Affine 0-Schur algebras and affine double flag varieties.

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Contents

Chapter 1

Introduction

The work of this thesis follows the geometric realisation of 0-Schur and 0-Hecke algebras given by Jensen and Su in [22] and is based on Lusztig's geometric approach to affine q-Schur algebras in [23].

To include in introduction: quantum groups

The first part of the introduction should discuss the context of the topic and mention references. Then proceed to describe the double flag variety realisations of (affine) q-Schur algebras in the next two sections.

1.1 The double flag variety approach to q-Schur algebras

1.2 The cyclic flags approach to affine q-Schur algebras

Affine q-Schur algebras arose from an affine analogue of quantum Schur-Weyl duality and as such are defined as the endomorphism algebra of a certain module over the affine Hecke algebra. The approach used in this thesis is based on Lusztig's construction using affine double flag varieties, which is itself an extension of a similar construction of finite type q-Schur algebras given by Beilinson, Lusztig and MacPherson in [1]. A good account of these two different realisations of affine q-Schur algebras can be found in the book [4] by Deng, Du and Fu.

Let $\mathbf{k}$ be a finite or algebraically closed field and let $n, r \geq 1$ be integers. Fix a free $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$-module V of rank r and let G be its automorphism group. A lattice in V is a $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$-submodule of V which is a free module of rank r over $\mathbf{k}[\varepsilon]$. The space of n-periodic flags in V is denoted $\mathcal{F}$ and consists of chains of lattices $L = (L_i)_{i\in\mathbb{Z}}$ in V such that $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. The group G acts naturally on $\mathcal{F}$ and the orbits are indexed by the set of compositions of r into n parts. Given a composition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of r into n parts, the corresponding G-orbit in $\mathcal{F}$ is

$$\mathcal{F}_\lambda = \{L \in \mathcal{F} : \dim(L_i/L_{i-1}) = \lambda_i \text{ for all } i \in \mathbb{Z}\}.$$

The diagonal action of G on $\mathcal{F}\times\mathcal{F}$ has orbits indexed by the set Λ_1 of *infinite periodic matrices*, which consists of the $\mathbb{Z}\times\mathbb{Z}$ matrices with non-negative integer entries $a_{i,j}$ such that $a_{i,j} = a_{i-n,j-n}$ for all $i, j \in \mathbb{Z}$ and the sum of the entries in any n consecutive rows or columns is r. The orbit corresponding to $A \in \Lambda_1$ is denoted $\mathcal{O}_A$ and is the set of pairs of flags (L, L') such that

$$\dim\left(\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}\right) = a_{i,j}$$

for all $i, j \in \mathbb{Z}$. There are polynomials $\gamma_{A,B,C} \in \mathbb{Z}[q]$ for $A, B, C \in \Lambda_1$ such that

$$\gamma_{A,B,C}(\#\mathbf{k}) = \#\{L' \in \mathcal{F} : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

for any finite field $\mathbf{k}$ and $(L, L'') \in \mathcal{O}_C$.

The affine q-Schur algebra $\hat{S}_q(n, r)$ is an associative algebra over $\mathbb{Z}[q]$, with a $\mathbb{Z}[q]$-basis

$$\{e_A : A \in \Lambda_1\}$$

and multiplication given by

$$e_A e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C} e_C$$

for $A, B \in \Lambda_1$.

Specialising the affine q-Schur algebra $\hat{S}_q(n, r)$ at $q = 0$ gives an associative $\mathbb{Z}$-algebra

$$\hat{S}_0(n, r) = \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}[q]} \hat{S}_q(n, r)$$

called the affine 0-Schur algebra.

Fundamental multiplication rules in $\hat{S}_q(n, r)$. Shifting in $\hat{S}_q(n, r)$. Define the quiver with relations for $\hat{S}_q(n, r)$ and define the homomorphism from the path algebra to $\hat{S}_q(n, r)$.

Define the slices of orbit products $X^L_{A,B}$. $X^L_{A,B}$ is an irreducible quasiprojective algebraic variety and is a union of finitely many G_L orbits, so there is a unique dense open orbit; so defining a generic product of orbits. Theorem: the generic product is associative, so there is a $\mathbb{Z}$-algebra $\hat{G}(n, r)$.

Give the quiver with relations for $\hat{G}(n, r)$; define the map from the path algebra. State that these are surjective over $\mathbb{Z}$, both generic product and the $q = 0$ form, and $\mathcal{Q}$. Define standard paths; state the conjecture that any path is congruent to a standard path when $r < n$. Image of a standard path in $\hat{S}_q(n, r)$ and $\hat{G}(n, r)$.

Proposition. *There is a unique open G_L-orbit in $X^L_{A,B}$.*

Proposition. *The generic algebra $\hat{G}(n, r)$ is an associative $\mathbb{Z}$-algebra with 1, with*

$$1 = \sum_{\lambda \in \Lambda_0} 1_\lambda$$

where

$$1_\lambda = e_{D_\lambda},$$

for each $\lambda \in \Lambda_0$.

Theorem. *If $r < n$ then ρ is a $\mathbb{Z}$-algebra isomorphism. Thus $\hat{G}(n, r)$ admits a presentation by the quiver Γ and the ideal of relations $\mathcal{J}$ in $\mathbb{Z}\Gamma$.*

Theorem. *When $r < n$, the map*

$$\Psi : \hat{G}(n, r) \to \hat{S}_0(n, r)$$

is an isomorphism of $\mathbb{Z}$-algebras.

The first chapter introduces the construction of affine q-Schur algebras via a convolution product of orbits in affine double flag varieties, which is due to Lusztig. The second chapter begins to study the structure of affine q-Schur algebras, including fundamental multiplication rules, some symmetries and a notion of periodic shifting as well as giving a quiver and a set of relations for the q-Schur algebra. In the third chapter, the geometry of affine flag varieties is studied further and is used to establish existence of a *generic product* of orbits, as has been shown in the finite case by Jensen and Su in [22]. Remarkably, the generic product of orbits is shown to be associative, which leads to the construction of an associative $\mathbb{Z}$-algebra called the *generic affine algebra.* The fourth chapter is dedicated to studying the relationship of the generic affine algebra to the affine 0-Schur algebra.

Chapter 2

Geometric approach to affine q-Schur algebras

2.1 The cyclic flags realisation of affine q-Schur algebras

Fix integers $n, r \geq 1$.

Definition 2.1.1. A *composition of* r *into* n *parts* is an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of non-negative integers whose sum equals r. Denote the set of compositions of r into n parts by Λ_0.

A composition $\lambda \in \Lambda_0$ is said to be sincere if $\lambda_i > 0$ for each $i \in \{1, \ldots, n\}$ and otherwise λ is said to be insincere.

For each $i \in \{1, \ldots, n\}$, let

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1},$$

where $\varepsilon_{n+1} = \varepsilon_1$.

Definition 2.1.2. Let Λ_1 be the set of matrices $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ with integer entries $a_{i,j}$ satisfying the following conditions:

i. $a_{i,j} \geq 0$ for each $i, j \in \mathbb{Z}$;

ii. each row or column has only finitely many non-zero entries;

iii. the sum of the entries in any n consecutive rows or columns equals r;

iv. $a_{i-n,j-n} = a_{i,j}$ for each $i, j \in \mathbb{Z}$.

These matrices are referred to as *infinite periodic matrices.*

Definition 2.1.3. Given $A \in \Lambda_1$, let $\mathrm{ro}(A)$ and $\mathrm{co}(A)$ be the compositions of r into n parts given by

$$\mathrm{ro}(A) = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \ldots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co}(A) = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \ldots, \sum_{i \in \mathbb{Z}} a_{i,n} \right).$$

The *source of* A is $\mathrm{co}(A)$ and the *target of* A is $\mathrm{ro}(A)$.

The row and column sums are finite since each row and column of A contains only finitely many nonzero entries, according to the definition of Λ_1.

For each $i, j \in \mathbb{Z}$, let $\mathcal{E}_{i,j}$ be the $\mathbb{Z} \times \mathbb{Z}$ ‘elementary periodic matrix’ with entries given by

$$(\mathcal{E}_{i,j})_{s,t} = 1$$

if $(s,t) = (i + cn, j + cn)$ for some $c \in \mathbb{Z}$ and $(\mathcal{E}_{i,j})_{s,t} = 0$ otherwise. Clearly $\mathcal{E}_{i,j} = \mathcal{E}_{i+n,j+n}$ for each $i, j \in \mathbb{Z}$.

Given $\lambda \in \Lambda_0$, let $D_\lambda \in \Lambda_1$ be the diagonal matrix with source and target λ, which is given by

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}. \tag{2.1.1}$$

2.1.1 Cyclic flags

Fix $n, r \in \mathbb{N}$ and let $\mathbf{k}$ be a field. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by ε, so $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free $\mathcal{S}$-module of rank r. Let G be the automorphism group of the $\mathcal{S}$-module V, so G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$.

Definition 2.1.4. A *lattice in* V is a $\mathcal{R}$-submodule L of V with

$$\mathcal{S} \otimes_{\mathcal{R}} L = V.$$

In particular, a lattice is an $\mathcal{R}$-submodule of V which is a free $\mathcal{R}$-module of rank r.

Lemma 2.1.5. *Let L be a lattice in V. $L/\varepsilon L$ is a torsion $\mathcal{R}$-module, where ε acts as zero and $L/\varepsilon L$ is an r-dimensional* $\mathbf{k}$*-vector space.*

Proof. L is a free $\mathcal{R}$-module of rank r, with $L \subset V$. Given an $\mathcal{R}$-basis $\{x_1, \ldots, x_r\}$ of L, $\{\varepsilon x_1, \ldots, \varepsilon x_r\}$ is an $\mathcal{R}$-basis of εL. Finally, the cosets $\{x_1 + \varepsilon L, \ldots, x_r + \varepsilon L\}$ give a basis for $L/\varepsilon L$ over $\mathcal{R}/\langle \varepsilon \rangle \cong \mathbf{k}$. $\square$

Observe that if L and L' are lattices in V, then both the sum $L + L'$ and the instersection $L \cap L'$ are lattices in V. Moreover,

$$g(L + L') = g(L) + g(L')$$
$$g(L \cap L') = g(L) \cap g(L')$$

for each $g \in G$.

Definition 2.1.6. A *cyclic flag* in V is a collection $(L_i)_{i \in \mathbb{Z}}$ of lattices in V with $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. The space of cyclic flags in V is denoted by $\mathcal{F} = \mathcal{F}(n, r)$.

The group $G = \mathrm{Aut}\,(V)$ acts on $\mathcal{F}$ by $(g \cdot L)_i = g(L_i)$ for each $i \in \mathbb{Z}$, $g \in G$ and $L \in \mathcal{F}$. The G-orbits in $\mathcal{F}$ are indexed by the set Λ_0 of compositions of r into n parts. In particular, the G-orbit in $\mathcal{F}$ corresponding to $\lambda \in \Lambda_0$ is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim \left(\frac{L_i}{L_{i-1}} \right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}.$$

Consider the space $\mathcal{F} \times \mathcal{F}$ of pairs of flags with the diagonal action of G, given by

$$g \cdot (L, L') = (gL, gL')$$

for $g \in G$ and $(L, L') \in \mathcal{F} \times \mathcal{F}$. Denote the G-orbit of (L, L') by $[L, L']$. The set of G-orbits in $\mathcal{F} \times \mathcal{F}$ admits a combinatorial description as described below.

Definition 2.1.7. The periodic characteristic matrix of a pair of cyclic flags $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the matrix $\mathrm{A}(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$ with entries

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j} \right)$$

for each $i, j \in \mathbb{Z}$.

The next lemma shows that the characteristic matrix of a pair of flags depends only on the G-orbit of the pair.

Lemma 2.1.8. *Given* $(L, L') \in \mathcal{F} \times \mathcal{F}$ *and* $g \in G$,

$$\mathrm{A}(gL, gL') = \mathrm{A}(L, L').$$

Proof. Write $A = \mathrm{A}(L, L')$ and $B = \mathrm{A}(gL, gL')$. For each $i, j \in \mathbb{Z}$, g induces a linear isomorphism

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \to \frac{g(L_i \cap L'_j)}{g(L_{i-1} \cap L'_j + L_i \cap L'_{j-1})},$$

so

$$\begin{aligned} b_{i,j} &= \dim \left(\frac{gL_i \cap gL'_j}{gL_{i-1} \cap gL'_j + gL_i \cap gL'_{j-1}} \right) \\ &= \dim \left(\frac{g(L_i \cap L'_j)}{g(L_{i-1} \cap L'_j + L_i \cap L'_{j-1})} \right) \\ &= \dim \left(\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right) \\ &= a_{i,j}, \end{aligned}$$

since the action of g commutes with sums and intersections of lattices. Therefore $A = B$ as claimed. ☐

The following result gives another useful set of expressions for the characteristic matrix.

Lemma 2.1.9. *For each* $i, j \in \mathbb{Z}$,

$$a_{i,j} = \dim \left(\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} \right).$$

Proof. Set $U = L_i \cap L'_j$ and $U' = L_{i-1} + L_i \cap L'_{j-1}$. Then $U + U' = L_{i-1} + L_i \cap L'_j$ and $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$. Applying the isomorphism theorems, $U + U'/U'$ is naturally isomorphic to $U/U \cap U'$ as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to $a_{i,j}$. ☐

Lemma 2.1.10. *For each* $(L, L') \in \mathcal{F} \times \mathcal{F}$, $\mathrm{A}(L, L') \in \Lambda_1$.

Proof. Let $(L, L') \in \mathcal{F} \times \mathcal{F}$. The *periodic characteristic matrix* $\mathrm{A}(L, L')$ is (n, n)-periodic since

$$\mathrm{A}(L, L')_{i-n,j-n} = \mathrm{A}(\varepsilon L, \varepsilon L')_{i,j} = \mathrm{A}(L, L')_{i,j}$$

for each $i, j \in \mathbb{Z}$.

For each $i \in \mathbb{Z}$ there is a chain of lattices

$$M_{i,j} = L_{i-1} + L_i \cap L'_j$$

for $j \in \mathbb{Z}$ such that $M_{i,j} = L_{i-1}$ for sufficiently small j and $M_{i,j} = L_i$ for sufficiently large j. The chain of lattices gives a filtration $M_{i,j}/L_{i-1 j \in \mathbb{Z}}$ of L_i/L_{i-1} where the dimensions of the factors in the filtration are

$$\begin{aligned}\dim\left(M_{i,j}/M_{i,j-1}\right) &= \dim\left(\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L_{j-1}}\right) \\ &= a_{i,j},\end{aligned}$$

using Lemma 2.1.9.

Let $\mu = |L|$. Then

$$\begin{aligned}\mu_i &= \dim\left(L_i/L_{i-1}\right) \\ &= \sum_{j \in \mathbb{Z}} a_{i,j},\end{aligned}$$

so the sum of the entries in rows 1 to n is

$$\mu_1 + \cdots + \mu_n = r$$

and therefore $\mathrm{A}(L, L') \in \Lambda_1$. □

Lemma 2.1.11. *Given a pair of flags $(L, L') \in \mathcal{F}^2$, the matrices $\mathrm{A}(L, L')$ and $\mathrm{A}(L', L)$ are related by the transpose. In particular, $\mathrm{A}(L, L')_{i,j} = \mathrm{A}(L', L)_{j,i}$ for each $i, j \in \mathbb{Z}$.*

Proof. By swapping the roles of i and j and swapping L and L' it is clear that $\mathrm{A}(L, L')_{i,j}$ and $\mathrm{A}(L', L)_{j,i}$ are both equal to the dimension of the $\mathbf{k}$-vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each $i, j \in \mathbb{Z}$. □

Lemma 2.1.12. *Given $A \in \Lambda_1$ and $(L, L') \in \mathcal{F} \times \mathcal{F}$ with $\mathrm{A}(L, L') = A$,*

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) = \sum_{s \le i, t > j} a_{s,t}$$

and

$$\dim\left(\frac{L'_j}{L_i \cap L'_j}\right) = \sum_{s > i, t \le j} a_{s,t},$$

for each $i, j \in \mathbb{Z}$.

Proof. For each $s, t \in \mathbb{Z}$ define a lattice

$$M_{s,t} = L_i \cap L'_j + L_{s-1} + L_s \cap L'_t.$$

Observe that $L_i \cap L'_j$ is a sublattice of each $M_{s,t}$ and when $s \leq i$, L_s is contained in L_i, so $M_{s,t}$ is a sublattice of L_i. The collection of lattices $(M_{s,t})_{s \leq i, t > j}$ are totally ordered by subset inclusion, as will be shown below, so give a chain of lattices each containing $L_i \cap L'_j$ and contained in L_i. This chain of lattices induces a filtration of $L_i / L_i \cap L'_j$ and it will be shown that the dimensions of the quotients are precisely $a_{s,t}$ for $s \leq i$ and $t > j$.

Let $s \leq i$ and $t \geq j$.

$$M_{s,t} \subset M_{s,t+1}$$

and

$$\begin{aligned} M_{s,j} &= L_i \cap L'_j + L_{s-1} + L_s \cap L'_j \\ &= L_i \cap L'_j + L_{s-1}. \end{aligned}$$

If t is sufficiently large, $L_s \subset L'_t$ and then

$$\begin{aligned} M_{s,t} &= L_i \cap L'_j + L_s \\ &= M_{s+1,j}. \end{aligned}$$

It follows that the collection of lattices is totally ordered, with $M_{s,t} \leq M_{u,v}$ if and only if $s < u$ or $s = u$ and $t \leq v$. Thus $L_i / L_i \cap L'_j$ has a filtration given by the spaces $M_{s,t} / L_i \cap L'_j$ for all $s \leq i$ and $t > j$.

$$\begin{aligned} M_{i,t} &= L_i \cap L'_j + L_{i-1} + L_i \cap L'_t \\ &= L_{i-1} + L_i \cap L'_t \end{aligned}$$

and if t is sufficiently large that $L_i \subset L'_t$, then $M_{i,t} = L_i$.

If s is sufficiently small that $L_s \subset L_i \cap L'_j$ then

$$\begin{aligned} M_{s,t} &= L_i \cap L'_j + L_{s-1} + L_s \cap L'_t \\ &= L_i \cap L'_j. \end{aligned}$$

$$\begin{aligned} \frac{M_{s,t}}{L_i \cap L'_j} &= \frac{L_i \cap L'_j + L_{s-1} + L_s \cap L'_t}{L_i \cap L'_j} \\ &= \frac{L_{s-1} + L_s \cap L'_t}{L_i \cap L'_j \cap (L_{s-1} + L_s \cap L'_t)} \\ &= \frac{L_{s-1} + L_s \cap L'_t}{L_s \cap L'_j} \end{aligned}$$

Then for each $s \leq i$ and $t > j$

$$\begin{aligned} \dim\left(\frac{M_{s,t}}{M_{s,t-1}}\right) &= \dim\left(\frac{L_{s-1} + L_s \cap L'_t}{L_{s-1} + L_s \cap L'_{t-1}}\right) \\ &= a_{s,t}, \end{aligned}$$

so

$$\dim\left(\frac{L_i}{L_i\cap L'_j}\right) = \sum_{s\le i, t>j} a_{s,t}.$$

To deduce the second formula observe that $(L', L) \in \mathcal{O}_{A^\top}$, by Lemma 2.1.11, so

$$\begin{aligned}\dim\left(\frac{L'_j}{L_i\cap L'_j}\right) &= \sum_{t\le j, s>i} a^\top_{t,s}\\ &= \sum_{s>i, t\le j} a_{s,t}.\end{aligned}$$

□

The following is a construction of a pair of flags corresponding to a matrix $A \in \Lambda_1$. Recall that V is the free $\mathcal{S}$-module $\mathcal{S}^r$ and let $V_{\mathbf{k}}$ be the underlying vector space together with the linear operator $\varepsilon\colon V_{\mathbf{k}} \to V_{\mathbf{k}}$.

Fix an r-dimensional subspace U of $V_{\mathbf{k}}$ such that

$$\mathcal{S}\otimes_{\mathbf{k}} U = V_{\mathbf{k}}$$

and write

$$U = \bigoplus_{i\in\{1,\dots,n\}, j\in\mathbb{Z}} M_{i,j},$$

given subspaces $M_{i,j}$ of dimension $a_{i,j}$. Then as vector spaces

$$V_{\mathbf{k}} = \bigoplus_{i\in\{1,\dots,n\}, j\in\mathbb{Z}} \bigoplus_{h\in\mathbb{Z}} \varepsilon^h M_{i,j}.$$

Define $M_{i,j}$ for $i, j \in \mathbb{Z}$ by setting

$$M_{i-cn, j-cn} = \varepsilon^c M_{i,j}$$

and define

$$L_i = \bigoplus_{s\le i, t\in\mathbb{Z}} M_{s,t}$$

and

$$L'_j = \bigoplus_{s\in\mathbb{Z}, t\le j} M_{s,t}$$

for each $i, j \in \mathbb{Z}$.

Each such L_i and L'_j is a direct sum of free $\mathbf{k}[\varepsilon]$-modules $\mathbf{k}[\varepsilon]M_{s,t}$ for $i-n < s \le i$ and $t \in \mathbb{Z}$, or $s \in \mathbb{Z}$ and $j-n < t \le j$ respectively, so each is a free $\mathbf{k}[\varepsilon]$-module of rank r and therefore is a lattice in V.

Observe that the vector space $L_{i-1} + L_i \cap L'_j$ is the direct sum of those $M_{s,t}$ such that $s < i$ or $s = i$ and $t \le j$, so

$$\dim\left(\frac{L_{i-1} + L_i\cap L'_j}{L_{i-1} + L_i\cap L'_{j-1}}\right) = \dim\left(M_{i,j}\right) = a_{i,j}$$

for each $i, j \in \mathbb{Z}$ and therefore $\mathrm{A}(L, L') = A$.

Lemma 2.1.13. *Mapping a pair of flags (L, L') to the characteristic matrix* $\mathrm{A}(L, L')$ *gives a bijection between the set of G-orbits in $\mathcal{F}\times\mathcal{F}$ and the set Λ_1.*

Proof. The construction of a pair of flags corresponding to a matrix preceding this lemma shows that this map is surjective.

Suppose (L, L') and (N, N') are pairs of flags with $\mathrm{A}(L, L') = \mathrm{A}(N, N', =)A$. There are decompositions of V which are adapted to (L, L') and (N, N') as below: There are subspaces $U_{i,j}$ of V for $i, j \in \mathbb{Z}$ such that the dimension of $U_{i,j}$ is $a_{i,j}$, $\varepsilon U_{i,j} = U_{i-n,j-n}$ and

$$V = \bigoplus_{i,j\in\mathbb{Z}} U_{i,j};$$

$$L_i = \bigoplus_{s\leq i, j\in\mathbb{Z}} U_{s,j}$$

for each $i \in \mathbb{Z}$;

$$L'_j = \bigoplus_{i\in\mathbb{Z}, t\leq j} U_{i,t}$$

for each $j \in \mathbb{Z}$.

There are subspaces $V_{i,j}$ of V for each $i, j \in \mathbb{Z}$ such that the dimension of $V_{i,j}$ is $a_{i,j}$, $\varepsilon V_{i,j} = V_{i-n,j-n}$ and

$$V = \bigoplus_{i,j\in\mathbb{Z}} V_{i,j};$$

$$N_i = \bigoplus_{s\leq i, j\in\mathbb{Z}} V_{s,j}$$

for each $i \in \mathbb{Z}$;

$$N'_j = \bigoplus_{i\in\mathbb{Z}, t\leq j} V_{i,t}$$

for each $j \in \mathbb{Z}$.

There exist $\mathbf{k}$-linear isomorphisms $g_{i,j} \colon U_{i,j} \to V_{i,j}$ for $i, j \in \mathbb{Z}$ such that $g_{i-n,j-n} = \varepsilon g_{i,j}\varepsilon^{-1}$. Then $g = (g_{i,j})_{i,j\in\mathbb{Z}}$ is a $\mathcal{S}$-linear automorphism of V with $g(L_i) = N_i$ and $g(L'_i) = N'_i$ for each $i \in \mathbb{Z}$, so $g(L, L') = (N, N')$. Therefore the map sending a G-orbit to its characteristic matrix is injective. $\square$

Lemma 2.1.14. *Given $(L, L') \in \mathcal{F}^2$, $L' \subset L$ if and only if $\mathrm{A}(L, L')_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i > j$.*

Proof. Suppose $L, L' \in \mathcal{F}$ with $L' \subset L$, meaning $L'_j \subset L_j$ for each $j \in \mathbb{Z}$. Then for $i > j$, $L_i \cap L'_j = L'_j$, $L_{i-1} \cap L'_j = L'_j$ and $L_i \cap L'_{j-1}$, which shows

$$\mathrm{A}(L, L')_{i,j} = \dim\left(\frac{L'_j}{L'_{j-1} + L'_j}\right) = 0$$

as required. Conversely, suppose $\mathrm{A}(L, L')$ is upper triangular, meaning $\mathrm{A}(L, L')_{i,j} = 0$ when $i > j$. Using Lemma 2.1.12,

$$\dim\left(\frac{L'_i}{L'_i \cap L_i}\right) = \sum_{s>i, t\leq i} a_{s,t} = 0,$$

so $L_i \cap L'_i = L'_i$ and thus $L'_i \subset L_i$ for each $i \in \mathbb{Z}$, as required. $\square$

Corollary 2.1.15. *Given $L, L' \in \mathcal{F}$, $L = L'$ if and only if $\mathrm{A}(L,L')_{i,j} = 0$ whenever $i \neq j$. In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L,L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each $\lambda \in \Lambda_0$.

