Tonk, Plonk and Plink
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TONK, PLONK AND PLINK[1]

By NUEL D. BELNAP

A. N. PRIOR has recently discussed[2] the connective *tonk*, where *tonk* is defined by specifying the role it plays in inference. Prior characterizes the role of *tonk* in inference by describing how it behaves as conclusion, and as premiss: (1) A ⊢ A-*tonk*-B, and (2) A-*tonk-B* ⊢ B (where we have used the sign ' ⊢ ' for deducibility). We are then led by the transitivity of deducibility to the validity of A ⊢ B, " which promises to banish *falsche Spitzfindigkeit* from Logic for ever."

A possible moral to be drawn is that connectives cannot be defined in terms of deducibility at all; that, for instance, it is illegitimate to define *and* as that connective such that (1) A-*and*-B ⊢ A, (2) A-*and*-B ⊢ B, and (3) A, B ⊢ A-*and*-B. We must first, so the moral goes, have a notion of what *and* means, independently of the role it plays as premiss and as conclusion. Truth-tables are one way of specifying this antecedent meaning; this seems to be the moral drawn by J. T. Stevenson.[3] There are good reasons, however, for defending the legitimacy of defining connections in terms of the roles they play in deductions.

It seems plain that throughout the whole texture of philosophy one can distinguish two modes of explanation: the analytic mode, which tends to explain wholes in terms of parts, and the synthetic mode, which explains parts in terms of the wholes or contexts in which they occur.[4] In logic, the analytic mode would be represented by Aristotle, who commences with terms as the ultimate atoms, explains propositions or judgments by means of these, syllogisms by means of the propositions which go to make them up, and finally ends with the notion of a science as a tissue of syllogisms. The analytic mode is also represented by the contemporary logician who first explains the meaning of complex sentences, by means of truth-tables, as a function of their parts, and then proceeds to give an account of correct inference in terms of the sentences occurring therein. The *locus classicus* of the application of the synthetic mode is, I suppose, Plato's treatment of justice in the *Republic*, where he defines the just man by reference to the larger context of the community. Among formal logicians, use of the synthetic mode in logic is illustrated by Kneale and Popper (cited by Prior), as well as by Jaskowski, Gentzen, Fitch, and Curry, all of these treating the meaning of connectives as

[1] This research was supported in part by the Office of Naval Research, Group Psychology Branch, Contract No. SAR/Nonr-609(16).

[2] 'The Runabout Inference-ticket', ANALYSIS 21.2, December 1960.

[3] 'Roundabout the Runabout Inference-ticket', ANALYSIS 21.6, June 1961. Cf. p. 127: "The important difference between the theory of analytic validity [Prior's phrase for what is here called the synthetic view] as it should be stated and as Prior stated it lies in the fact that he gives the meaning of connectives in terms of permissive rules, whereas they should be stated in terms of truth-function statements in a meta-language."

[4] I learned this way of looking at the matter from R. S. Brumbaugh.

arising from the role they play in the context of formal inference. It is equally well illustrated, I think, by aspects of Wittgenstein and those who learned from him to treat the meanings of words as arising from the role they play in the context of discourse. It seems to me nearly self-evident that employment of both modes of explanation is important and useful. It would therefore be truly a shame to see the synthetic mode in logic pass away as a result of a severe attack of tonktitis.

Suppose, then, that we wish to hold that it is after all possible to define connectives contextually, in terms of deducibility. How are we to prevent tonktitis? How are we to make good the claim that there is no connective such as *tonk*[1] though there is a connective such as *and* (where *tonk* and *and* are defined as above)?

It seems to me that the key to a solution[2] lies in observing that even on the synthetic view, we are not defining our connectives *ab initio*, but rather in terms of an *antecedently given context of deducibility*, concerning which we have some definite notions. By that I mean that before arriving at the problem of characterizing connectives, we have already made some assumptions about the nature of deducibility. That this is so can be seen immediately by observing Prior's use of the transitivity of deducibility in order to secure his ingenious result. But if we note that we already *have* some assumptions about the context of deducibility within which we are operating, it becomes apparent that by a too careless use of definitions, it is possible to create a situation in which we are forced to say things inconsistent with those assumptions.

