ACTA UNIVERSITATIS STOCKHOLMIENSIS
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DAG PRAWITZ

NATURAL DEDUCTION

A Proof-Theoretical Study

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CONTENTS

IV. NORMAL DEDUCTIONS IN INTUITIONISTIC LOGIC

§ 1. Theorem on normal deductions

Concerning the question of a normal form for deductions in the systems for intuitionistic and minimal logic, we note that applications of the $\vee$E- and $\exists$E-rules give formula occurrences immediately below each other of the same shape. A sequence of formula occurrences of the same shape obtained in that way will be called a *segment*. In a normal deduction we want to exclude the possibility that a segment begins with a consequence of an I-rule or the Λ_I-rule and ends with a major premiss of an E-rule. More precisely, we make the following definitions.

A *segment* in a deduction Π is a sequence $A_1, A_2, \ldots, A_n$ of consecutive formula occurrences in a thread in Π such that

1) A_1 is not the consequence of an application of $\vee$E or $\exists$E;

2) A_i, for each $i < n$, is a minor premiss of an application of $\vee$E or $\exists$E; and

3) A_n is not the minor premiss of an application of $\vee$E or $\exists$E.

Note that a single formula occurrence that is not the consequence or minor premiss of an application of $\vee$E or $\exists$E is a segment by this definition (clause 2 being vacuously satisfied), and that all formula occurrences in a segment are of the same shape.

A segment σ_1 is said to be *above* [*below*] a segment σ_2 if the formula occurrences in σ_1 stand above [below] the formula occurrences in σ_1.

A *maximum segment* is a segment that begins with a consequence of an application of an I-rule or the Λ_I-rule and ends with a major premiss of an E-rule. Note that a maximum formula as defined in Chapter III is a special case of a maximum segment.

By a *normal* deduction, I shall now and henceforth (if nothing else is said) mean a deduction that contains no maximum segment and no redundant applications of $\vee$E or $\exists$E. An application of $\vee$E or $\exists$E in a deduction is said to be redundant if it has a minor premiss at which no assumption is discharged; obviously, such applications are superfluous. (Note that a normal deduction in C′ is normal also according to this definition.)

We now have:

THEOREM 1. *If $\Gamma \vdash A$ holds in the system for intuitionstic or minimal logic, then there is a normal deduction in this system of A from Γ.*[1]

Proof. By the *degree* of a segment, I mean the degree of the formula that occurs in the segment; and by the *length* of a segment, the number of formula occurrences in the segment. The *induction value* of a deduction Π is defined as the pair $\langle d, l\rangle$ such that d is the highest degree of a maximum segment in Π or 0 if there is no such segment, and l is the sum of the lengths of maximum segments in Π of degree d. The induction value $\langle d', l'\rangle$ is *less* than the induction value $\langle d, l\rangle$ if and only if either (i) $d' < d$ or (ii) $d' = d$ and $l' < l$.

Let Π be a deduction in I or M of A from Γ that has only pure parameters, Let $v = \langle d, l\rangle$ be its induction value and assume that $d > 0$. We shall show that there is a deduction Π' in I or M (respectively) of A from Γ that has an induction value less than v. This proves the theorem, since the induction value is not changed by making the proper parameters pure and since it is trivial to remove redundant applications of $\vee$E and $\exists$E.

To prove the assertion, we choose a maximum segment σ of degree d in Π such that there is (i) no maximum segment of degree d above σ and (ii) no maximum segment of degree d that stands above or contains a formula occurrence side-connected with the last formula occurrence in σ.[2]

If σ is a maximum formula that is the consequence of an I-rule, then let Π' be a reduction of Π at σ as defined in Chapter II. It is easily seen that the induction value of Π' is less than v. The case when σ is a maximum formula and consequence of the $\curlywedge_I$-rule is treated simi-

[1] This theorem can be proved for the system C, too. Theorem 1 in Ch. III, however, cannot be extended to C.