2.1.2 A product of orbits

Given $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also $L \in \mathcal{F}_{\mathrm{ro}(A)}$, define the L-slices of $Y_{A,B}$ and $X_{A,B}$ respectively as

$$Y^L_{A,B} = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X^L_{A,B} = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

Remark 2.1.16. There are only finitely many G-orbits in $X_{A,B}$.

Lemma 2.1.17. *Given $A \in \Lambda_1$, $X_{D_\lambda,A} = \mathcal{O}_A$ if $\lambda = \mathrm{ro}(A)$ and $X_{A,D_\lambda} = \mathcal{O}_A$ if $\lambda = \mathrm{co}(A)$.*

Proof. Let $A \in \Lambda_1$ and set $\lambda = \mathrm{ro}(A)$. $Y_{D_\lambda,A}$ is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_{D_\lambda}$, thus $L = L'$ by Corollary 2.1.15, and $(L', L'') \in \mathcal{O}_A$. $X_{D_\lambda,A}$ is the projection of $Y_{D_\lambda,A}$, which equals $\mathcal{O}_A$.

Similarly, if $\lambda = \mathrm{co}(A)$, Y_{A,D_λ} is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_A$ and $L'' = L'$, so X_{A,D_λ} is exactly the orbit $\mathcal{O}_B$. ☐

2.1.3 Triple products

Given $A, B, C \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(B) = \mathrm{ro}(C)$ and $L \in \mathcal{F}_{\mathrm{ro}(A)}$, there are spaces $X_{A,B,C}$, $Y_{A,B,C}$ and their respective L-slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y^L_{A,B,C} = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X^L_{A,B,C} = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

2.1.4 Convolution algebras

Suppose $\mathbf{k}$ is a finite field and let q denote the number of elements of $\mathbf{k}$. Consider the set S of G-invariant functions $\mathcal{F}\times\mathcal{F} \to \mathbb{Z}$ with constructible support. S is a free $\mathbb{Z}$-module with a basis consisting of the indicator functions of the G-orbits in $\mathcal{F}\times\mathcal{F}$. Define an operation $\star$ on S as follows: for each $f, g \in S$, $f \star g \in S$ is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for $(L, L'') \in \mathcal{F}\times\mathcal{F}$.

$f \star g$ is well defined since the supports of f and g consist of finitely many G-orbits, so there are only finitely many $L' \in \mathcal{F}$ such that $f(L, L')g(L', L'') \neq 0$, given $(L, L'') \in \mathcal{F}\times\mathcal{F}$. $f \star g$ is constant on G-orbits and is supported on finitely many G-orbits, so $f \star g \in S$.

Lemma 2.1.18. *The set S together with the operation $\star$ is an associative $\mathbb{Z}$-algebra with identity element ι given by $\iota(L, L) = 1$ and $\iota(L, L') = 0$ for $L' \neq L$.*

Proof. Given $f, g, h \in S$ and $(L, L''') \in \mathcal{F} \times \mathcal{F}$,

$$\begin{aligned}((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''),\end{aligned}$$

thus $\star$ is associative. ι is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L')\iota(L', L'') = f(L, L''),$$

for each $f \in S$ and $(L, L'') \in \mathcal{F} \times \mathcal{F}$. □

Given $A \in \Lambda_1$, let $e_A \in S$ denote the indicator function of the orbit $\mathcal{O}_A$. S is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1\}$. There exist $\gamma_{A,B,C;\mathrm{q}} \in \mathbb{Z}$ for $A, B, C \in \Lambda_1$ such that

$$e_A \star e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C;\mathrm{q}} e_C$$

for each $A, B \in \Lambda_1$. Then

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L')e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},\end{aligned}$$

for any $(L, L'') \in \mathcal{O}_C$.

2.1.5 Affine q-Schur algebras

There exist polynomials $\gamma_{A,B,C} \in \mathbb{Z}[q]$ for $A, B, C \in \Lambda_1$ such that $\gamma_{A,B,C}(\mathrm{q}) = \gamma_{A,B,C;\mathrm{q}}$ for any prime power q, following [23, section 4]. The affine q-Schur algebra $\hat{S}_q(n, r)$ is a $\mathbb{Z}[q]$-algebra which is a free $\mathbb{Z}[q]$-module with basis $\{e_A : A \in \Lambda_1\}$ and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these 'universal polynomials' $\gamma_{A,B,C} \in \mathbb{Z}[q]$, it follows from Lemma 2.1.18 that $\hat{S}_q(n, r)$ is an associative $\mathbb{Z}[q]$-algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

2.2 Affine zero-Schur algebras

Fix integers $n, r \geq 1$.

Definition 2.2.1. The affine zero Schur algebra $\hat{S}_0(n,r)$ is the $\mathbb{Z}$-algebra

$$\hat{S}_0(n,r) = \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}[q]} \hat{S}_q(n,r).$$

Observe that $\hat{S}_0(n,r)$ is a free $\mathbb{Z}$-module, since $\hat{S}_q(n,r)$ is a free $\mathbb{Z}[q]$-module. Moreover, $\hat{S}_0(n,r)$ has a $\mathbb{Z}$-basis $\{e_A : A \in \Lambda_1\}$ with multiplication given by

$$e_A \cdot e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C}(0) e_C,$$

for each $A, B \in \Lambda_1$.

Chapter 3

Presenting affine q-Schur algebras

3.1 The distinguished basis

3.1.1 Elementary basis elements

Recall that $\mathcal{E}_{i,j}$, for $i, j \in \mathbb{Z}$ is the $\mathbb{Z} \times \mathbb{Z}$ elementary periodic matrix, given by

$$(\mathcal{E}_{i,j})_{s,t} = 1$$

if $(s,t) = (i + cn, j + cn)$ for some $c \in \mathbb{Z}$ and $(\mathcal{E}_{i,j})_{s,t} = 0$ otherwise.

Recall that the diagonal matrix with source and target λ is

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n},$$

as in Equation 2.1.1.

The corresponding basis elements e_{D_λ}, for $\lambda \in \Lambda_0$, are pairwise orthogonal idempotents in $\hat{S}_q(n,r)$ with

$$\sum_{\lambda \in \Lambda_0} e_{D_\lambda} = 1,$$

as a result of Lemma 2.1.17.

For each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and define

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}.$$

Also define, for each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$,

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and define

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}.$$

For each $i \in \{1, \ldots, n\}$, let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then

$$\mathrm{co}(E_{i,\lambda}) = \mathrm{co}(F_{i,\lambda}) = \lambda,$$

$$\mathrm{ro}(E_{i,\lambda}) = \lambda + \alpha_i$$

and

$$\mathrm{ro}(F_{i,\lambda}) = \lambda - \alpha_i.$$

3.1.2 Transpose involution

Let S be the $\mathbb{Z}[q]$-module automorphism of $\hat{S}_q(n,r)$ given by

$$S(e_A) = e_{A^\top},$$

for each $A \in \Lambda_1$.

Lemma 3.1.1. *The map S is a $\mathbb{Z}[q]$-algebra antihomomorphism of order* 2*. In particular,*

$$S(e_A e_B) = S(e_B)S(e_A)$$

for each $A, B \in \Lambda_1$.

Proof. Let $A, B, C \in \Lambda_1$ and let $\mathbf{k}$ be a finite field with $\mathrm{q} = \#\mathbf{k}$ elements. If $(L, L'') \in \mathcal{O}_C$ then $(L'', L) \in \mathcal{O}_{C^\top}$ and

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;\mathrm{q}}\end{aligned}$$

It follows that

$$S(e_A e_B) = S(e_B)S(e_A),$$

for each $A, B \in \Lambda_1$ and therefore S is a $\mathbb{Z}[q]$-algebra antihomomorphism. Moreover, $S \circ S$ is the identity map on $\hat{S}_q(n,r)$ since $(A^\top)^\top = A$. ☐

The action of S on E_i, F_i and 1_λ is as follows:

$$S(1_\lambda) = 1_\lambda$$

for each $\lambda \in \Lambda_0$,

$$S(E_{i,\lambda}) = F_{i,\lambda+\alpha_i}$$

for each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$, and

$$S(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}$$

for each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$.

In particular,

$$\begin{aligned}S(E_i) &= F_i, \\ S(F_i) &= E_i, \\ S(1_\lambda) &= 1_\lambda\end{aligned}$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$.

3.1.3 Fundamental multiplication rules

For each $m \in \mathbb{N}$, define the q-integer $[[m]] \in \mathbb{Z}[q]$ by

$$[[m]] = \frac{1-q^m}{1-q},$$

so that

$$\begin{aligned}
[[0]] &= 0 \\
[[1]] &= 1 \\
[[2]] &= 1+q \\
[[3]] &= 1+q+q^2
\end{aligned}$$

and

$$[[m]] = 1 + q + \cdots + q^{m-1}$$

for $m \geq 1$.

Lemma 3.1.2. *Given $A \in \Lambda_1$ and $i \in \{1, \ldots, n\}$ with $\mathrm{ro}(A)_{i+1} > 0$,*

$$E_i e_A = \sum_{p \in \mathbb{Z}: a_{i+1,p} > 0} q^{\sum_{j>p} a_{i,j}} [[a_{i,p}+1]] e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}}.$$

Given $A \in \Lambda_1$ and $i \in \{1, \ldots, n\}$ with $\mathrm{ro}(A)_i > 0$,

$$F_i e_A = \sum_{p \in \mathbb{Z}: a_{i,p} > 0} q^{\sum_{j<p} a_{i+1,j}} [[a_{i+1,p}+1]] e_{A+\mathcal{E}_{i+1,p}-\mathcal{E}_{i,p}}.$$

Note that these formulas are still valid in the cases $E_i e_A = 0$ and $F_i e_A = 0$. If the convention that $e_B = 0$ whenever B is not in Λ_1 is used, then the conditions on p in the above sums may be ignored.

Corollary 3.1.3. *Given $A \in \Lambda_1$ and $j \in \{1, \ldots, n\}$ with $\mathrm{co}(A)_{j+1} > 0$,*

$$e_A F_j = \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j}+1]] e_{A+\mathcal{E}_{p,j}-\mathcal{E}_{p,j+1}}.$$

Given $A \in \Lambda_1$ and $j \in \{1, \ldots, n\}$ with $\mathrm{co}(A)_j > 0$,

$$e_A E_j = \sum_{p \in \mathbb{Z}: a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1}+1]] e_{A+\mathcal{E}_{p,j+1}-\mathcal{E}_{p,j}}.$$

Proof.

$$\begin{aligned}
e_A F_j &= S(E_j e_{A^\top}) \\
&= S\left(\sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j}+1]] e_{A^\top + \mathcal{E}_{j,p} - \mathcal{E}_{j+1,p}} \right) \\
&= \sum_{p \in \mathbb{Z}: a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j}+1]] e_{A+\mathcal{E}_{p,j}-\mathcal{E}_{p,j+1}},
\end{aligned}$$

where the second equality comes from Lemma 3.1.2. Similarly,

$$\begin{aligned}
e_A E_j &= S(F_j e_{A^\top}) \\
&= S\left(\sum_{p\in\mathbb{Z}: a_{p,j}>0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1}+1]] e_{A^\top + \mathcal{E}_{j+1,p} - \mathcal{E}_{j,p}} \right) \\
&= \sum_{p\in\mathbb{Z}: a_{p,j}>0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1}+1]] e_{A + \mathcal{E}_{p,j+1} - \mathcal{E}_{p,j}}.
\end{aligned}$$

□

3.1.4 The hook order

For each $i, j \in \mathbb{Z}$, let $d_{i,j}$ and $\bar{d}_{i,j}$ be the maps from Λ_1 to $\mathbb{Z}$ given by

$$d_{i,j}(A) = \sum_{s\leq i, t>j} a_{s,t}$$

and

$$\bar{d}_{i,j}(A) = \sum_{s>i, t\leq j} a_{s,t}$$

for each $A \in \Lambda_1$.

Lemma 3.1.4. *For each $A \in \Lambda_1$ and $i, j \in \mathbb{Z}$, the following equations hold:*

$$d_{i,j}(A) - d_{i-1,j}(A) = \sum_{t>j} a_{i,t}$$
$$d_{i,j}(A) - d_{i,j-1}(A) = -\sum_{s\leq i} a_{s,j}$$

and

$$\bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) = -\sum_{t\leq j} a_{i,t}$$
$$\bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) = \sum_{s>i} a_{s,j}$$

Proof. Let $i, j \in \mathbb{Z}$ and $A \in \Lambda_1$. Then

$$d_{i,j}(A) - d_{i-1,j}(A) = \sum_{s\leq i, t>j} a_{s,t} - \sum_{s\leq i-1, t>j} a_{s,t} = \sum_{t>j} a_{i,t}$$

and

$$d_{i,j}(A) - d_{i,j-1}(A) = \sum_{s\leq i, t>j} a_{s,t} - \sum_{s\leq i, t>j-1} a_{s,t} = -\sum_{s\leq i} a_{s,j}.$$

Similarly,

$$\bar{d}_{i,j}(A) - \bar{d}_{i-1,j}(A) = \sum_{s>i, t\leq j} a_{s,t} - \sum_{s>i-1, t\leq j} a_{s,t} = -\sum_{t\leq j} a_{i,t}$$

and

$$\bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) = \sum_{s>i, t\leq j} a_{s,t} - \sum_{s>i, t\leq j-1} a_{s,t} = \sum_{s>i} a_{s,j}.$$

□

Lemma 3.1.5. *For each $A \in \Lambda_1$ and $i, j \in \mathbb{Z}$,*

$$a_{i,j} = d_{i,j-1}(A) - d_{i-1,j-1}(A) - d_{i,j}(A) + d_{i-1,j}(A)$$

and

$$a_{i,j} = \bar{d}_{i,j-1}(A) - \bar{d}_{i-1,j-1}(A) - \bar{d}_{i,j}(A) + \bar{d}_{i-1,j}(A).$$

Proof. As a result of Lemma 3.1.4,

$$\begin{aligned} d_{i,j-1}(A) - d_{i-1,j-1}(A) - d_{i,j}(A) + d_{i-1,j}(A) &= \sum_{t>j-1} a_{i,t} - \sum_{t>j} a_{i,t} \\ &= a_{i,j} \end{aligned}$$

and

$$\begin{aligned} \bar{d}_{i,j-1}(A) - \bar{d}_{i-1,j-1}(A) - \bar{d}_{i,j}(A) + \bar{d}_{i-1,j}(A) &= -\sum_{t\le j-1} a_{i,t} + \sum_{t\le j} a_{i,t} \\ &= a_{i,j}. \end{aligned}$$

□

Define a relation $\le$ on Λ_1 by $A \le B$ if and only if the following conditions are satisfied:

- $\mathrm{ro}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(A) = \mathrm{co}(B)$.
- For each $i, j \in \mathbb{Z}$, $d_{i,j}(A) \le d_{i,j}(B)$.
- For each $i, j \in \mathbb{Z}$, $\bar{d}_{i,j}(A) \le \bar{d}_{i,j}(B)$.

Lemma 3.1.6. *The relation $\le$ defines a partial order on Λ_1.*

Proof. It is clear that $\le$ is reflexive and transitive.

Suppose $A, B \in \Lambda_1$ with $A \le B$ and $B \le A$. Then $d_{i,j}(A) = d_{i,j}(B)$ for each $i, j \in \mathbb{Z}$ with $i \le j$, which shows $a_{s,t} = b_{s,t}$ whenever $s < t$, as a result of Lemma 3.1.5. Similarly, $\bar{d}_{i,j}(A) = \bar{d}_{i,j}(B)$ for each $i, j \in \mathbb{Z}$ with $i \ge j$, so $a_{s,t} = b_{s,t}$ whenever $s > t$. Moreover, $a_{i,i} = b_{i,i}$ for each $i \in \mathbb{Z}$, since $\mathrm{co}(A) = \mathrm{co}(B)$. Thus $A = B$, which shows $\le$ is antisymmetric and therefore $\le$ is a partial order on Λ_1. □

This partial order is sometimes called the hook order. The following lemma will be invoked later in induction arguments.

Lemma 3.1.7. *For any $A \in \Lambda_1$, the set $\{B \in \Lambda_1 : B \le A\}$ is finite.*

Proof. Let $B \in \Lambda_1$. Only finitely many of the $d_{i,j}(B)$ and $\bar{d}_{i,j}(B)$ are sufficient to determine B and $B \le A$ if and only if

$$0 \le d_{i,j}(B) \le d_{i,j}(A)$$

and

$$0 \le \bar{d}_{i,j}(B) \le \bar{d}_{i,j}(A)$$

for each $i, j \in \mathbb{Z}$, thus there are only finitely many possible values of $d_{i,j}(B)$ and $\bar{d}_{i,j}(B)$ provided $B \le A$. Therefore there are only finitely many $B \in \Lambda_1$ such that $B \le A$. □

Lemma 3.1.8. *The transpose operation on Λ_1 is order preserving. In particular, $B \leq A$ if and only if $B^\top \leq A^\top$.*

Proof. Suppose $A, B \in \Lambda_1$ with $B \leq A$. The condition $\mathrm{co}(A) = \mathrm{co}(B)$ and $\mathrm{ro}(A) = \mathrm{ro}(B)$ is preserved by the transpose operation.

For each $i, j \in \mathbb{Z}$,

$$d_{i,j}(A^\top) = \sum_{s \leq i, t > j} a_{t,s} = \bar{d}_{j,i}(A)$$

and

$$\bar{d}_{i,j}(A^\top) = \sum_{s > i, t \leq j} a_{t,s} = d_{j,i}(A).$$

It follows that $B^\top \leq A^\top$ and therefore the transpose is order preserving. $\square$

Lemma 3.1.9. *Suppose $A, B \in \Lambda_1$ with*

$$B = A + \mathcal{E}_{i,j} - \mathcal{E}_{s,j} + \mathcal{E}_{s,t} - \mathcal{E}_{i,t}$$

for some $i, j, s, t \in \mathbb{Z}$ with $i < s$ and $j < t$. Then $B < A$.

Proof. Let $p, q \in \mathbb{Z}$. Then

$$d_{p,q}(B) = \begin{cases} d_{p,q}(A) - 1 & : i \leq p < s \text{ and } j \leq q < t, \\ d_{p,q}(A) & : \text{otherwise}, \end{cases}$$

and

$$\bar{d}_{p,q}(B) = \begin{cases} d_{p,q}(A) - 1 & : i \leq p < s \text{ and } j \leq q < t, \\ d_{p,q}(A) & : \text{otherwise}, \end{cases}$$

which proves that $B < A$. $\square$

Let $A \in \Lambda_1$ and $i \in \{1, \ldots, n\}$ with $\mathrm{ro}(A)_{i+1} > 0$. Using the fundamental multiplication rules 3.1.2 and Lemma 3.1.9,

$$E_i e_A = \sum_{s=1}^{m} q^{\sum_{t > j_s} a_{i,t}} [[a_{i,j_s} + 1]] e_{A + \mathcal{E}_{i,j_s} - \mathcal{E}_{i+1,j_s}}$$

where $j_1, \ldots, j_m \in \mathbb{Z}$ with $j_1 < j_2 < \ldots < j_m$ and

$$\{j_1, \ldots, j_m\} = \{j \in \mathbb{Z} : a_{i+1,j} > 0\}.$$

The basis elements appearing in the above expression are totally ordered, with

$$A + \mathcal{E}_{i,j_s} - \mathcal{E}_{i+1,j_s} < A + \mathcal{E}_{i,j_{s+1}} - \mathcal{E}_{i+1,j_{s+1}}$$

for $s = 1, \ldots, m-1$. Thus the term with $s = m$ is the maximum.

The partial order on Λ_1 induces a partial order on the set of G-orbits in $\mathcal{F} \times \mathcal{F}$, such that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $A \leq B$. The following is a restatement of Lemma 2.1.12 and gives some geometric significance to the hook order on Λ_1.

Lemma 3.1.10. *Let $A \in \Lambda_1$ and $(L, L') \in \mathcal{O}_A$. Then*

$$\dim \left(\frac{L_i}{L_i \cap L'_j} \right) = d_{i,j}(A)$$

and

$$\dim \left(\frac{L'_j}{L_i \cap L'_j} \right) = \bar{d}_{i,j}(A),$$

for each $i, j \in \mathbb{Z}$.

3.1.5 Shifting

In this subsection it is shown that the operations on Λ_1 given by shifting up by one row or to the right by one column may be described by the action, on the left or right respectively, of an invertible element R of $\hat{S}_q(n,r)$.

For each $A \in \Lambda_1$ and $m \in \mathbb{Z}$, the row shift of A by m is the element $[m]A$ of Λ_1 given by

$$([m]A)_{i,j} = a_{i+m,j},$$

for each $i, j \in \mathbb{Z}$.

The column shift of A by m is the element $A[m]$ given by

$$(A[m])_{i,j} = a_{i,j+m},$$

for each $i, j \in \mathbb{Z}$.

For $\lambda \in \Lambda_0$ and $m \in \mathbb{Z}$, the translation of λ by m is the element $\lambda[m]$ of Λ_0 given by

$$(\lambda[m])_i = \lambda_{i+m},$$

for each $i \in \mathbb{Z}$, where the indices of λ are taken modulo n.

Example 3.1.11. Let $\lambda = (2, 1, 3)$. Then $\lambda[1] = (1, 3, 2)$, $\lambda[2] = (3, 2, 1)$ and $\lambda[3] = \lambda$.

For each $\lambda \in \Lambda_0$, define

$$\begin{aligned} R_\lambda &= e_{[1]D_\lambda} \\ &= e_{\lambda_1 \mathcal{E}_{0,1} + \cdots + \lambda_n \mathcal{E}_{n-1,n}} \end{aligned}$$

and let

$$R = \sum_{\lambda \in \Lambda_0} R_\lambda.$$

Recall that

$$\mathcal{O}_{D_\lambda} = \{(L, L) : L \in \mathcal{F}_\lambda\},$$

so

$$\mathcal{O}_{[m]D_\lambda} = \{([m]L, L) : L \in \mathcal{F}_\lambda\}$$

and

$$\mathcal{O}_{D_{\lambda[m]}} = \{(L, [m]L) : L \in \mathcal{F}_\lambda\}.$$

This leads to a simple rule for multiplication by R in terms of these shifts on matrices.

Lemma 3.1.12. *If $A \in \Lambda_1$ then*

$$Re_A = e_{[1]A}$$

and

$$e_A R = e_{A[-1]}.$$

Proof. Let $\mu = \mathrm{ro}(A)$ and $\lambda = \mathrm{co}(A)[-1]$. Observe that

$$\{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_{[1]D_\mu}, (L', L'') \in \mathcal{O}_A\} = \{(L'[1], L', L'') : (L', L'') \in \mathcal{O}_A\},$$

and the image under the projection onto the first and last components is

$$\{(L'[1], L'') : (L, L'') \in \mathcal{O}_A\} = \mathcal{O}_{[1]A}.$$

The coefficient of $\mathcal{O}_{[1]A}$ in the product Re_A is 1 since, for any $(N, N'') \in \mathcal{O}_{[1]A}$,

$$\{N' \in \mathcal{F} : (N, N') \in \mathcal{O}_{[1]D_\mu}, (N', N'') \in \mathcal{O}_A\} = \{N[-1]\},$$

so it follows that $Re_A = e_{[1]A}$.

To compute the product $e_A R$, consider

$$\{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_{[1]D_\lambda}\} = \{(L, L', L'[-1]) : (L, L') \in \mathcal{O}_A\}.$$

The image under the projection onto the first and last components is

$$\{(L, L'[-1]) : (L, L') \in \mathcal{O}_A\} = \mathcal{O}_{A[-1]}$$

and, for any $(N, N'') \in \mathcal{O}_{A[-1]}$,

$$\{N' \in \mathcal{F} : (N, N') \in \mathcal{O}_A, (N', N'') \in \mathcal{O}_{[1]D_\lambda}\} = \{N''[1]\}.$$

Therefore $e_A R = e_{A[-1]}$. □

Lemma 3.1.13. *The element R is invertible and*

$$RS(R) = S(R)R = 1.$$

In particular,

$$R^{-1} = \sum_{\lambda \in \Lambda_0} e_{[-1]D_\lambda}.$$

Proof. Recall that

$$S(R) = \sum_{\lambda \in \Lambda_0} e_{[-1]D_\lambda}.$$

Then it follows from Lemma 3.1.12 that

$$RS(R) = \sum_{\lambda \in \Lambda_0} e_{[1][-1]D_\lambda} = 1$$

and

$$\begin{aligned} S(R)R &= \sum_{\lambda \in \Lambda_0} e_{[-1]D_\lambda[-1]} \\ &= \sum_{\lambda \in \Lambda_0} e_{D_{(\lambda[-1])}} \\ &= 1. \end{aligned}$$

□

As a visual cue, acting on a basis element e_A on the left by R corresponds to moving the matrix A up by one row, while acting on the right by R corresponds to moving the matrix to the right by one column. Then conjugating by R corresponds to the composition of a shift to the left by one and a shift up by one, which is a shift by one along the diagonal, so conjugating by R^n leaves e_A invariant. Thus conjugation by R gives a $\mathbb{Z}[q]$-algebra automorphism of $\hat{S}_q(n,r)$ which has order n.

