Implication and the Algebra of Logic
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IV.—IMPLICATION AND THE ALGEBRA OF LOGIC.

By C. I. Lewis.

The development of the algebra of logic brings to light two somewhat startling theorems: (1) a false proposition implies any proposition, and (2) a true proposition is implied by any proposition. These are not the only theorems of the algebra which seem suspicious to common sense, but their sweeping generality has attracted particular attention. In themselves, they are neither mysterious sayings, nor great discoveries, nor gross absurdities. They exhibit only, in sharp outline, the meaning of "implies" which has been incorporated into the algebra. What this meaning is, what are its characteristics and limitations, and its relation to the "implies" of ordinary valid inference, it is the object of this paper briefly to indicate.

Such an attempt might be superfluous were it not that certain confusions of interpretation are involved, and that the expositors of the algebra of logic have not always taken pains to indicate that there is a difference between the algebraic and the ordinary meanings of implication. One may suspect that some of them would deny the divergence, or at least would maintain that the technical use is preferable and ought generally to be adopted. As a result, symbolic logic appears to the uninitiated somewhat as an *enfant terrible*, which intimidates one with its array of exact demonstrations, and demands the acceptance of incomprehensible results.

In the algebra of logic, 'p implies q' is defined to mean 'either p is false or q is true' [$(p \supset q) = ({\sim} p \vee q)$ Df.].[1] But

[1] I choose this form of the definition partly because it is the one used in the most economical development of the calculus of propositions—in the *Principia Mathematica* of Russell and Whitehead—and partly because of its convenience for the discussion in hand. Other defined equivalents of '$p \supset q$' are:—

(1) $p=pq$ (the assertion of p is equivalent to the assertion of p and q both);

(2) ${\sim}(p {\sim} q)$, or, $p {\sim} q = o$ (that 'p is true and q is false' is a false assertion; or, the proposition which asserts p and denies q is false); and

this last expression is equivocal. Implication is defined in terms of disjunction, but 'either—or' propositions may have at least three different meanings. One of these is ruled out when we understand that 'p or q'—either p is true or q is true—must not be taken to exclude the possibility that both p and q may be true. Disjunctions in the algebra do not signify mutual exclusion. If p be true, it is not implied that q is false. A convenient statement of this takes the form, "*At least one* of the propositions p and q is true". Two meanings of disjunction still remain. The implication of the algebra of logic bears the same relation to the one of these that the Aristotelian "implies" bears to the other. Hence the need of distinguishing carefully between these two sorts of disjunction.

Compare, if you will, the disjunctions: (1) Either Cæsar died or the moon is made of green cheese, and (2) Either Matilda does not love me or I am beloved. In both cases, at least one of the disjoined propositions *is* true. The difference between the two may be expressed in a variety of ways. The second disjunction is such that at least one of the disjoined propositions is "necessarily" true. Reject either of the possibilities and you thereby embrace the other. Suppose one of its propositions false and you are in consistency bound to suppose the other true. If either lemma *were* false, the other would, by the same token, be true. None of these statements will hold for the first disjunction. At least one of its propositions is, as a fact, true. But to suppose it false that Cæsar died, would not bind one to suppose the moon made of green cheese. If 'Cæsar died' *were* false, the moon would not necessarily be made of green cheese,—if conditions contrary to fact have any meaning at all. It is this last which the algebra is, according to its meaning of disjunction and implication, bound to deny.

The most significant distinction, however, remains to be noted. The second disjunction is such that its truth is in-

(3) $(p \vee q) = q$ ('either p is true or q is true' is equivalent to 'q is true').

It comes to the same thing in the end, whichever one of the four mentioned definitions of implication be chosen. Any one of them may be deduced as a theorem in a properly constructed system which adopts any other at the start. The choice depends solely upon the method of developing the particular system (see Whitehead, *Universal Algebra*, p. 40).