[2] That there exists a segment of this kind can easily be seen., e.g. as follows. Consider the set of maximum segments of degree d that satisfy clause (i). If σ_1 is a segment in this set for which clause (ii) does not hold, then there is obviously a segment σ_2 in this set that makes clause (ii) fail for σ_1. If clause (ii) also fails for σ_2, then we can find a third segment σ_3 in the set that makes clause (ii) fail for σ_2, etc. In this way, we obtain a sequence $\sigma_1, \sigma_2, \sigma_3, \ldots$ and clearly $\sigma_i \neq \sigma_j$ if $i \neq j$. Thus, we must finally get a segment in the set that also satisfies clause (ii).

larly (but is easier). If the length of σ is greater than one, we reduce the length of σ as follows. Let F be the formula that occurs in σ. There are two possible cases:

The last occurrence of F in σ is the consequence of $\vee E$. Then Π has the form shown to the left below, except that Σ_4 may instead stand to the left of F, but that case is quite symmetrical to the one below (Σ_4 may, of course, also be empty). Π' is to be as shown to the right below and is thus obtained by moving the application of $\vee$E one step down, so to say.

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ B\vee C\end{array}\quad\begin{array}{c}\Sigma_2\\ F\end{array}\quad\begin{array}{c}\Sigma_3\\ F\end{array}}{F}\qquad\Sigma_4}{(C)}\\ \Pi_5\end{array}\qquad\qquad\begin{array}{c}\dfrac{\begin{array}{c}\Sigma_1\\ B\vee C\end{array}\quad\dfrac{\begin{array}{c}\Sigma_2\\ F\end{array}\quad\Sigma_4}{C}\quad\dfrac{\begin{array}{c}\Sigma_3\\ F\end{array}\quad\Sigma_4}{C}}{(C)}\\ \Pi_5\end{array}$$

We note that the application of the E-rule in Π that has the last occurrence of F in σ as major premiss is replaced in Π' by two applications of the same form, but with the difference that the major premisses of these applications now depend also on the formulas B and C respectively. However, this is inessential since the restrictions concerning the formulas that a formula occurrence depends on do not concern major premisses of E-rules, (but only premisses of $\forall$I and minor premisses of $\exists$E). One can then see that Π' is still a deduction of A from Γ. That the induction value of Π' is less than v is obtained by applying clause (ii) in the assumptions about σ.

The last occurrence of F in σ is the consequence of $\exists$E. Then Π has the form shown to the left below, again with the possible exception that Σ_4 may stand to the right of F instead. Π' is to be as shown to the right below.

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ \exists xB\end{array}\quad\begin{array}{c}\Sigma_2\\ F\end{array}}{F}\qquad\Sigma_4}{(D)}\\ \Pi_5\end{array}\qquad\qquad\begin{array}{c}\dfrac{\begin{array}{c}\Sigma_1\\ \exists xB\end{array}\quad\dfrac{\begin{array}{c}\Sigma_2\\ F\end{array}\quad\Sigma_4}{D}}{(D)}\\ \Pi_5\end{array}$$

Note that the proper parameter of the application of $\exists$E in question does not occur in Σ_4 (clause 2 in the lemma on parameters). That Π' is still a deduction of A from Γ then follows as above. The induction value of Π' is clearly less than v.

§ 2. The form of normal deductions

What was said in Chapter III about branches in a normal deduction now holds if we take the segments, instead of the formula occurrences, as units and replace the branches with sequences of formula occurrences which are like branches except that the formula occurrence immediately succeeding a major premiss of an application α of $\vee$E or $\exists$E is an assumption discharged by α instead of the consequence of α. Such a sequence is called a *path* if it does not begin with an assumption discharged by an application of $\vee$E or $\exists$E. More precisely, we define:

A sequence $A_1, A_2, \ldots, A_n$ is a *path* in the deduction Π if and only if

1) A_1 is a top-formula in Π that is not discharged by an application of $\vee$E or $\exists$E; and

2) A_i, for each $i<n$, is not the minor premiss of an application of $\supset$E, and either (i) A_i is not the major premiss of $\vee$E or $\exists$E, and A_{i+1} is the formula occurrence immediately below A_i, or (ii) A_i is the major premiss of an application α of $\vee$E or $\exists$E, and A_{i+1} is an assumption discharged in Π by α; and

3) A_n is either a minor premiss of $\supset$E, the end-formula of Π, or a major premiss of an application α of $\vee$E or $\exists$E such that α does not discharge any assumptions.

