A geometric realisation of affine 0-Schur algebras.

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Contents

Chapter 1

Introduction

Chapter 2

Background: The double flag variety approach to q-Schur algebras

Chapter 3

The cyclic flags approach to affine q-Schur algebras

Fix natural numbers n and r.

Definition 3.0.1 (compositions). *A composition of r into n parts is an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of non-negative integers whose sum equals r. Denote the set of compositions of r into n parts by Λ_0.*

Definition 3.0.2 (infinite periodic matrices). *Let Λ_1 be the set of matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with integer entries $a_{i,j}$ satisfying the following conditions:*

- *$a_{i,j} \geq 0$ for each $i, j \in \mathbb{Z}$;*
- *each row or column has only finitely many non-zero entries;*
- *the sum of the entries in any n consecutive rows or columns equals r;*
- *$a_{i-n,j-n} = a_{i,j}$ for each $i, j \in \mathbb{Z}$.*

These matrices are referred to as infinite periodic matrices.

Definition 3.0.3 (source and target). *Given $A \in \Lambda_1$, let $\mathrm{ro}(A)$ and $\mathrm{co}(A)$ be the compositions of r into n parts given by*

$$\mathrm{ro}(A) = \left(\sum_{j\in\mathbb{Z}} a_{1,j}, \ldots, \sum_{j\in\mathbb{Z}} a_{n,j}\right)$$

and

$$\mathrm{co}(A) = \left(\sum_{i\in\mathbb{Z}} a_{i,1}, \ldots, \sum_{i\in\mathbb{Z}} a_{i,n}\right).$$

The source is $\mathrm{co}(A)$ and the target is $\mathrm{ro}(A)$.

These sums are finite since each row and column of A contains only finitely many nonzero entries, by definition of the set Λ_1.

Definition 3.0.4 (diagonal matrices). *Given $\lambda \in \Lambda_0$, let $D_\lambda \in \Lambda_1$ be the matrix given by $(D_\lambda)_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i \neq j$ and $(D_\lambda)_{i,i} = \lambda_i$ for $i \in \mathbb{Z}$; where the indices are taken modulo n.*

3.1 Cyclic flags

Fix $n, r \in \mathbb{N}$ and let $\mathbf{k}$ be a field. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by ε, so $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free $\mathcal{S}$-module of rank r. Let G be the automorphism group of the $\mathcal{S}$-module V, so G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. A lattice in V is a $\mathcal{R}$-submodule L of V with $\mathcal{S} \otimes_{\mathcal{R}} L = V$. In particular, a lattice is an $\mathcal{R}$-submodule of V which is a free $\mathcal{R}$-module of rank r.

Lemma 3.1.1. *Let L be a lattice in V. $L/\varepsilon L$ is a torsion $\mathcal{R}$-module, where ε acts as zero. $L/\varepsilon L$ is a free $\mathcal{R}/\langle\varepsilon\rangle$-module of rank r; that is, $L/\varepsilon L$ is an r-dimensional $\mathbf{k}$-vector space.*

Proof. L is a free $\mathcal{R}$-module of rank r, with $L \subset V$. Given an $\mathcal{R}$-basis $\{x_1, \dots, x_r\}$ of L, $\{\varepsilon x_1, \dots, \varepsilon x_r\}$ is an $\mathcal{R}$-basis of εL. Finally, the cosets $\{x_1 + \varepsilon L, \dots, x_r + \varepsilon L\}$ give a basis for $L/\varepsilon L$ over $\mathcal{R}/\langle\varepsilon\rangle \cong \mathbf{k}$. □

Let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of collections $(L_i)_{i\in\mathbb{Z}}$ of lattices in V with $L_i \subset L_{i+1}$ and $\varepsilon L_i = L_{i-n}$ for each $i \in \mathbb{Z}$. These collections of lattices in V are referred to as cyclic flags in V.

G acts on $\mathcal{F}$ by $(g\cdot L)_i = g(L_i)$ for each $i \in \mathbb{Z}$, $g \in G$ and $L \in \mathcal{F}$. The G-orbits in $\mathcal{F}$ are indexed by the set Λ_0 of compositions of r into n parts. In particular, the G-orbit in $\mathcal{F}$ corresponding to $\lambda \in \Lambda_0$ is

$$\mathcal{F}_\lambda = \left\{ L \in \mathcal{F} : \dim\left(\frac{L_i}{L_{i-1}}\right) = \lambda_i \text{ for each } i \in \mathbb{Z} \right\}$$

Definition 3.1.1. *The periodic characteristic matrix of a pair of cyclic flags $(L, L') \in \mathcal{F}\times\mathcal{F}$ is the matrix $\mathrm{A}(L, L') = (a_{i,j})_{i,j\in\mathbb{Z}}$ with entries*

$$a_{i,j} = \dim_{\mathbf{k}}\left(\frac{L_i \cap L'_j}{L_i \cap L'_{j-1} + L_{i-1} \cap L'_j}\right)$$

for each $i, j \in \mathbb{Z}$.

The diagonal action of G on $\mathcal{F}\times\mathcal{F}$ has orbits indexed by the set Λ_1 of infinite periodic matrices (see definition 3.0.2). The G-orbit corresponding to $A \in \Lambda_1$ is denoted $\mathcal{O}_A$ and consists of those pairs $(L, L') \in \mathcal{F}\times\mathcal{F}$ with periodic characteristic matrix $\mathrm{A}(L, L')$ equal to A.

Lemma 3.1.2 (alternative expression for characteristic matrix)**.** *Alternatively,*

$$a_{i,j} = \dim_{\mathbf{k}}\left(\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}}\right)$$

for each $i, j \in \mathbb{Z}$.

Proof. Set $U = L_i \cap L'_j$ and $U' = L_{i-1} + L_i \cap L'_{j-1}$. Then $U + U' = L_{i-1} + L_i \cap L'_j$ and $U \cap U' = L_i \cap L'_j \cap L_{i-1} + L_i \cap L'_{j-1}$. Applying the isomorphism theorems, $U + U'/U'$ is naturally isomorphic to $U/U \cap U'$ as a vector space. In particular,

$$\frac{L_{i-1} + L_i \cap L'_j}{L_{i-1} + L_i \cap L'_{j-1}} = \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}}$$

and thus the dimensions of these spaces are both equal to $a_{i,j}$. □

Lemma 3.1.3 (transposing characteristic matrix)**.** *Given a pair of flags $(L, L') \in \mathcal{F}^2$, the matrices $\mathrm{A}(L, L')$ and $\mathrm{A}(L', L)$ are related by the transpose. In particular, $\mathrm{A}(L, L')_{i,j} = \mathrm{A}(L', L)_{j,i}$ for each $i, j \in \mathbb{Z}$.*

Proof. By swapping the roles of i and j and swapping L and L' it is clear that $\mathrm{A}(L, L')_{i,j}$ and $\mathrm{A}(L', L)_{j,i}$ are both given by the dimension of the $\mathbf{k}$-vector space

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}},$$

for each $i, j \in \mathbb{Z}$. □

Lemma 3.1.4 (a codimension formula)**.** *Given $(L, L') \in \mathcal{F}^2$ and $i, j \in \mathbb{Z}$,*

$$\dim_{\mathbf{k}} \left(\frac{L_i}{L_i \cap L'_j} \right) = \sum_{s \le i, t > j} a_{s,t},$$

where $\mathrm{A}(L, L') = (a_{i,j})_{i,j \in \mathbb{Z}}$.

Proof. COMPLETE THIS PROOF □

Lemma 3.1.5 (nested flags)**.** *Given $(L, L') \in \mathcal{F}^2$, $L' \subset L$ if and only if $\mathrm{A}(L, L')_{i,j} = 0$ for $i, j \in \mathbb{Z}$ with $i > j$.*

Proof. Suppose $L, L' \in \mathcal{F}$ with $L' \subset L$, meaning $L'_j \subset L_j$ for each $j \in \mathbb{Z}$. Then for $i > j$, $L_i \cap L'_j = L'_j$, $L_{i-1} \cap L'_j = L'_j$ and $L_i \cap L'_{j-1}$, which shows

$$\mathrm{A}(L, L')_{i,j} = \dim_{\mathbf{k}} \left(\frac{L'_j}{L'_{j-1} + L'_j} \right) = 0$$

as required. Conversely, suppose $\mathrm{A}(L, L')$ is upper triangular, meaning $\mathrm{A}(L, L')_{i,j} = 0$ when $i > j$. Using Lemma 3.1.4,

$$\dim_{\mathbf{k}} \left(\frac{L'_i}{L'_i \cap L_i} \right) = \sum_{s > i, t \le i} a_{s,t} = 0,$$

so $L_i \cap L'_i = L'_i$ and thus $L'_i \subset L_i$ for each $i \in \mathbb{Z}$, as required. □

Corollary 3.1.6 (diagonal orbits)**.** *Given $L, L' \in \mathcal{F}$, $L = L'$ if and only if $\mathrm{A}(L, L')_{i,j} = 0$ whenever $i \neq j$. In particular,*

$$\mathcal{O}_{D_\lambda} = \{(L, L) \in \mathcal{F}^2 : L \in \mathcal{F}_\lambda\},$$

for each $\lambda \in \Lambda_0$.

3.1.1 A product of orbits

Given $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, define

$$Y_{A,B} = \{(L, L', L'') \in \mathcal{F}^3 : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\},$$

$$X_{A,B} = \{(L, L'') \in \mathcal{F}^2 : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}.$$

If also $L \in \mathcal{F}_{\mathrm{ro}(A)}$, define the L-slices of $Y_{A,B}$ and $X_{A,B}$ respectively as

$$Y^L_{A,B} = \{(L', L'') \in \mathcal{F}^2 : (L, L', L'') \in Y_{A,B}\},$$

$$X^L_{A,B} = \{L'' \in \mathcal{F} : (L, L'') \in X_{A,B}\}.$$

Observation 1. *There are only finitely many G-orbits in $X_{A,B}$.*

Lemma 3.1.7. *Given $A \in \Lambda_1$, $X_{D_\lambda,A} = \mathcal{O}_A$ if $\lambda = \mathrm{ro}(A)$ and $X_{A,D_\lambda} = \mathcal{O}_A$ if $\lambda = \mathrm{co}(A)$.*

Proof. Let $A \in \Lambda_1$ and set $\lambda = \mathrm{ro}(A)$. $Y_{D_\lambda,A}$ is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_{D_\lambda}$, thus $L = L'$ by Corollary 3.1.6, and $(L', L'') \in \mathcal{O}_A$. $X_{D_\lambda,A}$ is the projection of $Y_{D_\lambda,A}$, which equals $\mathcal{O}_A$.

Similarly, if $\lambda = \mathrm{co}(A)$, Y_{A,D_λ} is the set of triples $(L, L', L'') \in \mathcal{F}^3$ with $(L, L') \in \mathcal{O}_A$ and $L'' = L'$, so X_{A,D_λ} is exactly the orbit $\mathcal{O}_B$. $\square$

3.1.2 Triple products

Given $A, B, C \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(B) = \mathrm{ro}(C)$ and $L \in \mathcal{F}_{\mathrm{ro}(A)}$, there are spaces $X_{A,B,C}$, $Y_{A,B,C}$ and their respective L-slices, defined as follows:

$$Y_{A,B,C} = \{(L, L', L'', L''') \in \mathcal{F}^4 : (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$X_{A,B,C} = \{(L, L''') \in \mathcal{F}^2 : \exists (L', L'') \in \mathcal{O}_B \text{ with } (L, L') \in \mathcal{O}_A \text{ and } (L'', L''') \in \mathcal{O}_C\},$$

$$Y^L_{A,B,C} = \{(L', L'', L''') \in \mathcal{F}^3 : (L, L', L'', L''') \in Y_{A,B,C}\},$$

$$X^L_{A,B,C} = \{L''' \in \mathcal{F} : (L, L''') \in X_{A,B,C}\}.$$

3.2 Convolution algebras

Suppose $\mathbf{k}$ is a finite field and let q denote the number of elements of $\mathbf{k}$. Consider the set S of G-invariant functions $\mathcal{F} \times \mathcal{F} \to \mathbb{Z}$ with constructible support. S is a free $\mathbb{Z}$-module with a basis consisting of the indicator functions of the G-orbits in $\mathcal{F} \times \mathcal{F}$. Define an operation $\star$ on S as follows: for each $f, g \in S$, $f \star g \in S$ is given by

$$(f \star g)(L, L'') = \sum_{L' \in \mathcal{F}} f(L, L')g(L', L''),$$

for $(L, L'') \in \mathcal{F} \times \mathcal{F}$.

$f \star g$ is well defined since the supports of f and g consist of finitely many G-orbits, so there are only finitely many $L' \in \mathcal{F}$ such that $f(L, L')g(L', L'') \neq 0$, given $(L, L'') \in \mathcal{F} \times \mathcal{F}$. $f \star g$ is constant on G-orbits and is supported on finitely many G-orbits, so $f \star g \in S$.

Lemma 3.2.1. *The set S together with the operation $\star$ is an associative $\mathbb{Z}$-algebra with identity element ι given by $\iota(L, L) = 1$ and $\iota(L, L') = 0$ for $L' \neq L$.*

Proof. Given $f, g, h \in S$ and $(L, L''') \in \mathcal{F} \times \mathcal{F}$,

$$\begin{aligned} ((f \star g) \star h)(L, L''') &= \sum_{L''} (f \star g)(L, L'')h(L'', L''') \\ &= \sum_{L''} \sum_{L'} f(L, L')g(L', L'')h(L'', L''') \\ &= (f \star (g \star h))(L, L'''), \end{aligned}$$

thus $\star$ is associative. ι is the multiplicative identity since

$$(\iota \star f)(L, L'') = \sum_{L'} \iota(L, L')f(L', L'') = f(L, L'')$$

and

$$(f \star \iota)(L, L'') = \sum_{L'} f(L, L')\iota(L', L'') = f(L, L''),$$

for each $f \in S$ and $(L, L'') \in \mathcal{F} \times \mathcal{F}$. $\square$

Given $A \in \Lambda_1$, let $e_A \in S$ denote the indicator function of the orbit $\mathcal{O}_A$. S is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1\}$. There exist $\gamma_{A,B,C;\mathrm{q}} \in \mathbb{Z}$ for $A, B, C \in \Lambda_1$ such that

$$e_A \star e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C;\mathrm{q}} e_C$$

for each $A, B \in \Lambda_1$. Then

$$\begin{aligned} \gamma_{A,B,C;\mathrm{q}} &= (e_A \star e_B)(L, L'') \\ &= \sum_{L'} e_A(L, L')e_B(L', L'') \\ &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\}, \end{aligned}$$

for any $(L, L'') \in \mathcal{O}_C$.

