

A geometric realisation of affine 0-Schur algebras.

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## Chapter 1

# Introduction

## Chapter 2

**Background: The double flag variety  
approach to  $q$ -Schur algebras**

## Chapter 3

# The cyclic flags approach to affine $q$ -Schur algebras

Fix natural numbers  $n$  and  $r$ .

**Definition 3.0.1** (compositions). *A composition of  $r$  into  $n$  parts is an  $n$ -tuple  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$  of non-negative integers whose sum equals  $r$ . Denote the set of compositions of  $r$  into  $n$  parts by  $\Lambda_0$ .*

**Definition 3.0.2** (infinite periodic matrices). *Let  $\Lambda_1$  be the set of matrices  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$  with integer entries  $a_{i,j}$  satisfying the following conditions:*

- $a_{i,j} \geq 0$  for each  $i, j \in \mathbb{Z}$ ;
- each row or column has only finitely many non-zero entries;
- the sum of the entries in any  $n$  consecutive rows or columns equals  $r$ ;
- $a_{i-n,j-n} = a_{i,j}$  for each  $i, j \in \mathbb{Z}$ .

*These matrices are referred to as infinite periodic matrices.*

**Definition 3.0.3** (source and target). *Given  $A \in \Lambda_1$ , let  $\text{ro}(A)$  and  $\text{co}(A)$  be the compositions of  $r$  into  $n$  parts given by*

$$\text{ro}(A) = \left( \sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

*and*

$$\text{co}(A) = \left( \sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right).$$

*$A \in \Lambda_1$  is said to go from  $\text{co}(A)$  to  $\text{ro}(A)$ .*

**Definition 3.0.4** (diagonal matrices). *Given  $\lambda \in \Lambda_0$ , let  $D_\lambda \in \Lambda_1$  be the matrix given by  $(D_\lambda)_{i,j} = 0$  for  $i, j \in \mathbb{Z}$  with  $i \neq j$  and  $(D_\lambda)_{i,i} = \lambda_i$  for  $i \in \mathbb{Z}$ ; where the indices are taken modulo  $n$ .*

### 3.1 Cyclic flags

Fix  $n, r \in \mathbb{N}$  and let  $\mathbf{k}$  be a field. Let  $\mathcal{S}$  be the  $\mathbf{k}$ -algebra  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$  and let  $\mathcal{R}$  be the subalgebra generated by  $\varepsilon$ , so  $\mathcal{R} = \mathbf{k}[\varepsilon]$ . Let  $V$  be a free  $\mathcal{S}$ -module of rank  $r$ . Let  $G$  be the automorphism group of the  $\mathcal{S}$ -module  $V$ , so  $G$  is isomorphic to  $\mathrm{GL}_r(\mathcal{S})$ . A lattice in  $V$  is a  $\mathcal{R}$ -submodule  $L$  of  $V$  with  $\mathcal{S} \otimes_{\mathcal{R}} L = V$ . In particular, a lattice is an  $\mathcal{R}$ -submodule of  $V$  which is a free  $\mathcal{R}$ -module of rank  $r$ .

**Lemma 3.1.1.** *Let  $L$  be a lattice in  $V$ .  $L/\varepsilon L$  is a torsion  $\mathcal{R}$ -module, where  $\varepsilon$  acts as zero.  $L/\varepsilon L$  is a free  $\mathcal{R}/\langle \varepsilon \rangle$ -module of rank  $r$ ; that is,  $L/\varepsilon L$  is an  $r$ -dimensional  $\mathbf{k}$ -vector space.*

*Proof.*  $L$  is a free  $\mathcal{R}$ -module of rank  $r$ , with  $L \subset V$ . Given an  $\mathcal{R}$ -basis  $\{x_1, \dots, x_r\}$  of  $L$ ,  $\{\varepsilon x_1, \dots, \varepsilon x_r\}$  is an  $\mathcal{R}$ -basis of  $\varepsilon L$ . Finally, the cosets  $\{x_1 + \varepsilon L, \dots, x_r + \varepsilon L\}$  give a basis for  $L/\varepsilon L$  over  $\mathcal{R}/\langle \varepsilon \rangle \cong \mathbf{k}$ .  $\square$

Let  $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$  be the set of collections  $(L_i)_{i \in \mathbb{Z}}$  of lattices in  $V$  with  $L_i \subset L_{i+1}$  and  $\varepsilon L_i = L_{i-n}$  for each  $i \in \mathbb{Z}$ . These collections of lattices in  $V$  are referred to as cyclic flags in  $V$ .

$G$  acts on  $\mathcal{F}$  by  $(g \cdot L)_i = g(L_i)$  for each  $i \in \mathbb{Z}$ ,  $g \in G$  and  $L \in \mathcal{F}$ . The  $G$ -orbits in  $\mathcal{F}$  are indexed by the set  $\Lambda_0$  of compositions of  $r$  into  $n$  parts. In particular, the  $G$ -orbit in  $\mathcal{F}$  corresponding to  $\lambda \in \Lambda_0$  is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim \left( \frac{L_i}{L_{i-1}} \right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

**Definition 3.1.1.** *The periodic characteristic matrix of a pair of cyclic flags  $(L, L') \in \mathcal{F} \times \mathcal{F}$  is the matrix  $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$  with entries*

$$a_{i,j} = \dim_{\mathbf{k}} \left( \frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j} \right)$$

for each  $i, j \in \mathbb{Z}$ .

The diagonal action of  $G$  on  $\mathcal{F} \times \mathcal{F}$  has orbits indexed by the set  $\Lambda_1$  of infinite periodic matrices (see definition 3.0.2). The  $G$ -orbit corresponding to  $A \in \Lambda_1$  is denoted  $\mathcal{O}_A$  and consists of those pairs  $(L, L') \in \mathcal{F} \times \mathcal{F}$  with periodic characteristic matrix  $A(L, L')$  equal to  $A$ .

**Lemma 3.1.2.** *(alternative expression for characteristic matrix) Alternatively,*

$$a_{i,j} = \dim_{\mathbf{k}} \left( \frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} \right)$$

for each  $i, j \in \mathbb{Z}$ .

*Proof.* Set  $U = L_i \cap L'_j$  and  $U' = L_{i-1} + L_i \cap L'_{j-1}$ . Then  $U + U' = L_{i-1} + L_i \cap L'_j$  and  $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$ . Applying the isomorphism theorems,  $U + U'/U'$  is naturally isomorphic to  $U/U \cap U'$  as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to  $a_{i,j}$ .  $\square$

**Lemma 3.1.3** (transposing characteristic matrix). *Given a pair of flags  $(L, L') \in \mathcal{F}^2$ , the matrices  $A(L, L')$  and  $A(L', L)$  are related by the transpose. In particular,  $A(L, L')_{i,j} = A(L', L)_{j,i}$  for each  $i, j \in \mathbb{Z}$ .*

*Proof.* By swapping the roles of  $i$  and  $j$  and swapping  $L$  and  $L'$  it is clear that  $A(L, L')_{i,j}$  and  $A(L', L)_{j,i}$  are both given by the dimension of the  $\mathbf{k}$ -vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each  $i, j \in \mathbb{Z}$ . □

**Lemma 3.1.4** (a codimension formula). *Given  $(L, L') \in \mathcal{F}^2$  and  $i, j \in \mathbb{Z}$ ,*

$$\dim_{\mathbf{k}} \left( \frac{L_i}{L_i \cap L'_j} \right) = \sum_{s \leq i, t > j} a_{s,t},$$

where  $A(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$ .

*Proof.* **COMPLETE THIS PROOF** □

**Lemma 3.1.5** (nested flags). *Given  $(L, L') \in \mathcal{F}^2$ ,  $L' \subset L$  if and only if  $A(L, L')_{i,j} = 0$  for  $i, j \in \mathbb{Z}$  with  $i > j$ .*

*Proof.* Suppose  $L, L' \in \mathcal{F}$  with  $L' \subset L$ , meaning  $L'_j \subset L_j$  for each  $j \in \mathbb{Z}$ . Then for  $i > j$ ,  $L_i \cap L'_j = L'_j$ ,  $L_{i-1} \cap L'_j = L'_j$  and  $L_i \cap L'_{j-1}$ , which shows

$$A(L, L')_{i,j} = \dim_{\mathbf{k}} \left( \frac{L'_j}{L'_{j-1} + L'_j} \right) = 0$$

as required. Conversely, suppose  $A(L, L')$  is upper triangular, meaning  $A(L, L')_{i,j} = 0$  when  $i > j$ . Using Lemma 3.1.4,

$$\dim_{\mathbf{k}} \left( \frac{L'_i}{L'_i \cap L_i} \right) = \sum_{s > i, t \leq i} a_{s,t} = 0,$$

so  $L_i \cap L'_i = L'_i$  and thus  $L'_i \subset L_i$  for each  $i \in \mathbb{Z}$ , as required. □

**Corollary 3.1.6** (diagonal orbits). *Given  $L, L' \in \mathcal{F}$ ,  $L = L'$  if and only if  $A(L, L')_{i,j} = 0$  whenever  $i \neq j$ . In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L, L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each  $\lambda \in \Lambda_0$ .

### 3.1.1 A product of orbits

Given  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$ , define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also  $L \in \mathcal{F}_{\text{ro}(A)}$ , define the  $L$ -slices of  $Y_{A,B}$  and  $X_{A,B}$  respectively as

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X_{A,B}^L = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$



**Observation 1.** *There are only finitely many  $G$ -orbits in  $X_{A,B}$ .*

**Lemma 3.1.7.** *Given  $A \in \Lambda_1$ ,  $X_{D_\lambda, A} = \mathcal{O}_A$  if  $\lambda = \text{ro}(A)$  and  $X_{A, D_\lambda} = \mathcal{O}_A$  if  $\lambda = \text{co}(A)$ .*

*Proof.* Let  $A \in \Lambda_1$  and set  $\lambda = \text{ro}(A)$ .  $Y_{D_\lambda, A}$  is the set of triples  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_{D_\lambda}$ , thus  $L = L'$  by Corollary 3.1.6, and  $(L', L'') \in \mathcal{O}_A$ .  $X_{D_\lambda, A}$  is the projection of  $Y_{D_\lambda, A}$ , which equals  $\mathcal{O}_A$ .

Similarly, if  $\lambda = \text{co}(A)$ ,  $Y_{A, D_\lambda}$  is the set of triples  $(L, L', L'') \in \mathcal{F}^3$  with  $(L, L') \in \mathcal{O}_A$  and  $L'' = L'$ , so  $X_{A, D_\lambda}$  is exactly the orbit  $\mathcal{O}_B$ .  $\square$

### 3.1.2 Triple products

Given  $A, B, C \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$  and  $\text{co}(B) = \text{ro}(C)$  and  $L \in \mathcal{F}_{\text{ro}(A)}$ , there are spaces  $X_{A,B,C}$ ,  $Y_{A,B,C}$  and their respective  $L$ -slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y_{A,B,C}^L = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X_{A,B,C}^L = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

## 3.2 Convolution algebras

Suppose  $\mathbf{k}$  is a finite field and let  $q$  denote the number of elements of  $\mathbf{k}$ . Consider the set  $S$  of  $G$ -invariant functions  $\mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Z}$  with constructible support.  $S$  is a free  $\mathbb{Z}$ -module with a basis consisting of the indicator functions of the  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$ . Define an operation  $\star$  on  $S$  as follows: for each  $f, g \in S$ ,  $f \star g \in S$  is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ .

$f \star g$  is well defined since the supports of  $f$  and  $g$  consist of finitely many  $G$ -orbits, so there are only finitely many  $L' \in \mathcal{F}$  such that  $f(L, L')g(L', L'') \neq 0$ , given  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ .  $f \star g$  is constant on  $G$ -orbits and is supported on finitely many  $G$ -orbits, so  $f \star g \in S$ .

**Lemma 3.2.1.** *The set  $S$  together with the operation  $\star$  is an associative  $\mathbb{Z}$ -algebra with identity element  $\iota$  given by  $\iota(L, L) = 1$  and  $\iota(L, L') = 0$  for  $L' \neq L$ .*

*Proof.* Given  $f, g, h \in S$  and  $(L, L''') \in \mathcal{F} \times \mathcal{F}$ ,

$$\begin{aligned} ((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''), \end{aligned}$$

thus  $\star$  is associative.  $\iota$  is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L') \iota(L', L'') = f(L, L''),$$

for each  $f \in S$  and  $(L, L'') \in \mathcal{F} \times \mathcal{F}$ . □

Given  $A \in \Lambda_1$ , let  $e_A \in S$  denote the indicator function of the orbit  $\mathcal{O}_A$ .  $S$  is a free  $\mathbb{Z}$ -module with basis  $\{e_A : A \in \Lambda_1\}$ . There exist  $\gamma_{A,B,C;q} \in \mathbb{Z}$  for  $A, B, C \in \Lambda_1$  such that

$$e_A \star e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C;q} e_C$$

for each  $A, B \in \Lambda_1$ . Then

$$\begin{aligned} \gamma_{A,B,C;q} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L') e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}, \end{aligned}$$

for any  $(L, L'') \in \mathcal{O}_C$ .