Multiplication on the left by $S(R)$ sends e_A to $e_{[-1]A}$, while multiplication on the right by $S(R)$ sends e_A to $e_{A[1]}$.

Lemma 3.1.14. *For each* $\lambda \in \Lambda_0$,

$$R1_\lambda S(R) = 1_{[1]\lambda}$$

and, for each $i \in \{1, \ldots, n\}$,

$$RE_iS(R) = E_{i-1}$$

and

$$RF_iS(R) = F_{i-1}.$$

Proof. It follows from Lemma 3.1.12 and Lemma 3.1.13 that

$$Re_AS(R) = e_{[1]A[1]},$$

for each $A \in \Lambda_1$. In particular,

$$R1_\lambda S(R) = 1_{\lambda[1]}$$

for each $\lambda \in \Lambda_0$,

$$RE_{i,\lambda}S(R) = E_{i-1,\lambda[1]}$$

for each $(\lambda, i) \in \Lambda_0 \times \mathbb{Z}$ with $\lambda_{i+1} > 0$, and

$$RF_{i,\lambda}S(R) = F_{i-1,\lambda[1]}$$

for each $(\lambda, i) \in \Lambda_0 \times \mathbb{Z}$ with $\lambda_i > 0$.

It now follows that

$$RE_iS(R) = E_{i-1}$$

and

$$RF_iS(R) = F_{i-1}$$

as claimed. □

3.2 Quivers and relations

Assume n and r are integers with $n \geq 3$ and $r \geq 1$.

3.2.1 Relations in affine q-Schur algebras

Lemma 3.2.1. *If* $i, j \in \{1, \ldots, n\}$ *and* $i \neq j$, *then*

$$E_iF_j - F_jE_i = 0.$$

For each $i \in \{1, \ldots, n\}$,

$$E_iF_i - F_iE_i = \sum_{\lambda \in \Lambda_0} ([[\lambda_i]] - [[\lambda_{i+1}]])1_\lambda.$$

Proof. Denote e_A by $[A]$. Fix $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then

$$\begin{aligned} E_i F_j &= \sum_{\lambda \in \Lambda_0} E_i \left[D_\lambda + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j}\right] \\ &= \sum_{\lambda \in \Lambda_0} \left[D_\lambda + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j} + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}\right]. \end{aligned}$$

Observe that the nonzero terms in the above sum are those for which $\lambda_j > 0$ and $\lambda_{i+1} > 0$. Similarly,

$$\begin{aligned} F_j E_i &= \sum_{\lambda \in \Lambda_0} F_j \left[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}\right] \\ &= \sum_{\lambda \in \Lambda_0} \left[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j}\right], \end{aligned}$$

where the sum is taken over those λ such that $\lambda_{i+1} > 0$ and $\lambda_j > 0$. Therefore

$$E_i F_j - F_j E_i = 0.$$

Again using Lemma 3.1.2,

$$\begin{aligned} E_i F_i &= \sum_{\lambda \in \Lambda_0} E_i \left[D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}\right] \\ &= \sum_{\lambda \in \Lambda_0} \left[D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i} + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}\right] + [[\lambda_i]] \left[D_\lambda\right] \end{aligned}$$

and

$$\begin{aligned} F_i E_i &= \sum_{\lambda \in \Lambda_0} F_i \left[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}\right] \\ &= \sum_{\lambda \in \Lambda_0} \left[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}\right] + [[\lambda_{i+1}]] \left[D_\lambda\right]. \end{aligned}$$

Therefore

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0} ([[\lambda_i]] - [[\lambda_{i+1}]]) 1_\lambda,$$

as required. ☐

An explicit version of these relations will be given after defining some terminology. Given $\lambda \in \Lambda_0$ and $i \in \{1, \ldots, n\}$, say that λ is internal with respect to i if $\lambda - \alpha_i, \lambda + \alpha + i \in \Lambda_0$. Say that λ is initial with respect to i if $\lambda - \alpha_i \notin \Lambda_0$ and that λ is final with respect to i if $\lambda + \alpha_i \notin \Lambda_0$.

Then the expression for the commutator $[E_i, F_i]$ in Lemma 3.2.1 gives the following relations in $\hat{S}_q(n, r)$:

- If λ is internal with respect to i then

$$E_{i,\lambda-\alpha_i} F_{i,\lambda} - F_{i,\lambda+\alpha_i} E_{i,\lambda} = 0.$$

- If λ is initial with respect to i then

$$F_{i,\lambda+\alpha_i} E_{i,\lambda} - 1_\lambda = 0.$$

- If λ is final with respect to i then

$$E_{i,\lambda-\alpha_i}F_{i,\lambda}-1_\lambda=0.$$

Lemma 3.2.2. *The following relations hold in $\hat{S}_q(n,r)$, when $n\geq 3$:*

$$E_iE_j-E_jE_i=0$$

and

$$F_iF_j-F_jF_i=0$$

for $i,j\in\{1,\ldots,n\}$ such that $j\geq i+2$,

$$\begin{aligned}E_iE_{i+1}^2-(1+q)E_{i+1}E_iE_{i+1}+qE_{i+1}^2E_i&=0\\ E_i^2E_{i+1}-(1+q)E_iE_{i+1}E_i+qE_{i+1}E_i^2&=0\end{aligned}$$

and

$$\begin{aligned}F_{i+1}F_i^2-(1+q)F_iF_{i+1}F_i+qF_i^2F_{i+1}&=0\\ F_{i+1}^2F_i-(1+q)F_{i+1}F_iF_{i+1}+qF_iF_{i+1}^2&=0.\end{aligned}$$

for $i\in\{1,\ldots,n\}$.

Proof. Denote e_A by $[A]$.

$$\begin{aligned}E_iE_{i+1}^2=\sum_{\lambda\in\Lambda_0}&[[2]]\,[D_\lambda+\mathcal{E}_{i,i+2}+\mathcal{E}_{i+1,i+2}-2\mathcal{E}_{i+2,i+2}]\\ &+[[2]]\,[D_\lambda+\mathcal{E}_{i,i+1}-\mathcal{E}_{i+1,i+1}+2\mathcal{E}_{i+1,i+2}-2\mathcal{E}_{i+2,i+2}]\end{aligned}$$

$$\begin{aligned}E_{i+1}E_iE_{i+1}=\sum_{\lambda\in\Lambda_0}&[D_\lambda+\mathcal{E}_{i,i+2}+\mathcal{E}_{i+1,i+2}-2\mathcal{E}_{i+2,i+2}]\\ &+[[2]]\,[D_\lambda+\mathcal{E}_{i,i+1}-\mathcal{E}_{i+1,i+1}+2\mathcal{E}_{i+1,i+2}-2\mathcal{E}_{i+2,i+2}]\end{aligned}$$

$$E_{i+1}^2E_i=\sum_{\lambda\in\Lambda_0}[[2]]\,[D_\lambda+\mathcal{E}_{i,i+1}-\mathcal{E}_{i+1,i+1}+2\mathcal{E}_{i+1,i+2}-2\mathcal{E}_{i+2,i+2}]$$

$$E_iE_{i+1}^2-(1+q)E_{i+1}E_iE_{i+1}+qE_{i+1}^2E_i=\sum_{\lambda\in\Lambda_0}([[2]]-(1+q))\,[X_\lambda]+([[2]]-(1+q)[[2]]+q[[2]])\,[Y_\lambda]$$

where

$$X_\lambda=D_\lambda+\mathcal{E}_{i,i+2}+\mathcal{E}_{i+1,i+2}-2\mathcal{E}_{i+2,i+2}$$

and

$$Y_\lambda=D_\lambda+\mathcal{E}_{i,i+1}-\mathcal{E}_{i+1,i+1}+2\mathcal{E}_{i+1,i+2}-2\mathcal{E}_{i+2,i+2}.$$

It follows

$$E_iE_{i+1}^2-(1+q)E_{i+1}E_iE_{i+1}+qE_{i+1}^2E_i=0$$

and so

$$F_{i+1}^2 F_i - (1+q)F_{i+1}F_iF_{i+1} + qF_iF_{i+1}^2 = 0,$$

by applying the transpose involution to the first relation.

$$\begin{aligned} E_i^2E_{i+1} = \sum_{\lambda\in\Lambda_0} & [[2]]\,[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i,i+2} - \mathcal{E}_{i+2,i+2}] \\ & + [[2]]\,[D_\lambda + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i+2} - \mathcal{E}_{i+2,i+2}] \end{aligned}$$

$$\begin{aligned} E_iE_{i+1}E_i = \sum_{\lambda\in\Lambda_0} & [D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i,i+2} - \mathcal{E}_{i+2,i+2}] \\ & + [[2]]\,[D_\lambda + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i+2} - \mathcal{E}_{i+2,i+2}] \end{aligned}$$

$$E_{i+1}E_i^2 = \sum_{\lambda\in\Lambda_0} [[2]]\,[D_\lambda + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i+2} - \mathcal{E}_{i+2,i+2}]$$

So

$$E_i^2E_{i+1} - (1+q)E_iE_{i+1}E_i + qE_{i+1}E_i = \sum_{\lambda\in\Lambda_0} ([[2]] - (1+q))A_\lambda + ([[2]] - (1+q)[[2]] + q[[2]])B_\lambda,$$

where

$$A_\lambda = D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i,i+2} - \mathcal{E}_{i+2,i+2}$$

and

$$B_\lambda = D_\lambda + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i+2} - \mathcal{E}_{i+2,i+2}.$$

Therefore

$$E_i^2E_{i+1} - (1+q)E_iE_{i+1}E_i + qE_{i+1}E_i = 0$$

and

$$F_{i+1}F_i^2 - (1+q)F_iF_{i+1}F_i + qF_i^2F_{i+1} = 0,$$

where the second relation follows from the first by applying the transpose involution. □

Recall the result of Lemma 3.1.14, which gives relations involving R:

$$\begin{aligned} R1_\lambda R^{-1} &= 1_{\lambda[1]} \\ RE_{i,\lambda}R^{-1} &= E_{i-1,\lambda[1]} \\ RF_{i,\lambda}R^{-1} &= F_{i-1,\lambda[1]}. \end{aligned}$$

3.2.2 A quiver algebra

Define a quiver $\Gamma = \Gamma(n,r)$ associated with the affine q-Schur algebra $\hat{S}_q(n,r)$ as follows:

- The set of vertices is $\Gamma_0 = \Lambda_0$.

- The set of edges is Γ_1, consisting of edges

$$e_{i,\lambda}\colon \lambda \to \lambda + \alpha_i$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ such that $\lambda_{i+1} > 0$ and

$$f_{i,\lambda}\colon \lambda \to \lambda - \alpha_i$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ such that $\lambda_i > 0$.

The path $\mathbb{Z}[q]$-algebra of Γ is an associative $\mathbb{Z}[q]$-algebra with a unit, which has a $\mathbb{Z}[q]$-basis consisting of the paths in Γ, where the multiplication is defined by concatenation of paths. That is, if p and q are paths in Γ, then the product pq is the path 'q followed by p' if the target of q equals the source of p, or equals zero otherwise.

For each $\lambda \in \Lambda_0$, denote the constant path at λ by k_λ. These elements form a set of pairwise orthogonal idempotents and the multiplicative identity in $\mathbb{Z}[q]\Gamma$ is

$$\sum_{\lambda \in \Lambda_0} k_\lambda.$$

For each $i \in \{1, \ldots, n\}$, define

$$e_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} f_{i,\lambda}.$$

To avoid long subscripts, define $e_{i,\lambda} = 0$ if $\lambda_{i+1} = 0$ and define $f_{i,\lambda} = 0$ if $\lambda_i = 0$.

Let $I = I(n, r)$ be the ideal in $\mathbb{Z}[q]\Gamma$ generated by the following expressions:

$$\begin{gathered} e_i e_j - e_j e_i \\ f_i f_j - f_j f_i \end{gathered}$$

for $i, j \in \{1, \ldots, n\}$ such that $j \neq i \pm 1$,

$$\begin{aligned} e_i e_{i+1}^2 &- [[2]] e_{i+1} e_i e_{i+1} + q e_{i+1}^2 e_i \\ e_i^2 e_{i+1} &- [[2]] e_i e_{i+1} e_i + q e_{i+1} e_i^2 \\ f_{i+1}^2 f_i &- [[2]] f_{i+1} f_i f_{i+1} + q f_i f_{i+1}^2 \\ f_{i+1} f_i^2 &- [[2]] f_i f_{i+1} f_i + q f_i^2 f_{i+1} \end{aligned}$$

for $i \in \{1, \ldots, n\}$,

$$e_i f_j - f_j e_i$$

for $i, j \in \{1, \ldots, n\}$ such that $i \neq j$,

$$e_i f_i - f_i e_i - \sum_{\lambda \in \Lambda_0} ([[\lambda_i]] - [[\lambda_{i+1}]]) k_\lambda$$

for $i \in \{1, \ldots, n\}$.

3.2.3 Mapping to the q-Schur algebra

Lemma 3.2.3. *There is a* $\mathbb{Z}[q]$*-algebra homomorphism*

$$\phi\colon \mathbb{Z}[q]\Gamma/I \to \hat{S}_q(n,r)$$

defined by

$$\begin{aligned}\phi(e_{i,\lambda}+I) &= E_{i,\lambda},\\ \phi(f_{i,\lambda}+I) &= F_{i,\lambda},\\ \phi(k_\lambda+I) &= 1_\lambda,\end{aligned}$$

for $i \in \{1,\dots,n\}$ *and* $\lambda \in \Lambda_0$.

Proof. Lemma 3.2.1 and Lemma 3.2.2 shows that each equation defining the ideal I corresponds to a zero relation in $\hat{S}_q(n,r)$, so there is a unique homomorphism of $\mathbb{Z}[q]$-algebras given by

$$\begin{aligned}\phi(e_{i,\lambda}+I) &= E_{i,\lambda},\\ \phi(f_{i,\lambda}+I) &= F_{i,\lambda},\\ \phi(k_\lambda+I) &= 1_\lambda,\end{aligned}$$

for each $i \in \{1,\dots,n\}$ and $\lambda \in \Lambda_0$. □

In fact, ϕ is determined by its values on $e_1,\dots,e_n$, $f_1,\dots,f_n$ and k_λ for $\lambda \in \Lambda_0$. In order to describe the image of this map we introduce the notion of standard paths in Γ.

Definition 3.2.4. A path $p = k_\lambda p_1 \cdots p_h$ with

$$p_s = e_{i+s-1}^{\alpha_{i,s}} e_{i+s-2}^{\alpha_{i-1,s}} \cdots e_{i+s-n}^{\alpha_{i-n+1,s}},$$

for $s \in \{1,\dots,h\}$, is a *standard positive path* if

$$i = \max\{t : 1 \le t \le n, \lambda_t = 0\} - 1;$$

$$\alpha_{j,s} \ge \alpha_{j,s+1} \text{ for } s \in \{1,\dots,h-1\};$$
$$0 \le \alpha_{j,1} \le \lambda_j \text{ for } j \in \{1,\dots,n\}.$$

Definition 3.2.5. A path $p = k_\lambda p_1 \cdots p_h$ with

$$p_s = f_{i-s+1}^{\beta_{i,s}} f_{i-s+2}^{\beta_{i+1,s}} \cdots f_{i-s+n}^{\beta_{i+n-1,s}},$$

for $s \in \{1,\dots,h\}$, is a *standard negative path* if

$$i = \min\{t : 1 \le t \le n, \lambda_t = 0\};$$

$$\beta_{j,s} \ge \beta_{j,s+1} \text{ for } s \in \{1,\dots,h-1\};$$
$$0 \le \beta_{j,s} \le \lambda_{j+1} \text{ for } j \in \{1,\dots,n\}.$$

Remark 3.2.6. The subindex j in e_j and f_j and the subindex j in $\alpha_{j,s}$ is regarded as an element of $\mathbb{Z}/n\mathbb{Z}$.

Definition 3.2.7. A path $p = k_\lambda p^+ p^-$ is a *standard path* if p^+ is a standard positive path, p^- is a standard negative path and

$$\alpha_{j,s} + \beta_{j-1,s} \leq \lambda_j$$

for $j \in \{1, \dots, n\}$. Call p^+ the *positive part* of p and p^- the *negative part* of p.

Remark 3.2.8. If p is a standard path with $p = p'p''$ for some paths p' and p'', then p' is a standard path.

Observe that the definition of standard paths includes the constant paths k_λ for $\lambda \in \Lambda_0$.

Definition 3.2.9. Let $A \in \Lambda_1$. The *standard path for* A is the standard path $p_A = k_\lambda p^+ p^-$, where $\lambda = \mathrm{ro}(A)$, p^+ is the standard positive path given by

$$\alpha_{i,s} = \sum_{t \geq i+s} a_{i,t}$$

and p^- is the standard negative path given by

$$\beta_{i,s} = \sum_{t \leq i-s+1} a_{i+1,t}$$

for $i \in \{1, \dots, n\}$ and $s \geq 1$, respectively.

Lemma 3.2.10. *If p is a standard path then there is a unique element $A \in \Lambda_1$ such that $p = p_A$. Thus there is a bijection between the set of standard paths in Γ and Λ_1.*

Proof. The map

$$\Lambda_1 \to \{\text{ standard paths in } \Gamma\} : A \mapsto p_A$$

is injective since distinct elements of Λ_1 define distinct standard paths. Finally, if p is a standard path then $p = p_A$ where

$$\begin{aligned} a_{i,i+s} &= \alpha_{i,s} - \alpha_{i,s+1} \\ a_{i,i-s} &= \beta_{i-1,s} - \beta_{i-1,s+1} \\ a_{i,i} &= \mu_i - \alpha_{i,1} - \beta_{i-1,1} \end{aligned}$$

for all $i \in \{1, \dots, n\}$ and $s \geq 1$. □

Definition 3.2.11. Let $A \in \Lambda_1$. The *positive part* of A is the element $A^+ \in \Lambda_1$ with $\mathrm{ro}(A^+) = \mathrm{ro}(A)$ and off diagonal entries

$$\begin{aligned} a^+_{i,j} &= a_{i,j} \text{ if } i < j; \\ a^+_{i,j} &= 0 \text{ if } i > j, \end{aligned}$$

for $i, j \in \mathbb{Z}$.

The negative part of A is the element $A^- \in \Lambda_1$ with $\mathrm{co}(A^-) = \mathrm{co}(A)$ and off-diagonal entries

$$\begin{aligned} a^-_{i,j} &= 0 \text{ if } i < j; \\ a^-_{i,j} &= a_{i,j} \text{ if } i > j, \end{aligned}$$

for $i, j \in \mathbb{Z}$.

Recall that the G-orbit of a pair of flags (L, L') is denoted by $[L, L']$.

Lemma 3.2.12. *Let $A \in \Lambda_1$. If $(L, L') \in \mathcal{O}_A$, then $[L, L \cap L'] = \mathcal{O}_{A^+}$ and $[L \cap L', L'] = \mathcal{O}_{A^-}$.*

Proof. Let $B \in \Lambda_1$ with $\mathcal{O}_B = [L, L \cap L']$. The row sum of B is $|L| = \mathrm{ro}(A)$ and B is upper triangular since $L \cap L' \subset L$. For $i < j$,

$$\begin{aligned} b_{i,j} &= \dim \left(\frac{L_i \cap L_j \cap L'_j}{L_{i-1} \cap L_j \cap L'_j + L_i \cap L_{j-1} \cap L'_{j-1}} \right) \\ &= \dim \left(\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right) \\ &= a_{i,j}, \end{aligned}$$

so B is the positive part of A as in Definition 3.2.11. The transpose of the negative part of A is the positive part of the transpose of A, so it follows that $\mathcal{O}_{A^-} = [L \cap L', L']$. $\square$

Lemma 3.2.13. *Let $A \in \Lambda_1$ and let p be the standard path for A. The positive part of p is the standard path for A^+ and the negative part of p is the standard path for A^-.*

Proof. Write $p = k_\lambda p^+ p^- k_\mu$, where p^+ and p^- are the positive and negative parts of p respectively. The exponents $\alpha_{i,s}$ in p^+ are determined by the entries of A strictly above the diagonal, so the $\alpha_{i,s}$ are also the exponents in the standard path for A^+. It follows that $k_\lambda p^+$ is the standard path for A^+ since $\lambda = \mathrm{ro}(A) = \mathrm{ro}(A^+)$.

Similarly, the exponents in the standard path for A are given by the entries in A strictly below the diagonal and $\mu = \mathrm{co}(A) = \mathrm{co}(A^-)$, so $p^- k_\mu$ is the standard path for A^-. $\square$

Proposition 3.2.14. *Let $A \in \Lambda_1$ and let p be the standard path associated to A. Then*

$$\phi(p + I) = \left(\prod_{i \in \{1,\ldots,n\}, s \geq 1} [[\alpha_{i,s}]]_! [[\beta_{i,s}]]_! \right) e_A + \sum_{B \in \Lambda_1 : B < A} g_B e_B$$

Proof. THIS IS A REALLY FUNDAMENTAL PROPOSITION – PROVE IT! $\square$

3.2.4 Change of rings

The following is based on the change of rings for quiver presentations result, Lemma 5.2 in [22]. Let R and S be commutative rings and suppose $f \colon R \to S$ is a ring homomorphism with $f(1) = 1$. Let Σ be a quiver and let $I \subset R\Sigma$ be an ideal of relations in Σ. The homomorphism f defines an R-algebra structure on S with $r \cdot s = f(r)s$ for all $r \in R$ and $s \in S$. Let $\bar{f} \colon R\Sigma \to S\Sigma$ be the R-algebra homomorphism induced by f, which is given by

$$\bar{f}(rp) = f(r)p$$

for each path p in Σ and $r \in R$.

Applying the right exact functor $S \otimes_R -$ to the short exact sequence

$$0 \to I \xrightarrow{i} R\Sigma \to R\Sigma/I \to 0$$

of R-modules gives the exact sequence

$$S \otimes_R I \xrightarrow{1 \otimes i} S \otimes_R R\Sigma \to S \otimes_R R\Sigma/I \to 0$$

of S-modules.

Let $m\colon S \otimes_R R\Sigma \to S\Sigma$ be the S-algebra homomorphism given by

$$m(s \otimes (rp + I)) = sf(r)p + S\bar{f}(I),$$

for all $r \in R$, $s \in S$ and paths p in Γ. The S-algebra homomorphism $S\Sigma \to S \otimes_R R\Sigma$ given by sending sp to $s \otimes p$ is inverse to m, so m is an isomorphism of S-algebras. Observe that m is also R-linear, so is an isomorphism of R-algebras. The image of $m \circ (1 \otimes i)\colon S \otimes_R I \to S\Sigma$ is $S\bar{f}(I)$ since the image is spanned by elements of the form

$$\begin{aligned} m((1 \otimes i)(s \otimes x)) &= m(s \otimes x) \\ &= s\bar{f}(x), \end{aligned}$$

for $s \in S$ and $x \in I$.

Thus we have a commuting diagram of S-modules with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \ker \bar{m} \arrow[d] & \\
 & S \otimes_R I \arrow[r, "1 \otimes i"] \arrow[d] & S \otimes_R R\Sigma \arrow[r] \arrow[d, "m"] & S \otimes_R R\Sigma/I \arrow[r] \arrow[d, "\bar{m}"] & 0 \\
0 \arrow[r] & S\bar{f}(I) \arrow[r] \arrow[d] & S\Sigma \arrow[r] \arrow[d] & S\Sigma/S\bar{f}(I) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & \operatorname{coker} \bar{m} &
\end{tikzcd}$$

The morphism $\bar{m}$ is given by the universal property of cokernels and can be computed explicitly using the commuting diagram, with

$$\begin{aligned} \bar{m}(s \otimes (rp + I)) &= m(s \otimes rp) + S\bar{f}(I) \\ &= sf(r)p + S\bar{f}(I), \end{aligned}$$

for all $r \in R$, $s \in S$ and paths p in Σ.

Lemma 3.2.15. *[22] The morphism*

$$\bar{m}\colon S \otimes_R R\Sigma/I \to S\Sigma/S\bar{f}(I)$$

is both an isomorphism of R-algebras and an isomorphism of S-algebras.