3.3 Affine q-Schur algebras

There exist polynomials $\gamma_{A,B,C} \in \mathbb{Z}[q]$ for $A, B, C \in \Lambda_1$ such that $\gamma_{A,B,C}(\mathrm{q}) = \gamma_{A,B,C;\mathrm{q}}$ for any prime power q, following [32, section 4]. The affine q-Schur algebra $\hat{S}_q(n, r)$ is a $\mathbb{Z}[q]$-algebra which is a free $\mathbb{Z}[q]$-module with basis $\{e_A : A \in \Lambda_1\}$ and with multiplication given by

$$e_A e_B = \sum_C \gamma_{A,B,C} e_C.$$

Given the existence of these 'universal polynomials' $\gamma_{A,B,C} \in \mathbb{Z}[q]$, it follows from Lemma 3.2.1 that $\hat{S}_q(n, r)$ is an associative $\mathbb{Z}[q]$-algebra with multiplicative identity given by

$$1 = \sum_{\lambda \in \Lambda_0} e_{D_\lambda}.$$

Chapter 4

Presenting affine q-Schur algebras

4.1 Basic results and notation

4.1.1 Elementary basis elements

For each $i, j \in \mathbb{Z}$, let $\mathcal{E}_{i,j}$ be the $\mathbb{Z} \times \mathbb{Z}$ 'elementary periodic matrix' with entries given by

$$(\mathcal{E}_{i,j})_{s,t} = 1$$

if $(s,t) = (i+cn, j+cn)$ for some $c \in \mathbb{Z}$ and $(\mathcal{E}_{i,j})_{s,t} = 0$ otherwise. Clearly $\mathcal{E}_{i,j} = \mathcal{E}_{i+n,j+n}$ for each $i, j \in \mathbb{Z}$.

Recall from Definition 3.0.4 that the diagonal matrix associated to a composition $\lambda \in \Lambda_0$ is

$$D_\lambda = \lambda_1 \mathcal{E}_{1,1} + \cdots + \lambda_n \mathcal{E}_{n,n}.$$

The corresponding basis elements e_{D_λ}, for $\lambda \in \Lambda_0$, are pairwise orthogonal idempotents in $\hat{S}_q(n,r)$ with

$$\sum_{\lambda \in \Lambda_0} e_{D_\lambda} = 1,$$

as a result of Lemma 3.1.7.

For each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and define

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}.$$

Also define, for each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$,

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and define

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}.$$

For each $i \in \{1, \ldots, n\}$, let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then

$$\mathrm{co}(E_{i,\lambda}) = \mathrm{co}(F_{i,\lambda}) = \lambda,$$

$$\mathrm{ro}(E_{i,\lambda}) = \lambda + \alpha_i$$

and

$$\mathrm{ro}(F_{i,\lambda}) = \lambda - \alpha_i.$$

4.1.2 Transpose involution

Let S be the $\mathbb{Z}[q]$-module automorphism of $\hat{S}_q(n,r)$ given by

$$S(e_A) = e_{A^\top},$$

for each $A \in \Lambda_1$.

Lemma 4.1.1. *The map S is a $\mathbb{Z}[q]$-algebra antihomomorphism of order 2. In particular,*

$$S(e_A e_B) = S(e_B)S(e_A)$$

for each $A, B \in \Lambda_1$.

Proof. Let $A, B, C \in \Lambda_1$ and let $\mathbf{k}$ be a finite field with $\mathrm{q} = \#\mathbf{k}$ elements. If $(L, L'') \in \mathcal{O}_C$ then $(L'', L) \in \mathcal{O}_{C^\top}$ and

$$\begin{aligned}\gamma_{A,B,C;\mathrm{q}} &= \#\{L' : (L, L') \in \mathcal{O}_A \text{ and } (L', L'') \in \mathcal{O}_B\} \\ &= \#\{L' : (L'', L') \in \mathcal{O}_{B^\top} \text{ and } (L', L) \in \mathcal{O}_{A^\top}\} \\ &= \gamma_{B^\top, A^\top, C^\top;\mathrm{q}}\end{aligned}$$

It follows that

$$S(e_A e_B) = S(e_B)S(e_A),$$

for each $A, B \in \Lambda_1$ and therefore S is a $\mathbb{Z}[q]$-algebra antihomomorphism. Moreover, $S \circ S$ is the identity map on $\hat{S}_q(n,r)$ since $(A^\top)^\top = A$. □

The action of S on E_i, F_i and 1_λ is as follows:

$$S(1_\lambda) = 1_\lambda$$

for each $\lambda \in \Lambda_0$,

$$S(E_{i,\lambda}) = F_{i,\lambda+\alpha_i}$$

for each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$, and

$$S(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}$$

for each $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$.

In particular,

$$\begin{aligned}S(E_i) &= F_i, \\ S(F_i) &= E_i, \\ S(1_\lambda) &= 1_\lambda\end{aligned}$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$.

4.1.3 Fundamental multiplication rules

Lemma 4.1.2. *Given $A \in \Lambda_1$ and $i \in \{1, \ldots, n\}$ with* $\mathrm{ro}(A)_{i+1} > 0$,

$$E_i e_A = \sum_{p \in \mathbb{Z} : a_{i+1,p} > 0} q^{\sum_{j>p} a_{i,j}} [[a_{i,p} + 1]] e_{A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}}.$$

Given $A \in \Lambda_1$ and $i \in \{1, \ldots, n\}$ with $\mathrm{ro}(A)_i > 0$,

$$F_i e_A = \sum_{p \in \mathbb{Z} : a_{i,p} > 0} q^{\sum_{j<p} a_{i+1,j}} [[a_{i+1,p} + 1]] e_{A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}}.$$

Note that these formulas are still valid in the cases $E_i e_A = 0$ and $F_i e_A = 0$. If the convention that $e_B = 0$ whenever B is not in Λ_1 is used, then the conditions on p in the above sums may be ignored.

Corollary 4.1.3. *Given $A \in \Lambda_1$ and $j \in \{1, \ldots, n\}$ with* $\mathrm{co}(A)_{j+1} > 0$,

$$e_A F_j = \sum_{p \in \mathbb{Z} : a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A + \mathcal{E}_{p,j} - \mathcal{E}_{p,j+1}}.$$

Given $A \in \Lambda_1$ and $j \in \{1, \ldots, n\}$ with $\mathrm{co}(A)_j > 0$,

$$e_A E_j = \sum_{p \in \mathbb{Z} : a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A + \mathcal{E}_{p,j+1} - \mathcal{E}_{p,j}}.$$

Proof.

$$\begin{aligned}
e_A F_j &= S(E_j e_{A^\top}) \\
&= S\left(\sum_{p \in \mathbb{Z} : a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A^\top + \mathcal{E}_{j,p} - \mathcal{E}_{j+1,p}} \right) \\
&= \sum_{p \in \mathbb{Z} : a_{p,j+1} > 0} q^{\sum_{i>p} a_{i,j}} [[a_{p,j} + 1]] e_{A + \mathcal{E}_{p,j} - \mathcal{E}_{p,j+1}},
\end{aligned}$$

where the second equality comes from Lemma 4.1.2. Similarly,

$$\begin{aligned}
e_A E_j &= S(F_j e_{A^\top}) \\
&= S\left(\sum_{p \in \mathbb{Z} : a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A^\top + \mathcal{E}_{j+1,p} - \mathcal{E}_{j,p}} \right) \\
&= \sum_{p \in \mathbb{Z} : a_{p,j} > 0} q^{\sum_{i<p} a_{i,j+1}} [[a_{p,j+1} + 1]] e_{A + \mathcal{E}_{p,j+1} - \mathcal{E}_{p,j}}.
\end{aligned}$$

□

4.1.4 Shifting

In this subsection it is shown that the operations on Λ_1 given by shifting up by one row or to the right by one column may be described by the action, on the left or right respectively, of an invertible element R of $\hat{S}_q(n, r)$.

For each $A \in \Lambda_1$ and $m \in \mathbb{Z}$, the row shift of A by m is the element $[m]A$ of Λ_1 given by

$$([m]A)_{i,j} = a_{i+m,j},$$

for each $i, j \in \mathbb{Z}$.

The column shift of A by m is the element $A[m]$ given by

$$(A[m])_{i,j} = a_{i,j+m},$$

for each $i, j \in \mathbb{Z}$.

For each $\lambda \in \Lambda_0$, define

$$\begin{aligned} R_\lambda &= e_{[1]D_\lambda} \\ &= e_{\lambda_1 \mathcal{E}_{0,1} + \cdots + \lambda_n \mathcal{E}_{n-1,n}} \end{aligned}$$

and let

$$R = \sum_{\lambda \in \Lambda_0} R_\lambda.$$

Recall that

$$\mathcal{O}_{D_\lambda} = \{(L, L) : L \in \mathcal{F}_\lambda\},$$

so

$$\mathcal{O}_{[m]D_\lambda} = \{([m]L, L) : L \in \mathcal{F}_\lambda\}$$

and

$$\mathcal{O}_{D_\lambda[m]} = \{(L, [m]L) : L \in \mathcal{F}_\lambda\}.$$

This leads to a simple rule for multiplication by R in terms of these shifts on matrices.

Lemma 4.1.4. *If $A \in \Lambda_1$ then*

$$Re_A = e_{[1]A}$$

and

$$e_A R = e_{A[-1]}.$$

Proof. TYPE THIS PROOF □

As a visual cue, acting on a basis element e_A on the left by R corresponds to moving the matrix A up by the row, while acting on the right by R corresponds to moving the matrix to the right by one column. Then conjugating by R corresponds to the composition of a shift to the left by one and a shift up by one, which is a shift by one along the diagonal, so conjugating by R^n leaves e_A invariant. Thus conjugation by R gives a $\mathbb{Z}[q]$-algebra automorphism of $\hat{S}_q(n, r)$ which has order n.

Lemma 4.1.5. *The element R is invertible and*

$$RS(R) = S(R)R = 1.$$

In particular,

$$R^{-1} = \sum_{\lambda \in \Lambda_0} e_{[-1]D_\lambda}.$$

Proof. TYPE THIS PROOF. □

For $\lambda \in \Lambda_0$ and $m \in \mathbb{Z}$, let $[m]\lambda$ be the element in Λ_0 given by

$$([m]\lambda)_i = \lambda_{i+m},$$

for each $i \in \mathbb{Z}$, where the indices of λ are taken modulo n.

For example, if $\lambda = (2, 1, 3)$, then $[1]\lambda = (1, 3, 2)$.

Lemma 4.1.6. *For each $\lambda \in \Lambda_0$,*

$$R1_\lambda S(R) = 1_{[1]\lambda}$$

and, for each $i \in \{1, \ldots, n\}$,

$$RE_i S(R) = E_{i-1}$$

and

$$RF_i S(R) = F_{i-1}.$$

Proof. TYPE THIS PROOF. □

Although I can't be sure, I suspect that conjugation by R gives a realisation of the Auslander-Reiten translation on the nilpotent representations of a cyclic quiver determined by the upper triangular matrices in Λ_1. This is at least plausible since the A.R translation τ sends the simple representation at vertex i to the simple representation at vertex $i-1$, which is consistent with the conjugation by R, which sends E_i to E_{i-1}.

4.2 Relations

Note that $E_i^{r+1} = F_i^{r+1} = 0$ while

$$E_i^r = [r]_! e_{r\mathcal{E}_{i,i+1}}$$

and

$$F_i^r = [r]_! e_{r\mathcal{E}_{i+1,i}}.$$

Lemma 4.2.1 (quantum Serre relations: $n \geq 3$)**.** *Suppose $n \geq 3$. The following relations hold in $\hat{S}_q(n, r)$:*

$$E_i E_j - E_j E_i = 0$$

$$F_i F_j - F_j F_i = 0$$

unless $j = i \pm 1$;

$$E_i E_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2 E_i = 0$$
$$E_i^2 E_{i+1} - (1+q)E_iE_{i+1}E_i + qE_{i+1}E_i^2 = 0$$

and

$$F_{i+1}F_i^2 - (1+q)F_iF_{i+1}F_i + qF_i^2F_{i+1} = 0$$
$$F_{i+1}^2 F_i - (1+q)F_{i+1}F_iF_{i+1} + qF_iF_{i+1}^2 = 0.$$

Proof. Here we introduce temporary notation for the basis elements: Write $[A] = e_A$.

Take $\lambda \in \Lambda_0$.

$$E_iE_{i+1}^2 1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+2}] + [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

$$E_{i+1}E_iE_{i+1}1_\lambda = [D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}] + [2][D_\lambda + 2X_{i+1,i+1} + X_{i,i+1}]$$

$$E_{i+1}^2E_i1_\lambda = [2][D_\lambda + 2X_{i+1,i+2} + X_{i,i+1}]$$

Then

$$(E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i)1_\lambda = 0,$$

for each $\lambda \in \Lambda_0$. The relation $E_iE_{i+1}^2 - (1+q)E_{i+1}E_iE_{i+1} + qE_{i+1}^2E_i = 0$ then follows.

The relations between F_i and F_{i+1} may be obtained directly, as above, or by applying the transpose operator to the relations already derived: note that the two sets of relations are related by swapping E_i and F_i and reversing the order of multiplication. □

Lemma 4.2.2 (quantum Serre relations: $n = 2$)**.** *In the case $n = 2$, the quantum Serre relations will be of total degree 4. Look at the presentation of quantum groups for candidate relations. If that fails, brute force won't be too hard.*

Lemma 4.2.3. $[E_i, F_j] = 0$ *unless* $j = i$*.*

$$E_iF_i - F_iE_i = \sum_{\lambda\in\Lambda_0}([\lambda_i] - [\lambda_{i+1}])1_\lambda.$$

4.3 quivers with relations

Denote by Λ_0 the set of compositions of r into n parts. That is, Λ_0 is the set of $\alpha \in \mathbb{Z}^n$ with non-negative entries which sum to r. Let $\varepsilon_i \in \mathbb{Z}^n$ be the ith elementary vector and write $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for each $i \in \{1, \ldots, n\}$. Then $\lambda + \alpha_i \in \Lambda_0$ if $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ if $\lambda_i > 0$.

Let $\Gamma = \Gamma(n, r)$ be the quiver with set of vertices Λ_0, with the following arrows:

For $\lambda \in \Lambda_0$ and $i \in \{1, \ldots, n\}$, there is an arrow $e_{i,\lambda} : \lambda \to \lambda + \alpha_i$ if $\lambda_{i+1} > 0$ and there is an arrow $f_{i,\lambda} : \lambda \to \lambda - \alpha_i$ if $\lambda_i > 0$.

Denote by $\mathbb{Z}[q]\Gamma$ the path $\mathbb{Z}[q]$-algebra of Γ. Thus $\mathbb{Z}[q]\Gamma$ is a free $\mathbb{Z}[q]$-module with a basis given by the set of paths in Γ, with multiplication given by the concatenation of paths. If p starts where q ends, the product pq is the path q followed by p. Write $e_{i,\lambda} = 0$ unless $\lambda, \lambda + \alpha_i \in \Lambda_0$ and write $f_{i,\lambda} = 0$ unless $\lambda, \lambda - \alpha_i \in \Lambda_0$.

By construction, there is a homomorphism of $\mathbb{Z}[q]$-algebras

$$\phi\colon \mathbb{Z}[q]\Gamma \to \hat{S}_q(n, r)$$

given by

$$\phi(e_{i,\lambda}) = E_{i,\lambda}$$
$$\phi(f_{i,\lambda}) = F_{i,\lambda}$$
$$\phi(k_\lambda) = 1_\lambda,$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$.