### 3.3 Affine q-Schur algebras

There exist polynomials  $\gamma_{A,B,C} \in \mathbb{Z}[q]$  for  $A, B, C \in \Lambda_1$  such that  $\gamma_{A,B,C}(q) = \gamma_{A,B,C;q}$  for any prime power  $q$ , following [32, section 4]. The affine  $q$ -Schur algebra  $\hat{S}_q(n, r)$  is a  $\mathbb{Z}[q]$ -algebra which is a free  $\mathbb{Z}[q]$ -module with basis  $\{e_A : A \in \Lambda_1\}$  and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these ‘universal polynomials’  $\gamma_{A,B,C} \in \mathbb{Z}[q]$ , it follows from Lemma 3.2.1 that  $\hat{S}_q(n, r)$  is an associative  $\mathbb{Z}[q]$ -algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

## Chapter 4

# Presenting affine q-Schur algebras

### 4.1 Basic results and notation

#### 4.1.1 Elementary matrices

For each  $i, j \in \mathbb{Z}$ , let  $\mathcal{E}_{i,j}$  be the  $\mathbb{Z} \times \mathbb{Z}$  ‘elementary periodic matrix’ with entries given by  $(\mathcal{E}_{i,j})_{s,t} = 1$  if  $(s, t) = (i + cn, j + cn)$  for some  $c \in \mathbb{Z}$  and  $(\mathcal{E}_{i,j})_{s,t} = 0$  otherwise. Clearly  $\mathcal{E}_{i,j} = \mathcal{E}_{i+n,j+n}$  for each  $i, j \in \mathbb{Z}$ . Recall from Definition 3.0.4 that the diagonal matrix associated to a composition  $\lambda \in \Lambda_0$  is

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}.$$

$\{e_{D_\lambda} : \lambda \in \Lambda_0\}$  is a set of pairwise orthogonal idempotents in  $\hat{S}_q(n, r)$  with  $\sum_{\lambda \in \Lambda_0} e_{D_\lambda} = 1$ , as a result of Lemma 3.1.7.

Given  $i \in [1, n]$  and  $\lambda \in \Lambda_0$  with  $\lambda_{i+1} > 0$ , define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and define

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}.$$

Given  $i \in [1, n]$  and  $\lambda \in \Lambda_0$  with  $\lambda_i > 0$ , define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and define

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}$$

#### 4.1.2 Transpose involution

**Lemma 4.1.1.** *Transposition gives a homomorphism of  $\mathbb{Z}[q]$ -modules  $\top : \hat{S}_q(n, r) \rightarrow \hat{S}_q(n, r)$  with  $\top(e_A) = e_{A^\top}$ ,  $\top \circ \top = 1$  and  $\top(e_A e_B) = \top(e_B) \top(e_A)$ .*

*Proof.* Let  $A, B, C \in \Lambda_1$  and let  $\mathbf{k}$  be a finite field with  $q = \#\mathbf{k}$  elements. If  $(L, L'') \in \mathcal{O}_C$  then  $(L'', L) \in \mathcal{O}_{C^\top}$  and

$$\begin{aligned} \gamma_{A,B,C;q} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;q} \end{aligned}$$

It then follows that  $\top(e_A e_B) = \top(e_B) \top(e_A)$ .  $\square$

The transpose relates the  $E_i$ ,  $F_i$  and  $1_\lambda$  in the following way:  $\top(E_{i,\lambda}) = F_{i,\lambda}$ ,  $\top(F_{i,\lambda}) = E_{i,\lambda-\varepsilon_i+\varepsilon_{i+1}}$  and  $\top(1_\lambda) = 1_\lambda$ . In particular,  $\top(E_i) = F_i$  and  $\top(F_i) = E_i$ .

### 4.1.3 A multiplication rule

**Lemma 4.1.2.** *Given  $A \in \Lambda_1$  and  $i \in [1, n]$  with  $\text{ro}(A)_{i+1} > 0$ ,*

$$E_i e_A = \sum_{p \in \mathbb{Z}: a_{i+1,p} > 0} q^{\sum_{j>p} a_{i,j}} [[a_{i,p} + 1]] e_{A+\varepsilon_{i,p}-\varepsilon_{i+1,p}}.$$

*Given  $A \in \Lambda_1$  and  $i \in [1, n]$  with  $\text{ro}(A)_i > 0$ ,*

$$F_i e_A = \sum_{p \in \mathbb{Z}: a_{i,p} > 0} q^{\sum_{j<p} a_{i+1,j}} [[a_{i+1,p} + 1]] e_{A+\varepsilon_{i+1,p}-\varepsilon_{i,p}}.$$

Note that these formulas are still valid in the cases  $E_i e_A = 0$  and  $F_i e_A = 0$ , provided it is understood that  $e_B = 0$  whenever  $B \notin \Lambda_1$ . There are similar formulas for right multiplication by  $E_i$  and  $F_i$ , obtained by applying the transpose involution to the above.

**Corollary 4.1.3.** *Given  $A \in \Lambda_1$  and  $j \in [1, n]$  with  $\text{co}(A)_{j+1} > 0$ ,*

$$e_A F_j = \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\varepsilon_{p,j}-\varepsilon_{p,j+1}}.$$

*Given  $A \in \Lambda_1$  and  $j \in [1, n]$  with  $\text{co}(A)_j > 0$ ,*

$$e_A E_j = \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A+\varepsilon_{p,j+1}-\varepsilon_{p,j}}.$$

*Proof.*

$$\begin{aligned} e_A F_j &= \top(E_j e_{A^\top}) \\ &= \top \left( \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A^\top+\varepsilon_{j,p}-\varepsilon_{j+1,p}} \right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A+\varepsilon_{p,j}-\varepsilon_{p,j+1}}, \end{aligned}$$

where the second equality comes from Lemma 4.1.2. Similarly,

$$\begin{aligned} e_A E_j &= \top(F_j e_{A^\top}) \\ &= \top \left( \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A^\top+\varepsilon_{j+1,p}-\varepsilon_{j,p}} \right) \\ &= \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A+\varepsilon_{p,j+1}-\varepsilon_{p,j}}. \end{aligned}$$

$\square$

## 4.2 Relations

Note that  $E_i^{r+1} = F_i^{r+1} = 0$  while

$$E_i^r = [r]! e_{r\mathcal{E}_{i,i+1}}$$

and

$$F_i^r = [r]! e_{r\mathcal{E}_{i+1,i}}.$$

**Lemma 4.2.1** (quantum Serre relations:  $n \geq 3$ ). *Suppose  $n \geq 3$ . The following relations hold in  $\hat{S}_q(n, r)$ :*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless  $j = i \pm 1$ ;

$$E_i E_{i+1}^2 - (1+q)E_{i+1}E_i E_{i+1} + qE_{i+1}^2 E_i = 0$$

$$E_i^2 E_{i+1} - (1+q)E_i E_{i+1} E_i + qE_{i+1} E_i^2 = 0$$

and

$$F_{i+1} F_i^2 - (1+q)F_i F_{i+1} F_i + qF_i^2 F_{i+1} = 0$$

$$F_{i+1}^2 F_i - (1+q)F_{i+1} F_i F_{i+1} + qF_i F_{i+1}^2 = 0.$$

*Proof.* Here we introduce temporary notation for the basis elements: Write  $[A] = e_A$ .

Take  $\lambda \in \Lambda_0$ .

$$E_i E_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1} E_i E_{i+1} 1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2 E_i 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then

$$(E_i E_{i+1}^2 - (1+q)E_{i+1}E_i E_{i+1} + qE_{i+1}^2 E_i) 1_\lambda = 0,$$

for each  $\lambda \in \Lambda_0$ . The relation  $E_i E_{i+1}^2 - (1+q)E_{i+1}E_i E_{i+1} + qE_{i+1}^2 E_i = 0$  then follows.

The relations between  $F_i$  and  $F_{i+1}$  may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping  $E_i$  and  $F_i$  and reversing the order of multiplication.  $\square$

**Lemma 4.2.2** (quantum Serre relations:  $n = 2$ ). *In the case  $n = 2$ , the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

**Lemma 4.2.3.**  $[E_i, F_j] = 0$  unless  $j = i$ .

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}]) 1_\lambda.$$

For  $\lambda \in \Lambda_0$ , let  $R_\lambda = e_{\lambda_1 \varepsilon_{0,1} + \dots + \lambda_n \varepsilon_{n-1,n}}$ . Write  $R = \sum_{\lambda \in \Lambda_0} R_\lambda$ . Note  $R_\lambda = R 1_\lambda$ . Given  $A \in \Lambda_1$  and  $m \in \mathbb{Z}$ , let  $A[m] \in \Lambda_1$  be given by  $A[m]_{i,j} = a_{i,j+m}$  and let  $A^{[m]}$  be given by  $A^{[m]}_{i,j} = a_{i+m,j}$  for each  $i \in \mathbb{Z}$ .

**Lemma 4.2.4** (Shifting). *If  $A \in \Lambda_1$  then*

$$Re_A = e_{A[\pm 1]}$$

and

$$e_A R = e_{A[\pm 1]}.$$

Conjugation by  $R$  gives an automorphism  $\rho$  of  $\hat{S}_q(n, r)$  satisfying  $\rho^n = 1$ .

### 4.3 quivers with relations

Denote by  $\Lambda_0$  the set of compositions of  $r$  into  $n$  parts. That is,  $\Lambda_0$  is the set of  $\alpha \in \mathbb{Z}^n$  with non-negative entries which sum to  $r$ . Let  $\varepsilon_i \in \mathbb{Z}^n$  be the  $i$ th elementary vector and write  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for each  $i \in [1, n]$ . Then  $\lambda + \alpha_i \in \Lambda_0$  if  $\lambda_{i+1} > 0$  and  $\lambda - \alpha_i \in \Lambda_0$  if  $\lambda_i > 0$ .

Let  $\Gamma = \Gamma(n, r)$  be the quiver with set of vertices  $\Lambda_0$ , with the following arrows:

For  $\lambda \in \Lambda_0$  and  $i \in [1, n]$ , there is an arrow  $e_{i,\lambda} : \lambda \rightarrow \lambda + \alpha_i$  if  $\lambda_{i+1} > 0$  and there is an arrow  $f_{i,\lambda} : \lambda \rightarrow \lambda - \alpha_i$  if  $\lambda_i > 0$ .

Denote by  $\mathbb{Z}[q]\Gamma$  the path  $\mathbb{Z}[q]$ -algebra of  $\Gamma$ . Thus  $\mathbb{Z}[q]\Gamma$  is a free  $\mathbb{Z}[q]$ -module with a basis given by the set of paths in  $\Gamma$ , with multiplication given by the concatenation of paths. If  $p$  starts where  $q$  ends, the product  $pq$  is the path  $q$  followed by  $p$ . Write  $e_{i,\lambda} = 0$  unless  $\lambda, \lambda + \alpha_i \in \Lambda_0$  and write  $f_{i,\lambda} = 0$  unless  $\lambda, \lambda - \alpha_i \in \Lambda_0$ .

By construction, there is a homomorphism of  $\mathbb{Z}[q]$ -algebras

$$\phi : \mathbb{Z}[q]\Gamma \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda, \end{aligned}$$

for  $i \in [1, n]$  and  $\lambda \in \Lambda_0$ .

The image of  $\phi$  is the subalgebra of  $\hat{S}_q(n, r)$  generated by  $E_i, F_i$  for  $i \in [1, n]$  and  $1_\lambda$  for  $\lambda \in \Lambda_0$ , since  $E_{i,\lambda} = E_i 1_\lambda$  and  $F_{i,\lambda} = F_i 1_\lambda$ , while  $E_i = \sum_\lambda E_{i,\lambda}$  and  $F_i = \sum_\lambda F_{i,\lambda}$ . In general  $\phi$  is not surjective, so this does not always lead to a presentation of  $\hat{S}_q(n, r)$ .

#### 4.3.1 Exceptional case $n=2$ .

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the  $q$ -Schur algebra.

### 4.3.2 Typical case.

Suppose  $n \geq 3$ . Then  $\Gamma = \Gamma(n, r)$  has vertex set  $\Lambda_0$ .

Define  $e_i, f_i \in \mathbb{Z}[q]\Gamma(n, r)$  by

$$e_i = \sum_{\lambda \in \Lambda_0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0} f_{i,\lambda},$$

with the convention  $e_{i,\lambda} = 0$  unless  $\lambda_{i+1} > 0$  and  $f_{i,\lambda} = 0$  unless  $\lambda_i > 0$ . Let  $k_\lambda$  denote the constant path at vertex  $\lambda$ .  $\{k_\lambda : \lambda \in \Lambda_0\}$  is a set of pairwise orthogonal idempotents in  $\mathbb{Z}[q]\Gamma(n, r)$ .

Let  $I(n, r) \subset \mathbb{Z}[q]\Gamma(n, r)$  be the ideal generated by the expressions

$$\begin{aligned} & e_i e_{i+1}^2 - (1+q)e_{i+1}e_i e_{i+1} + qe_{i+1}^2 e_i \\ & e_i^2 e_{i+1} - (1+q)e_i e_{i+1} e_i + qe_{i+1} e_i^2 \\ & f_{i+1} f_i^2 - (1+q)f_i f_{i+1} f_i + qf_i^2 f_{i+1} \\ & f_{i+1}^2 f_i - (1+q)f_{i+1} f_i f_{i+1} + qf_i f_{i+1}^2 \\ & e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}]) k_\lambda \end{aligned}$$

Recall that a relation is a  $\mathbb{Z}[q]$ -linear combination of paths with common start and end vertices. The relations involving paths  $\lambda \rightarrow \mu$  are given by  $1_\mu \text{expr} 1_\lambda$ , for each of the above expressions.