The situation is thus exactly analogous to that, pointed out by Peano, which occurs when one attempts to define an operation, ' ? ', on rational numbers as follows:

$$\left(\frac{a}{b} ? \frac{c}{d}\right) =_{\text{df}} \frac{a+c}{b+d}.$$

This definition is inadmissible precisely because it has consequences which contradict prior assumptions; for, as can easily be shown, admitting this definition would lead to (say) $\frac{2}{3} = \frac{3}{5}$.

In short, we can distinguish between the admissibility of the definition of *and* and the inadmissibility of *tonk* on the grounds of consistency —*i.e.*, consistency with antecedent assumptions. We can give a precise account of the requirement of consistency from the synthetic point of view as follows.

[1] That there is no meaningful proposition expressed by A-*tonk*-B; that there is no meaningful sentence A-*tonk*-B—distinctions suggested by these alternative modes of expression are irrelevant. Not myself being a victim of eidophobia, I will continue to use language which treats the connective-word ' *tonk* ' as standing for the putative propositional connective, *tonk*. It is equally irrelevant whether we take the sign $\vdash$ as representing a syntactic concept of deducibility or a semantic concept of logical consequence.

[2] Prior's note is a gem, reminding one of Lewis Carroll's ' What the Tortoise said to Achilles '. And as for the latter, so for the former, I suspect that no solution will ever be universally accepted as *the* solution.

(1) We consider some characterization of deducibility, which may be treated as a formal system, *i.e.*, as a set of axioms and rules involving the sign of deducibility, '$\vdash$', where '$A_1, \ldots, A_n \vdash B$' is read 'B is deducible from $A_1, \ldots, A_n$.' For definiteness, we shall choose as our characterization the structural rules of Gentzen:

Axiom. $A \vdash A$

Rules.	*Weakening:*	from $A_1, \ldots, A_n \vdash C$ to infer $A_1, \ldots, A_n, B \vdash C$
	Permutation:	from $A_1, \ldots, A_i, A_{i+1}, \ldots, A_n \vdash B$ to infer $A_1, \ldots, A_{i+1}, A_i, \ldots, A_n \vdash B$.
	Contraction:	from $A_1, \ldots, A_n, A_n \vdash B$ to infer $A_1, \ldots, A_n \vdash B$
	Transitivity:	from $A_1, \ldots, A_m \vdash B$ and $C_1, \ldots, C_n, B \vdash D$ to infer $A_1, \ldots, A_m, C_1, \ldots, C_n \vdash D$.

In accordance with the opinions of experts (or even perhaps on more substantial grounds) we may take this little system as expressing all and only the universally valid statements and rules expressible in the given notation: it completely determines the context.

(2) We may consider the proposed definition of some connective, say *plonk*, as an *extension* of the formal system characterizing deducibility, and an extension in two senses. (a) The notion of sentence is extended by introducing A-*plonk*-B as a sentence, whenever A and B are sentences. (b) We add some axioms or rules governing A-*plonk*-B as occurring as one of the premisses or as conclusion of a deducibility-statement. These axioms or rules constitute our definition of *plonk* in terms of the role it plays in inference.

(3) We may now state the demand for the consistency of the definition of the new connective, *plonk*, as follows: the extension must be *conservative*[1]; *i.e.*, although the extension may well have new deducibility-statements, these new statements will all involve *plonk*. The extension will not have any new deducibility-statements which do not involve *plonk* itself. It will not lead to any deducibility-statement $A_1, \ldots, A_n \vdash B$ not containing *plonk*, unless that statement is already provable in the absence of the *plonk*-axioms and *plonk*-rules. The justification for unpacking the demand for consistency in terms of conservativeness is precisely our antecedent assumption that we already had *all* the universally valid deducibility-statements not involving any special connectives.

So the trouble with the definition of *tonk* given by Prior is that it is inconsistent. It gives us an extension of our original characterization of deducibility which is not conservative, since in the extension (but not in the original) we have, for arbitrary A and B, $A \vdash B$. Adding a tonkish role to the deducibility-context would be like adding to cricket a player whose role was so specified as to make it impossible to distinguish winning from losing.

[1] The notion of conservative extensions is due to Emil Post.

Hence, given that our characterization of deducibility is taken as complete, we may with propriety say ' There is no such connective as *tonk*'; just as we say that there is no operation, '?', on rational numbers such that $\left(\frac{a}{b} ? \frac{c}{d}\right) = \frac{a+c}{b+d}$. On the other hand, it is easily shown that the extension got by adding *and* is conservative, and we may hence say ' There *is* a connective having these properties '.