The image of ϕ is the subalgebra of $\hat{S}_q(n, r)$ generated by E_i, F_i for $i \in \{1, \ldots, n\}$ and 1_λ for $\lambda \in \Lambda_0$, since $E_{i,\lambda} = E_i1_\lambda$ and $F_{i,\lambda} = F_i1_\lambda$, while $E_i = \sum_\lambda E_{i,\lambda}$ and $F_i = \sum_\lambda F_{i,\lambda}$. In general ϕ is not surjective, so this does not always lead to a presentation of $\hat{S}_q(n, r)$.

4.3.1 Exceptional case n=2.

Describe the quiver.

Define an ideal of relations in the path algebra.

Write down the homomorphism from the bound quiver algebra to the q-Schur algebra.

4.3.2 Typical case.

Suppose $n \geq 3$. Then $\Gamma = \Gamma(n, r)$ has vertex set Λ_0.

Define $e_i, f_i \in \mathbb{Z}[q]\Gamma(n, r)$ by

$$e_i = \sum_{\lambda \in \Lambda_0} e_{i,\lambda}$$

and

$$f_i = \sum_{\lambda \in \Lambda_0} f_{i,\lambda},$$

with the convention $e_{i,\lambda} = 0$ unless $\lambda_{i+1} > 0$ and $f_{i,\lambda} = 0$ unless $\lambda_i > 0$. Let k_λ denote the constant path at vertex λ. $\{k_\lambda : \lambda \in \Lambda_0\}$ is a set of pairwise orthogonal idempotents in $\mathbb{Z}[q]\Gamma(n, r)$.

Let $I(n, r) \subset \mathbb{Z}[q]\Gamma(n, r)$ be the ideal generated by the expressions

$$e_i e_{i+1}^2 - (1 + q)e_{i+1} e_i e_{i+1} + q e_{i+1}^2 e_i$$

$$e_i^2 e_{i+1} - (1 + q)e_i e_{i+1} e_i + q e_{i+1} e_i^2$$

$$f_{i+1} f_i^2 - (1 + q)f_i f_{i+1} f_i + q f_i^2 f_{i+1}$$

$$f_{i+1}^2 f_i - (1 + q)f_{i+1} f_i f_{i+1} + q f_i f_{i+1}^2$$

$$e_i f_j - f_j e_i - \delta_{i,j} \sum_{\lambda \in \Lambda_0} ([\lambda_i] - [\lambda_{i+1}])k_\lambda$$

Recall that a relation is a $\mathbb{Z}[q]$-linear combination of paths with common start and end vertices. The relations involving paths $\lambda \to \mu$ are given by $1_\mu \mathrm{expr} 1_\lambda$, for each of the above expressions.

Lemma 4.3.1. *There is a homomorphism of $\mathbb{Z}[q]$-algebras*

$$\phi \colon \mathbb{Z}[q]\Gamma(n, r)/I(n, r) \to \hat{S}_q(n, r)$$

given by

$$\phi(e_{i,\lambda}) = E_{i,\lambda}$$

$$\phi(f_{i,\lambda}) = F_{i,\lambda}$$

$$\phi(k_\lambda) = 1_\lambda.$$

Chapter 5

A generic affine algebra

5.1 Introduction

Assume $\mathbf{k} = \mathbb{C}$ and fix $n, r \geq 1$. Let $\mathcal{S}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon, \varepsilon^{-1}]$ and let $\mathcal{R}$ be the subalgebra generated by ε, namely $\mathcal{R} = \mathbf{k}[\varepsilon]$. Let V be a free $\mathcal{S}$-module of rank r and let $\mathcal{F} = \mathcal{F}_{\mathbf{k}}(n, r)$ be the set of n-periodic cyclic flags in V; so $\mathcal{F}$ consists of collections $L = (L_i)_{i \in \mathbb{Z}}$ of $\mathcal{R}$-lattices in V with $L_i \subset L_{i+1}$ for $i \in \mathbb{Z}$ and $\varepsilon L_i = L_{i-n}$ for $i \in \mathbb{Z}$.

Let G be the group of $\mathcal{S}$-module automorphisms of V. Thus G is isomorphic to $\mathrm{GL}_r(\mathcal{S})$. G acts on $\mathcal{F}$ with orbits $\{\mathcal{F}_\lambda : \lambda \in \Lambda_0\}$, where Λ_0 is the set of compositions of r into n parts, as in Definition 3.0.1.

The diagonal action of G on $\mathcal{F} \times \mathcal{F}$ has orbits $\{\mathcal{O}_A : A \in \Lambda_1\}$, where $\mathcal{O}_A$ consists of those pairs of flags with periodic characteristic matrix equal to A. Definitions of the periodic characteristic matrix and the set Λ_1 are given in Definition 3.1.1 and Definition 3.0.2 respectively.

Recall that the periodic characteristic matrix of a pair $(L, L') \in \mathcal{F} \times \mathcal{F}$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix $A = (a_{i,j})_{i,j \in \mathbb{Z}}$, with

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \right)$$

for each $i, j \in \mathbb{Z}$.

Recall that ro and co are the maps $\Lambda_1 \to \Lambda_0$ given by

$$\mathrm{ro}(A) = \left(\sum_{j \in \mathbb{Z}} a_{1,j}, \ldots, \sum_{j \in \mathbb{Z}} a_{n,j} \right)$$

and

$$\mathrm{co}(A) = \left(\sum_{i \in \mathbb{Z}} a_{i,1}, \ldots, \sum_{i \in \mathbb{Z}} a_{i,n} \right)$$

for each $A \in \Lambda_1$. Given $A \in \Lambda_1$, write $A\colon \mathrm{co}(A) \to \mathrm{ro}(A)$.

The purpose of this chapter is to define an associative $\mathbb{Z}$-algebra with a multiplicative basis by defining a modified form of the product in the affine q-Schur algebra. In particular, given $A, B \in \Lambda_1$, the orbit product

$$X_{A,B} = \{(L, L'') \in \mathcal{F} \times \mathcal{F} : \exists L' \in \mathcal{F} \text{ with } (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_B\}$$

consists of finitely many G-orbits and it will be shown that there is a unique 'generic' orbit in $X_{A,B}$, denoted $\mathcal{O}_{A*B}$, with the property that

$$\dim\left(\frac{L_i}{L_i \cap L''_j}\right) \leq \dim\left(\frac{N_i}{N_i \cap N''_j}\right)$$

and

$$\dim\left(\frac{L''_j}{L_i \cap L''_j}\right) \leq \dim\left(\frac{N''_j}{N_i \cap N''_j}\right)$$

for all $i, j \in \mathbb{Z}$, $(N, N'') \in \mathcal{O}_{A*B}$ and $(L, L'') \in X_{A,B}$. It will be shown that the above 'generic product' of orbits is associative, so the free $\mathbb{Z}$-module on the set of G-orbits in $\mathcal{F}\times\mathcal{F}$ with $\mathbb{Z}$-bilinear multiplication given by

$$\mathcal{O}_A * \mathcal{O}_B = \mathcal{O}_{A*B},$$

for each $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, and

$$\mathcal{O}_A * \mathcal{O}_B = 0$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) \neq \mathrm{ro}(B)$, is an associative $\mathbb{Z}$-algebra with multiplicative identity given by

$$\sum_{\lambda\in\Lambda_0} \mathcal{O}_{D_\lambda},$$

where D_λ is the diagonal matrix with $\mathrm{co}(D_\lambda) = \lambda$. The resulting $\mathbb{Z}$-algebra is called the *generic affine algebra* (of rank r and period n), denoted $\hat{G}(n, r)$.

5.2 A combinatorial partial order

For each $i, j \in \mathbb{Z}$, let $d_{i,j}$ and $\bar{d}_{i,j}$ be the maps from Λ_1 to Λ_0 given by

$$d_{i,j}(A) = \sum_{s\leq i, t>j} a_{s,t}$$

and

$$\bar{d}_{i,j}(A) = \sum_{s>i, t\leq j} a_{s,t}$$

for each $A \in \Lambda_1$.

Lemma 5.2.1. *For each $A \in \Lambda_1$ and $i, j \in \mathbb{Z}$, the following equations hold:*

$$d_{i,j}(A) - d_{i-1,j}(A) = \sum_{t>j} a_{i,t}$$
$$d_{i,j}(A) - d_{i,j-1}(A) = -\sum_{s\leq i} a_{s,j}$$

and

$$\bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) = -\sum_{t\leq j} a_{i,t}$$
$$\bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) = \sum_{s>i} a_{s,j}$$

Proof. Let $i, j \in \mathbb{Z}$ and $A \in \Lambda_1$. Then

$$d_{i,j}(A) - d_{i-1,j}(A) = \sum_{s \le i, t > j} a_{s,t} - \sum_{s \le i-1, t > j} a_{s,t} = \sum_{t > j} a_{i,t}$$

and

$$d_{i,j}(A) - d_{i,j-1}(A) = \sum_{s \le i, t > j} a_{s,t} - \sum_{s \le i, t > j-1} a_{s,t} = -\sum_{s \le i} a_{s,j}.$$

Similarly,

$$\bar{d}_{i,j}(A) - \bar{d}_{i-1,j}(A) = \sum_{s > i, t \le j} a_{s,t} - \sum_{s > i-1, t \le j} a_{s,t} = -\sum_{t \le j} a_{i,t}$$

and

$$\bar{d}_{i,j}(A) - \bar{d}_{i,j-1}(A) = \sum_{s > i, t \le j} a_{s,t} - \sum_{s > i, t \le j-1} a_{s,t} = \sum_{s > i} a_{s,j}.$$

□

Lemma 5.2.2. *For each $A \in \Lambda_1$ and $i, j \in \mathbb{Z}$,*

$$a_{i,j} = d_{i,j-1}(A) - d_{i-1,j-1}(A) - d_{i,j}(A) + d_{i-1,j}(A)$$

and

$$a_{i,j} = \bar{d}_{i,j-1}(A) - \bar{d}_{i-1,j-1}(A) - \bar{d}_{i,j}(A) + \bar{d}_{i-1,j}(A).$$

Proof. As a result of Lemma 5.2.1,

$$\begin{aligned} d_{i,j-1}(A) - d_{i-1,j-1}(A) - d_{i,j}(A) + d_{i-1,j}(A) &= \sum_{t > j-1} a_{i,t} - \sum_{t > j} a_{i,t} \\ &= a_{i,j} \end{aligned}$$

and

$$\begin{aligned} \bar{d}_{i,j-1}(A) - \bar{d}_{i-1,j-1}(A) - \bar{d}_{i,j}(A) + \bar{d}_{i-1,j}(A) &= -\sum_{t \le j-1} a_{i,t} + \sum_{t \le j} a_{i,t} \\ &= a_{i,j}. \end{aligned}$$

□

Define a relation $\le$ on Λ_1 by $A \le B$ if and only if the following conditions are satisfied:

- $\mathrm{ro}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(A) = \mathrm{co}(B)$.
- For each $i, j \in \mathbb{Z}$, $d_{i,j}(A) \le d_{i,j}(B)$.
- For each $i, j \in \mathbb{Z}$, $\bar{d}_{i,j}(A) \le \bar{d}_{i,j}(B)$.

Lemma 5.2.3. *The relation $\le$ defines a partial order on Λ_1.*

Proof. It is clear that $\le$ is reflexive and transitive.

Suppose $A, B \in \Lambda_1$ with $A \le B$ and $B \le A$. Then $d_{i,j}(A) = d_{i,j}(B)$ for each $i, j \in \mathbb{Z}$ with $i \le j$, which shows $a_{s,t} = b_{s,t}$ whenever $s < t$, as a result of Lemma 5.2.2. Similarly, $\bar{d}_{i,j}(A) = \bar{d}_{i,j}(B)$ for each $i, j \in \mathbb{Z}$ with $i \ge j$, so $a_{s,t} = b_{s,t}$ whenever $s > t$. Moreover, $a_{i,i} = b_{i,i}$ for each $i \in \mathbb{Z}$, since $\mathrm{co}(A) = \mathrm{co}(B)$. Thus $A = B$, which shows $\le$ is antisymmetric and therefore $\le$ is a partial order on Λ_1. □

Lemma 5.2.4. *The transpose operation on Λ_1 is order preserving. In particular, $B \leq A$ if and only if $B^\top \leq A^\top$.*

Proof. Suppose $A, B \in \Lambda_1$ with $B \leq A$. The condition $\mathrm{co}(A) = \mathrm{co}(B)$ and $\mathrm{ro}(A) = \mathrm{ro}(B)$ is preserved by the transpose operation.

For each $i, j \in \mathbb{Z}$,

$$d_{i,j}(A^\top) = \sum_{s \leq i, t > j} a_{t,s} = \bar{d}_{j,i}(A)$$

and

$$\bar{d}_{i,j}(A^\top) = \sum_{s > i, t \leq j} a_{t,s} = d_{j,i}(A).$$

It follows that $B^\top \leq A^\top$ and therefore the transpose is order preserving. □

The partial order on Λ_1 induces a partial order on the set of G-orbits in $\mathcal{F} \times \mathcal{F}$, such that $\mathcal{O}_A \leq \mathcal{O}_B$ if and only if $A \leq B$. The following lemma is rephrased from Lemma 3.1.4 and gives some geometric significance to the partial order on Λ_1.

Lemma 5.2.5. *Let $A \in \Lambda_1$ and $(L, L') \in \mathcal{O}_A$. Then*

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) = d_{i,j}(A)$$

and

$$\dim\left(\frac{L'_j}{L_i \cap L'_j}\right) = \bar{d}_{i,j}(A),$$

for each $i, j \in \mathbb{Z}$.

Proof. This is a rephrasing of Lemma 3.1.4. □

5.3 Grassmannians and related varieties

Here we collect a few elementary results on Grassmannians and some related varieties. In this section, let V be an n-dimensional $\mathbf{k}$-vector space and let $0 \leq d \leq n$ be an integer. There is a linear map

$$\phi^{(d)} \colon \Lambda^d(V) \to \mathrm{Hom}(V, \Lambda^{d+1}(V))$$

given by

$$\phi^{(d)}(\alpha)(v) = \alpha \wedge v$$

for $\alpha \in \Lambda^d(V)$ and $v \in V$. The kernel of $\phi^{(d)}(\alpha)$ is the space of divisors of α,

$$D_\alpha = \{v \in V : \alpha \wedge v = 0\}.$$

An element $\alpha \in \Lambda^d(V)$ is said to be totally decomposable if $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_d$, where $\alpha_1, \ldots, \alpha_d \in V$ are linearly independent. The dimension of D_α is at most d and $\dim(D_\alpha) = d$ precisely when α is totally decomposable. Consequently, the rank of $\phi^{(d)}(\alpha)$ is at least $n - d$ and α is totally decomposable if and only if $\mathrm{rank}\,\phi^{(d)}(\alpha) \leq n - d$, which holds if and only if the $(n-d+1) \times (n-d+1)$-minors of a matrix of $\phi^{(d)}(\alpha)$ are all zero.

Lemma 5.3.1. $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ *is a projective variety, for each* $d_1, d_2, a \in \mathbb{N}$ *with* $d_1, d_2, a \leq n$.