As an *example*, we have three paths in the deduction of p. 19, namely

(i) the sequence of the formula occurrences marked (1), (3), left occurrence of (4), (7), (8), (9), (10), and

(ii) the sequence of those marked (1), (3), right occurrence of (4), and

(iii) the thread that starts with the formula occurrence marked (2).

In normal deductions, the last formula occurrence in a path is always a minor premiss of $\supset$E or the end-formula of the deduction, since the possibility of applications of $\vee$E and $\exists$E that do not discharge any assumption is then excluded.

Every path π can obviously be uniquely divided into consecutive segments (usually consisting of just one formula occurrence). In other words, π is the concatenation of a sequence of segments and can be written

$$A_{1.1}, A_{1.2}, \ldots, A_{1.n_1}, A_{2.1}, A_{2.2}, \ldots, A_{2.n_2}, \ldots, A_{k.1}, A_{k.2}, \ldots, A_{k.n_k},$$

where $A_{i.1}, A_{i.2}, \ldots, A_{i.n_i}$ for each $i \leqslant k$ is a segment σ_i in Π. The sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$ will be called *the sequence of segments in π*.

It will be convenient to say that a segment σ is a *top-segment* [*end-segment*] or is a *consequence* [(*major* or *minor*) *premiss*] of an application α of an inference rule when the first [last] formula occurrence in σ is a top-formula [the end-formula] or a consequence [(major or minor) premiss] of α respectively. However, this terminology will be used only in phrases such as "the segment σ is a consequence of α" or "the segment σ that is premiss of α"; when not explicitly speaking about segments, we still mean by the premiss or the consequence of an application of a rule the formula occurrence in question.

THEOREM 2. *Let* Π *be a normal deduction in* I *or* M, *let* π *be a path in* Π, *and let* $\sigma_1, \sigma_2, \ldots, \sigma_n$ *be the sequence of segments in* π. *Then there is a segment* σ_i, *called the* minimum segment *in* π, *which separates two (possibly empty) parts of* π, *called the* E-part *and* I-part *of* π, *with the properties:*

1) *For each* σ_j *in the E-part (i.e.* $j < i$*) it holds that* σ_j *is a major premiss of an E-rule and that the formula occurring in* σ_j *is a subformula of the one occurring in* σ_{j+1}.

2) σ_i, *provided that* $i \neq n$, *is a premiss of an I-rule or of the* $\curlywedge_I$*-rule.*[1]

3) *For each* σ_j *in the I-part, except the last one, (i.e.,* $i < j < n$*) it holds that* σ_j *is a premiss of an I-rule and that the formula occurring in* σ_j *is a subformula of the one occurring in* σ_{j+1}.

We assign an *order* to the paths in the same way as to the branches. A path of order 0 is said to be a *main path*. We then prove as in Chapter II:

COROLLARY 1. (Subformula principle.) *Every formula occurring in a normal deduction in* I *or* M *of* A *from* Γ *is a subformula of* A *or of some formula of* Γ.

This corollary can be strengthened as in Chapter III. To clause 2) in the definition of positive and negative subformula, we add:

B and C are positive [negative] subformulas of A when $B \vee C$ is one, and so is B^x_t when $\exists x B$ is.

We then have:

[1] In the case of minimal logic, σ_i can of course be a premiss of an I-rule only; the formula occurring in σ_i is then always a subformula of the one occurring in σ_{i+1} (provided that $i \neq n$).

COROLLARY 2. *Corollary 2 in Chapter II holds also for* I *and* M *but now with the following simplifications:*

(i) *The second part of clause 1) concerning* $\curlywedge_C$ *is left out.*

(ii) *In the case of minimal logic, the phrase "if different from* $\curlywedge$*" (in clause 2) is left out.*

As a lemma for later theorems, we record also the following properties concerning branches in normal deductions in I and M (it can be obtained as a corollary from Theorem 2 or proved directly in the same way as that theorem):

CORALLARY 3. *Let* Π *be a normal deduction in* I *or* M *and let* A_1, $A_2, \ldots, A_n$ *be a branch in* Π *that contains no minor premisses of* $\vee$E *or* $\exists$E. *Then there is a formula occurrence* A_i *such that:*

1) A_j, *for each* $j < i$, *is a major premiss of an E-rule;*

2) A_i, *provided* $i \neq n$, *is a premiss of an I-rule or the* $\curlywedge_I$*-rule;*

3) A_j, *for each* $i < j < n$, *is a premiss of an I-rule and is a subformula of* A_{j+1}.