The symbolism which will be used in the paper is that of the *Principia Mathematica* with slight modifications. The letters, p, q, stand for propositions or 'propositional functions'. $\supset$ signifies 'implies'. $\vee$ is the sign of disjunction. $\sim p$ may be read 'not-p' or 'the negation of p' or 'p is false'. Similarly p may be read as written or as 'p is true'.

dependent of the truth of either member considered separately. Or, more accurately, its truth can be known, while it is still problematic *which* of its lemmas is the true one. It has a truth which is prior to the determination of the facts in question. The truth of 'Either Cæsar died or the moon is made of green cheese' has not this purely logical or formal character. It lacks this independence of facts. Its contradiction would not surprise a logical mind unacquainted with history.

It requires careful analysis to separate these two meanings of 'either—or' propositions, though their main features may seem sufficiently distinct. We may call disjunctions like (1), whose truth cannot be known apart from the facts, extensional disjunctions; those of the type of (2), whose truth can be known while it is still problematic which member is true,—or whether both are true,—we may call intensional. These two may be further distinguished by considering their negatives. If one take 'Either Cæsar died or the moon is made of green cheese' to be a false statement, one may mean thereby that a certain relation is falsely asserted of the two propositions 'Cæsar died' and 'the moon is made of green cheese'. If Smith asserts, "Either my name is not Smith or this is my hat," one might reply: "No, you are wrong; there may be other Smiths in the hall with names in their hats". One does not deny that Smith knows his own name, or that this is his hat. One denies only that his statement exhausts the possibilities. The negative of intensional disjunction is, thus, the negation of the disjunctive relation itself and not the negation of either member. To take another example: (3) Either London is in England or Paris is in France; one may deny that any "necessary" disjunction is here involved, though either half of the statement by itself is true. If either member of the disjunction *were* false, the truth or falsity of the other would not thereby be affected. Perhaps we cannot ever be certain of the possibility of such a contrary-to-fact condition. Still we *know what we mean* when we suppose it. The negation of intensional disjunction is, then, the negation of a logical relation of propositions, and is entirely consistent with the truth of one or both of the disjoined assertions. One denies only that the disjunction has any truth apart from the facts in question.

The negative of extensional disjunction is the denial of *both* its members. Taking 'either—or' in their reference to extension, neither (1) nor (3) can be denied. The truth of extensional disjunction is secured by the truth of either member, regardless of "logical connexions," and the negation of extensional disjunction accordingly negates both the disjoined propositions.

That the meaning of disjunction in the algebra of logic must consistently be confined to the extensional, follows from the fact that, in the algebra, the negative of a disjunction is the negation of both its members $[\backsim(p \vee q) = \backsim p \backsim q]$,[1] and the negative of a "product"—*e.g.* 'both p and q are true'—is the disjunction of the negatives of its factors $[\backsim(pq) = (\backsim p \vee \backsim q)]$.[1] Every intensional disjunction is also extensional, or, more accurately, the intensional disjunction of p and q implies their extensional disjunction also. But the reverse does not hold. Of every intensional disjunction, at least one member *is* true; but not every 'either . . . or' proposition with at least one true member is an intensional disjunction. If, however, any one suppose that the algebra can treat of intensional disjunctions "because they are a special class of extensional disjunctions," let him consider the fact that, in negating *any* disjunction, the algebra negates both its members. If p and q are disjoined both extensionally and intensionally, still the algebra treats only of their extensional disjunction. That this is not mere "logic chopping" will appear when we come to convert disjunctions into implications.

Before leaving the subject of disjunctions, we should note two further characters of the intensional variety. A genuine intensional disjunction does not, of course, suffer any alteration of its logical nature if one of its members is known to be false, or one known to be true, or when both things are known. 'Either Matilda does not love me or I am beloved' loses none of its intensional character if it is discovered that Matilda does not, in fact, love me, or that I am actually beloved. In argument, one produces a dilemma[2] for the purpose of introducing later the falsity of one member and thus proving the truth of the other. The dilemma has the same meaning to the speaker who knows its solution and to the hearer who does not. Its character as intensional disjunction

[1] De Morgan's theorem.