Proof. Using the snake lemma on the above commuting diagram gives an exact sequence of S-modules

$$0 \to \ker \bar{m} \to 0 \to 0 \to \operatorname{coker} \bar{m} \to 0,$$

so $\ker \bar{m}$ and $\operatorname{coker} \bar{m}$ are both zero and therefore $\bar{m}$ is an isomorphism of S-algebras. Moreover, $\bar{m}$ is R-linear, so is also an isomorphism of R-algebras. $\square$

Recall that the q-integers are given by

$$[[0]] = 0$$

and

$$[[m]] = 1 + q + \cdots + q^{m-1} = 1 + q[[m-1]]$$

for $m \in \mathbb{Z}$ with $m \geq 1$. For $m \in \mathbb{N}$, define the q-factorial

$$[[m]]_! = \prod_{a=1}^{m} [[a]].$$

Given integers a and b with $0 < a < b$,

$$[[b]] - [[a]] = q^a [[b-a]]$$

and the product $[[a]][[b]]$ can be computed recursively as follows:

$$\begin{aligned} [[a]][[b]] &= (1 + q[[a-1]])(1 + q[[b-1]]) \\ &= 1 + q([[a-1]] + [[b-1]] + q[[a-1]][[b-1]]). \end{aligned}$$

The set of q-integers is not multiplicatively closed since, for example $[[2]]^2 = 1 + 2q + q^2$, but the set $1 + q\mathbb{Z}[q]$ is multiplicatively closed and contains the q-integers. Let $\mathcal{Q}$ be the localisation of $\mathbb{Z}[q]$ at the set of elements of the form $1 + qf$ for $f \in \mathbb{Z}[q]$, so $\mathcal{Q}$ is the subring of $\mathbb{Q}(q)$ given by

$$\mathcal{Q} = \left\{ \frac{f}{1 + qg} : f, g \in \mathbb{Z}[q] \right\}.$$

Observe that $\mathbb{Z}[q]$ is a subring of $\mathcal{Q}$, so $\mathcal{Q}$ is a $\mathbb{Z}[q]$-algebra. The $\mathcal{Q}$-form of the affine q-Schur algebra $\hat{S}_q(n, r)$ is defined to be the $\mathcal{Q}$-algebra

$$\hat{S}_{\mathcal{Q}}(n, r) = \mathcal{Q} \otimes_{\mathbb{Z}[q]} \hat{S}_q(n, r).$$

Lemma 3.2.16. *The $\mathcal{Q}$-algebra homomorphism*

$$\bar{m} \colon \mathcal{Q} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q]\Gamma/I \to \mathcal{Q}\Gamma/\mathcal{Q}I$$

given by

$$\bar{m}(s \otimes (hp + I)) = shp + \mathcal{Q}I,$$

for all $s \in \mathcal{Q}$, $h \in \mathbb{Z}[q]$ and paths p in Γ, is an isomorphism.

Proof. Applying Lemma 3.2.15 for the inclusion $\mathbb{Z}[q] \hookrightarrow \mathcal{Q}$ proves that $\bar{m}$ is an isomorphism of $\mathcal{Q}$-algebras and an isomorphism of $\mathbb{Z}[q]$-algebras. □

Let $\phi_{\mathcal{Q}}$ be the $\mathcal{Q}$-algebra homomorphism

$$\phi_{\mathcal{Q}} = (1 \otimes \phi) \circ \bar{m}^{-1} \colon \mathcal{Q}\Gamma/\mathcal{Q}I \to \hat{S}_{\mathcal{Q}}(n, r),$$

which is given by

$$\begin{aligned} \phi_{\mathcal{Q}}(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi_{\mathcal{Q}}(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi_{\mathcal{Q}}(k_\lambda) &= 1_\lambda, \end{aligned}$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$.

Proposition 3.2.17. *If $r < n$ then $\phi_{\mathcal{Q}}$ is surjective.*

Proof. Fix $A \in \Lambda_1$ and let p be the standard path in Γ corresponding to A. Then

$$\phi_{\mathcal{Q}}(p + \mathcal{Q}I) = \sum_{B:B\leq A} g_B e_B$$

for some $g_B \in \mathbb{Z}[q]$, where

$$g_A = \prod_{i\in\{1,\dots,n\},s\geq 1} [[\alpha_{i,s}]]_![[\beta_{i,s}]]_!,$$

by Proposition 3.2.14. The coefficient of the leading term g_A is a unit in $\mathcal{Q}$, so

$$e_A = \phi_{\mathcal{Q}}(g_A^{-1}p + \mathcal{Q}I) - \sum_{B:B<A} g_B g_A^{-1} e_B.$$

There are only finitely many $B \in \Lambda_1$ with $B < A$ and for each such B, e_B admits a similar expression, which shows that e_A can be expressed as the image of a $\mathcal{Q}$-linear combination of the standard paths corresponding to the matrices B with $B \leq A$ and therefore $\phi_{\mathcal{Q}}$ is surjective. □

Conjecture 3.2.18. If $r < n$, the quiver with relations (Γ, I) gives a presentation of $\hat{S}_{\mathcal{Q}}(n, r)$ over $\mathcal{Q}$.

Ideas for proof. The only thing that remains to be shown is that the map from the quiver algebra is injective, since Proposition 3.2.17 shows that this map is surjective.

I hope to deduce this from the presentation of the affine generic algebra by tensoring the surjective map between $\mathcal{Q}$-forms of the path algebra and q-Schur algbebra with the $\mathcal{Q}$-algebra $\mathcal{Q}/(q)$ and observing this map is an isomorphism of $\mathbb{Z}$-algebras. □

3.3 Relations for the n=2 case

In this section we give relations in $\hat{S}_q(2, r)$. Compare with the relations given in [REFERENCE - a double hall algebra approach to affine quantum schur weyl theory p.13] in the presentation of quantum affine $\mathfrak{sl}_n$.

Lemma 3.3.1. *The following equations hold in $\hat{S}_q(2, r)$:*

$$\begin{aligned}
qE_1E_2^3 - [[3]]E_2E_1E_2^2 + [[3]]E_2^2E_1E_2 - qE_2^3E_1 &= 0\\
qE_1^3E_2 - [[3]]E_1^2E_2E_1 + [[3]]E_1E_2E_1^2 - qE_2E_1^3 &= 0\\
qF_2F_1^3 - [[3]]F_1F_2F_1^2 + [[3]]F_1^2F_2F_1 - qF_1^3F_2 &= 0\\
qF_2^3F_1 - [[3]]F_2^2F_1F_2 + [[3]]F_2F_1F_2^2 - qF_1F_2^3 &= 0.
\end{aligned}$$

Proof. It suffices to prove the first of these relations holds, since the second relation is obtained by applying the shifting automorphism of $\hat{S}_q(n, r)$ given by conjugation by R, which sends E_1 to E_2 and E_2 to E_1, and then the last two relations are obtained by applying the transpose operator S on $\hat{S}_q(n, r)$, which sends E_i to F_i (for $i = 1, 2$) and reverses the order of multplication.

Next, the first relation will be established by an explicit computation using the fundamental multiplication rules 3.1.3.

Write

$$\begin{aligned} W &= D_\lambda + \mathcal{E}_{1,2} - \mathcal{E}_{1,1} + 3\mathcal{E}_{2,3} - 3\mathcal{E}_{2,2} \\ X &= D_\lambda + 2\mathcal{E}_{2,3} + \mathcal{E}_{2,4} - 3\mathcal{E}_{2,2} \\ Y &= D_\lambda + \mathcal{E}_{1,3} - \mathcal{E}_{1,1} + 2\mathcal{E}_{2,3} - 2\mathcal{E}_{2,2} \\ Z &= D_\lambda + \mathcal{E}_{2,3} + \mathcal{E}_{2,5} - 2\mathcal{E}_{2,2}. \end{aligned}$$

$$E_1E_2^3 = \sum_{\lambda\in\Lambda_0} [[2]][[3]]e_W + [[2]][[3]]e_Y$$

$$E_2E_1E_2^2 = \sum_{\lambda\in\Lambda_0} [[2]][[3]]e_W + [[2]]e_X + [[2]]^2e_Y + [[2]]e_Z$$

$$E_2^2E_1E_2 = \sum_{\lambda\in\Lambda_0} [[2]][[3]]e_W + [[2]]^2e_X + [[2]]e_Y + [[2]]e_Z$$

$$E_2^3E_1 = \sum_{\lambda\in\Lambda_0} [[2]]3e_W + [[2]][[3]]e_X$$

Thus

$$\begin{aligned} qE_1E_2^3 - [[3]]E_2E_1E_2^2 + [[3]]E_2^2E_1E_2 - qE_2^3E_1 &= [[2]][[3]](q - [[3]] + [[3]] - q)e_W \\ &+ [[2]][[3]](-1 + [[2]] - q)e_X \\ &+ [[2]][[3]](q - [[2]] + 1)e_Y \\ &+ ([[2]][[3]] - [[2]][[3]])e_Z, \end{aligned}$$

which proves that the first relation holds and hence all the relations hold. □

Chapter 4

A generic affine algebra

4.1 Introduction

Assume $\mathbf{k} = \mathbb{C}$ and fix $n, r \geq 1$. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by ε, namely $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free $\mathcal{S}$-module of rank r and let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of n-periodic cyclic flags in V; so $\mathcal{F}$ consists of collections $L = (L_i)_{i\in\mathbb{Z}}$ of $\mathcal{R}$-lattices in V with $L_i \subset L_{i+1}$ for $i \in \mathbb{Z}$ and $\varepsilon L_i = L_{i-n}$ for $i \in \mathbb{Z}$.

Let G be the group of $\mathcal{S}$-module automorphisms of V. Thus G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. G acts on $\mathcal{F}$ with orbits $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0\}$, where Λ_0 is the set of compositions of r into n parts, as in Definition 2.1.1.

The diagonal action of G on $\mathcal{F}\times\mathcal{F}$ has orbits $\{\mathcal{O}_A : A \in \Lambda_1\}$, where $\mathcal{O}_A$ consists of those pairs of flags with periodic characteristic matrix equal to A. Definitions of the periodic characteristic matrix and the set Λ_1 are given in Definition 2.1.7 and Definition 2.1.2 respectively.

Recall that the periodic characteristic matrix of a pair $(L, L') \in \mathcal{F}\times\mathcal{F}$ is the $\mathbb{Z}\times\mathbb{Z}$ matrix $A = (a_{i,j})_{i,j\in\mathbb{Z}}$, with

$$a_{i,j} = \dim\left(\frac{L_i \cap L'_j}{L_{i-1}\cap L'_j + L_i \cap L'_{j-1}}\right)$$

for each $i, j \in \mathbb{Z}$.

Recall that ro and co are the maps $\Lambda_1 \to \Lambda_0$ given by

$$\mathrm{ro}(A) = \left(\sum_{j\in\mathbb{Z}} a_{1,j}, \ldots, \sum_{j\in\mathbb{Z}} a_{n,j}\right)$$

and

$$\mathrm{co}(A) = \left(\sum_{i\in\mathbb{Z}} a_{i,1}, \ldots, \sum_{i\in\mathbb{Z}} a_{i,n}\right)$$

for each $A \in \Lambda_1$. Given $A \in \Lambda_1$, write $A\colon \mathrm{co}(A) \to \mathrm{ro}(A)$.

The purpose of this chapter is to define an associative $\mathbb{Z}$-algebra with a multiplicative basis by defining a modified form of the product in the affine q-Schur algebra. In particular, given $A, B \in \Lambda_1$, the orbit product

$$X_{A,B} = \{(L, L'') \in \mathcal{F}\times\mathcal{F} : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

consists of finitely many G-orbits and it will be shown that there is a unique 'generic' orbit in $X_{A,B}$, denoted $\mathcal{O}_{A*B}$, with the property that

$$\dim\left(\frac{L_i}{L_i \cap L''_j}\right) \leq \dim\left(\frac{N_i}{N_i \cap N''_j}\right)$$

and

$$\dim\left(\frac{L''_j}{L_i \cap L''_j}\right) \leq \dim\left(\frac{N''_j}{N_i \cap N''_j}\right)$$

for all $i, j \in \mathbb{Z}$, $(N, N'') \in \mathcal{O}_{A*B}$ and $(L, L'') \in X_{A,B}$. It will be shown that the above 'generic product' of orbits is associative, so the free $\mathbb{Z}$-module on the set of G-orbits in $\mathcal{F} \times \mathcal{F}$ with $\mathbb{Z}$-bilinear multiplication given by

$$\mathcal{O}_A * \mathcal{O}_B = \mathcal{O}_{A*B},$$

for each $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, and

$$\mathcal{O}_A * \mathcal{O}_B = 0$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) \neq \mathrm{ro}(B)$, is an associative $\mathbb{Z}$-algebra with multiplicative identity given by

$$\sum_{\lambda \in \Lambda_0} \mathcal{O}_{D_\lambda},$$

where D_λ is the diagonal matrix with $\mathrm{co}(D_\lambda) = \lambda$. The resulting $\mathbb{Z}$-algebra is called the *generic affine algebra* (of rank r and period n), denoted $\hat{G}(n, r)$.

4.2 Grassmannians and related varieties

Here we collect a few elementary results on Grassmannians and some related varieties. In this section, let V be an n-dimensional $\mathbf{k}$-vector space and let $0 \leq d \leq n$ be an integer. There is a linear map

$$\phi^{(d)} \colon \Lambda^d(V) \to \mathrm{Hom}(V, \Lambda^{d+1}(V))$$

given by

$$\phi^{(d)}(\alpha)(v) = \alpha \wedge v$$

for $\alpha \in \Lambda^d(V)$ and $v \in V$. The kernel of $\phi^{(d)}(\alpha)$ is the space of divisors of α,

$$D_\alpha = \{v \in V : \alpha \wedge v = 0\}.$$

An element $\alpha \in \Lambda^d(V)$ is said to be totally decomposable if $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_d$, where $\alpha_1, \ldots, \alpha_d \in V$ are linearly independent. The dimension of D_α is at most d and $\dim(D_\alpha) = d$ precisely when α is totally decomposable. Consequently, the rank of $\phi^{(d)}(\alpha)$ is at least $n - d$ and α is totally decomposable if and only if $\mathrm{rank}\, \phi^{(d)}(\alpha) \leq n - d$, which holds if and only if the $(n-d+1) \times (n-d+1)$-minors of a matrix of $\phi^{(d)}(\alpha)$ are all zero.

Lemma 4.2.1. *$\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a projective variety, for each $d_1, d_2, a \in \mathbb{N}$ with $d_1, d_2, a \leq n$.*

Proof. As above, there is a linear map $\Psi\colon \Lambda^{d_1}V \oplus \Lambda^{d_2}V \to \mathrm{Hom}(V, \Lambda^{d_1+1}(V) \oplus \Lambda^{d_2+1}(V))$ given by $\Psi(\alpha,\beta)(v) = (\alpha\wedge v, \beta\wedge v)$. Given $\alpha \in \Lambda^{d_1}(V)$ and $\beta \in \Lambda^{d_2}(V)$, the kernel of $\Psi(\alpha,\beta)$ is $D_\alpha \cap D_\beta$ and so the rank of $\Psi(\alpha,\beta)$ is $n - \dim(D_\alpha \cap D_\beta)$.

Let $U_i \in \mathrm{Gr}_{d_i}(V)$ and suppose $p_i(U_i) = [\alpha_i]$, where p_i is the Plücker embedding of $\mathrm{Gr}_{d_i}(V)$ in $\mathbb{P}(\Lambda^{d_i}(V))$, so $U_i = D_{\alpha_i} = \ker \phi^{(d_i)}(\alpha)$. Therefore the kernel of $\Psi(\alpha_1,\alpha_2)$ is $U_1 \cap U_2$, so the condition that $\dim(U_1 \cap U_2) \geq a$ is equivalent to the condition that $\Psi(\alpha_1,\alpha_2)$ has rank at most $n-a$. After fixing a basis of V, this condition is given by the vanishing of the $(n-a+1)\times(n-a+1)$ minors of the matrix of $\Psi(\alpha_1,\alpha_2)$ with respect to this basis. Therefore $\{(U_1,U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a closed subset of the product of Grassmannians $\mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V)$, so is a projective variety.

More precisely, the entries of a matrix of $\Psi(\alpha_1,\alpha_2)$ are homogeneous polynomials of degree 1 in the Plücker coordinates on $\mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V)$ since Ψ is linear and so the minors of $\Psi(\alpha_1,\alpha_2)$ are also homogeneous polynomials in the Plücker coordinates. □

Lemma 4.2.2. *Let V be an n-dimensional vector space over $\mathbf{k}$ and let $d_1, d_2, a \in \mathbb{N}$ with $d_1, d_2, a \leq n$. The following hold:*

1. $\{(U_1,U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) = a\}$ *is a quasiprojective variety;*
2. $\{(U_1,U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : U_1 \subset U_2\}$ *is a projective variety;*
3. *Given $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a projective variety;*
4. *Given $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) = a\}$ is a quasiprojective variety;*
5. *Given $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : U_1 \subset U_2\}$ is a projective variety;*
6. *Given $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : U_2 \subset U_1\}$ is a projective variety.*

Proof. Let X_i denote the space in statement i of the lemma. To emphasise the dependence of X_i on a, write $X_{i,a}$.

X_1 is a quasiprojective variety since it is equal to the intersection of the projective variety $\{(U_1,U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ with the open set $\{(U_1,U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \leq a\}$.

Given $(U_1,U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V)$, $U_1 \subset U_2$ if and only if $\dim(U_1 \cap U_2) \geq d_1$, so Lemma 4.2.1 shows X_2 is a projective variety.

Let $\pi_i\colon \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) \to \mathrm{Gr}_{d_i}(V)$ be the projection map onto the i-th factor, for $i = 1,2$. The completeness property of projective varieties ensures that π_i is a closed morphism. Observe that

$$\begin{aligned} X_3 &= \{U_1 \in \mathrm{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\} \\ &= \pi_1(\{(U_1,W) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap W) \geq a\} \cap \pi_2^{-1}\{U_2\}). \end{aligned}$$

The fibre of π_2 over U_2 is closed, so the intersection of the fibre with the variety from Lemma 4.2.1 is closed and then the image of this intersection under π_1 is closed. This shows X_3 is a projective variety.

X_4 is a quasiprojective variety since it is the complement of the subvariety $X_{3,a+1}$ in $X_{3,a}$. Finally, *5-6* follow as special cases of *3* since $X_5 = X_{3,d_1}$ and $X_6 = X_{3,d_2}$. □

4.3 Geometry of affine flag varieties

Given $L \in \mathcal{F}$, $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0$ define

$$\Pi_{N,\lambda}(L) = \{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi^a_{N,\lambda}(L) = \left\{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0, \dim\left(\frac{\varepsilon^{-N}L_0}{L'_0}\right) = a\right\}.$$

Lemma 4.3.1. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$ and $\lambda \in \Lambda_0$,*

$$\Pi_{N,\lambda}(L) = \bigcup_{a:0\le a\le 2Nr} \Pi^a_{N,\lambda}(L).$$

Proof. If $L' \in \Pi_{N,\lambda}(L)$ then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and the $\mathbf{k}[\varepsilon]$-module $\varepsilon^{-N}L_0/L'_0$ is naturally isomorphic to $(\varepsilon^{-N}L_0/\varepsilon^N L_0)/(L'_0/\varepsilon^N L_0)$, so

$$\dim\left(\frac{\varepsilon^{-N}L_0}{L'_0}\right) \le \dim\left(\frac{\varepsilon^{-N}L_0}{\varepsilon^N L_0}\right) = 2Nr.$$

□

Lemma 4.3.2. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \le a \le 2Nr$, $\Pi^a_{N,\lambda}(L)$ is a projective algebraic variety.*

Proof. Let W be the $\mathbf{k}[\varepsilon]$-module $\varepsilon^{-1-N}L_0/\varepsilon^N L_0$, which has dimension $(2N+1)r$ over $\mathbf{k}$. Let $d_i = 2Nr - a + \lambda_1 + \cdots + \lambda_i$ for each $i = 1, \ldots, n$. The correspondence between submodules of $\varepsilon^{-1-N}L_0$ which contain $\varepsilon^N L_0$ and submodules of $\varepsilon^{-1-N}L_0/\varepsilon^N L_0$ determines a map

$$\rho\colon \Pi^a_{N,\lambda}(L) \to \mathrm{Gr}_{d_1}(W) \times \cdots \times \mathrm{Gr}_{d_n}(W),$$

with $\rho(L') = (L'_1/\varepsilon^N L_0, \ldots, L'_n/\varepsilon^N L_0)$.

Let $\mathcal{X}$ be the space of $(U_1, \ldots, U_n) \in \mathrm{Gr}_{d_1}(W) \times \cdots \times \mathrm{Gr}_{d_n}(W)$ with $U_i \subset U_{i+1}$ for $i = 1, \ldots, n-1$ and $\varepsilon U_n \subset U_1$. Lemma 4.2.2 shows that each of these conditions is closed, so $\mathcal{X}$ is a closed subset of $\mathrm{Gr}_{d_1}(W) \times \cdots \times \mathrm{Gr}_{d_n}(W)$, therefore $\mathcal{X}$ is a projective algebraic variety.

The image of ρ is contained in $\mathcal{X}$ since

$$\varepsilon L'_n/\varepsilon^N L_0 = L'_0/\varepsilon^N L_0 \subset L'_1/\varepsilon^N L_0 \subset \cdots \subset L'_n/\varepsilon^N L_0.$$

Suppose $(U_1, \ldots, U_n) \in \mathcal{X}$. Then U_i is a $\mathbf{k}[\varepsilon]$-module, since $\varepsilon U_i \subset \varepsilon U_n \subset U_1 \subset U_i$, for each $i = 1, \ldots, n$, so U_i lifts uniquely to a $\mathbf{k}[\varepsilon]$-module L'_i with $\varepsilon^N L_0 \subset L'_i \subset \varepsilon^{-1-N}L_0$. Therefore $L'_1, \ldots, L'_n$ are $\mathbf{k}[\varepsilon]$-lattices with $L_i \subset L_{i+1}$ for $i = 1, \ldots, n-1$ and $\varepsilon L'_n \subset L'_1$, with

$$\dim\left(\varepsilon^{-1-N}L_0/L'_n\right) = \dim\left(W/W_n\right) = (2N+1)r - d_n = a$$

and

$$\dim\left(L'_i/L'_{i-1}\right) = \dim\left(W_i/W_{i-1}\right) = d_i - d_{i-1} = \lambda_i,$$

for each $i = 2, \ldots, n$. Therefore there is a unique $L' \in \Pi^a_{N,\lambda}(L)$ such that $\rho(L') = (W_1, \ldots, W_n)$, where L' is given by $L'_{i+cn} = \varepsilon^{-c}L'_i$ for $i = 1, \ldots, n$ and $c \in \mathbb{Z}$. It follows ρ is injective and $\operatorname{im}\rho = \mathcal{X}$, which is a projective variety, so $\Pi^a_{N,\lambda}(L)$ is a projective variety. □

Lemma 4.3.3. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \le a \le 2Nr$, $\Pi^a_{N,\lambda}(L)$ is closed in $\Pi^{a+r}_{N+1,\lambda}(L)$.*

Proof. If $L' \in \Pi^a_{N,\lambda}(L)$, then $\varepsilon^{N+1}L_0 \subset \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0 \subset \varepsilon^{-(N+1)}L_0$ and

$$\dim\left(\frac{\varepsilon^{-(1+n)}L_0}{L'_0}\right) = \dim\left(\frac{L_0}{\varepsilon L_0}\right) + \dim\left(\frac{\varepsilon^{-N}L_0}{L'_0}\right) = r + a,$$

which shows that $\Pi^a_{N,\lambda}(L) \subset \Pi^{a+r}_{N+1,\lambda}(L)$. For $L' \in \Pi^{a+r}_{N+1,\lambda}(L)$, if additionally $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$, then

$$\dim\left(\frac{\varepsilon^{-(N+1)}L_0}{L'_0}\right) = r + \dim\left(\frac{\varepsilon^{-N}L_0}{L'_0}\right),$$

which shows $L' \in \Pi^a_{N,\lambda}(L)$. Therefore $\Pi^a_{N,\lambda}(L)$ is the subspace of $\Pi^{a+r}_{N+1,\lambda}(L)$ defined by the two closed conditions $\varepsilon^N L_0 \subset L'_0$ and $L'_0 \subset \varepsilon^{-N}L_0$, using Lemma 4.2.2. □

Lemma 4.3.4. *Let $\lambda \in \Lambda_0$, $M, N \in \mathbb{N}$, $L, \tilde{L} \in \mathcal{F}$, $0 \le a \le 2Nr$, $0 \le b \le 2Mr$. $\Pi^a_{N,\lambda}(L) \cap \Pi^b_{M,\lambda}(\tilde{L})$ is a closed set in $\Pi^a_{N,\lambda}(L)$. In particular, if the intersection is nonempty it is a projective algebraic variety.*

Proof. Observe that $\Pi^a_{N,\lambda}(L) \cap \Pi^b_{M,\lambda}(\tilde{L})$ is the subset of $\Pi^a_{N,\lambda}(L)$ defined by the additional conditions that $\varepsilon^M \tilde{L}_0 \subset L'_0$ and $L'_0 \subset \varepsilon^{-M}\tilde{L}_0$, so is a closed subset of $\Pi^a_{N,\lambda}(L)$, using 4.2.2. □

Lemma 4.3.5. *Suppose $L \in \mathcal{F}$, $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0$ with $a \le 2Nr$. For each $g \in G$, the natural map (restriction of the action map) $\Pi^a_{N,\lambda}(L) \to \Pi^a_{N,\lambda}(gL)$ is an isomorphism of projective varieties.*

Proof. If $L' \in \Pi^a_{N,\lambda}(L)$, then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$ and so $\varepsilon^N g(L_0) \subset g(L'_0) \subset \varepsilon^{-N}g(L_0)$, so $gL' \in \Pi^a_{N,\lambda}(L)$. Thus g and g^{-1} induce mutually inverse morphisms of varieties $g\colon \Pi^a_{N,\lambda}(L) \to \Pi^a_{N,\lambda}(gL)$ and $g^{-1}\colon \Pi^a_{N,\lambda}(gL) \to \Pi^a_{N,\lambda}(L)$. □

4.3.1 Action through an algebraic group

Let W be the $\mathbb{C}[\varepsilon]$-module $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$. ε^{2N+1} acts as zero on W and $\mathbb{C}[\varepsilon]/\langle\varepsilon^{2N+1}\rangle \otimes_{\mathbb{C}[\varepsilon]} W$ is a free $\mathbb{C}[\varepsilon]/\langle\varepsilon^{2N+1}\rangle$-module of rank r. In particular, W is a complex vector space of dimension $(2N+1)r$.

