. the sum of the entries in any  $n$  consecutive rows or columns equals  $r$ ;
- iv.  $a_{i-n,j-n} = a_{i,j}$  for each  $i, j \in \mathbb{Z}$ .

These matrices are referred to as *infinite periodic matrices*.

**Definition 2.1.3.** Given  $A \in \Lambda_1$ , let  $\text{ro}(A)$  and  $\text{co}(A)$  be the compositions of  $r$  into  $n$  parts given by

$$\text{ro}(A) = \left( \sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\text{co}(A) = \left( \sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right).$$

The *source* of  $A$  is  $\text{co}(A)$  and the *target* of  $A$  is  $\text{ro}(A)$ .

The row and column sums are finite since each row and column of  $A$  contains only finitely many nonzero entries, according to the definition of  $\Lambda_1$ .

For each  $i, j \in \mathbb{Z}$ , let  $\mathcal{E}_{i,j}$  be the  $\mathbb{Z} \times \mathbb{Z}$  ‘elementary periodic matrix’ with entries given by

$$(\mathcal{E}_{i,j})_{s,t} = 1$$

if  $(s, t) = (i + cn, j + cn)$  for some  $c \in \mathbb{Z}$  and  $(\mathcal{E}_{i,j})_{s,t} = 0$  otherwise. Clearly  $\mathcal{E}_{i,j} = \mathcal{E}_{i+n,j+n}$  for each  $i, j \in \mathbb{Z}$ .

Given  $\lambda \in \Lambda_0$ , let  $D_\lambda \in \Lambda_1$  be the diagonal matrix with source and target  $\lambda$ , which is given by

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}. \quad (2.1.1)$$

### 2.1.1 Cyclic flags

Fix  $n, r \in \mathbb{N}$  and let  $\mathbf{k}$  be a field. Let  $\mathcal{S}$  be the  $\mathbf{k}$ -algebra  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$  and let  $\mathcal{R}$  be the subalgebra generated by  $\varepsilon$ , so  $\mathcal{R} = \mathbf{k}[\varepsilon]$ . Let  $V$  be a free  $\mathcal{S}$ -module of rank  $r$ . Let  $G$  be the automorphism group of the  $\mathcal{S}$ -module  $V$ , so  $G$  is isomorphic to  $\mathrm{GL}_r(\mathcal{S})$ .

**Definition 2.1.4.** A *lattice* in  $V$  is a  $\mathcal{R}$ -submodule  $L$  of  $V$  with

$$\mathcal{S} \otimes_{\mathcal{R}} L = V.$$

In particular, a lattice is an  $\mathcal{R}$ -submodule of  $V$  which is a free  $\mathcal{R}$ -module of rank  $r$ .

**Lemma 2.1.5.** Let  $L$  be a lattice in  $V$ .  $L/\varepsilon L$  is a torsion  $\mathcal{R}$ -module, where  $\varepsilon$  acts as zero and  $L/\varepsilon L$  is an  $r$ -dimensional  $\mathbf{k}$ -vector space.

*Proof.*  $L$  is a free  $\mathcal{R}$ -module of rank  $r$ , with  $L \subset V$ . Given an  $\mathcal{R}$ -basis  $\{x_1, \dots, x_r\}$  of  $L$ ,  $\{\varepsilon x_1, \dots, \varepsilon x_r\}$  is an  $\mathcal{R}$ -basis of  $\varepsilon L$ . Finally, the cosets  $\{x_1 + \varepsilon L, \dots, x_r + \varepsilon L\}$  give a basis for  $L/\varepsilon L$  over  $\mathcal{R}/\langle \varepsilon \rangle \cong \mathbf{k}$ .  $\square$

Observe that if  $L$  and  $L'$  are lattices in  $V$ , then both the sum  $L + L'$  and the intersection  $L \cap L'$  are lattices in  $V$ . Moreover,

$$\begin{aligned} g(L + L') &= g(L) + g(L') \\ g(L \cap L') &= g(L) \cap g(L') \end{aligned}$$

for each  $g \in G$ .

**Definition 2.1.6.** A *cyclic flag* in  $V$  is a collection  $(L_i)_{i \in \mathbb{Z}}$  of lattices in  $V$  with  $L_i \subset L_{i+1}$  and  $\varepsilon L_i = L_{i-n}$  for each  $i \in \mathbb{Z}$ . The space of cyclic flags in  $V$  is denoted by  $\mathcal{F} = \mathcal{F}(n, r)$ .

The group  $G = \mathrm{Aut}(V)$  acts on  $\mathcal{F}$  by  $(g \cdot L)_i = g(L_i)$  for each  $i \in \mathbb{Z}$ ,  $g \in G$  and  $L \in \mathcal{F}$ . The  $G$ -orbits in  $\mathcal{F}$  are indexed by the set  $\Lambda_0$  of compositions of  $r$  into  $n$  parts. In particular, the  $G$ -orbit in  $\mathcal{F}$  corresponding to  $\lambda \in \Lambda_0$  is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim \left( \frac{L_i}{L_{i-1}} \right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}.$$

Consider the space  $\mathcal{F} \times \mathcal{F}$  of pairs of flags with the diagonal action of  $G$ , given by

$$g \cdot (L, L') = (gL, gL')$$

for  $g \in G$  and  $(L, L') \in \mathcal{F} \times \mathcal{F}$ . Denote the  $G$ -orbit of  $(L, L')$  by  $[L, L']$ . The set of  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$  admits a combinatorial description as described below.

**Definition 2.1.7.** The periodic characteristic matrix of a pair of cyclic flags  $(L, L') \in \mathcal{F} \times \mathcal{F}$  is the matrix  $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$  with entries

$$a_{i,j} = \dim \left( \frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j} \right)$$

for each  $i, j \in \mathbb{Z}$ .

The next lemma shows that the characteristic matrix of a pair of flags depends only on the  $G$ -orbit of the pair.

**Lemma 2.1.8.** *Given  $(L, L') \in \mathcal{F} \times \mathcal{F}$  and  $g \in G$ ,*

$$A(gL, gL') = A(L, L').$$

*Proof.* Write  $A = A(L, L')$  and  $B = A(gL, gL')$ . For each  $i, j \in \mathbb{Z}$ ,  $g$  induces a linear isomorphism

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \rightarrow \frac{g(L_i \cap L'_j)}{g(L_{i-1} \cap L'_j + L_i \cap L'_{j-1})},$$

so

$$\begin{aligned} b_{i,j} &= \dim \left( \frac{gL_i \cap gL'_j}{gL_{i-1} \cap gL'_j + gL_i \cap gL'_{j-1}} \right) \\ &= \dim \left( \frac{g(L_i \cap L'_j)}{g(L_{i-1} \cap L'_j + L_i \cap L'_{j-1})} \right) \\ &= \dim \left( \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right) \\ &= a_{i,j}, \end{aligned}$$

since the action of  $g$  commutes with sums and intersections of lattices. Therefore  $A = B$  as claimed.  $\square$

The following result gives another useful set of expressions for the characteristic matrix.

**Lemma 2.1.9.** *For each  $i, j \in \mathbb{Z}$ ,*

$$a_{i,j} = \dim \left( \frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} \right).$$

*Proof.* Set  $U = L_i \cap L'_j$  and  $U' = L_{i-1} + L_i \cap L'_{j-1}$ . Then  $U + U' = L_{i-1} + L_i \cap L'_j$  and  $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$ . Applying the isomorphism theorems,  $U + U'/U'$  is naturally isomorphic to  $U/U \cap U'$  as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to  $a_{i,j}$ .  $\square$

**Lemma 2.1.10.** *For each  $(L, L') \in \mathcal{F} \times \mathcal{F}$ ,  $A(L, L') \in \Lambda_1$ .*

*Proof.* Let  $(L, L') \in \mathcal{F} \times \mathcal{F}$ . The *periodic characteristic matrix*  $A(L, L')$  is  $(n, n)$ -periodic since

$$A(L, L')_{i-n, j-n} = A(\varepsilon L, \varepsilon L')_{i, j} = A(L, L')_{i, j}$$

for each  $i, j \in \mathbb{Z}$ .

For each  $i \in \mathbb{Z}$  there is a chain of lattices

$$M_{i, j} = L_{i-1} + L_i \cap L'_j$$

for  $j \in \mathbb{Z}$  such that  $M_{i, j} = L_{i-1}$  for sufficiently small  $j$  and  $M_{i, j} = L_i$  for sufficiently large  $j$ . The chain of lattices gives a filtration  $M_{i, j}/L_{i-1}$  of  $L_i/L_{i-1}$  where the dimensions of the factors in the filtration are

$$\begin{aligned} \dim(M_{i, j}/M_{i, j-1}) &= \dim\left(\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}}\right) \\ &= a_{i, j}, \end{aligned}$$

using Lemma 2.1.9.

Let  $\mu = |L|$ . Then

$$\begin{aligned} \mu_i &= \dim(L_i/L_{i-1}) \\ &= \sum_{j \in \mathbb{Z}} a_{i, j}, \end{aligned}$$

so the sum of the entries in rows 1 to  $n$  is

$$\mu_1 + \cdots + \mu_n = r$$

and therefore  $A(L, L') \in \Lambda_1$ . □

**Lemma 2.1.11.** *Given a pair of flags  $(L, L') \in \mathcal{F}^2$ , the matrices  $A(L, L')$  and  $A(L', L)$  are related by the transpose. In particular,  $A(L, L')_{i, j} = A(L', L)_{j, i}$  for each  $i, j \in \mathbb{Z}$ .*

*Proof.* By swapping the roles of  $i$  and  $j$  and swapping  $L$  and  $L'$  it is clear that  $A(L, L')_{i, j}$  and  $A(L', L)_{j, i}$  are both equal to the dimension of the  $\mathbf{k}$ -vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each  $i, j \in \mathbb{Z}$ . □

**Lemma 2.1.12.** *Given  $A \in \Lambda_1$  and  $(L, L') \in \mathcal{F} \times \mathcal{F}$  with  $A(L, L') = A$ ,*

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) = \sum_{s \leq i, t > j} a_{s, t}$$

and

$$\dim\left(\frac{L'_j}{L_i \cap L'_j}\right) = \sum_{s > i, t \leq j} a_{s, t},$$

for each  $i, j \in \mathbb{Z}$ .

*Proof.* For each  $s, t \in \mathbb{Z}$  define a lattice

$$M_{s,t} = L_i \cap L'_j + L_{s-1} + L_s \cap L'_t.$$

Observe that  $L_i \cap L'_j$  is a sublattice of each  $M_{s,t}$  and when  $s \leq i$ ,  $L_s$  is contained in  $L_i$ , so  $M_{s,t}$  is a sublattice of  $L_i$ . The collection of lattices  $(M_{s,t})_{s \leq i, t > j}$  are totally ordered by subset inclusion, as will be shown below, so give a chain of lattices each containing  $L_i \cap L'_j$  and contained in  $L_i$ . This chain of lattices induces a filtration of  $L_i/L_i \cap L'_j$  and it will be shown that the dimensions of the quotients are precisely  $a_{s,t}$  for  $s \leq i$  and  $t > j$ .

Let  $s \leq i$  and  $t \geq j$ .

$$M_{s,t} \subset M_{s,t+1}$$

and

$$\begin{aligned} M_{s,j} &= L_i \cap L'_j + L_{s-1} + L_s \cap L'_j \\ &= L_i \cap L'_j + L_{s-1}. \end{aligned}$$

If  $t$  is sufficiently large,  $L_s \subset L'_t$  and then

$$\begin{aligned} M_{s,t} &= L_i \cap L'_j + L_s \\ &= M_{s+1,j}. \end{aligned}$$

It follows that the collection of lattices is totally ordered, with  $M_{s,t} \leq M_{u,v}$  if and only if  $s < u$  or  $s = u$  and  $t \leq v$ . Thus  $L_i/L_i \cap L'_j$  has a filtration given by the spaces  $M_{s,t}/L_i \cap L'_j$  for all  $s \leq i$  and  $t > j$ .

$$\begin{aligned} M_{i,t} &= L_i \cap L'_j + L_{i-1} + L_i \cap L'_t \\ &= L_{i-1} + L_i \cap L'_t \end{aligned}$$

and if  $t$  is sufficiently large that  $L_i \subset L'_t$ , then  $M_{i,t} = L_i$ .

If  $s$  is sufficiently small that  $L_s \subset L_i \cap L'_j$  then

$$\begin{aligned} M_{s,t} &= L_i \cap L'_j + L_{s-1} + L_s \cap L'_t \\ &= L_i \cap L'_j. \end{aligned}$$

$$\begin{aligned} \frac{M_{s,t}}{L_i \cap L'_j} &= \frac{L_i \cap L'_j + L_{s-1} + L_s \cap L'_t}{L_i \cap L'_j} \\ &= \frac{L_{s-1} + L_s \cap L'_t}{L_i \cap L'_j \cap (L_{s-1} + L_s \cap L'_t)} \\ &= \frac{L_{s-1} + L_s \cap L'_t}{L_s \cap L'_j} \end{aligned}$$

Then for each  $s \leq i$  and  $t > j$

$$\begin{aligned} \dim \left( \frac{M_{s,t}}{M_{s,t-1}} \right) &= \dim \left( \frac{L_{s-1} + L_s \cap L'_t}{L_{s-1} + L_s \cap L'_{t-1}} \right) \\ &= a_{s,t}, \end{aligned}$$

so

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) = \sum_{s \leq i, t > j} a_{s,t}.$$

To deduce the second formula observe that  $(L', L) \in \mathcal{O}_{A^\top}$ , by Lemma 2.1.11, so

$$\begin{aligned} \dim \left( \frac{L'_j}{L_i \cap L'_j} \right) &= \sum_{t \leq j, s > i} a_{t,s}^\top \\ &= \sum_{s > i, t \leq j} a_{s,t}. \end{aligned}$$

□

The following is a construction of a pair of flags corresponding to a matrix  $A \in \Lambda_1$ . Recall that  $V$  is the free  $\mathcal{S}$ -module  $\mathcal{S}^r$  and let  $V_{\mathbf{k}}$  be the underlying vector space together with the linear operator  $\varepsilon: V_{\mathbf{k}} \rightarrow V_{\mathbf{k}}$ .

Fix an  $r$ -dimensional subspace  $U$  of  $V_{\mathbf{k}}$  such that

$$\mathcal{S} \otimes_{\mathbf{k}} U = V_{\mathbf{k}}$$

and write

$$U = \bigoplus_{i \in \{1, \dots, n\}, j \in \mathbb{Z}} M_{i,j},$$

given subspaces  $M_{i,j}$  of dimension  $a_{i,j}$ . Then as vector spaces

$$V_{\mathbf{k}} = \bigoplus_{i \in \{1, \dots, n\}, j \in \mathbb{Z}} \bigoplus_{h \in \mathbb{Z}} \varepsilon^h M_{i,j}.$$

Define  $M_{i,j}$  for  $i, j \in \mathbb{Z}$  by setting

$$M_{i-cn, j-cn} = \varepsilon^c M_{i,j}$$

and define

$$L_i = \bigoplus_{s \leq i, t \in \mathbb{Z}} M_{s,t}$$

and

$$L'_j = \bigoplus_{s \in \mathbb{Z}, t \leq j} M_{s,t}$$

for each  $i, j \in \mathbb{Z}$ .

Each such  $L_i$  and  $L'_j$  is a direct sum of free  $\mathbf{k}[\varepsilon]$ -modules  $\mathbf{k}[\varepsilon]M_{s,t}$  for  $i - n < s \leq i$  and  $t \in \mathbb{Z}$ , or  $s \in \mathbb{Z}$  and  $j - n < t \leq j$  respectively, so each is a free  $\mathbf{k}[\varepsilon]$ -module of rank  $r$  and therefore is a lattice in  $V$ .

Observe that the vector space  $L_{i-1} + L_i \cap L'_j$  is the direct sum of those  $M_{s,t}$  such that  $s < i$  or  $s = i$  and  $t \leq j$ , so

$$\dim \left( \frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} \right) = \dim(M_{i,j}) = a_{i,j}$$

for each  $i, j \in \mathbb{Z}$  and therefore  $A(L, L') = A$ .

**Lemma 2.1.13.** *Mapping a pair of flags  $(L, L')$  to the characteristic matrix  $A(L, L')$  gives a bijection between the set of  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$  and the set  $\Lambda_1$ .*

*Proof.* The construction of a pair of flags corresponding to a matrix preceding this lemma shows that this map is surjective.

Suppose  $(L, L')$  and  $(N, N')$  are pairs of flags with  $A(L, L') = A(N, N', \varepsilon)A$ . There are decompositions of  $V$  which are adapted to  $(L, L')$  and  $(N, N')$  as below: There are subspaces  $U_{i,j}$  of  $V$  for  $i, j \in \mathbb{Z}$  such that the dimension of  $U_{i,j}$  is  $a_{i,j}$ ,  $\varepsilon U_{i,j} = U_{i-n, j-n}$  and

$$V = \bigoplus_{i,j \in \mathbb{Z}} U_{i,j};$$

$$L_i = \bigoplus_{s \leq i, j \in \mathbb{Z}} U_{s,j}$$

for each  $i \in \mathbb{Z}$ ;

$$L'_j = \bigoplus_{i \in \mathbb{Z}, t \leq j} U_{i,t}$$

for each  $j \in \mathbb{Z}$ .

There are subspaces  $V_{i,j}$  of  $V$  for each  $i, j \in \mathbb{Z}$  such that the dimension of  $V_{i,j}$  is  $a_{i,j}$ ,  $\varepsilon V_{i,j} = V_{i-n, j-n}$  and

$$V = \bigoplus_{i,j \in \mathbb{Z}} V_{i,j};$$

$$N_i = \bigoplus_{s \leq i, j \in \mathbb{Z}} V_{s,j}$$

for each  $i \in \mathbb{Z}$ ;

$$N'_j = \bigoplus_{i \in \mathbb{Z}, t \leq j} V_{i,t}$$

for each  $j \in \mathbb{Z}$ .

There exist  $\mathbf{k}$ -linear isomorphisms  $g_{i,j}: U_{i,j} \rightarrow V_{i,j}$  for  $i, j \in \mathbb{Z}$  such that  $g_{i-n, j-n} = \varepsilon g_{i,j} \varepsilon^{-1}$ . Then  $g = (g_{i,j})_{i,j \in \mathbb{Z}}$  is a  $\mathcal{S}$ -linear automorphism of  $V$  with  $g(L_i) = N_i$  and  $g(L'_i) = N'_i$  for each  $i \in \mathbb{Z}$ , so  $g(L, L') = (N, N')$ . Therefore the map sending a  $G$ -orbit to its characteristic matrix is injective.  $\square$

**Lemma 2.1.14.** *Given  $(L, L') \in \mathcal{F}^2$ ,  $L' \subset L$  if and only if  $A(L, L')_{i,j} = 0$  for  $i, j \in \mathbb{Z}$  with  $i > j$ .*

*Proof.* Suppose  $L, L' \in \mathcal{F}$  with  $L' \subset L$ , meaning  $L'_j \subset L_j$  for each  $j \in \mathbb{Z}$ . Then for  $i > j$ ,  $L_i \cap L'_j = L'_j$ ,  $L_{i-1} \cap L'_j = L'_j$  and  $L_i \cap L'_{j-1}$ , which shows

$$A(L, L')_{i,j} = \dim \left( \frac{L'_j}{L'_{j-1} + L'_j} \right) = 0$$

as required. Conversely, suppose  $A(L, L')$  is upper triangular, meaning  $A(L, L')_{i,j} = 0$  when  $i > j$ . Using Lemma 2.1.12,

$$\dim \left( \frac{L'_i}{L'_i \cap L_i} \right) = \sum_{s > i, t \leq i} a_{s,t} = 0,$$

so  $L_i \cap L'_i = L'_i$  and thus  $L'_i \subset L_i$  for each  $i \in \mathbb{Z}$ , as required.  $\square$

**Corollary 2.1.15.** *Given  $L, L' \in \mathcal{F}$ ,  $L = L'$  if and only if  $A(L, L')_{i,j} = 0$  whenever  $i \neq j$ . In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L, L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each  $\lambda \in \Lambda_0$ .

### 2.1.2 A product of orbits

Given  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$ , define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also  $L \in \mathcal{F}_{\text{ro}(A)}$ , define the  $L$ -slices of  $Y_{A,B}$  and  $X_{A,B}$  respectively as

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X_{A,B}^L = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

**Remark 2.1.16.** There are only finitely many  $G$ -orbits in  $X_{A,B}$ .

**Lemma 2.1.17.** *Given  $A \in \Lambda_1$ ,  $X_{D_\lambda, A} = \mathcal{O}_A$  if  $\lambda = \text{ro}(A)$  and  $X_{A, D_\lambda} = \mathcal{O}_A$  if  $\lambda = \text{co}(A)$ .*

*Proof.* Let  $A \in \Lambda_1$  and set  $\lambda = \text{ro}(A)$ .  $Y_{D_\lambda, A}$  is the set of triples  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_{D_\lambda}$ , thus  $L = L'$  by Corollary 2.1.15, and  $(L', L'') \in \mathcal{O}_A$ .  $X_{D_\lambda, A}$  is the projection of  $Y_{D_\lambda, A}$ , which equals  $\mathcal{O}_A$ .

Similarly, if  $\lambda = \text{co}(A)$ ,  $Y_{A, D_\lambda}$  is the set of triples  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_A$  and  $L'' = L'$ , so  $X_{A, D_\lambda}$  is exactly the orbit  $\mathcal{O}_B$ .  $\square$

### 2.1.3 Triple products

Given  $A, B, C \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$  and  $\text{co}(B) = \text{ro}(C)$  and  $L \in \mathcal{F}_{\text{ro}(A)}$ , there are spaces  $X_{A,B,C}$ ,  $Y_{A,B,C}$  and their respective  $L$ -slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y_{A,B,C}^L = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X_{A,B,C}^L = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

### 2.1.4 Convolution algebras

Suppose  $\mathbf{k}$  is a finite field and let  $q$  denote the number of elements of  $\mathbf{k}$ . Consider the set  $S$  of  $G$ -invariant functions  $\mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Z}$  with constructible support.  $S$  is a free  $\mathbb{Z}$ -module with a basis consisting of the indicator functions of the  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$ . Define an operation  $\star$  on  $S$  as follows: for each  $f, g \in S$ ,  $f \star g \in S$  is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ .

$f \star g$  is well defined since the supports of  $f$  and  $g$  consist of finitely many  $G$ -orbits, so there are only finitely many  $L' \in \mathcal{F}$  such that  $f(L, L')g(L', L'') \neq 0$ , given  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ .  $f \star g$  is constant on  $G$ -orbits and is supported on finitely many  $G$ -orbits, so  $f \star g \in S$ .



**Lemma 2.1.18.** *The set  $S$  together with the operation  $\star$  is an associative  $\mathbb{Z}$ -algebra with identity element  $\iota$  given by  $\iota(L, L) = 1$  and  $\iota(L, L') = 0$  for  $L' \neq L$ .*

*Proof.* Given  $f, g, h \in S$  and  $(L, L''') \in \mathcal{F} \times \mathcal{F}$ ,

$$\begin{aligned} ((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'') h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L') g(L', L'') h(L'', L''') \\ &= (f \star (g \star h))(L, L'''), \end{aligned}$$

thus  $\star$  is associative.  $\iota$  is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L') f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L') \iota(L', L'') = f(L, L''),$$

for each  $f \in S$  and  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ . □

Given  $A \in \Lambda_1$ , let  $e_A \in S$  denote the indicator function of the orbit  $\mathcal{O}_A$ .  $S$  is a free  $\mathbb{Z}$ -module with basis  $\{e_A : A \in \Lambda_1\}$ . There exist  $\gamma_{A,B,C;q} \in \mathbb{Z}$  for  $A, B, C \in \Lambda_1$  such that

$$e_A \star e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C;q} e_C$$

for each  $A, B \in \Lambda_1$ . Then

$$\begin{aligned} \gamma_{A,B,C;q} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L') e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}, \end{aligned}$$

for any  $(L, L'') \in \mathcal{O}_C$ .

### 2.1.5 Affine $q$ -Schur algebras

There exist polynomials  $\gamma_{A,B,C} \in \mathbb{Z}[q]$  for  $A, B, C \in \Lambda_1$  such that  $\gamma_{A,B,C}(q) = \gamma_{A,B,C;q}$  for any prime power  $q$ , following [24, section 4]. The affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  is a  $\mathbb{Z}[q]$ -algebra which is a free  $\mathbb{Z}[q]$ -module with basis  $\{e_A : A \in \Lambda_1\}$  and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these ‘universal polynomials’  $\gamma_{A,B,C} \in \mathbb{Z}[q]$ , it follows from Lemma 2.1.18 that  $\hat{S}_q(n, r)$  is an associative  $\mathbb{Z}[q]$ -algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

## 2.2 Affine zero-Schur algebras

Fix integers  $n, r \geq 1$ .

**Definition 2.2.1.** The affine zero Schur algebra  $\hat{S}_0(n, r)$  is the  $\mathbb{Z}$ -algebra

$$\hat{S}_0(n, r) = \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}[q]} \hat{S}_q(n, r).$$

Observe that  $\hat{S}_0(n, r)$  is a free  $\mathbb{Z}$ -module, since  $\hat{S}_q(n, r)$  is a free  $\mathbb{Z}[q]$ -module. Moreover,  $\hat{S}_0(n, r)$  has a  $\mathbb{Z}$ -basis  $\{e_A : A \in \Lambda_1\}$  with multiplication given by

$$e_A \cdot e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C}(0) e_C,$$

for each  $A, B \in \Lambda_1$ .

## Chapter 3

# Presenting affine q-Schur algebras

### 3.1 The distinguished basis

#### 3.1.1 Elementary basis elements

Recall that  $\mathcal{E}_{i,j}$ , for  $i, j \in \mathbb{Z}$  is the  $\mathbb{Z} \times \mathbb{Z}$  elementary periodic matrix, given by

$$(\mathcal{E}_{i,j})_{s,t} = 1$$

if  $(s, t) = (i + cn, j + cn)$  for some  $c \in \mathbb{Z}$  and  $(\mathcal{E}_{i,j})_{s,t} = 0$  otherwise.

Recall that the diagonal matrix with source and target  $\lambda$  is

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n},$$

as in Equation 2.1.1.

The corresponding basis elements  $e_{D_\lambda}$ , for  $\lambda \in \Lambda_0$ , are pairwise orthogonal idempotents in  $\hat{S}_q(n, r)$  with

$$\sum_{\lambda \in \Lambda_0} e_{D_\lambda} = 1,$$

as a result of Lemma 2.1.17.