Proof. As above, there is a linear map $\Psi\colon \Lambda^{d_1} V \oplus \Lambda^{d_2} V \to \mathrm{Hom}(V, \Lambda^{d_1+1}(V) \oplus \Lambda^{d_2+1}(V))$ given by $\Psi(\alpha, \beta)(v) = (\alpha \wedge v, \beta \wedge v)$. Given $\alpha \in \Lambda^{d_1}(V)$ and $\beta \in \Lambda^{d_2}(V)$, the kernel of $\Psi(\alpha, \beta)$ is $D_\alpha \cap D_\beta$ and so the rank of $\Psi(\alpha, \beta)$ is $n - \dim(D_\alpha \cap D_\beta)$.

Let $U_i \in \mathrm{Gr}_{d_i}(V)$ and suppose $p_i(U_i) = [\alpha_i]$, where p_i is the Plücker embedding of $\mathrm{Gr}_{d_i}(V)$ in $\mathbb{P}(\Lambda^{d_i}(V))$, so $U_i = D_{\alpha_i} = \ker \phi^{(d_i)}(\alpha)$. Therefore the kernel of $\Psi(\alpha_1, \alpha_2)$ is $U_1 \cap U_2$, so the condition that $\dim(U_1 \cap U_2) \geq a$ is equivalent to the condition that $\Psi(\alpha_1, \alpha_2)$ has rank at most $n - a$. After fixing a basis of V, this condition is given by the vanishing of the $(n - a + 1) \times (n - a + 1)$ minors of the matrix of $\Psi(\alpha_1, \alpha_2)$ with respect to this basis. Therefore $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ is a closed subset of the product of Grassmannians $\mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V)$, so is a projective variety.

More precisely, the entries of a matrix of $\Psi(\alpha_1, \alpha_2)$ are homogeneous polynomials of degree 1 in the Plücker coordinates on $\mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V)$ since Ψ is linear and so the minors of $\Psi(\alpha_1, \alpha_2)$ are also homogeneous polynomials in the Plücker coordinates. □

Lemma 5.3.2. *Let* V *be an* n*-dimensional vector space over* $\mathbf{k}$ *and let* $d_1, d_2, a \in \mathbb{N}$ *with* $d_1, d_2, a \leq n$. *The following hold:*

1. $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) = a\}$ *is a quasiprojective variety;*
2. $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : U_1 \subset U_2\}$ *is a projective variety;*
3. *Given* $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\}$ *is a projective variety;*
4. *Given* $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) = a\}$ *is a quasiprojective variety;*
5. *Given* $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : U_1 \subset U_2\}$ *is a projective variety;*
6. *Given* $U_2 \in \mathrm{Gr}_{d_2}(V)$, $\{U_1 \in \mathrm{Gr}_{d_1}(V) : U_2 \subset U_1\}$ *is a projective variety.*

Proof. Let X_i denote the space in statement i of the lemma. To emphasise the dependence of X_i on a, write $X_{i,a}$.

X_1 is a quasiprojective variety since it is equal to the intersection of the projective variety $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \geq a\}$ with the open set $\{(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap U_2) \leq a\}$.

Given $(U_1, U_2) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V)$, $U_1 \subset U_2$ if and only if $\dim(U_1 \cap U_2) \geq d_1$, so Lemma 5.3.1 shows X_2 is a projective variety.

Let $\pi_i\colon \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) \to \mathrm{Gr}_{d_i}(V)$ be the projection map onto the i-th factor, for $i = 1, 2$. The completeness property of projective varieties ensures that π_i is a closed morphism. Observe that

$$\begin{aligned} X_3 &= \{U_1 \in \mathrm{Gr}_{d_1}(V) : \dim(U_1 \cap U_2) \geq a\} \\ &= \pi_1(\{(U_1, W) \in \mathrm{Gr}_{d_1}(V) \times \mathrm{Gr}_{d_2}(V) : \dim(U_1 \cap W) \geq a\} \cap \pi_2^{-1}\{U_2\}). \end{aligned}$$

The fibre of π_2 over U_2 is closed, so the intersection of the fibre with the variety from Lemma 5.3.1 is closed and then the image of this intersection under π_1 is closed. This shows X_3 is a projective variety.

X_4 is a quasiprojective variety since it is the complement of the subvariety $X_{3,a+1}$ in $X_{3,a}$. Finally, *5-6* follow as special cases of *3* since $X_5 = X_{3,d_1}$ and $X_6 = X_{3,d_2}$. □

5.4 Geometry of affine flag varieties

Given $L \in \mathcal{F}$, $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0$ define

$$\Pi_{N,\lambda}(L) = \{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0\}.$$

and

$$\Pi^a_{N,\lambda}(L) = \left\{L' \in \mathcal{F}_\lambda : \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0, \dim\left(\frac{\varepsilon^{-N} L_0}{L'_0}\right) = a\right\}.$$

Lemma 5.4.1. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$ and $\lambda \in \Lambda_0$,*

$$\Pi_{N,\lambda}(L) = \bigcup_{a:0\leq a\leq 2Nr} \Pi^a_{N,\lambda}(L).$$

Proof. If $L' \in \Pi_{N,\lambda}(L)$ then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and the $\mathbf{k}[\varepsilon]$-module $\varepsilon^{-N}L_0/L'_0$ is naturally isomorphic to $(\varepsilon^{-N}L_0/\varepsilon^N L_0)/(L'_0/\varepsilon^N L_0)$, so

$$\dim_{\mathbf{k}}\left(\frac{\varepsilon^{-N}L_0}{L'_0}\right) \leq \dim_{\mathbf{k}}\left(\frac{\varepsilon^{-N}L_0}{\varepsilon^N L_0}\right) = 2Nr.$$

□

Lemma 5.4.2. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \leq a \leq 2Nr$, $\Pi^a_{N,\lambda}(L)$ is a projective algebraic variety.*

Proof. Let W be the $\mathbf{k}[\varepsilon]$-module $\varepsilon^{-1-N}L_0/\varepsilon^N L_0$, which has dimension $(2N+1)r$ over $\mathbf{k}$. Let $d_i = 2Nr - a + \lambda_1 + \cdots + \lambda_i$ for each $i = 1, \ldots, n$. The correspondence between submodules of $\varepsilon^{-1-N}L_0$ which contain $\varepsilon^N L_0$ and submodules of $\varepsilon^{-1-N}L_0/\varepsilon^N L_0$ determines a map

$$\rho\colon \Pi^a_{N,\lambda}(L) \to \mathrm{Gr}_{d_1}(W) \times \cdots \times \mathrm{Gr}_{d_n}(W),$$

with $\rho(L') = (L'_1/\varepsilon^N L_0, \ldots, L'_n/\varepsilon^N L_0)$.

Let $\mathcal{X}$ be the space of $(U_1, \ldots, U_n) \in \mathrm{Gr}_{d_1}(W) \times \cdots \times \mathrm{Gr}_{d_n}(W)$ with $U_i \subset U_{i+1}$ for $i = 1, \ldots, n-1$ and $\varepsilon U_n \subset U_1$. Lemma 5.3.2 shows that each of these conditions is closed, so $\mathcal{X}$ is a closed subset of $\mathrm{Gr}_{d_1}(W) \times \cdots \times \mathrm{Gr}_{d_n}(W)$, therefore $\mathcal{X}$ is a projective algebraic variety.

The image of ρ is contained in $\mathcal{X}$ since

$$\varepsilon L'_n/\varepsilon^N L_0 = L'_0/\varepsilon^N L_0 \subset L'_1/\varepsilon^N L_0 \subset \cdots \subset L'_n/\varepsilon^N L_0.$$

Suppose $(U_1, \ldots, U_n) \in \mathcal{X}$. Then U_i is a $\mathbf{k}[\varepsilon]$-module, since $\varepsilon U_i \subset \varepsilon U_n \subset U_1 \subset U_i$, for each $i = 1, \ldots, n$, so U_i lifts uniquely to a $\mathbf{k}[\varepsilon]$-module L'_i with $\varepsilon^N L_0 \subset L'_i \subset \varepsilon^{-1-N}L_0$. Therefore $L'_1, \ldots, L'_n$ are $\mathbf{k}[\varepsilon]$-lattices with $L_i \subset L_{i+1}$ for $i = 1, \ldots, n-1$ and $\varepsilon L'_n \subset L'_1$, with

$$\dim\left(\varepsilon^{-1-N}L_0/L'_n\right) = \dim\left(W/W_n\right) = (2N+1)r - d_n = a$$

and

$$\dim\left(L'_i/L'_{i-1}\right) = \dim\left(W_i/W_{i-1}\right) = d_i - d_{i-1} = \lambda_i,$$

for each $i = 2, \ldots, n$. Therefore there is a unique $L' \in \Pi^a_{N,\lambda}(L)$ such that $\rho(L') = (W_1, \ldots, W_n)$, where L' is given by $L'_{i+cn} = \varepsilon^{-c}L'_i$ for $i = 1, \ldots, n$ and $c \in \mathbb{Z}$. It follows ρ is injective and $\operatorname{im}\rho = \mathcal{X}$, which is a projective variety, so $\Pi^a_{N,\lambda}(L)$ is a projective variety. □

Lemma 5.4.3. *Given $L \in \mathcal{F}$, $N \in \mathbb{N}$, $\lambda \in \Lambda_0$ and $a \in \mathbb{N}$ with $0 \leq a \leq 2Nr$, $\Pi^a_{N,\lambda}(L)$ is closed in $\Pi^{a+r}_{N+1,\lambda}(L)$.*

Proof. If $L' \in \Pi^a_{N,\lambda}(L)$, then $\varepsilon^{N+1}L_0 \subset \varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0 \subset \varepsilon^{-(N+1)}L_0$ and

$$\dim\left(\frac{\varepsilon^{-(1+n)}L_0}{L'_0}\right) = \dim\left(\frac{L_0}{\varepsilon L_0}\right) + \dim\left(\frac{\varepsilon^{-N}L_0}{L'_0}\right) = r + a,$$

which shows that $\Pi^a_{N,\lambda}(L) \subset \Pi^{a+r}_{N+1,\lambda}(L)$. For $L' \in \Pi^{a+r}_{N+1,\lambda}(L)$, if additionally $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$, then

$$\dim\left(\frac{\varepsilon^{-(N+1)}L_0}{L'_0}\right) = r + \dim\left(\frac{\varepsilon^{-N}L_0}{L'_0}\right),$$

which shows $L' \in \Pi^a_{N,\lambda}(L)$. Therefore $\Pi^a_{N,\lambda}(L)$ is the subspace of $\Pi^{a+r}_{N+1,\lambda}(L)$ defined by the two closed conditions $\varepsilon^N L_0 \subset L'_0$ and $L'_0 \subset \varepsilon^{-N}L_0$, using Lemma 5.3.2. □

Lemma 5.4.4. *Let $\lambda \in \Lambda_0$, $M, N \in \mathbb{N}$, $L, \tilde{L} \in \mathcal{F}$, $0 \leq a \leq 2Nr$, $0 \leq b \leq 2Mr$. $\Pi^a_{N,\lambda}(L) \cap \Pi^b_{M,\lambda}(\tilde{L})$ is a closed set in $\Pi^a_{N,\lambda}(L)$. In particular, if the intersection is nonempty it is a projective algebraic variety.*

Proof. Observe that $\Pi^a_{N,\lambda}(L) \cap \Pi^b_{M,\lambda}(\tilde{L})$ is the subset of $\Pi^a_{N,\lambda}(L)$ defined by the additional conditions that $\varepsilon^M \tilde{L}_0 \subset L'_0$ and $L'_0 \subset \varepsilon^{-M}\tilde{L}_0$, so is a closed subset of $\Pi^a_{N,\lambda}(L)$, using 5.3.2. □

Lemma 5.4.5. *Suppose $L \in \mathcal{F}$, $N, a \in \mathbb{N}$ and $\lambda \in \Lambda_0$ with $a \leq 2Nr$. For each $g \in G$, the natural map (restriction of the action map) $\Pi^a_{N,\lambda}(L) \to \Pi^a_{N,\lambda}(gL)$ is an isomorphism of projective varieties.*

Proof. If $L' \in \Pi^a_{N,\lambda}(L)$, then $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N}L_0$ and so $\varepsilon^N g(L_0) \subset g(L'_0) \subset \varepsilon^{-N}g(L_0)$, so $gL' \in \Pi^a_{N,\lambda}(L)$. Thus g and g^{-1} induce mutually inverse morphisms of varieties $g\colon \Pi^a_{N,\lambda}(L) \to \Pi^a_{N,\lambda}(gL)$ and $g^{-1}\colon \Pi^a_{N,\lambda}(gL) \to \Pi^a_{N,\lambda}(L)$. □

5.4.1 Action through an algebraic group

Let W be the $\mathbb{C}[\varepsilon]$-module $\varepsilon^{-(1+N)}L_0/\varepsilon^N L_0$. ε^{2N+1} acts as zero on W and $\mathbb{C}[\varepsilon]/\langle\varepsilon^{2N+1}\rangle \otimes_{\mathbb{C}[\varepsilon]} W$ is a free $\mathbb{C}[\varepsilon]/\langle\varepsilon^{2N+1}\rangle$-module of rank r. In particular, W is a complex vector space of dimension $(2N+1)r$.

Each element $g \in G_L$ determines an endomorphism $\overline{g}$ of W, given by

$$\overline{g}(x + \varepsilon^N L_0) = g(x) + \varepsilon^N L_0,$$

for each $x \in \varepsilon^{-1-N}L_0$. Given $g, h \in G_L$, $\overline{gh} = \overline{g}\overline{h}$ and so $\bar{g}$ is an automorphism of W with $\overline{g}^{-1} = \overline{g^{-1}}$. Therefore the map $\bar{\ }\colon G_L \mapsto \mathrm{GL}(W)$ given by $g \mapsto \overline{g}$ is a group homomorphism with kernel

$$H_{N,L} \coloneqq \{g \in G_L : \overline{g} = 1\},$$

which consists of those $g \in G_L$ such that

$$g(x) - x \in \varepsilon^N L_0$$

for each $x \in \varepsilon^{-1-N}L_0$. Thus $G_L/H_{N,L}$ may be identified with a subgroup of $\mathrm{GL}(W)$.

Lemma 5.4.6. *$G_L/H_{N,L}$ is a connected algebraic group.*

Proof. As a result of the first isomorphism theorem, $G_L/H_{N,L}$ is isomorphic to the image of G_L in $\mathrm{GL}(W)$, which will be described explicitly by equations in the coordinate functions on $\mathrm{GL}(W)$, with respect to a fixed basis of W.

Let $\{\tilde{x}_1, \ldots, \tilde{x}_r\}$ be a basis of L_n/L_0 over $\mathbb{C}$ which is adapted to the flag

$$L_1/L_0 \subset \cdots \subset L_{n-1}/L_0 \subset L_n/L_0,$$

so that

$$L_i/L_0 = \langle \tilde{x}_1, \ldots, \tilde{x}_{\lambda_1 + \cdots \lambda_i} \rangle$$

for each $i \in \{1, \ldots, n\}$. Fix $x_1, \ldots, x_r \in L_n$ such that $\tilde{x}_i = x_i + L_0$ for each $i = 1, \ldots, r$, then

$$L_i = L_0 + \langle x_1, \ldots, x_{\lambda_1 + \cdots + \lambda_i} \rangle$$

for $i = 1, \ldots, r$.