Each element $g \in G_L$ determines an endomorphism $\overline{g}$ of W, given by

$$\overline{g}(x + \varepsilon^N L_0) = g(x) + \varepsilon^N L_0,$$

for each $x \in \varepsilon^{-1-N}L_0$. Given $g, h \in G_L$, $\overline{gh} = \overline{g}\overline{h}$ and so $\bar{g}$ is an automorphism of W with $\overline{g}^{-1} = \overline{g^{-1}}$. Therefore the map $\bar{\ }\colon G_L \mapsto \mathrm{GL}(W)$ given by $g \mapsto \overline{g}$ is a group homomorphism with kernel

$$H_{N,L} \coloneqq \{g \in G_L : \overline{g} = 1\},$$

which consists of those $g \in G_L$ such that

$$g(x) - x \in \varepsilon^N L_0$$

for each $x \in \varepsilon^{-1-N}L_0$. Thus $G_L/H_{N,L}$ may be identified with a subgroup of $\mathrm{GL}(W)$.

Lemma 4.3.6. *$G_L/H_{N,L}$ is a connected algebraic group.*

Proof. As a result of the first isomorphism theorem, $G_L/H_{N,L}$ is isomorphic to the image of G_L in $\mathrm{GL}(W)$, which will be described explicitly by equations in the coordinate functions on $\mathrm{GL}(W)$, with respect to a fixed basis of W.

Let $\{\tilde{x}_1, \dots, \tilde{x}_r\}$ be a basis of L_n/L_0 over $\mathbb{C}$ which is adapted to the flag

$$L_1/L_0 \subset \cdots \subset L_{n-1}/L_0 \subset L_n/L_0,$$

so that

$$L_i/L_0 = \langle \tilde{x}_1, \dots, \tilde{x}_{\lambda_1 + \cdots \lambda_i} \rangle$$

for each $i \in \{1, \dots, n\}$. Fix $x_1, \dots, x_r \in L_n$ such that $\tilde{x}_i = x_i + L_0$ for each $i = 1, \dots, r$, then

$$L_i = L_0 + \langle x_1, \dots, x_{\lambda_1 + \cdots + \lambda_i} \rangle$$

for $i = 1, \dots, r$.

Then W has a $\mathbb{C}$-basis $\{y_j : 1 \leq j \leq (2N+1)r\}$ given by

$$y_{i+cr} = \varepsilon^{-c+N} x_i$$

for each $i \in \{1, \dots, r\}$ and $c \in \{0, \dots, 2N\}$. Observe that $\varepsilon y_i = 0$ for $i \in \{1, \dots, r\}$ and $\varepsilon y_i = y_{i-r}$ for $r < i \leq (2N+1)r$.

The coordinate functions on $\mathrm{GL}(W)$ with respect to this choice of basis are the maps

$$\gamma_{i,j} \colon \mathrm{GL}(W) \to \mathbb{C}$$

for $i, j \in \mathbb{Z}$ with $1 \leq i, j \leq (2N+1)r$, given by

$$g(y_j) = \sum_i \gamma_{ij}(g) y_i,$$

for each $j = 1, \dots, (2N+1)r$.

The image of G_L in $\mathrm{GL}(W)$ is the subgroup defined by the conditions

$$\gamma_{i,j} = \gamma_{i-r,j-r}$$

for each $i, j \in \{r+1, \dots, (2N+1)r\}$ and

$$\gamma_{i,j} = 0$$

for each $i, j \in \{1, \dots, (2N+1)r\}$ with $i > \lambda_1 + \cdots + \lambda_s$ and $j \leq \lambda_1 + \cdots + \lambda_s$ for some $s \in \{1, \dots, r\}$. This shows that the image of G_L in $\mathrm{GL}(W)$ is a connected algebraic group and therefore $G_L/H_{N,L}$ is a connected algebraic group.

With respect to the basis $\{y_i : i \in \{1, \dots, (2N+1)r\}\}$, the image of G_L in $\mathrm{GL}(W)$ consists of matrices of the form

$$\begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{2N} \\ 0 & A_0 & A_1 & \cdots & A_{2N-1} \\ 0 & 0 & A_0 & \cdots & A_{2N-2} \\ & & & & \\ 0 & 0 & 0 & \cdots & A_0 \end{pmatrix}$$

where $A_0 \in \mathcal{P}_\lambda$ and $A_1, \dots, A_{2N} \in M_r(\mathbb{C})$, where $\mathcal{P}_\lambda$ is the parabolic subgroup of $\mathrm{GL}_r(\mathbb{C})$ which is the stabiliser of the flag

$$L_1/L_0 \subset \cdots \subset L_{n-1}/L_0 \subset L_n/L_0.$$

□

Given $g \in G$, the map $G_L \to G_{gL}$ sending h to ghg^{-1} is a group isomorphism which descends to an isomorphism of algebraic groups $G_L/H_{N,L} \to G_{gL}/H_{N,gL}$. Thus we have a commuting diagram of morphisms of varieties, where the vertical arrows are isomorphisms:

$$\begin{array}{ccc} G_L/H_{N,L} \times \Pi^a_{N,\lambda}(L) & \longrightarrow & \Pi^a_{N,\lambda}(L) \\ \downarrow & & \downarrow \\ G_{gL}/H_{N,gL} \times \Pi^a_{N,\lambda}(gL) & \longrightarrow & \Pi^a_{N,\lambda}(gL) \end{array}$$

4.3.2 Incidence in affine flag varieties

Lemma 4.3.7. *Given $N, a, b, c \in \mathbb{N}$, $\lambda, \mu \in \Lambda_0$, $L \in \mathcal{F}$ and $i, j \in \mathbb{Z}$,*

$$\left\{(L', L'') \in \Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L) : \dim\left(\frac{L'_i}{L'_i \cap L''_j}\right) \le c\right\}$$

is a closed set in the projective variety $\Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L)$.

Proof. There is $M \ge N$ so that $\varepsilon^M L_0 \subset L'_i \subset \varepsilon^{-M} L_0$ and $\varepsilon^M L_0 \subset L''_j \subset \varepsilon^{-M} L_0$. Let $a' = a + (M - N)r$ and $b' = b + (M - N)r$. Lemma 4.3.3 shows that $\Pi^a_{N,\lambda}(L)$ is a subvariety of $\Pi^{a'}_{M,\lambda}(L)$, so $\Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L)$ is a subvariety of $\Pi^{a'}_{M,\lambda}(L) \times \Pi^{b'}_{M,\mu}(L)$.

The fact that

$$\dim\left(\frac{L'_i}{L'_i \cap L''_j}\right) = \dim\left(\frac{L'_i/\varepsilon^M L_0}{L'_i/\varepsilon^M L_0 \cap L''_j/\varepsilon^M L_0}\right),$$

together with Lemma 4.3.2 and Lemma 4.2.1, shows that

$$\left\{(L', L'') \in \Pi^{a'}_{M,\lambda}(L) \times \Pi^{b'}_{M,\mu}(L) : \dim\left(\frac{L'_i}{L'_i \cap L''_j}\right) \le c\right\}$$

is closed, so the intersection with $\Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L)$ is closed. $\square$

Lemma 4.3.8. *Given $N, a, c \in \mathbb{N}$, $\lambda \in \Lambda_0$, $L \in \mathcal{F}$ and $i, j \in \mathbb{Z}$,*

$$\left\{L' \in \Pi^a_{N,\lambda}(L) : \dim\left(\frac{L_i}{L_i \cap L'_j}\right) \le c\right\}$$

and

$$\left\{L' \in \Pi^a_{N,\lambda}(L) : \dim\left(\frac{L'_j}{L_i \cap L'_j}\right) \le c\right\}$$

are closed sets in $\Pi^a_{N,\lambda}(L)$.

Proof. This is a result of Lemma 4.2.2, since

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) = \dim\left(\frac{L_i/\varepsilon^M L_0}{L_i/\varepsilon^M L_0 \cap L'_j/\varepsilon^M L_0}\right),$$

where $M \ge N$ is chosen so that $\varepsilon^M L_0 \subset L_i \subset \varepsilon^{-M} L_0$ and $\varepsilon^M L_0 \subset L'_j \subset \varepsilon^{-M} L_0$ for each $L' \in \Pi^a_{N,\lambda}(L)$. $\square$

4.4 Geometry of orbits

Let $A \in \Lambda_1$ and $L \in \mathcal{F}_{\mathrm{ro}(A)}$ and write $\lambda = \mathrm{co}(A)$. Recall that

$$X_A^L = \{L' \in \mathcal{F}_\lambda : (L, L') \in \mathcal{O}_A\}.$$

Lemma 4.4.1. *There is $N \in \mathbb{N}$ such that $X_A^L \subset \Pi^a_{N,\lambda}(L)$, where $a = d_{nN,0}A$.*

Proof. There is $N \in \mathbb{N}$ so that $a_{i,j} = 0$ whenever $|j - i| > nN$. If $(L, L') \in \mathcal{O}_A$ then

$$\dim\left(\frac{L'_0}{L'_0 \cap \varepsilon^{-N} L_0}\right) = \dim\left(\frac{L'_0}{L'_0 \cap L_{nN}}\right) = \sum_{s>nN, t\leq 0} a_{s,t} = 0,$$

so it follows $L'_0 \subset \varepsilon^{-N} L_0$. Similarly,

$$\dim\left(\frac{\varepsilon^N L_0}{\varepsilon^N L_0 \cap L'_0}\right) = \dim\left(\frac{L_{-nN}}{L_{-nN} \cap L'_0}\right) = \sum_{s\leq -nN, t>0} a_{s,t} = 0,$$

which shows $\varepsilon^N L_0 \subset L'_0$. Moreover,

$$\dim\left(\frac{\varepsilon^{-N} L_0}{L'_0}\right) = \dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N} L_0 \cap L'_0}\right) = \sum_{s\leq nN, t>0} a_{s,t} = d_{nN,0}(A),$$

as a result of Lemma 2.1.12. □

Assume $N \in \mathbb{N}$ is chosen so that $X_A^L \subset \Pi^a_{N,\lambda}(L)$, where $a = d_{nN,0}A$, as in Lemma 4.4.1.

Lemma 4.4.2. *X_A^L is a locally closed subset of $\Pi^a_{N,\lambda}(L)$. In particular, X_A^L is a quasiprojective variety.*

Proof. If $L' \in \Pi^a_{N,\lambda}(L)$ then

$$L_{-Nn} = \varepsilon^N L_0 \subset L'_0 \subset L'_1 \subset L'_n \subset \varepsilon^{-1-N} L_0 = L_{(N+1)n}.$$

Therefore X_A^L is the subset of $\Pi^a_{N,\lambda}(L)$ defined by the conditions $\dim(L_i/L_i \cap L'_j) = d_{i,j}A$ for $i : -Nn \leq i < j$ and $\dim(L'_j/L_i \cap L'_j) = \bar{d}_{i,j}A$ for $i : j < i \leq (N+1)n$, for $j = 1, \ldots, n$.

The set of $L' \in \Pi^a_{N,\lambda}(L)$ with $\dim(L_i/L_i \cap L'_j) \leq d_{i,j}A$ for $j = 1, \ldots, n$ and $i : -Nn \leq i < j$ and $\dim(L'_j/L_i \cap L'_j) \leq \bar{d}_{i,j}A$ for $j = 1, \ldots, n$ and $i : j < i \leq (N+1)n$ is a closed subset of $\Pi^a_{N,\lambda}(L)$, as a result of Lemma 4.3.8.

On the other hand, the set of $L' \in \Pi^a_{N,\lambda}(L)$ satisfying the conditions $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}A$ (for $i < j$) and $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}A$ (for $i > j$) is open in $\Pi^a_{N,\lambda}(L)$ since the complement is closed, as a result of Lemma 4.3.8.

Therefore X_A^L is the intersection of an open set and a closed set in $\Pi^a_{N,\lambda}(L)$, so X_A^L is locally closed. It follows that X_A^L is an open subset of the projective variety $\overline{X_A^L}$, so is a quasiprojective variety as claimed. □

Lemma 4.4.3. *X_A^L is irreducible.*

Proof. For any $L' \in X_A^L$, $X_A^L = G_L/H_{N,L} \cdot L'$. Lemma 4.3.6 shows that $G_L/H_{N,L}$ is a connected algebraic group which acts algebraically on $\Pi^a_{N,\lambda}(L)$. The image of $G_L/H_{N,L}$ under the morphism $g \mapsto gL'$ equals X_A^L, which shows X_A^L is irreducible since $G_L/H_{N,L}$ is irreducible. □

Consequently, $\overline{X_A^L}$ is an irreducible projective variety and the action of $G_L/H_{N,L}$ on $\Pi^a_{N,\lambda}(L)$ restricts to an algebraic group action on $\overline{X_A^L}$ for which there are finitely many orbits. In particular, $\overline{X_A^L} \setminus X_A^L$ is a union of finitely many orbits which are so-called degenerations of the orbit X_A^L.

4.5 Geometry of orbit products

Let $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$ and write $\lambda = \mathrm{co}(A)$ and $\mu = \mathrm{co}(B)$. Fix $L \in \mathcal{F}_{\mathrm{ro}(A)}$. Recall

$$Y^L_{A,B} = \{(L', L'') \in \mathcal{F}_\lambda \times \mathcal{F}_\mu : L' \in X^L_A, L'' \in X^{L'}_B\}$$

and

$$X^L_{A,B} = \{L'' \in \mathcal{F}_\mu : \exists L' \in X^L_A \text{ with } L'' \in X^{L'}_B\}$$

Lemma 4.5.1. *There is $N \in \mathbb{N}$ such that*

$$Y^L_{A,B} \subset \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L),$$

where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$.

Proof. There is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$ for each $(L', L'') \in Y^L_{A,B}$, using Lemma 4.4.1. Set $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$.

Then for any $(L', L'') \in Y^L_{A,B}$,

$$\varepsilon^{2N} L_0 \subset \varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0 \subset \varepsilon^{-2N} L_0$$

and

$$\begin{aligned}
\dim\left(\frac{\varepsilon^{-2N} L_0}{L''_0}\right) &= \dim\left(\frac{\varepsilon^{-N} L'_0}{L''_0}\right) + \dim\left(\frac{\varepsilon^{-2N} L_0}{\varepsilon^{-N} L'_0}\right) \\
&= \dim\left(\frac{\varepsilon^{-N} L'_0}{L''_0}\right) + \dim\left(\frac{\varepsilon^{-N} L_0}{L'_0}\right) \\
&= a + b,
\end{aligned}$$

as a result of Lemma 2.1.12, so $(L', L'') \in \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ as required. □

Now assume $N \in \mathbb{N}$ is chosen so that $Y^L_{A,B} \subset \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$, where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$, using Lemma 4.5.1.

Lemma 4.5.2. *$Y^L_{A,B}$ is a locally closed subset of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$. In particular, $Y^L_{A,B}$ is a quasiprojective variety.*

Proof. $Y^L_{A,B}$ is the subset of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ consisting of those (L', L'') satisfying the following conditions: $\dim(L_i/L_i \cap L'_j) = d_{i,j}(A)$ for $i < j$, $\dim(L'_j/L_i \cap L'_j) = \bar{d}_{i,j}(A)$ for $i > j$, $\dim(L'_i/L'_i \cap L''_j) = d_{i,j}(B)$ for $i < j$ and $\dim(L''_j/L'_i \cap L''_j) = \bar{d}_{i,j}(B)$. Only finitely many conditions are required to define $Y^L_{A,B}$ since there are only finitely many nonzero entries in A and B modulo the (n, n)-periodicity.

The conditions $\dim(L_i/L_i \cap L'_j) \leq d_{i,j}(A)$, $\dim(L'_i/L'_i \cap L''_j) \leq d_{i,j}(B)$, $\dim(L'_j/L_i \cap L'_j) \leq \bar{d}_{i,j}(A)$ and $\dim(L''_j/L'_i \cap L''_j) \leq \bar{d}_{i,j}(B)$ define closed subsets of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ for each $i, j \in \mathbb{Z}$, as a result of Lemma 4.3.7 and Lemma 4.3.8.

On the other hand, the conditions $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}(A)$, $\dim(L'_i/L'_i \cap L''_j) \geq d_{i,j}(B)$, $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}(A)$ and $\dim(L''_j/L'_i \cap L''_j) \geq \bar{d}_{i,j}(B)$ define open subsets of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ for each $i, j \in \mathbb{Z}$, using Lemma 4.3.7 and Lemma 4.3.8.

Therefore $Y^L_{A,B}$ is the intersection of finitely many open sets and finitely many closed sets in $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$, so $Y^L_{A,B}$ is locally closed. In particular, $Y^L_{A,B}$ is a quasiprojective variety. □

Lemma 4.5.3. *For any $L' \in X_A^L$, $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$.*

Proof. Let $L' \in X_A^L$, then $\{L'\} \times X_B^{L'}$ is contained in $Y_{A,B}^L$ and G_L acts on $Y_{A,B}^L$, so $G_L \cdot (\{L'\} \times X_B^{L'})$ is contained in $Y_{A,B}^L$. If $(N', N'') \in Y_{A,B}^L$, then $N' = \sigma L'$ for some $\sigma \in G_L$, since $N' \in X_A^L$. Then $(N', N'') = \sigma(L', \sigma^{-1}N'')$ and $\sigma^{-1}N'' \in X_B^{\sigma^{-1}N'} = X_B^{L'}$, so $(N', N'') \in \sigma \cdot (\{L'\} \times X_B^{L'})$. Therefore $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$ as claimed. □

Proposition 4.5.4. *$Y_{A,B}^L$ is irreducible.*

Proof. Let $L' \in X_A^L$. $G_L/H_{2N,L}$ is a connected algebraic group acting algebraically on $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ by Lemma 4.3.6. $X_B^{L'}$ is an irreducible locally closed subset of $\Pi_{2N,\mu}^{a+b}(L)$, so $\{L'\} \times X_B^{L'}$ is an irreducible locally closed set in $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$. $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'}) = G_L/H_{2N,L} \cdot (\{L'\} \times X_B^{L'})$, by Lemma 4.5.3, so it follows that $Y_{A,B}^L$ is irreducible. □

Let p_2 be the projection onto the second factor $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \to \Pi_{2N,\mu}^{a+b}(L)$. p_2 is a closed morphism since $\Pi_{N,\lambda}^a(L)$ is a projective variety and therefore complete, by Lemma 4.3.2. Therefore $p_2(\overline{Y_{A,B}^L}) = \overline{X_{A,B}^L}$, since $p_2(Y_{A,B}^L) = X_{A,B}^L$.

Lemma 4.5.5. *$X_{A,B}^L$ is irreducible and constructible.*

Proof. Proposition 4.5.4 shows that $Y_{A,B}^L$ is irreducible and locally closed, so it follows $X_{A,B}^L$ is irreducible and constructible, since $X_{A,B}^L = p_2(Y_{A,B}^L)$. □

Proposition 4.5.6. *There is a unique open G_L-orbit in $X_{A,B}^L$.*

Proof. $X_{A,B}^L$ consists of finitely many G_L-orbits and is an irreducible topological space, by Lemma 4.5.5. Consequently, X_C^L is dense in $X_{A,B}^L$ for some $C \in {\Lambda_1}^{A,B}$. Lemma 4.4.2 shows that X_C^L is locally closed in $X_{A,B}^L$, so X_C^L is open in $\overline{X_C^L} = X_{A,B}^L$. Irreducibility of $X_{A,B}^L$ shows that there is a unique open G_L-orbit, since two nonempty open sets in $X_{A,B}^L$ intersect nontrivially, thus any two open G_L orbits in $X_{A,B}^L$ coincide. □

Let $A * B \in \Lambda_1$ denote the matrix corresponding to the dense open G_L-orbit in $X_{A,B}^L$, so $\overline{X_{A*B}^L} = \overline{X_{A,B}^L}$.

4.6 Degenerations of orbits and the combinatorial partial order

Proposition 4.6.1. *Let $A, B \in \Lambda_1$ with $\mathrm{ro}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(A) = \mathrm{co}(B)$. If $X_B^L \subset \overline{X_A^L}$ for some $L \in \mathcal{F}_{\mathrm{ro}(A)}$ then $B \leq A$ with respect to the hook order.*

Proof. Let $\lambda = \mathrm{co}(A)$, $\mu = \mathrm{ro}(A)$ and fix $L \in \mathcal{F}_\mu$. Assume $N \in \mathbb{N}$ is chosen so that $X_A^L \subset \Pi_{N,\lambda}^a(L)$ and $X_B^L \subset \Pi_{N,\lambda}^b(L)$, where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$. Then X_A^L is an open subset of the projective variety consisting of those $L' \in \Pi_{N,\lambda}^a(L)$ such that

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) \leq d_{i,j}(A)$$

and

$$\dim\left(\frac{L'_j}{L_i\cap L'_j}\right)\leq \bar{d}_{i,j}(A),$$

for all $i,j\in\mathbb{Z}$.

Assume $X^L_B\subset\overline{X^L_A}$, then

$$d_{i,j}(B)=\dim\left(\frac{L_i}{L_i\cap L'_j}\right)\leq d_{i,j}(A)$$

and

$$\bar{d}_{i,j}(B)=\dim\left(\frac{L'_j}{L_i\cap L'_j}\right)\leq \bar{d}_{i,j}(A),$$

for each $i,j\in\mathbb{Z}$, for any $L'\in X^L_B$. So $B\leq A$ if $X^L_B\leq\overline{X^L_A}$. □

Remark 4.6.2. In practice it seems that the converse of Proposition 4.6.1 is true, so that the closure order and the hook order are the same, although I have not been able to find a proof.

Corollary 4.6.3. *The maximum in* ${\Lambda_1}^{A,B}$ *is* $A*B$.

4.7 Associativity of the generic product

Let $A,B,C\in\Lambda_1$ with $\mathrm{co}(A)=\mathrm{ro}(B)$ and $\mathrm{co}(B)=\mathrm{ro}(C)$ and fix $L\in\mathcal{F}_{\mathrm{ro}(A)}$. Write $\lambda=\mathrm{co}(A)$, $\mu=\mathrm{co}(B)$ and $\nu=\mathrm{co}(C)$. Define

$$Y^L_{A,B,C}=\left\{(L',L'',L''')\in\mathcal{F}^3 : L'\in X^L_A, L''\in X^{L'}_B, L'''\in X^{L''}_C\right\}$$

and

$$X^L_{A,B,C}=\left\{L'''\in\mathcal{F} : \exists(L',L'')\in\mathcal{F}^2 \text{ with } (L',L'',L''')\in Y^L_{A,B,C}\right\}.$$

Lemma 4.7.1. *There is* $N\in\mathbb{N}$ *such that* $Y^L_{A,B,C}$ *is contained in* $\Pi^a_{N,\lambda}(L)\times\Pi^{a+b}_{2N,\mu}(L)\times\Pi^{a+b+c}_{3N,\nu}(L)$, *where* $a=d_{nN,0}(A)$, $b=d_{nN,0}(B)$ *and* $c=d_{nN,0}(C)$.

Proof. Lemma 4.4.1 shows that there is $N\in\mathbb{N}$ such that $\varepsilon^N L_0\subset L'_0\subset\varepsilon^{-N}L_0$, $\varepsilon^N L'_0\subset L''_0\subset\varepsilon^{-N}L'_0$ and $\varepsilon^N L''_0\subset L'''_0\subset\varepsilon^{-N}L''_0$ for each $(L',L'',L''')\in Y^L_{A,B,C}$. Using the proof of Lemma 4.5.1, it follows $L''\in\Pi^{a+b}_{2N,\mu}(L)$ and $L'''\in\Pi^{b+c}_{2N,\nu}(L')\subset\Pi^{a+b+c}_{3N,\nu}(L)$. □

Assume $N\in\mathbb{N}$ is chosen so that $Y^L_{A,B,C}\subset\Pi^a_{N,\lambda}(L)\times\Pi^{a+b}_{2N,\mu}(L)\times\Pi^{a+b+c}_{3N,\nu}(L)$, where $a=d_{nN,0}(A)$, $b=d_{nN,0}(B)$ and $c=d_{nN,0}(C)$, as in Lemma 4.7.1.