For each  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_{i+1} > 0$ , define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and define

$$E_i = \sum_{\lambda \in \Lambda_0: \lambda_{i+1} > 0} E_{i,\lambda}.$$

Also define, for each  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_i > 0$ ,

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and define

$$F_i = \sum_{\lambda \in \Lambda_0: \lambda_i > 0} F_{i,\lambda}.$$

For each  $i \in \{1, \dots, n\}$ , let  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ . Then

$$\text{co}(E_{i,\lambda}) = \text{co}(F_{i,\lambda}) = \lambda,$$

$$\text{ro}(E_{i,\lambda}) = \lambda + \alpha_i$$

and

$$\text{ro}(F_{i,\lambda}) = \lambda - \alpha_i.$$

### 3.1.2 Transpose involution

Let  $S$  be the  $\mathbb{Z}[q]$ -module automorphism of  $\hat{S}_q(n, r)$  given by

$$S(e_A) = e_{A^\top},$$

for each  $A \in \Lambda_1$ .

**Lemma 3.1.1.** *The map  $S$  is a  $\mathbb{Z}[q]$ -algebra antihomomorphism of order 2. In particular,*

$$S(e_A e_B) = S(e_B) S(e_A)$$

for each  $A, B \in \Lambda_1$ .

*Proof.* Let  $A, B, C \in \Lambda_1$  and let  $\mathbf{k}$  be a finite field with  $q = \#\mathbf{k}$  elements. If  $(L, L'') \in \mathcal{O}_C$  then  $(L'', L) \in \mathcal{O}_{C^\top}$  and

$$\begin{aligned} \gamma_{A,B,C;q} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;q} \end{aligned}$$

It follows that

$$S(e_A e_B) = S(e_B) S(e_A),$$

for each  $A, B \in \Lambda_1$  and therefore  $S$  is a  $\mathbb{Z}[q]$ -algebra antihomomorphism. Moreover,  $S \circ S$  is the identity map on  $\hat{S}_q(n, r)$  since  $(A^\top)^\top = A$ .  $\square$

The action of  $S$  on  $E_i$ ,  $F_i$  and  $1_\lambda$  is as follows:

$$S(1_\lambda) = 1_\lambda$$

for each  $\lambda \in \Lambda_0$ ,

$$S(E_{i,\lambda}) = F_{i,\lambda+\alpha_i}$$

for each  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_{i+1} > 0$ , and

$$S(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}$$

for each  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_i > 0$ .

In particular,

$$\begin{aligned} S(E_i) &= F_i, \\ S(F_i) &= E_i, \\ S(1_\lambda) &= 1_\lambda \end{aligned}$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$ .

### 3.1.3 Fundamental multiplication rules

For each  $m \in \mathbb{N}$ , define the  $q$ -integer  $[[m]] \in \mathbb{Z}[q]$  by

$$[[m]] = \frac{1 - q^m}{1 - q},$$

so that

$$\begin{aligned} [[0]] &= 0 \\ [[1]] &= 1 \\ [[2]] &= 1 + q \\ [[3]] &= 1 + q + q^2 \end{aligned}$$

and

$$[[m]] = 1 + q + \cdots + q^{m-1}$$

for  $m \geq 1$ .

**Lemma 3.1.2.** *Given  $A \in \Lambda_1$  and  $i \in \{1, \dots, n\}$  with  $\text{ro}(A)_{i+1} > 0$ ,*

$$E_i e_A = \sum_{p \in \mathbb{Z}: a_{i+1,p} > 0} q^{\sum_{j>p} a_{i,j}} [[a_{i,p} + 1]] e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}}.$$

*Given  $A \in \Lambda_1$  and  $i \in \{1, \dots, n\}$  with  $\text{ro}(A)_i > 0$ ,*

$$F_i e_A = \sum_{p \in \mathbb{Z}: a_{i,p} > 0} q^{\sum_{j<p} a_{i+1,j}} [[a_{i+1,p} + 1]] e_{A+\mathcal{E}_{i+1,p}-\mathcal{E}_{i,p}}.$$

Note that these formulas are still valid in the cases  $E_i e_A = 0$  and  $F_i e_A = 0$ . If the convention that  $e_B = 0$  whenever  $B$  is not in  $\Lambda_1$  is used, then the conditions on  $p$  in the above sums may be ignored.

**Corollary 3.1.3.** *Given  $A \in \Lambda_1$  and  $j \in \{1, \dots, n\}$  with  $\text{co}(A)_{j+1} > 0$ ,*

$$e_A F_j = \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\mathcal{E}_{p,j}-\mathcal{E}_{p,j+1}}.$$

*Given  $A \in \Lambda_1$  and  $j \in \{1, \dots, n\}$  with  $\text{co}(A)_j > 0$ ,*

$$e_A E_j = \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A+\mathcal{E}_{p,j+1}-\mathcal{E}_{p,j}}.$$

*Proof.*

$$\begin{aligned} e_A F_j &= S(E_j e_{A^\top}) \\ &= S \left( \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A^\top + \mathcal{E}_{j,p} - \mathcal{E}_{j+1,p}} \right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\mathcal{E}_{p,j}-\mathcal{E}_{p,j+1}}, \end{aligned}$$

where the second equality comes from Lemma 3.1.2. Similarly,

$$\begin{aligned}
e_A E_j &= S(F_j e_{A^\top}) \\
&= S\left(\sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i < p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A^\top + \mathcal{E}_{j+1,p} - \mathcal{E}_{j,p}}\right) \\
&= \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i < p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A + \mathcal{E}_{p,j+1} - \mathcal{E}_{p,j}}.
\end{aligned}$$

□

### 3.1.4 The hook order

For each  $i, j \in \mathbb{Z}$ , let  $d_{i,j}$  and  $\bar{d}_{i,j}$  be the maps from  $\Lambda_1$  to  $\mathbb{Z}$  given by

$$d_{i,j}(A) = \sum_{s \leq i, t > j} a_{s,t}$$

and

$$\bar{d}_{i,j}(A) = \sum_{s > i, t \leq j} a_{s,t}$$

for each  $A \in \Lambda_1$ .

**Lemma 3.1.4.** *For each  $A \in \Lambda_1$  and  $i, j \in \mathbb{Z}$ , the following equations hold:*

$$\begin{aligned}
d_{i,j}(A) - d_{i-1,j}(A) &= \sum_{t > j} a_{i,t} \\
d_{i,j}(A) - d_{i,j-1}(A) &= -\sum_{s \leq i} a_{s,j}
\end{aligned}$$

and

$$\begin{aligned}
\bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) &= -\sum_{t \leq j} a_{i,t} \\
\bar{d}_{i,j}(A) - \bar{d}_{i-1,j}(A) &= \sum_{s > i} a_{s,j}
\end{aligned}$$

*Proof.* Let  $i, j \in \mathbb{Z}$  and  $A \in \Lambda_1$ . Then

$$d_{i,j}(A) - d_{i-1,j}(A) = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i-1, t > j} a_{s,t} = \sum_{t > j} a_{i,t}$$

and

$$d_{i,j}(A) - d_{i,j-1}(A) = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i, t > j-1} a_{s,t} = -\sum_{s \leq i} a_{s,j}.$$

Similarly,

$$\bar{d}_{i,j}(A) - \bar{d}_{i-1,j}(A) = \sum_{s > i, t \leq j} a_{s,t} - \sum_{s > i-1, t \leq j} a_{s,t} = -\sum_{t \leq j} a_{i,t}$$

and

$$\bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) = \sum_{s > i, t \leq j} a_{s,t} - \sum_{s > i, t \leq j-1} a_{s,t} = \sum_{s > i} a_{s,j}.$$

□

**Lemma 3.1.5.** For each  $A \in \Lambda_1$  and  $i, j \in \mathbb{Z}$ ,

$$a_{i,j} = d_{i,j-1}(A) - d_{i-1,j-1}(A) - d_{i,j}(A) + d_{i-1,j}(A)$$

and

$$a_{i,j} = \bar{d}_{i,j-1}(A) - \bar{d}_{i-1,j-1}(A) - \bar{d}_{i,j}(A) + \bar{d}_{i-1,j}(A).$$

*Proof.* As a result of Lemma 3.1.4,

$$\begin{aligned} d_{i,j-1}(A) - d_{i-1,j-1}(A) - d_{i,j}(A) + d_{i-1,j}(A) &= \sum_{t>j-1} a_{i,t} - \sum_{t>j} a_{i,t} \\ &= a_{i,j} \end{aligned}$$

and

$$\begin{aligned} \bar{d}_{i,j-1}(A) - \bar{d}_{i-1,j-1}(A) - \bar{d}_{i,j}(A) + \bar{d}_{i-1,j}(A) &= - \sum_{t \leq j-1} a_{i,t} + \sum_{t \leq j} a_{i,t} \\ &= a_{i,j}. \end{aligned}$$

□

Define a relation  $\leq$  on  $\Lambda_1$  by  $A \leq B$  if and only if the following conditions are satisfied:

- $\text{ro}(A) = \text{ro}(B)$  and  $\text{co}(A) = \text{co}(B)$ .
- For each  $i, j \in \mathbb{Z}$ ,  $d_{i,j}(A) \leq d_{i,j}(B)$ .
- For each  $i, j \in \mathbb{Z}$ ,  $\bar{d}_{i,j}(A) \leq \bar{d}_{i,j}(B)$ .

**Lemma 3.1.6.** The relation  $\leq$  defines a partial order on  $\Lambda_1$ .

*Proof.* It is clear that  $\leq$  is reflexive and transitive.

Suppose  $A, B \in \Lambda_1$  with  $A \leq B$  and  $B \leq A$ . Then  $d_{i,j}(A) = d_{i,j}(B)$  for each  $i, j \in \mathbb{Z}$  with  $i \leq j$ , which shows  $a_{s,t} = b_{s,t}$  whenever  $s < t$ , as a result of Lemma 3.1.5. Similarly,  $\bar{d}_{i,j}(A) = \bar{d}_{i,j}(B)$  for each  $i, j \in \mathbb{Z}$  with  $i \geq j$ , so  $a_{s,t} = b_{s,t}$  whenever  $s > t$ . Moreover,  $a_{i,i} = b_{i,i}$  for each  $i \in \mathbb{Z}$ , since  $\text{co}(A) = \text{co}(B)$ . Thus  $A = B$ , which shows  $\leq$  is antisymmetric and therefore  $\leq$  is a partial order on  $\Lambda_1$ . □

This partial order is sometimes called the hook order. The following lemma will be invoked later in induction arguments.

**Lemma 3.1.7.** For any  $A \in \Lambda_1$ , the set  $\{B \in \Lambda_1 : B \leq A\}$  is finite.

*Proof.* Let  $B \in \Lambda_1$ . Only finitely many of the  $d_{i,j}(B)$  and  $\bar{d}_{i,j}(B)$  are sufficient to determine  $B$  and  $B \leq A$  if and only if

$$0 \leq d_{i,j}(B) \leq d_{i,j}(A)$$

and

$$0 \leq \bar{d}_{i,j}(B) \leq \bar{d}_{i,j}(A)$$

for each  $i, j \in \mathbb{Z}$ , thus there are only finitely many possible values of  $d_{i,j}(B)$  and  $\bar{d}_{i,j}(B)$  provided  $B \leq A$ . Therefore there are only finitely many  $B \in \Lambda_1$  such that  $B \leq A$ . □

**Lemma 3.1.8.** *The transpose operation on  $\Lambda_1$  is order preserving. In particular,  $B \leq A$  if and only if  $B^\top \leq A^\top$ .*

*Proof.* Suppose  $A, B \in \Lambda_1$  with  $B \leq A$ . The condition  $\text{co}(A) = \text{co}(B)$  and  $\text{ro}(A) = \text{ro}(B)$  is preserved by the transpose operation.

For each  $i, j \in \mathbb{Z}$ ,

$$d_{i,j}(A^\top) = \sum_{s \leq i, t > j} a_{t,s} = \bar{d}_{j,i}(A)$$

and

$$\bar{d}_{i,j}(A^\top) = \sum_{s > i, t \leq j} a_{t,s} = d_{j,i}(A).$$

It follows that  $B^\top \leq A^\top$  and therefore the transpose is order preserving.  $\square$

**Lemma 3.1.9.** *Suppose  $A, B \in \Lambda_1$  with*

$$B = A + \mathcal{E}_{i,j} - \mathcal{E}_{s,j} + \mathcal{E}_{s,t} - \mathcal{E}_{i,t}$$

*for some  $i, j, s, t \in \mathbb{Z}$  with  $i < s$  and  $j < t$ . Then  $B < A$ .*

*Proof.* Let  $p, q \in \mathbb{Z}$ . Then

$$d_{p,q}(B) = \begin{cases} d_{p,q}(A) - 1 & : i \leq p < s \text{ and } j \leq q < t, \\ d_{p,q}(A) & : \text{otherwise,} \end{cases}$$

and

$$\bar{d}_{p,q}(B) = \begin{cases} \bar{d}_{p,q}(A) - 1 & : i \leq p < s \text{ and } j \leq q < t, \\ \bar{d}_{p,q}(A) & : \text{otherwise,} \end{cases}$$

which proves that  $B < A$ .  $\square$

Let  $A \in \Lambda_1$  and  $i \in \{1, \dots, n\}$  with  $\text{ro}(A)_{i+1} > 0$ . Using the fundamental multiplication rules 3.1.2 and Lemma 3.1.9,

$$E_i e_A = \sum_{s=1}^m q^{\sum_{t>j_s} a_{i,t}} [[a_{i,j_s} + 1]] e_{A + \mathcal{E}_{i,j_s} - \mathcal{E}_{i+1,j_s}}$$

where  $j_1, \dots, j_m \in \mathbb{Z}$  with  $j_1 < j_2 < \dots < j_m$  and

$$\{j_1, \dots, j_m\} = \{j \in \mathbb{Z} : a_{i+1,j} > 0\}.$$

The basis elements appearing in the above expression are totally ordered, with

$$A + \mathcal{E}_{i,j_s} - \mathcal{E}_{i+1,j_s} < A + \mathcal{E}_{i,j_{s+1}} - \mathcal{E}_{i+1,j_{s+1}}$$

for  $s = 1, \dots, m-1$ . Thus the term with  $s = m$  is the maximum.

The partial order on  $\Lambda_1$  induces a partial order on the set of  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$ , such that  $\mathcal{O}_A \leq \mathcal{O}_B$  if and only if  $A \leq B$ . The following is a restatement of Lemma 2.1.12 and gives some geometric significance to the hook order on  $\Lambda_1$ .



**Lemma 3.1.10.** *Let  $A \in \Lambda_1$  and  $(L, L') \in \mathcal{O}_A$ . Then*

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}(A)$$

and

$$\dim \left( \frac{L'_j}{L_i \cap L'_j} \right) = \bar{d}_{i,j}(A),$$

for each  $i, j \in \mathbb{Z}$ .

### 3.1.5 Shifting

In this subsection it is shown that the operations on  $\Lambda_1$  given by shifting up by one row or to the right by one column may be described by the action, on the left or right respectively, of an invertible element  $R$  of  $\hat{S}_q(n, r)$ .

For each  $A \in \Lambda_1$  and  $m \in \mathbb{Z}$ , the row shift of  $A$  by  $m$  is the element  $[m]A$  of  $\Lambda_1$  given by

$$([m]A)_{i,j} = a_{i+m,j},$$

for each  $i, j \in \mathbb{Z}$ .

The column shift of  $A$  by  $m$  is the element  $A[m]$  given by

$$(A[m])_{i,j} = a_{i,j+m},$$

for each  $i, j \in \mathbb{Z}$ .

For  $\lambda \in \Lambda_0$  and  $m \in \mathbb{Z}$ , the translation of  $\lambda$  by  $m$  is the element  $\lambda[m]$  of  $\Lambda_0$  given by

$$(\lambda[m])_i = \lambda_{i+m},$$

for each  $i \in \mathbb{Z}$ , where the indices of  $\lambda$  are taken modulo  $n$ .

**Example 3.1.11.** Let  $\lambda = (2, 1, 3)$ . Then  $\lambda[1] = (1, 3, 2)$ ,  $\lambda[2] = (3, 2, 1)$  and  $\lambda[3] = \lambda$ .

For each  $\lambda \in \Lambda_0$ , define

$$\begin{aligned} R_\lambda &= e_{[1]D_\lambda} \\ &= e_{\lambda_1 \mathcal{E}_{0,1} + \dots + \lambda_n \mathcal{E}_{n-1,n}} \end{aligned}$$

and let

$$R = \sum_{\lambda \in \Lambda_0} R_\lambda.$$

Recall that

$$\mathcal{O}_{D_\lambda} = \{(L, L) : L \in \mathcal{F}_\lambda\},$$

so

$$\mathcal{O}_{[m]D_\lambda} = \{([m]L, L) : L \in \mathcal{F}_\lambda\}$$

and

$$\mathcal{O}_{D_\lambda[m]} = \{(L, [m]L) : L \in \mathcal{F}_\lambda\}.$$

This leads to a simple rule for multiplication by  $R$  in terms of these shifts on matrices.

**Lemma 3.1.12.** *If  $A \in \Lambda_1$  then*

$$Re_A = e_{[1]A}$$

and

$$e_AR = e_{A[-1]}.$$

*Proof.* Let  $\mu = \text{ro}(A)$  and  $\lambda = \text{co}(A)[-1]$ . Observe that

$$\{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_{[1]D_\mu}, (L', L'') \in \mathcal{O}_A\} = \{(L'[1], L', L'') : (L', L'') \in \mathcal{O}_A\},$$

and the image under the projection onto the first and last components is

$$\{(L'[1], L'') : (L', L'') \in \mathcal{O}_A\} = \mathcal{O}_{[1]A}.$$

The coefficient of  $\mathcal{O}_{[1]A}$  in the product  $Re_A$  is 1 since, for any  $(N, N'') \in \mathcal{O}_{[1]A}$ ,

$$\{N' \in \mathcal{F} : (N, N') \in \mathcal{O}_{[1]D_\mu}, (N', N'') \in \mathcal{O}_A\} = \{N[-1]\},$$

so it follows that  $Re_A = e_{[1]A}$ .

To compute the product  $e_AR$ , consider

$$\{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_{[1]D_\lambda}\} = \{(L, L', L'[-1]) : (L, L') \in \mathcal{O}_A\}.$$

The image under the projection onto the first and last components is

$$\{(L, L'[-1]) : (L, L') \in \mathcal{O}_A\} = \mathcal{O}_{A[-1]}$$

and, for any  $(N, N'') \in \mathcal{O}_{A[-1]}$ ,

$$\{N' \in \mathcal{F} : (N, N') \in \mathcal{O}_A, (N', N'') \in \mathcal{O}_{[1]D_\lambda}\} = \{N''[1]\}.$$

Therefore  $e_AR = e_{A[-1]}$ . □

**Lemma 3.1.13.** *The element  $R$  is invertible and*

$$RS(R) = S(R)R = 1.$$

*In particular,*

$$R^{-1} = \sum_{\lambda \in \Lambda_0} e_{[-1]D_\lambda}.$$

*Proof.* Recall that

$$S(R) = \sum_{\lambda \in \Lambda_0} e_{[-1]D_\lambda}.$$

Then it follows from Lemma 3.1.12 that

$$RS(R) = \sum_{\lambda \in \Lambda_0} e_{[1][-1]D_\lambda} = 1$$

and

$$\begin{aligned} S(R)R &= \sum_{\lambda \in \Lambda_0} e_{[-1]D_\lambda[-1]} \\ &= \sum_{\lambda \in \Lambda_0} e_{D(\lambda[-1])} \\ &= 1. \end{aligned}$$

□

As a visual cue, acting on a basis element  $e_A$  on the left by  $R$  corresponds to moving the matrix  $A$  up by one row, while acting on the right by  $R$  corresponds to moving the matrix to the right by one column. Then conjugating by  $R$  corresponds to the composition of a shift to the left by one and a shift up by one, which is a shift by one along the diagonal, so conjugating by  $R^n$  leaves  $e_A$  invariant. Thus conjugation by  $R$  gives a  $\mathbb{Z}[q]$ -algebra automorphism of  $\hat{S}_q(n, r)$  which has order  $n$ .

Multiplication on the left by  $S(R)$  sends  $e_A$  to  $e_{[-1]A}$ , while multiplication on the right by  $S(R)$  sends  $e_A$  to  $e_{A[1]}$ .

**Lemma 3.1.14.** *For each  $\lambda \in \Lambda_0$ ,*

$$R1_\lambda S(R) = 1_{[1]\lambda}$$

*and, for each  $i \in \{1, \dots, n\}$ ,*

$$RE_i S(R) = E_{i-1}$$

*and*

$$RF_i S(R) = F_{i-1}.$$

*Proof.* It follows from Lemma 3.1.12 and Lemma 3.1.13 that

$$Re_A S(R) = e_{[1]A[1]},$$

for each  $A \in \Lambda_1$ . In particular,

$$R1_\lambda S(R) = 1_{\lambda[1]}$$

for each  $\lambda \in \Lambda_0$ ,

$$RE_{i,\lambda} S(R) = E_{i-1,\lambda[1]}$$

for each  $(\lambda, i) \in \Lambda_0 \times \mathbb{Z}$  with  $\lambda_{i+1} > 0$ , and

$$RF_{i,\lambda} S(R) = F_{i-1,\lambda[1]}$$

for each  $(\lambda, i) \in \Lambda_0 \times \mathbb{Z}$  with  $\lambda_i > 0$ .

It now follows that

$$RE_i S(R) = E_{i-1}$$

and

$$RF_i S(R) = F_{i-1}$$

as claimed. □

## 3.2 Quivers and relations

Assume  $n$  and  $r$  are integers with  $n \geq 3$  and  $r \geq 1$ .

### 3.2.1 Relations in affine q-Schur algebras

**Lemma 3.2.1.** *If  $i, j \in \{1, \dots, n\}$  and  $i \neq j$ , then*

$$E_i F_j - F_j E_i = 0.$$

*For each  $i \in \{1, \dots, n\}$ ,*

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0} ([[\lambda_i]] - [[\lambda_{i+1}]]) 1_\lambda.$$

*Proof.* Denote  $e_A$  by  $[A]$ . Fix  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ . Then

$$\begin{aligned} E_i F_j &= \sum_{\lambda \in \Lambda_0} E_i [D_\lambda + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j} + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}]. \end{aligned}$$

Observe that the nonzero terms in the above sum are those for which  $\lambda_j > 0$  and  $\lambda_{i+1} > 0$ . Similarly,

$$\begin{aligned} F_j E_i &= \sum_{\lambda \in \Lambda_0} F_j [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j}], \end{aligned}$$

where the sum is taken over those  $\lambda$  such that  $\lambda_{i+1} > 0$  and  $\lambda_j > 0$ . Therefore

$$E_i F_j - F_j E_i = 0.$$

Again using Lemma 3.1.2,

$$\begin{aligned} E_i F_i &= \sum_{\lambda \in \Lambda_0} E_i [D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i} + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}] + [[\lambda_i]] [D_\lambda] \end{aligned}$$

and

$$\begin{aligned} F_i E_i &= \sum_{\lambda \in \Lambda_0} F_i [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}] + [[\lambda_{i+1}]] [D_\lambda]. \end{aligned}$$

Therefore

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0} ([[\lambda_i]] - [[\lambda_{i+1}]]) 1_\lambda,$$

as required.  $\square$

An explicit version of these relations will be given after defining some terminology. Given  $\lambda \in \Lambda_0$  and  $i \in \{1, \dots, n\}$ , say that  $\lambda$  is internal with respect to  $i$  if  $\lambda - \alpha_i, \lambda + \alpha_i \in \Lambda_0$ . Say that  $\lambda$  is initial with respect to  $i$  if  $\lambda - \alpha_i \notin \Lambda_0$  and that  $\lambda$  is final with respect to  $i$  if  $\lambda + \alpha_i \notin \Lambda_0$ .

Then the expression for the commutator  $[E_i, F_i]$  in Lemma 3.2.1 gives the following relations in  $\hat{S}_q(n, r)$ :

- If  $\lambda$  is internal with respect to  $i$  then

$$E_{i, \lambda - \alpha_i} F_{i, \lambda} - F_{i, \lambda + \alpha_i} E_{i, \lambda} = 0.$$

- If  $\lambda$  is initial with respect to  $i$  then

$$F_{i, \lambda + \alpha_i} E_{i, \lambda} - 1_\lambda = 0.$$

- If  $\lambda$  is final with respect to  $i$  then

$$E_{i,\lambda-\alpha_i}F_{i,\lambda} - 1_\lambda = 0.$$

**Lemma 3.2.2.** *The following relations hold in  $\hat{S}_q(n, r)$ , when  $n \geq 3$ :*

$$E_i E_j - E_j E_i = 0$$

and

$$F_i F_j - F_j F_i = 0$$

for  $i, j \in \{1, \dots, n\}$  such that  $j \geq i + 2$ ,

$$\begin{aligned} E_i E_{i+1}^2 - (1+q)E_{i+1}E_i E_{i+1} + qE_{i+1}^2 E_i &= 0 \\ E_i^2 E_{i+1} - (1+q)E_i E_{i+1} E_i + qE_{i+1} E_i^2 &= 0 \end{aligned}$$

and

$$\begin{aligned} F_{i+1} F_i^2 - (1+q)F_i F_{i+1} F_i + qF_i^2 F_{i+1} &= 0 \\ F_{i+1}^2 F_i - (1+q)F_{i+1} F_i F_{i+1} + qF_i F_{i+1}^2 &= 0. \end{aligned}$$

for  $i \in \{1, \dots, n\}$ .