Then W has a $\mathbb{C}$-basis $\{y_j : 1 \leq j \leq (2N+1)r\}$ given by

$$y_{i+cr} = \varepsilon^{-c+N} x_i$$

for each $i \in \{1, \ldots, r\}$ and $c \in \{0, \ldots, 2N\}$. Observe that $\varepsilon y_i = 0$ for $i \in \{1, \ldots, r\}$ and $\varepsilon y_i = y_{i-r}$ for $r < i \leq (2N+1)r$.

The coordinate functions on $\mathrm{GL}(W)$ with respect to this choice of basis are the maps

$$\gamma_{i,j} \colon \mathrm{GL}(W) \to \mathbb{C}$$

for $i, j \in \mathbb{Z}$ with $1 \leq i, j \leq (2N+1)r$, given by

$$g(y_j) = \sum_i \gamma_{ij}(g) y_i,$$

for each $j = 1, \ldots, (2N+1)r$.

The image of G_L in $\mathrm{GL}(W)$ is the subgroup defined by the conditions

$$\gamma_{i,j} = \gamma_{i-r,j-r}$$

for each $i, j \in \{r+1, \ldots, (2N+1)r\}$ and

$$\gamma_{i,j} = 0$$

for each $i, j \in \{1, \ldots, (2N+1)r\}$ with $i > \lambda_1 + \cdots + \lambda_s$ and $j \leq \lambda_1 + \cdots + \lambda_s$ for some $s \in \{1, \ldots, r\}$. This shows that the image of G_L in $\mathrm{GL}(W)$ is a connected algebraic group and therefore $G_L/H_{N,L}$ is a connected algebraic group.

With respect to the basis $\{y_i : i \in \{1, \ldots, (2N+1)r\}\}$, the image of G_L in $\mathrm{GL}(W)$ consists of matrices of the form

$$\begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{2N} \\ 0 & A_0 & A_1 & \cdots & A_{2N-1} \\ 0 & 0 & A_0 & \cdots & A_{2N-2} \\ & & & & \\ 0 & 0 & 0 & \cdots & A_0 \end{pmatrix}$$

where $A_0 \in \mathcal{P}_\lambda$ and $A_1, \ldots, A_{2N} \in M_r(\mathbb{C})$, where $\mathcal{P}_\lambda$ is the parabolic subgroup of $\mathrm{GL}_r(\mathbb{C})$ which is the stabiliser of the flag

$$L_1/L_0 \subset \cdots \subset L_{n-1}/L_0 \subset L_n/L_0.$$

□

Given $g \in G$, the map $G_L \to G_{gL}$ sending h to ghg^{-1} is a group isomorphism which descends to an isomorphism of algebraic groups $G_L/H_{N,L} \to G_{gL}/H_{N,gL}$. Thus we have a commuting diagram of morphisms of varieties, where the vertical arrows are isomorphisms:

$$\begin{tikzcd} G_L/H_{N,L} \times \Pi^a_{N,\lambda}(L) \arrow[r] \arrow[d] & \Pi^a_{N,\lambda}(L) \arrow[d] \\ G_{gL}/H_{N,gL} \times \Pi^a_{N,\lambda}(gL) \arrow[r] & \Pi^a_{N,\lambda}(gL) \end{tikzcd}$$

5.4.2 Incidence in affine flag varieties

Lemma 5.4.7. *Given $N, a, b, c \in \mathbb{N}$, $\lambda, \mu \in \Lambda_0$, $L \in \mathcal{F}$ and $i, j \in \mathbb{Z}$,*

$$\left\{ (L', L'') \in \Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L) : \dim\left(\frac{L'_i}{L'_i \cap L''_j} \right) \leq c \right\}$$

is a closed set in the projective variety $\Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L)$.

Proof. There is $M \geq N$ so that $\varepsilon^M L_0 \subset L'_i \subset \varepsilon^{-M} L_0$ and $\varepsilon^M L_0 \subset L''_j \subset \varepsilon^{-M} L_0$. Let $a' = a + (M-N)r$ and $b' = b + (M-N)r$. Lemma 5.4.3 shows that $\Pi^a_{N,\lambda}(L)$ is a subvariety of $\Pi^{a'}_{M,\lambda}(L)$, so $\Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L)$ is a subvariety of $\Pi^{a'}_{M,\lambda}(L) \times \Pi^{b'}_{M,\mu}(L)$.

The fact that

$$\dim\left(\frac{L'_i}{L'_i \cap L''_j} \right) = \dim\left(\frac{L'_i/\varepsilon^M L_0}{L'_i/\varepsilon^M L_0 \cap L''_j/\varepsilon^M L_0} \right),$$

together with Lemma 5.4.2 and Lemma 5.3.1, shows that

$$\left\{ (L', L'') \in \Pi^{a'}_{M,\lambda}(L) \times \Pi^{b'}_{M,\mu}(L) : \dim\left(\frac{L'_i}{L'_i \cap L''_j} \right) \leq c \right\}$$

is closed, so the intersection with $\Pi^a_{N,\lambda}(L) \times \Pi^b_{N,\mu}(L)$ is closed. □

Lemma 5.4.8. *Given $N, a, c \in \mathbb{N}$, $\lambda \in \Lambda_0$, $L \in \mathcal{F}$ and $i, j \in \mathbb{Z}$,*

$$\left\{ L' \in \Pi^a_{N,\lambda}(L) : \dim\left(\frac{L_i}{L_i \cap L'_j} \right) \leq c \right\}$$

and

$$\left\{ L' \in \Pi^a_{N,\lambda}(L) : \dim\left(\frac{L'_j}{L_i \cap L'_j} \right) \leq c \right\}$$

are closed sets in $\Pi^a_{N,\lambda}(L)$.

Proof. This is a result of Lemma 5.3.2, since

$$\dim\left(\frac{L_i}{L_i \cap L'_j} \right) = \dim\left(\frac{L_i/\varepsilon^M L_0}{L_i/\varepsilon^M L_0 \cap L'_j/\varepsilon^M L_0} \right),$$

where $M \geq N$ is chosen so that $\varepsilon^M L_0 \subset L_i \subset \varepsilon^{-M} L_0$ and $\varepsilon^M L_0 \subset L'_j \subset \varepsilon^{-M} L_0$ for each $L' \in \Pi^a_{N,\lambda}(L)$. □

5.5 Geometry of orbits

Let $A \in \Lambda_1$ and $L \in \mathcal{F}_{\mathrm{ro}(A)}$ and write $\lambda = \mathrm{co}(A)$. Recall that

$$X_A^L = \{L' \in \mathcal{F}_\lambda : (L, L') \in \mathcal{O}_A\}.$$

Lemma 5.5.1. *There is $N \in \mathbb{N}$ such that $X_A^L \subset \Pi_{N,\lambda}^a(L)$, where $a = d_{nN,0}A$.*

Proof. There is $N \in \mathbb{N}$ so that $a_{i,j} = 0$ whenever $|j - i| > nN$. If $(L, L') \in \mathcal{O}_A$ then

$$\dim\left(\frac{L_0'}{L_0' \cap \varepsilon^{-N} L_0}\right) = \dim\left(\frac{L_0'}{L_0' \cap L_{nN}}\right) = \sum_{s > nN, t \le 0} a_{s,t} = 0,$$

so it follows $L_0' \subset \varepsilon^{-N} L_0$. Similarly,

$$\dim\left(\frac{\varepsilon^N L_0}{\varepsilon^N L_0 \cap L_0'}\right) = \dim\left(\frac{L_{-nN}}{L_{-nN} \cap L_0'}\right) = \sum_{s \le -nN, t > 0} a_{s,t} = 0,$$

which shows $\varepsilon^N L_0 \subset L_0'$. Moreover,

$$\dim\left(\frac{\varepsilon^{-N} L_0}{L_0'}\right) = \dim\left(\frac{\varepsilon^{-N} L_0}{\varepsilon^{-N} L_0 \cap L_0'}\right) = \sum_{s \le nN, t > 0} a_{s,t} = d_{nN,0}(A),$$

as a result of Lemma 5.2.5. □

Assume $N \in \mathbb{N}$ is chosen so that $X_A^L \subset \Pi_{N,\lambda}^a(L)$, where $a = d_{nN,0}A$, as in Lemma 5.5.1.

Lemma 5.5.2. *X_A^L is a locally closed subset of $\Pi_{N,\lambda}^a(L)$. In particular, X_A^L is a quasiprojective variety.*

Proof. If $L' \in \Pi_{N,\lambda}^a(L)$ then

$$L_{-Nn} = \varepsilon^N L_0 \subset L_0' \subset L_1' \subset L_n' \subset \varepsilon^{-1-N} L_0 = L_{(N+1)n}.$$

Therefore X_A^L is the subset of $\Pi_{N,\lambda}^a(L)$ defined by the conditions $\dim(L_i/L_i \cap L_j') = d_{i,j}A$ for $i : -Nn \le i < j$ and $\dim(L_j'/L_i \cap L_j') = \bar{d}_{i,j}A$ for $i : j < i \le (N+1)n$, for $j = 1, \ldots, n$.

The set of $L' \in \Pi_{N,\lambda}^a(L)$ with $\dim(L_i/L_i \cap L_j') \le d_{i,j}A$ for $j = 1, \ldots, n$ and $i : -Nn \le i < j$ and $\dim(L_j'/L_i \cap L_j') \le \bar{d}_{i,j}A$ for $j = 1, \ldots, n$ and $i : j < i \le (N+1)n$ is a closed subset of $\Pi_{N,\lambda}^a(L)$, as a result of Lemma 5.4.8.

On the other hand, the set of $L' \in \Pi_{N,\lambda}^a(L)$ satisfying the conditions $\dim(L_i/L_i \cap L_j') \ge d_{i,j}A$ (for $i < j$) and $\dim(L_j'/L_i \cap L_j') \ge \bar{d}_{i,j}A$ (for $i > j$) is open in $\Pi_{N,\lambda}^a(L)$ since the complement is closed, as a result of Lemma 5.4.8.

Therefore X_A^L is the intersection of an open set and a closed set in $\Pi_{N,\lambda}^a(L)$, so X_A^L is locally closed. It follows that X_A^L is an open subset of the projective variety $\overline{X_A^L}$, so is a quasiprojective variety as claimed. □

Lemma 5.5.3. *X_A^L is irreducible.*

Proof. For any $L' \in X_A^L$, $X_A^L = G_L/H_{N,L} \cdot L'$. Lemma 5.4.6 shows that $G_L/H_{N,L}$ is a connected algebraic group which acts algebraically on $\Pi_{N,\lambda}^a(L)$. The image of $G_L/H_{N,L}$ under the morphism $g \mapsto gL'$ equals X_A^L, which shows X_A^L is irreducible since $G_L/H_{N,L}$ is irreducible. □

Consequently, $\overline{X_A^L}$ is an irreducible projective variety and the action of $G_L/H_{N,L}$ on $\Pi_{N,\lambda}^a(L)$ restricts to an algebraic group action on $\overline{X_A^L}$ for which there are finitely many orbits. In particular, $\overline{X_A^L} \setminus X_A^L$ is a union of finitely many orbits which are so-called degenerations of the orbit X_A^L.

5.6 Geometry of orbit products

Let $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$ and write $\lambda = \mathrm{co}(A)$ and $\mu = \mathrm{co}(B)$. Fix $L \in \mathcal{F}_{\mathrm{ro}(A)}$. Recall

$$Y^L_{A,B} = \{(L', L'') \in \mathcal{F}_\lambda \times \mathcal{F}_\mu : L' \in X^L_A, L'' \in X^{L'}_B\}$$

and

$$X^L_{A,B} = \{L'' \in \mathcal{F}_\mu : \exists L' \in X^L_A \text{ with } L'' \in X^{L'}_B\}$$

Lemma 5.6.1. *There is $N \in \mathbb{N}$ such that*

$$Y^L_{A,B} \subset \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L),$$

where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$.

Proof. There is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$ and $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$ for each $(L', L'') \in Y^L_{A,B}$, using Lemma 5.5.1. Set $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$.

Then for any $(L', L'') \in Y^L_{A,B}$,

$$\varepsilon^{2N} L_0 \subset \varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0 \subset \varepsilon^{-2N} L_0$$

and

$$\begin{aligned} \dim\left(\frac{\varepsilon^{-2N} L_0}{L''_0}\right) &= \dim\left(\frac{\varepsilon^{-N} L'_0}{L''_0}\right) + \dim\left(\frac{\varepsilon^{-2N} L_0}{\varepsilon^{-N} L'_0}\right) \\ &= \dim\left(\frac{\varepsilon^{-N} L'_0}{L''_0}\right) + \dim\left(\frac{\varepsilon^{-N} L_0}{L'_0}\right) \\ &= a + b, \end{aligned}$$

as a result of Lemma 5.2.5, so $(L', L'') \in \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ as required. □

Now assume $N \in \mathbb{N}$ is chosen so that $Y^L_{A,B} \subset \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$, where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$, using Lemma 5.6.1.

Lemma 5.6.2. *$Y^L_{A,B}$ is a locally closed subset of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$. In particular, $Y^L_{A,B}$ is a quasiprojective variety.*

Proof. $Y^L_{A,B}$ is the subset of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ consisting of those (L', L'') satisfying the following conditions: $\dim(L_i/L_i \cap L'_j) = d_{i,j}(A)$ for $i < j$, $\dim(L'_j/L_i \cap L'_j) = \bar{d}_{i,j}(A)$ for $i > j$, $\dim(L'_i/L'_i \cap L''_j) = d_{i,j}(B)$ for $i < j$ and $\dim(L''_j/L'_i \cap L''_j) = \bar{d}_{i,j}(B)$. Only finitely many conditions are required to define $Y^L_{A,B}$ since there are only finitely many nonzero entries in A and B modulo the (n,n)-periodicity.

The conditions $\dim(L_i/L_i \cap L'_j) \leq d_{i,j}(A)$, $\dim(L'_i/L'_i \cap L''_j) \leq d_{i,j}(B)$, $\dim(L'_j/L_i \cap L'_j) \leq \bar{d}_{i,j}(A)$ and $\dim(L''_j/L'_i \cap L''_j) \leq \bar{d}_{i,j}(B)$ define closed subsets of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ for each $i, j \in \mathbb{Z}$, as a result of Lemma 5.4.7 and Lemma 5.4.8.

On the other hand, the conditions $\dim(L_i/L_i \cap L'_j) \geq d_{i,j}(A)$, $\dim(L'_i/L'_i \cap L''_j) \geq d_{i,j}(B)$, $\dim(L'_j/L_i \cap L'_j) \geq \bar{d}_{i,j}(A)$ and $\dim(L''_j/L'_i \cap L''_j) \geq \bar{d}_{i,j}(B)$ define open subsets of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$ for each $i, j \in \mathbb{Z}$, using Lemma 5.4.7 and Lemma 5.4.8.