Lemma 4.7.2. $Y^L_{A,B,C}$ *is a locally closed subset of* $\Pi^a_{N,\lambda}(L)\times\Pi^{a+b}_{2N,\mu}(L)\times\Pi^{a+b+c}_{3N,\nu}(L)$. *In particular,* $Y^L_{A,B,C}$ *is a quasiprojective variety.*

Proof. Write $\Pi = \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi_{3N,\nu}(L)$. Then $Y^L_{A,B,C}$ consists of those $(L', L'', L''') \in \Pi$ satisfying the following conditions:

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) = d_{i,j}(A), \tag{4.7.1}$$

$$\dim\left(\frac{L'_i}{L'_i \cap L''_j}\right) = d_{i,j}(B), \tag{4.7.2}$$

$$\dim\left(\frac{L''_i}{L''_i \cap L'''_j}\right) = d_{i,j}(C), \tag{4.7.3}$$

for $(i,j) \in \{1,\dots,n\} \times \mathbb{Z}$ with $i < j < (N+1)n$, and

$$\dim\left(\frac{L'_j}{L_i \cap L'_j}\right) = \bar{d}_{i,j}(A), \tag{4.7.4}$$

$$\dim\left(\frac{L''_j}{L'_i \cap L''_j}\right) = \bar{d}_{i,j}(B), \tag{4.7.5}$$

$$\dim\left(\frac{L'''_j}{L''_i \cap L'''_j}\right) = \bar{d}_{i,j}(C), \tag{4.7.6}$$

for $(i,j) \in \{1,\dots,n\} \times \mathbb{Z}$ with $-Nn < j < i$.

For $i < j$, the conditions

$$\dim\left(L_i/L_i \cap L'_j\right) \le d_{i,j}(A),$$
$$\dim\left(L'_i/L'_i \cap L''_j\right) \le d_{i,j}(B)$$

and

$$\dim\left(L''_i/L''_i \cap L'''_j\right) \le d_{i,j}(C)$$

define closed subsets of Π, by Lemma 4.3.7. For $i > j$, the conditions

$$\dim\left(L'_j/L_i \cap L'_j\right) \le \bar{d}_{i,j}(A),$$
$$\dim\left(L''_j/L'_i \cap L''_j\right) \le \bar{d}_{i,j}(B)$$

and

$$\dim\left(L'''_j/L''_i \cap L'''_j\right) \le \bar{d}_{i,j}(C)$$

also define closed subsets of Π.

On the other hand, the conditions $\dim\left(L_i/L_i \cap L'_j\right) \ge d_{i,j}(A)$, $\dim\left(L'_i/L'_i \cap L''_j\right) \ge d_{i,j}(B)$ and $\dim\left(L''_i/L''_i \cap L'''_j\right) \ge d_{i,j}(C)$ for $i < j$ define open subsets of Π. Similarly, the conditions $\dim\left(L'_j/L_i \cap L'_j\right) \ge \bar{d}_{i,j}(A)$, $\dim\left(L''_j/L'_i \cap L''_j\right) \ge \bar{d}_{i,j}(B)$ and $\dim\left(L'''_j/L''_i \cap L'''_j\right) \ge \bar{d}_{i,j}(C)$ for $i > j$ define open subsets of Π.

Therefore $Y^L_{A,B,C}$ is the intersection of finitely many closed sets in Π with finitely many open subsets of Π, so $Y^L_{A,B,C}$ is locally closed. In particular, $Y^L_{A,B,C}$ is a quasiprojective variety. $\square$

Lemma 4.7.3. *For any* $(L', L'', L''') \in Y^L_{A,B,C}$*,*

$$Y^L_{A,B,C} = \left\{\alpha \cdot (L', \beta L'', \beta\gamma L''') : \alpha \in G_L, \beta \in G_{L'}, \gamma \in G_{L''}\right\}.$$

In particular,

$$Y^L_{A,B,C} = G_L \cdot \left(\{L'\} \times Y^{L'}_{B,C}\right)$$

for each $L' \in X^L_A$*.*

Proof. Let $(L', L'', L''') \in Y^L_{A,B,C}$. Given $\alpha \in G_L$, $\beta \in G_{L'}$ and $\gamma \in G_{L''}$, $(\alpha L', \alpha\beta L'', \alpha\beta\gamma L''')$ is in $Y^L_{A,B,C}$ since

$$\begin{aligned}
(L, \alpha L') &= \alpha(L, L') \in \mathcal{O}_A \\
(\alpha L', \alpha\beta L'') &= \alpha\beta(L', L'') \in \mathcal{O}_B \\
(\alpha\beta L'', \alpha\beta\gamma L''') &= \alpha\beta\gamma(L'', L''') \in \mathcal{O}_C
\end{aligned}$$

For each $(N', N'', N''')Y^L_{A,B,C}$ there exist $\sigma_1, \sigma_2, \sigma_3 \in G$ with

$$\begin{aligned}
(L, N') &= \sigma_1(L, L') \\
(N', N'') &= \sigma_2(L', L'') \\
(N'', N''') &= \sigma_3(L'', L''').
\end{aligned}$$

Let $\alpha = \sigma_1$, $\beta = \sigma_1^{-1}\sigma_2$ and $\gamma = \sigma_2^{-1}\sigma_3$, so $\sigma_2 = \alpha\beta$ and $\sigma_3 = \alpha\beta\gamma$. It follows that

$$(N', N'', N''') = (\alpha L', \alpha\beta L'', \alpha\beta\gamma L'''),$$

which proves the first claim. The second claim follows from the first since $(L'', L''') \in Y^{L'}_{B,C}$ and therefore

$$Y^{L'}_{B,C} = \{(\beta L'', \beta\gamma L''') : \beta \in G_{L'}, \gamma \in G_{L''}\},$$

as required. □

Proposition 4.7.4. $Y^L_{A,B,C}$ *is irreducible.*

Proof. Write

$$\Pi = \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi^{a+b+c}_{3N,\nu}(L).$$

Lemma 4.3.2 shows that Π is a projective algebraic variety and Lemma 4.3.6 shows that $G_L/H_{3N,L}$ is a connected algebraic group acting algebraically on Π by the diagonal action.

Let $L' \in X^L_A$. As a result of Lemma 4.7.3

$$\begin{aligned}
Y^L_{A,B,C} &= G_L \cdot (\{L'\} \times Y^{L'}_{B,C}) \\
&= G_L/H_{3N,L} \cdot (\{L'\} \times Y^{L'}_{B,C}).
\end{aligned}$$

Proposition 4.5.4 shows that $Y^{L'}_{B,C}$ is irreducible, so $\{L'\} \times Y^{L'}_{B,C}$ is irreducible. The image of $\{L'\} \times Y^{L'}_{B,C}$ under the action of $G_L/H_{3N,L}$ is irreducible, since $G_L/H_{3N,L}$ is connected and therefore irreducible. Therefore $Y^L_{A,B,C}$ is irreducible. □

Let p_3 be the projection of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi^{a+b+c}_{3N,\nu}(L)$ onto the third factor. By the completeness property of projective varieties, p_3 is a closed morphism. The image of $Y^L_{A,B,C}$ under p_3 is $X^L_{A,B,C}$, so $p_3(\overline{Y^L_{A,B,C}}) = \overline{X^L_{A,B,C}}$.

Lemma 4.7.5. *$X^L_{A,B,C}$ is irreducible and constructible.*

Proof. Lemma 4.7.2 and Proposition 4.7.4 show that $Y^L_{A,B,C}$ is locally closed and irreducible. It follows $X^L_{A,B}$ is irreducible and constructible, since $X^L_{A,B,C}$ is the image of $Y^L_{A,B,C}$ under the morphism p_3. □

Lemma 4.7.6. *There is a unique open and dense G_L-orbit in $X^L_{A,B,C}$.*

Proof. There are only finitely many G_L-orbits in $X^L_{A,B,C}$. In particular,

$$X^L_{A,B,C} = \bigcup_{D\in\Lambda_1{}^{A,B}} X^L_{D,C} = \bigcup_{D\in\Lambda_1{}^{A,B}} \bigcup_{D'\in\Lambda_1{}^{D,C}} X^L_{D'}$$

and

$$\overline{X^L_{A,B,C}} = \bigcup_{D\in\Lambda_1{}^{A,B}} \bigcup_{D'\in\Lambda_1{}^{D,C}} \overline{X^L_{D'}}.$$

There is $D \in \Lambda_1$ such that $\overline{X^L_D} = \overline{X^L_{A,B,C}}$, since $X^L_{A,B,C}$ is irreducible, by Lemma 4.7.5. By Lemma 4.4.2, X^L_D is open in $\overline{X^L_D} = \overline{X^L_{A,B,C}}$, so X^L_D is open in $X^L_{A,B,C}$.

If X^L_D and $X^L_{D'}$ are open in $X^L_{A,B,C}$, then X^L_D and $X^L_{D'}$ have nonempty intersection since $X^L_{A,B,C}$ is irreducible, then $X^L_D = X^L_{D'}$. □

Lemma 4.7.7. *$p_3^{-1}(X^L_{A*B,C})$ is open in $\overline{Y^L_{A,B,C}}$.*

Proof. Projection onto the second component is a closed morphism of varieties $p_2 \colon \overline{Y^L_{A,B,C}} \to \overline{X^L_{A,B}}$ with $p_2(Y^L_{A,B,C}) = X^L_{A,B}$. It follows that $p_3^{-1}(X^L_{A*B,C})$ is open in $\overline{Y^L_{A,B,C}}$ since $p_3^{-1}(X^L_{A*B,C}) = p_2^{-1}(X^L_{A*B})$ and X^L_{A*B} is open in $\overline{X^L_{A,B}}$. □

Lemma 4.7.8. *$p_3^{-1}(X^L_{A,B*C})$ is open in $\overline{Y^L_{A,B,C}}$.*

Proof. $p_3^{-1}(X^L_{A,B*C})$ consists of those $(L', L'', L''') \in \overline{Y^L_{A,B,C}}$ such that $\dim\left(L'_i/L'_i \cap L'''_j\right) \geq d_{i,j}(B*C)$ for $i < j$ and $\dim\left(L'''_j/L'_i \cap L'''_j\right) \geq \bar{d}_{i,j}(B*C)$ for $i > j$. Each of these conditions defines an open subset of $\overline{Y^L_{A,B,C}}$ as a result of Lemma 4.3.7 and only finitely many conditions are required to determine $p_3^{-1}(X^L_{A,B*C})$, as before. Therefore $p_3^{-1}(X^L_{A,B*C})$ is the intersection of finitely many open sets in $\overline{Y^L_{A,B,C}}$, so is open as claimed. □

Proposition 4.7.9. $X^L_{A*(B*C)} = X^L_{(A*B)*C}$.

Proof. The unique open G_L-orbit in $X^L_{A*B,C}$ is $X^L_{(A*B)*C}$, so $p_3^{-1}(X^L_{(A*B)*C})$ is open in $p_3^{-1}(X^L_{A*B,C})$. Lemma 4.7.7 shows that $p_3^{-1}(X^L_{A*B,C})$ is open in $\overline{Y^L_{A,B,C}}$, so $p_3^{-1}(X^L_{(A*B)*C})$ is open in $\overline{Y^L_{A,B,C}}$.

Similarly, $X^L_{A*(B*C)}$ is open in $X^L_{A,B*C}$, so $p_3^{-1}(X^L_{A*(B*C)})$ is open in $p_3^{-1}(X^L_{A,B*C})$. Lemma 4.7.8 shows that $p_3^{-1}(X^L_{A,B*C})$ is open in $\overline{Y^L_{A,B,C}}$, so it follows $p_3^{-1}(X^L_{A*(B*C)})$ is open in $\overline{Y^L_{A,B,C}}$.

Therefore $f^{-1}(X^L_{A*(B*C)})$ has nonempty intersection with $f^{-1}(X^L_{(A*B)*C})$, since $Y^L_{A,B,C}$ is irreducible by Proposition 4.7.4. It follows that the G_L-orbits $X^L_{A*(B*C)}$ and $X^L_{(A*B)*C}$ have nonempty intersection and therefore $X^L_{A*(B*C)}$ equals $X^L_{(A*B)*C}$. □

4.8 The generic affine algebra

The generic affine algebra of rank r and period n, denoted by $\hat{G}(n,r)$, is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1\}$ and $\mathbb{Z}$-bilinear multiplication given by

$$e_A * e_B = e_{A*B}$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, and

$$e_A * e_B = 0$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) \neq \mathrm{ro}(B)$.

Proposition 4.8.1. *The generic algebra $\hat{G}(n,r)$ is an associative $\mathbb{Z}$-algebra with 1, with*

$$1 = \sum_{\lambda \in \Lambda_0} 1_\lambda$$

where

$$1_\lambda = e_{D_\lambda},$$

for each $\lambda \in \Lambda_0$.

Proof. Let $A, B, C \in \Lambda_1$. If $\mathrm{co}(A) \neq \mathrm{ro}(B)$ or $\mathrm{co}(B) \neq \mathrm{ro}(C)$, then

$$(e_A * e_B) * e_C = 0 = e_A * (e_B * e_C),$$

so we may now suppose $\mathrm{co}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(B) = \mathrm{ro}(C)$.

As a result of Proposition 4.7.9,

$$\begin{aligned}(e_A * e_B) * e_C &= e_{(A*B)*C} \\ &= e_{A*(B*C)} \\ &= e_A * (e_B * e_C),\end{aligned}$$

so it follows $\hat{G}(n,r)$ is an associative $\mathbb{Z}$-algebra.

The expression for the multiplicative identity follows from Lemma 2.1.17, since

$$e_A * \left(\sum_{\lambda \in \Lambda_0} 1_\lambda\right) = e_A * 1_{\mathrm{co}(A)} = e_A$$

and

$$\left(\sum_{\lambda \in \Lambda_0} 1_\lambda\right) * e_A = 1_{\mathrm{ro}(A)} * e_A = e_A,$$

for each $A \in \Lambda_1$. ☐

4.8.1 A categorical perspective

Proposition 4.8.2. *The following constitutes a small category: the set of objects is* Λ_0 *and the set of morphisms is* Λ_1*. Given compositions* $\lambda, \mu \in \Lambda_0$*, the morphisms with source* μ *and target* λ *are those matrices* $A \in \Lambda_1$ *with* $\mathrm{co}(A) = \mu$ *and* $\mathrm{ro}(A) = \lambda$*. Given* $\lambda, \mu, \nu \in \Lambda_0$ *and* $A, B \in \Lambda_1$ *with* $\mathrm{co}(B) = \nu$*,* $\mathrm{ro}(B) = \mu = \mathrm{co}(A)$ *and* $\mathrm{ro}(A) = \lambda$*, their composition is* $A * B$*, with source* $\mathrm{co}(A * B) = \mathrm{co}(B) = \nu$ *and target* $\mathrm{ro}(A * B) = \mathrm{ro}(A) = \lambda$*.*

Proof. Proposition 4.7.9 shows that the generic product $*$ is associative. For each object $\lambda \in \Lambda_0$, the identity morphism $\lambda \to \lambda$ is the diagonal matrix D_λ. $\square$

Then the generic affine algebra $\hat{G}(n, r)$ may be realised as the $\mathbb{Z}$-algebra of this category. Observe that there are only finitely many objects in this category and distinct objects are non-isomorphic, so the isomorphism classes in this category are in one to one correspondence with Λ_0. The $\mathbb{Z}$-algebra of this category is the free $\mathbb{Z}$-module on Λ_1 with $\mathbb{Z}$-bilinear multiplication given by the generic product $*$.

Chapter 5

Towards a realisation of affine zero Schur algebras

The purpose of this chapter is to study the link between the generic affine algebra $\hat{G}(n,r)$ and the affine 0-Schur algebra $\hat{S}_0(n,r)$.

The main result is the construction of an isomorphism of $\mathbb{Z}$-algebras from $\hat{G}(n,r)$ to $\hat{S}_0(n,r)$ such that $E_i \mapsto E_i$, $F_i \mapsto F_i$ and $1_\lambda \mapsto 1_\lambda$, in the case that $n, r \geq 1$ with $r < n$.

5.1 Preliminary results on the generic affine algebra

Recall that the generic affine algebra $\hat{G}(n,r)$ is an associative $\mathbb{Z}$-algebra with a multiplicative basis $\{e_A : A \in \Lambda_1\}$ over $\mathbb{Z}$, where

$$e_A * e_B = e_{A*B}$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, and

$$e_A * e_B = 0$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) \neq \mathrm{ro}(B)$.

5.1.1 Elementary basis elements

For $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ such that $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and let

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}$$

for each $i \in \{1, \ldots, n\}$

For $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ such that $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and let

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}$$

for each $i \in \{1, \ldots, n\}$.

Lemma 5.1.1. *Let $i \in \{1, \dots, n\}$ and $A \in \Lambda_1$ and write $\mu = \mathrm{ro}(A)$.*

If $\mu_{i+1} = 0$ then $E_i * e_A = 0$. If $\mu_{i+1} > 0$, then

$$E_i * e_A = e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}},$$

where

$$p = \max\{j \in \mathbb{Z} : a_{i+1,j} > 0\}.$$

If $\mu_i = 0$ then $F_i * e_A = 0$. If $\mu_i > 0$ then

$$F_i * e_A = e_{A+\mathcal{E}_{i+1,q}-\mathcal{E}_{i,q}},$$

where

$$q = \min\{j \in \mathbb{Z} : a_{i,j} > 0\}.$$

Proof. Suppose $\mu_{i+1} > 0$. Recall that the corresponding product in the affine q-Schur algebra $\hat{S}_q(n, r)$ is

$$E_i \cdot e_A = \sum_{j \in \mathbb{Z}: a_{i+1,j} > 0} q^{\sum_{t>j} a_{i,t}} [[a_{i,j} + 1]] e_{A+\mathcal{E}_{i,j}-\mathcal{E}_{i+1,j}},$$

by Lemma 3.1.2.

Suppose $B \in \Lambda_1$ with $B = A + \mathcal{E}_{i,j} - \mathcal{E}_{i+1,j}$ for some $j \in \mathbb{Z}$. For $s \in \{1, \dots, n\}$ and $t \in \mathbb{Z}$,

$$d_{s,t}(B) = \begin{cases} d_{s,t}(A) + 1 & : s = i \text{ and } t < j, \\ d_{s,t}(A) & : \text{ otherwise,} \end{cases}$$

and

$$\bar{d}_{s,t}(B) = \begin{cases} \bar{d}_{s,t}(A) - 1 & : s = i \text{ and } t \geq j, \\ \bar{d}_{s,t}(A) & : \text{ otherwise.} \end{cases}$$

It follows that if $j' < j$, then

$$A + \mathcal{E}_{i,j'} - \mathcal{E}_{i+1,j'} < A + \mathcal{E}_{i,j} - \mathcal{E}_{i+1,j}.$$

Therefore, the product in $\hat{G}(n, r)$ is given by

$$E_i * e_A = e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}},$$

where

$$p = \max\{j \in \mathbb{Z} : a_{i+1,j} > 0\}.$$

The argument for the action of F_i is similar, but there is a pleasing symmetry in the two proofs.

Now suppose $\mu_i > 0$. Using Lemma 3.1.2,

$$F_i \cdot e_A = \sum_{j \in \mathbb{Z}: a_{i,j} > 0} q^{\sum_{t<j} a_{i+1,t}} [[a_{i+1,j} + 1]] e_{A+\mathcal{E}_{i+1,j}-\mathcal{E}_{i,j}},$$

in $\hat{S}_q(n, r)$.

Suppose $B \in \Lambda_1$ with $B = A + \mathcal{E}_{i+1,j} - \mathcal{E}_{i,j}$, for some $j \in \mathbb{Z}$. Then for $i \in \{1, \dots, n\}$ and $j \in \mathbb{Z}$,

$$d_{s,t}(B) = \begin{cases} d_{s,t}(A) - 1 & : s = i \text{ and } t < j, \\ d_{s,t}(A) & : \text{ otherwise,} \end{cases}$$

and

$$\bar{d}_{s,t}(B) = \begin{cases} \bar{d}_{s,t}(A) + 1 & : s = i \text{ and } t \geq j, \\ \bar{d}_{s,t}(A) & : \text{ otherwise.} \end{cases}$$

Then if $j' < j$ it follows

$$A + \mathcal{E}_{i+1,j'} - \mathcal{E}_{i,j'} > A + \mathcal{E}_{i+1,j} - \mathcal{E}_{i,j},$$

so the terms with nonzero coefficients in the product $F_i \cdot e_A$ are totally ordered and the maximum is

$$F_i * e_A = e_{A+\mathcal{E}_{i+1,q}-\mathcal{E}_{i,q}},$$

where $q = \min\{j \in \mathbb{Z} : a_{i,j} > 0\}$. □

5.1.2 Transpose involution

Let S be the $\mathbb{Z}$-module automorphism of $\hat{G}(n,r)$ given by

$$S(e_A) = e_{A^\top}$$

for each $A \in \Lambda_1$.

Lemma 5.1.2. *The map S is a $\mathbb{Z}$-algebra antihomomorphism. In particular,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A,$$

for each $A, B \in \Lambda_1$.

Proof. Lemma 3.1.8 show that the transpose preserves the partial order on Λ_1 and so

$$(B * A)^\top = A^\top * B^\top,$$

using Lemma 3.1.1. □

For any $A \in \Lambda_1$,

$$S(S(e_A)) = e_{(A^\top)^\top} = e_A,$$

so $S \circ S$ is the identity map on $\hat{S}_q(n,r)$.

For each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$,

$$S(E_{i,\lambda}) = F_{i,\lambda+\alpha_i},$$

for each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$,

$$S(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}, \text{ and}$$

and

$$S(1_\lambda) = 1_\lambda,$$

for each $\lambda \in \Lambda_0$.

Lemma 5.1.3. *Let $i \in \{1, \ldots, n\}$ and $A \in \Lambda_1$ and write $\lambda = \mathrm{co}(A)$.*

If $\lambda_j = 0$ then $e_A * E_j = 0$. If $\lambda_j > 0$ then

$$e_A * E_j = e_{A+\mathcal{E}_{p,j+1}-\mathcal{E}_{p,j}},$$

where

$$p = \min\{i \in \mathbb{Z} : a_{i,j} > 0\}.$$

If $\lambda_{j+1} = 0$ then $e_A * F_j = 0$. If $\lambda_{j+1} > 0$ then

$$e_A * F_j = e_{A+\mathcal{E}_{p',j}-\mathcal{E}_{p',j+1}},$$

where

$$p' = \max\{i \in \mathbb{Z} : a_{i,j+1} > 0\}.$$

Proof. This follows immediately on applying the transpose involution to the formulas for the action of E_i and F_i on the left given in Lemma 5.1.1.

Equally, this result can be proven directly using the formulas for the action of E_i and F_i on the right in Lemma 3.1.3, as in the proof of Lemma 5.1.1. □

5.1.3 Shifting and periodicity

For each $\lambda \in \Lambda_0$, define

$$R_\lambda = e_{[1]D_\lambda} = e_{\lambda_1 \mathcal{E}_{0,1}+\cdots+\lambda_n \mathcal{E}_{n-1,n}}$$

and set

$$R = \sum_{\lambda \in \Lambda_0} R_\lambda.$$

Lemma 5.1.4. *For each $A \in \Lambda_1$,*

$$R * e_A = e_{[1]A}$$

and

$$e_A * R = e_{A[-1]}.$$

Proof. Lemma 3.1.12 shows that the same formulas hold in $\hat{S}_q(n,r)$, then the result follows for the generic multiplication $*$, since each product $R * e_A$ and $e_A * R$ is supported on one orbit, so the generic multiplication and the product on $\hat{S}_q(n,r)$ are the same in this instance. □

Observe that

$$\begin{aligned} S(R_\lambda) &= e_{\lambda_1 \mathcal{E}_{1,0}+\cdots+\lambda_n \mathcal{E}_{n,n-1}} \\ &= e_{[-1]D_{[1]\lambda}} \end{aligned}$$

so

$$S(R) = \sum_{\lambda \in \Lambda_0} e_{[-1]D_\lambda}.$$

Lemma 5.1.5. *The element R of $\hat{G}(n,r)$ is invertible, with*

$$R * S(R) = 1 = S(R) * R.$$

Proof. Lemma 5.1.4 shows that

$$\begin{aligned} R * S(R)1_\lambda &= Re_{[-1]D_{[1]\lambda}} \\ &= e_{D_{[1]\lambda}} \\ &= 1_{[1]\lambda} \end{aligned}$$

for each $\lambda \in \Lambda_0$, so

$$R * S(R) = 1.$$

Similarly,

$$\begin{aligned} S(R) * R &= \sum_{\lambda \in \Lambda_0} e_{D_\lambda[1]} * R \\ &= \sum_{\lambda \in \Lambda_0} e_{D_\lambda} \\ &= 1. \end{aligned}$$

$\square$

Let τ be the $\mathbb{Z}$-algebra automorphism of $\hat{G}(n,r)$ given by conjugation by R, so

$$\begin{aligned} \tau(e_A) &= R * e_A * S(R) \\ &= R * e_A * R^{-1}, \end{aligned}$$

for each $A \in \Lambda_1$.

Observe that τ has order n, by the (n,n)-periodicity condition on Λ_1.

As in Lemma 3.1.14, it follows from Lemma 5.1.4 that

$$\tau(E_{i,\lambda}) = E_{i-1,[1]\lambda}$$

for $i \in \{1, \dots, r\}$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$,

$$\tau(F_{i,\lambda}) = F_{i-1,[1]\lambda}$$

for $i \in \{1, \dots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$, and

$$\tau(1_\lambda) = 1_{[1]\lambda}$$

for $\lambda \in \Lambda_0$.