*Proof.* Denote  $e_A$  by  $[A]$ .

$$\begin{aligned} E_i E_{i+1}^2 &= \sum_{\lambda \in \Lambda_0} [[2]] [D_\lambda + \mathcal{E}_{i,i+2} + \mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2}] \\ &\quad + [[2]] [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + 2\mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2}] \end{aligned}$$

$$\begin{aligned} E_{i+1} E_i E_{i+1} &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{i,i+2} + \mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2}] \\ &\quad + [[2]] [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + 2\mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2}] \end{aligned}$$

$$E_{i+1}^2 E_i = \sum_{\lambda \in \Lambda_0} [[2]] [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + 2\mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2}]$$

$$E_i E_{i+1}^2 - (1+q)E_{i+1}E_i E_{i+1} + qE_{i+1}^2 E_i = \sum_{\lambda \in \Lambda_0} ([[2]] - (1+q)) [X_\lambda] + ([[2]] - (1+q)[[2]] + q[[2]]) [Y_\lambda]$$

where

$$X_\lambda = D_\lambda + \mathcal{E}_{i,i+2} + \mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2}$$

and

$$Y_\lambda = D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + 2\mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2}.$$

It follows

$$E_i E_{i+1}^2 - (1+q)E_{i+1}E_i E_{i+1} + qE_{i+1}^2 E_i = 0$$

and so

$$F_{i+1}^2 F_i - (1+q) F_{i+1} F_i F_{i+1} + q F_i F_{i+1}^2 = 0,$$

by applying the transpose involution to the first relation.

$$\begin{aligned} E_i^2 E_{i+1} &= \sum_{\lambda \in \Lambda_0} [[2]] [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i,i+2} - \mathcal{E}_{i+2,i+2}] \\ &\quad + [[2]] [D_\lambda + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i+2} - \mathcal{E}_{i+2,i+2}] \end{aligned}$$

$$\begin{aligned} E_i E_{i+1} E_i &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i,i+2} - \mathcal{E}_{i+2,i+2}] \\ &\quad + [[2]] [D_\lambda + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i+2} - \mathcal{E}_{i+2,i+2}] \end{aligned}$$

$$E_{i+1} E_i^2 = \sum_{\lambda \in \Lambda_0} [[2]] [D_\lambda + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i+2} - \mathcal{E}_{i+2,i+2}]$$

So

$$E_i^2 E_{i+1} - (1+q) E_i E_{i+1} E_i + q E_{i+1} E_i = \sum_{\lambda \in \Lambda_0} ([2] - (1+q)) A_\lambda + ([2] - (1+q)[2] + q[[2]]) B_\lambda,$$

where

$$A_\lambda = D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i,i+2} - \mathcal{E}_{i+2,i+2}$$

and

$$B_\lambda = D_\lambda + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i+2} - \mathcal{E}_{i+2,i+2}.$$

Therefore

$$E_i^2 E_{i+1} - (1+q) E_i E_{i+1} E_i + q E_{i+1} E_i = 0$$

and

$$F_{i+1} F_i^2 - (1+q) F_i F_{i+1} F_i + q F_i^2 F_{i+1} = 0,$$

where the second relation follows from the first by applying the transpose involution.  $\square$

Recall the result of Lemma 3.1.14, which gives relations involving  $R$ :

$$\begin{aligned} R 1_\lambda R^{-1} &= 1_{\lambda[1]} \\ R E_{i,\lambda} R^{-1} &= E_{i-1,\lambda[1]} \\ R F_{i,\lambda} R^{-1} &= F_{i-1,\lambda[1]}. \end{aligned}$$

### 3.2.2 A quiver algebra

Define a quiver  $\Gamma = \Gamma(n, r)$  associated with the affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  as follows:

- The set of vertices is  $\Gamma_0 = \Lambda_0$ .

- The set of edges is  $\Gamma_1$ , consisting of edges

$$e_{i,\lambda}: \lambda \rightarrow \lambda + \alpha_i$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  such that  $\lambda_{i+1} > 0$  and

$$f_{i,\lambda}: \lambda \rightarrow \lambda - \alpha_i$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  such that  $\lambda_i > 0$ .

The path  $\mathbb{Z}[q]$ -algebra of  $\Gamma$  is an associative  $\mathbb{Z}[q]$ -algebra with a unit, which has a  $\mathbb{Z}[q]$ -basis consisting of the paths in  $\Gamma$ , where the multiplication is defined by concatenation of paths. That is, if  $p$  and  $q$  are paths in  $\Gamma$ , then the product  $pq$  is the path ‘ $q$  followed by  $p$ ’ if the target of  $q$  equals the source of  $p$ , or equals zero otherwise.

For each  $\lambda \in \Lambda_0$ , denote the constant path at  $\lambda$  by  $k_\lambda$ . These elements form a set of pairwise orthogonal idempotents and the multiplicative identity in  $\mathbb{Z}[q]\Gamma$  is

$$\sum_{\lambda \in \Lambda_0} k_\lambda.$$

For each  $i \in \{1, \dots, n\}$ , define

$$e_i = \sum_{\lambda \in \Lambda_0: \lambda_{i+1} > 0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0: \lambda_i > 0} f_{i,\lambda}.$$

To avoid long subscripts, define  $e_{i,\lambda} = 0$  if  $\lambda_{i+1} = 0$  and define  $f_{i,\lambda} = 0$  if  $\lambda_i = 0$ .

Let  $I = I(n, r)$  be the ideal in  $\mathbb{Z}[q]\Gamma$  generated by the following expressions:

$$\begin{aligned} e_i e_j - e_j e_i \\ f_i f_j - f_j f_i \end{aligned}$$

for  $i, j \in \{1, \dots, n\}$  such that  $j \neq i \pm 1$ ,

$$\begin{aligned} e_i e_{i+1}^2 - [[2]] e_{i+1} e_i e_{i+1} + q e_{i+1}^2 e_i \\ e_i^2 e_{i+1} - [[2]] e_i e_{i+1} e_i + q e_{i+1} e_i^2 \\ f_{i+1}^2 f_i - [[2]] f_{i+1} f_i f_{i+1} + q f_i f_{i+1}^2 \\ f_{i+1} f_i^2 - [[2]] f_i f_{i+1} f_i + q f_i^2 f_{i+1} \end{aligned}$$

for  $i \in \{1, \dots, n\}$ ,

$$e_i f_j - f_j e_i$$

for  $i, j \in \{1, \dots, n\}$  such that  $i \neq j$ ,

$$e_i f_i - f_i e_i - \sum_{\lambda \in \Lambda_0} ([[\lambda_i]] - [[\lambda_{i+1}]]) k_\lambda$$

for  $i \in \{1, \dots, n\}$ .

### 3.2.3 Mapping to the q-Schur algebra

**Lemma 3.2.3.** *There is a  $\mathbb{Z}[q]$ -algebra homomorphism*

$$\phi: \mathbb{Z}[q]\Gamma/I \rightarrow \hat{S}_q(n, r)$$

*defined by*

$$\begin{aligned}\phi(e_{i,\lambda} + I) &= E_{i,\lambda}, \\ \phi(f_{i,\lambda} + I) &= F_{i,\lambda}, \\ \phi(k_\lambda + I) &= 1_\lambda,\end{aligned}$$

*for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$ .*

*Proof.* Lemma 3.2.1 and Lemma 3.2.2 shows that each equation defining the ideal  $I$  corresponds to a zero relation in  $\hat{S}_q(n, r)$ , so there is a unique homomorphism of  $\mathbb{Z}[q]$ -algebras given by

$$\begin{aligned}\phi(e_{i,\lambda} + I) &= E_{i,\lambda}, \\ \phi(f_{i,\lambda} + I) &= F_{i,\lambda}, \\ \phi(k_\lambda + I) &= 1_\lambda,\end{aligned}$$

for each  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$ . □

In fact,  $\phi$  is determined by its values on  $e_1, \dots, e_n$ ,  $f_1, \dots, f_n$  and  $k_\lambda$  for  $\lambda \in \Lambda_0$ . In order to describe the image of this map we introduce the notion of standard paths in  $\Gamma$ .

**Definition 3.2.4.** A path  $p = k_\lambda p_1^+ \cdots p_h^+$  with

$$p_s^+ = e_{i+s-1}^{\alpha_{i,s}} e_{i+s-2}^{\alpha_{i-1,s}} \cdots e_{i+n-s}^{\alpha_{i-n+1,s}},$$

for  $s \in \{1, \dots, h\}$ , is a *standard positive path* if

$$\begin{aligned}\alpha_{j,s} &\geq \alpha_{j,s+1} \text{ for } s \in \{1, \dots, h-1\}; \\ 0 &\leq \alpha_{j,1} \leq \lambda_j \text{ for } j \in \{1, \dots, n\}. \\ i &= \max\{t : 1 \leq t \leq n, \alpha_{t,1} = 0\} - 1;\end{aligned}$$

**Definition 3.2.5.** A path  $p = k_\lambda p_1^- \cdots p_h^-$  with

$$p_s^- = f_{i-s+1}^{\beta_{i,s}} f_{i-s+2}^{\beta_{i+1,s}} \cdots f_{i+n-s}^{\beta_{i+n-1,s}},$$

for  $s \in \{1, \dots, h\}$ , is a *standard negative path* if

$$\begin{aligned}\beta_{j,s} &\geq \beta_{j,s+1} \text{ for } s \in \{1, \dots, h-1\}; \\ 0 &\leq \beta_{j,1} \leq \lambda_{j+1} \text{ for } j \in \{1, \dots, n\}. \\ i &= \min\{t : 1 \leq t \leq n, \beta_{t-1,1} = 0\};\end{aligned}$$

**Remark 3.2.6.** The subindex  $j$  in  $e_j$  and  $f_j$  and the subindex  $j$  in  $\alpha_{j,s}$  is regarded as an element of  $\mathbb{Z}/n\mathbb{Z}$ .



**Definition 3.2.7.** A path  $p = k_\lambda p^+ k_\mu p^-$  is a *standard path* if  $k_\lambda p^+$  is a standard positive path,  $k_\mu p^-$  is a standard negative path and the exponents satisfy the conditions

$$\alpha_{j,s} + \beta_{j-1,s} \leq \lambda_j$$

for  $j \in \{1, \dots, n\}$ . Call  $p^+$  the *positive part* of  $p$  and  $p^-$  the *negative part* of  $p$ .

**Remark 3.2.8.** If  $p$  is a standard path with  $p = p'p''$  for some paths  $p'$  and  $p''$ , then  $p'$  is a standard path.

Observe that the definition of standard paths includes the constant paths  $k_\lambda$  for  $\lambda \in \Lambda_0$ .

**Definition 3.2.9.** Let  $A \in \Lambda_1$ . The *standard path for  $A$*  is the standard path  $p_A = k_\lambda p^+ p^-$ , where  $\lambda = \text{ro}(A)$ ,  $p^+$  is the standard positive path given by

$$\alpha_{i,s} = \sum_{t \geq i+s} a_{i,t}$$

and  $p^-$  is the standard negative path given by

$$\beta_{i,s} = \sum_{t \leq i-s+1} a_{i+1,t}$$

for  $i \in \{1, \dots, n\}$  and  $s \geq 1$ , respectively.

**Lemma 3.2.10.** *If  $p$  is a standard path then there is a unique element  $A \in \Lambda_1$  such that  $p = p_A$ . Thus there is a bijection between the set of standard paths in  $\Gamma$  and  $\Lambda_1$ .*

*Proof.* The map

$$\Lambda_1 \rightarrow \{\text{standard paths in } \Gamma\} : A \mapsto p_A$$

is injective since distinct elements of  $\Lambda_1$  define distinct standard paths. Finally, if  $p$  is a standard path then  $p = p_A$  where

$$\begin{aligned} a_{i,i+s} &= \alpha_{i,s} - \alpha_{i,s+1} \\ a_{i,i-s} &= \beta_{i-1,s} - \beta_{i-1,s+1} \\ a_{i,i} &= \mu_i - \alpha_{i,1} - \beta_{i-1,1} \end{aligned}$$

for all  $i \in \{1, \dots, n\}$  and  $s \geq 1$ . □

**Definition 3.2.11.** Let  $A \in \Lambda_1$ . The *positive part* of  $A$  is the element  $A^+ \in \Lambda_1$  with  $\text{ro}(A^+) = \text{ro}(A)$  and off diagonal entries

$$\begin{aligned} a_{i,j}^+ &= a_{i,j} \text{ if } i < j; \\ a_{i,j}^+ &= 0 \text{ if } i > j, \end{aligned}$$

for  $i, j \in \mathbb{Z}$ .

The negative part of  $A$  is the element  $A^- \in \Lambda_1$  with  $\text{co}(A^-) = \text{co}(A)$  and off-diagonal entries

$$\begin{aligned} a_{i,j}^- &= 0 \text{ if } i < j; \\ a_{i,j}^- &= a_{i,j} \text{ if } i > j, \end{aligned}$$

for  $i, j \in \mathbb{Z}$ .

Recall that the  $G$ -orbit of a pair of flags  $(L, L')$  is denoted by  $[L, L']$ .

**Lemma 3.2.12.** *Let  $A \in \Lambda_1$ . If  $(L, L') \in \mathcal{O}_A$ , then  $[L, L \cap L'] = \mathcal{O}_{A^+}$  and  $[L \cap L', L'] = \mathcal{O}_{A^-}$ .*

*Proof.* Let  $B \in \Lambda_1$  with  $\mathcal{O}_B = [L, L \cap L']$ . The row vector of  $B$  is  $|L| = \text{ro}(A)$  and  $B$  is upper triangular since  $L \cap L' \subset L$ . For  $i < j$ ,

$$\begin{aligned} b_{i,j} &= \dim \left( \frac{L_i \cap L_j \cap L'_j}{L_{i-1} \cap L_j \cap L'_j + L_i \cap L_{j-1} \cap L'_{j-1}} \right) \\ &= \dim \left( \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right) \\ &= a_{i,j}, \end{aligned}$$

so  $B$  is the positive part of  $A$  as in Definition 3.2.11. The transpose of the negative part of  $A$  is the positive part of the transpose of  $A$ , so it follows that  $\mathcal{O}_{A^-} = [L \cap L', L']$ .  $\square$

**Lemma 3.2.13.** *Let  $A \in \Lambda_1$  and let  $p$  be the standard path for  $A$ . The positive part of  $p$  is the standard path for  $A^+$  and the negative part of  $p$  is the standard path for  $A^-$ .*

*Proof.* Write  $p = k_\lambda p^+ p^- k_\mu$ , where  $p^+$  and  $p^-$  are the positive and negative parts of  $p$  respectively. The exponents  $\alpha_{i,s}$  in  $p^+$  are determined by the entries of  $A$  strictly above the diagonal, so the  $\alpha_{i,s}$  are also the exponents in the standard path for  $A^+$ . It follows that  $k_\lambda p^+$  is the standard path for  $A^+$  since  $\lambda = \text{ro}(A) = \text{ro}(A^+)$ .

Similarly, the exponents in the standard path for  $A$  are given by the entries in  $A$  strictly below the diagonal and  $\mu = \text{co}(A) = \text{co}(A^-)$ , so  $p^- k_\mu$  is the standard path for  $A^-$ .  $\square$

**Proposition 3.2.14.** *Let  $A \in \Lambda_1$  and let  $p$  be the standard path corresponding to  $A$ . Then*

$$\phi(p + I) = \left( \prod_{i \in \{1, \dots, n\}, s \geq 1} [[\alpha_{i,s}]]! [[\beta_{i,s}]]! \right) e_A + \sum_{B \in \Lambda_1: B < A} g_B e_B \quad (3.2.1)$$

*Proof.* Proceed by induction on the length of  $p$ . If the length of  $p$  is zero then  $p = k_\lambda$  and  $A = D_\lambda$  for some  $\lambda \in \Lambda_0$ , then 3.2.1 holds since  $\phi(k_\lambda + I) = 1_\lambda$ . Assume  $p$  has positive length and the formula holds for standard paths of smaller length.

First suppose  $p = p_1^+ \cdots p_h^+$  is a standard positive path, where  $h = \max\{j - i : a_{i,j} > 0\}$ . Factoring out the first arrow  $p = p' e_j$  and  $p'$  is a standard positive path with length less than that of  $p$ . The exponent of  $e_j$  in  $p_h^+$  is  $\alpha_{j-h+1,h} = a_{j-h+1,j+1}$  and so  $a_{j-h,j} = 0$  since  $e_j$  is the first arrow in  $p$  and the exponent of  $e_{j-1}$  in  $p_h^+$  is zero. Then  $p'$  is the standard path corresponding to  $B = A + \mathcal{E}_{j+1-h,j} - \mathcal{E}_{j+1-h,j+1}$  so

$$\phi(p' + I) = \frac{1}{[[\alpha_{j-h+1,h}]]} \left( \prod_{i \in \{1, \dots, n\}, s \geq 1} [[\alpha_{i,s}]]! \right) e_B + \sum_{C \in \Lambda_1: C < B} g'_C e_C,$$

using the inductive hypothesis. Using the fundamental multiplication rules

$$e_B E_j = [[\alpha_{j-h+1,h}]] e_A + \sum_{s > j+1-h: b_{s,j} > 0} q^{\sum_{t < s} b_{t,j+1}} [[b_{s,j+1} + 1]] e_{B + \mathcal{E}_{s,j+1} - \mathcal{E}_{s,j}}.$$

For each  $C < B$ ,  $c_{s,t} = 0$  if  $t - s > h$  and  $c_{j-h,j} = 0$ , so the product  $e_C E_j$  is a  $\mathbb{Z}[q]$ -linear combination of the terms  $e_{C+\mathcal{E}_{s,j+1}-\mathcal{E}_{s,j}}$  for  $s \geq j+1-h$ , which are totally ordered with respect to the hook order and the maximum term  $C+\mathcal{E}_{j+1-h,j+1}-\mathcal{E}_{j-h,j}$  is strictly less than  $B+\mathcal{E}_{j+1-h,j+1}-\mathcal{E}_{j-h,h} = A$ . Therefore

$$\begin{aligned}\phi(p+I) &= \phi(p'+I)E_j \\ &= \left( \prod_{i \in \{1, \dots, n\}, s \geq 1} [[\alpha_{i,s}]]! \right) e_A + \sum_{B \in \Lambda_1: B < A} g_B e_B,\end{aligned}$$

which completes the case where  $p$  is a standard positive path.

Now suppose the negative part of  $p$  is nontrivial, so  $p = p' f_j$  for some  $j$  and a standard path  $p' = p_1^- \cdots p_h^-$ , where  $h = \max\{i - j : a_{i,j} > 0\}$ . Since  $f_j$  is the first arrow in  $p$  the exponent of  $f_{j+1}$  in  $p_h^-$  is zero, so  $0 = \beta_{j+h,h} = a_{j+1+h,j+1}$ . The matrix corresponding to the standard path  $p'$  is  $B = A + \mathcal{E}_{j+h,j+1} - \mathcal{E}_{j+h,j}$  so

$$\phi(p'+I) = \frac{1}{[[\beta_{j+h-1,h}]]} \left( \prod_{i \in \{1, \dots, n\}, s \geq 1} [[\alpha_{i,s}]]! [[\beta_{i,s}]]! \right) e_B + \sum_{C \in \Lambda_1: C < B} g'_C e_C$$

by the inductive hypothesis. Using the fundamental multiplication rules

$$e_B F_j = [[\beta_{j+h-1,h}]] e_A + \sum_{C < A} g''_C e_C.$$

For each  $C < B$ ,  $c_{s,t} = 0$  if  $s - t > h$  and  $c_{j+h+1,j+1} = 0$ , so the product  $e_C F_j$  is a  $\mathbb{Z}[q]$ -linear combination of the terms  $e_{C+\mathcal{E}_{s,j}-\mathcal{E}_{s,j+1}}$  for  $s \leq j+h$ , which are all strictly smaller than  $B + \mathcal{E}_{j+h,j} - \mathcal{E}_{j+h,j+1} = A$ . Therefore

$$\begin{aligned}\phi(p+\mathcal{J}) &= \phi(p'+\mathcal{J})F_j \\ &= \left( \prod_{i \in \{1, \dots, n\}, s \geq 1} [[\alpha_{i,s}]]! [[\beta_{i,s}]]! \right) e_A + \sum_{B < A} g_B e_B\end{aligned}$$

for some  $g_B \in \mathbb{Z}[q]$ . □

### 3.2.4 Change of rings

The following is based on the change of rings for quiver presentations result, Lemma 5.2 in [23]. Let  $R$  and  $S$  be commutative rings and suppose  $f: R \rightarrow S$  is a ring homomorphism with  $f(1) = 1$ . Let  $\Sigma$  be a quiver and let  $I \subset R\Sigma$  be an ideal of relations in  $\Sigma$ . The homomorphism  $f$  defines an  $R$ -algebra structure on  $S$  with  $r \cdot s = f(r)s$  for all  $r \in R$  and  $s \in S$ . Let  $\bar{f}: R\Sigma \rightarrow S\Sigma$  be the  $R$ -algebra homomorphism induced by  $f$ , which is given by

$$\bar{f}(rp) = f(r)p$$

for each path  $p$  in  $\Sigma$  and  $r \in R$ .

Applying the right exact functor  $S \otimes_R -$  to the short exact sequence

$$0 \rightarrow I \xrightarrow{i} R\Sigma \rightarrow R\Sigma/I \rightarrow 0$$

of  $R$ -modules gives the exact sequence

$$S \otimes_R I \xrightarrow{1 \otimes i} S \otimes_R R\Sigma \rightarrow S \otimes_R R\Sigma/I \rightarrow 0$$

of  $S$ -modules.

Let  $m: S \otimes_R R\Sigma \rightarrow S\Sigma$  be the  $S$ -algebra homomorphism given by

$$m(s \otimes (rp + I)) = sf(r)p + S\bar{f}(I),$$

for all  $r \in R$ ,  $s \in S$  and paths  $p$  in  $\Gamma$ . The  $S$ -algebra homomorphism  $S\Sigma \rightarrow S \otimes_R R\Sigma$  given by sending  $sp$  to  $s \otimes p$  is inverse to  $m$ , so  $m$  is an isomorphism of  $S$ -algebras. Observe that  $m$  is also  $R$ -linear, so is an isomorphism of  $R$ -algebras. The image of  $m \circ (1 \otimes i): S \otimes_R I \rightarrow S\Sigma$  is  $S\bar{f}(I)$  since the image is spanned by elements of the form

$$\begin{aligned} m((1 \otimes i)(s \otimes x)) &= m(s \otimes x) \\ &= s\bar{f}(x), \end{aligned}$$

for  $s \in S$  and  $x \in I$ .

Thus we have a commuting diagram of  $S$ -modules with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & \ker \bar{m} & & \\ & & \downarrow & & \downarrow & & \\ & S \otimes_R I & \xrightarrow{1 \otimes i} & S \otimes_R R\Sigma & \rightarrow & S \otimes_R R\Sigma/I & \rightarrow 0 \\ & \downarrow & & \downarrow m & & \downarrow \bar{m} & \\ 0 & \rightarrow & S\bar{f}(I) & \rightarrow & S\Sigma & \rightarrow & S\Sigma/S\bar{f}(I) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & \text{coker } \bar{m} \end{array}$$

The morphism  $\bar{m}$  is given by the universal property of cokernels and can be computed explicitly using the commuting diagram, with

$$\begin{aligned} \bar{m}(s \otimes (rp + I)) &= m(s \otimes rp) + S\bar{f}(I) \\ &= sf(r)p + S\bar{f}(I), \end{aligned}$$

for all  $r \in R$ ,  $s \in S$  and paths  $p$  in  $\Sigma$ .

**Lemma 3.2.15.** [23] *The morphism*

$$\bar{m}: S \otimes_R R\Sigma/I \rightarrow S\Sigma/S\bar{f}(I)$$

*is both an isomorphism of  $R$ -algebras and an isomorphism of  $S$ -algebras.*

*Proof.* Using the snake lemma on the above commuting diagram gives an exact sequence of  $S$ -modules

$$0 \rightarrow \ker \bar{m} \rightarrow 0 \rightarrow 0 \rightarrow \operatorname{coker} \bar{m} \rightarrow 0,$$

so  $\ker \bar{m}$  and  $\operatorname{coker} \bar{m}$  are both zero and therefore  $\bar{m}$  is an isomorphism of  $S$ -algebras. Moreover,  $\bar{m}$  is  $R$ -linear, so is also an isomorphism of  $R$ -algebras.  $\square$

Recall that the  $q$ -integers are given by

$$[[0]] = 0$$

and

$$[[m]] = 1 + q + \cdots + q^{m-1} = 1 + q[[m-1]]$$

for  $m \in \mathbb{Z}$  with  $m \geq 1$ . For  $m \in \mathbb{N}$ , define the  $q$ -factorial

$$[[m]]! = \prod_{a=1}^m [[a]].$$

Given integers  $a$  and  $b$  with  $0 < a < b$ ,

$$[[b]] - [[a]] = q^a [[b-a]]$$

and the product  $[[a]][[b]]$  can be computed recursively as follows:

$$\begin{aligned} [[a]][[b]] &= (1 + q[[a-1]])(1 + q[[b-1]]) \\ &= 1 + q([a-1] + [b-1] + q[a-1][b-1]). \end{aligned}$$

The set of  $q$ -integers is not multiplicatively closed since, for example  $[[2]]^2 = 1 + 2q + q^2$ , but the set  $1 + q\mathbb{Z}[q]$  is multiplicatively closed and contains the  $q$ -integers. Let  $\mathcal{Q}$  be the localisation of  $\mathbb{Z}[q]$  at the set of elements of the form  $1 + qf$  for  $f \in \mathbb{Z}[q]$ , so  $\mathcal{Q}$  is the subring of  $\mathbb{Q}(q)$  given by

$$\mathcal{Q} = \left\{ \frac{f}{1 + qg} : f, g \in \mathbb{Z}[q] \right\}.$$

Observe that  $\mathbb{Z}[q]$  is a subring of  $\mathcal{Q}$ , so  $\mathcal{Q}$  is a  $\mathbb{Z}[q]$ -algebra. The  $\mathcal{Q}$ -form of the affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  is defined to be the  $\mathcal{Q}$ -algebra

$$\hat{S}_{\mathcal{Q}}(n, r) = \mathcal{Q} \otimes_{\mathbb{Z}[q]} \hat{S}_q(n, r).$$

**Lemma 3.2.16.** *The  $\mathcal{Q}$ -algebra homomorphism*

$$\bar{m}: \mathcal{Q} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q]\Gamma/I \rightarrow \mathcal{Q}\Gamma/\mathcal{Q}I$$

*given by*

$$\bar{m}(s \otimes (hp + I)) = shp + \mathcal{Q}I,$$

*for all  $s \in \mathcal{Q}$ ,  $h \in \mathbb{Z}[q]$  and paths  $p$  in  $\Gamma$ , is an isomorphism.*

*Proof.* Applying Lemma 3.2.15 for the inclusion  $\mathbb{Z}[q] \hookrightarrow \mathcal{Q}$  proves that  $\bar{m}$  is an isomorphism of  $\mathcal{Q}$ -algebras and an isomorphism of  $\mathbb{Z}[q]$ -algebras.  $\square$

Let  $\phi_{\mathcal{Q}}$  be the  $\mathcal{Q}$ -algebra homomorphism

$$\phi_{\mathcal{Q}} = (1 \otimes \phi) \circ \bar{m}^{-1} : \mathcal{Q}\Gamma/\mathcal{Q}I \rightarrow \hat{S}_{\mathcal{Q}}(n, r),$$

which is given by

$$\begin{aligned}\phi_{\mathcal{Q}}(e_{i,\lambda} + \mathcal{Q}I) &= E_{i,\lambda} \\ \phi_{\mathcal{Q}}(f_{i,\lambda} + \mathcal{Q}I) &= F_{i,\lambda} \\ \phi_{\mathcal{Q}}(k_{\lambda} + \mathcal{Q}I) &= 1_{\lambda},\end{aligned}$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$ .