**Lemma 4.3.1.** *There is a homomorphism of  $\mathbb{Z}[q]$ -algebras*

$$\phi: \mathbb{Z}[q]\Gamma(n, r)/I(n, r) \rightarrow \hat{S}_q(n, r)$$

given by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda. \end{aligned}$$

## Chapter 5

# A generic affine algebra

### 5.1 Introduction

Assume  $\mathbf{k} = \mathbb{C}$  and fix  $n, r \geq 1$ . Let  $\mathcal{S}$  be the  $\mathbf{k}$ -algebra  $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$  and let  $\mathcal{R}$  be the subalgebra generated by  $\varepsilon$ , namely  $\mathcal{R} = \mathbf{k}[\varepsilon]$ . Let  $V$  be a free  $\mathcal{S}$ -module of rank  $r$  and let  $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$  be the set of  $n$ -periodic cyclic flags in  $V$ ; so  $\mathcal{F}$  consists of collections  $L = (L_i)_{i \in \mathbb{Z}}$  of  $\mathcal{R}$ -lattices in  $V$  with  $L_i \subset L_{i+1}$  for  $i \in \mathbb{Z}$  and  $\varepsilon L_i = L_{i-n}$  for  $i \in \mathbb{Z}$ .

Let  $G$  be the group of  $\mathcal{S}$ -module automorphisms of  $V$ . Thus  $G$  is isomorphic to  $\mathrm{GL}_r(\mathcal{S})$ .  $G$  acts on  $\mathcal{F}$  with orbits  $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0\}$ , where  $\Lambda_0$  is the set of compositions of  $r$  into  $n$  parts, as in Definition 3.0.1.

The diagonal action of  $G$  on  $\mathcal{F} \times \mathcal{F}$  has orbits  $\{\mathcal{O}_A : A \in \Lambda_1\}$ , where  $\mathcal{O}_A$  consists of those pairs of flags with periodic characteristic matrix equal to  $A$ . Definitions of the periodic characteristic matrix and the set  $\Lambda_1$  are given in Definition 3.1.1 and Definition 3.0.2 respectively.

Recall that the periodic characteristic matrix of a pair  $(L, L') \in \mathcal{F} \times \mathcal{F}$  is the  $\mathbb{Z} \times \mathbb{Z}$  matrix  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ , with

$$a_{i,j} = \dim \left( \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right)$$

for each  $i, j \in \mathbb{Z}$ .

Recall that  $\mathrm{ro}$  and  $\mathrm{co}$  are the maps  $\Lambda_1 \rightarrow \Lambda_0$  given by

$$\mathrm{ro}(A) = \left( \sum_{j \in \mathbb{Z}} a_{1,j}, \dots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co}(A) = \left( \sum_{i \in \mathbb{Z}} a_{i,1}, \dots, \sum_{i \in \mathbb{Z}} a_{i,n} \right)$$

for each  $A \in \Lambda_1$ . Given  $A \in \Lambda_1$ , write  $A: \mathrm{co}(A) \rightarrow \mathrm{ro}(A)$ .

The purpose of this chapter is to define an associative  $\mathbb{Z}$ -algebra with a multiplicative basis by defining a modified form of the product in the affine  $q$ -Schur algebra. In particular, given  $A, B \in \Lambda_1$ , the orbit product

$$X_{A,B} = \{(L, L'') \in \mathcal{F} \times \mathcal{F} : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$



consists of finitely many  $G$ -orbits and it will be shown that there is a unique ‘generic’ orbit in  $X_{A,B}$ , denoted  $\mathcal{O}_{A*B}$ , with the property that

$$\dim \left( \frac{L_i}{L_i \cap L_j''} \right) \leq \dim \left( \frac{N_i}{N_i \cap N_j''} \right)$$

and

$$\dim \left( \frac{L_j''}{L_i \cap L_j''} \right) \leq \dim \left( \frac{N_j''}{N_i \cap N_j''} \right)$$

for all  $i, j \in \mathbb{Z}$ ,  $(N, N'') \in \mathcal{O}_{A*B}$  and  $(L, L'') \in X_{A,B}$ . It will be shown that the above ‘generic product’ of orbits is associative, so the free  $\mathbb{Z}$ -module on the set of  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$  with  $\mathbb{Z}$ -bilinear multiplication given by

$$\mathcal{O}_A * \mathcal{O}_B = \mathcal{O}_{A*B},$$

for each  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$ , and

$$\mathcal{O}_A * \mathcal{O}_B = 0$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) \neq \text{ro}(B)$ , is an associative  $\mathbb{Z}$ -algebra with multiplicative identity given by

$$\sum_{\lambda \in \Lambda_0} \mathcal{O}_{D_\lambda},$$

where  $D_\lambda$  is the diagonal matrix with  $\text{co}(D_\lambda) = \lambda$ . The resulting  $\mathbb{Z}$ -algebra is called the *generic affine algebra* (of rank  $r$  and period  $n$ ), denoted  $\hat{G}(n, r)$ .

## 5.2 A combinatorial partial order

For each  $i, j \in \mathbb{Z}$ , let  $d_{i,j}$  and  $\bar{d}_{i,j}$  be the maps from  $\Lambda_1$  to  $\Lambda_0$  given by

$$d_{i,j}(A) = \sum_{s \leq i, t > j} a_{s,t}$$

and

$$\bar{d}_{i,j}(A) = \sum_{s > i, t \leq j} a_{s,t}$$

for each  $A \in \Lambda_1$ .

**Lemma 5.2.1.** *For each  $A \in \Lambda_1$  and  $i, j \in \mathbb{Z}$ , the following equations hold:*

$$\begin{aligned} d_{i,j}(A) - d_{i-1,j}(A) &= \sum_{t > j} a_{i,t} \\ d_{i,j}(A) - d_{i,j-1}(A) &= - \sum_{s \leq i} a_{s,j} \end{aligned}$$

and

$$\begin{aligned} \bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) &= - \sum_{t \leq j} a_{i,t} \\ \bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) &= \sum_{s > i} a_{s,j} \end{aligned}$$

*Proof.* Let  $i, j \in \mathbb{Z}$  and  $A \in \Lambda_1$ . Then

$$d_{i,j}(A) - d_{i-1,j}(A) = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i-1, t > j} a_{s,t} = \sum_{t > j} a_{i,t}$$

and

$$d_{i,j}(A) - d_{i,j-1}(A) = \sum_{s \leq i, t > j} a_{s,t} - \sum_{s \leq i, t > j-1} a_{s,t} = - \sum_{s \leq i} a_{s,j}.$$

Similarly,

$$\bar{d}_{i,j}(A) - \bar{d}_{i-1,j}(A) = \sum_{s > i, t \leq j} a_{s,t} - \sum_{s > i-1, t \leq j} a_{s,t} = - \sum_{t \leq j} a_{i,t}$$

and

$$\bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) = \sum_{s > i, t \leq j} a_{s,t} - \sum_{s > i, t \leq j-1} a_{s,t} = \sum_{s > i} a_{s,j}.$$

□

**Lemma 5.2.2.** For each  $A \in \Lambda_1$  and  $i, j \in \mathbb{Z}$ ,

$$a_{i,j} = d_{i,j-1}(A) - d_{i-1,j-1}(A) - d_{i,j}(A) + d_{i-1,j}(A)$$

and

$$a_{i,j} = \bar{d}_{i,j-1}(A) - \bar{d}_{i-1,j-1}(A) - \bar{d}_{i,j}(A) + \bar{d}_{i-1,j}(A).$$

*Proof.* As a result of Lemma 5.2.1,

$$\begin{aligned} d_{i,j-1}(A) - d_{i-1,j-1}(A) - d_{i,j}(A) + d_{i-1,j}(A) &= \sum_{t > j-1} a_{i,t} - \sum_{t > j} a_{i,t} \\ &= a_{i,j} \end{aligned}$$

and

$$\begin{aligned} \bar{d}_{i,j-1}(A) - \bar{d}_{i-1,j-1}(A) - \bar{d}_{i,j}(A) + \bar{d}_{i-1,j}(A) &= - \sum_{t \leq j-1} a_{i,t} + \sum_{t \leq j} a_{i,t} \\ &= a_{i,j}. \end{aligned}$$

□

Define a relation  $\leq$  on  $\Lambda_1$  by  $A \leq B$  if and only if the following conditions are satisfied:

- $\text{ro}(A) = \text{ro}(B)$  and  $\text{co}(A) = \text{co}(B)$ .
- For each  $i, j \in \mathbb{Z}$ ,  $d_{i,j}(A) \leq d_{i,j}(B)$ .
- For each  $i, j \in \mathbb{Z}$ ,  $\bar{d}_{i,j}(A) \leq \bar{d}_{i,j}(B)$ .

**Lemma 5.2.3.** The relation  $\leq$  defines a partial order on  $\Lambda_1$ .

*Proof.* It is clear that  $\leq$  is reflexive and transitive.

Suppose  $A, B \in \Lambda_1$  with  $A \leq B$  and  $B \leq A$ . Then  $d_{i,j}(A) = d_{i,j}(B)$  for each  $i, j \in \mathbb{Z}$  with  $i \leq j$ , which shows  $a_{s,t} = b_{s,t}$  whenever  $s < t$ , as a result of Lemma 5.2.2. Similarly,  $\bar{d}_{i,j}(A) = \bar{d}_{i,j}(B)$  for each  $i, j \in \mathbb{Z}$  with  $i \geq j$ , so  $a_{s,t} = b_{s,t}$  whenever  $s > t$ . Moreover,  $a_{i,i} = b_{i,i}$  for each  $i \in \mathbb{Z}$ , since  $\text{co}(A) = \text{co}(B)$ . Thus  $A = B$ , which shows  $\leq$  is antisymmetric and therefore  $\leq$  is a partial order on  $\Lambda_1$ . □

**Lemma 5.2.4.** *The transpose operation on  $\Lambda_1$  is order preserving. In particular,  $B \leq A$  if and only if  $B^\top \leq A^\top$ .*

*Proof.* Suppose  $A, B \in \Lambda_1$  with  $B \leq A$ . The condition  $\text{co}(A) = \text{co}(B)$  and  $\text{ro}(A) = \text{ro}(B)$  is preserved by the transpose operation.

For each  $i, j \in \mathbb{Z}$ ,

$$d_{i,j}(A^\top) = \sum_{s \leq i, t > j} a_{t,s} = \bar{d}_{j,i}(A)$$

and

$$\bar{d}_{i,j}(A^\top) = \sum_{s > i, t \leq j} a_{t,s} = d_{j,i}(A).$$

It follows that  $B^\top \leq A^\top$  and therefore the transpose is order preserving.  $\square$

The partial order on  $\Lambda_1$  induces a partial order on the set of  $G$ -orbits in  $\mathcal{F} \times \mathcal{F}$ , such that  $\mathcal{O}_A \leq \mathcal{O}_B$  if and only if  $A \leq B$ . The following lemma is rephrased from Lemma 3.1.4 and gives some geometric significance to the partial order on  $\Lambda_1$ .

**Lemma 5.2.5.** *Let  $A \in \Lambda_1$  and  $(L, L') \in \mathcal{O}_A$ . Then*

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}(A)$$

and

$$\dim \left( \frac{L'_j}{L_i \cap L'_j} \right) = \bar{d}_{i,j}(A),$$

for each  $i, j \in \mathbb{Z}$ .

*Proof.* This is a rephrasing of Lemma 3.1.4.  $\square$

### 5.3 Grassmannians and related varieties

Here we collect a few elementary results on Grassmannians and some related varieties. In this section, let  $V$  be an  $n$ -dimensional  $\mathbf{k}$ -vector space and let  $0 \leq d \leq n$  be an integer. There is a linear map

$$\phi^{(d)}: \Lambda^d(V) \rightarrow \text{Hom}(V, \Lambda^{d+1}(V))$$

given by

$$\phi^{(d)}(\alpha)(v) = \alpha \wedge v$$

for  $\alpha \in \Lambda^d(V)$  and  $v \in V$ . The kernel of  $\phi^{(d)}(\alpha)$  is the space of divisors of  $\alpha$ ,

$$D_\alpha = \{v \in V : \alpha \wedge v = 0\}.$$

An element  $\alpha \in \Lambda^d(V)$  is said to be totally decomposable if  $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_d$ , where  $\alpha_1, \dots, \alpha_d \in V$  are linearly independent. The dimension of  $D_\alpha$  is at most  $d$  and  $\dim(D_\alpha) = d$  precisely when  $\alpha$  is totally decomposable. Consequently, the rank of  $\phi^{(d)}(\alpha)$  is at least  $n - d$  and  $\alpha$  is totally decomposable if and only if  $\text{rank } \phi^{(d)}(\alpha) \leq n - d$ , which holds if and only if the  $(n - d + 1) \times (n - d + 1)$ -minors of a matrix of  $\phi^{(d)}(\alpha)$  are all zero.

**Lemma 5.3.1.**  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$  is a projective variety, for each  $d_1, d_2, a \in \mathbb{N}$  with  $d_1, d_2, a \leq n$ .

*Proof.* As above, there is a linear map  $\Psi: \Lambda^{d_1}V \oplus \Lambda^{d_2}V \rightarrow \text{Hom}(V, \Lambda^{d_1+1}(V) \oplus \Lambda^{d_2+1}(V))$  given by  $\Psi(\alpha, \beta)(v) = (\alpha \wedge v, \beta \wedge v)$ . Given  $\alpha \in \Lambda^{d_1}(V)$  and  $\beta \in \Lambda^{d_2}(V)$ , the kernel of  $\Psi(\alpha, \beta)$  is  $D_\alpha \cap D_\beta$  and so the rank of  $\Psi(\alpha, \beta)$  is  $n - \dim(D_\alpha \cap D_\beta)$ .