Therefore $Y^L_{A,B}$ is the intersection of finitely many open sets and finitely many closed sets in $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L)$, so $Y^L_{A,B}$ is locally closed. In particular, $Y^L_{A,B}$ is a quasiprojective variety. □

Lemma 5.6.3. *For any $L' \in X_A^L$, $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$.*

Proof. Let $L' \in X_A^L$, then $\{L'\} \times X_B^{L'}$ is contained in $Y_{A,B}^L$ and G_L acts on $Y_{A,B}^L$, so $G_L \cdot (\{L'\} \times X_B^{L'})$ is contained in $Y_{A,B}^L$. If $(N', N'') \in Y_{A,B}^L$, then $N' = \sigma L'$ for some $\sigma \in G_L$, since $N' \in X_A^L$. Then $(N', N'') = \sigma(L', \sigma^{-1}N'')$ and $\sigma^{-1}N'' \in X_B^{\sigma^{-1}N'} = X_B^{L'}$, so $(N', N'') \in \sigma \cdot (\{L'\} \times X_B^{L'})$. Therefore $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'})$ as claimed. □

Proposition 5.6.4. *$Y_{A,B}^L$ is irreducible.*

Proof. Let $L' \in X_A^L$. $G_L/H_{2N,L}$ is a connected algebraic group acting algebraically on $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$ by Lemma 5.4.6. $X_B^{L'}$ is an irreducible locally closed subset of $\Pi_{2N,\mu}^{a+b}(L)$, so $\{L'\} \times X_B^{L'}$ is an irreducible locally closed set in $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L)$. $Y_{A,B}^L = G_L \cdot (\{L'\} \times X_B^{L'}) = G_L/H_{2N,L} \cdot (\{L'\} \times X_B^{L'})$, by Lemma 5.6.3, so it follows that $Y_{A,B}^L$ is irreducible. □

Let p_2 be the projection onto the second factor $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \to \Pi_{2N,\mu}^{a+b}(L)$. p_2 is a closed morphism since $\Pi_{N,\lambda}^a(L)$ is a projective variety and therefore complete, by Lemma 5.4.2. Therefore $p_2(\overline{Y_{A,B}^L}) = \overline{X_{A,B}^L}$, since $p_2(Y_{A,B}^L) = X_{A,B}^L$.

Lemma 5.6.5. *$X_{A,B}^L$ is irreducible and constructible.*

Proof. Proposition 5.6.4 shows that $Y_{A,B}^L$ is irreducible and locally closed, so it follows $X_{A,B}^L$ is irreducible and constructible, since $X_{A,B}^L = p_2(Y_{A,B}^L)$. □

Proposition 5.6.6. *There is a unique open G_L-orbit in $X_{A,B}^L$.*

Proof. $X_{A,B}^L$ consists of finitely many G_L-orbits and is an irreducible topological space, by Lemma 5.6.5. Consequently, X_C^L is dense in $X_{A,B}^L$ for some $C \in {\Lambda_1}^{A,B}$. Lemma 5.5.2 shows that X_C^L is locally closed in $X_{A,B}^L$, so X_C^L is open in $\overline{X_C^L} = X_{A,B}^L$. Irreducibility of $X_{A,B}^L$ shows that there is a unique open G_L-orbit, since two nonempty open sets in $X_{A,B}^L$ intersect nontrivially, thus any two open G_L orbits in $X_{A,B}^L$ coincide. □

Let $A * B \in \Lambda_1$ be the matrix corresponding to the dense open G_L-orbit in $X_{A,B}^L$, so $\overline{X_{A*B}^L} = \overline{X_{A,B}^L}$.

5.7 Degenerations of orbits and the combinatorial partial order

Theorem 5.7.1. *Let $A, B \in \Lambda_1$ with $\mathrm{ro}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(A) = \mathrm{co}(B)$, then $B \leq A$ if and only if $X_B^L \subset \overline{X_A^L}$ for any $L \in \mathcal{F}_{\mathrm{ro}(A)}$.*

Proof. Let $\lambda = \mathrm{co}(A)$, $\mu = \mathrm{ro}(A)$ and fix $L \in \mathcal{F}_\mu$. Assume $N \in \mathbb{N}$ is chosen so that $X_A^L \subset \Pi_{N,\lambda}^a(L)$ and $X_B^L \subset \Pi_{N,\lambda}^b(L)$, where $a = d_{nN,0}(A)$ and $b = d_{nN,0}(B)$. Then X_A^L is an open subset of the projective variety consisting of those $L' \in \Pi_{N,\lambda}^a(L)$ such that

$$\dim\left(\frac{L_i}{L_i \cap L_j'}\right) \leq d_{i,j}(A)$$

and

$$\dim\left(\frac{L'_j}{L_i \cap L'_j}\right) \leq \bar{d}_{i,j}(A),$$

for all $i, j \in \mathbb{Z}$.

Assume $X_B^L \subset \overline{X_A^L}$, then

$$d_{i,j}(B) = \dim\left(\frac{L_i}{L_i \cap L'_j}\right) \leq d_{i,j}(A)$$

and

$$\bar{d}_{i,j}(B) = \dim\left(\frac{L'_j}{L_i \cap L'_j}\right) \leq \bar{d}_{i,j}(A),$$

for each $i, j \in \mathbb{Z}$, for any $L' \in X_B^L$. So $B \leq A$ if $X_B^L \leq \overline{X_A^L}$.

Conversely, suppose $A \leq B$.

□

Corollary 5.7.2. *The maximum in* ${\Lambda_1}^{A,B}$ *is* $A * B$.

5.8 Associativity of the generic product

Let $A, B, C \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(B) = \mathrm{ro}(C)$ and fix $L \in \mathcal{F}_{\mathrm{ro}(A)}$. Write $\lambda = \mathrm{co}(A)$, $\mu = \mathrm{co}(B)$ and $\nu = \mathrm{co}(C)$. Define

$$Y_{A,B,C}^L = \left\{(L', L'', L''') \in \mathcal{F}^3 : L' \in X_A^L, L'' \in X_B^{L'}, L''' \in X_C^{L''}\right\}$$

and

$$X_{A,B,C}^L = \left\{L''' \in \mathcal{F} : \exists (L', L'') \in \mathcal{F}^2 \text{ with } (L', L'', L''') \in Y_{A,B,C}^L\right\}.$$

Lemma 5.8.1. *There is* $N \in \mathbb{N}$ *such that* $Y_{A,B,C}^L$ *is contained in* $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$*, where* $a = d_{nN,0}(A)$*,* $b = d_{nN,0}(B)$ *and* $c = d_{nN,0}(C)$*.*

Proof. Lemma 5.5.1 shows that there is $N \in \mathbb{N}$ such that $\varepsilon^N L_0 \subset L'_0 \subset \varepsilon^{-N} L_0$, $\varepsilon^N L'_0 \subset L''_0 \subset \varepsilon^{-N} L'_0$ and $\varepsilon^N L''_0 \subset L'''_0 \subset \varepsilon^{-N} L''_0$ for each $(L', L'', L''') \in Y_{A,B,C}^L$. Using the proof of Lemma 5.6.1, it follows $L'' \in \Pi_{2N,\mu}^{a+b}(L)$ and $L''' \in \Pi_{2N,\nu}^{b+c}(L') \subset \Pi_{3N,\nu}^{a+b+c}(L)$. □

Assume $N \in \mathbb{N}$ is chosen so that $Y_{A,B,C}^L \subset \Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$, where $a = d_{nN,0}(A)$, $b = d_{nN,0}(B)$ and $c = d_{nN,0}(C)$, as in Lemma 5.8.1.

Lemma 5.8.2. $Y_{A,B,C}^L$ *is a locally closed subset of* $\Pi_{N,\lambda}^a(L) \times \Pi_{2N,\mu}^{a+b}(L) \times \Pi_{3N,\nu}^{a+b+c}(L)$*. In particular,* $Y_{A,B,C}^L$ *is a quasiprojective variety.*

Proof. Write $\Pi = \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi_{3N,\nu}(L)$. Then $Y^L_{A,B,C}$ consists of those $(L', L'', L''') \in \Pi$ satisfying the following conditions:

$$\dim\left(\frac{L_i}{L_i \cap L'_j}\right) = d_{i,j}(A), \tag{5.1}$$

$$\dim\left(\frac{L'_i}{L'_i \cap L''_j}\right) = d_{i,j}(B), \tag{5.2}$$

$$\dim\left(\frac{L''_i}{L''_i \cap L'''_j}\right) = d_{i,j}(C), \tag{5.3}$$

for $(i,j) \in \{1,\dots,n\} \times \mathbb{Z}$ with $i < j < (N+1)n$, and

$$\dim\left(\frac{L'_j}{L_i \cap L'_j}\right) = \bar{d}_{i,j}(A), \tag{5.4}$$

$$\dim\left(\frac{L''_j}{L'_i \cap L''_j}\right) = \bar{d}_{i,j}(B), \tag{5.5}$$

$$\dim\left(\frac{L'''_j}{L''_i \cap L'''_j}\right) = \bar{d}_{i,j}(C), \tag{5.6}$$

for $(i,j) \in \{1,\dots,n\} \times \mathbb{Z}$ with $-Nn < j < i$.

For $i < j$, the conditions

$$\dim\left(L_i/L_i \cap L'_j\right) \le d_{i,j}(A),$$
$$\dim\left(L'_i/L'_i \cap L''_j\right) \le d_{i,j}(B)$$

and

$$\dim\left(L''_i/L''_i \cap L'''_j\right) \le d_{i,j}(C)$$

define closed subsets of Π, by Lemma 5.4.7. For $i > j$, the conditions

$$\dim\left(L'_j/L_i \cap L'_j\right) \le \bar{d}_{i,j}(A),$$
$$\dim\left(L''_j/L'_i \cap L''_j\right) \le \bar{d}_{i,j}(B)$$

and

$$\dim\left(L'''_j/L''_i \cap L'''_j\right) \le \bar{d}_{i,j}(C)$$

also define closed subsets of Π.

On the other hand, the conditions $\dim\left(L_i/L_i \cap L'_j\right) \ge d_{i,j}(A)$, $\dim\left(L'_i/L'_i \cap L''_j\right) \ge d_{i,j}(B)$ and $\dim\left(L''_i/L''_i \cap L'''_j\right) \ge d_{i,j}(C)$ for $i < j$ define open subsets of Π. Similarly, the conditions $\dim\left(L'_j/L_i \cap L'_j\right) \ge \bar{d}_{i,j}(A)$, $\dim\left(L''_j/L'_i \cap L''_j\right) \ge \bar{d}_{i,j}(B)$ and $\dim\left(L'''_j/L''_i \cap L'''_j\right) \ge \bar{d}_{i,j}(C)$ for $i > j$ define open subsets of Π.

Therefore $Y^L_{A,B,C}$ is the intersection of finitely many closed sets in Π with finitely many open subsets of Π, so $Y^L_{A,B,C}$ is locally closed. In particular, $Y^L_{A,B,C}$ is a quasiprojective variety. $\square$

Lemma 5.8.3. *For any* $(L', L'', L''') \in Y^L_{A,B,C}$,

$$Y^L_{A,B,C} = \left\{ \alpha \cdot (L', \beta L'', \beta\gamma L''') : \alpha \in G_L, \beta \in G_{L'}, \gamma \in G_{L''} \right\}.$$

In particular,

$$Y^L_{A,B,C} = G_L \cdot \left(\{L'\} \times Y^{L'}_{B,C} \right)$$

for each $L' \in X^L_A$.

Proof. Let $(L', L'', L''') \in Y^L_{A,B,C}$. Given $\alpha \in G_L$, $\beta \in G_{L'}$ and $\gamma \in G_{L''}$, $(\alpha L', \alpha\beta L'', \alpha\beta\gamma L''')$ is in $Y^L_{A,B,C}$ since

$$\begin{aligned}
(L, \alpha L') &= \alpha(L, L') \in \mathcal{O}_A \\
(\alpha L', \alpha\beta L'') &= \alpha\beta(L', L'') \in \mathcal{O}_B \\
(\alpha\beta L'', \alpha\beta\gamma L''') &= \alpha\beta\gamma(L'', L''') \in \mathcal{O}_C
\end{aligned}$$

For each $(N', N'', N''') Y^L_{A,B,C}$ there exist $\sigma_1, \sigma_2, \sigma_3 \in G$ with

$$\begin{aligned}
(L, N') &= \sigma_1(L, L') \\
(N', N'') &= \sigma_2(L', L'') \\
(N'', N''') &= \sigma_3(L'', L''').
\end{aligned}$$

Let $\alpha = \sigma_1$, $\beta = \sigma_1^{-1}\sigma_2$ and $\gamma = \sigma_2^{-1}\sigma_3$, so $\sigma_2 = \alpha\beta$ and $\sigma_3 = \alpha\beta\gamma$. It follows that

$$(N', N'', N''') = (\alpha L', \alpha\beta L'', \alpha\beta\gamma L'''),$$

which proves the first claim. The second claim follows from the first since $(L'', L''') \in Y^{L'}_{B,C}$ and therefore

$$Y^{L'}_{B,C} = \{(\beta L'', \beta\gamma L''') : \beta \in G_{L'}, \gamma \in G_{L''}\},$$

as required. □

Proposition 5.8.4. $Y^L_{A,B,C}$ *is irreducible.*

Proof. Write

$$\Pi = \Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi^{a+b+c}_{3N,\nu}(L).$$

Lemma 5.4.2 shows that Π is a projective algebraic variety and Lemma 5.4.6 shows that $G_L/H_{3N,L}$ is a connected algebraic group acting algebraically on Π by the diagonal action.

Let $L' \in X^L_A$. As a result of Lemma 5.8.3

$$\begin{aligned}
Y^L_{A,B,C} &= G_L \cdot (\{L'\} \times Y^{L'}_{B,C}) \\
&= G_L/H_{3N,L} \cdot (\{L'\} \times Y^{L'}_{B,C}).
\end{aligned}$$

Proposition 5.6.4 shows that $Y^{L'}_{B,C}$ is irreducible, so $\{L'\} \times Y^{L'}_{B,C}$ is irreducible. The image of $\{L'\} \times Y^{L'}_{B,C}$ under the action of $G_L/H_{3N,L}$ is irreducible, since $G_L/H_{3N,L}$ is connected and therefore irreducible. Therefore $Y^L_{A,B,C}$ is irreducible. □

Let p_3 be the projection of $\Pi^a_{N,\lambda}(L) \times \Pi^{a+b}_{2N,\mu}(L) \times \Pi^{a+b+c}_{3N,\nu}(L)$ onto the third factor. By the completeness property of projective varieties, p_3 is a closed morphism. The image of $Y^L_{A,B,C}$ under p_3 is $X^L_{A,B,C}$, so $p_3(\overline{Y^L_{A,B,C}}) = \overline{X^L_{A,B,C}}$.

Lemma 5.8.5. *$X^L_{A,B,C}$ is irreducible and constructible.*

Proof. Lemma 5.8.2 and Proposition 5.8.4 show that $Y^L_{A,B,C}$ is locally closed and irreducible. It follows $X^L_{A,B}$ is irreducible and constructible, since $X^L_{A,B,C}$ is the image of $Y^L_{A,B,C}$ under the morphism p_3. □

Lemma 5.8.6. *There is a unique open and dense G_L-orbit in $X^L_{A,B,C}$.*

Proof. There are only finitely many G_L-orbits in $X^L_{A,B,C}$. In particular,

$$X^L_{A,B,C} = \bigcup_{D \in \Lambda_1{}^{A,B}} X^L_{D,C} = \bigcup_{D \in \Lambda_1{}^{A,B}} \bigcup_{D' \in \Lambda_1{}^{D,C}} X^L_{D'}$$

and

$$\overline{X^L_{A,B,C}} = \bigcup_{D \in \Lambda_1{}^{A,B}} \bigcup_{D' \in \Lambda_1{}^{D,C}} \overline{X^L_{D'}}.$$

There is $D \in \Lambda_1$ such that $\overline{X^L_D} = \overline{X^L_{A,B,C}}$, since $X^L_{A,B,C}$ is irreducible, by Lemma 5.8.5. By Lemma 5.5.2, X^L_D is open in $\overline{X^L_D} = \overline{X^L_{A,B,C}}$, so X^L_D is open in $X^L_{A,B,C}$.