[2] A dilemma *may be* an intensional disjunction with the restriction that its members cannot be true together. Extensional disjunctions admit of the same limitation while still remaining extensional. This last type, however, do not appear in discourse except as mere truisms or as figures of speech. Example (1)—Either Cæsar died or the moon is made of green cheese—belongs to this class. It is a truism, or—if meant to be taken as *intensional* disjunction—hyperbole. Thus we might have distinguished four types of disjunction instead of three. But the important division is that of intensional in general from extensional in general.

is attested by the fact that the hearer can know its truth before knowing its solution,—and by the further fact that both speaker and hearer, after reaching the solution, are still bound by the condition which turns out to be contrary to fact. If the true member *were* false, the other would necessarily be true.

Again, intensional disjunction is not restricted to the purely formal or *a priori* type of (2). Suppose a wholly reliable weather forecast for the 16th of the month to be "Warm". This implies that (4) either to-day is not the 16th or the weather is warm. On the supposition made, this is an intensional disjunction. One might know its truth even if one could not find a calendar and were suffering from chills and fever. But strike out the initial assumption and the disjunction becomes, if still true, extensional. Knowledge of its truth now depends upon verification of one or both of its members. We may say that extensional disjunction concerns actualities; intensional disjunction, possibilities. But one or more facts being given, the possibilities are thereby narrowed, and an intensional disjunction which is not *a priori* may be *implied*.

As has been said, intensional disjunction bears the same relation to inferential or "strict" implication[1] that extensional disjunction bears to the algebraic or "material" implication. Intensional disjunctions when converted into implications, according to the equivalence which the algebra states, become strict implications. Extensional disjunctions, by the same rule, produce material implications. In either case 'p implies q' is equivalent to 'either p is false or q is true,'—to 'either not-p or q'. $[(p \supset q) = (-p \vee q)$ Df.]. Taking the intensional disjunctions: if we let p represent 'Matilda loves me' and q 'I am beloved,' example (2) states exactly 'either not-p or q'. 'Either Matilda does not love me or I am beloved' is equivalent to 'Matilda loves me implies that I am beloved'. Since 'either . . . or' states a reversible relation, we may equally well let p represent 'I am beloved,' and q 'Matilda does not love me'. 'Not-p implies q' will then read: 'I am not beloved' implies that 'Matilda does not love me'. By the same process, (4)—'Either to-day is not the 16th or the weather is warm'—may be transformed into, 'To-day is the 16th implies that the weather is warm,' and, 'The weather is not warm implies that to-day is not the 16th'.

[1] We may call this kind of implication "strict" at least in the sense that its meaning is narrower than that of the algebraic implication.

Remembering that (4) is an intensional disjunction only in the light of a certain presupposition, we may observe that, in this case also, intensional disjunction produces strict implication.

If p and q are intensionally disjoined, then,—whether p, or q, or both are, in fact, true,—if p were false, q would be true. The negation-of-p implies q in the ordinary meaning of 'implies'. Also if q were false, p would be true; p can *validly be inferred* from the proposition which negates q.

Examples (1) and (3) are extensional disjunctions. If we let p represent 'Cæsar died' and q represent 'The moon is made of green cheese,' 'not-p implies q' will read, 'Cæsar did not die' implies that 'the moon is made of green cheese'. Interchanging p and q above—since 'either . . . or' is reversible—we have, 'The moon is not made of green cheese' implies that 'Cæsar died'. Thus we get the implications of the algebra. The former of these is a good example of the sense in which a false proposition implies anything; the latter well illustrates how a true proposition may be implied by any proposition. By the same method (3) 'either London is in England or Paris is in France' gives us, 'London is not in England' implies that 'Paris is in France,' and, 'Paris is not in France' implies that 'London is in England'. Each of these last may be regarded as a case of a false proposition implying any proposition and, at the same time, of a true proposition being implied by any. *Any two true propositions whatever* might have been substituted for 'London is in England' and 'Paris is in France'; the implications would have resulted in the same way. The denial of the one would imply the other; the denial of the other, the one.