Let  $U_i \in \text{Gr}_{d_i}(V)$  and suppose  $p_i(U_i) = [\alpha_i]$ , where  $p_i$  is the Plücker embedding of  $\text{Gr}_{d_i}(V)$  in  $\mathbb{P}(\Lambda^{d_i}(V))$ , so  $U_i = D_{\alpha_i} = \ker \phi^{(d_i)}(\alpha_i)$ . Therefore the kernel of  $\Psi(\alpha_1, \alpha_2)$  is  $U_1 \cap U_2$ , so the condition that  $\dim(U_1 \cap U_2) \geq a$  is equivalent to the condition that  $\Psi(\alpha_1, \alpha_2)$  has rank at most  $n - a$ . After fixing a basis of  $V$ , this condition is given by the vanishing of the  $(n - a + 1) \times (n - a + 1)$  minors of the matrix of  $\Psi(\alpha_1, \alpha_2)$  with respect to this basis. Therefore  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$  is a closed subset of the product of Grassmannians  $\text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V)$ , so is a projective variety.

More precisely, the entries of a matrix of  $\Psi(\alpha_1, \alpha_2)$  are homogeneous polynomials of degree 1 in the Plücker coordinates on  $\text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V)$  since  $\Psi$  is linear and so the minors of  $\Psi(\alpha_1, \alpha_2)$  are also homogeneous polynomials in the Plücker coordinates.  $\square$

**Lemma 5.3.2.** Let  $V$  be an  $n$ -dimensional vector space over  $\mathbf{k}$  and let  $d_1, d_2, a \in \mathbb{N}$  with  $d_1, d_2, a \leq n$ . The following hold:

1.  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) = a\}$  is a quasiprojective variety;
2.  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : U_1 \subset U_2\}$  is a projective variety;
3. Given  $U_2 \in \text{Gr}_{d_2}(V)$ ,  $\{U_1 \in \text{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\}$  is a projective variety;
4. Given  $U_2 \in \text{Gr}_{d_2}(V)$ ,  $\{U_1 \in \text{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) = a\}$  is a quasiprojective variety;
5. Given  $U_2 \in \text{Gr}_{d_2}(V)$ ,  $\{U_1 \in \text{Gr}_{d_1}(V) : U_1 \subset U_2\}$  is a projective variety;
6. Given  $U_2 \in \text{Gr}_{d_2}(V)$ ,  $\{U_1 \in \text{Gr}_{d_1}(V) : U_2 \subset U_1\}$  is a projective variety.

*Proof.* Let  $X_i$  denote the space in statement  $i$  of the lemma. To emphasise the dependence of  $X_i$  on  $a$ , write  $X_{i,a}$ .

$X_1$  is a quasiprojective variety since it is equal to the intersection of the projective variety  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$  with the open set  $\{(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \leq a\}$ .

Given  $(U_1, U_2) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V)$ ,  $U_1 \subset U_2$  if and only if  $\dim(U_1 \cap U_2) \geq d_1$ , so Lemma 5.3.1 shows  $X_2$  is a projective variety.

Let  $\pi_i: \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) \rightarrow \text{Gr}_{d_i}(V)$  be the projection map onto the  $i$ -th factor, for  $i = 1, 2$ . The completeness property of projective varieties ensures that  $\pi_i$  is a closed morphism. Observe that

$$\begin{aligned} X_3 &= \{U_1 \in \text{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\} \\ &= \pi_1(\{(U_1, W) \in \text{Gr}_{d_1}(V) \times \text{Gr}_{d_2}(V) : \dim(U_1 \cap W) \geq a\} \cap \pi_2^{-1}\{U_2\}). \end{aligned}$$

The fibre of  $\pi_2$  over  $U_2$  is closed, so the intersection of the fibre with the variety from Lemma 5.3.1 is closed and then the image of this intersection under  $\pi_1$  is closed. This shows  $X_3$  is a projective variety.

$X_4$  is a quasiprojective variety since it is the complement of the subvariety  $X_{3,a+1}$  in  $X_{3,a}$ . Finally, 5-6 follow as special cases of 3 since  $X_5 = X_{3,d_1}$  and  $X_6 = X_{3,d_2}$ .  $\square$

## 5.4 Geometry of affine flag varieties

Given  $L \in \mathcal{F}$ ,  $N, a \in \mathbb{N}$  and  $\lambda \in \Lambda_0$  define

$$\Pi_{N,\lambda}(L) = \{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi_{N,\lambda}^a(L) = \left\{ L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0, \dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) = a \right\}.$$

**Lemma 5.4.1.** *Given  $L \in \mathcal{F}$ ,  $N \in \mathbb{N}$  and  $\lambda \in \Lambda_0$ ,*

$$\Pi_{N,\lambda}(L) = \bigcup_{a: 0 \leq a \leq 2Nr} \Pi_{N,\lambda}^a(L).$$

*Proof.* If  $L' \in \Pi_{N,\lambda}(L)$  then  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$  and the  $\mathbf{k}[\varepsilon]$ -module  $\varepsilon^{-N} L_0 / L'_0$  is naturally isomorphic to  $(\varepsilon^{-N} L_0 / \varepsilon^N L_0) / (L'_0 / \varepsilon^N L_0)$ , so

$$\dim_{\mathbf{k}} \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) \leq \dim_{\mathbf{k}} \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^N L_0} \right) = 2Nr.$$

□

**Lemma 5.4.2.** *Given  $L \in \mathcal{F}$ ,  $N \in \mathbb{N}$ ,  $\lambda \in \Lambda_0$  and  $a \in \mathbb{N}$  with  $0 \leq a \leq 2Nr$ ,  $\Pi_{N,\lambda}^a(L)$  is a projective algebraic variety.*

*Proof.* Let  $W$  be the  $\mathbf{k}[\varepsilon]$ -module  $\varepsilon^{-1-N} L_0 / \varepsilon^N L_0$ , which has dimension  $(2N+1)r$  over  $\mathbf{k}$ . Let  $d_i = 2Nr - a + \lambda_1 + \dots + \lambda_i$  for each  $i = 1, \dots, n$ . The correspondence between submodules of  $\varepsilon^{-1-N} L_0$  which contain  $\varepsilon^N L_0$  and submodules of  $\varepsilon^{-1-N} L_0 / \varepsilon^N L_0$  determines a map

$$\rho: \Pi_{N,\lambda}^a(L) \rightarrow \mathrm{Gr}_{d_1}(W) \times \dots \times \mathrm{Gr}_{d_n}(W),$$

with  $\rho(L') = (L'_1 / \varepsilon^N L_0, \dots, L'_n / \varepsilon^N L_0)$ .

Let  $\mathcal{X}$  be the space of  $(U_1, \dots, U_n) \in \mathrm{Gr}_{d_1}(W) \times \dots \times \mathrm{Gr}_{d_n}(W)$  with  $U_i \subset U_{i+1}$  for  $i = 1, \dots, n-1$  and  $\varepsilon U_n \subset U_1$ . Lemma 5.3.2 shows that each of these conditions is closed, so  $\mathcal{X}$  is a closed subset of  $\mathrm{Gr}_{d_1}(W) \times \dots \times \mathrm{Gr}_{d_n}(W)$ , therefore  $\mathcal{X}$  is a projective algebraic variety.

The image of  $\rho$  is contained in  $\mathcal{X}$  since

$$\varepsilon L'_n / \varepsilon^N L_0 = L'_0 / \varepsilon^N L_0 \subset L'_1 / \varepsilon^N L_0 \subset \dots \subset L'_n / \varepsilon^N L_0.$$

Suppose  $(U_1, \dots, U_n) \in \mathcal{X}$ . Then  $U_i$  is a  $\mathbf{k}[\varepsilon]$ -module, since  $\varepsilon U_i \subset \varepsilon U_n \subset U_1 \subset U_i$ , for each  $i = 1, \dots, n$ , so  $U_i$  lifts uniquely to a  $\mathbf{k}[\varepsilon]$ -module  $L'_i$  with  $\varepsilon^N L_0 \subset L'_i \subset \varepsilon^{-1-N} L_0$ . Therefore  $L'_1, \dots, L'_n$  are  $\mathbf{k}[\varepsilon]$ -lattices with  $L_i \subset L_{i+1}$  for  $i = 1, \dots, n-1$  and  $\varepsilon L'_n \subset L'_1$ , with

$$\dim(\varepsilon^{-1-N} L_0 / L'_n) = \dim(W / W_n) = (2N+1)r - d_n = a$$

and

$$\dim(L'_i / L'_{i-1}) = \dim(W_i / W_{i-1}) = d_i - d_{i-1} = \lambda_i,$$

for each  $i = 2, \dots, n$ . Therefore there is a unique  $L' \in \Pi_{N,\lambda}^a(L)$  such that  $\rho(L') = (W_1, \dots, W_n)$ , where  $L'$  is given by  $L'_{i+cn} = \varepsilon^{-c} L'_i$  for  $i = 1, \dots, n$  and  $c \in \mathbb{Z}$ . It follows  $\rho$  is injective and  $\mathrm{im} \rho = \mathcal{X}$ , which is a projective variety, so  $\Pi_{N,\lambda}^a(L)$  is a projective variety. □

**Lemma 5.4.3.** *Given  $L \in \mathcal{F}$ ,  $N \in \mathbb{N}$ ,  $\lambda \in \Lambda_0$  and  $a \in \mathbb{N}$  with  $0 \leq a \leq 2Nr$ ,  $\Pi_{N,\lambda}^a(L)$  is closed in  $\Pi_{N+1,\lambda}^{a+r}(L)$ .*

*Proof.* If  $L' \in \Pi_{N,\lambda}^a(L)$ , then  $\varepsilon^{N+1}L_0 \subset \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0 \subset \varepsilon^{-(N+1)}L_0$  and

$$\dim \left( \frac{\varepsilon^{-(1+n)}L_0}{L'_0} \right) = \dim \left( \frac{L_0}{\varepsilon L_0} \right) + \dim \left( \frac{\varepsilon^{-N}L_0}{L'_0} \right) = r + a,$$

which shows that  $\Pi_{N,\lambda}^a(L) \subset \Pi_{N+1,\lambda}^{a+r}(L)$ . For  $L' \in \Pi_{N+1,\lambda}^{a+r}(L)$ , if additionally  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$ , then

$$\dim \left( \frac{\varepsilon^{-(N+1)}L_0}{L'_0} \right) = r + \dim \left( \frac{\varepsilon^{-N}L_0}{L'_0} \right),$$

which shows  $L' \in \Pi_{N,\lambda}^a(L)$ . Therefore  $\Pi_{N,\lambda}^a(L)$  is the subspace of  $\Pi_{N+1,\lambda}^{a+r}(L)$  defined by the two closed conditions  $\varepsilon^N L_0 \subset L'_0$  and  $L'_0 \subset \varepsilon^{-N}L_0$ , using Lemma 5.3.2.  $\square$

**Lemma 5.4.4.** *Let  $\lambda \in \Lambda_0$ ,  $M, N \in \mathbb{N}$ ,  $L, \tilde{L} \in \mathcal{F}$ ,  $0 \leq a \leq 2Nr$ ,  $0 \leq b \leq 2Mr$ .  $\Pi_{N,\lambda}^a(L) \cap \Pi_{M,\lambda}^b(\tilde{L})$  is a closed set in  $\Pi_{N,\lambda}^a(L)$ . In particular, if the intersection is nonempty it is a projective algebraic variety.*

*Proof.* Observe that  $\Pi_{N,\lambda}^a(L) \cap \Pi_{M,\lambda}^b(\tilde{L})$  is the subset of  $\Pi_{N,\lambda}^a(L)$  defined by the additional conditions that  $\varepsilon^M \tilde{L}_0 \subset L'_0$  and  $L'_0 \subset \varepsilon^{-M} \tilde{L}_0$ , so is a closed subset of  $\Pi_{N,\lambda}^a(L)$ , using 5.3.2.  $\square$

**Lemma 5.4.5.** *Suppose  $L \in \mathcal{F}$ ,  $N, a \in \mathbb{N}$  and  $\lambda \in \Lambda_0$  with  $a \leq 2Nr$ . For each  $g \in G$ , the natural map (restriction of the action map)  $\Pi_{N,\lambda}^a(L) \rightarrow \Pi_{N,\lambda}^a(gL)$  is an isomorphism of projective varieties.*

*Proof.* If  $L' \in \Pi_{N,\lambda}^a(L)$ , then  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$  and so  $\varepsilon^N g(L_0) \subset g(L'_0) \subset \varepsilon^{-N}g(L_0)$ , so  $gL' \in \Pi_{N,\lambda}^a(gL)$ . Thus  $g$  and  $g^{-1}$  induce mutually inverse morphisms of varieties  $g: \Pi_{N,\lambda}^a(L) \rightarrow \Pi_{N,\lambda}^a(gL)$  and  $g^{-1}: \Pi_{N,\lambda}^a(gL) \rightarrow \Pi_{N,\lambda}^a(L)$ .  $\square$

### 5.4.1 Action through an algebraic group

Let  $W$  be the  $\mathbb{C}[\varepsilon]$ -module  $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$ .  $\varepsilon^{2N+1}$  acts as zero on  $W$  and  $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle \otimes_{\mathbb{C}[\varepsilon]} W$  is a free  $\mathbb{C}[\varepsilon]/\langle \varepsilon^{2N+1} \rangle$ -module of rank  $r$ . In particular,  $W$  is a complex vector space of dimension  $(2N+1)r$ .

Each element  $g \in G_L$  determines an endomorphism  $\bar{g}$  of  $W$ , given by

$$\bar{g}(x + \varepsilon^N L_0) = g(x) + \varepsilon^N L_0,$$

for each  $x \in \varepsilon^{-(1+N)}L_0$ . Given  $g, h \in G_L$ ,  $\overline{gh} = \bar{g}\bar{h}$  and so  $\bar{g}$  is an automorphism of  $W$  with  $\bar{g}^{-1} = \overline{g^{-1}}$ . Therefore the map  $\bar{\cdot}: G_L \mapsto \text{GL}(W)$  given by  $g \mapsto \bar{g}$  is a group homomorphism with kernel

$$H_{N,L} := \{g \in G_L : \bar{g} = 1\},$$

which consists of those  $g \in G_L$  such that

$$g(x) - x \in \varepsilon^N L_0$$

for each  $x \in \varepsilon^{-(1+N)}L_0$ . Thus  $G_L/H_{N,L}$  may be identified with a subgroup of  $\text{GL}(W)$ .