Finally, we note the following corollary, whose proof is immediate from Corollary 1 by inspection of the inference rules:

COROLLARY 4. (Separation theorem.)[1] *If* Π *is a normal deduction in* I *or* M *of* A *from* Γ, *then the only inference rules that are applied in* Π *are the inference rules in* I *or* M *for the logical constants that occur in* A *or in some formula of* Γ.

§ 3. Some further corollaries

We note some further results for the systems for intuitionistic and minimal logic that can be obtained as corollaries from the theorem on normal deductions. Some of them are already well known from other or similar proofs but are obtained here by relatively easy and uniform applications of Theorems 1 and 2. What is said in this section is to be

[1] A similar theorem for the calculus of sequents is an immediate corollary of Gentzen's Hauptsatz (cf. App. A). For a system of intuitionistic logic of axiomatic type, a similar result was stated by Wajsberg [1] and proved by Curry [1]. An algebraic proof of the separation theorem for the sentential part of intuitionistic logic has recently been given by Horn [1].

understood as referring to the system for intuitionistic logic or, alternatively, to the system for minimal logic.[1]

COROLLARY 5. *The interpolation theorem (Corollary 5 in Chapter III) holds also for intuitionistic and minimal logic.*[2]

The proof is obtained from the proof of Corollary 5 in Chapter III by adding cases for $\vee$ and $\exists$. (They become more or less dual to the ones for & and $\forall$.) In case II, we consider the $\curlywedge_I$-rule instead of the $\curlywedge_C$-rule. In case III, we consider a thread τ that contains no minor premiss (instead of the main branch β) and apply Corallary 3 (instead of Theorem 3).

COROLLARY 6. *If $\Gamma \vdash A \vee B$, then either $\Gamma \vdash A$ or $\Gamma \vdash B$, provided that no formula of Γ has a strictly positive subformula that contains $\vee$ as principal sign.*[3]

Proof. Let Π be a normal deduction of $A \vee B$ from Γ. We first show that there is exactly one end-segment σ in Π. If there were two end-segments, they must contain a minor premiss of $\vee$E. Consider the corresponding major premiss $C \vee D$ and let π be a path to which $C \vee D$ belongs. The first formula occurrence F in π cannot be discharged in Π (since $C \vee D$ belongs to the E-part of π and there is no premiss of $\supset$I below $C \vee D$) and hence $F \in \Gamma$. But by Corollary 2, $C \vee D$ is then a strictly positive subformula of F contrary to the assumptions about Γ.

σ must further be consequence of an application α of an I-rule (i.e. $\vee$I) or of the $\curlywedge$-rule; otherwise, it is a minimum segment of the

[1] The proofs are usually given with the case of intuitionistic logic in mind and could be simplified in case of minimal logic.

[2] First proved by Schütte [3] (cf. note 1 on p. 46).

[3] For the case when Γ is empty, the theorem was first stated (without proof and for the propositional calculus only) by Gödel [2] and proved by Gentzen [3] (p. 407) as a corollary of his Hauptsatz. Other proofs are found in e.g., McKinsey-Tarski [1] and Harrop [1].

With the present restriction of Γ, the theorem was proved by Harrop [2] for the intuitionistic propositional calculus and for the intuitionistic elementary number theory; in the latter case provided that all the formulas involved are closed (or, in our terminology, that they do not contain individual parameters). Harrop's proof is not constructive in the case of number theory. Kleene [2] extends this result to intuitionistic predicate logic and considers also a weaker but not effective restriction on Γ.