If X^L_D and $X^L_{D'}$ are open in $X^L_{A,B,C}$, then X^L_D and $X^L_{D'}$ have nonempty intersection since $X^L_{A,B,C}$ is irreducible, then $X^L_D = X^L_{D'}$. □

Lemma 5.8.7. *$p_3^{-1}(X^L_{A*B,C})$ is open in $\overline{Y^L_{A,B,C}}$.*

Proof. Projection onto the second component is a closed morphism of varieties $p_2 \colon \overline{Y^L_{A,B,C}} \to \overline{X^L_{A,B}}$ with $p_2(Y^L_{A,B,C}) = X^L_{A,B}$. It follows that $p_3^{-1}(X^L_{A*B,C})$ is open in $\overline{Y^L_{A,B,C}}$ since $p_3^{-1}(X^L_{A*B,C}) = p_2^{-1}(X^L_{A*B})$ and X^L_{A*B} is open in $\overline{X^L_{A,B}}$. □

Lemma 5.8.8. *$p_3^{-1}(X^L_{A,B*C})$ is open in $\overline{Y^L_{A,B,C}}$.*

Proof. $p_3^{-1}(X^L_{A,B*C})$ consists of those $(L', L'', L''') \in \overline{Y^L_{A,B,C}}$ such that $\dim \left(L'_i/L'_i \cap L'''_j\right) \geq d_{i,j}(B * C)$ for $i < j$ and $\dim \left(L'''_j/L'_i \cap L'''_j\right) \geq \bar{d}_{i,j}(B * C)$ for $i > j$. Each of these conditions defines an open subset of $\overline{Y^L_{A,B,C}}$ as a result of Lemma 5.4.7 and only finitely many conditions are required to determine $p_3^{-1}(X^L_{A,B*C})$, as before. Therefore $p_3^{-1}(X^L_{A,B*C})$ is the intersection of finitely many open sets in $\overline{Y^L_{A,B,C}}$, so is open as claimed. □

Proposition 5.8.9. $X^L_{A*(B*C)} = X^L_{(A*B)*C}$.

Proof. The unique open G_L-orbit in $X^L_{A*B,C}$ is $X^L_{(A*B)*C}$, so $p_3^{-1}(X^L_{(A*B)*C})$ is open in $p_3^{-1}(X^L_{A*B,C})$. Lemma 5.8.7 shows that $p_3^{-1}(X^L_{A*B,C})$ is open in $\overline{Y^L_{A,B,C}}$, so $p_3^{-1}(X^L_{(A*B)*C})$ is open in $\overline{Y^L_{A,B,C}}$.

Similarly, $X^L_{A*(B*C)}$ is open in $X^L_{A,B*C}$, so $p_3^{-1}(X^L_{A*(B*C)})$ is open in $p_3^{-1}(X^L_{A,B*C})$. Lemma 5.8.8 shows that $p_3^{-1}(X^L_{A,B*C})$ is open in $\overline{Y^L_{A,B,C}}$, so it follows $p_3^{-1}(X^L_{A*(B*C)})$ is open in $\overline{Y^L_{A,B,C}}$.

Therefore $f^{-1}(X^L_{A*(B*C)})$ has nonempty intersection with $f^{-1}(X^L_{(A*B)*C})$, since $Y^L_{A,B,C}$ is irreducible by Proposition 5.8.4. It follows that the G_L-orbits $X^L_{A*(B*C)}$ and $X^L_{(A*B)*C}$ have nonempty intersection and therefore $X^L_{A*(B*C)}$ equals $X^L_{(A*B)*C}$. □

5.9 The generic affine algebra

The generic affine algebra of rank r and period n, denoted by $\hat{G}(n, r)$, is a free $\mathbb{Z}$-module with basis $\{e_A : A \in \Lambda_1\}$ and $\mathbb{Z}$-bilinear multiplication given by

$$e_A * e_B = e_{A*B}$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, and

$$e_A * e_B = 0$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) \neq \mathrm{ro}(B)$.

Proposition 5.9.1. *The generic algebra $\hat{G}(n, r)$ is an associative $\mathbb{Z}$-algebra with 1, with*

$$1 = \sum_{\lambda \in \Lambda_0} 1_\lambda$$

where

$$1_\lambda = e_{D_\lambda},$$

for each $\lambda \in \Lambda_0$.

Proof. Let $A, B, C \in \Lambda_1$. If $\mathrm{co}(A) \neq \mathrm{ro}(B)$ or $\mathrm{co}(B) \neq \mathrm{ro}(C)$, then

$$(e_A * e_B) * e_C = 0 = e_A * (e_B * e_C),$$

so we may now suppose $\mathrm{co}(A) = \mathrm{ro}(B)$ and $\mathrm{co}(B) = \mathrm{ro}(C)$.

As a result of Proposition 5.8.9,

$$\begin{aligned}(e_A * e_B) * e_C &= e_{(A*B)*C} \\ &= e_{A*(B*C)} \\ &= e_A * (e_B * e_C),\end{aligned}$$

so it follows $\hat{G}(n, r)$ is an associative $\mathbb{Z}$-algebra.

The expression for the multiplicative identity follows from Lemma 3.1.7, since

$$e_A * \left(\sum_{\lambda \in \Lambda_0} 1_\lambda \right) = e_A * 1_{\mathrm{co}(A)} = e_A$$

and

$$\left(\sum_{\lambda \in \Lambda_0} 1_\lambda \right) * e_A = 1_{\mathrm{ro}(A)} * e_A = e_A,$$

for each $A \in \Lambda_1$. $\square$

5.9.1 A categorical perspective

Proposition 5.9.2. *The following constitutes a small category: the set of objects is* Λ_0 *and the set of morphisms is* Λ_1*. Given compositions* $\lambda, \mu \in \Lambda_0$*, the morphisms with source* μ *and target* λ *are those matrices* $A \in \Lambda_1$ *with* $\mathrm{co}(A) = \mu$ *and* $\mathrm{ro}(A) = \lambda$*. Given* $\lambda, \mu, \nu \in \Lambda_0$ *and* $A, B \in \Lambda_1$ *with* $\mathrm{co}(B) = \nu$*,* $\mathrm{ro}(B) = \mu = \mathrm{co}(A)$ *and* $\mathrm{ro}(A) = \lambda$*, their composition is* $A * B$*, with source* $\mathrm{co}(A * B) = \mathrm{co}(B) = \nu$ *and target* $\mathrm{ro}(A * B) = \mathrm{ro}(A) = \lambda$*.*

Proof. Proposition 5.8.9 shows that the generic product $*$ is associative. For each object $\lambda \in \Lambda_0$, the identity morphism $\lambda \to \lambda$ is the diagonal matrix D_λ. $\square$

Then the generic affine algebra $\hat{G}(n, r)$ may be realised as the $\mathbb{Z}$-algebra of this category. Observe that there are only finitely many objects in this category and distinct objects are non-isomorphic, so the isomorphism classes in this category are in one to one correspondence with Λ_0. The $\mathbb{Z}$-algebra of this category is the free $\mathbb{Z}$-module on Λ_1 with $\mathbb{Z}$-bilinear multiplication given by the generic product $*$.

Chapter 6

A realisation of affine zero Schur algebras

The purpose of this chapter is to study the link between the generic affine algebra $\hat{G}(n,r)$ to the affine 0-Schur algebra $\hat{S}_0(n,r)$.

The main result is the construction of an isomorphism of $\mathbb{Z}$-algebras from $\hat{G}(n,r)$ to $\hat{S}_0(n,r)$ such that $E_i \mapsto E_i$, $F_j \mapsto F_j$ and $1_\lambda \mapsto 1_\lambda$, in the case that $n, r \geq 1$ with $r < n$.

6.1 Preliminary results on the generic affine algebra

Recall that the generic affine algebra $\hat{G}(n,r)$ is an associative $\mathbb{Z}$-algebra with a multiplicative basis $\{e_A : A \in \Lambda_1\}$ over $\mathbb{Z}$, where

$$e_A * e_B = e_{A*B}$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) = \mathrm{ro}(B)$, and

$$e_A * e_B = 0$$

for $A, B \in \Lambda_1$ with $\mathrm{co}(A) \neq \mathrm{ro}(B)$.

6.1.1 Elementary basis elements

For $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ such that $\lambda_{i+1} > 0$, define

$$E_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i,i+1} - \mathcal{E}_{i+1,i+1}}$$

and let

$$E_i = \sum_{\lambda \in \Lambda_0 : \lambda_{i+1} > 0} E_{i,\lambda}$$

for each $i \in \{1, \ldots, n\}$

For $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ such that $\lambda_i > 0$, define

$$F_{i,\lambda} = e_{D_\lambda + \mathcal{E}_{i+1,i} - \mathcal{E}_{i,i}}$$

and let

$$F_i = \sum_{\lambda \in \Lambda_0 : \lambda_i > 0} F_{i,\lambda}$$

for each $i \in \{1, \ldots, n\}$.

Lemma 6.1.1. *Let $i \in \{1, \ldots, n\}$ and $A \in \Lambda_1$ and write $\mu = \mathrm{ro}(A)$.*
If $\mu_{i+1} = 0$ then $E_i * e_A = 0$. If $\mu_{i+1} > 0$, then

$$E_i * e_A = e_{A+\mathcal{E}_{i,p}-\mathcal{E}_{i+1,p}},$$

where

$$p = \max\{j \in \mathbb{Z} : a_{i+1,j} > 0\}.$$

If $\mu_i = 0$ then $F_i e_A = 0$. If $\mu_i > 0$ then

$$F_i * e_A = e_{A+\mathcal{E}_{i+1,q}-\mathcal{E}_{i,q}},$$

where

$$q = \min\{j \in \mathbb{Z} : a_{i,j} > 0\}.$$

Proof. TYPE PROOF. □

Lemma 6.1.2. *Let $i \in \{1, \ldots, n\}$ and $A \in \Lambda_1$ and write $\lambda = \mathrm{co}(A)$.*
If $\lambda_j = 0$ then $e_A * E_j = 0$. If $\lambda_j > 0$ then

$$e_A * E_j = e_{A+\mathcal{E}_{p,j+1}-\mathcal{E}_{p,j}},$$

where

$$p = \min\{i \in \mathbb{Z} : a_{i,j} > 0\}.$$

If $\lambda_{j+1} = 0$ then $e_A * F_j = 0$. If $\lambda_{j+1} > 0$ then

$$e_A * F_j = e_{A+\mathcal{E}_{q,j}-\mathcal{E}_{q,j+1}},$$

where

$$q = \max\{i \in \mathbb{Z} : a_{i,j+1} > 0\}.$$

Proof. TYPE PROOF . □

6.1.2 Transpose involution

Let S be the $\mathbb{Z}$-module automorphism of $\hat{G}(n, r)$ given by

$$S(e_A) = e_{A^\top}$$

for each $A \in \Lambda_1$.

Lemma 6.1.3. *The map S is a $\mathbb{Z}$-algebra antihomomorphism. In particular,*

$$e_{A^\top} * e_{B^\top} = e_B * e_A,$$

for each $A, B \in \Lambda_1$.

Proof. Lemma 5.2.4 show that the transpose preserves the partial order on Λ_1 and so

$$(B * A)^\top = A^\top * B^\top,$$

using Lemma 4.1.1. □

For any $A \in \Lambda_1$,

$$S(S(e_A)) = e_{(A^\top)^\top} = e_A,$$

so $S \circ S$ is the identity map on $\hat{S}_q(n,r)$.

For each $i \in \{1,\ldots,n\}$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$,

$$S(E_{i,\lambda}) = F_{i,\lambda+\alpha_i},$$

for each $i \in \{1,\ldots,n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$,

$$S(F_{i,\lambda}) = E_{i,\lambda-\alpha_i}, \text{ and}$$

and

$$S(1_\lambda) = 1_\lambda,$$

for each $\lambda \in \Lambda_0$.

6.1.3 Shifting and periodicity

For each $\lambda \in \Lambda_0$, define

$$R_\lambda = e_{[1]D_\lambda} = e_{\lambda_1\mathcal{E}_{0,1}+\cdots+\lambda_n\mathcal{E}_{n-1,n}}$$

and set

$$R = \sum_{\lambda\in\Lambda_0} R_\lambda.$$

Lemma 6.1.4. *For each $A \in \Lambda_1$,*

$$R * e_A = e_{[1]A}$$

and

$$e_A * R = e_{A[-1]}.$$

Proof. Lemma 4.1.4 shows that the same formulas hold in $\hat{S}_q(n,r)$, then the result follows for the generic multiplication $*$, since each product $R * e_A$ and $e_A * R$ is supported on one orbit, so the generic multiplication and the product on $\hat{S}_q(n,r)$ are the same in this instance. $\square$

Observe that

$$\begin{aligned} S(R_\lambda) &= e_{\lambda_1\mathcal{E}_{1,0}+\cdots+\lambda_n\mathcal{E}_{n,n-1}} \\ &= e_{[-1]D_{[1]\lambda}} \end{aligned}$$

so

$$S(R) = \sum_{\lambda\in\Lambda_0} e_{[-1]D_\lambda}.$$

Lemma 6.1.5. *The element R of $\hat{G}(n,r)$ is invertible, with*

$$R * S(R) = 1 = S(R) * R.$$

Proof. Lemma 6.1.4 shows that

$$\begin{aligned} R * S(R)1_\lambda &= Re_{[-1]D_{[1]\lambda}} \\ &= e_{D_{[1]\lambda}} \\ &= 1_{[1]\lambda} \end{aligned}$$

for each $\lambda \in \Lambda_0$, so

$$R * S(R) = 1.$$

Similarly,

$$\begin{aligned} S(R) * R &= \sum_{\lambda \in \Lambda_0} e_{D_\lambda [1]} * R \\ &= \sum_{\lambda \in \Lambda_0} e_{D_\lambda} \\ &= 1. \end{aligned}$$

$\square$

Let τ be the $\mathbb{Z}$-algebra automorphism of $\hat{G}(n,r)$ given by conjugation by R, so

$$\begin{aligned} \tau(e_A) &= R * e_A * S(R) \\ &= R * e_A * R^{-1}, \end{aligned}$$

for each $A \in \Lambda_1$.

Then τ has order n, since

$$\tau^n(e_A) = e_{[n]A[n]} = e_A,$$

by the n-periodicity condition on Λ_1.

As in Lemma 4.1.6, it follows from Lemma 6.1.4 that

$$\tau(E_{i,\lambda}) = E_{i-1,[1]\lambda}$$

for $i \in \{1, \ldots, r\}$ and $\lambda \in \Lambda_0$ with $\lambda_{i+1} > 0$,

$$\tau(F_{i,\lambda}) = F_{i-1,[1]\lambda}$$

for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ with $\lambda_i > 0$, and

$$\tau(1_\lambda) = 1_{[1]\lambda}$$

for $\lambda \in \Lambda_0$.