In order that it may be clearer that implication has, in the algebra, *no other significance* than that exemplified by the transformations of (1) and (3), let us consider what is involved in *denying* the algebraic implication relation. Take any false proposition, p—*e.g.* 'Rome is still burning'—and any true one, q—*e.g.* 'Christmas is coming'. At once the extensional disjunction 'either p is false or q is true' is satisfied —by the falsity of p alone, or by the truth of q alone,—and it follows that p implies q. $[(p \supset q) = (-p \vee q)$ Df.$]$. To deny that p implies q is to deny the equivalent disjunction 'Either p is false or q is true'. $[-(p \supset q) = -(-p \vee q)]$. To deny this disjunction is, according to the algebra, to deny the truth of both its members, *i.e.* to assert p and deny q. $[-(-p \vee q) = p - q]$. Thus, if one would deny that 'Rome is still burning' implies 'Christmas is coming,' one must assert that Rome still burns and deny the advent of Christmas.

Or we may take any two problematic propositions, as (p) 'Swift married Stella,' and (q) 'There are other universes beyond ours'. At once we can assert, according to the algebra, that if p does not imply q, q implies p. If p is false, that alone satisfies the extensional 'Either p is false or q is true' and proves that p implies q. Similarly if q is true. If p is true and q is false—the only situation for which p does not imply q—then it is at once doubly certain that q implies p. Of any two false propositions, each implies the other; and similarly, of any two true propositions, each is implied by the other. If one of two propositions is false and the other true, the former implies the latter. Either 'Swift married Stella' implies that 'there are other universes beyond ours,' or 'there are other universes beyond ours' implies that 'Swift married Stella'. And there is an even chance that the implication is mutual. Indeed the algebra of logic allows us to make these assertions prior to all knowledge of the content of p and q and apart from any consideration of what would ordinarily be called their logical import.

Most theorems in the algebra admit of being exemplified within the field of strict implications and intensional disjunctions. Aside from those which involve the negative of a disjunction, there are only a few which do not. All of these are the results of a single assumption of the calculus of propositions, the so-called principle of addition. This principle states that p implies 'either p or q'—if p is true, then either p is true or any other proposition, q, is true. [$p \supset (p \vee q)$]. We have already noted it in observing that an extensional disjunction is satisfied simply by the fact that one of its members is true. That this principle is formally false for intensional disjunctions is apparent when we note that—in the strict sense of implies—p does not imply that if p *were* false, any other proposition q would necessarily be true. From the fact that to-day is Monday, we cannot infer that if to-day were not Monday, the corn crop would be destroyed.

Assuming 'p implies (either p or q),' the proof that a false proposition implies any proposition is short and easy. Substituting not-p (p is false) for p, we have, 'not-p implies (either not-p or q)'. Replacing 'either not-p or q' by its defined equivalent 'p implies q,' not-p implies that 'p implies q,'—'p is false' implies that 'p implies any other proposition, q'. If or when p *is* false, the consequence 'p

implies q' follows. A false proposition implies anything. Resuming the proof in symbols: Addition— $p \supset (p \vee q)$. Substituting $-p$ for p throughout, $-p \supset (-p \vee q)$. $(-p \vee q) = (p \supset q)$, by definition. $\therefore -p \supset (p \supset q)$. The proof that a true proposition implies any proposition requires one additional principle—that disjunctions are reversible. $[(p \vee q) = (q \vee p)]$. Assuming that '$p$ implies (either p or q),' we may reverse the disjunction and get 'p implies (either q or p). Substituting not-q for q, 'p implies (either not-q or p)'. Replacing this disjunction by its equivalent 'q implies p,' the result is, 'p implies that q implies p'. If p is true, it is also true that any other proposition, q, implies p. A true proposition is implied by any proposition. Addition— $p \supset (p \vee q)$. $(p \vee q) = (q \vee p)$. Substituting $-q$ for q, $p \supset (-q \vee p)$. $(-q \vee p) = (q \supset p)$, by definition. $\therefore p \supset (q \supset p)$.