paths to which it belongs, and we then obtain as above that the first formula that occurs in any of these paths belongs to Γ and has $A \vee B$ as strictly positive subformula. It follows that the premiss of α is A, B, or $\curlywedge$. For simplicity, we may assume that it is A or B, since instead of inferring $A \vee B$ directly from $\curlywedge$, we can first infer A, e.g. It is then clear how to obtain a deduction of A or B respectively from Γ: We just refrain from inferring $A \vee B$ by the application α and substitute A or B, respectively, for all occurrences of $A \vee B$ in σ. In more detail for the case when σ contains more than one element: Let (Σ/A) be the subdeduction of Π that has the major premiss of α as end-formula. Then the given deduction and the new deduction have respectively the forms

$$\dfrac{\dfrac{\Sigma_n}{\exists x_n C_n} \qquad \begin{array}{c} \dfrac{\dfrac{\Sigma_2}{\exists x_2 C_2} \qquad \dfrac{\dfrac{\Sigma_1}{\exists x_1 C_1} \qquad \dfrac{\begin{array}{c}\Sigma\\ A\end{array}}{A \vee B}}{A \vee B}}{A \vee B} \\ \vdots \\ A \vee B \end{array}}{A \vee B} \qquad\qquad \dfrac{\dfrac{\Sigma_n}{\exists x_n C_n} \qquad \begin{array}{c} \dfrac{\dfrac{\Sigma_2}{\exists x_2 C_2} \qquad \dfrac{\dfrac{\Sigma_1}{\exists x_1 C_1} \qquad \begin{array}{c}\Sigma\\ A\end{array}}{A}}{A} \\ \vdots \\ A \end{array}}{A}$$

COROLLARY 7.[1] *Let $t_1, t_2, \ldots, t_n$ ($n \geq 0$) be all the terms that occur in $\exists xA$ or in some formula of Γ and let there be no formula of Γ that has a strictly positive subformula containing $\exists$ as principal sign.*

We then have:

(i) *For $n > 0$: If $\Gamma \vdash \exists xA$, then $\vdash A^x_{t_1} \vee A^x_{t_2} \vee \ldots \vee A^x_{t_n}$.*

(ii) *For $n > 0$ and provided that no formula of Γ has a strictly positive subformula that contains $\vee$ as principal sign:*
If $\Gamma \vdash \exists xA$, then $\Gamma \vdash A^x_{t_i}$ for some $i \leq n$.

(iii) *For $n = 0$: If $\Gamma \vdash \exists xA$, then $\Gamma \vdash \forall xA$.*

Proof. Let Π be a normal deduction of $\exists xA$ from Γ. By the lemma on parameters, we can assume that the proper parameters are pure and different from the terms $t_1, t_2, \ldots, t_n$ that occur in $\exists xA$ or in some formula of Γ. Let σ be any end-segment in Π. We then have that

[1] For the case when Γ is empty, part (ii) of the corollary can be obtained from Gentzen [3].

A result similar to part (ii) of the corollary is obtained by Harrop [2] for intuitionistic elementary number theory (the term t_i is then a numeral). Kleene [2] and [3] prove part (ii) of the corollary also for predicate logic and consider also a weaker but not effective restriction on Γ.

(1) σ contains no minor premiss of $\exists$E

and that σ is the consequence of an application α of an I-rule (i.e. $\exists$I) or the $\curlywedge_{\mathrm{I}}$-rule; otherwise, as in the proof of Corollary 6, we obtain that a formula of Γ has a strictly positve subformula that contains $\exists$ as principal sign. Hence, the major premiss of α has the form A^x_u or $\curlywedge$. For simplicity, we may assume that it has the form A^x_u, since instead of inferring $\exists xA$ directly from $\curlywedge$ we can first infer A^x_u. If u is not one of $t_1, t_2, \ldots, t_n$, we now replace all occurrences of u in Π by one of the terms t_i (e.g. t_1) in case $n > 0$ and by a parameter a not occurring in Π in case $n = 0$. We repeat this process for every end-segment of Π so that every one comes to stand immediately below a formula occurrence of one of the two forms $A^x_{t_i}$ (for some $i \leqslant n$) and A^x_a. Then, we still have a deduction Π' of $\exists xA$ from Γ (note that neither of u, a, and t_i ($i \leqslant n$) is a proper parameter in Π).

We first prove part (ii) of the corollary. In this case, we know from the proof of Corollary 6 that there is exactly one end-segment σ in Π'. Hence, by (1), σ consists of just the end-formula $\exists xA$ of Π', and leaving out this end-formula, we have a deduction of $A^x_{t_i}$ from Γ for some $i \leqslant n$.