**Proposition 3.2.17.** *If  $r < n$  then  $\phi_{\mathcal{Q}}$  is surjective.*

*Proof.* Fix  $A \in \Lambda_1$  and let  $p$  be the standard path in  $\Gamma$  corresponding to  $A$ . Then

$$\phi_{\mathcal{Q}}(p + \mathcal{Q}I) = \sum_{B: B \leq A} g_B e_B$$

for some  $g_B \in \mathbb{Z}[q]$ , where

$$g_A = \prod_{i \in \{1, \dots, n\}, s \geq 1} [[\alpha_{i,s}]]! [[\beta_{i,s}]]!,$$

by Proposition 3.2.14. The coefficient of the leading term  $g_A$  is a unit in  $\mathcal{Q}$ , so

$$e_A = \phi_{\mathcal{Q}}(g_A^{-1}p + \mathcal{Q}I) - \sum_{B: B < A} g_B g_A^{-1} e_B.$$

There are only finitely many  $B \in \Lambda_1$  with  $B < A$  and for each such  $B$ ,  $e_B$  admits a similar expression, which shows that  $e_A$  can be expressed as the image of a  $\mathcal{Q}$ -linear combination of the standard paths corresponding to the matrices  $B$  with  $B \leq A$  and therefore  $\phi_{\mathcal{Q}}$  is surjective.  $\square$

**Conjecture 3.2.18.** If  $r < n$ , the quiver with relations  $(\Gamma, I)$  gives a presentation of  $\hat{S}_{\mathcal{Q}}(n, r)$  over  $\mathcal{Q}$ .

*Ideas for proof.* The only thing that remains to be shown is that the map from the quiver algebra is injective, since Proposition 3.2.17 shows that this map is surjective.

I hope to deduce this from the presentation of the affine generic algebra by tensoring the surjective map between  $\mathcal{Q}$ -forms of the path algebra and  $q$ -Schur algebra with the  $\mathcal{Q}$ -algebra  $\mathcal{Q}/(q)$  and observing this map is an isomorphism of  $\mathbb{Z}$ -algebras.  $\square$

### 3.3 Relations for the $n=2$ case

In this section we give relations in  $\hat{S}_q(2, r)$ . Compare with the relations given in [\[REFERENCE - a double hall algebra approach to affine quantum schur weyl theory p.13\]](#) in the presentation of quantum affine  $\mathfrak{sl}_n$ .

**Lemma 3.3.1.** *The following equations hold in  $\hat{S}_q(2, r)$ :*

$$\begin{aligned}qE_1E_2^3 - [[3]]E_2E_1E_2^2 + [[3]]E_2^2E_1E_2 - qE_2^3E_1 &= 0 \\ qE_1^3E_2 - [[3]]E_1^2E_2E_1 + [[3]]E_1E_2E_1^2 - qE_2E_1^3 &= 0 \\ qF_2F_1^3 - [[3]]F_1F_2F_1^2 + [[3]]F_1^2F_2F_1 - qF_1^3F_2 &= 0 \\ qF_2^3F_1 - [[3]]F_2^2F_1F_2 + [[3]]F_2F_1F_2^2 - qF_1F_2^3 &= 0.\end{aligned}$$

*Proof.* It suffices to prove the first of these relations holds, since the second relation is obtained by applying the shifting automorphism of  $\hat{S}_q(n, r)$  given by conjugation by  $R$ , which sends  $E_1$  to  $E_2$  and  $E_2$  to  $E_1$ , and then the last two relations are obtained by applying the transpose operator  $S$  on  $\hat{S}_q(n, r)$ , which sends  $E_i$  to  $F_i$  (for  $i = 1, 2$ ) and reverses the order of multiplication.

Next, the first relation will be established by an explicit computation using the fundamental multiplication rules 3.1.3.

Write

$$\begin{aligned} W &= D_\lambda + \mathcal{E}_{1,2} - \mathcal{E}_{1,1} + 3\mathcal{E}_{2,3} - 3\mathcal{E}_{2,2} \\ X &= D_\lambda + 2\mathcal{E}_{2,3} + \mathcal{E}_{2,4} - 3\mathcal{E}_{2,2} \\ Y &= D_\lambda + \mathcal{E}_{1,3} - \mathcal{E}_{1,1} + 2\mathcal{E}_{2,3} - 2\mathcal{E}_{2,2} \\ Z &= D_\lambda + \mathcal{E}_{2,3} + \mathcal{E}_{2,5} - 2\mathcal{E}_{2,2}. \end{aligned}$$

$$E_1 E_2^3 = \sum_{\lambda \in \Lambda_0} [[2]][[3]]e_W + [[2]][[3]]e_Y$$

$$E_2 E_1 E_2^2 = \sum_{\lambda \in \Lambda_0} [[2]][[3]]e_W + [[2]]e_X + [[2]]^2 e_Y + [[2]]e_Z$$

$$E_2^2 E_1 E_2 = \sum_{\lambda \in \Lambda_0} [[2]][[3]]e_W + [[2]]^2 e_X + [[2]]e_Y + [[2]]e_Z$$

$$E_2^3 E_1 = \sum_{\lambda \in \Lambda_0} [[2]]3e_W + [[2]][[3]]e_X$$

Thus

$$\begin{aligned} qE_1 E_2^3 - [[3]]E_2 E_1 E_2^2 + [[3]]E_2^2 E_1 E_2 - qE_2^3 E_1 &= [[2]][[3]](q - [[3]] + [[3]] - q)e_W \\ &\quad + [[2]][[3]](-1 + [[2]] - q)e_X \\ &\quad + [[2]][[3]](q - [[2]] + 1)e_Y \\ &\quad + ([[2]][[3]] - [[2]][[3]])e_Z, \end{aligned}$$

which proves that the first relation holds and hence all the relations hold.  $\square$

## Chapter 4

# A generic affine algebra

### 4.1 Introduction

Assume  $\mathbf{k} = \mathbb{C}$  and fix  $n, r \geq 1$ . Let  $\mathcal{S}$  be the  $\mathbf{k}$ -algebra  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$  and let  $\mathcal{R}$  be the subalgebra generated by  $\varepsilon$ , namely  $\mathcal{R} = \mathbf{k}[\varepsilon]$ . Let  $V$  be a free  $\mathcal{S}$ -module of rank  $r$  and let  $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$  be the set of  $n$ -periodic cyclic flags in  $V$ ; so  $\mathcal{F}$  consists of collections  $L = (L_i)_{i \in \mathbb{Z}}$  of  $\mathcal{R}$ -lattices in  $V$  with  $L_i \subset L_{i+1}$  for  $i \in \mathbb{Z}$  and  $\varepsilon L_i = L_{i-n}$  for  $i \in \mathbb{Z}$ .

Let  $G$  be the group of  $\mathcal{S}$ -module automorphisms of  $V$ . Thus  $G$  is isomorphic to  $\mathrm{GL}_r(\mathcal{S})$ .  $G$  acts on  $\mathcal{F}$  with orbits  $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0\}$ , where  $\Lambda_0$  is the set of compositions of  $r$  into  $n$  parts, as in Definition 2.1.1.

The diagonal action of  $G$  on  $\mathcal{F} \times \mathcal{F}$  has orbits  $\{\mathcal{O}_A : A \in \Lambda_1\}$ , where  $\mathcal{O}_A$  consists of those pairs of flags with periodic characteristic matrix equal to  $A$ . Definitions of the periodic characteristic matrix and the set  $\Lambda_1$  are given in Definition 2.1.7 and Definition 2.1.2 respectively.

Recall that the periodic characteristic matrix of a pair  $(L, L') \in \mathcal{F} \times \mathcal{F}$  is the  $\mathbb{Z} \times \mathbb{Z}$  matrix  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ , with

$$a_{i,j} = \dim \left( \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right)$$

for each  $i, j \in \mathbb{Z}$ .

Recall that  $\mathrm{ro}$  and  $\mathrm{co}$  are the maps  $\Lambda_1 \rightarrow \Lambda_0$  given by

$$\mathrm{ro}(A) = \left( \sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co}(A) = \left( \sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right)$$

for each  $A \in \Lambda_1$ . Given  $A \in \Lambda_1$ , write  $A: \mathrm{co}(A) \rightarrow \mathrm{ro}(A)$ .

The purpose of this chapter is to define an associative  $\mathbb{Z}$ -algebra with a multiplicative basis by defining a modified form of the product in the affine  $q$ -Schur algebra. In particular, given  $A, B \in \Lambda_1$ , the orbit product

$$X_{A,B} = \{(L, L'') \in \mathcal{F} \times \mathcal{F} : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$



consists of finitely many  $G$ -orbits and it will be shown that there is a unique ‘generic’ orbit in  $X_{A,B}$ , denoted by  $\mathcal{O}_{A*B}$ , with the property that

$$\dim \left( \frac{L_i}{L_i \cap L_j''} \right) \leq \dim \left( \frac{N_i}{N_i \cap N_j''} \right)$$

and

$$\dim \left( \frac{L_j''}{L_i \cap L_j''} \right) \leq \dim \left( \frac{N_j''}{N_i \cap N_j''} \right)$$

for all  $i, j \in \mathbb{Z}$ ,  $(N, N'') \in \mathcal{O}_{A*B}$  and  $(L, L'') \in X_{A,B}$ . It will be shown that the above ‘generic product’ of orbits is associative, so the free  $\mathbb{Z}$ -module on the set of  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$  with  $\mathbb{Z}$ -bilinear multiplication given by

$$\mathcal{O}_A * \mathcal{O}_B = \mathcal{O}_{A*B},$$

for each  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$ , and

$$\mathcal{O}_A * \mathcal{O}_B = 0$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) \neq \text{ro}(B)$ , is an associative  $\mathbb{Z}$ -algebra with multiplicative identity given by

$$\sum_{\lambda \in \Lambda_0} \mathcal{O}_{D_\lambda},$$

where  $D_\lambda$  is the diagonal matrix with  $\text{co}(D_\lambda) = \lambda$ . The resulting  $\mathbb{Z}$ -algebra is called the *generic affine algebra* (of rank  $r$  and period  $n$ ), denoted by  $\hat{G}(n, r)$ .

## 4.2 Grassmannians and related varieties

Here we collect a few elementary results on Grassmannians and some related varieties. In this section, let  $V$  be an  $n$ -dimensional  $\mathbf{k}$ -vector space and let  $0 \leq d \leq n$  be an integer. There is a linear map

$$\phi^{(d)} : \Lambda^d(V) \rightarrow \text{Hom}(V, \Lambda^{d+1}(V))$$

given by

$$\phi^{(d)}(\alpha)(v) = \alpha \wedge v$$

for  $\alpha \in \Lambda^d(V)$  and  $v \in V$ . The kernel of  $\phi^{(d)}(\alpha)$  is the space of divisors of  $\alpha$ ,

$$D_\alpha = \{v \in V : \alpha \wedge v = 0\}.$$

An element  $\alpha \in \Lambda^d(V)$  is said to be totally decomposable if  $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_d$ , where  $\alpha_1, \dots, \alpha_d \in V$  are linearly independent. The dimension of  $D_\alpha$  is at most  $d$  and  $\dim(D_\alpha) = d$  precisely when  $\alpha$  is totally decomposable. Consequently, the rank of  $\phi^{(d)}(\alpha)$  is at least  $n - d$  and  $\alpha$  is totally decomposable if and only if  $\text{rank } \phi^{(d)}(\alpha) \leq n - d$ , which holds if and only if the  $(n - d + 1) \times (n - d + 1)$ -minors of a matrix of  $\phi^{(d)}(\alpha)$  are all zero.

**Lemma 4.2.1.**  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$  is a projective variety, for each  $d_1, d_2, a \in \mathbb{N}$  with  $d_1, d_2, a \leq n$ .

*Proof.* As above, there is a linear map  $\Psi: \Lambda^{d_1}V \oplus \Lambda^{d_2}V \rightarrow \text{Hom}(V, \Lambda^{d_1+1}(V) \oplus \Lambda^{d_2+1}(V))$  given by  $\Psi(\alpha, \beta)(v) = (\alpha \wedge v, \beta \wedge v)$ . Given  $\alpha \in \Lambda^{d_1}(V)$  and  $\beta \in \Lambda^{d_2}(V)$ , the kernel of  $\Psi(\alpha, \beta)$  is  $D_\alpha \cap D_\beta$  and so the rank of  $\Psi(\alpha, \beta)$  is  $n - \dim(D_\alpha \cap D_\beta)$ .

Let  $U_i \in \text{Gr}_{d_i}(V)$  and suppose  $p_i(U_i) = [\alpha_i]$ , where  $p_i$  is the Plücker embedding of  $\text{Gr}_{d_i}(V)$  in  $\mathbb{P}(\Lambda^{d_i}(V))$ , so  $U_i = D_{\alpha_i} = \ker \phi^{(d_i)}(\alpha)$ . Therefore the kernel of  $\Psi(\alpha_1, \alpha_2)$  is  $U_1 \cap U_2$ , so the condition that  $\dim(U_1 \cap U_2) \geq a$  is equivalent to the condition that  $\Psi(\alpha_1, \alpha_2)$  has rank at most  $n - a$ . After fixing a basis of  $V$ , this condition is given by the vanishing of the  $(n - a + 1) \times (n - a + 1)$  minors of the matrix of  $\Psi(\alpha_1, \alpha_2)$  with respect to this basis. Therefore  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$  is a closed subset of the product of Grassmannians  $\text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V)$ , so is a projective variety.

More precisely, the entries of a matrix of  $\Psi(\alpha_1, \alpha_2)$  are homogeneous polynomials of degree 1 in the Plücker coordinates on  $\text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V)$  since  $\Psi$  is linear and so the minors of  $\Psi(\alpha_1, \alpha_2)$  are also homogeneous polynomials in the Plücker coordinates.  $\square$

**Lemma 4.2.2.** *Let  $V$  be an  $n$ -dimensional vector space over  $\mathbf{k}$  and let  $d_1, d_2, a \in \mathbb{N}$  with  $d_1, d_2, a \leq n$ . The following hold:*

1.  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) = a\}$  is a quasiprojective variety;
2.  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : U_1 \subset U_2\}$  is a projective variety;
3. Given  $U_2 \in \text{Gr}_{d_2}(V)$ ,  $\{U_1 \in \text{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\}$  is a projective variety;
4. Given  $U_2 \in \text{Gr}_{d_2}(V)$ ,  $\{U_1 \in \text{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) = a\}$  is a quasiprojective variety;
5. Given  $U_2 \in \text{Gr}_{d_2}(V)$ ,  $\{U_1 \in \text{Gr}_{d_1}(V) : U_1 \subset U_2\}$  is a projective variety;
6. Given  $U_2 \in \text{Gr}_{d_2}(V)$ ,  $\{U_1 \in \text{Gr}_{d_1}(V) : U_2 \subset U_1\}$  is a projective variety.

*Proof.* Let  $X_i$  denote the space in statement  $i$  of the lemma. To emphasise the dependence of  $X_i$  on  $a$ , write  $X_{i,a}$ .

$X_1$  is a quasiprojective variety since it is equal to the intersection of the projective variety  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$  with the open set  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \leq a\}$ .

Given  $(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V)$ ,  $U_1 \subset U_2$  if and only if  $\dim(U_1 \cap U_2) \geq d_1$ , so Lemma 4.2.1 shows  $X_2$  is a projective variety.

Let  $\pi_i: \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) \rightarrow \text{Gr}_{d_i}(V)$  be the projection map onto the  $i$ -th factor, for  $i = 1, 2$ . The completeness property of projective varieties ensures that  $\pi_i$  is a closed morphism. Observe that

$$\begin{aligned} X_3 &= \{U_1 \in \text{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\} \\ &= \pi_1(\{(U_1, W) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap W) \geq a\} \cap \pi_2^{-1}\{U_2\}). \end{aligned}$$

The fibre of  $\pi_2$  over  $U_2$  is closed, so the intersection of the fibre with the variety from Lemma 4.2.1 is closed and then the image of this intersection under  $\pi_1$  is closed. This shows  $X_3$  is a projective variety.

$X_4$  is a quasiprojective variety since it is the complement of the subvariety  $X_{3,a+1}$  in  $X_{3,a}$ . Finally, 5-6 follow as special cases of 3 since  $X_5 = X_{3,d_1}$  and  $X_6 = X_{3,d_2}$ .  $\square$

### 4.3 Geometry of affine flag varieties

Given  $L \in \mathcal{F}$ ,  $N, a \in \mathbb{N}$  and  $\lambda \in \Lambda_0$  define

$$\Pi_{N,\lambda}(L) = \{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi_{N,\lambda}^a(L) = \left\{ L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0, \dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) = a \right\}.$$

**Lemma 4.3.1.** *Given  $L \in \mathcal{F}$ ,  $N \in \mathbb{N}$  and  $\lambda \in \Lambda_0$ ,*

$$\Pi_{N,\lambda}(L) = \bigcup_{a: 0 \leq a \leq 2Nr} \Pi_{N,\lambda}^a(L).$$

*Proof.* If  $L' \in \Pi_{N,\lambda}(L)$  then  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$  and the  $\mathbf{k}[\varepsilon]$ -module  $\varepsilon^{-N} L_0 / L'_0$  is naturally isomorphic to  $(\varepsilon^{-N} L_0 / \varepsilon^N L_0) / (L'_0 / \varepsilon^N L_0)$ , so

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) \leq \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0} \right) = 2Nr.$$

□

**Lemma 4.3.2.** *Given  $L \in \mathcal{F}$ ,  $N \in \mathbb{N}$ ,  $\lambda \in \Lambda_0$  and  $a \in \mathbb{N}$  with  $0 \leq a \leq 2Nr$ ,  $\Pi_{N,\lambda}^a(L)$  is a projective algebraic variety.*

*Proof.* Let  $W$  be the  $\mathbf{k}[\varepsilon]$ -module  $\varepsilon^{-1-N} L_0 / \varepsilon^N L_0$ , which has dimension  $(2N+1)r$  over  $\mathbf{k}$ . Let  $d_i = 2Nr - a + \lambda_1 + \dots + \lambda_i$  for each  $i = 1, \dots, n$ . The correspondence between submodules of  $\varepsilon^{-1-N} L_0$  which contain  $\varepsilon^N L_0$  and submodules of  $\varepsilon^{-1-N} L_0 / \varepsilon^N L_0$  determines a map

$$\rho: \Pi_{N,\lambda}^a(L) \rightarrow \mathrm{Gr}_{d_1}(W) \times \dots \times \mathrm{Gr}_{d_n}(W),$$

with  $\rho(L') = (L'_1 / \varepsilon^N L_0, \dots, L'_n / \varepsilon^N L_0)$ .

Let  $\mathcal{X}$  be the space of  $(U_1, \dots, U_n) \in \mathrm{Gr}_{d_1}(W) \times \dots \times \mathrm{Gr}_{d_n}(W)$  with  $U_i \subset U_{i+1}$  for  $i = 1, \dots, n-1$  and  $\varepsilon U_n \subset U_1$ . Lemma 4.2.2 shows that each of these conditions is closed, so  $\mathcal{X}$  is a closed subset of  $\mathrm{Gr}_{d_1}(W) \times \dots \times \mathrm{Gr}_{d_n}(W)$ , therefore  $\mathcal{X}$  is a projective algebraic variety.

The image of  $\rho$  is contained in  $\mathcal{X}$  since

$$\varepsilon L'_n / \varepsilon^N L_0 = L'_0 / \varepsilon^N L_0 \subset L'_1 / \varepsilon^N L_0 \subset \dots \subset L'_n / \varepsilon^N L_0.$$

Suppose  $(U_1, \dots, U_n) \in \mathcal{X}$ . Then  $U_i$  is a  $\mathbf{k}[\varepsilon]$ -module, since  $\varepsilon U_i \subset \varepsilon U_n \subset U_1 \subset U_i$ , for each  $i = 1, \dots, n$ , so  $U_i$  lifts uniquely to a  $\mathbf{k}[\varepsilon]$ -module  $L'_i$  with  $\varepsilon^N L_0 \subset L'_i \subset \varepsilon^{-1-N} L_0$ . Therefore  $L'_1, \dots, L'_n$  are  $\mathbf{k}[\varepsilon]$ -lattices with  $L_i \subset L_{i+1}$  for  $i = 1, \dots, n-1$  and  $\varepsilon L'_n \subset L'_1$ , with

$$\dim(\varepsilon^{-1-N} L_0 / L'_n) = \dim(W / W_n) = (2N+1)r - d_n = a$$

and

$$\dim(L'_i / L'_{i-1}) = \dim(W_i / W_{i-1}) = d_i - d_{i-1} = \lambda_i,$$

for each  $i = 2, \dots, n$ . Therefore there is a unique  $L' \in \Pi_{N,\lambda}^a(L)$  such that  $\rho(L') = (W_1, \dots, W_n)$ , where  $L'$  is given by  $L'_{i+cn} = \varepsilon^{-c} L'_i$  for  $i = 1, \dots, n$  and  $c \in \mathbb{Z}$ . It follows  $\rho$  is injective and  $\mathrm{im} \rho = \mathcal{X}$ , which is a projective variety, so  $\Pi_{N,\lambda}^a(L)$  is a projective variety. □

**Lemma 4.3.3.** *Given  $L \in \mathcal{F}$ ,  $N \in \mathbb{N}$ ,  $\lambda \in \Lambda_0$  and  $a \in \mathbb{N}$  with  $0 \leq a \leq 2Nr$ ,  $\Pi_{N,\lambda}^a(L)$  is closed in  $\Pi_{N+1,\lambda}^{a+r}(L)$ .*

*Proof.* If  $L' \in \Pi_{N,\lambda}^a(L)$ , then  $\varepsilon^{N+1}L_0 \subset \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0 \subset \varepsilon^{-(N+1)}L_0$  and

$$\dim \left( \frac{\varepsilon^{-(1+n)}L_0}{L'_0} \right) = \dim \left( \frac{L_0}{\varepsilon L_0} \right) + \dim \left( \frac{\varepsilon^{-N}L_0}{L'_0} \right) = r + a,$$

which shows that  $\Pi_{N,\lambda}^a(L) \subset \Pi_{N+1,\lambda}^{a+r}(L)$ . For  $L' \in \Pi_{N+1,\lambda}^{a+r}(L)$ , if additionally  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$ , then

$$\dim \left( \frac{\varepsilon^{-(N+1)}L_0}{L'_0} \right) = r + \dim \left( \frac{\varepsilon^{-N}L_0}{L'_0} \right),$$

which shows  $L' \in \Pi_{N,\lambda}^a(L)$ . Therefore  $\Pi_{N,\lambda}^a(L)$  is the subspace of  $\Pi_{N+1,\lambda}^{a+r}(L)$  defined by the two closed conditions  $\varepsilon^N L_0 \subset L'_0$  and  $L'_0 \subset \varepsilon^{-N}L_0$ , using Lemma 4.2.2.  $\square$

**Lemma 4.3.4.** *Let  $\lambda \in \Lambda_0$ ,  $M, N \in \mathbb{N}$ ,  $L, \tilde{L} \in \mathcal{F}$ ,  $0 \leq a \leq 2Nr$ ,  $0 \leq b \leq 2Mr$ .  $\Pi_{N,\lambda}^a(L) \cap \Pi_{M,\lambda}^b(\tilde{L})$  is a closed set in  $\Pi_{N,\lambda}^a(L)$ . In particular, if the intersection is nonempty it is a projective algebraic variety.*

*Proof.* Observe that  $\Pi_{N,\lambda}^a(L) \cap \Pi_{M,\lambda}^b(\tilde{L})$  is the subset of  $\Pi_{N,\lambda}^a(L)$  defined by the additional conditions that  $\varepsilon^M \tilde{L}_0 \subset L'_0$  and  $L'_0 \subset \varepsilon^{-M} \tilde{L}_0$ , so is a closed subset of  $\Pi_{N,\lambda}^a(L)$ , using 4.2.2.  $\square$

**Lemma 4.3.5.** *Suppose  $L \in \mathcal{F}$ ,  $N, a \in \mathbb{N}$  and  $\lambda \in \Lambda_0$  with  $a \leq 2Nr$ . For each  $g \in G$ , the natural map (restriction of the action map)  $\Pi_{N,\lambda}^a(L) \rightarrow \Pi_{N,\lambda}^a(gL)$  is an isomorphism of projective varieties.*

*Proof.* If  $L' \in \Pi_{N,\lambda}^a(L)$ , then  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$  and so  $\varepsilon^N g(L_0) \subset g(L'_0) \subset \varepsilon^{-N}g(L_0)$ , so  $gL' \in \Pi_{N,\lambda}^a(gL)$ . Thus  $g$  and  $g^{-1}$  induce mutually inverse morphisms of varieties  $g: \Pi_{N,\lambda}^a(L) \rightarrow \Pi_{N,\lambda}^a(gL)$  and  $g^{-1}: \Pi_{N,\lambda}^a(gL) \rightarrow \Pi_{N,\lambda}^a(L)$ .  $\square$

### 4.3.1 Action through an algebraic group

Let  $W$  be the  $\mathbb{C}[\varepsilon]$ -module  $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$ .  $\varepsilon^{2N+1}$  acts as zero on  $W$  and  $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle \otimes_{\mathbb{C}[\varepsilon]} W$  is a free  $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle$ -module of rank  $r$ . In particular,  $W$  is a complex vector space of dimension  $(2N+1)r$ .