In particular,

$$\begin{aligned} \tau(E_i) &= E_{i-1} \\ \tau(F_i) &= F_{i-1} \end{aligned}$$

for $i \in \{1, \dots, r\}$.

As earlier, I can not be sure but I think this map τ is related to the Auslander-Reiten translation on the isomorphism classes of nilpotent representations of the cyclic quiver on n vertices. The result that $\tau(E_i) = E_{i-1}$ is consistent with the fact the A.R translation sends the simple representation at vertex i to the simple representation at vertex $i-1$.

5.2 Multiplicative bases in affine zero Schur algebras: motivating example

Recall that the affine 0-Schur algebra $\hat{S}_0(n,r)$ is defined to be the associative $\mathbb{Z}$-algebra

$$\hat{S}_0(n,r) = \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}[q]} \hat{S}_q(n,r).$$

In particular, $\hat{S}_0(n,r)$ has a $\mathbb{Z}$-basis

$$\{e_A : A \in \Lambda_1\}$$

with $\mathbb{Z}$-bilinear product given by

$$e_A e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C}(0) e_C$$

for $A, B, C \in \Lambda_1$; where $\gamma_{A,B,C} \in \mathbb{Z}[q]$ are the structure polynomials of the affine q-Schur algebra $\hat{S}_q(n,r)$ with respect to this distinguished basis.

The multiplicative identity in $\hat{S}_0(n,r)$ is

$$\sum_{\lambda \in \Lambda_0} 1_\lambda.$$

The result of the shifting lemma, Lemma 3.1.12, also holds in $\hat{S}_0(n,r)$. In particular,

$$Re_A = e_{[1]A}$$

and

$$e_A R = e_{A[-1]},$$

for each $A \in \Lambda_1$.

Now assume $r = 1$, so

$$\Lambda_1(n,1) = \{\mathcal{E}_{i,j} : (i,j) \in \mathbb{Z} \times \{1, \ldots, n\}\}$$

and

$$\Lambda_0(n,1) = \{\varepsilon_n, \ldots, \varepsilon_1\}.$$

Lemma 5.2.1. *The distinguished basis $\{e_A : A \in \Lambda_1(n,1)\}$ is a multiplicative basis of $\hat{S}_0(n,1)$. More precisely,*

$$e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{j,k}} = e_{\mathcal{E}_{i,k}}$$

for $i, j, k \in \mathbb{Z}$, and

$$e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{k,l}} = 0$$

for $i, j, k, l \in \mathbb{Z}$ with $j \neq k$ modulo n.

Proof. Let $i, j \in \mathbb{Z}$. Lemma 3.1.12 shows that

$$e_{\mathcal{E}_{i,j}} = R^{j-i} 1_{\varepsilon_j},$$

where the subscript of ε_j is taken modulo n.

If $i, j, k, l \in \mathbb{Z}$ with $j \neq k$ modulo n, then

$$\mathrm{co}(\mathcal{E}_{i,j}) = \varepsilon_j \neq \varepsilon_k = \mathrm{ro}(\mathcal{E}_{k,l}),$$

so

$$e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{k,l}} = 0.$$

Finally, let $i, j, k \in \mathbb{Z}$. Then

$$\begin{aligned} e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{j,k}} &= R^{j-i} 1_{\varepsilon_j} R^{k-j} 1_{\varepsilon_k} \\ &= R^{j-i} R^{k-j} 1_{\varepsilon_k} \\ &= R^{k-i} 1_{\varepsilon_k} \\ &= e_{\mathcal{E}_{i,k}}. \end{aligned}$$

This proves that the basis $\{e_A : A \in \Lambda_1(n, 1)\}$ of $\hat{S}_0(n, 1)$ is a multiplicative basis. □

This result also shows that the product in $\hat{S}_0(n, 1)$ is the same as the generic product, since

$$e_A e_B = e_{A*B}$$

if $\mathrm{co}(A) = \mathrm{ro}(B)$, and

$$e_A e_B = 0$$

if $\mathrm{co}(A) \neq \mathrm{ro}(B)$, for $A, B \in \Lambda_1(n, 1)$.

Corollary 5.2.2. *For each integer $n \geq 1$,*

$$\hat{S}_0(n, 1) = \hat{G}(n, 1).$$

Proof. This is a consequence of Lemma 5.2.1 and the comment which follows the proof. □

5.3 Aperiodicity in the generic affine algebra

Definition 5.3.1. An element $A \in \Lambda_1$ is aperiodic if for each $l \in \mathbb{Z} \setminus \{0\}$ there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$.

An element of $\hat{G}(n, r)$ is said to be aperiodic if it is a $\mathbb{Z}$-linear combination of basis elements e_A corresponding to the aperiodic elements in Λ_1.

For example, the diagonal matrix D_λ is aperiodic so 1_λ is aperiodic, for any $\lambda \in \Lambda_0$. The elementary basis elements $E_{i,\lambda}$ and $F_{i,\lambda}$ introduced earlier are also aperiodic.

When $r < n$, any element $A \in \Lambda_1$ is aperiodic since $\mathrm{co}(A)$ is insincere and therefore A has a zero column.

Lemma 5.3.2. *Suppose $A \in \Lambda_1$ is aperiodic and write $\mu = \mathrm{ro}(A)$. If $\mu_{i+1} > 0$, then $E_i * e_A$ is aperiodic. If $\mu_i > 0$, then $F_i * e_A$ is aperiodic.*

Proof. Let $A \in \Lambda_1$ be aperiodic and let $\mu = \mathrm{ro}(A)$.

Suppose $\mu_{i+1} > 0$. There is $p \in \mathbb{Z}$ such that $a_{i+1,p} > 0$ and $a_{i+1,p'} = 0$ whenever $p' > p$. Lemma 3.1.2 shows that $E_i * e_A = e_B$, where $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p - i, p - i - 1\}$, then $b_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $b_{s,s+l} = a_{s,s+l} = 0$, since A is aperiodic. If $l = p - i$, then $b_{i+1,i+1+l} = b_{i+1,p+1} = a_{i+1,p+1} = 0$, by maximality of p. If

$l = p-i-1$, there is $s \neq i+1$ such that $a_{s,s+l} = 0$, since A is aperiodic and $a_{i+1,i+1+l} = a_{i+1,p} > 0$, so $b_{s,s+l} = a_{s,s+l} = 0$. Therefore, $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ is aperiodic.

Suppose $\mu_i > 0$. Lemma 3.1.2 shows that $F_i * e_A = e_C$ where $C = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$ and $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p-i, p-i-1\}$ then $c_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $c_{s,s+p} = a_{s,s+p} = 0$, by aperiodicity of A. If $l = p-i$, then $a_{i,i+l} = a_{i,p} > 0$, so there is $s \neq i$ such that $a_{s,s+l} = 0$. Then $c_{s,s+l} = a_{s,s+l} = 0$. Finally, if $l = p-i-1$, then $c_{i,i+l} = a_{i,p-1} = 0$ by minimality of p. Thus C is aperiodic as required. □

Suppose $\lambda \in \Lambda_0$ and

$$\omega = \omega_1 \cdots \omega_m,$$

where

$$\omega_1, \ldots, \omega_m \in \{E_1, \ldots, E_n\} \cup \{F_1, \ldots, F_n\}.$$

Either $\omega * 1_\lambda = 0$ or $\omega * 1_\lambda = e_A$ for some $A \in \Lambda_1$, where A is aperiodic, as a result of Lemma 5.3.2.

The next step is to prove a converse of this result. It will be shown that each of the aperiodic basis elements e_A in $\hat{G}(n,r)$ can be expressed in the form $\omega 1_\lambda$, where ω is a word in $E_1, \ldots E_n$ and $F_1, \ldots, F_n$ and $\lambda = \mathrm{co}(A)$. This will be proven by induction on the *weight* of a matrix by showing how any aperiodic basis element can be written as the product of some E_i or F_i with an aperiodic basis element of strictly smaller weight.

Definition 5.3.3. For each $A \in \Lambda_1$, define the *weight of A* to be the non negative integer

$$\mathrm{wt}(A) = \sum_{i \in \{1,\ldots,n\}, j \in \mathbb{Z}} |j-i| a_{i,j}.$$

Observe that

$$\mathrm{wt}(A) = \sum_{[i,j]: i<j} (j-i) a_{i,j} + \sum_{[i,j]: i>j} (i-j) a_{i,j}.$$

Also write $\mathrm{wt}(e_A) = \mathrm{wt}(A)$. Then 1_λ has weight 0, and $E_{i,\lambda}$ and $F_{i,\lambda}$ have weight 1. In fact, the converse also holds: If $\mathrm{wt}(A) = 0$ then $e_A = 1_\lambda$ where $\lambda = \mathrm{co}(A)$, and if $\mathrm{wt}(A) = 1$ then e_A is $E_{i,\lambda}$ or $F_{i,\lambda}$ for some i, where $\lambda = \mathrm{co}(A)$.

Lemma 5.3.4. *Let $A \in \Lambda_1$ and write $\mu = \mathrm{ro}(A)$. Suppose $\mu_{i+1} > 0$ and set*

$$p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}.$$

If $p > i$ then

$$\mathrm{wt}(E_i * e_A) = 1 + \mathrm{wt}(e_A)$$

and if $p \leq i$ then

$$\mathrm{wt}(E_i * e_A) = -1 + \mathrm{wt}(e_A).$$

Proof. Lemma 5.1.1 shows that

$$E_i * e_A = e_{A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}}$$

so

$$\mathrm{wt}(E_i * e_A) - \mathrm{wt}(e_A) = |p-i| - |p-i-1|,$$

which equals 1 if $p > i$ and equals -1 if $p \leq i$. □

Lemma 5.3.5. *Let $A \in \Lambda_1$ and $\mu = \mathrm{ro}(A)$. Suppose $i \in \{1, \ldots, n\}$ is such that $\mu_i > 0$ and let*

$$q = \min\{q' \in \mathbb{Z} : a_{i,q'} > 0\}.$$

If $q \leq i$ then

$$\mathrm{wt}(F_i * e_A) = \mathrm{wt}(e_A) + 1$$

and if $q > i$ then

$$\mathrm{wt}(F_i * e_A) = \mathrm{wt}(e_A) - 1.$$

Proof. Again using Lemma 5.1.1,

$$F_i * e_A = e_{A+\mathcal{E}_{i+1,q}-\mathcal{E}_{i,q}},$$

so

$$\mathrm{wt}(F_i * e_A) - \mathrm{wt}(e_A) = |q - i - 1| - |q - i|,$$

which equals -1 if $q > i$ and equals 1 if $q \leq i$. □

Lemma 5.3.6. *If $A \in \Lambda_1$ is aperiodic, then*

$$e_A = \omega_1 \cdots \omega_m 1_\lambda$$

for some

$$\omega_1, \ldots, \omega_m \in \{E_1, \ldots, E_n\} \cup \{F_1, \ldots, F_n\},$$

where $\lambda = \mathrm{co}(A)$ and $m = \mathrm{wt}(A)$.

Proof. The proof uses induction on the weight of A.

If $\mathrm{wt}(A) = 0$ then $A = D_\lambda$, where $\lambda = \mathrm{co}(A)$, so

$$e_A = 1_\lambda.$$

Assume $\mathrm{wt}(A) > 0$. Then A has at least one nonzero entry which is not on the diagonal.

Suppose the upper part of A is nonzero and set

$$h^+ = \max\{j - i : a_{i,j} \neq 0\}.$$

There is $i \in \{1, \ldots, n\}$ such that $a_{i,i+h^+} > 0$ and $a_{i+1,i+1+h^+} = 0$, using the aperiodicity property of A. Let p be the smallest integer such that $p > i$, $a_{i,p} > 0$ and $a_{i+1,j} = 0$ for $j > p$.

Then

$$e_A = E_i * e_B$$

where $B = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$. Moreover, B is aperiodic and

$$\mathrm{wt}(B) = \mathrm{wt}(A) - 1,$$

using Lemma 5.3.4.

Next suppose the lower part of A is nonzero and set

$$h^- = \max\{i - j : a_{i,j} > 0\}.$$

There is $i \in \{1, \ldots, n\}$ such that $a_{i,i-h^-} = 0$ and $a_{i+1,i+1-h^-} > 0$, by the aperiodicity property of A. Let q be the largest integer such that $q < i + 1$, $a_{i+1,q} > 0$ and $a_{i,j} = 0$ for $j < q$. Then $q \geq i - h^-$ and

$$e_A = F_i e_B,$$

where

$$B = A + \mathcal{E}_{i,q} - \mathcal{E}_{i+1,q}.$$

Observe B is aperiodic and

$$\mathrm{wt}(B) = \mathrm{wt}(A) - 1,$$

by Lemma 5.3.5.

Therefore, if $\mathrm{wt}(A) > 0$ there exists an aperiodic element $B \in \Lambda_1$ with

$$\mathrm{wt}(B) = \mathrm{wt}(A) - 1$$

and such that

$$e_A = \omega e_B$$

for some $\omega \in \{E_1, \ldots, E_n\} \cup \{F_1, \ldots, F_n\}$.

It follows that any aperiodic basis element e_A is the product of a word of length $\mathrm{wt}(A)$ in $E_1, \ldots, E_n$ and $F_1, \ldots, F_n$ with the idempotent 1_λ, where $\lambda = \mathrm{co}(A)$. □

Proposition 5.3.7. *The subalgebra of $\hat{G}(n,r)$ generated by E_i and F_i for $i \in \{1, \ldots, n\}$ and 1_λ for $\lambda \in \Lambda_0$ has $\mathbb{Z}$-basis*

$$\{e_A : A \in \Lambda_1 \textit{ is aperiodic.}\}.$$

Proof. By definition, this subalgebra is spanned by the nonzero products in E_i and F_i for $i \in \{1, \ldots, n\}$ and 1_λ for $\lambda \in \Lambda_0$, which are exactly the aperiodic basis elements, by Lemma 5.3.2 and Lemma 5.3.6. □

Lemma 5.3.8. *In the case $r < n$, $\hat{G}(n,r)$ is generated by E_i and F_i for $i \in \{1, \ldots, n\}$ and 1_λ for $\lambda \in \Lambda_0$.*

Proof. When $r < n$, any $A \in \Lambda_1$ is aperiodic since $\mathrm{co}(A)$ has a zero entry, so A has a column of zero entries. Therefore each of the basis elements e_A in $\hat{G}(n,r)$ may be written as a product of the E_i, F_i and 1_λ, using Proposition 5.3.7. □

5.4 Quiver presentation of the generic affine algebra.

Let n and r be integers with $n \geq 3$ and $r \geq 1$. Let $\Gamma = \Gamma(n,r)$ be the quiver associated to the affine q-Schur algebra $\hat{S}_q(n,r)$, as defined in Section 3.2.2.

Recall that Γ is the quiver with set of vertices $\Gamma_0 = \Lambda_0$ and set of arrows $\Gamma_1 = \Gamma_1^+ \cup \Gamma_1^-$, where Γ_1^+ consists of the arrows

$$e_{i,\lambda} \colon \lambda \to \lambda + \alpha_i \text{ for } (i,\lambda) \in \{1, \ldots, n\} \times \Lambda_0 \text{ with } \lambda_{i+1} > 0,$$

and Γ_1^- consists of the arrows

$$f_{i,\lambda} \colon \lambda \to \lambda - \alpha_i \text{ for } (i,\lambda) \in \{1, \ldots, n\} \times \Lambda_0 \text{ with } \lambda_i > 0.$$

Recall that the path $\mathbb{Z}$-algebra of Γ is an associative $\mathbb{Z}$-algebra with a $\mathbb{Z}$-basis consisting of the paths in Γ and with multiplication defined by concatenation of paths. If p and q are paths in Γ then the product pq is the path q followed by p if the target of q equals the source of p, otherwise pq equals zero.

For each $i \in \{1, \ldots, n\}$, define

$$e_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda\in\Lambda_0:\lambda_i>0} f_{i,\lambda}.$$

Let $\mathcal{J}$ be the ideal in $\mathbb{Z}\Gamma$ generated by the following expressions, which are obtained from the relations in the q-Schur algebra by setting q equal to 0:

$$e_ie_j - e_je_i,$$
$$f_if_j - f_jf_i$$

for $i, j \in \{1, \ldots, n\}$ with $j > i + 1$;

$$e_ie_{i+1}^2 - e_{i+1}e_ie_{i+1},$$
$$e_i^2e_{i+1} - e_ie_{i+1}e_i,$$
$$f_{i+1}^2f_i - f_{i+1}f_if_{i+1},$$
$$f_{i+1}f_i^2 - f_if_{i+1}f_i$$

for $i \in \{1, \ldots, n\}$;

$$e_if_j - f_je_i$$

for $i, j \in \{1, \ldots, n\}$ with $i < j$;

$$e_if_i - f_ie_i - \sum_{\lambda\in\Lambda_0} c_{i,\lambda}k_\lambda$$

for $i \in \{1, \ldots, n\}$, where

$$c_{i,\lambda} = \begin{cases} 1 & : \text{ if } \lambda_{i+1} = 0, \lambda_i > 0 \\ 0 & : \text{ if } \lambda_i > 0, \lambda_{i+1} > 0 \\ -1 & : \text{ if } \lambda_i = 0, \lambda_{i+1} > 0. \end{cases}$$

Multiplying each expression above with the idempotents k_λ for $\lambda \in \Lambda_0$ gives a relation involving paths with common source and target vertices, thus $\mathcal{J}$ is an ideal of $\mathbb{Z}$-linear relations in Γ.

The ideal $\mathcal{J}$ in $\mathbb{Z}\Gamma$ is generated by the following set of relations:

$$e_{i,\lambda+\alpha_j}e_{j,\lambda} - e_{j,\lambda+\alpha_i}e_{i,\lambda},$$
$$f_{i,\lambda-\alpha_j}f_{j,\lambda} - f_{j,\lambda-\alpha_i}f_{i,\lambda},$$

for $i, j \in \{1, \ldots, n\}$ with $j > i + 1$;

$$e_{i,\lambda+2\alpha_{i+1}}e_{i+1,\lambda+\alpha_{i+1}}e_{i+1,\lambda} - e_{i+1,\lambda+\alpha_i+\alpha_{i+1}}e_{i,\lambda+\alpha_{i+1}}e_{i+1,\lambda},$$
$$e_{i,\lambda+\alpha_i+\alpha_{i+1}}e_{i,\lambda+\alpha_{i+1}}e_{i+1,\lambda} - e_{i,\lambda+\alpha_i+\alpha_{i+1}}e_{i+1,\lambda+\alpha_i}e_{i,\lambda},$$
$$f_{i+1,\lambda-\alpha_i-\alpha_{i+1}}f_{i+1,\lambda-\alpha_i}f_{i,\lambda} - f_{i+1,\lambda-\alpha_i-\alpha_{i+1}}f_{i,\lambda-\alpha_{i+1}}f_{i+1,\lambda},$$
$$f_{i+1,\lambda-2\alpha_i}f_{i,\lambda-\alpha_i}f_{i,\lambda} - f_{i,\lambda-\alpha_i-\alpha_{i+1}}f_{i+1,\lambda-\alpha_i}f_{i,\lambda},$$

for $i \in \{1, \ldots, n\}$;

$$e_{i,\lambda-\alpha_j}f_{j,\lambda} - f_{j,\lambda+\alpha_i}e_{i,\lambda}$$

for $i, j \in \{1, \ldots, n\}$ with $i \neq j$;

$$e_{i,\lambda-\alpha_i}f_{i,\lambda} - f_{i,\lambda+\alpha_i}e_{i,\lambda}$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$ and $\lambda_{i+1} > 0$;

$$e_{i,\lambda-\alpha_i}f_{i,\lambda} - k_\lambda$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ such that $\lambda_i > 0$ and $\lambda_{i+1} = 0$;

$$f_{i,\lambda+\alpha_i} e_{i,\lambda} - k_\lambda$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i = 0$ and $\lambda_{i+1} > 0$.

Lemma 5.4.1. *The following equations hold in the generic affine algebra $\hat{G}(n, r)$:*

$$\begin{aligned} E_i E_j &= E_j E_i \\ F_i F_j &= F_j F_i \end{aligned}$$

for $i, j \in \{1, \ldots, n\}$ with $|j - i| \neq 1$;

$$\begin{aligned} E_i E_{i+1}^2 &= E_{i+1} E_i E_{i+1} \\ E_i^2 E_{i+1} &= E_i E_{i+1} E_i \\ F_{i+1}^2 F_i &= F_{i+1} F_i F_{i+1} \\ F_{i+1} F_i^2 &= F_i F_{i+1} F_i \end{aligned}$$

for $i \in \{1, \ldots, n\}$;

$$E_i F_j = F_j E_i$$

for $i, j \in \{1, \ldots, n\}$ with $i \neq j$;

$$E_i F_i - F_i E_i = \sum_{\lambda \in \Lambda_0} c_{i,\lambda} 1_\lambda$$

for $i \in \{1, \ldots, n\}$.

Proof. Suppose $i, j \in \{1, \ldots, n\}$ with $j > i + 1$, so $\{i, i+1\}$ and $\{j, j+1\}$ are disjoint, then

$$\begin{aligned} E_i E_j &= \sum_{\lambda \in \Lambda_0} E_i \left[D_\lambda + \mathcal{E}_{j,j+1} - \mathcal{E}_{j+1,j+1} \right] \\ &= \sum_{\lambda \in \Lambda_0} \left[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{j,j+1} - \mathcal{E}_{j+1,j+1} \right] \\ &= E_j E_i \end{aligned}$$

Then applying the transpose involution yields the second equation:

$$F_i F_j - F_j F_i = -S([E_i, E_j]) = 0.$$

Using the fundamental multiplication rules 5.1.1 and 5.1.3, for each $i \{1, \ldots, n\}$,

$$\begin{aligned} E_i E_{i+1}^2 &= \sum_{\lambda \in \Lambda_0} E_i \left[D_\lambda + 2\mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2} \right] \\ &= \sum_{\lambda \in \Lambda_0} \left[D_\lambda + 2\mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2} + \mathcal{E}_{i,i+2} - \mathcal{E}_{i+1,i+2} \right] \\ &= \sum_{\lambda \in \Lambda_0} \left[D_\lambda + \mathcal{E}_{i,i+2} + \mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2} \right] \end{aligned}$$

and

$$\begin{aligned}E_{i+1}E_iE_{i+1} &= \sum_{\lambda\in\Lambda_0} E_{i+1}\left[D_\lambda + \mathcal{E}_{i,i+2} - \mathcal{E}_{i+2,i+2}\right] \\ &= \sum_{\lambda\in\Lambda_0} \left[D_\lambda + \mathcal{E}_{i,i+2} + \mathcal{E}_{i+1,i+2} - 2\mathcal{E}_{i+2,i+2}\right],\end{aligned}$$

so $E_iE_{i+1}^2 = E_{i+1}E_iE_{i+1}$.

$$\begin{aligned}E_i^2E_{i+1} &= \sum_{\mu\in\Lambda_0} \left[D_\mu + 2\mathcal{E}_{i,i+1} - 2\mathcal{E}_{i,i}\right] E_{i+1} \\ &= \sum_{\mu\in\Lambda_0} \left[D_\mu + \mathcal{E}_{i,i+2} + \mathcal{E}_{i,i+1} - 2\mathcal{E}_{i,i}\right]\end{aligned}$$

and

$$\begin{aligned}E_iE_{i+1}E_i &= \sum_{\mu\in\Lambda_0} \left[D_\mu + \mathcal{E}_{i,i+2} - \mathcal{E}_{i,i}\right] E_i \\ &= \sum_{\mu\in\Lambda_0} \left[D_\mu + \mathcal{E}_{i,i+2} + \mathcal{E}_{i,i+1} - 2\mathcal{E}_{i,i}\right],\end{aligned}$$

so $E_i^2E_{i+1} = E_iE_{i+1}E_i$.

The relations between F_i and F_{i+1} may be deduced using the transpose involution as follows:

$$F_{i+1}^2F_i = S(E_iE_{i+1}^2) = S(E_{i+1}E_iE_{i+1}) = F_{i+1}F_iF_{i+1}$$

and

$$F_{i+1}F_i^2 = S(E_i^2E_{i+1}) = S(E_iE_{i+1}E_i) = F_iF_{i+1}F_i.$$

Suppose $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then

$$\begin{aligned}E_iF_j &= \sum_{\lambda\in\Lambda_0} E_i\left[D_\lambda + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j}\right] \\ &= \sum_{\lambda\in\Lambda_0} \left[D_\lambda + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j} + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}\right]\end{aligned}$$

and

$$\begin{aligned}F_jE_i &= \sum_{\lambda\in\Lambda_0} F_j\left[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}\right] \\ &= \sum_{\lambda\in\Lambda_0} \left[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{j+1,j} - \mathcal{E}_{j,j}\right],\end{aligned}$$

so $E_iF_j = F_jE_i$.