In case (iii), every end-segment in Π' stands immediately below a major premiss A^x_a of an application of $\exists$I. As in the proof of Corollary 6 —except that we may now have several end-segments and a series of applications of $\vee$E instead of $\exists$E—we get a deduction Π^* of A^x_a from Γ by leaving out these applications of $\exists$I and substituting A^x_a for all occurrences of $\exists xA$ in the end-segments of Π'. As a does not occur in any assumptions on which the end-formula A^x_a of Π^* depends, we can then apply $\forall$I and obtain the desired deduction: $(\Pi^*/\forall xA)$.

In case (i), finally, every end-segment in Π' stands immediately below a premiss $A^x_{t_i}$ of an application of $\exists$I. We leave out these applications of $\exists$I, insert instead a series of applications of $\vee$I ending with $A^x_{t_1} \vee A^x_{t_2} \vee \ldots \vee A^x_{t_n}$, and substitute this disjunction for all occurrences of $\exists xA$ in the end-segments of Π'. Because of (1), this does not conflict with the restrictions on $\exists$E, and we thus have the desired deduction.

COROLLARY 8. *Let C be a formula which does not contain any occurrence of $\supset$, and let Γ be $\{A_1 \supset B_1, A_2 \supset B_2, \ldots, A_n \supset B_n\}$. If $\Gamma \vdash C$, then $\Gamma \vdash A_i$ for some $i \leqslant n$.*[1]

[1] Cf. Wajsberg [1].

Proof. Let Π be a normal deduction of C from Γ and let τ be a thread (main branch) in Π that contains no minor premiss. As C contains no occurrence of $\supset$, it follows from Corollary 3 that no assumption is discharged in Π at a formula occurrence in τ. Hence, the first formula occurrence in τ has the form $A_i \supset B_i$ (for some $i \leqslant n$), and Π has the form

$$\frac{\begin{array}{c}\Sigma\\ A_i\end{array} \quad A_i \supset B_i}{\begin{array}{c}[B_i]\\ \Pi'\end{array}}$$

where A_i depends on formulas of Γ only. Hence, (Σ/A_i) is a deduction of A_i from Γ.

For the next corollary, I shall say that a binary sentential connective [quantifier] γ is (*strongly*) *definable* in the system S if for every formula A and B there is a formula C not containing γ such that $\vdash_S (A\gamma B) \equiv C$ $[\vdash_S \gamma x A \equiv C]$.[1] I shall say that $\curlywedge$ is (strongly) definable in S if there is a formula A not containing $\curlywedge$ such that $\vdash_S \curlywedge \equiv A$.

There is also a weaker sense in which a logical constant may be definable in a system S, namely when we can uniformly transform every formula containing γ to a formula not containing γ without changing provability in S. More precisely, I shall say that a binary sentential connective [quantifier] γ is *weakly definable* in S if there is a transformation that transforms any pseudo-formula A to a pseudo-formula A^* not containing γ in such a way that:

(i) A^* can be obtained from A as the result of successively replacing parts of the form $(B\,\gamma\, C)$ $[\gamma x B]$ by $(B\,\gamma\, C)^*$ $[(\gamma x B)^*]$;[2] and

(ii) $\vdash_S A^*$ if and only if $\vdash_S A$.

Similarly, I shall say that $\curlywedge$ is weakly definable in S if there is a formula B not containing $\curlywedge$ such that if A^* is obtained from A by replacing every occurrence of $\curlywedge$ with B, then $\vdash_S A^*$ if and only if $\vdash_S A$.

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[1] Actual definitions of course also satisfy the stronger requirement that C is determined uniformly by A and B [by A] (e.g. by a schema). However, I shall here be concerned with showing non-definability, and that will be possible without imposing such a requirement.

[2] Again, most transformations of this kind that have actually been used are uniform in a stronger sense than required by (i) (cf. note 1 above).

Of course, strong definability implies weak definability. Conversely, it can rather easily be seen that weak definability of one of the constants &, v, and E in the system for intuitionistic or minimal logic implies strong definability. However, $\curlywedge$ is weakly, but not strongly, definable in the system for minimal logic. Let P be a o-place predicate parameter and let A^* be the result of replacing every occurrence of $\curlywedge$ in A with P. It can then easily be seen that $\vdash_M A^*$ if and only if $\vdash_M A$.