Each element  $g \in G_L$  determines an endomorphism  $\bar{g}$  of  $W$ , given by

$$\bar{g}(x + \varepsilon^N L_0) = g(x) + \varepsilon^N L_0,$$

for each  $x \in \varepsilon^{-(1+N)}L_0$ . Given  $g, h \in G_L$ ,  $\overline{gh} = \bar{g}\bar{h}$  and so  $\bar{g}$  is an automorphism of  $W$  with  $\bar{g}^{-1} = \overline{g^{-1}}$ . Therefore the map  $\bar{\cdot}: G_L \mapsto \text{GL}(W)$  given by  $g \mapsto \bar{g}$  is a group homomorphism with kernel

$$H_{N,L} := \{g \in G_L : \bar{g} = 1\},$$

which consists of those  $g \in G_L$  such that

$$g(x) - x \in \varepsilon^N L_0$$

for each  $x \in \varepsilon^{-(1+N)}L_0$ . Thus  $G_L/H_{N,L}$  may be identified with a subgroup of  $\text{GL}(W)$ .

**Lemma 4.3.6.**  $G_L/H_{N,L}$  is a connected algebraic group.

*Proof.* As a result of the first isomorphism theorem,  $G_L/H_{N,L}$  is isomorphic to the image of  $G_L$  in  $\text{GL}(W)$ , which will be described explicitly by equations in the coordinate functions on  $\text{GL}(W)$ , with respect to a fixed basis of  $W$ .

Let  $\{\tilde{x}_1, \dots, \tilde{x}_r\}$  be a basis of  $L_n/L_0$  over  $\mathbb{C}$  which is adapted to the flag

$$L_1/L_0 \subset \dots \subset L_{n-1}/L_0 \subset L_n/L_0,$$

so that

$$L_i/L_0 = \langle \tilde{x}_1, \dots, \tilde{x}_{\lambda_1 + \dots + \lambda_i} \rangle$$

for each  $i \in \{1, \dots, n\}$ . Fix  $x_1, \dots, x_r \in L_n$  such that  $\tilde{x}_i = x_i + L_0$  for each  $i = 1, \dots, r$ , then

$$L_i = L_0 + \langle x_1, \dots, x_{\lambda_1 + \dots + \lambda_i} \rangle$$

for  $i = 1, \dots, r$ .

Then  $W$  has a  $\mathbb{C}$ -basis  $\{y_j : 1 \leq j \leq (2N+1)r\}$  given by

$$y_{i+cr} = \varepsilon^{-c+N} x_i$$

for each  $i \in \{1, \dots, r\}$  and  $c \in \{0, \dots, 2N\}$ . Observe that  $\varepsilon y_i = 0$  for  $i \in \{1, \dots, r\}$  and  $\varepsilon y_i = y_{i-r}$  for  $r < i \leq (2N+1)r$ .

The coordinate functions on  $\text{GL}(W)$  with respect to this choice of basis are the maps

$$\gamma_{i,j} : \text{GL}(W) \rightarrow \mathbb{C}$$

for  $i, j \in \mathbb{Z}$  with  $1 \leq i, j \leq (2N+1)r$ , given by

$$g(y_j) = \sum_i \gamma_{ij}(g) y_i,$$

for each  $j = 1, \dots, (2N+1)r$ .

The image of  $G_L$  in  $\text{GL}(W)$  is the subgroup defined by the conditions

$$\gamma_{i,j} = \gamma_{i-r,j-r}$$

for each  $i, j \in \{r+1, \dots, (2N+1)r\}$  and

$$\gamma_{i,j} = 0$$

for each  $i, j \in \{1, \dots, (2N+1)r\}$  with  $i > \lambda_1 + \dots + \lambda_s$  and  $j \leq \lambda_1 + \dots + \lambda_s$  for some  $s \in \{1, \dots, r\}$ . This shows that the image of  $G_L$  in  $\text{GL}(W)$  is a connected algebraic group and therefore  $G_L/H_{N,L}$  is a connected algebraic group.

With respect to the basis  $\{y_i : i \in \{1, \dots, (2N+1)r\}\}$ , the image of  $G_L$  in  $\text{GL}(W)$  consists of matrices of the form

$$\begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{2N} \\ 0 & A_0 & A_1 & \cdots & A_{2N-1} \\ 0 & 0 & A_0 & \cdots & A_{2N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_0 \end{pmatrix}$$

where  $A_0 \in \mathcal{P}_\lambda$  and  $A_1, \dots, A_{2N} \in M_r(\mathbb{C})$ , where  $\mathcal{P}_\lambda$  is the parabolic subgroup of  $\text{GL}_r(\mathbb{C})$  which is the stabiliser of the flag

$$L_1/L_0 \subset \dots \subset L_{n-1}/L_0 \subset L_n/L_0.$$

□

Given  $g \in G$ , the map  $G_L \rightarrow G_{gL}$  sending  $h$  to  $ghg^{-1}$  is a group isomorphism which descends to an isomorphism of algebraic groups  $G_L/H_{N,L} \rightarrow G_{gL}/H_{N,gL}$ . Thus we have a commuting diagram of morphisms of varieties, where the vertical arrows are isomorphisms:

$$\begin{array}{ccc} G_L/H_{N,L} \times \Pi_{N,\lambda}^a(L) & \longrightarrow & \Pi_{N,\lambda}^a(L) \\ \downarrow & & \downarrow \\ G_{gL}/H_{N,gL} \times \Pi_{N,\lambda}^a(gL) & \longrightarrow & \Pi_{N,\lambda}^a(gL) \end{array}$$

### 4.3.2 Incidence in affine flag varieties

**Lemma 4.3.7.** *Given  $N, a, b, c \in \mathbb{N}$ ,  $\lambda, \mu \in \Lambda_0$ ,  $L \in \mathcal{F}$  and  $i, j \in \mathbb{Z}$ ,*

$$\left\{ (L', L'') \in \Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L) : \dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) \leq c \right\}$$

*is a closed set in the projective variety  $\Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L)$ .*

*Proof.* There is  $M \geq N$  so that  $\varepsilon^M L_0 \subset L'_i \subset \varepsilon^{-M} L_0$  and  $\varepsilon^M L_0 \subset L''_j \subset \varepsilon^{-M} L_0$ . Let  $a' = a + (M - N)r$  and  $b' = b + (M - N)r$ . Lemma 4.3.3 shows that  $\Pi_{N,\lambda}^a(L)$  is a subvariety of  $\Pi_{M,\lambda}^{a'}(L)$ , so  $\Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L)$  is a subvariety of  $\Pi_{M,\lambda}^{a'}(L) \times \Pi_{M,\mu}^{b'}(L)$ .

The fact that

$$\dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) = \dim \left( \frac{L'_i/\varepsilon^M L_0}{L'_i/\varepsilon^M L_0 \cap L''_j/\varepsilon^M L_0} \right),$$

together with Lemma 4.3.2 and Lemma 4.2.1, shows that

$$\left\{ (L', L'') \in \Pi_{M,\lambda}^{a'}(L) \times \Pi_{M,\mu}^{b'}(L) : \dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) \leq c \right\}$$

is closed, so the intersection with  $\Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L)$  is closed. □

**Lemma 4.3.8.** *Given  $N, a, c \in \mathbb{N}$ ,  $\lambda \in \Lambda_0$ ,  $L \in \mathcal{F}$  and  $i, j \in \mathbb{Z}$ ,*

$$\left\{ L' \in \Pi_{N,\lambda}^a(L) : \dim \left( \frac{L_i}{L_i \cap L'_j} \right) \leq c \right\}$$

*and*

$$\left\{ L' \in \Pi_{N,\lambda}^a(L) : \dim \left( \frac{L'_j}{L_i \cap L'_j} \right) \leq c \right\}$$

*are closed sets in  $\Pi_{N,\lambda}^a(L)$ .*

*Proof.* This is a result of Lemma 4.2.2, since

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) = \dim \left( \frac{L_i/\varepsilon^M L_0}{L_i/\varepsilon^M L_0 \cap L'_j/\varepsilon^M L_0} \right),$$

where  $M \geq N$  is chosen so that  $\varepsilon^M L_0 \subset L_i \subset \varepsilon^{-M} L_0$  and  $\varepsilon^M L_0 \subset L'_j \subset \varepsilon^{-M} L_0$  for each  $L' \in \Pi_{N,\lambda}^a(L)$ . □

## 4.4 Geometry of orbits

Let  $A \in \Lambda_1$  and  $L \in \mathcal{F}_{\text{ro}(A)}$  and write  $\lambda = \text{co}(A)$ . Recall that

$$X_A^L = \{L' \in \mathcal{F}_\lambda : (L, L') \in \mathcal{O}_A\}.$$

**Lemma 4.4.1.** *There is  $N \in \mathbb{N}$  such that  $X_A^L \subset \Pi_{N,\lambda}^a(L)$ , where  $a = d_{nN,0}A$ .*

*Proof.* There is  $N \in \mathbb{N}$  so that  $a_{i,j} = 0$  whenever  $|j - i| > nN$ . If  $(L, L') \in \mathcal{O}_A$  then

$$\dim \left( \frac{L'_0}{L'_0 \cap \varepsilon^{-N} L_0} \right) = \dim \left( \frac{L'_0}{L'_0 \cap L_{nN}} \right) = \sum_{s > nN, t \leq 0} a_{s,t} = 0,$$

so it follows  $L'_0 \subset \varepsilon^{-N} L_0$ . Similarly,

$$\dim \left( \frac{\varepsilon^N L_0}{\varepsilon^N L_0 \cap L'_0} \right) = \dim \left( \frac{L_{-nN}}{L_{-nN} \cap L'_0} \right) = \sum_{s \leq -nN, t > 0} a_{s,t} = 0,$$

which shows  $\varepsilon^N L_0 \subset L'_0$ . Moreover,

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) = \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^{-N} L_0 \cap L'_0} \right) = \sum_{s \leq nN, t > 0} a_{s,t} = d_{nN,0}(A),$$

as a result of Lemma 2.1.12. □

Assume  $N \in \mathbb{N}$  is chosen so that  $X_A^L \subset \Pi_{N,\lambda}^a(L)$ , where  $a = d_{nN,0}A$ , as in Lemma 4.4.1.

**Lemma 4.4.2.**  *$X_A^L$  is a locally closed subset of  $\Pi_{N,\lambda}^a(L)$ . In particular,  $X_A^L$  is a quasiprojective variety.*

*Proof.* If  $L' \in \Pi_{N,\lambda}^a(L)$  then

$$L_{-Nn} = \varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset L'_n \subset \varepsilon^{-1-N} L_0 = L_{(N+1)n}.$$

Therefore  $X_A^L$  is the subset of  $\Pi_{N,\lambda}^a(L)$  defined by the conditions  $\dim(L_i/L_i \cap L'_j) = d_{i,j}A$  for  $i : -Nn \leq i < j$  and  $\dim(L'_j/L_i \cap L'_j) = \bar{d}_{i,j}A$  for  $i : j < i \leq (N+1)n$ , for  $j = 1, \dots, n$ .

The set of  $L' \in \Pi_{N,\lambda}^a(L)$  with  $\dim(L_i/L_i \cap L'_j) \leq d_{i,j}A$  for  $j = 1, \dots, n$  and  $i : -Nn \leq i < j$  and  $\dim(L'_j/L_i \cap L'_j) \leq \bar{d}_{i,j}A$  for  $j = 1, \dots, n$  and  $i : j < i \leq (N+1)n$  is a closed subset of  $\Pi_{N,\lambda}^a(L)$ , as a result of Lemma 4.3.8.

On the other hand, the set of  $L' \in \Pi_{N,\lambda}^a(L)$  satisfying the conditions  $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}A$  (for  $i < j$ ) and  $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}A$  (for  $i > j$ ) is open in  $\Pi_{N,\lambda}^a(L)$  since the complement is closed, as a result of Lemma 4.3.8.

Therefore  $X_A^L$  is the intersection of an open set and a closed set in  $\Pi_{N,\lambda}^a(L)$ , so  $X_A^L$  is locally closed. It follows that  $X_A^L$  is an open subset of the projective variety  $\overline{X_A^L}$ , so is a quasiprojective variety as claimed. □

**Lemma 4.4.3.**  *$X_A^L$  is irreducible.*

*Proof.* For any  $L' \in X_A^L$ ,  $X_A^L = G_L/H_{N,L} \cdot L'$ . Lemma 4.3.6 shows that  $G_L/H_{N,L}$  is a connected algebraic group which acts algebraically on  $\Pi_{N,\lambda}^a(L)$ . The image of  $G_L/H_{N,L}$  under the morphism  $g \mapsto gL'$  equals  $X_A^L$ , which shows  $X_A^L$  is irreducible since  $G_L/H_{N,L}$  is irreducible. □

Consequently,  $\overline{X_A^L}$  is an irreducible projective variety and the action of  $G_L/H_{N,L}$  on  $\Pi_{N,\lambda}^a(L)$  restricts to an algebraic group action on  $\overline{X_A^L}$  for which there are finitely many orbits. In particular,  $\overline{X_A^L} \setminus X_A^L$  is a union of finitely many orbits which are so-called degenerations of the orbit  $X_A^L$ .

## 4.5 Geometry of orbit products

Let  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$  and write  $\lambda = \text{co}(A)$  and  $\mu = \text{co}(B)$ . Fix  $L \in \mathcal{F}_{\text{ro}(A)}$ . Recall

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}_\lambda \times \mathcal{F}_\mu : L' \in X_A^L, L'' \in X_B^{L'}\}$$

and

$$X_{A,B}^L = \{L'' \in \mathcal{F}_\mu : \exists L' \in X_A^L \text{ with } L'' \in X_B^{L'}\}$$

**Lemma 4.5.1.** *There is  $N \in \mathbb{N}$  such that*

$$Y_{A,B}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L),$$

where  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ .

*Proof.* There is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$  and  $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$  for each  $(L', L'') \in Y_{A,B}^L$ , using Lemma 4.4.1. Set  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ .

Then for any  $(L', L'') \in Y_{A,B}^L$ ,

$$\varepsilon^{2N} L_0 \subset \varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0 \subset \varepsilon^{-2N} L_0$$

and

$$\begin{aligned} \dim \left( \frac{\varepsilon^{-2N} L_0}{L''_0} \right) &= \dim \left( \frac{\varepsilon^{-N} L'_0}{L''_0} \right) + \dim \left( \frac{\varepsilon^{-2N} L_0}{\varepsilon^{-N} L'_0} \right) \\ &= \dim \left( \frac{\varepsilon^{-N} L'_0}{L''_0} \right) + \dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) \\ &= a + b, \end{aligned}$$

as a result of Lemma 2.1.12, so  $(L', L'') \in \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  as required.  $\square$

Now assume  $N \in \mathbb{N}$  is chosen so that  $Y_{A,B}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ , where  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ , using Lemma 4.5.1.

**Lemma 4.5.2.**  *$Y_{A,B}^L$  is a locally closed subset of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ . In particular,  $Y_{A,B}^L$  is a quasiprojective variety.*

*Proof.*  $Y_{A,B}^L$  is the subset of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  consisting of those  $(L', L'')$  satisfying the following conditions:  $\dim(L_i/L_i \cap L'_j) = d_{i,j}(A)$  for  $i < j$ ,  $\dim(L'_j/L_i \cap L'_j) = \bar{d}_{i,j}(A)$  for  $i > j$ ,  $\dim(L'_i/L'_i \cap L''_j) = d_{i,j}(B)$  for  $i < j$  and  $\dim(L''_j/L'_i \cap L''_j) = \bar{d}_{i,j}(B)$ . Only finitely many conditions are required to define  $Y_{A,B}^L$  since there are only finitely many nonzero entries in  $A$  and  $B$  modulo the  $(n, n)$ -periodicity.

The conditions  $\dim(L_i/L_i \cap L'_j) \leq d_{i,j}(A)$ ,  $\dim(L'_j/L'_i \cap L''_j) \leq d_{i,j}(B)$ ,  $\dim(L'_j/L_i \cap L'_j) \leq \bar{d}_{i,j}(A)$  and  $\dim(L''_j/L'_i \cap L''_j) \leq \bar{d}_{i,j}(B)$  define closed subsets of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  for each  $i, j \in \mathbb{Z}$ , as a result of Lemma 4.3.7 and Lemma 4.3.8.

On the other hand, the conditions  $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}(A)$ ,  $\dim(L'_j/L'_i \cap L''_j) \geq d_{i,j}(B)$ ,  $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}(A)$  and  $\dim(L''_j/L'_i \cap L''_j) \geq \bar{d}_{i,j}(B)$  define open subsets of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  for each  $i, j \in \mathbb{Z}$ , using Lemma 4.3.7 and Lemma 4.3.8.

Therefore  $Y_{A,B}^L$  is the intersection of finitely many open sets and finitely many closed sets in  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ , so  $Y_{A,B}^L$  is locally closed. In particular,  $Y_{A,B}^L$  is a quasiprojective variety.  $\square$



**Lemma 4.5.3.** *For any  $L' \in X_A^L$ ,  $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$ .*

*Proof.* Let  $L' \in X_A^L$ , then  $\{L'\} \times X_B^{L'}$  is contained in  $Y_{A,B}^L$  and  $G_L$  acts on  $Y_{A,B}^L$ , so  $G_L \cdot (\{L'\} \times X_B^{L'})$  is contained in  $Y_{A,B}^L$ . If  $(N', N'') \in Y_{A,B}^L$ , then  $N' = \sigma L'$  for some  $\sigma \in G_L$ , since  $N' \in X_A^L$ . Then  $(N', N'') = \sigma(L', \sigma^{-1}N'')$  and  $\sigma^{-1}N'' \in X_B^{\sigma^{-1}N'} = X_B^{L'}$ , so  $(N', N'') \in \sigma \cdot (\{L'\} \times X_B^{L'})$ . Therefore  $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$  as claimed.  $\square$

**Proposition 4.5.4.**  *$Y_{A,B}^L$  is irreducible.*

*Proof.* Let  $L' \in X_A^L$ .  $G_L/H_{2N,L}$  is a connected algebraic group acting algebraically on  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  by Lemma 4.3.6.  $X_B^{L'}$  is an irreducible locally closed subset of  $\Pi_{2N,\mu}^{a+b}(L)$ , so  $\{L'\} \times X_B^{L'}$  is an irreducible locally closed set in  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ .  $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'}) = G_L/H_{2N,L} \cdot (\{L'\} \times X_B^{L'})$ , by Lemma 4.5.3, so it follows that  $Y_{A,B}^L$  is irreducible.  $\square$

Let  $p_2$  be the projection onto the second factor  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \rightarrow \Pi_{2N,\mu}^{a+b}(L)$ .  $p_2$  is a closed morphism since  $\Pi_{N,\lambda}^a(L)$  is a projective variety and therefore complete, by Lemma 4.3.2. Therefore  $p_2(\overline{Y_{A,B}^L}) = \overline{X_{A,B}^L}$ , since  $p_2(Y_{A,B}^L) = X_{A,B}^L$ .

**Lemma 4.5.5.**  *$X_{A,B}^L$  is irreducible and constructible.*

*Proof.* Proposition 4.5.4 shows that  $Y_{A,B}^L$  is irreducible and locally closed, so it follows  $X_{A,B}^L$  is irreducible and constructible, since  $X_{A,B}^L = p_2(Y_{A,B}^L)$ .  $\square$

**Proposition 4.5.6.** *There is a unique open  $G_L$ -orbit in  $X_{A,B}^L$ .*

*Proof.*  $X_{A,B}^L$  consists of finitely many  $G_L$ -orbits and is an irreducible topological space, by Lemma 4.5.5. Consequently,  $X_C^L$  is dense in  $X_{A,B}^L$  for some  $C \in \Lambda_1^{A,B}$ . Lemma 4.4.2 shows that  $X_C^L$  is locally closed in  $X_{A,B}^L$ , so  $X_C^L$  is open in  $\overline{X_C^L} = X_{A,B}^L$ . Irreducibility of  $X_{A,B}^L$  shows that there is a unique open  $G_L$ -orbit, since two nonempty open sets in  $X_{A,B}^L$  intersect nontrivially, thus any two open  $G_L$  orbits in  $X_{A,B}^L$  coincide.  $\square$

Let  $A * B \in \Lambda_1$  denote the matrix corresponding to the dense open  $G_L$ -orbit in  $X_{A,B}^L$ , so  $\overline{X_{A*B}^L} = \overline{X_{A,B}^L}$ .

## 4.6 Degenerations of orbits and the combinatorial partial order

**Proposition 4.6.1.** *Let  $A, B \in \Lambda_1$  with  $\text{ro}(A) = \text{ro}(B)$  and  $\text{co}(A) = \text{co}(B)$ . If  $X_B^L \subset \overline{X_A^L}$  for some  $L \in \mathcal{F}_{\text{ro}(A)}$  then  $B \leq A$  with respect to the hook order.*

*Proof.* Let  $\lambda = \text{co}(A)$ ,  $\mu = \text{ro}(A)$  and fix  $L \in \mathcal{F}_\mu$ . Assume  $N \in \mathbb{N}$  is chosen so that  $X_A^L \subset \Pi_{N,\lambda}^a(L)$  and  $X_B^L \subset \Pi_{N,\lambda}^b(L)$ , where  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ . Then  $X_A^L$  is an open subset of the projective variety consisting of those  $L' \in \Pi_{N,\lambda}^a(L)$  such that

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) \leq d_{i,j}(A)$$

and

$$\dim \left( \frac{L'_j}{L_i \cap L'_j} \right) \leq \bar{d}_{i,j}(A),$$

for all  $i, j \in \mathbb{Z}$ .

Assume  $X_B^L \subset \overline{X_A^L}$ , then

$$d_{i,j}(B) = \dim \left( \frac{L_i}{L_i \cap L'_j} \right) \leq d_{i,j}(A)$$

and

$$\bar{d}_{i,j}(B) = \dim \left( \frac{L'_j}{L_i \cap L'_j} \right) \leq \bar{d}_{i,j}(A),$$

for each  $i, j \in \mathbb{Z}$ , for any  $L' \in X_B^L$ . So  $B \leq A$  if  $X_B^L \subseteq \overline{X_A^L}$ .  $\square$

**Remark 4.6.2.** In practice it seems that the converse of Proposition 4.6.1 is true, so that the closure order and the hook order are the same, although I have not been able to find a proof.

**Corollary 4.6.3.** *The maximum in  $\Lambda_1^{A,B}$  is  $A * B$ .*

## 4.7 Associativity of the generic product

Let  $A, B, C \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$  and  $\text{co}(B) = \text{ro}(C)$  and fix  $L \in \mathcal{F}_{\text{ro}(A)}$ . Write  $\lambda = \text{co}(A)$ ,  $\mu = \text{co}(B)$  and  $\nu = \text{co}(C)$ . Define

$$Y_{A,B,C}^L = \left\{ (L', L'', L''') \in \mathcal{F}^3 : L' \in X_A^L, L'' \in X_B^{L'}, L''' \in X_C^{L''} \right\}$$

and

$$X_{A,B,C}^L = \left\{ L''' \in \mathcal{F} : \exists (L', L'') \in \mathcal{F}^2 \text{ with } (L', L'', L''') \in Y_{A,B,C}^L \right\}.$$

**Lemma 4.7.1.** *There is  $N \in \mathbb{N}$  such that  $Y_{A,B,C}^L$  is contained in  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$ , where  $a = d_{nN,0}(A)$ ,  $b = d_{nN,0}(B)$  and  $c = d_{nN,0}(C)$ .*

*Proof.* Lemma 4.4.1 shows that there is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ ,  $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$  and  $\varepsilon^N L''_0 \subset L'''_0 \subset \varepsilon^{-N} L''_0$  for each  $(L', L'', L''') \in Y_{A,B,C}^L$ . Using the proof of Lemma 4.5.1, it follows  $L'' \in \Pi_{2N,\mu}^{a+b}(L)$  and  $L''' \in \Pi_{2N,\nu}^{b+c}(L') \subset \Pi_{3N,\nu}^{a+b+c}(L)$ .  $\square$

Assume  $N \in \mathbb{N}$  is chosen so that  $Y_{A,B,C}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$ , where  $a = d_{nN,0}(A)$ ,  $b = d_{nN,0}(B)$  and  $c = d_{nN,0}(C)$ , as in Lemma 4.7.1.

**Lemma 4.7.2.**  *$Y_{A,B,C}^L$  is a locally closed subset of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$ . In particular,  $Y_{A,B,C}^L$  is a quasiprojective variety.*

*Proof.* Write  $\Pi = \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}(L)$ . Then  $Y_{A,B,C}^L$  consists of those  $(L', L'', L''') \in \Pi$  satisfying the following conditions:

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}(A), \quad (4.7.1)$$

$$\dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) = d_{i,j}(B), \quad (4.7.2)$$

$$\dim \left( \frac{L''_i}{L''_i \cap L'''_j} \right) = d_{i,j}(C), \quad (4.7.3)$$

for  $(i, j) \in \{1, \dots, n\} \times \mathbb{Z}$  with  $i < j < (N+1)n$ , and

$$\dim \left( \frac{L'_j}{L_i \cap L'_j} \right) = \bar{d}_{i,j}(A), \quad (4.7.4)$$

$$\dim \left( \frac{L''_j}{L'_i \cap L''_j} \right) = \bar{d}_{i,j}(B), \quad (4.7.5)$$

$$\dim \left( \frac{L'''_j}{L''_i \cap L'''_j} \right) = \bar{d}_{i,j}(C), \quad (4.7.6)$$

for  $(i, j) \in \{1, \dots, n\} \times \mathbb{Z}$  with  $-Nn < j < i$ .

For  $i < j$ , the conditions

$$\begin{aligned} \dim(L_i/L_i \cap L'_j) &\leq d_{i,j}(A), \\ \dim(L'_i/L'_i \cap L''_j) &\leq d_{i,j}(B) \end{aligned}$$

and

$$\dim(L''_i/L''_i \cap L'''_j) \leq d_{i,j}(C)$$

define closed subsets of  $\Pi$ , by Lemma 4.3.7. For  $i > j$ , the conditions

$$\begin{aligned} \dim(L'_j/L_i \cap L'_j) &\leq \bar{d}_{i,j}(A), \\ \dim(L''_j/L'_i \cap L''_j) &\leq \bar{d}_{i,j}(B) \end{aligned}$$

and

$$\dim(L'''_j/L''_i \cap L'''_j) \leq \bar{d}_{i,j}(C)$$

also define closed subsets of  $\Pi$ .

On the other hand, the conditions  $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}(A)$ ,  $\dim(L'_i/L'_i \cap L''_j) \geq d_{i,j}(B)$  and  $\dim(L''_i/L''_i \cap L'''_j) \geq d_{i,j}(C)$  for  $i < j$  define open subsets of  $\Pi$ . Similarly, the conditions  $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}(A)$ ,  $\dim(L''_j/L'_i \cap L''_j) \geq \bar{d}_{i,j}(B)$  and  $\dim(L'''_j/L''_i \cap L'''_j) \geq \bar{d}_{i,j}(C)$  for  $i > j$  define open subsets of  $\Pi$ .