It is good to keep in mind that the question of the existence of a connective having such and such properties is relative to our characterization of deducibility. If we had initially allowed $A \vdash B$ (!), there would have been no objection to *tonk*, since the extension would then have been conservative. Also, there would have been no inconsistency had we omitted from our characterization of deducibility the rule of transitivity.

The mathematical analogy leads us to ask if we ought not also to add *uniqueness*[1] as a requirement for connectives introduced by definitions in terms of deducibility (although clearly this requirement is not as essential as the first, or at least not in the same way). Suppose, for example, that I propose to define a connective *plonk* by specifying that B ⊢ A-*plonk*-B. The extension is easily shown to be conservative, and we may, therefore, say ' There is a connective having these properties '. But is there only one? It seems rather odd to say we have defined *plonk* unless we can show that A-*plonk*-B is a function of A and B, *i.e.*, given A and B, there is only one proposition A-*plonk*-B. But what do we mean by uniqueness when operating from a synthetic, contextualist point of view? Clearly that at most *one* inferential role is permitted by the characterization of *plonk*; *i.e.*, that there cannot be two connectives which share the characterization given to *plonk* but which otherwise sometimes play different roles. Formally put, uniqueness means that if exactly the same properties are ascribed to some other connective, say *plink*, then A-*plink*-B will play exactly the same role in inference as A-*plonk*-B, both as premiss and as conclusion. To say that *plonk* (characterized thus and so) describes a unique way of combining A and B is to say that if *plink* is given a characterization formally identical to that of *plonk*, then

(1) $A_1, \ldots,$ B-*plonk*-C$, \ldots, A_n \vdash D$ if and only if $A_1, \ldots,$ B-*plink*-C$, \ldots, A_n \vdash D$ and

(2) $A_1, \ldots, A_n \vdash$ B-*plonk*-C if and only if $A_1, \ldots, A_n \vdash$ B-*plink*-C.

Whether or not we can show this will depend, of course, not only on the properties ascribed to the connectives, but also on the properties ascribed to deducibility. Given the characterization of deducibility above, it is sufficient and necessary that B-*plonk*-C ⊢ B-*plink*-C, and conversely.

[1]Application to connectives of the notions of existence and uniqueness was suggested to me by a lecture of H. Hiż.

Harking back now to the definition of *plonk* by: B ⊢ A-*plonk*-B, it is easy to show that *plonk* is *not* unique; that given only: B ⊢ A-*plonk*-B, and B ⊢ A-*plink*-B, we cannot show that *plonk* and *plink* invariably play the same role in inference. Hence, the possibility arises that *plonk* and *plink* stand for different connectives: the conditions on *plonk* do not determine a unique connective. On the other hand, if we introduce a connective, *et*, with the same characterization as *and*, it will turn out that A-*and*-B and A-*et*-B play exactly the same role in inference. The conditions on *and* therefore do determine a unique connective.

Though it is difficult to draw a moral from Prior's delightful note without being plonking, I suppose we might put it like this: one *can* define connectives in terms of deducibility, but one bears the onus of proving at least consistency (existence); and if one wishes further to talk about *the* connective (instead of *a* connective) satisfying certain conditions, it is necessary to prove uniqueness as well. But it is not necessary to have an antecedent idea of the independent meaning of the connective.

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UNQUANTIFIED INDUCTIVE GENERALIZATIONS

By J. E. Llewelyn

J. O. NELSON argues that the ordinary grammar of inductive generalizations is not amenable to quantification and that this grammar is preferable to the one that, in his view, logicians have foisted upon them (Analysis, January 1962). It seems to me that his diagnosis departs from instead of returning to ordinary grammar.

1. As an example of an unquantified inductive generalization (henceforth U.I.G.) he gives 'Ants attack spiders'. This, he says, is an ambiguous proposition. "At the point of asserting the proposition, do we mean by it all or some ants? And if 'all', all ants at present on earth or all ants present and future? And if 'some', merely the ant we have observed or others in addition, and what proportion of them? We do not mean any of these things *as yet*. It is still up to us to establish or decide which of these interpretations, if any, our assertion is to have"