The existence of these two theorems in the algebra brings to light the most severe limitation of the algebraic or material implication. One of the important practical uses of implication is the testing of hypotheses whose truth or falsity is problematic. The algebraic implication has no application here. If the hypothesis happens to be false, it implies anything you please. If one find facts, x, y, z, otherwise unexpected but suggested by the hypothesis, the truth of these facts is implied by one's hypothesis, whether that hypothesis be true or not—since any true proposition is implied by all others. In other words, no proposition could be verified by its logical consequences. If the proposition be false, it has these "consequences" anyway. Similarly, no contrary-to-fact condition could have any logical significance, whether one happen to know that it *is* contrary to fact or not. For if the fact *is* otherwise, the proposition which states the supposition implies anything and everything. In the ordinary and "proper" use of implies certain conclusions can validly be inferred from contrary-to-fact suppositions, while certain others cannot. Hypotheses whose truth is problematic have logical consequences *which are independent of its truth or falsity*. These are the vital distinctions of the ordinary meaning of "implies"—for which 'p implies q' is equivalent to 'q can validly be inferred from p'—from that implication which figures in the algebra.

That the definition of implication in terms of extensional disjunction is in accord with any ordinary or useful meaning of the term can hardly be maintained with success. There can be, however, with regard to such a definition, no question of truth or falsity in the ordinary sense. As one of the assumptions or conventions of the calculus of propositions, the definition represents only the exact statement of the way in which expressions are to be equated or substituted for one another. Provided it is possible so to equate them without contradiction, it is meaningless to call the equations untrue. We may, however, object to the definition on the ground that a more useful one is possible; and especially will this be the case when the system in question is one, like logic, which we wish to apply in some field of practical human endeavour. The present calculus of propositions is untrue in the sense in which non-Euclidean geometry is untrue; and we may reproach the logician who disregards our needs as the ancients might have reproached Euclid had he busied himself too exclusively with the consequences of a different parallel postulate.

Nothing that has preceded should be taken to imply that the algebra of logic is necessarily unequal to the task of symbolising such logical processes as those of inference and proof, or the more general processes which the algebra itself has the value of bringing to light. Our conclusions militate not against symbolic logic in general, but against the calculus of propositions in its present form. As a matter of fact, a few simple changes would remove all the "absurdities" from the present calculus and bring it into agreement with the strict meaning of implication. The principle of addition—p implies 'either p is true or q is true'—is the only one of an economical set of postulates of the present calculus[1] which is false for the intensional meaning of disjunction and, consequently, for strict implication. If this were removed, and disjunction confined—as a matter of interpretation—to the intensional variety, we should be well on our way to a new calculus. One other change would be necessary. The equivalence of "products" with the negatives of disjunctions and of the negatives of products with disjunctions [$pq = \backsim(\backsim p \vee \backsim q)$, and, $\backsim(pq) = (\backsim p \vee \backsim q)$] is inconsistent with the exclusion of purely extensional disjunctions.[2] The product pq—'p and q are both true'—would, accordingly, appear as a new indefinable, though capable of clear interpretation. In place of the principle of addition, the principle of simplification—'p and q are both true' implies 'p is true' [$pq \supset p$]—would be assumed. 'Addition' could no longer be deduced from it, as at present, when the negatives of disjunctions and products had no symbolic equivalents.[3] A careful analysis of what these

[1] See those of the *Principia Mathematica*, pp. 98-101.
[2] De Morgan's Theorem holds only for extensional disjunction.
[3] Both would still have important *implications*.

changes involve leads one to discover certain ambiguities and confusions which exist even in what ordinarily passes for sound reasoning.

An alternative and more fruitful method of developing the calculus of strict implication would be to retain both extensional and intensional disjunction, symbolise them differently, and define implication in terms of intensional disjunction only. The extensional disjunction would now have its negative in a product, as at present, and the principle of addition could be retained, but only for extensional disjunction. As a consequence, such theorems as 'a false proposition implies any proposition' would still not appear, but the principle of simplification [$pq \supset p$] could be deduced instead of being assumed. This second mode of development would produce a calculus which retained all the theorems of the present one which hold for the ordinary meaning of implication, and would reject automatically those which appear to the uninitiated as "absurd". It would also be much wealthier in theorems than the present calculus, because of the fact that the intensional disjunction of p and q implies their extensional disjunction also, though not *vice versa*. And, owing to the distinction of these two meanings of 'either . . . or' propositions, this calculus would prove a valuable instrument of logical analysis. Its primary advantage over any present system lies in the fact that its meaning of implication is precisely that of ordinary inference and proof.