Therefore  $Y_{A,B,C}^L$  is the intersection of finitely many closed sets in  $\Pi$  with finitely many open subsets of  $\Pi$ , so  $Y_{A,B,C}^L$  is locally closed. In particular,  $Y_{A,B,C}^L$  is a quasiprojective variety.  $\square$

**Lemma 4.7.3.** For any  $(L', L'', L''') \in Y_{A,B,C}^L$ ,

$$Y_{A,B,C}^L = \{ \alpha \cdot (L', \beta L'', \beta \gamma L''') : \alpha \in G_L, \beta \in G_{L'}, \gamma \in G_{L''} \}.$$

In particular,

$$Y_{A,B,C}^L = G_L \cdot (\{L'\} \times Y_{B,C}^{L'})$$

for each  $L' \in X_A^L$ .

*Proof.* Let  $(L', L'', L''') \in Y_{A,B,C}^L$ . Given  $\alpha \in G_L$ ,  $\beta \in G_{L'}$  and  $\gamma \in G_{L''}$ ,  $(\alpha L', \alpha \beta L'', \alpha \beta \gamma L''')$  is in  $Y_{A,B,C}^L$  since

$$\begin{aligned} (L, \alpha L') &= \alpha(L, L') \in \mathcal{O}_A \\ (\alpha L', \alpha \beta L'') &= \alpha \beta(L', L'') \in \mathcal{O}_B \\ (\alpha \beta L'', \alpha \beta \gamma L''') &= \alpha \beta \gamma(L'', L''') \in \mathcal{O}_C \end{aligned}$$

For each  $(N', N'', N''') \in Y_{A,B,C}^L$  there exist  $\sigma_1, \sigma_2, \sigma_3 \in G$  with

$$\begin{aligned} (L, N') &= \sigma_1(L, L') \\ (N', N'') &= \sigma_2(L', L'') \\ (N'', N''') &= \sigma_3(L'', L'''). \end{aligned}$$

Let  $\alpha = \sigma_1$ ,  $\beta = \sigma_1^{-1} \sigma_2$  and  $\gamma = \sigma_2^{-1} \sigma_3$ , so  $\sigma_2 = \alpha \beta$  and  $\sigma_3 = \alpha \beta \gamma$ . It follows that

$$(N', N'', N''') = (\alpha L', \alpha \beta L'', \alpha \beta \gamma L'''),$$

which proves the first claim. The second claim follows from the first since  $(L'', L''') \in Y_{B,C}^{L'}$  and therefore

$$Y_{B,C}^{L'} = \{(\beta L'', \beta \gamma L''') : \beta \in G_{L'}, \gamma \in G_{L''}\},$$

as required.  $\square$

**Proposition 4.7.4.**  $Y_{A,B,C}^L$  is irreducible.

*Proof.* Write

$$\Pi = \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L).$$

Lemma 4.3.2 shows that  $\Pi$  is a projective algebraic variety and Lemma 4.3.6 shows that  $G_L/H_{3N,L}$  is a connected algebraic group acting algebraically on  $\Pi$  by the diagonal action.

Let  $L' \in X_A^L$ . As a result of Lemma 4.7.3

$$\begin{aligned} Y_{A,B,C}^L &= G_L \cdot (\{L'\} \times Y_{B,C}^{L'}) \\ &= G_L/H_{3N,L} \cdot (\{L'\} \times Y_{B,C}^{L'}). \end{aligned}$$

Proposition 4.5.4 shows that  $Y_{B,C}^{L'}$  is irreducible, so  $\{L'\} \times Y_{B,C}^{L'}$  is irreducible. The image of  $\{L'\} \times Y_{B,C}^{L'}$  under the action of  $G_L/H_{3N,L}$  is irreducible, since  $G_L/H_{3N,L}$  is connected and therefore irreducible. Therefore  $Y_{A,B,C}^L$  is irreducible.  $\square$

Let  $p_3$  be the projection of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$  onto the third factor. By the completeness property of projective varieties,  $p_3$  is a closed morphism. The image of  $Y_{A,B,C}^L$  under  $p_3$  is  $X_{A,B,C}^L$ , so  $p_3(\overline{Y_{A,B,C}^L}) = \overline{X_{A,B,C}^L}$ .

**Lemma 4.7.5.**  $X_{A,B,C}^L$  is irreducible and constructible.

*Proof.* Lemma 4.7.2 and Proposition 4.7.4 show that  $Y_{A,B,C}^L$  is locally closed and irreducible. It follows  $X_{A,B}^L$  is irreducible and constructible, since  $X_{A,B,C}^L$  is the image of  $Y_{A,B,C}^L$  under the morphism  $p_3$ .  $\square$

**Lemma 4.7.6.** There is a unique open and dense  $G_L$ -orbit in  $X_{A,B,C}^L$ .

*Proof.* There are only finitely many  $G_L$ -orbits in  $X_{A,B,C}^L$ . In particular,

$$X_{A,B,C}^L = \bigcup_{D \in \Lambda_1^{A,B}} X_{D,C}^L = \bigcup_{D \in \Lambda_1^{A,B}} \bigcup_{D' \in \Lambda_1^{D,C}} X_{D'}^L$$

and

$$\overline{X_{A,B,C}^L} = \bigcup_{D \in \Lambda_1^{A,B}} \bigcup_{D' \in \Lambda_1^{D,C}} \overline{X_{D'}^L}.$$

There is  $D \in \Lambda_1$  such that  $\overline{X_D^L} = \overline{X_{A,B,C}^L}$ , since  $X_{A,B,C}^L$  is irreducible, by Lemma 4.7.5. By Lemma 4.4.2,  $X_D^L$  is open in  $\overline{X_D^L} = \overline{X_{A,B,C}^L}$ , so  $X_D^L$  is open in  $X_{A,B,C}^L$ .

If  $X_D^L$  and  $X_{D'}^L$  are open in  $X_{A,B,C}^L$ , then  $X_D^L$  and  $X_{D'}^L$  have nonempty intersection since  $X_{A,B,C}^L$  is irreducible, then  $X_D^L = X_{D'}^L$ .  $\square$

**Lemma 4.7.7.**  $p_3^{-1}(X_{A*B,C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ .

*Proof.* Projection onto the second component is a closed morphism of varieties  $p_2: \overline{Y_{A,B,C}^L} \rightarrow \overline{X_{A,B}^L}$  with  $p_2(Y_{A,B,C}^L) = X_{A,B}^L$ . It follows that  $p_3^{-1}(X_{A*B,C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$  since  $p_3^{-1}(X_{A*B,C}^L) = p_2^{-1}(X_{A*B}^L)$  and  $X_{A*B}^L$  is open in  $\overline{X_{A,B}^L}$ .  $\square$

**Lemma 4.7.8.**  $p_3^{-1}(X_{A,B*C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ .

*Proof.*  $p_3^{-1}(X_{A,B*C}^L)$  consists of those  $(L', L'', L''') \in \overline{Y_{A,B,C}^L}$  such that  $\dim(L'_i/L'_i \cap L''_j) \geq d_{i,j}(B * C)$  for  $i < j$  and  $\dim(L'''_j/L'_i \cap L''_j) \geq \bar{d}_{i,j}(B * C)$  for  $i > j$ . Each of these conditions defines an open subset of  $\overline{Y_{A,B,C}^L}$  as a result of Lemma 4.3.7 and only finitely many conditions are required to determine  $p_3^{-1}(X_{A,B*C}^L)$ , as before. Therefore  $p_3^{-1}(X_{A,B*C}^L)$  is the intersection of finitely many open sets in  $\overline{Y_{A,B,C}^L}$ , so is open as claimed.  $\square$

**Proposition 4.7.9.**  $X_{A*(B*C)}^L = X_{(A*B)*C}^L$ .

*Proof.* The unique open  $G_L$ -orbit in  $X_{A*B,C}^L$  is  $X_{(A*B)*C}^L$ , so  $p_3^{-1}(X_{(A*B)*C}^L)$  is open in  $p_3^{-1}(X_{A*B,C}^L)$ . Lemma 4.7.7 shows that  $p_3^{-1}(X_{A*B,C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ , so  $p_3^{-1}(X_{(A*B)*C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ .

Similarly,  $X_{A*(B*C)}^L$  is open in  $X_{A,B*C}^L$ , so  $p_3^{-1}(X_{A*(B*C)}^L)$  is open in  $p_3^{-1}(X_{A,B*C}^L)$ . Lemma 4.7.8 shows that  $p_3^{-1}(X_{A,B*C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ , so it follows  $p_3^{-1}(X_{A*(B*C)}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ .

Therefore  $f^{-1}(X_{A*(B*C)}^L)$  has nonempty intersection with  $f^{-1}(X_{(A*B)*C}^L)$ , since  $Y_{A,B,C}^L$  is irreducible by Proposition 4.7.4. It follows that the  $G_L$ -orbits  $X_{A*(B*C)}^L$  and  $X_{(A*B)*C}^L$  have nonempty intersection and therefore  $X_{A*(B*C)}^L$  equals  $X_{(A*B)*C}^L$ .  $\square$

## 4.8 The generic affine algebra

The generic affine algebra of rank  $r$  and period  $n$ , denoted by  $\hat{G}(n, r)$ , is a free  $\mathbb{Z}$ -module with basis  $\{e_A : A \in \Lambda_1\}$  and  $\mathbb{Z}$ -bilinear multiplication given by

$$e_A * e_B = e_{A*B}$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$ , and

$$e_A * e_B = 0$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) \neq \text{ro}(B)$ .

**Theorem 4.8.1.** *The generic algebra  $\hat{G}(n, r)$  is an associative  $\mathbb{Z}$ -algebra with 1, with*

$$1 = \sum_{\lambda \in \Lambda_0} 1_\lambda$$

where

$$1_\lambda = e_{D_\lambda},$$

for each  $\lambda \in \Lambda_0$ .

*Proof.* Let  $A, B, C \in \Lambda_1$ . If  $\text{co}(A) \neq \text{ro}(B)$  or  $\text{co}(B) \neq \text{ro}(C)$ , then

$$(e_A * e_B) * e_C = 0 = e_A * (e_B * e_C),$$

so we may now suppose  $\text{co}(A) = \text{ro}(B)$  and  $\text{co}(B) = \text{ro}(C)$ .

As a result of Proposition 4.7.9,

$$\begin{aligned} (e_A * e_B) * e_C &= e_{(A*B)*C} \\ &= e_{A*(B*C)} \\ &= e_A * (e_B * e_C), \end{aligned}$$

so it follows  $\hat{G}(n, r)$  is an associative  $\mathbb{Z}$ -algebra.

The expression for the multiplicative identity follows from Lemma 2.1.17, since

$$e_A * \left( \sum_{\lambda \in \Lambda_0} 1_\lambda \right) = e_A * 1_{\text{co}(A)} = e_A$$

and

$$\left( \sum_{\lambda \in \Lambda_0} 1_\lambda \right) * e_A = 1_{\text{ro}(A)} * e_A = e_A,$$

for each  $A \in \Lambda_1$ . □

### 4.8.1 A categorical perspective

**Proposition 4.8.2.** *The following constitutes a small category: the set of objects is  $\Lambda_0$  and the set of morphisms is  $\Lambda_1$ . Given compositions  $\lambda, \mu \in \Lambda_0$ , the morphisms with source  $\mu$  and target  $\lambda$  are those matrices  $A \in \Lambda_1$  with  $\text{co}(A) = \mu$  and  $\text{ro}(A) = \lambda$ . Given  $\lambda, \mu, \nu \in \Lambda_0$  and  $A, B \in \Lambda_1$  with  $\text{co}(B) = \nu$ ,  $\text{ro}(B) = \mu = \text{co}(A)$  and  $\text{ro}(A) = \lambda$ , their composition is  $A * B$ , with source  $\text{co}(A * B) = \text{co}(B) = \nu$  and target  $\text{ro}(A * B) = \text{ro}(A) = \lambda$ .*

*Proof.* Proposition 4.7.9 shows that the generic product  $*$  is associative. For each object  $\lambda \in \Lambda_0$ , the identity morphism  $\lambda \rightarrow \lambda$  is the diagonal matrix  $D_\lambda$ .  $\square$

Then the generic affine algebra  $\hat{G}(n, r)$  may be realised as the  $\mathbb{Z}$ -algebra of this category. Observe that there are only finitely many objects in this category and distinct objects are non-isomorphic, so the isomorphism classes in this category are in one to one correspondence with  $\Lambda_0$ . The  $\mathbb{Z}$ -algebra of this category is the free  $\mathbb{Z}$ -module on  $\Lambda_1$  with  $\mathbb{Z}$ -bilinear multiplication given by the generic product  $*$ .

## Chapter 5

# Towards a realisation of affine zero Schur algebras

The purpose of this chapter is to study the link between the generic affine algebra  $\hat{G}(n, r)$  and the affine 0-Schur algebra  $\hat{S}_0(n, r)$ .

The main result is the construction of an isomorphism of  $\mathbb{Z}$ -algebras from  $\hat{G}(n, r)$  to  $\hat{S}_0(n, r)$  such that  $E_i \mapsto E_i$ ,  $F_i \mapsto F_i$  and  $1_\lambda \mapsto 1_\lambda$ , in the case that  $n, r \geq 1$  with  $r < n$ .

### 5.1 Preliminary results on the generic affine algebra

Recall that the generic affine algebra  $\hat{G}(n, r)$  is an associative  $\mathbb{Z}$ -algebra with a multiplicative basis  $\{e_A : A \in \Lambda_1\}$  over  $\mathbb{Z}$ , where

$$e_A * e_B = e_{A*B}$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$ , and

$$e_A * e_B = 0$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) \neq \text{ro}(B)$ .

#### 5.1.1 Elementary basis elements

For  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  such that  $\lambda_{i+1} > 0$ , define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and let

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}$$

for each  $i \in \{1, \dots, n\}$

For  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  such that  $\lambda_i > 0$ , define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and let

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}$$

for each  $i \in \{1, \dots, n\}$ .



**Lemma 5.1.1.** *Let  $i \in \{1, \dots, n\}$  and  $A \in \Lambda_1$  and write  $\mu = \text{ro}(A)$ .*

*If  $\mu_{i+1} = 0$  then  $E_i * e_A = 0$ . If  $\mu_{i+1} > 0$ , then*

$$E_i * e_A = e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}},$$

where

$$p = \max\{j \in \mathbb{Z} : a_{i+1,j} > 0\}.$$

*If  $\mu_i = 0$  then  $F_i * e_A = 0$ . If  $\mu_i > 0$  then*

$$F_i * e_A = e_{A+\mathcal{E}_{i+1,q}-\mathcal{E}_{i,q}},$$

where

$$q = \min\{j \in \mathbb{Z} : a_{i,j} > 0\}.$$

*Proof.* Suppose  $\mu_{i+1} > 0$ . Recall that the corresponding product in the affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  is

$$E_i \cdot e_A = \sum_{j \in \mathbb{Z} : a_{i+1,j} > 0} q^{\sum_{t > j} a_{i,t}} [[a_{i,j} + 1]] e_{A+\mathcal{E}_{i,j}-\mathcal{E}_{i+1,j}},$$

by Lemma 3.1.2.

Suppose  $B \in \Lambda_1$  with  $B = A + \mathcal{E}_{i,j} - \mathcal{E}_{i+1,j}$  for some  $j \in \mathbb{Z}$ . For  $s \in \{1, \dots, n\}$  and  $t \in \mathbb{Z}$ ,

$$d_{s,t}(B) = \begin{cases} d_{s,t}(A) + 1 & : s = i \text{ and } t < j, \\ d_{s,t}(A) & : \text{otherwise,} \end{cases}$$

and

$$\bar{d}_{s,t}(B) = \begin{cases} \bar{d}_{s,t}(A) - 1 & : s = i \text{ and } t \geq j, \\ \bar{d}_{s,t}(A) & : \text{otherwise.} \end{cases}$$

It follows that if  $j' < j$ , then

$$A + \mathcal{E}_{i,j'} - \mathcal{E}_{i+1,j'} < A + \mathcal{E}_{i,j} - \mathcal{E}_{i+1,j}.$$

Therefore, the product in  $\hat{G}(n, r)$  is given by

$$E_i * e_A = e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}},$$

where

$$p = \max\{j \in \mathbb{Z} : a_{i+1,j} > 0\}.$$

The argument for the action of  $F_i$  is similar, but there is a pleasing symmetry in the two proofs.

Now suppose  $\mu_i > 0$ . Using Lemma 3.1.2,

$$F_i \cdot e_A = \sum_{j \in \mathbb{Z} : a_{i,j} > 0} q^{\sum_{t < j} a_{i+1,t}} [[a_{i+1,j} + 1]] e_{A+\mathcal{E}_{i+1,j}-\mathcal{E}_{i,j}},$$

in  $\hat{S}_q(n, r)$ .

Suppose  $B \in \Lambda_1$  with  $B = A + \mathcal{E}_{i+1,j} - \mathcal{E}_{i,j}$ , for some  $j \in \mathbb{Z}$ . Then for  $i \in \{1, \dots, n\}$  and  $j \in \mathbb{Z}$ ,

$$d_{s,t}(B) = \begin{cases} d_{s,t}(A) - 1 & : s = i \text{ and } t < j, \\ d_{s,t}(A) & : \text{otherwise,} \end{cases}$$

and

$$\bar{d}_{s,t}(B) = \begin{cases} \bar{d}_{s,t}(A) + 1 & : s = i \text{ and } t \geq j, \\ \bar{d}_{s,t}(A) & : \text{otherwise.} \end{cases}$$

Then if  $j' < j$  it follows

$$A + \mathcal{E}_{i+1,j'} - \mathcal{E}_{i,j'} > A + \mathcal{E}_{i+1,j} - \mathcal{E}_{i,j},$$

so the terms with nonzero coefficients in the product  $F_i \cdot e_A$  are totally ordered and the maximum is

$$F_i * e_A = e_{A + \mathcal{E}_{i+1,q} - \mathcal{E}_{i,q}},$$

where  $q = \min\{j \in \mathbb{Z} : a_{i,j} > 0\}$ . □

### 5.1.2 Transpose involution

Let  $S$  be the  $\mathbb{Z}$ -module automorphism of  $\hat{G}(n, r)$  given by

$$S(e_A) = e_{A^\top}$$

for each  $A \in \Lambda_1$ .

**Lemma 5.1.2.** *The map  $S$  is a  $\mathbb{Z}$ -algebra antihomomorphism. In particular,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A,$$

for each  $A, B \in \Lambda_1$ .

*Proof.* Lemma 3.1.8 show that the transpose preserves the partial order on  $\Lambda_1$  and so

$$(B * A)^\top = A^\top * B^\top,$$

using Lemma 3.1.1. □

For any  $A \in \Lambda_1$ ,

$$S(S(e_A)) = e_{(A^\top)^\top} = e_A,$$

so  $S \circ S$  is the identity map on  $\hat{S}_q(n, r)$ .

For each  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_{i+1} > 0$ ,

$$S(E_{i,\lambda}) = F_{i,\lambda+\alpha_i},$$

for each  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_i > 0$ ,

$$S(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}, \text{ and}$$

and

$$S(1_\lambda) = 1_\lambda,$$

for each  $\lambda \in \Lambda_0$ .

**Lemma 5.1.3.** *Let  $i \in \{1, \dots, n\}$  and  $A \in \Lambda_1$  and write  $\lambda = \text{co}(A)$ .*

*If  $\lambda_j = 0$  then  $e_A * E_j = 0$ . If  $\lambda_j > 0$  then*

$$e_A * E_j = e_{A + \varepsilon_{p,j+1} - \varepsilon_{p,j}},$$

*where*

$$p = \min\{i \in \mathbb{Z} : a_{i,j} > 0\}.$$

*If  $\lambda_{j+1} = 0$  then  $e_A * F_j = 0$ . If  $\lambda_{j+1} > 0$  then*

$$e_A * F_j = e_{A + \varepsilon_{p',j} - \varepsilon_{p',j+1}},$$

*where*

$$p' = \max\{i \in \mathbb{Z} : a_{i,j+1} > 0\}.$$

*Proof.* This follows immediately on applying the transpose involution to the formulas for the action of  $E_i$  and  $F_i$  on the left given in Lemma 5.1.1.

Equally, this result can be proven directly using the formulas for the action of  $E_i$  and  $F_i$  on the right in Lemma 3.1.3, as in the proof of Lemma 5.1.1.  $\square$

### 5.1.3 Shifting and periodicity

For each  $\lambda \in \Lambda_0$ , define

$$R_\lambda = e_{[1]D_\lambda} = e_{\lambda_1 \varepsilon_{0,1} + \dots + \lambda_n \varepsilon_{n-1,n}}$$

and set

$$R = \sum_{\lambda \in \Lambda_0} R_\lambda.$$

**Lemma 5.1.4.** *For each  $A \in \Lambda_1$ ,*

$$R * e_A = e_{[1]A}$$

*and*

$$e_A * R = e_{A[-1]}.$$

*Proof.* Lemma 3.1.12 shows that the same formulas hold in  $\hat{S}_q(n, r)$ , then the result follows for the generic multiplication  $*$ , since each product  $R * e_A$  and  $e_A * R$  is supported on one orbit, so the generic multiplication and the product on  $\hat{S}_q(n, r)$  are the same in this instance.  $\square$

Observe that

$$\begin{aligned} S(R_\lambda) &= e_{\lambda_1 \varepsilon_{1,0} + \dots + \lambda_n \varepsilon_{n,n-1}} \\ &= e_{[-1]D_{[1]\lambda}} \end{aligned}$$

so

$$S(R) = \sum_{\lambda \in \Lambda_0} e_{[-1]D_\lambda}.$$

**Lemma 5.1.5.** *The element  $R$  of  $\hat{G}(n, r)$  is invertible, with*

$$R * S(R) = 1 = S(R) * R.$$

*Proof.* Lemma 5.1.4 shows that

$$\begin{aligned} R * S(R)1_\lambda &= Re_{[-1]D_{[1]}\lambda} \\ &= e_{D_{[1]}\lambda} \\ &= 1_{[1]\lambda} \end{aligned}$$

for each  $\lambda \in \Lambda_0$ , so

$$R * S(R) = 1.$$

Similarly,

$$\begin{aligned} S(R) * R &= \sum_{\lambda \in \Lambda_0} e_{D_\lambda[1]} * R \\ &= \sum_{\lambda \in \Lambda_0} e_{D_\lambda} \\ &= 1. \end{aligned}$$

□

Let  $\tau$  be the  $\mathbb{Z}$ -algebra automorphism of  $\hat{G}(n, r)$  given by conjugation by  $R$ , so

$$\begin{aligned} \tau(e_A) &= R * e_A * S(R) \\ &= R * e_A * R^{-1}, \end{aligned}$$

for each  $A \in \Lambda_1$ .

Observe that  $\tau$  has order  $n$ , by the  $(n, n)$ -periodicity condition on  $\Lambda_1$ .

As in Lemma 3.1.14, it follows from Lemma 5.1.4 that

$$\tau(E_{i,\lambda}) = E_{i-1,[1]\lambda}$$

for  $i \in \{1, \dots, r\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_{i+1} > 0$ ,

$$\tau(F_{i,\lambda}) = F_{i-1,[1]\lambda}$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_i > 0$ , and

$$\tau(1_\lambda) = 1_{[1]\lambda}$$

for  $\lambda \in \Lambda_0$ .

In particular,

$$\begin{aligned} \tau(E_i) &= E_{i-1} \\ \tau(F_i) &= F_{i-1} \end{aligned}$$

for  $i \in \{1, \dots, r\}$ .

As earlier, I can not be sure but I think this map  $\tau$  is related to the Auslander-Reiten translation on the isomorphism classes of nilpotent representations of the cyclic quiver on  $n$  vertices. The result that  $\tau(E_i) = E_{i-1}$  is consistent with the fact the A.R translation sends the simple representation at vertex  $i$  to the simple representation at vertex  $i - 1$ .

## 5.2 Multiplicative bases in affine zero Schur algebras: motivating example

Recall that the affine 0-Schur algebra  $\hat{S}_0(n, r)$  is defined to be the associative  $\mathbb{Z}$ -algebra

$$\hat{S}_0(n, r) = \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}[q]} \hat{S}_q(n, r).$$

In particular,  $\hat{S}_0(n, r)$  has a  $\mathbb{Z}$ -basis

$$\{e_A : A \in \Lambda_1\}$$

with  $\mathbb{Z}$ -bilinear product given by

$$e_A e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C}(0) e_C$$

for  $A, B, C \in \Lambda_1$ ; where  $\gamma_{A,B,C} \in \mathbb{Z}[q]$  are the structure polynomials of the affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  with respect to this distinguished basis.

The multiplicative identity in  $\hat{S}_0(n, r)$  is

$$\sum_{\lambda \in \Lambda_0} 1_\lambda.$$

The result of the shifting lemma, Lemma 3.1.12, also holds in  $\hat{S}_0(n, r)$ . In particular,

$$Re_A = e_{[1]A}$$

and

$$e_A R = e_{A[-1]},$$

for each  $A \in \Lambda_1$ .

Now assume  $r = 1$ , so

$$\Lambda_1(n, 1) = \{\mathcal{E}_{i,j} : (i, j) \in \mathbb{Z} \times \{1, \dots, n\}\}$$

and

$$\Lambda_0(n, 1) = \{\varepsilon_n, \dots, \varepsilon_1\}.$$

**Lemma 5.2.1.** *The distinguished basis  $\{e_A : A \in \Lambda_1(n, 1)\}$  is a multiplicative basis of  $\hat{S}_0(n, 1)$ . More precisely,*

$$e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{j,k}} = e_{\mathcal{E}_{i,k}}$$

for  $i, j, k \in \mathbb{Z}$ , and

$$e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{k,l}} = 0$$

for  $i, j, k, l \in \mathbb{Z}$  with  $j \neq k$  modulo  $n$ .

*Proof.* Let  $i, j \in \mathbb{Z}$ . Lemma 3.1.12 shows that

$$e_{\mathcal{E}_{i,j}} = R^{j-i} 1_{\varepsilon_j},$$

where the subscript of  $\varepsilon_j$  is taken modulo  $n$ .