**Lemma 5.4.6.**  $G_L/H_{N,L}$  is a connected algebraic group.

*Proof.* As a result of the first isomorphism theorem,  $G_L/H_{N,L}$  is isomorphic to the image of  $G_L$  in  $\mathrm{GL}(W)$ , which will be described explicitly by equations in the coordinate functions on  $\mathrm{GL}(W)$ , with respect to a fixed basis of  $W$ .

Let  $\{\tilde{x}_1, \dots, \tilde{x}_r\}$  be a basis of  $L_n/L_0$  over  $\mathbb{C}$  which is adapted to the flag

$$L_1/L_0 \subset \dots \subset L_{n-1}/L_0 \subset L_n/L_0,$$

so that

$$L_i/L_0 = \langle \tilde{x}_1, \dots, \tilde{x}_{\lambda_1 + \dots + \lambda_i} \rangle$$

for each  $i \in \{1, \dots, n\}$ . Fix  $x_1, \dots, x_r \in L_n$  such that  $\tilde{x}_i = x_i + L_0$  for each  $i = 1, \dots, r$ , then

$$L_i = L_0 + \langle x_1, \dots, x_{\lambda_1 + \dots + \lambda_i} \rangle$$

for  $i = 1, \dots, r$ .

Then  $W$  has a  $\mathbb{C}$ -basis  $\{y_j : 1 \leq j \leq (2N+1)r\}$  given by

$$y_{i+cr} = \varepsilon^{-c+N} x_i$$

for each  $i \in \{1, \dots, r\}$  and  $c \in \{0, \dots, 2N\}$ . Observe that  $\varepsilon y_i = 0$  for  $i \in \{1, \dots, r\}$  and  $\varepsilon y_i = y_{i-r}$  for  $r < i \leq (2N+1)r$ .

The coordinate functions on  $\mathrm{GL}(W)$  with respect to this choice of basis are the maps

$$\gamma_{i,j} : \mathrm{GL}(W) \rightarrow \mathbb{C}$$

for  $i, j \in \mathbb{Z}$  with  $1 \leq i, j \leq (2N+1)r$ , given by

$$g(y_j) = \sum_i \gamma_{ij}(g) y_i,$$

for each  $j = 1, \dots, (2N+1)r$ .

The image of  $G_L$  in  $\mathrm{GL}(W)$  is the subgroup defined by the conditions

$$\gamma_{i,j} = \gamma_{i-r,j-r}$$

for each  $i, j \in \{r+1, \dots, (2N+1)r\}$  and

$$\gamma_{i,j} = 0$$

for each  $i, j \in \{1, \dots, (2N+1)r\}$  with  $i > \lambda_1 + \dots + \lambda_s$  and  $j \leq \lambda_1 + \dots + \lambda_s$  for some  $s \in \{1, \dots, r\}$ . This shows that the image of  $G_L$  in  $\mathrm{GL}(W)$  is a connected algebraic group and therefore  $G_L/H_{N,L}$  is a connected algebraic group.

With respect to the basis  $\{y_i : i \in \{1, \dots, (2N+1)r\}\}$ , the image of  $G_L$  in  $\mathrm{GL}(W)$  consists of matrices of the form

$$\begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{2N} \\ 0 & A_0 & A_1 & \cdots & A_{2N-1} \\ 0 & 0 & A_0 & \cdots & A_{2N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_0 \end{pmatrix}$$

where  $A_0 \in \mathcal{P}_\lambda$  and  $A_1, \dots, A_{2N} \in M_r(\mathbb{C})$ , where  $\mathcal{P}_\lambda$  is the parabolic subgroup of  $\mathrm{GL}_r(\mathbb{C})$  which is the stabiliser of the flag

$$L_1/L_0 \subset \dots \subset L_{n-1}/L_0 \subset L_n/L_0.$$

□

Given  $g \in G$ , the map  $G_L \rightarrow G_{gL}$  sending  $h$  to  $ghg^{-1}$  is a group isomorphism which descends to an isomorphism of algebraic groups  $G_L/H_{N,L} \rightarrow G_{gL}/H_{N,gL}$ . Thus we have a commuting diagram of morphisms of varieties, where the vertical arrows are isomorphisms:

$$\begin{array}{ccc} G_L/H_{N,L} \times \Pi_{N,\lambda}^a(L) & \longrightarrow & \Pi_{N,\lambda}^a(L) \\ \downarrow & & \downarrow \\ G_{gL}/H_{N,gL} \times \Pi_{N,\lambda}^a(gL) & \longrightarrow & \Pi_{N,\lambda}^a(gL) \end{array}$$

#### 5.4.2 Incidence in affine flag varieties

**Lemma 5.4.7.** *Given  $N, a, b, c \in \mathbb{N}$ ,  $\lambda, \mu \in \Lambda_0$ ,  $L \in \mathcal{F}$  and  $i, j \in \mathbb{Z}$ ,*

$$\left\{ (L', L'') \in \Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L) : \dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) \leq c \right\}$$

*is a closed set in the projective variety  $\Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L)$ .*

*Proof.* There is  $M \geq N$  so that  $\varepsilon^M L_0 \subset L'_i \subset \varepsilon^{-M} L_0$  and  $\varepsilon^M L_0 \subset L''_j \subset \varepsilon^{-M} L_0$ . Let  $a' = a + (M - N)r$  and  $b' = b + (M - N)r$ . Lemma 5.4.3 shows that  $\Pi_{N,\lambda}^a(L)$  is a subvariety of  $\Pi_{M,\lambda}^{a'}(L)$ , so  $\Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L)$  is a subvariety of  $\Pi_{M,\lambda}^{a'}(L) \times \Pi_{M,\mu}^{b'}(L)$ .

The fact that

$$\dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) = \dim \left( \frac{L'_i/\varepsilon^M L_0}{L'_i/\varepsilon^M L_0 \cap L''_j/\varepsilon^M L_0} \right),$$

together with Lemma 5.4.2 and Lemma 5.3.1, shows that

$$\left\{ (L', L'') \in \Pi_{M,\lambda}^{a'}(L) \times \Pi_{M,\mu}^{b'}(L) : \dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) \leq c \right\}$$

is closed, so the intersection with  $\Pi_{N,\lambda}^a(L) \times \Pi_{N,\mu}^b(L)$  is closed. □

**Lemma 5.4.8.** *Given  $N, a, c \in \mathbb{N}$ ,  $\lambda \in \Lambda_0$ ,  $L \in \mathcal{F}$  and  $i, j \in \mathbb{Z}$ ,*

$$\left\{ L' \in \Pi_{N,\lambda}^a(L) : \dim \left( \frac{L_i}{L_i \cap L'_j} \right) \leq c \right\}$$

*and*

$$\left\{ L' \in \Pi_{N,\lambda}^a(L) : \dim \left( \frac{L'_j}{L_i \cap L'_j} \right) \leq c \right\}$$

*are closed sets in  $\Pi_{N,\lambda}^a(L)$ .*

*Proof.* This is a result of Lemma 5.3.2, since

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) = \dim \left( \frac{L_i/\varepsilon^M L_0}{L_i/\varepsilon^M L_0 \cap L'_j/\varepsilon^M L_0} \right),$$

where  $M \geq N$  is chosen so that  $\varepsilon^M L_0 \subset L_i \subset \varepsilon^{-M} L_0$  and  $\varepsilon^M L_0 \subset L'_j \subset \varepsilon^{-M} L_0$  for each  $L' \in \Pi_{N,\lambda}^a(L)$ . □



## 5.5 Geometry of orbits

Let  $A \in \Lambda_1$  and  $L \in \mathcal{F}_{\text{ro}(A)}$  and write  $\lambda = \text{co}(A)$ . Recall that

$$X_A^L = \{L' \in \mathcal{F}_\lambda : (L, L') \in \mathcal{O}_A\}.$$

**Lemma 5.5.1.** *There is  $N \in \mathbb{N}$  such that  $X_A^L \subset \Pi_{N,\lambda}^a(L)$ , where  $a = d_{nN,0}A$ .*

*Proof.* There is  $N \in \mathbb{N}$  so that  $a_{i,j} = 0$  whenever  $|j - i| > nN$ . If  $(L, L') \in \mathcal{O}_A$  then

$$\dim \left( \frac{L'_0}{L'_0 \cap \varepsilon^{-N} L_0} \right) = \dim \left( \frac{L'_0}{L'_0 \cap L_{nN}} \right) = \sum_{s > nN, t \leq 0} a_{s,t} = 0,$$

so it follows  $L'_0 \subset \varepsilon^{-N} L_0$ . Similarly,

$$\dim \left( \frac{\varepsilon^N L_0}{\varepsilon^N L_0 \cap L'_0} \right) = \dim \left( \frac{L_{-nN}}{L_{-nN} \cap L'_0} \right) = \sum_{s \leq -nN, t > 0} a_{s,t} = 0,$$

which shows  $\varepsilon^N L_0 \subset L'_0$ . Moreover,

$$\dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) = \dim \left( \frac{\varepsilon^{-N} L_0}{\varepsilon^{-N} L_0 \cap L'_0} \right) = \sum_{s \leq nN, t > 0} a_{s,t} = d_{nN,0}(A),$$

as a result of Lemma 5.2.5. □

Assume  $N \in \mathbb{N}$  is chosen so that  $X_A^L \subset \Pi_{N,\lambda}^a(L)$ , where  $a = d_{nN,0}A$ , as in Lemma 5.5.1.

**Lemma 5.5.2.**  *$X_A^L$  is a locally closed subset of  $\Pi_{N,\lambda}^a(L)$ . In particular,  $X_A^L$  is a quasiprojective variety.*

*Proof.* If  $L' \in \Pi_{N,\lambda}^a(L)$  then

$$L_{-Nn} = \varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset L'_n \subset \varepsilon^{-1-N} L_0 = L_{(N+1)n}.$$

Therefore  $X_A^L$  is the subset of  $\Pi_{N,\lambda}^a(L)$  defined by the conditions  $\dim(L_i/L_i \cap L'_j) = d_{i,j}A$  for  $i : -Nn \leq i < j$  and  $\dim(L'_j/L_i \cap L'_j) = \bar{d}_{i,j}A$  for  $i : j < i \leq (N+1)n$ , for  $j = 1, \dots, n$ .

The set of  $L' \in \Pi_{N,\lambda}^a(L)$  with  $\dim(L_i/L_i \cap L'_j) \leq d_{i,j}A$  for  $j = 1, \dots, n$  and  $i : -Nn \leq i < j$  and  $\dim(L'_j/L_i \cap L'_j) \leq \bar{d}_{i,j}A$  for  $j = 1, \dots, n$  and  $i : j < i \leq (N+1)n$  is a closed subset of  $\Pi_{N,\lambda}^a(L)$ , as a result of Lemma 5.4.8.

On the other hand, the set of  $L' \in \Pi_{N,\lambda}^a(L)$  satisfying the conditions  $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}A$  (for  $i < j$ ) and  $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}A$  (for  $i > j$ ) is open in  $\Pi_{N,\lambda}^a(L)$  since the complement is closed, as a result of Lemma 5.4.8.

Therefore  $X_A^L$  is the intersection of an open set and a closed set in  $\Pi_{N,\lambda}^a(L)$ , so  $X_A^L$  is locally closed. It follows that  $X_A^L$  is an open subset of the projective variety  $\overline{X_A^L}$ , so is a quasiprojective variety as claimed. □

**Lemma 5.5.3.**  *$X_A^L$  is irreducible.*

*Proof.* For any  $L' \in X_A^L$ ,  $X_A^L = G_L/H_{N,L} \cdot L'$ . Lemma 5.4.6 shows that  $G_L/H_{N,L}$  is a connected algebraic group which acts algebraically on  $\Pi_{N,\lambda}^a(L)$ . The image of  $G_L/H_{N,L}$  under the morphism  $g \mapsto gL'$  equals  $X_A^L$ , which shows  $X_A^L$  is irreducible since  $G_L/H_{N,L}$  is irreducible. □

Consequently,  $\overline{X_A^L}$  is an irreducible projective variety and the action of  $G_L/H_{N,L}$  on  $\Pi_{N,\lambda}^a(L)$  restricts to an algebraic group action on  $\overline{X_A^L}$  for which there are finitely many orbits. In particular,  $\overline{X_A^L} \setminus X_A^L$  is a union of finitely many orbits which are so-called degenerations of the orbit  $X_A^L$ .

## 5.6 Geometry of orbit products

Let  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$  and write  $\lambda = \text{co}(A)$  and  $\mu = \text{co}(B)$ . Fix  $L \in \mathcal{F}_{\text{ro}(A)}$ . Recall

$$Y_{A,B}^L = \{(L', L'') \in \mathcal{F}_\lambda \times \mathcal{F}_\mu : L' \in X_A^L, L'' \in X_B^{L'}\}$$

and

$$X_{A,B}^L = \{L'' \in \mathcal{F}_\mu : \exists L' \in X_A^L \text{ with } L'' \in X_B^{L'}\}$$

**Lemma 5.6.1.** *There is  $N \in \mathbb{N}$  such that*

$$Y_{A,B}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L),$$

where  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ .

*Proof.* There is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$  and  $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$  for each  $(L', L'') \in Y_{A,B}^L$ , using Lemma 5.5.1. Set  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ .

Then for any  $(L', L'') \in Y_{A,B}^L$ ,

$$\varepsilon^{2N} L_0 \subset \varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0 \subset \varepsilon^{-2N} L_0$$

and

$$\begin{aligned} \dim \left( \frac{\varepsilon^{-2N} L_0}{L''_0} \right) &= \dim \left( \frac{\varepsilon^{-N} L'_0}{L''_0} \right) + \dim \left( \frac{\varepsilon^{-2N} L_0}{\varepsilon^{-N} L'_0} \right) \\ &= \dim \left( \frac{\varepsilon^{-N} L'_0}{L''_0} \right) + \dim \left( \frac{\varepsilon^{-N} L_0}{L'_0} \right) \\ &= a + b, \end{aligned}$$

as a result of Lemma 5.2.5, so  $(L', L'') \in \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  as required.  $\square$

Now assume  $N \in \mathbb{N}$  is chosen so that  $Y_{A,B}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ , where  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ , using Lemma 5.6.1.

**Lemma 5.6.2.**  *$Y_{A,B}^L$  is a locally closed subset of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ . In particular,  $Y_{A,B}^L$  is a quasiprojective variety.*

*Proof.*  $Y_{A,B}^L$  is the subset of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  consisting of those  $(L', L'')$  satisfying the following conditions:  $\dim(L_i/L_i \cap L'_j) = d_{i,j}(A)$  for  $i < j$ ,  $\dim(L'_j/L_i \cap L'_j) = \bar{d}_{i,j}(A)$  for  $i > j$ ,  $\dim(L'_i/L'_i \cap L''_j) = d_{i,j}(B)$  for  $i < j$  and  $\dim(L''_j/L'_i \cap L''_j) = \bar{d}_{i,j}(B)$ . Only finitely many conditions are required to define  $Y_{A,B}^L$  since there are only finitely many nonzero entries in  $A$  and  $B$  modulo the  $(n, n)$ -periodicity.

The conditions  $\dim(L_i/L_i \cap L'_j) \leq d_{i,j}(A)$ ,  $\dim(L'_j/L'_i \cap L''_j) \leq d_{i,j}(B)$ ,  $\dim(L'_j/L_i \cap L'_j) \leq \bar{d}_{i,j}(A)$  and  $\dim(L''_j/L'_i \cap L''_j) \leq \bar{d}_{i,j}(B)$  define closed subsets of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  for each  $i, j \in \mathbb{Z}$ , as a result of Lemma 5.4.7 and Lemma 5.4.8.

On the other hand, the conditions  $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}(A)$ ,  $\dim(L'_j/L'_i \cap L''_j) \geq d_{i,j}(B)$ ,  $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}(A)$  and  $\dim(L''_j/L'_i \cap L''_j) \geq \bar{d}_{i,j}(B)$  define open subsets of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  for each  $i, j \in \mathbb{Z}$ , using Lemma 5.4.7 and Lemma 5.4.8.

Therefore  $Y_{A,B}^L$  is the intersection of finitely many open sets and finitely many closed sets in  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ , so  $Y_{A,B}^L$  is locally closed. In particular,  $Y_{A,B}^L$  is a quasiprojective variety.  $\square$

**Lemma 5.6.3.** *For any  $L' \in X_A^L$ ,  $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$ .*

*Proof.* Let  $L' \in X_A^L$ , then  $\{L'\} \times X_B^{L'}$  is contained in  $Y_{A,B}^L$  and  $G_L$  acts on  $Y_{A,B}^L$ , so  $G_L \cdot (\{L'\} \times X_B^{L'})$  is contained in  $Y_{A,B}^L$ . If  $(N', N'') \in Y_{A,B}^L$ , then  $N' = \sigma L'$  for some  $\sigma \in G_L$ , since  $N' \in X_A^L$ . Then  $(N', N'') = \sigma(L', \sigma^{-1}N'')$  and  $\sigma^{-1}N'' \in X_B^{\sigma^{-1}N'} = X_B^{L'}$ , so  $(N', N'') \in \sigma \cdot (\{L'\} \times X_B^{L'})$ . Therefore  $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$  as claimed.  $\square$

**Proposition 5.6.4.**  *$Y_{A,B}^L$  is irreducible.*

*Proof.* Let  $L' \in X_A^L$ .  $G_L/H_{2N,L}$  is a connected algebraic group acting algebraically on  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$  by Lemma 5.4.6.  $X_B^{L'}$  is an irreducible locally closed subset of  $\Pi_{2N,\mu}^{a+b}(L)$ , so  $\{L'\} \times X_B^{L'}$  is an irreducible locally closed set in  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ .  $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'}) = G_L/H_{2N,L} \cdot (\{L'\} \times X_B^{L'})$ , by Lemma 5.6.3, so it follows that  $Y_{A,B}^L$  is irreducible.  $\square$

Let  $p_2$  be the projection onto the second factor  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \rightarrow \Pi_{2N,\mu}^{a+b}(L)$ .  $p_2$  is a closed morphism since  $\Pi_{N,\lambda}^a(L)$  is a projective variety and therefore complete, by Lemma 5.4.2. Therefore  $p_2(\overline{Y_{A,B}^L}) = \overline{X_{A,B}^L}$ , since  $p_2(Y_{A,B}^L) = X_{A,B}^L$ .

**Lemma 5.6.5.**  *$X_{A,B}^L$  is irreducible and constructible.*

*Proof.* Proposition 5.6.4 shows that  $Y_{A,B}^L$  is irreducible and locally closed, so it follows  $X_{A,B}^L$  is irreducible and constructible, since  $X_{A,B}^L = p_2(Y_{A,B}^L)$ .  $\square$

**Proposition 5.6.6.** *There is a unique open  $G_L$ -orbit in  $X_{A,B}^L$ .*

*Proof.*  $X_{A,B}^L$  consists of finitely many  $G_L$ -orbits and is an irreducible topological space, by Lemma 5.6.5. Consequently,  $X_C^L$  is dense in  $X_{A,B}^L$  for some  $C \in \Lambda_1^{A,B}$ . Lemma 5.5.2 shows that  $X_C^L$  is locally closed in  $X_{A,B}^L$ , so  $X_C^L$  is open in  $\overline{X_C^L} = X_{A,B}^L$ . Irreducibility of  $X_{A,B}^L$  shows that there is a unique open  $G_L$ -orbit, since two nonempty open sets in  $X_{A,B}^L$  intersect nontrivially, thus any two open  $G_L$  orbits in  $X_{A,B}^L$  coincide.  $\square$

Let  $A * B \in \Lambda_1$  be the matrix corresponding to the dense open  $G_L$ -orbit in  $X_{A,B}^L$ , so  $\overline{X_{A*B}^L} = \overline{X_{A,B}^L}$ .

## 5.7 Degenerations of orbits and the combinatorial partial order

**Theorem 5.7.1.** *Let  $A, B \in \Lambda_1$  with  $\text{ro}(A) = \text{ro}(B)$  and  $\text{co}(A) = \text{co}(B)$ , then  $B \leq A$  if and only if  $X_B^L \subset \overline{X_A^L}$  for any  $L \in \mathcal{F}_{\text{ro}(A)}$ .*

*Proof.* Let  $\lambda = \text{co}(A)$ ,  $\mu = \text{ro}(A)$  and fix  $L \in \mathcal{F}_\mu$ . Assume  $N \in \mathbb{N}$  is chosen so that  $X_A^L \subset \Pi_{N,\lambda}^a(L)$  and  $X_B^L \subset \Pi_{N,\lambda}^b(L)$ , where  $a = d_{nN,0}(A)$  and  $b = d_{nN,0}(B)$ . Then  $X_A^L$  is an open subset of the projective variety consisting of those  $L' \in \Pi_{N,\lambda}^a(L)$  such that

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) \leq d_{i,j}(A)$$

and

$$\dim \left( \frac{L'_j}{L_i \cap L'_j} \right) \leq \bar{d}_{i,j}(A),$$

for all  $i, j \in \mathbb{Z}$ .

Assume  $X_B^L \subset \overline{X_A^L}$ , then

$$d_{i,j}(B) = \dim \left( \frac{L_i}{L_i \cap L'_j} \right) \leq d_{i,j}(A)$$

and

$$\bar{d}_{i,j}(B) = \dim \left( \frac{L'_j}{L_i \cap L'_j} \right) \leq \bar{d}_{i,j}(A),$$

for each  $i, j \in \mathbb{Z}$ , for any  $L' \in X_B^L$ . So  $B \leq A$  if  $X_B^L \leq \overline{X_A^L}$ .

**Conversely, suppose  $A \leq B$ .**

□

**Corollary 5.7.2.** *The maximum in  $\Lambda_1^{A,B}$  is  $A * B$ .*

## 5.8 Associativity of the generic product

Let  $A, B, C \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$  and  $\text{co}(B) = \text{ro}(C)$  and fix  $L \in \mathcal{F}_{\text{ro}(A)}$ . Write  $\lambda = \text{co}(A)$ ,  $\mu = \text{co}(B)$  and  $\nu = \text{co}(C)$ . Define

$$Y_{A,B,C}^L = \left\{ (L', L'', L''') \in \mathcal{F}^3 : L' \in X_A^L, L'' \in X_B^L, L''' \in X_C^L \right\}$$

and

$$X_{A,B,C}^L = \left\{ L''' \in \mathcal{F} : \exists (L', L'') \in \mathcal{F}^2 \text{ with } (L', L'', L''') \in Y_{A,B,C}^L \right\}.$$

**Lemma 5.8.1.** *There is  $N \in \mathbb{N}$  such that  $Y_{A,B,C}^L$  is contained in  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$ , where  $a = d_{nN,0}(A)$ ,  $b = d_{nN,0}(B)$  and  $c = d_{nN,0}(C)$ .*

*Proof.* Lemma 5.5.1 shows that there is  $N \in \mathbb{N}$  such that  $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ ,  $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$  and  $\varepsilon^N L''_0 \subset L'''_0 \subset \varepsilon^{-N} L''_0$  for each  $(L', L'', L''') \in Y_{A,B,C}^L$ . Using the proof of Lemma 5.6.1, it follows  $L'' \in \Pi_{2N,\mu}^{a+b}(L)$  and  $L''' \in \Pi_{2N,\nu}^{b+c}(L') \subset \Pi_{3N,\nu}^{a+b+c}(L)$ . □

Assume  $N \in \mathbb{N}$  is chosen so that  $Y_{A,B,C}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$ , where  $a = d_{nN,0}(A)$ ,  $b = d_{nN,0}(B)$  and  $c = d_{nN,0}(C)$ , as in Lemma 5.8.1.

**Lemma 5.8.2.**  *$Y_{A,B,C}^L$  is a locally closed subset of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$ . In particular,  $Y_{A,B,C}^L$  is a quasiprojective variety.*

*Proof.* Write  $\Pi = \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}(L)$ . Then  $Y_{A,B,C}^L$  consists of those  $(L', L'', L''') \in \Pi$  satisfying the following conditions:

$$\dim \left( \frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}(A), \quad (5.1)$$

$$\dim \left( \frac{L'_i}{L'_i \cap L''_j} \right) = d_{i,j}(B), \quad (5.2)$$

$$\dim \left( \frac{L''_i}{L''_i \cap L'''_j} \right) = d_{i,j}(C), \quad (5.3)$$

for  $(i, j) \in [1, n] \times \mathbb{Z}$  with  $i < j < (N+1)n$ , and

$$\dim \left( \frac{L'_j}{L_i \cap L'_j} \right) = \bar{d}_{i,j}(A), \quad (5.4)$$

$$\dim \left( \frac{L''_j}{L'_i \cap L''_j} \right) = \bar{d}_{i,j}(B), \quad (5.5)$$

$$\dim \left( \frac{L'''_j}{L''_i \cap L'''_j} \right) = \bar{d}_{i,j}(C), \quad (5.6)$$

for  $(i, j) \in [1, n] \times \mathbb{Z}$  with  $-Nn < j < i$ .

For  $i < j$ , the conditions

$$\begin{aligned} \dim(L_i/L_i \cap L'_j) &\leq d_{i,j}(A), \\ \dim(L'_i/L'_i \cap L''_j) &\leq d_{i,j}(B) \end{aligned}$$

and

$$\dim(L''_i/L''_i \cap L'''_j) \leq d_{i,j}(C)$$

define closed subsets of  $\Pi$ , by Lemma 5.4.7. For  $i > j$ , the conditions

$$\begin{aligned} \dim(L'_j/L_i \cap L'_j) &\leq \bar{d}_{i,j}(A), \\ \dim(L''_j/L'_i \cap L''_j) &\leq \bar{d}_{i,j}(B) \end{aligned}$$

and

$$\dim(L'''_j/L''_i \cap L'''_j) \leq \bar{d}_{i,j}(C)$$

also define closed subsets of  $\Pi$ .

On the other hand, the conditions  $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}(A)$ ,  $\dim(L'_i/L'_i \cap L''_j) \geq d_{i,j}(B)$  and  $\dim(L''_i/L''_i \cap L'''_j) \geq d_{i,j}(C)$  for  $i < j$  define open subsets of  $\Pi$ . Similarly, the conditions  $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}(A)$ ,  $\dim(L''_j/L'_i \cap L''_j) \geq \bar{d}_{i,j}(B)$  and  $\dim(L'''_j/L''_i \cap L'''_j) \geq \bar{d}_{i,j}(C)$  for  $i > j$  define open subsets of  $\Pi$ .

Therefore  $Y_{A,B,C}^L$  is the intersection of finitely many closed sets in  $\Pi$  with finitely many open subsets of  $\Pi$ , so  $Y_{A,B,C}^L$  is locally closed. In particular,  $Y_{A,B,C}^L$  is a quasiprojective variety.  $\square$

**Lemma 5.8.3.** For any  $(L', L'', L''') \in Y_{A,B,C}^L$ ,

$$Y_{A,B,C}^L = \{ \alpha \cdot (L', \beta L'', \beta \gamma L''') : \alpha \in G_L, \beta \in G_{L'}, \gamma \in G_{L''} \}.$$

In particular,

$$Y_{A,B,C}^L = G_L \cdot (\{L'\} \times Y_{B,C}^{L'})$$

for each  $L' \in X_A^L$ .