In particular,

$$\begin{aligned} \tau(E_i) &= E_{i-1} \\ \tau(F_i) &= F_{i-1} \end{aligned}$$

for $i \in \{1, \ldots, r\}$.

As earlier, I can not be sure but I think this map τ is related to the Auslander-Reiten translation on the isomorphism classes of nilpotent representations of the cyclic quiver on n vertices. The result that $\tau(E_i) = E_{i-1}$ is consistent with the fact the A.R translation sends the simple representation at vertex i to the simple representation at vertex $i-1$.

6.2 Multiplicative bases in affine zero Schur algebras: motivating example

Recall that the affine 0-Schur algebra $\hat{S}_0(n,r)$ is an associative $\mathbb{Z}$-algebra with a $\mathbb{Z}$-basis

$$\{e_A : A \in \Lambda_1\}$$

and with $\mathbb{Z}$-bilinear product given by

$$e_A e_B = \sum_{C \in \Lambda_1} \gamma_{A,B,C}(0) e_C,$$

with multiplicative identity

$$\sum_{\lambda \in \Lambda_0} 1_\lambda.$$

The result of the shifting lemma, Lemma 4.1.4, also holds in $\hat{S}_0(n,r)$. In particular,

$$Re_A = e_{[1]A}$$

and

$$e_A R = e_{A[-1]},$$

for each $A \in \Lambda_1$.

Now assume $r = 1$, so

$$\Lambda_1(n,1) = \{\mathcal{E}_{i,j} : (i,j) \in \mathbb{Z} \times \{1, \ldots, n\}\}$$

and

$$\Lambda_0(n,1) = \{\varepsilon_n, \ldots, \varepsilon_1\}.$$

Lemma 6.2.1. *The distinguished basis $\{e_A : A \in \Lambda_1(n,1)\}$ is a multiplicative basis of $\hat{S}_0(n,1)$. More precisely,*

$$e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{j,k}} = e_{\mathcal{E}_{i,k}}$$

for $i,j,k \in \mathbb{Z}$, and

$$e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{k,l}} = 0$$

for $i,j,k,l \in \mathbb{Z}$ with $j \neq k$ modulo n.

Proof. Let $i,j \in \mathbb{Z}$. Lemma 4.1.4 shows that

$$e_{\mathcal{E}_{i,j}} = R^{j-i} 1_{\varepsilon_j},$$

where the subscript of ε_j is taken modulo n.

If $i,j,k,l \in \mathbb{Z}$ with $j \neq k$ modulo n, then

$$\mathrm{co}(\mathcal{E}_{i,j}) = \varepsilon_j \neq \varepsilon_k = \mathrm{ro}(\mathcal{E}_{k,l}),$$

so

$$e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{k,l}} = 0.$$

Finally, let $i, j, k \in \mathbb{Z}$. Then

$$\begin{aligned} e_{\mathcal{E}_{i,j}} e_{\mathcal{E}_{j,k}} &= R^{j-i} 1_{\varepsilon_j} R^{k-j} 1_{\varepsilon_k} \\ &= R^{j-i} R^{k-j} 1_{\varepsilon_k} \\ &= R^{k-i} 1_{\varepsilon_k} \\ &= e_{\mathcal{E}_{i,k}}. \end{aligned}$$

This proves that the basis $\{e_A : A \in \Lambda_1(n, 1)\}$ of $\hat{S}_0(n, 1)$ is a multiplicative basis. □

This result also shows that the product in $\hat{S}_0(n, 1)$ is the same as the generic product, since

$$e_A e_B = e_{A*B}$$

if $\mathrm{co}(A) = \mathrm{ro}(B)$, and

$$e_A e_B = 0$$

if $\mathrm{co}(A) \neq \mathrm{ro}(B)$, for $A, B \in \Lambda_1(n, 1)$.

Corollary 6.2.2. *For each integer $n \geq 1$,*

$$\hat{S}_0(n, 1) = \hat{G}(n, 1).$$

Proof. This is a consequence of Lemma 6.2.1 and the comment which follows the proof. □

6.3 Aperiodicity in the generic affine algebra

[SECTION NEEDS CHECKING.]

Definition 6.3.1. *(aperiodicity) $A \in \Lambda_1$ is aperiodic if for each $l \in \mathbb{Z} \setminus \{0\}$ there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$. If $r < n$ then and $A \in \Lambda_1$ is aperiodic. Linear combinations of the basis elements corresponding to aperiodic matrices are also said to be aperiodic - if A is aperiodic, we say e_A is aperiodic.*

Lemma 6.3.1. *Let $A \in \Lambda_1$ and write $\lambda = \mathrm{ro}(A)$. If A is aperiodic and $\lambda_{i+1} > 0$, then $E_i * e_A$ is aperiodic. If A is aperiodic and $\lambda_i > 0$, then $F_i * e_A$ is aperiodic.*

Proof. Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_{i+1} > 0$, where $\lambda = \mathrm{ro}(A)$. There is $p \in \mathbb{Z}$ such that $a_{i+1,p} > 0$ and $a_{i+1,p'} = 0$ whenever $p' > p$. Lemma 6.1.1 shows that $E_i * e_A = e_B$, where $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p - i, p - i - 1\}$, then $b_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $b_{s,s+l} = a_{s,s+l} = 0$, since A is aperiodic. If $l = p - i$, then $b_{i+1,i+1+l} = b_{i+1,p+1} = a_{i+1,p+1} = 0$, by maximality of p. If $l = p - i - 1$, there is $s \neq i + 1$ such that $a_{s,s+l} = 0$, since A is aperiodic and $a_{i+1,i+1+l} = a_{i+1,p} > 0$, so $b_{s,s+l} = a_{s,s+l} = 0$. Therefore, $B = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$ is aperiodic.

Suppose $A \in \Lambda_1$ is aperiodic and $\lambda_i > 0$, where $\lambda = \mathrm{ro}(A)$. Lemma 6.1.1 shows that $F_i * e_A = e_C$ where $C = A + \mathcal{E}_{i+1,p} - \mathcal{E}_{i,p}$ and $p = \min\{p' \in \mathbb{Z} : a_{i,p'} > 0\}$. Let $l \in \mathbb{Z} \setminus \{0\}$. If $l \notin \{p - i, p - i - 1\}$ then $c_{s,s+l} = a_{s,s+l}$ for each $s \in \mathbb{Z}$, so there is $s \in \mathbb{Z}$ such that $c_{s,s+p} = a_{s,s+p} = 0$, by aperiodicity of A. If $l = p - i$, then $a_{i,i+l} = a_{i,p} > 0$, so there is $s \neq i$ such that $a_{s,s+l} = 0$. Then $c_{s,s+l} = a_{s,s+l} = 0$. Finally, if $l = p - i - 1$, then $c_{i,i+l} = a_{i,p-1} = 0$ by minimality of p. Thus C is aperiodic as required. □

Definition 6.3.2. *(Weight function) Define the weight function* $\mathrm{wt}\colon \Lambda_1 \to \mathbb{Z}$ *by*

$$\mathrm{wt}\, A = \sum_{i\in\{1,\ldots,n\}, j\in\mathbb{Z}} |j-i| a_{i,j}$$

for each $A \in \Lambda_1$*. The sum is taken over a transversal of the set of congruence classes of* (i,j) *modulo* (n,n) *for* $i,j \in \mathbb{Z}$*.*

Lemma 6.3.2. *Let* $A \in \Lambda_1$ *and write* $\lambda = \mathrm{ro}(A)$*. Suppose* $\lambda_{i+1} > 0$ *and set* $p = \max\{p' \in \mathbb{Z} : a_{i+1,p'} > 0\}$*. If* $p > i$ *then* $\mathrm{wt}\, e_{i,\lambda} * A = 1 + \mathrm{wt}\, A$*. If* $p \le i$ *then* $\mathrm{wt}\, e_{i,\lambda} * A = -1 + \mathrm{wt}\, A$*. Suppose* $\lambda_i > 0$ *and set* $q = \min\{q' \in \mathbb{Z} : a_{i,q'} > 0\}$*. If* $q \le i$ *then* $\mathrm{wt}\, f_{i,\lambda} * A = 1 + \mathrm{wt}\, A$*. If* $q > i$ *then* $\mathrm{wt}\, f_{i,\lambda} * A = -1 + \mathrm{wt}\, A$*.*

Proof. Lemma 6.1.1 shows that $e_i A = A + \mathcal{E}_{i,p} - \mathcal{E}_{i+1,p}$, so $\mathrm{wt}\, e_i A - \mathrm{wt}\, A = |p-i| - |p-i-1|$, which equals 1 if $p > i$ and equals -1 if $p \le i$. Similarly, $f_i A = A + \mathcal{E}_{i+1,q} - \mathcal{E}_{i,q}$ by Lemma 6.1.1, so $\mathrm{wt}\, f_i A - \mathrm{wt}\, A = |q-i-1| - |q-i|$, which equals -1 if $q > i$ and equals 1 if $q \le i$. □

Lemma 6.3.3. *If* $A \in \Lambda_1$ *is aperiodic, then* e_A *may be obtained from* $1_{\mathrm{co}(A)}$ *by finitely many applications of* E_i *and* F_i *for* $i \in \{1,\ldots,n\}$*.*

Proposition 6.3.4. *The* $\mathbb{Z}$*-subalgebra of* $\hat{G}(n,r)$ *generated by* $E_{i,\lambda}$*,* $F_{i,\lambda}$ *and* 1_λ *has* $\mathbb{Z}$*-basis*

$$\{e_A : A \in \Lambda_1 \text{ is aperiodic.}\}.$$

Proof. COMPLETE THIS. □

6.4 Presentation of the generic affine algebra.

generators should be a subset of a basis.

define the quiver here

write down the algebra homomorphism from the path algebra to the generic algebra.

using results of the previous section, when $r < n$, this map is surjective.

for general r and n, the image is spanned by the aperiodic basis elements.

an ideal of relations for the case $n \ge 3$. Prove the relations hold and give notaton for the ideal they generate in the path algebra.

prove this ideal is the kernel of the presentation.

conclude the main theorem: isomorphism between $\hat{G}(n,r)$ and $\hat{S}_0(n,r)$ when $r < n$.

consequence: multiplicative basis in $\hat{G}(n,r)$.

Recall that Λ_0 denotes the set of compositions of r into n parts. That is, Λ_0 is the set of tuples $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n$ with each λ_i non-negative and $\lambda_1 + \cdots + \lambda_n = r$. Given $i \in \{1,\ldots,n\}$, let $\varepsilon_i = (0,\ldots,1,\ldots,0) \in \mathbb{Z}^n$ be the i–th elementary vector and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then given $\lambda \in \Lambda_0$, we have $\lambda + \alpha_i \in \Lambda_0$ provided $\lambda_{i+1} > 0$ and $\lambda - \alpha_i \in \Lambda_0$ provided $\lambda_i > 0$.

Let $\Gamma = \Gamma(n,r)$ be the quiver with set of vertices Λ_0 with arrows $e_{i,\lambda}\colon \lambda \to \lambda + \alpha_i$ (if $\lambda_{i+1} > 0$) and $f_{i,\lambda}\colon \lambda \to \lambda - \alpha_i$ (if $\lambda_i > 0$). Thus there are no arrows between λ and μ unless $\lambda = \mu \pm \alpha_i$ for some $i \in \{1,\ldots,n\}$.

If $n \ge 3$ then neighbouring vertices are connected by two arrows, one of each direction. In the case $n = 2$, neighbouring vertices are joined by four arrows, two of each direction. The $\mathbb{Z}\Gamma$ denote the path $\mathbb{Z}$ algebra of Γ. By construction of Γ, there is a $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}\Gamma \to \hat{G}(n,r)$ with $e_{i,\lambda} \mapsto E_{i,\lambda}$, $f_{i,\lambda} \mapsto F_{i,\lambda}$ and $k_\lambda = 1_\lambda$. We aim to describe the image and kernel of the morphism to give a presentation of the generic algebra by a quiver with relations, when possible. In general, we should obtain a presentation of a subalgebra of the generic algebra consisting of the so-called aperiodic elements (c.f. [32]).

6.4.1 Relations

Suppose $n \geq 3$.

Lemma 6.4.1. *The following relations hold in $\hat{G}(n,r)$:*

$$E_iE_j - E_jE_i = 0$$

$$F_iF_j - F_jF_i = 0$$

unless $|j-i| = 1$.

$$E_iE_{i+1}^2 - E_{i+1}E_iE_{i+1} = 0$$
$$E_i^2E_{i+1} - E_iE_{i+1}E_i = 0$$
$$F_{i+1}F_i^2 - F_iF_{i+1}F_i = 0$$
$$F_{i+1}^2F_i - F_{i+1}F_iF_{i+1} = 0$$

$$E_iF_j - F_jE_i = 0$$

unless $j = i$.

$$E_iFi - F_iE_i + \sum_{\lambda:\lambda_i=0,\lambda_{i+1}>0} 1_\lambda - \sum_{\lambda:\lambda_i>0,\lambda_{i+1}=0} 1_\lambda = 0.$$

6.4.2 The period 2 case

In this case, the quiver $\Gamma(2,r)$ has vertices $\Lambda_0(2,r) = \{(0,r),(1,r-1),\ldots,(r,0)\}$; adjacent vertices are connected by two pairs of arrows with opposite orientation: (e_1,f_1) and (e_2,f_2). The relations arising from $\hat{G}(2,r)$ are of a more complicated form - in particular, the serre relations of total degree 3 will not hold in this case - so this case will be treated separately and at a later date.

However, the presentation of the subalgebra of $\hat{G}(2,r)$ spanned by the aperiodic distinguished basis elements by a quiver with relations is not strictly needed for our study here, since we are considering the case where $r < n$, and we have already seen concretely why $\hat{G}(2,1)$ and $\hat{S}_0(2,1)$ are the same.

Chapter 7

Further directions

7.1 Further results on affine zero Schur algebras

[1] Investigate link between this generic product and the generic extension of representations. Shifting to the non-negative subalgebra to do computations purely in terms of generic extensions of quiver representations.

7.2 Deformed group algebras of symmetric groups

[2] Degenerate group algebras of symmetric groups: write down a presentation of the degenerate group algebras, with generators given by the transpositions, or 2-cycles. Type up the computations done for degenerate group algebras for S_3 and S_4. Formulate propositions for the general case: the transpositions generate the degenerate group algebra; lemma: 'these' relations hold; these generators and relations give a presentation of the degenerate group algebras.

Terminology: deformed group algebra.

7.3 back matter

[1] $Y^L_{A,B}$ is the image of $G_L \times G_{L'}$ under the action map $(\alpha, \beta) \mapsto \alpha\beta \cdot (L', L'')$, for any $(L', L'') \in Y^L_{A,B}$. Lemma 5.4.6 shows that $G_L/H_{N,L}$ is a connected algebraic group. Moreover, $G_{L'}/H_{2N,L}$ is an irreducible affine variety, so $G_L/H_{N,L} \times G_{L'}/H_{2N,L}$ is an irreducible affine variety. It follows that $Y^L_{A,B}$ is irreducible and constructible.

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