If  $i, j, k, l \in \mathbb{Z}$  with  $j \neq k$  modulo  $n$ , then

$$\text{co}(\mathcal{E}_{i,j}) = \varepsilon_j \neq \varepsilon_k = \text{ro}(\mathcal{E}_{k,l}),$$

so

$$e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{k,l}} = 0.$$

Finally, let  $i, j, k \in \mathbb{Z}$ . Then

$$\begin{aligned} e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{j,k}} &= R^{j-i} 1_{\varepsilon_j} R^{k-j} 1_{\varepsilon_k} \\ &= R^{j-i} R^{k-j} 1_{\varepsilon_k} \\ &= R^{k-i} 1_{\varepsilon_k} \\ &= e_{\mathcal{E}_{i,k}}. \end{aligned}$$

This proves that the basis  $\{e_A : A \in \Lambda_1(n, 1)\}$  of  $\hat{S}_0(n, 1)$  is a multiplicative basis.  $\square$

This result also shows that the product in  $\hat{S}_0(n, 1)$  is the same as the generic product, since

$$e_A e_B = e_{A*B}$$

if  $\text{co}(A) = \text{ro}(B)$ , and

$$e_A e_B = 0$$

if  $\text{co}(A) \neq \text{ro}(B)$ , for  $A, B \in \Lambda_1(n, 1)$ .

**Corollary 5.2.2.** *For each integer  $n \geq 1$ ,*

$$\hat{S}_0(n, 1) = \hat{G}(n, 1).$$

*Proof.* This is a consequence of Lemma 5.2.1 and the comment which follows the proof.  $\square$

### 5.3 Aperiodicity in the generic affine algebra

**Definition 5.3.1.** An element  $A \in \Lambda_1$  is aperiodic if for each  $l \in \mathbb{Z} \setminus \{0\}$  there exists  $i \in \mathbb{Z}$  such that  $a_{i,i+l} = 0$ .

An element of  $\hat{G}(n, r)$  is said to be aperiodic if it is a  $\mathbb{Z}$ -linear combination of basis elements  $e_A$  corresponding to the aperiodic elements in  $\Lambda_1$ .

For example, the diagonal matrix  $D_\lambda$  is aperiodic so  $1_\lambda$  is aperiodic, for any  $\lambda \in \Lambda_0$ . The elementary basis elements  $E_{i,\lambda}$  and  $F_{i,\lambda}$  introduced earlier are also aperiodic.

When  $r < n$ , any element  $A \in \Lambda_1$  is aperiodic since  $\text{co}(A)$  is insincere and therefore  $A$  has a zero column.

**Lemma 5.3.2.** *Suppose  $A \in \Lambda_1$  is aperiodic and write  $\mu = \text{ro}(A)$ . If  $\mu_{i+1} > 0$ , then  $E_i * e_A$  is aperiodic. If  $\mu_i > 0$ , then  $F_i * e_A$  is aperiodic.*

*Proof.* Let  $A \in \Lambda_1$  be aperiodic and let  $\mu = \text{ro}(A)$ .

Suppose  $\mu_{i+1} > 0$ . There is  $p \in \mathbb{Z}$  such that  $a_{i+1,p} > 0$  and  $a_{i+1,p'} = 0$  whenever  $p' > p$ . Lemma 3.1.2 shows that  $E_i * e_A = e_B$ , where  $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ . Let  $l \in \mathbb{Z} \setminus \{0\}$ . If  $l \notin \{p-i, p-i-1\}$ , then  $b_{s,s+l} = a_{s,s+l}$  for each  $s \in \mathbb{Z}$ , so there is  $s \in \mathbb{Z}$  such that  $b_{s,s+l} = a_{s,s+l} = 0$ , since  $A$  is aperiodic. If  $l = p-i$ , then  $b_{i+1,i+1+l} = b_{i+1,p+1} = a_{i+1,p+1} = 0$ , by maximality of  $p$ . If

$l = p - i - 1$ , there is  $s \neq i + 1$  such that  $a_{s,s+l} = 0$ , since  $A$  is aperiodic and  $a_{i+1,i+1+l} = a_{i+1,p} > 0$ , so  $b_{s,s+l} = a_{s,s+l} = 0$ . Therefore,  $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$  is aperiodic.

Suppose  $\mu_i > 0$ . Lemma 3.1.2 shows that  $F_i * e_A = e_C$  where  $C = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$  and  $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$ . Let  $l \in \mathbb{Z} \setminus \{0\}$ . If  $l \notin \{p - i, p - i - 1\}$  then  $c_{s,s+l} = a_{s,s+l}$  for each  $s \in \mathbb{Z}$ , so there is  $s \in \mathbb{Z}$  such that  $c_{s,s+p} = a_{s,s+p} = 0$ , by aperiodicity of  $A$ . If  $l = p - i$ , then  $a_{i,i+l} = a_{i,p} > 0$ , so there is  $s \neq i$  such that  $a_{s,s+l} = 0$ . Then  $c_{s,s+l} = a_{s,s+l} = 0$ . Finally, if  $l = p - i - 1$ , then  $c_{i,i+l} = a_{i,p-1} = 0$  by minimality of  $p$ . Thus  $C$  is aperiodic as required.  $\square$

Suppose  $\lambda \in \Lambda_0$  and

$$\omega = \omega_1 \cdots \omega_m,$$

where

$$\omega_1, \dots, \omega_m \in \{E_1, \dots, E_n\} \cup \{F_1, \dots, F_n\}.$$

Either  $\omega * 1_\lambda = 0$  or  $\omega * 1_\lambda = e_A$  for some  $A \in \Lambda_1$ , where  $A$  is aperiodic, as a result of Lemma 5.3.2.

The next step is to prove a converse of this result. It will be shown that each of the aperiodic basis elements  $e_A$  in  $\hat{G}(n, r)$  can be expressed in the form  $\omega 1_\lambda$ , where  $\omega$  is a word in  $E_1, \dots, E_n$  and  $F_1, \dots, F_n$  and  $\lambda = \text{co}(A)$ . This will be proven by induction on the *weight* of a matrix by showing how any aperiodic basis element can be written as the product of some  $E_i$  or  $F_i$  with an aperiodic basis element of strictly smaller weight.

**Definition 5.3.3.** For each  $A \in \Lambda_1$ , define the *weight* of  $A$  to be the non negative integer

$$\text{wt}(A) = \sum_{i \in \{1, \dots, n\}, j \in \mathbb{Z}} |j - i| a_{i,j}.$$

Observe that

$$\text{wt}(A) = \sum_{[i,j]: i < j} (j - i) a_{i,j} + \sum_{[i,j]: i > j} (i - j) a_{i,j}.$$

Also write  $\text{wt}(e_A) = \text{wt}(A)$ . Then  $1_\lambda$  has weight 0, and  $E_{i,\lambda}$  and  $F_{i,\lambda}$  have weight 1. In fact, the converse also holds: If  $\text{wt}(A) = 0$  then  $e_A = 1_\lambda$  where  $\lambda = \text{co}(A)$ , and if  $\text{wt}(A) = 1$  then  $e_A$  is  $E_{i,\lambda}$  or  $F_{i,\lambda}$  for some  $i$ , where  $\lambda = \text{co}(A)$ .

**Lemma 5.3.4.** Let  $A \in \Lambda_1$  and write  $\mu = \text{ro}(A)$ . Suppose  $\mu_{i+1} > 0$  and set

$$p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}.$$

If  $p > i$  then

$$\text{wt}(E_i * e_A) = 1 + \text{wt}(e_A)$$

and if  $p \leq i$  then

$$\text{wt}(E_i * e_A) = -1 + \text{wt}(e_A).$$

*Proof.* Lemma 5.1.1 shows that

$$E_i * e_A = e_{A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}}$$

so

$$\text{wt}(E_i * e_A) - \text{wt}(e_A) = |p - i| - |p - i - 1|,$$

which equals 1 if  $p > i$  and equals -1 if  $p \leq i$ .  $\square$

**Lemma 5.3.5.** Let  $A \in \Lambda_1$  and  $\mu = \text{ro}(A)$ . Suppose  $i \in \{1, \dots, n\}$  is such that  $\mu_i > 0$  and let

$$q = \min\{q' \in \mathbb{Z} : a_{i,q'} > 0\}.$$

If  $q \leq i$  then

$$\text{wt}(F_i * e_A) = \text{wt}(e_A) + 1$$

and if  $q > i$  then

$$\text{wt}(F_i * e_A) = \text{wt}(e_A) - 1.$$

*Proof.* Again using Lemma 5.1.1,

$$F_i * e_A = e_{A + \mathcal{E}_{i+1,q} - \mathcal{E}_{i,q}},$$

so

$$\text{wt}(F_i * e_A) - \text{wt}(e_A) = |q - i - 1| - |q - i|,$$

which equals  $-1$  if  $q > i$  and equals  $1$  if  $q \leq i$ . □

**Lemma 5.3.6.** If  $A \in \Lambda_1$  is aperiodic, then

$$e_A = \omega_1 \cdots \omega_m 1_\lambda$$

for some

$$\omega_1, \dots, \omega_m \in \{E_1, \dots, E_n\} \cup \{F_1, \dots, F_n\},$$

where  $\lambda = \text{co}(A)$  and  $m = \text{wt}(A)$ .

*Proof.* The proof uses induction on the weight of  $A$ .

If  $\text{wt}(A) = 0$  then  $A = D_\lambda$ , where  $\lambda = \text{co}(A)$ , so

$$e_A = 1_\lambda.$$

Assume  $\text{wt}(A) > 0$ . Then  $A$  has at least one nonzero entry which is not on the diagonal.

Suppose the upper part of  $A$  is nonzero and set

$$h^+ = \max\{j - i : a_{i,j} \neq 0\}.$$

There is  $i \in \{1, \dots, n\}$  such that  $a_{i,i+h^+} > 0$  and  $a_{i+1,i+1+h^+} = 0$ , using the aperiodicity property of  $A$ . Let  $p$  be the smallest integer such that  $p > i$ ,  $a_{i,p} > 0$  and  $a_{i+1,j} = 0$  for  $j > p$ .

Then

$$e_A = E_i * e_B$$

where  $B = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$ . Moreover,  $B$  is aperiodic and

$$\text{wt}(B) = \text{wt}(A) - 1,$$

using Lemma 5.3.4.

Next suppose the lower part of  $A$  is nonzero and set

$$h^- = \max\{i - j : a_{i,j} > 0\}.$$

There is  $i \in \{1, \dots, n\}$  such that  $a_{i,i-h^-} = 0$  and  $a_{i+1,i+1-h^-} > 0$ , by the aperiodicity property of  $A$ . Let  $q$  be the largest integer such that  $q < i + 1$ ,  $a_{i+1,q} > 0$  and  $a_{i,j} = 0$  for  $j < q$ . Then  $q \geq i - h^-$  and

$$e_A = F_i e_B,$$



where

$$B = A + \mathcal{E}_{i,q} - \mathcal{E}_{i+1,q}.$$

Observe  $B$  is aperiodic and

$$\text{wt}(B) = \text{wt}(A) - 1,$$

by Lemma 5.3.5.

Therefore, if  $\text{wt}(A) > 0$  there exists an aperiodic element  $B \in \Lambda_1$  with

$$\text{wt}(B) = \text{wt}(A) - 1$$

and such that

$$e_A = \omega e_B$$

for some  $\omega \in \{E_1, \dots, E_n\} \cup \{F_1, \dots, F_n\}$ .

It follows that any aperiodic basis element  $e_A$  is the product of a word of length  $\text{wt}(A)$  in  $E_1, \dots, E_n$  and  $F_1, \dots, F_n$  with the idempotent  $1_\lambda$ , where  $\lambda = \text{co}(A)$ .  $\square$

**Proposition 5.3.7.** *The subalgebra of  $\hat{G}(n, r)$  generated by  $E_i$  and  $F_i$  for  $i \in \{1, \dots, n\}$  and  $1_\lambda$  for  $\lambda \in \Lambda_0$  has  $\mathbb{Z}$ -basis*

$$\{e_A : A \in \Lambda_1 \text{ is aperiodic}\}.$$

*Proof.* By definition, this subalgebra is spanned by the nonzero products in  $E_i$  and  $F_i$  for  $i \in \{1, \dots, n\}$  and  $1_\lambda$  for  $\lambda \in \Lambda_0$ , which are exactly the aperiodic basis elements, by Lemma 5.3.2 and Lemma 5.3.6.  $\square$

**Lemma 5.3.8.** *In the case  $r < n$ ,  $\hat{G}(n, r)$  is generated by  $E_i$  and  $F_i$  for  $i \in \{1, \dots, n\}$  and  $1_\lambda$  for  $\lambda \in \Lambda_0$ .*

*Proof.* When  $r < n$ , any  $A \in \Lambda_1$  is aperiodic since  $\text{co}(A)$  has a zero entry, so  $A$  has a column of zero entries. Therefore each of the basis elements  $e_A$  in  $\hat{G}(n, r)$  may be written as a product of the  $E_i$ ,  $F_i$  and  $1_\lambda$ , using Proposition 5.3.7.  $\square$

## 5.4 Quiver presentation of the generic affine algebra.

Let  $n$  and  $r$  be integers with  $n \geq 3$  and  $r \geq 1$ . Let  $\Gamma = \Gamma(n, r)$  be the quiver associated to the affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$ , as defined in Section 3.2.2.

Recall that  $\Gamma$  is the quiver with set of vertices  $\Gamma_0 = \Lambda_0$  and set of arrows  $\Gamma_1 = \Gamma_1^+ \cup \Gamma_1^-$ , where  $\Gamma_1^+$  consists of the arrows

$$e_{i,\lambda}: \lambda \rightarrow \lambda + \alpha_i \text{ for } (i, \lambda) \in \{1, \dots, n\} \times \Lambda_0 \text{ with } \lambda_{i+1} > 0,$$

and  $\Gamma_1^-$  consists of the arrows

$$f_{i,\lambda}: \lambda \rightarrow \lambda - \alpha_i \text{ for } (i, \lambda) \in \{1, \dots, n\} \times \Lambda_0 \text{ with } \lambda_i > 0.$$

Recall that the path  $\mathbb{Z}$ -algebra of  $\Gamma$  is an associative  $\mathbb{Z}$ -algebra with a  $\mathbb{Z}$ -basis consisting of the paths in  $\Gamma$  and with multiplication defined by concatenation of paths. If  $p$  and  $q$  are paths in  $\Gamma$  then the product  $pq$  is the path  $q$  followed by  $p$  if the target of  $q$  equals the source of  $p$ , otherwise  $pq$  equals zero.

For each  $i \in \{1, \dots, n\}$ , define

$$e_i = \sum_{\lambda \in \Lambda_0: \lambda_{i+1} > 0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0: \lambda_i > 0} f_{i,\lambda}.$$

Let  $\mathcal{J}$  be the ideal in  $\mathbb{Z}\Gamma$  generated by the following expressions, which are obtained from the relations in the  $q$ -Schur algebra by setting  $q$  equal to 0:

$$\begin{aligned} e_i e_j - e_j e_i, \\ f_i f_j - f_j f_i \end{aligned}$$

for  $i, j \in \{1, \dots, n\}$  with  $j > i + 1$ ;

$$\begin{aligned} e_i e_{i+1}^2 - e_{i+1} e_i e_{i+1}, \\ e_i^2 e_{i+1} - e_i e_{i+1} e_i, \\ f_{i+1}^2 f_i - f_{i+1} f_i f_{i+1}, \\ f_{i+1} f_i^2 - f_i f_{i+1} f_i \end{aligned}$$

for  $i \in \{1, \dots, n\}$ ;

$$e_i f_j - f_j e_i$$

for  $i, j \in \{1, \dots, n\}$  with  $i < j$ ;

$$e_i f_i - f_i e_i - \sum_{\lambda \in \Lambda_0} c_{i,\lambda} k_\lambda$$

for  $i \in \{1, \dots, n\}$ , where

$$c_{i,\lambda} = \begin{cases} 1 & : \text{if } \lambda_{i+1} = 0, \lambda_i > 0 \\ 0 & : \text{if } \lambda_i > 0, \lambda_{i+1} > 0 \\ -1 & : \text{if } \lambda_i = 0, \lambda_{i+1} > 0. \end{cases}$$

Multiplying each expression above with the idempotents  $k_\lambda$  for  $\lambda \in \Lambda_0$  gives a relation involving paths with common source and target vertices, thus  $\mathcal{J}$  is an ideal of  $\mathbb{Z}$ -linear relations in  $\Gamma$ .

The ideal  $\mathcal{J}$  in  $\mathbb{Z}\Gamma$  is generated by the following set of relations:

$$\begin{aligned} e_{i,\lambda+\alpha_j} e_{j,\lambda} - e_{j,\lambda+\alpha_i} e_{i,\lambda}, \\ f_{i,\lambda-\alpha_j} f_{j,\lambda} - f_{j,\lambda-\alpha_i} f_{i,\lambda}, \end{aligned}$$

for  $i, j \in \{1, \dots, n\}$  with  $j > i + 1$ ;

$$\begin{aligned} e_{i,\lambda+2\alpha_{i+1}} e_{i+1,\lambda+\alpha_{i+1}} e_{i+1,\lambda} - e_{i+1,\lambda+\alpha_i+\alpha_{i+1}} e_{i,\lambda+\alpha_{i+1}} e_{i+1,\lambda}, \\ e_{i,\lambda+\alpha_i+\alpha_{i+1}} e_{i,\lambda+\alpha_{i+1}} e_{i+1,\lambda} - e_{i,\lambda+\alpha_i+\alpha_{i+1}} e_{i+1,\lambda+\alpha_i} e_{i,\lambda}, \\ f_{i+1,\lambda-\alpha_i-\alpha_{i+1}} f_{i+1,\lambda-\alpha_i} f_{i,\lambda} - f_{i+1,\lambda-\alpha_i-\alpha_{i+1}} f_{i,\lambda-\alpha_{i+1}} f_{i+1,\lambda}, \\ f_{i+1,\lambda-2\alpha_i} f_{i,\lambda-\alpha_i} f_{i,\lambda} - f_{i,\lambda-\alpha_i-\alpha_{i+1}} f_{i+1,\lambda-\alpha_i} f_{i,\lambda}, \end{aligned}$$

for  $i \in \{1, \dots, n\}$ ;

$$e_{i,\lambda-\alpha_j} f_{j,\lambda} - f_{j,\lambda+\alpha_i} e_{i,\lambda}$$

for  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ ;

$$e_{i,\lambda-\alpha_i} f_{i,\lambda} - f_{i,\lambda+\alpha_i} e_{i,\lambda}$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_i > 0$  and  $\lambda_{i+1} > 0$ ;

$$e_{i,\lambda-\alpha_i} f_{i,\lambda} - k_\lambda$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  such that  $\lambda_i > 0$  and  $\lambda_{i+1} = 0$ ;

$$f_{i, \lambda + \alpha_i} e_{i, \lambda} - k_\lambda$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  with  $\lambda_i = 0$  and  $\lambda_{i+1} > 0$ .

**Lemma 5.4.1.** *The following equations hold in the generic affine algebra  $\hat{G}(n, r)$ :*

$$E_i E_j = E_j E_i$$

$$F_i F_j = F_j F_i$$

for  $i, j \in \{1, \dots, n\}$  with  $|j - i| \neq 1$ ;

$$E_i E_{i+1}^2 = E_{i+1} E_i E_{i+1}$$

$$E_i^2 E_{i+1} = E_i E_{i+1} E_i$$

$$F_{i+1}^2 F_i = F_{i+1} F_i F_{i+1}$$

$$F_{i+1} F_i^2 = F_i F_{i+1} F_i$$

for  $i \in \{1, \dots, n\}$ ;

$$E_i F_j = F_j E_i$$

for  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ ;

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0} c_{i, \lambda} 1_\lambda$$

for  $i \in \{1, \dots, n\}$ .

*Proof.* Suppose  $i, j \in \{1, \dots, n\}$  with  $j > i + 1$ , so  $\{i, i + 1\}$  and  $\{j, j + 1\}$  are disjoint, then

$$\begin{aligned} E_i E_j &= \sum_{\lambda \in \Lambda_0} E_i [D_\lambda + \mathcal{E}_{j, j+1} - \mathcal{E}_{j+1, j+1}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{i, i+1} - \mathcal{E}_{i+1, i+1} + \mathcal{E}_{j, j+1} - \mathcal{E}_{j+1, j+1}] \\ &= E_j E_i \end{aligned}$$

Then applying the transpose involution yields the second equation:

$$F_i F_j - F_j F_i = -S([E_i, E_j]) = 0.$$

Using the fundamental multiplication rules 5.1.1 and 5.1.3, for each  $i \in \{1, \dots, n\}$ ,

$$\begin{aligned} E_i E_{i+1}^2 &= \sum_{\lambda \in \Lambda_0} E_i [D_\lambda + 2\mathcal{E}_{i+1, i+2} - 2\mathcal{E}_{i+2, i+2}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + 2\mathcal{E}_{i+1, i+2} - 2\mathcal{E}_{i+2, i+2} + \mathcal{E}_{i, i+2} - \mathcal{E}_{i+1, i+2}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{i, i+2} + \mathcal{E}_{i+1, i+2} - 2\mathcal{E}_{i+2, i+2}] \end{aligned}$$

and

$$\begin{aligned} E_{i+1}E_iE_{i+1} &= \sum_{\lambda \in \Lambda_0} E_{i+1} [D_\lambda + \mathcal{E}_{i,i+2} - \mathcal{E}_{i+2,i+2}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{i,i+2} + \mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2}], \end{aligned}$$

so  $E_iE_{i+1}^2 = E_{i+1}E_iE_{i+1}$ .

$$\begin{aligned} E_i^2E_{i+1} &= \sum_{\mu \in \Lambda_0} [D_\mu + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i,i}] E_{i+1} \\ &= \sum_{\mu \in \Lambda_0} [D_\mu + \mathcal{E}_{i,i+2} + \mathcal{E}_{i,i+1} - 2\mathcal{E}_{i,i}] \end{aligned}$$

and

$$\begin{aligned} E_iE_{i+1}E_i &= \sum_{\mu \in \Lambda_0} [D_\mu + \mathcal{E}_{i,i+2} - \mathcal{E}_{i,i}] E_i \\ &= \sum_{\mu \in \Lambda_0} [D_\mu + \mathcal{E}_{i,i+2} + \mathcal{E}_{i,i+1} - 2\mathcal{E}_{i,i}], \end{aligned}$$

so  $E_i^2E_{i+1} = E_iE_{i+1}E_i$ .

The relations between  $F_i$  and  $F_{i+1}$  may be deduced using the transpose involution as follows:

$$F_{i+1}^2F_i = S(E_iE_{i+1}^2) = S(E_{i+1}E_iE_{i+1}) = F_{i+1}F_iF_{i+1}$$

and

$$F_{i+1}F_i^2 = S(E_i^2E_{i+1}) = S(E_iE_{i+1}E_i) = F_iF_{i+1}F_i.$$

Suppose  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ . Then

$$\begin{aligned} E_iF_j &= \sum_{\lambda \in \Lambda_0} E_i [D_\lambda + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j} + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}] \end{aligned}$$

and

$$\begin{aligned} F_jE_i &= \sum_{\lambda \in \Lambda_0} F_j [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}] \\ &= \sum_{\lambda \in \Lambda_0} [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j}], \end{aligned}$$

so  $E_iF_j = F_jE_i$ .

Finally, for  $i \in \{1, \dots, n\}$ ,

$$E_iF_i = \sum_{\lambda: \lambda_i > 0, \lambda_{i+1} = 0} 1_\lambda + \sum_{\lambda: \lambda_i > 0, \lambda_{i+1} > 0} [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}]$$

and

$$F_i E_i = \sum_{\lambda: \lambda_i=0, \lambda_{i+1}>0} 1_\lambda + \sum_{\lambda: \lambda_i>0, \lambda_{i+1}>0} [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}],$$

so

$$\begin{aligned} E_i F_i - F_i E_i &= \sum_{\lambda: \lambda_i>0, \lambda_{i+1}=0} 1_\lambda - \sum_{\lambda: \lambda_i=0, \lambda_{i+1}>0} 1_\lambda \\ &= \sum_{\lambda \in \Lambda_0} c_{i,\lambda} 1_\lambda. \end{aligned}$$

□

Lemma 5.4.1 shows that there is a homomorphism of  $\mathbb{Z}$ -algebras

$$\rho: \mathbb{Z}\Gamma/\mathcal{J} \rightarrow \hat{G}(n, r)$$

defined by

$$\begin{aligned} \rho(k_\lambda + \mathcal{J}) &= 1_\lambda \\ \rho(e_{i,\lambda} + \mathcal{J}) &= E_{i,\lambda} \\ \rho(f_{i,\lambda} + \mathcal{J}) &= F_{i,\lambda}, \end{aligned}$$

for all  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$ . Thus  $\hat{G}(n, r)$  may also be regarded as an algebra over  $\mathbb{Z}\Gamma$  where the action of a path  $p$  is given by

$$e_A \cdot p = e_A \rho(p + \mathcal{J})$$

for all  $A \in \Lambda_1$ .

**Proposition 5.4.2.** *The image of  $\rho$  is spanned by the aperiodic basis elements. If  $r < n$  then  $\rho$  is surjective.*

*Proof.* The image of  $\rho$  is the subalgebra of  $\hat{G}(n, r)$  generated by  $E_i$  and  $F_i$  for  $i \in \{1, \dots, n\}$  and  $1_\lambda$  for  $\lambda \in \Lambda_0$ , which has  $\mathbb{Z}$ -basis

$$\{e_A : A \in \Lambda_1, A \text{ is aperiodic.}\},$$

using Proposition 5.3.7. If  $r < n$  then every  $A \in \Lambda_1$  is aperiodic, since  $A$  must contain a zero row or column. Therefore  $\rho$  is surjective when  $r < n$ . □

### 5.4.1 Standard paths

Recall the definition of standard paths in  $\Gamma$ , from Definition 3.2.7. There is a bijection between the set of standard paths in  $\Gamma$  and the standard monomial basis in  $\hat{G}(n, r)$  indexed by  $\Lambda_1$ , using Lemma 3.2.10.

The expression for the standard path of  $A$  is derived by contracting the rows of  $A$  so that each step produces zero entries on the highest or lowest diagonal, yielding the element  $D_\lambda$  where  $\lambda = \text{ro}(A)$  after finitely many steps. Computing the image of a standard path in  $\hat{G}(n, r)$  by computing the segments from left to right constructs  $e_A$  slice by slice. In particular, the segment  $p_s^+$  produces the diagonal at level  $s$  while the segment  $p_s^-$  produces the diagonal at level  $-s$ . In order to describe this process precisely we now give some notation for the row contractions of  $A$ .