*Proof.* Let  $(L', L'', L''') \in Y_{A,B,C}^L$ . Given  $\alpha \in G_L$ ,  $\beta \in G_{L'}$  and  $\gamma \in G_{L''}$ ,  $(\alpha L', \alpha \beta L'', \alpha \beta \gamma L''')$  is in  $Y_{A,B,C}^L$  since

$$\begin{aligned} (L, \alpha L') &= \alpha(L, L') \in \mathcal{O}_A \\ (\alpha L', \alpha \beta L'') &= \alpha \beta(L', L'') \in \mathcal{O}_B \\ (\alpha \beta L'', \alpha \beta \gamma L''') &= \alpha \beta \gamma(L'', L''') \in \mathcal{O}_C \end{aligned}$$

For each  $(N', N'', N''') \in Y_{A,B,C}^L$  there exist  $\sigma_1, \sigma_2, \sigma_3 \in G$  with

$$\begin{aligned} (L, N') &= \sigma_1(L, L') \\ (N', N'') &= \sigma_2(L', L'') \\ (N'', N''') &= \sigma_3(L'', L'''). \end{aligned}$$

Let  $\alpha = \sigma_1$ ,  $\beta = \sigma_1^{-1} \sigma_2$  and  $\gamma = \sigma_2^{-1} \sigma_3$ , so  $\sigma_2 = \alpha \beta$  and  $\sigma_3 = \alpha \beta \gamma$ . It follows that

$$(N', N'', N''') = (\alpha L', \alpha \beta L'', \alpha \beta \gamma L'''),$$

which proves the first claim. The second claim follows from the first since  $(L'', L''') \in Y_{B,C}^{L'}$  and therefore

$$Y_{B,C}^{L'} = \{(\beta L'', \beta \gamma L''') : \beta \in G_{L'}, \gamma \in G_{L''}\},$$

as required.  $\square$

**Proposition 5.8.4.**  $Y_{A,B,C}^L$  is irreducible.

*Proof.* Write

$$\Pi = \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L).$$

Lemma 5.4.2 shows that  $\Pi$  is a projective algebraic variety and Lemma 5.4.6 shows that  $G_L/H_{3N,L}$  is a connected algebraic group acting algebraically on  $\Pi$  by the diagonal action.

Let  $L' \in X_A^L$ . As a result of Lemma 5.8.3

$$\begin{aligned} Y_{A,B,C}^L &= G_L \cdot (\{L'\} \times Y_{B,C}^{L'}) \\ &= G_L/H_{3N,L} \cdot (\{L'\} \times Y_{B,C}^{L'}). \end{aligned}$$

Proposition 5.6.4 shows that  $Y_{B,C}^{L'}$  is irreducible, so  $\{L'\} \times Y_{B,C}^{L'}$  is irreducible. The image of  $\{L'\} \times Y_{B,C}^{L'}$  under the action of  $G_L/H_{3N,L}$  is irreducible, since  $G_L/H_{3N,L}$  is connected and therefore irreducible. Therefore  $Y_{A,B,C}^L$  is irreducible.  $\square$

Let  $p_3$  be the projection of  $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$  onto the third factor. By the completeness property of projective varieties,  $p_3$  is a closed morphism. The image of  $Y_{A,B,C}^L$  under  $p_3$  is  $X_{A,B,C}^L$ , so  $p_3(\overline{Y_{A,B,C}^L}) = \overline{X_{A,B,C}^L}$ .

**Lemma 5.8.5.**  $X_{A,B,C}^L$  is irreducible and constructible.

*Proof.* Lemma 5.8.2 and Proposition 5.8.4 show that  $Y_{A,B,C}^L$  is locally closed and irreducible. It follows  $X_{A,B}^L$  is irreducible and constructible, since  $X_{A,B,C}^L$  is the image of  $Y_{A,B,C}^L$  under the morphism  $p_3$ .  $\square$

**Lemma 5.8.6.** There is a unique open and dense  $G_L$ -orbit in  $X_{A,B,C}^L$ .

*Proof.* There are only finitely many  $G_L$ -orbits in  $X_{A,B,C}^L$ . In particular,

$$X_{A,B,C}^L = \bigcup_{D \in \Lambda_1^{A,B}} X_{D,C}^L = \bigcup_{D \in \Lambda_1^{A,B}} \bigcup_{D' \in \Lambda_1^{D,C}} X_{D'}^L$$

and

$$\overline{X_{A,B,C}^L} = \bigcup_{D \in \Lambda_1^{A,B}} \bigcup_{D' \in \Lambda_1^{D,C}} \overline{X_{D'}^L}.$$

There is  $D \in \Lambda_1$  such that  $\overline{X_D^L} = \overline{X_{A,B,C}^L}$ , since  $X_{A,B,C}^L$  is irreducible, by Lemma 5.8.5. By Lemma 5.5.2,  $X_D^L$  is open in  $\overline{X_D^L} = \overline{X_{A,B,C}^L}$ , so  $X_D^L$  is open in  $X_{A,B,C}^L$ .

If  $X_D^L$  and  $X_{D'}^L$  are open in  $X_{A,B,C}^L$ , then  $X_D^L$  and  $X_{D'}^L$  have nonempty intersection since  $X_{A,B,C}^L$  is irreducible, then  $X_D^L = X_{D'}^L$ .  $\square$

**Lemma 5.8.7.**  $p_3^{-1}(X_{A*B,C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ .

*Proof.* Projection onto the second component is a closed morphism of varieties  $p_2: \overline{Y_{A,B,C}^L} \rightarrow \overline{X_{A,B}^L}$  with  $p_2(Y_{A,B,C}^L) = X_{A,B}^L$ . It follows that  $p_3^{-1}(X_{A*B,C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$  since  $p_3^{-1}(X_{A*B,C}^L) = p_2^{-1}(X_{A*B}^L)$  and  $X_{A*B}^L$  is open in  $\overline{X_{A,B}^L}$ .  $\square$

**Lemma 5.8.8.**  $p_3^{-1}(X_{A,B*C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ .

*Proof.*  $p_3^{-1}(X_{A,B*C}^L)$  consists of those  $(L', L'', L''') \in \overline{Y_{A,B,C}^L}$  such that  $\dim(L'_i/L'_i \cap L_j''') \geq d_{i,j}(B * C)$  for  $i < j$  and  $\dim(L_j'''/L'_i \cap L_j''') \geq \bar{d}_{i,j}(B * C)$  for  $i > j$ . Each of these conditions defines an open subset of  $\overline{Y_{A,B,C}^L}$  as a result of Lemma 5.4.7 and only finitely many conditions are required to determine  $p_3^{-1}(X_{A,B*C}^L)$ , as before. Therefore  $p_3^{-1}(X_{A,B*C}^L)$  is the intersection of finitely many open sets in  $\overline{Y_{A,B,C}^L}$ , so is open as claimed.  $\square$

**Proposition 5.8.9.**  $X_{A*(B*C)}^L = X_{(A*B)*C}^L$ .

*Proof.* The unique open  $G_L$ -orbit in  $X_{A*B,C}^L$  is  $X_{(A*B)*C}^L$ , so  $p_3^{-1}(X_{(A*B)*C}^L)$  is open in  $p_3^{-1}(X_{A*B,C}^L)$ . Lemma 5.8.7 shows that  $p_3^{-1}(X_{A*B,C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ , so  $p_3^{-1}(X_{(A*B)*C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ .

Similarly,  $X_{A*(B*C)}^L$  is open in  $X_{A,B*C}^L$ , so  $p_3^{-1}(X_{A*(B*C)}^L)$  is open in  $p_3^{-1}(X_{A,B*C}^L)$ . Lemma 5.8.8 shows that  $p_3^{-1}(X_{A,B*C}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ , so it follows  $p_3^{-1}(X_{A*(B*C)}^L)$  is open in  $\overline{Y_{A,B,C}^L}$ .

Therefore  $f^{-1}(X_{A*(B*C)}^L)$  has nonempty intersection with  $f^{-1}(X_{(A*B)*C}^L)$ , since  $Y_{A,B,C}^L$  is irreducible by Proposition 5.8.4. It follows that the  $G_L$ -orbits  $X_{A*(B*C)}^L$  and  $X_{(A*B)*C}^L$  have nonempty intersection and therefore  $X_{A*(B*C)}^L$  equals  $X_{(A*B)*C}^L$ .  $\square$

## 5.9 The generic affine algebra

The generic affine algebra of rank  $r$  and period  $n$ , denoted by  $\hat{G}(n, r)$ , is a free  $\mathbb{Z}$ -module with basis  $\{e_A : A \in \Lambda_1\}$  and  $\mathbb{Z}$ -bilinear multiplication given by

$$e_A * e_B = e_{A*B}$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$ , and

$$e_A * e_B = 0$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) \neq \text{ro}(B)$ .

**Proposition 5.9.1.** *The generic algebra  $\hat{G}(n, r)$  is an associative  $\mathbb{Z}$ -algebra with 1, with*

$$1 = \sum_{\lambda \in \Lambda_0} 1_\lambda$$

where

$$1_\lambda = e_{D_\lambda},$$

for each  $\lambda \in \Lambda_0$ .

*Proof.* Let  $A, B, C \in \Lambda_1$ . If  $\text{co}(A) \neq \text{ro}(B)$  or  $\text{co}(B) \neq \text{ro}(C)$ , then

$$(e_A * e_B) * e_C = 0 = e_A * (e_B * e_C),$$

so we may now suppose  $\text{co}(A) = \text{ro}(B)$  and  $\text{co}(B) = \text{ro}(C)$ .

As a result of Proposition 5.8.9,

$$\begin{aligned} (e_A * e_B) * e_C &= e_{(A*B)*C} \\ &= e_{A*(B*C)} \\ &= e_A * (e_B * e_C), \end{aligned}$$

so it follows  $\hat{G}(n, r)$  is an associative  $\mathbb{Z}$ -algebra.

The expression for the multiplicative identity follows from Lemma 3.1.7, since

$$e_A * \left( \sum_{\lambda \in \Lambda_0} 1_\lambda \right) = e_A * 1_{\text{co}(A)} = e_A$$

and

$$\left( \sum_{\lambda \in \Lambda_0} 1_\lambda \right) * e_A = 1_{\text{ro}(A)} * e_A = e_A,$$

for each  $A \in \Lambda_1$ . □



### 5.9.1 A categorical perspective

**Proposition 5.9.2.** *The following constitutes a small category: the set of objects is  $\Lambda_0$  and the set of morphisms is  $\Lambda_1$ . Given compositions  $\lambda, \mu \in \Lambda_0$ , the morphisms with source  $\mu$  and target  $\lambda$  are those matrices  $A \in \Lambda_1$  with  $\text{co}(A) = \mu$  and  $\text{ro}(A) = \lambda$ . Given  $\lambda, \mu, \nu \in \Lambda_0$  and  $A, B \in \Lambda_1$  with  $\text{co}(B) = \nu$ ,  $\text{ro}(B) = \mu = \text{co}(A)$  and  $\text{ro}(A) = \lambda$ , their composition is  $A * B$ , with source  $\text{co}(A * B) = \text{co}(B) = \nu$  and target  $\text{ro}(A * B) = \text{ro}(A) = \lambda$ .*

*Proof.* Proposition 5.8.9 shows that the generic product  $*$  is associative. For each object  $\lambda \in \Lambda_0$ , the identity morphism  $\lambda \rightarrow \lambda$  is the diagonal matrix  $D_\lambda$ .  $\square$

Then the generic affine algebra  $\hat{G}(n, r)$  may be realised as the  $\mathbb{Z}$ -algebra of this category. Observe that there are only finitely many objects in this category and distinct objects are non-isomorphic, so the isomorphism classes in this category are in one to one correspondence with  $\Lambda_0$ . The  $\mathbb{Z}$ -algebra of this category is the free  $\mathbb{Z}$ -module on  $\Lambda_1$  with  $\mathbb{Z}$ -bilinear multiplication given by the generic product  $*$ .

## Chapter 6

# A realisation of affine zero Schur algebras

The purpose of this chapter is to study the link between the generic affine algebra  $\hat{G}(n, r)$  to the affine 0-Schur algebra  $\hat{S}_0(n, r)$ .

The main result is the construction of an isomorphism of  $\mathbb{Z}$ -algebras from  $\hat{G}(n, r)$  to  $\hat{S}_0(n, r)$  such that  $E_i \mapsto E_i$ ,  $F_j \mapsto F_j$  and  $1_\lambda \mapsto 1_\lambda$ , in the case that  $n, r \geq 1$  with  $r < n$ .

### 6.1 Preliminary results on the generic affine algebra

Recall that the generic affine algebra  $\hat{G}(n, r)$  is an associative  $\mathbb{Z}$ -algebra with a multiplicative basis  $\{e_A : A \in \Lambda_1\}$  over  $\mathbb{Z}$ , where

$$e_A * e_B = e_{A*B}$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) = \text{ro}(B)$ , and

$$e_A * e_B = 0$$

for  $A, B \in \Lambda_1$  with  $\text{co}(A) \neq \text{ro}(B)$ .

#### 6.1.1 Elementary basis elements

For  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  such that  $\lambda_{i+1} > 0$ , define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and let

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}$$

for each  $i \in \{1, \dots, n\}$

For  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda_0$  such that  $\lambda_i > 0$ , define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and let

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}$$

for each  $i \in \{1, \dots, n\}$ .