Finally, for $i \in \{1, \ldots, n\}$,

$$E_iF_i = \sum_{\lambda:\lambda_i>0,\lambda_{i+1}=0} 1_\lambda + \sum_{\lambda:\lambda_i>0,\lambda_{i+1}>0} \left[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}\right]$$

and

$$F_iE_i = \sum_{\lambda:\lambda_i=0,\lambda_{i+1}>0} 1_\lambda + \sum_{\lambda:\lambda_i>0,\lambda_{i+1}>0} \left[D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1} + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}\right],$$

so

$$\begin{aligned} E_iF_i - F_iE_i &= \sum_{\lambda:\lambda_i>0,\lambda_{i+1}=0} 1_\lambda - \sum_{\lambda:\lambda_i=0,\lambda_{i+1}>0} 1_\lambda \\ &= \sum_{\lambda\in\Lambda_0} c_{i,\lambda}1_\lambda. \end{aligned}$$

□

Lemma 5.4.1 shows that there is a homomorphism of $\mathbb{Z}$-algebras

$$\rho\colon \mathbb{Z}\Gamma/\mathcal{J} \to \hat{G}(n,r)$$

defined by

$$\begin{aligned} \rho(k_\lambda + \mathcal{J}) &= 1_\lambda \\ \rho(e_{i,\lambda} + \mathcal{J}) &= E_{i,\lambda} \\ \rho(f_{i,\lambda} + \mathcal{J}) &= F_{i,\lambda}, \end{aligned}$$

for all $i \in \{1,\dots,n\}$ and $\lambda \in \Lambda_0$. Thus $\hat{G}(n,r)$ may also be regarded as an algebra over $\mathbb{Z}\Gamma$ where the action of a path p is given by

$$e_A \cdot p = e_A\rho(p + \mathcal{J})$$

for all $A \in \Lambda_1$.

Proposition 5.4.2. *The image of ρ is spanned by the aperiodic basis elements. If $r < n$ then ρ is surjective.*

Proof. The image of ρ is the subalgebra of $\hat{G}(n,r)$ generated by E_i and F_i for $i \in \{1,\dots,n\}$ and 1_λ for $\lambda \in \Lambda_0$, which has $\mathbb{Z}$-basis

$$\{e_A : A \in \Lambda_1, A \text{ is aperiodic.}\},$$

using Proposition 5.3.7. If $r < n$ then every $A \in \Lambda_1$ is aperiodic, since A must contain a zero row or column. Therefore ρ is surjective when $r < n$. □

5.4.1 Standard paths

Recall the definition of standard paths in Γ, from Definition 3.2.7. There is a bijection between the set of standard paths in Γ and the standard monomial basis in $\hat{G}(n,r)$ indexed by Λ_1, using Lemma 3.2.10.

The expression for the standard path of A is derived by contracting the rows of A so that each step produces zero entries on the highest or lowest diagonal, yielding the element D_λ where $\lambda = \mathrm{ro}(A)$ after finitely many steps. Computing the image of a standard path in $\hat{G}(n,r)$ by computing the segments from left to right constructs e_A slice by slice. In particular, the segment p_s^+ produces the diagonal at level s while the segment p_s^- produces the diagonal at level $-s$. In order to describe this process precisely we now give some notation for the row contractions of A.

Given $A \in \Lambda_1$ and $s \geq 1$ define elements $(s)A$ and $A(s)$ in Λ_1 by

$$((s)A)_{i,j} = \begin{cases} a_{i,j} & \text{if } i-j < s, \\ 0 & \text{if } i-j > s, \\ \sum_{t \leq j} a_{i,t} & \text{if } i-j = s \end{cases}$$

and

$$(A(s))_{i,j} = \begin{cases} a_{i,j} & \text{if } j-i < s, \\ 0 & \text{if } j-i > s, \\ \sum_{t \geq j} a_{i,t} & \text{if } j-i = s \end{cases}$$

for $i, j \in \mathbb{Z}$. Observe that $(0)A(0) = D_\lambda$ where $\lambda = \mathrm{ro}(A)$; $(0)A$ is upper triangular and coincides with A above the diagonal; $A(0)$ is lower triangular and coincides with A below the diagonal; $\mathrm{ro}((s)A) = \mathrm{ro}(A)$ and $\mathrm{ro}(A(s)) = \mathrm{ro}(A)$. Also define the *height* of A as

$$\mathrm{ht}(A) = \max\{|j-i| : i, j \in \mathbb{Z}, a_{i,j} > 0\}$$

so that $(h)A = A$ and $A(h) = A$ for $h \geq \mathrm{ht}(A)$.

Lemma 5.4.3. *Let $A \in \Lambda_1$ and let $p = k_\lambda p_1^+ \cdots p_h^+ p_1^- \cdots p_h^-$ be the standard path for A. Then*

$$e_{A(s-1)} \cdot p_s^+ = e_{A(s)}$$

and

$$e_{(s-1)A} \cdot p_s^- = e_{(s)A}$$

for each $s \in \{1, \ldots, h\}$.

Proof. Let $B = A(s-1) \cdot p_s^+$. Using the fundamental multiplication rules in $\hat{G}(n,r)$, Lemma 5.1.3, it follows that

$$B = A(s-1) + \sum_{i \in \{1,\ldots,n\}} \alpha_{i,s}(\mathcal{E}_{i,i+s} - \mathcal{E}_{i,i+s-1}).$$

So $b_{i,j} = a_{i,j}$ if $j - i < s - 1$,

$$\begin{aligned} b_{i,i+s-1} &= \alpha_{i,s-1} - \alpha_{i,s} \\ &= a_{i,i+s-1} \end{aligned}$$

and

$$b_{i,i+s} = \alpha_{i,s},$$

which proves that $B = A(s)$.

Similarly, let $B = (s-1)A \cdot p_s^-$. Using Lemma 5.1.3 it follows that

$$B = (s-1)A + \sum_{i \in \{1,\ldots,n\}} \beta_{i-1,s}(\mathcal{E}_{i,i-s} - \mathcal{E}_{i,i-s+1}).$$

So $b_{i,j} = a_{i,j}$ if $i - j < s - 1$,

$$\begin{aligned} b_{i,i-s+1} &= \beta_{i-1,s-1} - \beta_{i-1,s} \\ &= a_{i,i-s+1} \end{aligned}$$

and

$$b_{i,i-s} = \beta_{i-1,s},$$

which proves $B = (s)A$. □

Lemma 5.4.4. *Let $A \in \Lambda_1$ and let p be the standard path for A. Then*

$$\rho(p + \mathcal{J}) = e_A.$$

Proof. Let $A \in \Lambda_1$, $\lambda = \mathrm{ro}(A)$, $\mu = \mathrm{co}(A)$, $h = \mathrm{ht}(A)$ and let $p = k_\lambda p_1^+ \cdots p_h^+ p_1^- \cdots p_h^- k_\mu$ be the standard path for A.

The standard path for $(0)A$ is $k_\lambda p_1^+ \cdots p_h^+$, by Lemma 3.2.13, so

$$\begin{aligned} e_{(0)A} &= e_{(0)A(h)} \\ &= e_{(0)A(0)} \cdot p_1^+ \cdots p_h^+ \end{aligned}$$

by repeatedly applying Lemma 5.4.3. Similarly,

$$\begin{aligned} e_A &= e_{(h)A} \\ &= e_{(0)A} \cdot p_1^- \cdots p_h^-, \end{aligned}$$

since p is the standard path for A. Therefore

$$\begin{aligned} e_A &= e_{(0)A(0)} \cdot p_1^+ \cdots p_h^+ p_1^- \cdots p_h^- \\ &= e_{D_\lambda} \cdot p \\ &= \rho(p + \mathcal{J}). \end{aligned}$$

□

Remark 5.4.5. The result of Lemma 5.4.4 gives another way to see that the homomorphism ρ from the quiver algebra to $\hat{G}(n, r)$ is surjective provided $r < n$. When $r \geq n$, the image of the quiver algebra in $\hat{G}(n, r)$ is spanned by the aperiodic basis elements, by Proposition 5.4.2.

Recall the definition of the positive and negative parts A^+ and A^- of a matrix $A \in \Lambda_1$, as in Definition 3.2.11.

Lemma 5.4.6. *Let $A \in \Lambda_1$. Then*

$$e_A = e_{A^+} e_{A^-}$$

and in terms of G-orbits,

$$[L, L'] = [L, L \cap L'][L \cap L', L'].$$

Proof. Let p be the standard path for A. Then $p = p^+ p^-$ where p^+ is the standard path for A^+ and p^- is the standard path for A^-, by Lemma 3.2.13. Then Lemma 5.4.4 proves that

$$\begin{aligned} e_A &= \rho(p + \mathcal{J}) \\ &= \rho(p^+ + \mathcal{J}) \rho(p^- + \mathcal{J}) \\ &= e_{A^+} e_{A^-}. \end{aligned}$$

The second part then follows from Lemma 3.2.12 which states that $\mathcal{O}_{A^+} = [L, L \cap L']$ and $\mathcal{O}_{A^-} = [L \cap L', L']$ for any $(L, L') \in \mathcal{O}_A$. □

Definition 5.4.7. A path is said to be *reduced* if it is not equivalent to a shorter path.

Lemma 5.4.8. *A standard path is reduced.*

Proof. If p is a standard positive or negative path then p is reduced, since the relations only involving the edges $e_i : i \in \{1, \ldots, n\}$ or $f_i : i \in \{1, \ldots, n\}$ are homogeneous polynomials, so any equivalent path is of the same length.

Now suppose $p = k_\lambda p^+ p^- k_\mu$ is a standard path for a standard positive path $k_\lambda p^+$ and a standard negative path $p^- k_\mu$. The number of arrows in p is

$$l = \sum_{i \in \{1,\ldots,n\}, s \geq 1} \alpha_{i,s} + \beta_{i,s}.$$

Let A be the matrix corresponding to the standard path p, so that $p = p_A$ as in Lemma 3.2.10. The minimum number of E_i and F_i in an expression for e_A in $\hat{G}(n,r)$ is

$$\mathrm{wt}(A) = \sum_{i \in \{1,\ldots,n\}, j \in \mathbb{Z}} |j - i| a_{i,j},$$

using Lemma 5.3.6, so $l \geq \mathrm{wt}(A)$.

Recall that

$$\alpha_{i,s} = \sum_{t \geq s} a_{i,i+t}$$

and

$$\beta_{i-1,s} = \sum_{t \geq s} a_{i,i-s},$$

so

$$\sum_{s \geq 1} \alpha_{i,s} = \sum_{s \geq 1} s a_{i,i+s}$$

and

$$\sum_{s \geq 1} \beta_{i-1,s} = \sum_{s \geq 1} s a_{i,i-s}.$$

Therefore

$$\begin{aligned} l &= \sum_{i \in \{1,\ldots,n\}, s \geq 1} \alpha_{i,s} + \beta_{i,s} \\ &= \sum_{i \in \{1,\ldots,n\}, s \geq 1} s(a_{i,i+s} + a_{i,i-s}) \\ &= \mathrm{wt}(A), \end{aligned}$$

which proves that p is reduced. $\square$

The next result gives a more general form of the 0-Serre relations in the quiver algebra for $\hat{G}(n,r)$, which will be useful in transforming a path into a standard path.

Lemma 5.4.9. *Let $t > s \geq 0$ be integers. Then*

$$e_i^s e_{i+1}^s e_i - e_i^{s+1} e_{i+1}^s = 0$$

and

$$e_i^s e_{i+1}^t - e_{i+1}^{t-s} e_i^s e_{i+1}^s = 0$$

in $\mathbb{Z}\Gamma/\mathcal{J}$, for each $i \in \{1, \ldots, n\}$.

Proof. COMPLETE THIS PROOF. □

Corollary 5.4.10. *Let $t > s \geq 0$ be integers. Then*

$$f_{i+1}^s f_i^{s+1} - f_i f_{i+1}^s f_i^s = 0$$

and

$$f_{i+1}^s f_i^s f_{i+1}^{t-s} - f_{i+1}^t f_i^s = 0$$

in $\mathbb{Z}\Gamma/\mathcal{J}$, for each $i \in \{1, \ldots, n\}$.

Proof. Applying the transpose involution to the relations in Lemma 5.4.9 yields these relations, since e_i is mapped to f_i and the order of multiplication is reversed. □

Lemma 5.4.11. *Let p be a standard path in Γ. If q is a path in Γ with $q = pe_i$ or $q = pf_i$ for some $i \in \{1, \ldots, n\}$, then q is congruent to a standard path modulo $\mathcal{J}$.*

Proof. IMPORTANT: need to prove this beyond doubt.

First suppose p is a positive standard path

$$p = k_\mu p_1 \cdots p_{h^+} k_\lambda,$$

where

$$p_s = e_{i_0+s-1}^{\alpha_{i_0,s}} \cdots e_{i_0+s-n+1}^{\alpha_{i_0-n+2,s}}$$

for $s = 1, \ldots, h^+$, such that $\alpha_{i,s} \geq \alpha_{i,s+1}$ for each i, s and $\alpha_{i_0+1,s} = 0$ for all s.

The index i in e_i and $\alpha_{i,s}$ is taken modulo n. Observe that $\alpha_{i,s}$ is the exponent of e_{i+s-1} in p_s and so the exponent of e_i in p_s is $\alpha_{i-s+1,s}$. Even when p is a positive path there are many cases to consider.

For the first case, if the exponent of e_{j-1} in p_{h^+}, which is α_{j-h^+,h^+}, is nonzero then $q = pe_j$ is a standard path.

For the second case, suppose the exponent of e_{j-1} in p_{h^+} is zero and the exponent of e_j in p_{h^+} is strictly less than the exponent of e_{j-1} in p_{h^+-1}. [continue...]

For the third case, suppose the exponent of e_{j-1} in p_{h^+} is zero and the exponent of e_j in p_{h^+} equals the exponent of e_{j-1} in p_{h^+-1}. [continue...][I'm stuck on this case.]

□

Proposition 5.4.12. *When $r < n$, any path in Γ is congruent to a standard path modulo $\mathcal{J}$.*

Proof. Let p be a path in Γ and proceed by induction on the length of p. If p has length zero then $p = k_\mu$ for some $\mu \in \Lambda_0$, so p is a standard path. If p has length one then $p = k_\mu e_i$ or $p = k_\mu f_i$ for some $\mu \in \Lambda_0$ and $i \in \{1, \ldots, n\}$.

Suppose p has length at least two and that any strictly shorter path is congruent to a standard path. Pulling out the first arrow, write $p = p' e_i$ or $p = p' f_i$ for some $i \in \{1, \ldots, n\}$. Using the inductive hypothesis we may assume p' is a standard path, so it follows from Lemma 5.4.11 that p is congruent to a standard path.

Note: The lemma on extending standard paths still needs to be proven for this proof to be complete. □

Theorem 5.4.13. *If $r < n$ then ρ is a $\mathbb{Z}$-algebra isomorphism. Thus $\hat{G}(n, r)$ admits a presentation by the quiver Γ and the ideal of relations $\mathcal{J}$ in $\mathbb{Z}\Gamma$.*

Proof. Under the assumption $r < n$, ρ is a surjective homomorphism of $\mathbb{Z}$-algebras, by Proposition 5.4.2.

Suppose p and p' are paths in Γ and $A \in \Lambda_1$ with

$$\rho(p + \mathcal{J}) = \rho(p' + \mathcal{J}) = e_A.$$

Proposition 5.4.12 shows that p and p' are both congruent modulo $\mathcal{J}$ to the standard path corresponding to A, so p and p' are congruent modulo $\mathcal{J}$. Therefore ρ is injective, so is an isomorphism of $\mathbb{Z}$-algebras as claimed. □

5.5 The isomorphism result

This section gives a realisation of the affine 0-Schur algebra by the generic affine algebra in the case that $r < n$. Recall that the affine 0-Schur algebra $\hat{S}_0(n,r)$ is defined to be the $\mathbb{Z}$-algebra

$$\hat{S}_0(n,r) = \mathbb{Z}[q]/(q) \otimes_{\mathbb{Z}} [q]\hat{S}_q(n,r).$$

The inclusion of $\mathbb{Z}[q]$ into $\mathcal{Q}$ sending f to $f/1$ gives an isomorphism of $\mathbb{Z}$ algebras

$$\mathbb{Z}[q]/q\mathbb{Z}[q] \to \mathcal{Q}/q\mathcal{Q} : a + q\mathbb{Z}[q] \mapsto a + q\mathcal{Q},$$

and both are isomorphic to $\mathbb{Z}$ itself. Therefore

$$\hat{S}_0(n,r) = \mathcal{Q}/q\mathcal{Q} \otimes_{\mathcal{Q}} \hat{S}_{\mathcal{Q}}(n,r)$$

Let $n, r \geq 1$ with $r < n$. Recall

$$\phi \colon \mathbb{Z}[q]\Gamma/I \to \hat{S}_q(n,r)$$

is the homomorphism of $\mathbb{Z}[q]$-algebras defined by

$$\begin{aligned} \phi(e_{i,\lambda}) &= E_{i,\lambda} \\ \phi(f_{i,\lambda}) &= F_{i,\lambda} \\ \phi(k_\lambda) &= 1_\lambda, \end{aligned}$$

for $i \in \{1, \dots, n\}$ and $\lambda \in \Lambda_0$. Let ϕ_0 be the $\mathbb{Z}$-algebra homomorphism

$$\phi_0 = \mathcal{Q}/(q) \otimes_{\mathcal{Q}} \phi_{\mathcal{Q}} \colon \mathbb{Z}\Gamma/\mathcal{J} \to \hat{S}_0(n,r).$$

Let ρ be the $\mathbb{Z}$-algebra isomorphism

$$\rho \colon \mathbb{Z}\Gamma/\mathcal{J} \to \hat{G}(n,r)$$

from Theorem 5.4.13.

Let Ψ be the $\mathbb{Z}$-algebra homomorphism

$$\Psi = \phi_0 \circ \rho^{-1} \colon \hat{G}(n,r) \to \hat{S}_0(n,r) \tag{5.5.1}$$

with

$$\begin{aligned} \Psi(E_{i,\lambda}) &= E_{i,\lambda} \\ \Psi(F_{i,\lambda}) &= F_{i,\lambda} \\ \Psi(1_\lambda) &= 1_\lambda, \end{aligned}$$

for $i \in \{1, \dots, n\}$ and $\lambda \in \Lambda_0$.

Proposition 5.5.1. *The map Ψ is surjective.*

Proof. Proposition 3.2.17 shows that

$$\phi_{\mathcal{Q}} \colon \mathcal{Q}\Gamma/\mathcal{Q}I \to \hat{S}_{\mathcal{Q}}(n,r)$$

is a surjective $\mathcal{Q}$-algebra homomorphism, so

$$\phi_0 = \mathcal{Q}/(q) \otimes_{\mathcal{Q}} \phi_{\mathcal{Q}} \colon \mathbb{Z}\Gamma/\mathcal{J} \to \hat{S}_0(n,r)$$

is a surjective $\mathbb{Z}$-algebra homomorphism, using right exactness of tensor products. It follows that Ψ is surjective since $\Psi \circ \rho = \phi_0$ and ρ is an isomorphism of $\mathbb{Z}$-algebras. $\square$

Lemma 5.5.2. *For each $A \in \Lambda_1$,*

$$\Psi(e_A) = e_A + \sum_{B : B < A} c_B e_B$$

for some $c_B \in \mathbb{N}$.

Proof. Fix $A \in \Lambda_1$ and let p be the standard path for A as in Definition 3.2.9, so that

$$\rho(p + \mathcal{J}) = e_A,$$

using Lemma 5.4.4.

Then

$$\begin{aligned} \Psi(e_A) &= \phi_0(p + \mathcal{J}) \\ &= \sum_{B \in \Lambda_1 : B \leq A} g_B(0) e_B, \end{aligned}$$

for some $g_B \in \mathbb{Z}[q]$, where

$$\begin{aligned} g_A(0) &= \left(\prod_{i \in \{1,\dots,n\}, s \geq 1} [[\alpha_{i,s}]]_! [[\beta_{i,s}]]_! \right)_{q=0} \\ &= 1, \end{aligned}$$

by Proposition 3.2.14. $\square$

Theorem 5.5.3. *When $r < n$, the map*

$$\Psi \colon \hat{G}(n,r) \to \hat{S}_0(n,r)$$

is an isomorphism of $\mathbb{Z}$-algebras.

Proof. Proposition 5.5.1 shows that Ψ is a surjective $\mathbb{Z}$-algebra homomorphism.

To prove that Ψ is injective, suppose x is a nonzero element of $\hat{G}(n,r)$ and write

$$x = \sum_{A \in \lambda_1} c_A e_A$$

and let $\Xi = \{A \in \Lambda_1 : c_A \neq 0\}$. Fix a maximal element D in Ξ, which exists since Ξ is nonempty and finite. Then

$$\begin{aligned}\Psi(x) &= \sum_{A\in\Xi} c_A \Psi(e_A) \\ &= \sum_{A\in\Xi}\left(c_A e_A + \sum_{B\in\Xi:B<A} c_{A,B} e_B\right) \\ &= c_D e_D + \sum_{A\in\Xi:A\neq D} c'_A e_A\end{aligned}$$

by Lemma 5.5.2, so $\Psi(x) \neq 0$, which proves that Ψ is injective and therefore Ψ is an isomorphism of $\mathbb{Z}$-algebras. □

5.6 The period 2 case

In the case $n = 2$ the quiver $\Gamma = \Gamma(2, r)$ associated to $\hat{G}(2, r)$ is consists of $r + 1$ vertices (totally ordered) with two pairs of edges between adjacent vertices, (e_1, f_1) and (e_2, f_2).

The following equations are a $q = 0$ form of the q-Serre relations in Lemma 3.3.1:

Lemma 5.6.1. *The following equations hold in $\hat{G}(2, r)$, for $i \in \mathbb{Z}/2\mathbb{Z}$:*

$$\begin{aligned}E_i E_{i+1} E_i^2 &= E_i^2 E_{i+1} E_i \\ F_i F_{i+1} F_i^2 &= F_i^2 F_{i+1} F_i.\end{aligned}$$

Proof.

$$\begin{aligned}E_1E_2E_1^2 &= \sum_{\mu\in\Lambda_0} [D_\mu + \mathcal{E}_{1,3} - \mathcal{E}_{1,1}]\, E_1^2 \\ &= \sum_{\mu\in\Lambda_0} [D_\mu + \mathcal{E}_{1,4} - \mathcal{E}_{1,1}]\, E_1 \\ &= \sum_{\mu\in\Lambda_0} [D_\mu + \mathcal{E}_{1,4} + \mathcal{E}_{1,2} - 2\mathcal{E}_{1,1}]\end{aligned}$$

and

$$\begin{aligned}E_1^2E_2E_1 &= \sum_{\mu\in\Lambda_0} [D_\mu + 2\mathcal{E}_{1,2} - 2\mathcal{E}_{1,1}]\, E_2E_1 \\ &= \sum_{\mu\in\Lambda_0} [D_\mu + \mathcal{E}_{1,3} + \mathcal{E}_{1,2} - 2\mathcal{E}_{1,1}]\, E_1 \\ &= \sum_{\mu\in\Lambda_0} [D_\mu + \mathcal{E}_{1,4} + \mathcal{E}_{1,2} - 2\mathcal{E}_{1,1}]\,,\end{aligned}$$

so $E_1E_2E_1^2 = E_1^2E_2E_1$.

Recall that conjugation by R defines an automorphism τ of $\hat{G}(n, r)$ of degree 2, with $\tau(E_1) = E_2$ and $\tau(E_2) = E_1$, so

$$E_2E_1E_2^2 - E_2^2E_1E_2 = \tau(E_1E_2E_1^2 - E_1^2E_2E_1) = 0.$$

Finally, the equations involving F_i and F_{i+1} follow by applying the transpose involution:

$$F_iF_{i+1}F_i^2 - F_i^2F_{i+1}F_i = S(E_i^2E_{i+1}E_i - E_iE_{i+1}E_i^2) = 0,$$

for $i \in \{1, 2\}$. $\square$

Chapter 6

Conclusion

6.1 The case of large r

When $r \geq n$ extra relations are needed in order to transform any path to a standard path using the relations, thus proving injectivity of the quiver presentation. These are thought to be of the form

$$e_i^2 e_{i-1} \cdots e_{i+1} e_i = e_i e_{i-1} \cdots e_{i+1} e_i^2$$

and

$$f_i^2 f_{i+1} \cdots f_{i-1} f_i = f_i f_{i+1} \cdots f_{i-1} f_i^2$$

for $i \in \{1, \dots, n\}$ and there are likely to be more general relations with arbitrary exponents for $e_{i-1}, \dots, e_{i+1}$ and $f_{i+1}, \dots, f_{i-1}$ respectively.

Further research could focus on the relation between the generic affine algebra $\hat{G}(n, r)$ and the affine zero Schur algebra $\hat{S}_0(n, r)$ when $r \geq n$. The case where $n \leq r < 2n$ appears to be tractable by including the shifting element R in the set of generators for each algebra and in this case I still expect the two algebras to be isomorphic, though the case of general r seems to be very difficult.

6.2 Further results on affine zero Schur algebras

Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

6.3 Combinatorial characterisation of degenerations

The degeneration order on orbits in $\mathcal{F} \times \mathcal{F}$ implies the hook order on Λ_1. Through examples it seems that these two orders are in fact equivalent, but a proof has so far been elusive.

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