Given  $A \in \Lambda_1$  and  $s \geq 1$  define elements  $(s)A$  and  $A(s)$  in  $\Lambda_1$  by

$$((s)A)_{i,j} = \begin{cases} a_{i,j} & \text{if } i - j < s, \\ 0 & \text{if } i - j > s, \\ \sum_{t \leq j} a_{i,t} & \text{if } i - j = s \end{cases}$$

and

$$(A(s))_{i,j} = \begin{cases} a_{i,j} & \text{if } j - i < s, \\ 0 & \text{if } j - i > s, \\ \sum_{t \geq j} a_{i,t} & \text{if } j - i = s \end{cases}$$

for  $i, j \in \mathbb{Z}$ . Observe that  $(0)A(0) = D_\lambda$  where  $\lambda = \text{ro}(A)$ ;  $(0)A$  is upper triangular and coincides with  $A$  above the diagonal;  $A(0)$  is lower triangular and coincides with  $A$  below the diagonal;  $\text{ro}((s)A) = \text{ro}(A)$  and  $\text{ro}(A(s)) = \text{ro}(A)$ . Also define the *height* of  $A$  as

$$\text{ht}(A) = \max\{|j - i| : i, j \in \mathbb{Z}, a_{i,j} > 0\}$$

so that  $(h)A = A$  and  $A(h) = A$  for  $h \geq \text{ht}(A)$ .

**Lemma 5.4.3.** *Let  $A \in \Lambda_1$  and let  $p = k_\lambda p_1^+ \cdots p_h^+ p_1^- \cdots p_h^-$  be the standard path for  $A$ . Then*

$$e_{A(s-1)} \cdot p_s^+ = e_{A(s)}$$

and

$$e_{(s-1)A} \cdot p_s^- = e_{(s)A}$$

for each  $s \in \{1, \dots, h\}$ .

*Proof.* Let  $B = A(s-1) \cdot p_s^+$ . Using the fundamental multiplication rules in  $\hat{G}(n, r)$ , Lemma 5.1.3, it follows that

$$B = A(s-1) + \sum_{i \in \{1, \dots, n\}} \alpha_{i,s} (\mathcal{E}_{i,i+s} - \mathcal{E}_{i,i+s-1}).$$

So  $b_{i,j} = a_{i,j}$  if  $j - i < s - 1$ ,

$$\begin{aligned} b_{i,i+s-1} &= \alpha_{i,s-1} - \alpha_{i,s} \\ &= a_{i,i+s-1} \end{aligned}$$

and

$$b_{i,i+s} = \alpha_{i,s},$$

which proves that  $B = A(s)$ .

Similarly, let  $B = (s-1)A \cdot p_s^-$ . Using Lemma 5.1.3 it follows that

$$B = (s-1)A + \sum_{i \in \{1, \dots, n\}} \beta_{i-1,s} (\mathcal{E}_{i,i-s} - \mathcal{E}_{i,i-s+1}).$$

So  $b_{i,j} = a_{i,j}$  if  $i - j < s - 1$ ,

$$\begin{aligned} b_{i,i-s+1} &= \beta_{i-1,s-1} - \beta_{i-1,s} \\ &= a_{i,i-s+1} \end{aligned}$$

and

$$b_{i,i-s} = \beta_{i-1,s},$$

which proves  $B = (s)A$ . □

**Lemma 5.4.4.** *Let  $A \in \Lambda_1$  and let  $p$  be the standard path for  $A$ . Then*

$$\rho(p + \mathcal{J}) = e_A.$$

*Proof.* Let  $A \in \Lambda_1$ ,  $\lambda = \text{ro}(A)$ ,  $\mu = \text{co}(A)$ ,  $h = \text{ht}(A)$  and let  $p = k_\lambda p_1^+ \cdots p_h^+ p_1^- \cdots p_h^- k_\mu$  be the standard path for  $A$ .

The standard path for  $(0)A$  is  $k_\lambda p_1^+ \cdots p_h^+$ , by Lemma 3.2.13, so

$$\begin{aligned} e_{(0)A} &= e_{(0)A(h)} \\ &= e_{(0)A(0)} \cdot p_1^+ \cdots p_h^+ \end{aligned}$$

by repeatedly applying Lemma 5.4.3. Similarly,

$$\begin{aligned} e_A &= e_{(h)A} \\ &= e_{(0)A} \cdot p_1^- \cdots p_h^-, \end{aligned}$$

since  $p$  is the standard path for  $A$ . Therefore

$$\begin{aligned} e_A &= e_{(0)A(0)} \cdot p_1^+ \cdots p_h^+ p_1^- \cdots p_h^- \\ &= e_{D_\lambda} \cdot p \\ &= \rho(p + \mathcal{J}). \end{aligned}$$

□

**Remark 5.4.5.** The result of Lemma 5.4.4 gives another way to see that the homomorphism  $\rho$  from the quiver algebra to  $\hat{G}(n, r)$  is surjective provided  $r < n$ . When  $r \geq n$ , the image of the quiver algebra in  $\hat{G}(n, r)$  is spanned by the aperiodic basis elements, by Proposition 5.4.2.

Recall the definition of the positive and negative parts  $A^+$  and  $A^-$  of a matrix  $A \in \Lambda_1$ , as in Definition 3.2.11.

**Lemma 5.4.6.** *Let  $A \in \Lambda_1$ . Then*

$$e_A = e_{A^+} e_{A^-}$$

*and in terms of  $G$ -orbits,*

$$[L, L'] = [L, L \cap L'] [L \cap L', L'].$$

*Proof.* Let  $p$  be the standard path for  $A$ . Then  $p = p^+ p^-$  where  $p^+$  is the standard path for  $A^+$  and  $p^-$  is the standard path for  $A^-$ , by Lemma 3.2.13. Then Lemma 5.4.4 proves that

$$\begin{aligned} e_A &= \rho(p + \mathcal{J}) \\ &= \rho(p^+ + \mathcal{J}) \rho(p^- + \mathcal{J}) \\ &= e_{A^+} e_{A^-}. \end{aligned}$$

The second part then follows from Lemma 3.2.12 which states that  $\mathcal{O}_{A^+} = [L, L \cap L']$  and  $\mathcal{O}_{A^-} = [L \cap L', L']$  for any  $(L, L') \in \mathcal{O}_A$ . □

**Definition 5.4.7.** A path is said to be *reduced* if it is not equivalent to a shorter path.

**Lemma 5.4.8.** *A standard path is reduced.*

*Proof.* If  $p$  is a standard positive or negative path then  $p$  is reduced, since the relations only involving the edges  $e_i : i \in \{1, \dots, n\}$  or  $f_i : i \in \{1, \dots, n\}$  are homogeneous polynomials, so any equivalent path is of the same length.

Now suppose  $p = k_\lambda p^+ p^- k_\mu$  is a standard path for a standard positive path  $k_\lambda p^+$  and a standard negative path  $p^- k_\mu$ . The number of arrows in  $p$  is

$$l = \sum_{i \in \{1, \dots, n\}, s \geq 1} \alpha_{i,s} + \beta_{i,s}.$$

Let  $A$  be the matrix corresponding to the standard path  $p$ , so that  $p = p_A$  as in Lemma 3.2.10. The minimum number of  $E_i$  and  $F_i$  in an expression for  $e_A$  in  $\hat{G}(n, r)$  is

$$\text{wt}(A) = \sum_{i \in \{1, \dots, n\}, j \in \mathbb{Z}} |j - i| a_{i,j},$$

using Lemma 5.3.6, so  $l \geq \text{wt}(A)$ .

Recall that

$$\alpha_{i,s} = \sum_{t \geq s} a_{i,i+t}$$

and

$$\beta_{i-1,s} = \sum_{t \geq s} a_{i,i-s},$$

so

$$\sum_{s \geq 1} \alpha_{i,s} = \sum_{s \geq 1} s a_{i,i+s}$$

and

$$\sum_{s \geq 1} \beta_{i-1,s} = \sum_{s \geq 1} s a_{i,i-s}.$$

Therefore

$$\begin{aligned} l &= \sum_{i \in \{1, \dots, n\}, s \geq 1} \alpha_{i,s} + \beta_{i,s} \\ &= \sum_{i \in \{1, \dots, n\}, s \geq 1} s(a_{i,i+s} + a_{i,i-s}) \\ &= \text{wt}(A), \end{aligned}$$

which proves that  $p$  is reduced. □

The next result gives a more general form of the 0-Serre relations in the quiver algebra for  $\hat{G}(n, r)$ , which will be useful in transforming a path into a standard path.

**Lemma 5.4.9.** *Let  $t > s \geq 0$  be integers. Then*

$$e_i^s e_{i+1}^s e_i - e_i^{s+1} e_{i+1}^s = 0 \tag{5.4.1}$$

$$e_i^s e_{i+1}^t - e_{i+1}^{t-s} e_i^s e_{i+1}^s = 0 \tag{5.4.2}$$

in  $\mathbb{Z}\Gamma/\mathcal{J}$ , for each  $i \in \{1, \dots, n\}$ .



*Proof.* First we prove 5.4.1. If  $s = 0$  the result is a tautology and if  $s = 1$  this is the usual 0-Serre relation. Suppose  $s > 1$  and equation 5.4.1 holds for smaller values. By repeatedly using the 0-Serre relations

$$e_i^2 e_{i+1} - e_i e_{i+1} e_i = 0$$

it follows that

$$e_i^s e_{i+1} = e_i e_{i+1} e_i^{s-1} \quad (5.4.3)$$

and so

$$\begin{aligned} e_i^s e_{i+1}^s e_i &= e_i e_{i+1} e_i^{s-1} e_{i+1}^{s-1} e_i \quad (\text{by 5.4.3.}) \\ &= e_i e_{i+1} e_i^s e_{i+1}^{s-1} \quad (\text{by induction.}) \\ &= e_i^{s+1} e_{i+1} e_{i+1}^{s-1} \quad (\text{by 5.4.3.}) \\ &= e_i^{s+1} e_{i+1}^s. \end{aligned}$$

Now we prove 5.4.2. If  $s = 0$  the result is clear. Suppose  $s > 0$  and the result of 5.4.2 holds for smaller values. Using the 0-Serre relations repeatedly gives

$$e_i e_{i+1}^t = e_{i+1}^{t-1} e_i e_{i+1} \quad (5.4.4)$$

and so

$$\begin{aligned} e_i^s e_{i+1}^t &= e_i^{s-1} e_{i+1}^{t-1} e_i e_{i+1} \quad (\text{by 5.4.4.}) \\ &= e_{i+1}^{t-s} e_i^{s-1} e_{i+1}^{s-1} e_i e_{i+1} \quad (\text{by induction.}) \\ &= e_{i+1}^{t-s} e_i^s e_{i+1}^{s-1} e_{i+1} \quad (\text{by 5.4.1.}) \\ &= e_{i+1}^{t-s} e_i^s e_{i+1}^s. \end{aligned}$$

□

**Corollary 5.4.10.** *Let  $t > s \geq 0$  be integers. Then*

$$f_{i+1}^s f_i^{s+1} - f_i f_{i+1}^s f_i^s = 0$$

and

$$f_{i+1}^s f_i^s f_{i+1}^{t-s} - f_{i+1}^t f_i^s = 0$$

in  $\mathbb{Z}\Gamma/\mathcal{J}$ , for each  $i \in \{1, \dots, n\}$ .

*Proof.* Applying the transpose involution to the relations in Lemma 5.4.9 yields these relations, since  $e_i$  is mapped to  $f_i$  and the order of multiplication is reversed. □

**Lemma 5.4.11.** *Assume that  $r < n$  and let  $p$  be a nonzero path. Then  $p$  does not contain a cyclic section*

$$c = e_i^{a_i} e_{i-1}^{a_{i-1}} \cdots e_{i+1}^{a_{i+1}}$$

with the sum of the exponents  $\sum_{j \neq i} a_j > r$ .

*Proof.* Suppose  $p$  contains such a cyclic section  $c$  and write  $p = p' k_\lambda c k_\mu p''$ . As  $p$  is nonzero it follows that

$$\mu = \lambda + a_i(\varepsilon_{i+1} - \varepsilon_i) + a_{i-1}(\varepsilon_i - \varepsilon_{i-1}) + \cdots + a_{i+1}(\varepsilon_{i+2} - \varepsilon_{i+1})$$

so  $a_j \leq \lambda_j$  for  $j \neq i+1$  and  $a_{i+1} \leq \lambda_{i+1} + a_i$ . Then

$$\sum_{j \neq i} a_j \leq \sum_j \lambda_j = r,$$

contradicting the hypothesis that such a cyclic section exists. □

**Lemma 5.4.12.** *Let  $p$  be a standard path in  $\Gamma$ . If  $q$  is a path in  $\Gamma$  with  $q = pe_i$  or  $q = pf_i$  for some  $i \in \{1, \dots, n\}$ , then  $q$  is congruent to a standard path modulo  $\mathcal{J}$ .*

*Proof.* **IMPORTANT: need to prove this beyond doubt.**

First suppose  $p$  is a positive standard path

$$p = k_\mu p_1 \cdots p_{h^+} k_\lambda,$$

where

$$p_s = e_{i_0+s-1}^{\alpha_{i_0,s}} \cdots e_{i_0+s-n+1}^{\alpha_{i_0-n+2,s}}$$

for  $s = 1, \dots, h^+$ , such that  $\alpha_{i,s} \geq \alpha_{i,s+1}$  for each  $i, s$  and  $\alpha_{i_0+1,s} = 0$  for all  $s$ .

The index  $i$  in  $e_i$  and  $\alpha_{i,s}$  is taken modulo  $n$ . Observe that  $\alpha_{i,s}$  is the exponent of  $e_{i+s-1}$  in  $p_s$  and so the exponent of  $e_i$  in  $p_s$  is  $\alpha_{i-s+1,s}$ . Even when  $p$  is a positive path there are many cases to consider.

For the first case, if the exponent of  $e_{j-1}$  in  $p_{h^+}$ , which is  $\alpha_{j-h^+,h^+}$ , is nonzero then  $q = pe_j$  is a standard path.

For the second case, suppose the exponent of  $e_{j-1}$  in  $p_{h^+}$  is zero and the exponent of  $e_j$  in  $p_{h^+}$  is strictly less than the exponent of  $e_{j-1}$  in  $p_{h^+-1}$ . [continue...]

For the third case, suppose the exponent of  $e_{j-1}$  in  $p_{h^+}$  is zero and the exponent of  $e_j$  in  $p_{h^+}$  equals the exponent of  $e_{j-1}$  in  $p_{h^+-1}$ . [continue...][I'm stuck on this case.]

□

**Proposition 5.4.13.** *When  $r < n$ , any path in  $\Gamma$  is congruent to a standard path modulo  $\mathcal{J}$ .*

*Proof.* Let  $p$  be a path in  $\Gamma$  and proceed by induction on the length of  $p$ . If  $p$  has length zero then  $p = k_\mu$  for some  $\mu \in \Lambda_0$ , so  $p$  is a standard path. If  $p$  has length one then  $p = k_\mu e_i$  or  $p = k_\mu f_i$  for some  $\mu \in \Lambda_0$  and  $i \in \{1, \dots, n\}$ .

Suppose  $p$  has length at least two and that any strictly shorter path is congruent to a standard path. Pulling out the first arrow, write  $p = p'e_i$  or  $p = p'f_i$  for some  $i \in \{1, \dots, n\}$ . Using the inductive hypothesis we may assume  $p'$  is a standard path, so it follows from Lemma 5.4.12 that  $p$  is congruent to a standard path.

**Note:** The lemma on extending standard paths still needs to be proven for this proof to be complete. □

**Theorem 5.4.14.** *If  $r < n$  then  $\rho$  is a  $\mathbb{Z}$ -algebra isomorphism. Thus  $\hat{G}(n, r)$  admits a presentation by the quiver  $\Gamma$  and the ideal of relations  $\mathcal{J}$  in  $\mathbb{Z}\Gamma$ .*

*Proof.* Under the assumption  $r < n$ ,  $\rho$  is a surjective homomorphism of  $\mathbb{Z}$ -algebras, by Proposition 5.4.2.

Suppose  $p$  and  $p'$  are paths in  $\Gamma$  and  $A \in \Lambda_1$  with

$$\rho(p + \mathcal{J}) = \rho(p' + \mathcal{J}) = e_A.$$

Proposition 5.4.13 shows that  $p$  and  $p'$  are both congruent modulo  $\mathcal{J}$  to the standard path corresponding to  $A$ , so  $p$  and  $p'$  are congruent modulo  $\mathcal{J}$ . Therefore  $\rho$  is injective, so is an isomorphism of  $\mathbb{Z}$ -algebras as claimed. □

## 5.5 The isomorphism result

This section gives a realisation of the affine 0-Schur algebra by the generic affine algebra in the case that  $r < n$ . Recall that the affine 0-Schur algebra  $\hat{S}_0(n, r)$  is defined to be the  $\mathbb{Z}$ -algebra

$$\hat{S}_0(n, r) = \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}} [q]\hat{S}_q(n, r).$$

The inclusion of  $\mathbb{Z}[q]$  into  $\mathcal{Q}$  sending  $f$  to  $f/1$  gives an isomorphism of  $\mathbb{Z}$  algebras

$$\mathbb{Z}[q]/q\mathbb{Z}[q] \rightarrow \mathcal{Q}/q\mathcal{Q} : a + q\mathbb{Z}[q] \mapsto a + q\mathcal{Q},$$

and both are isomorphic to  $\mathbb{Z}$  itself. Therefore

$$\hat{S}_0(n, r) = \mathcal{Q}/q\mathcal{Q} \otimes_{\mathcal{Q}} \hat{S}_q(n, r)$$

Let  $n, r \geq 1$  with  $r < n$ . Recall

$$\phi: \mathbb{Z}[q]\Gamma/I \rightarrow \hat{S}_q(n, r)$$

is the homomorphism of  $\mathbb{Z}[q]$ -algebras defined by

$$\begin{aligned}\phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda,\end{aligned}$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$ . Let  $\phi_0$  be the  $\mathbb{Z}$ -algebra homomorphism

$$\phi_0 = \mathcal{Q}/(q) \otimes_{\mathcal{Q}} \phi: \mathbb{Z}\Gamma/\mathcal{J} \rightarrow \hat{S}_0(n, r).$$

Let  $\rho$  be the  $\mathbb{Z}$ -algebra isomorphism

$$\rho: \mathbb{Z}\Gamma/\mathcal{J} \rightarrow \hat{G}(n, r)$$

from Theorem 5.4.14.

Let  $\Psi$  be the  $\mathbb{Z}$ -algebra homomorphism

$$\Psi = \phi_0 \circ \rho^{-1}: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r) \tag{5.5.1}$$

with

$$\begin{aligned}\Psi(E_{i,\lambda}) &= E_{i,\lambda} \\ \Psi(F_{i,\lambda}) &= F_{i,\lambda} \\ \Psi(1_\lambda) &= 1_\lambda,\end{aligned}$$

for  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$ .

**Proposition 5.5.1.** *The map  $\Psi$  is surjective.*

*Proof.* Proposition 3.2.17 shows that

$$\phi_{\mathcal{Q}}: \mathcal{Q}\Gamma/\mathcal{Q}I \rightarrow \hat{S}_{\mathcal{Q}}(n, r)$$

is a surjective  $\mathcal{Q}$ -algebra homomorphism, so

$$\phi_0 = \mathcal{Q}/(q) \otimes_{\mathcal{Q}} \phi_{\mathcal{Q}}: \mathbb{Z}\Gamma/\mathcal{J} \rightarrow \hat{S}_0(n, r)$$

is a surjective  $\mathbb{Z}$ -algebra homomorphism, using right exactness of tensor products. It follows that  $\Psi$  is surjective since  $\Psi \circ \rho = \phi_0$  and  $\rho$  is an isomorphism of  $\mathbb{Z}$ -algebras.  $\square$

**Lemma 5.5.2.** *For each  $A \in \Lambda_1$ ,*

$$\Psi(e_A) = e_A + \sum_{B:B < A} c_B e_B$$

*for some  $c_B \in \mathbb{N}$ .*

*Proof.* Fix  $A \in \Lambda_1$  and let  $p$  be the standard path for  $A$  as in Definition 3.2.9, so that

$$\rho(p + \mathcal{J}) = e_A,$$

using Lemma 5.4.4.

Then

$$\begin{aligned} \Psi(e_A) &= \phi_0(p + \mathcal{J}) \\ &= \sum_{B \in \Lambda_1: B \leq A} g_B(0) e_B, \end{aligned}$$

for some  $g_B \in \mathbb{Z}[q]$ , where

$$\begin{aligned} g_A(0) &= \left( \prod_{i \in \{1, \dots, n\}, s \geq 1} [[\alpha_{i,s}]]! [[\beta_{i,s}]]! \right)_{q=0} \\ &= 1, \end{aligned}$$

by Proposition 3.2.14. □

**Theorem 5.5.3.** *When  $r < n$ , the map*

$$\Psi: \hat{G}(n, r) \rightarrow \hat{S}_0(n, r)$$

*is an isomorphism of  $\mathbb{Z}$ -algebras.*

*Proof.* Proposition 5.5.1 shows that  $\Psi$  is a surjective  $\mathbb{Z}$ -algebra homomorphism.

To prove that  $\Psi$  is injective, suppose  $x$  is a nonzero element of  $\hat{G}(n, r)$  and write

$$x = \sum_{A \in \Lambda_1} c_A e_A$$

and let  $\Xi = \{A \in \Lambda_1 : c_A \neq 0\}$ . Fix a maximal element  $D$  in  $\Xi$ , which exists since  $\Xi$  is nonempty and finite. Then

$$\begin{aligned} \Psi(x) &= \sum_{A \in \Xi} c_A \Psi(e_A) \\ &= \sum_{A \in \Xi} \left( c_A e_A + \sum_{B \in \Xi: B < A} c_{A,B} e_B \right) \\ &= c_D e_D + \sum_{A \in \Xi: A \neq D} c'_A e_A \end{aligned}$$

by Lemma 5.5.2, so  $\Psi(x) \neq 0$ , which proves that  $\Psi$  is injective and therefore  $\Psi$  is an isomorphism of  $\mathbb{Z}$ -algebras. □

## 5.6 The period 2 case

In the case  $n = 2$  the quiver  $\Gamma = \Gamma(2, r)$  associated to  $\hat{G}(2, r)$  consists of  $r + 1$  vertices (totally ordered) with two pairs of edges between adjacent vertices,  $(e_1, f_1)$  and  $(e_2, f_2)$ .

The following equations are a  $q = 0$  form of the  $q$ -Serre relations in Lemma 3.3.1:

**Lemma 5.6.1.** *The following equations hold in  $\hat{G}(2, r)$ , for  $i \in \mathbb{Z}/2\mathbb{Z}$ :*

$$\begin{aligned} E_i E_{i+1} E_i^2 &= E_i^2 E_{i+1} E_i \\ F_i F_{i+1} F_i^2 &= F_i^2 F_{i+1} F_i. \end{aligned}$$

*Proof.*

$$\begin{aligned} E_1 E_2 E_1^2 &= \sum_{\mu \in \Lambda_0} [D_\mu + \mathcal{E}_{1,3} - \mathcal{E}_{1,1}] E_1^2 \\ &= \sum_{\mu \in \Lambda_0} [D_\mu + \mathcal{E}_{1,4} - \mathcal{E}_{1,1}] E_1 \\ &= \sum_{\mu \in \Lambda_0} [D_\mu + \mathcal{E}_{1,4} + \mathcal{E}_{1,2} - 2\mathcal{E}_{1,1}] \end{aligned}$$

and

$$\begin{aligned} E_1^2 E_2 E_1 &= \sum_{\mu \in \Lambda_0} [D_\mu + 2\mathcal{E}_{1,2} - 2\mathcal{E}_{1,1}] E_2 E_1 \\ &= \sum_{\mu \in \Lambda_0} [D_\mu + \mathcal{E}_{1,3} + \mathcal{E}_{1,2} - 2\mathcal{E}_{1,1}] E_1 \\ &= \sum_{\mu \in \Lambda_0} [D_\mu + \mathcal{E}_{1,4} + \mathcal{E}_{1,2} - 2\mathcal{E}_{1,1}], \end{aligned}$$

so  $E_1 E_2 E_1^2 = E_1^2 E_2 E_1$ .

Recall that conjugation by  $R$  defines an automorphism  $\tau$  of  $\hat{G}(n, r)$  of degree 2, with  $\tau(E_1) = E_2$  and  $\tau(E_2) = E_1$ , so

$$E_2 E_1 E_2^2 - E_2^2 E_1 E_2 = \tau(E_1 E_2 E_1^2 - E_1^2 E_2 E_1) = 0.$$

Finally, the equations involving  $F_i$  and  $F_{i+1}$  follow by applying the transpose involution:

$$F_i F_{i+1} F_i^2 - F_i^2 F_{i+1} F_i = S(E_i^2 E_{i+1} E_i - E_i E_{i+1} E_i^2) = 0,$$

for  $i \in \{1, 2\}$ . □

## Chapter 6

# Conclusion

### 6.1 The case of large $r$

When  $r \geq n$  extra relations are needed in order to transform any path to a standard path using the relations, thus proving injectivity of the quiver presentation. These are thought to be of the form

$$e_i^2 e_{i-1} \cdots e_{i+1} e_i = e_i e_{i-1} \cdots e_{i+1} e_i^2$$

and

$$f_i^2 f_{i+1} \cdots f_{i-1} f_i = f_i f_{i+1} \cdots f_{i-1} f_i^2$$

for  $i \in \{1, \dots, n\}$  and there are likely to be more general relations with arbitrary exponents for  $e_{i-1}, \dots, e_{i+1}$  and  $f_{i+1}, \dots, f_{i-1}$  respectively.

Further research could focus on the relation between the generic affine algebra  $\hat{G}(n, r)$  and the affine zero Schur algebra  $\hat{S}_0(n, r)$  when  $r \geq n$ . The case where  $n \leq r < 2n$  appears to be tractable by including the shifting element  $R$  in the set of generators for each algebra and in this case I still expect the two algebras to be isomorphic, though the case of general  $r$  seems to be very difficult.

### 6.2 Further results on affine zero Schur algebras

Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

### 6.3 Combinatorial characterisation of degenerations

The degeneration order on orbits in  $\mathcal{F} \times \mathcal{F}$  implies the hook order on  $\Lambda_1$ . Through examples it seems that these two orders are in fact equivalent, but a proof has so far been elusive.

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