**Lemma 6.1.1.** *Let  $i \in \{1, \dots, n\}$  and  $A \in \Lambda_1$  and write  $\mu = \text{ro}(A)$ .*

*If  $\mu_{i+1} = 0$  then  $E_i * e_A = 0$ . If  $\mu_{i+1} > 0$ , then*

$$E_i * e_A = e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}},$$

*where*

$$p = \max\{j \in \mathbb{Z} : a_{i+1,j} > 0\}.$$

*If  $\mu_i = 0$  then  $F_i e_A = 0$ . If  $\mu_i > 0$  then*

$$F_i * e_A = e_{A+\mathcal{E}_{i+1,q}-\mathcal{E}_{i,q}},$$

*where*

$$q = \min\{j \in \mathbb{Z} : a_{i,j} > 0\}.$$

**Lemma 6.1.2.** *Let  $i \in \{1, \dots, n\}$  and  $A \in \Lambda_1$  and write  $\lambda = \text{co}(A)$ .*

*If  $\lambda_j = 0$  then  $e_A * E_j = 0$ . If  $\lambda_j > 0$  then*

$$e_A * E_j = e_{A+\mathcal{E}_{p,j+1}-\mathcal{E}_{p,j}},$$

*where*

$$p = \min\{i \in \mathbb{Z} : a_{i,j} > 0\}.$$

*If  $\lambda_{j+1} = 0$  then  $e_A * F_j = 0$ . If  $\lambda_{j+1} > 0$  then*

$$e_A F_j = e_{A+\mathcal{E}_{q,j}-\mathcal{E}_{q,j+1}},$$

*where*

$$q = \max\{i \in \mathbb{Z} : a_{i,j+1} > 0\}.$$

### 6.1.2 Shifting and periodicity

DEFINE THE ELEMENTS  $R$  AND  $R^{-1}$ , GIVE MULTIPLICATION RULES.

### 6.1.3 Transpose involution

Let  $S$  be the  $\mathbb{Z}$ -module automorphism of  $\hat{G}(n, r)$  given by

$$S e_A = e_{A^\top}$$

for each  $A \in \Lambda_1$ .

**Lemma 6.1.3.** *The map  $S$  is a  $\mathbb{Z}$ -algebra antihomomorphism. In particular,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A,$$

*for each  $A, B \in \Lambda_1$ .*

*Proof.* Lemma 5.2.4 show that the transpose preserves the partial order on  $\Lambda_1$  and so

$$(B * A)^\top = A^\top * B^\top,$$

using Lemma 4.1.1. □

Moreover,

$$\begin{aligned} S(E_{i,\lambda}) &= F_{i,\lambda+\alpha_i}, \\ S(F_{i,\lambda}) &= E_{i,\lambda-\alpha_i}, \text{ and} \\ S(1_\lambda) &= 1_\lambda. \end{aligned}$$

## 6.2 Multiplicative bases in affine zero Schur algebras: motivating example

IN THIS SECTION IT WILL BE SHOWN THAT  $\hat{S}_0(n, 1) = \hat{G}(n, 1)$ .

In particular, there is an isomorphism of  $\mathbb{Z}$ -algebras  $\hat{G}(2, 1) \rightarrow \hat{S}_0(2, 1)$  which sends  $e_A$  to  $e_A$  for each  $A \in \Lambda_1$ .

## 6.3 Presentation of the generic algebra.

Recall that  $\Lambda_0$  denotes the set of compositions of  $r$  into  $n$  parts. That is,  $\Lambda_0$  is the set of tuples  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$  with each  $\lambda_i$  non-negative and  $\lambda_1 + \dots + \lambda_n = r$ . Given  $i \in [1, n]$ , let  $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$  be the  $i$ -th elementary vector and let  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ . Then given  $\lambda \in \Lambda_0$ , we have  $\lambda + \alpha_i \in \Lambda_0$  provided  $\lambda_{i+1} > 0$  and  $\lambda - \alpha_i \in \Lambda_0$  provided  $\lambda_i > 0$ .

Let  $\Gamma = \Gamma(n, r)$  be the quiver with set of vertices  $\Lambda_0$  with arrows  $e_{i,\lambda}: \lambda \rightarrow \lambda + \alpha_i$  (if  $\lambda_{i+1} > 0$ ) and  $f_{i,\lambda}: \lambda \rightarrow \lambda - \alpha_i$  (if  $\lambda_i > 0$ ). Thus there are no arrows between  $\lambda$  and  $\mu$  unless  $\lambda = \mu \pm \alpha_i$  for some  $i \in [1, n]$ .

If  $n \geq 3$  then neighbouring vertices are connected by two arrows, one of each direction. In the case  $n = 2$ , neighbouring vertices are joined by four arrows, two of each direction. The  $\mathbb{Z}\Gamma$  denote the path  $\mathbb{Z}$  algebra of  $\Gamma$ . By construction of  $\Gamma$ , there is a  $\mathbb{Z}$ -algebra homomorphism  $\mathbb{Z}\Gamma \rightarrow \hat{G}(n, r)$  with  $e_{i,\lambda} \mapsto E_{i,\lambda}$ ,  $f_{i,\lambda} \mapsto F_{i,\lambda}$  and  $k_\lambda = 1_\lambda$ . We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [32]).

**Definition 6.3.1.** (*aperiodicity*)  $A \in \Lambda_1$  is aperiodic if for each  $l \in \mathbb{Z} \setminus \{0\}$  there exists  $i \in \mathbb{Z}$  such that  $a_{i,i+l} = 0$ . If  $r < n$  then and  $A \in \Lambda_1$  is aperiodic. Linear combinations of the basis elements corresponding to aperiodic matrices are also said to be aperiodic - if  $A$  is aperiodic, we say  $e_A$  is aperiodic.

**Lemma 6.3.1.** Let  $A \in \Lambda_1$  and write  $\lambda = \text{ro}(A)$ . If  $A$  is aperiodic and  $\lambda_{i+1} > 0$ , then  $E_i * e_A$  is aperiodic. If  $A$  is aperiodic and  $\lambda_i > 0$ , then  $F_i * e_A$  is aperiodic.

*Proof.* Suppose  $A \in \Lambda_1$  is aperiodic and  $\lambda_{i+1} > 0$ , where  $\lambda = \text{ro}(A)$ . There is  $p \in \mathbb{Z}$  such that  $a_{i+1,p} > 0$  and  $a_{i+1,p'} = 0$  whenever  $p' > p$ . Lemma 6.1.1 shows that  $E_i * e_A = e_B$ , where  $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ . Let  $l \in \mathbb{Z} \setminus \{0\}$ . If  $l \notin \{p-i, p-i-1\}$ , then  $b_{s,s+l} = a_{s,s+l}$  for each  $s \in \mathbb{Z}$ , so there is  $s \in \mathbb{Z}$  such that  $b_{s,s+l} = a_{s,s+l} = 0$ , since  $A$  is aperiodic. If  $l = p-i$ , then  $b_{i+1,i+1+l} = b_{i+1,p+1} = a_{i+1,p+1} = 0$ , by maximality of  $p$ . If  $l = p-i-1$ , there is  $s \neq i+1$  such that  $a_{s,s+l} = 0$ , since  $A$  is aperiodic and  $a_{i+1,i+1+l} = a_{i+1,p} > 0$ , so  $b_{s,s+l} = a_{s,s+l} = 0$ . Therefore,  $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$  is aperiodic.

Suppose  $A \in \Lambda_1$  is aperiodic and  $\lambda_i > 0$ , where  $\lambda = \text{ro}(A)$ . Lemma 6.1.1 shows that  $F_i * e_A = e_C$  where  $C = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$  and  $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$ . Let  $l \in \mathbb{Z} \setminus \{0\}$ . If  $l \notin \{p-i, p-i-1\}$  then  $c_{s,s+l} = a_{s,s+l}$  for each  $s \in \mathbb{Z}$ , so there is  $s \in \mathbb{Z}$  such that  $c_{s,s+l} = a_{s,s+l} = 0$ , by aperiodicity of  $A$ . If  $l = p-i$ , then  $a_{i,i+l} = a_{i,p} > 0$ , so there is  $s \neq i$  such that  $a_{s,s+l} = 0$ . Then  $c_{s,s+l} = a_{s,s+l} = 0$ . Finally, if  $l = p-i-1$ , then  $c_{i,i+l} = a_{i,p-1} = 0$  by minimality of  $p$ . Thus  $C$  is aperiodic as required.  $\square$

**Definition 6.3.2.** (*Weight function*) Define the weight function  $\text{wt}: \Lambda_1 \rightarrow \mathbb{Z}$  by

$$\text{wt } A = \sum_{i \in [1, n], j \in \mathbb{Z}} |j - i| a_{i,j}$$

for each  $A \in \Lambda_1$ . The sum is taken over a transversal of the set of congruence classes of  $(i, j)$  modulo  $(n, n)$  for  $i, j \in \mathbb{Z}$ .

**Lemma 6.3.2.** *Let  $A \in \Lambda_1$  and write  $\lambda = \text{ro}(A)$ . Suppose  $\lambda_{i+1} > 0$  and set  $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$ . If  $p > i$  then  $\text{wt } e_{i,\lambda} * A = 1 + \text{wt } A$ . If  $p \leq i$  then  $\text{wt } e_{i,\lambda} * A = -1 + \text{wt } A$ . Suppose  $\lambda_i > 0$  and set  $q = \min\{q' \in \mathbb{Z} : a_{i,q'} > 0\}$ . If  $q \leq i$  then  $\text{wt } f_{i,\lambda} * A = 1 + \text{wt } A$ . If  $q > i$  then  $\text{wt } f_{i,\lambda} * A = -1 + \text{wt } A$ .*

*Proof.* Lemma 6.1.1 shows that  $e_i A = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ , so  $\text{wt } e_i A - \text{wt } A = |p - i| - |p - i - 1|$ , which equals 1 if  $p > i$  and equals  $-1$  if  $p \leq i$ . Similarly,  $f_i A = A + \mathcal{E}_{i+1,q} - \mathcal{E}_{i,q}$  by Lemma 6.1.1, so  $\text{wt } f_i A - \text{wt } A = |q - i - 1| - |q - i|$ , which equals  $-1$  if  $q > i$  and equals 1 if  $q \leq i$ .  $\square$

**Lemma 6.3.3.** *If  $A \in \Lambda_1$  is aperiodic, then  $e_A$  may be obtained from  $1_{\text{co}(A)}$  by finitely many applications of  $E_i$  and  $F_i$  for  $i \in [1, n]$ .*

**Proposition 6.3.4.** *The  $\mathbb{Z}$ -subalgebra of  $\hat{G}(n, r)$  generated by  $E_{i,\lambda}$ ,  $F_{i,\lambda}$  and  $1_\lambda$  has  $\mathbb{Z}$ -basis*

$$\{e_A : A \in \Lambda_1 \text{ is aperiodic}\}.$$

*Proof.* [COMPLETE THIS.](#)  $\square$

### 6.3.1 Relations

Suppose  $n \geq 3$ .

**Lemma 6.3.5.** *The following relations hold in  $\hat{G}(n, r)$ :*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

*unless  $|j - i| = 1$ .*

$$E_i E_{i+1}^2 - E_{i+1} E_i E_{i+1} = 0$$

$$E_i^2 E_{i+1} - E_i E_{i+1} E_i = 0$$

$$F_{i+1} F_i^2 - F_i F_{i+1} F_i = 0$$

$$F_{i+1}^2 F_i - F_{i+1} F_i F_{i+1} = 0$$

$$E_i F_j - F_j E_i = 0$$

*unless  $j = i$ .*

$$E_i F_i - F_i E_i + \sum_{\lambda: \lambda_i=0, \lambda_{i+1}>0} 1_\lambda - \sum_{\lambda: \lambda_i>0, \lambda_{i+1}=0} 1_\lambda = 0.$$

### 6.3.2 The period 2 case

In this case, the quiver  $\Gamma(2, r)$  has vertices  $\Lambda_0[2, r] = \{(0, r), (1, r-1), \dots, (r, 0)\}$ ; adjacent vertices are connected by two pairs of arrows with opposite orientation:  $(e_1, f_1)$  and  $(e_2, f_2)$ . The relations arising from  $\hat{G}(2, r)$  are of a more complicated form - in particular, the serre relations of total degree 3 will not hold in this case - so this case will be treated separately and at a later date.

Although, the presentation of the subalgebra of  $\hat{G}(2, r)$  spanned by the aperiodic distinguished basis elements by a quiver with relations is not strictly needed for our study here, since we are considering the case where  $r < n$ , and we have already seen concretely why  $\hat{G}(2, 1)$  and  $\hat{S}_0(2, 1)$  are the same.

## Chapter 7

# Further directions

### 7.1 Further results on affine zero Schur algebras

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

### 7.2 Deformed group algebras of symmetric groups

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algebras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for  $S_3$  and  $S_4$ . Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: ‘these’ relations hold; these generators and relations give a presentation of the degenerate group algebras.

Terminology: deformed group algebra.

### 7.3 back matter

[1]  $Y_{A,B}^L$  is the image of  $G_L \times G_{L'}$  under the action map  $(\alpha, \beta) \mapsto \alpha\beta \cdot (L', L'')$ , for any  $(L', L'') \in Y_{A,B}^L$ . Lemma 5.4.6 shows that  $G_L/H_{N,L}$  is a connected algebraic group. Moreover,  $G_{L'}/H_{2N,L}$  is an irreducible affine variety, so  $G_L/H_{N,L} \times G_{L'}/H_{2N,L}$  is an irreducible affine variety. It follows that  $Y_{A,B}^L$  is irreducible and constructible.

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