I shall say that a logical constant is (*weakly*) *independent* [*strongly independent*] in S if γ is not strongly [weakly] definable in S.

CORALLARY 9. *All the logical constants are strongly independent in the system of intuitionistic logic. In the system for minimal logic,* $\curlywedge$ *is weakly independent, and all the other logical constants are strongly independent.*[1]

For the proofs below, we assume—contrary to what is to be proved—that there is a transformation (*) as required in the definition of weak definability.

Independence of &. Let P and Q be different o-place predicate parameters, and let $(P\&Q)^* = A$. Since $\vdash (P\&Q)\supset P$ and $\vdash (P\&Q)\supset Q$, we have $\vdash A\supset P$ and $\vdash A\supset Q$, and hence $A\vdash P$ and $A\vdash Q$. We prove the following general lemma:

If A lacks occurrences of & and $P \neq Q$, then $A\vdash P$ and $A\vdash Q$ cannot both hold unless $A\vdash \curlywedge$.

A contradiction is then obtained, e.g. as follows: $\vdash P\supset(Q\supset(P\&Q))$, hence $\vdash P\supset(Q\supset A)$ and, by the lemma, $\{P, Q\}\vdash \curlywedge$. The latter can easily be seen to be false by application of Theorems 1 and 2.

To prove the lemma, let Π_1 [Π_2] be a normal deduction of P [Q] from A, and let π_1 be a main path in Π_1. Thus, π_1 begins with an occurrence of A and ends with an occurrence of P. As P is atomic, the minimum segment of π_1 consists either of occurrences of P, in which case the I-part of π_1 is empty, or of occurrences of $\curlywedge$, in which case the I-part of π_1 contains occurrences of P only. Now consider any two paths in normal deductions that begin with assumptions of the same form lacking occurrences of &, and consider the formulas that

[1] That none of the constants &, v, $\supset$, and $\sim$ can be (strongly) defined in terms of the other three is known from Wajsberg [1] and McKinsey [1]. As the present corollary shows, they are still independent (also in a stronger sense) when first order quantifiers are added. However, in second order systems, we shall see that &, v, $\exists$, and $\curlywedge$ are (strongly) definable in terms of $\supset$ and $\forall$ (Ch. V).

occur in the successive segments in these paths. It is then easily verified that the first place at which these formulas can differ—disregarding differences with respect to individual parameters—is either at segments that immediately succeed segments that are major premisses of $\vee$E or at segments one of which belongs to the I-part. Thus, if π_2 is a main path in Π_2 such that the choices with respect to successors to segments that are major premisses of $\vee$E are made in the same way as in π_1, then the formula occurring in the minimum segment of π_2 must be identical with the formula occurring in the minimum segment of π_1. As $P \neq Q$, it follows that every main path in Π_1 [Π_2] has a minimum segment that consists of occurrences of $\curlywedge$ and that is a premiss of an application of the $\curlywedge_I$-rule, having P [Q] as consequence. Leaving out these applications of the $\curlywedge_I$-rule in Π_1 (cf. the proof of Corollary 6 or 7) and substituting $\curlywedge$ for P in all end-segments of Π_1, we obtain a deduction of $\curlywedge$ from A.

Independence of $\vee$. Let P and Q be different 0-place parameters, and let $(P \vee Q)^* = A$. Let R be a 0-place parameter that does not occur in A. Since

$$\vdash (P \vee Q) \supset ((P \supset R) \& (Q \supset R) \supset R)$$

it follows that $A \vdash (P \supset R) \& (Q \supset R) \supset R$ and hence that

$$A \vdash P \vee Q \tag{1}$$

(substitute $(P \vee Q)$ for R). Corollary 6 applied to (1) gives that

$$\text{either } A \vdash P \text{ or } A \vdash Q. \tag{2}$$

But, on the other hand, $\vdash P \supset (P \vee Q)$ and $\vdash Q \supset (P \vee Q)$, and hence $P \vdash A$ and $Q \vdash A$. Combined with (2), this is absurd, since, by easy application of Theorems 1 and 2, it can be seen that neither $P \vdash Q$ nor $Q \vdash P$.

Independence of $\supset$. Immediate, since a formula not containing $\supset$ cannot be provable. (That this is so can be seen from Theorem 2, because if the assumption with which a main path in a normal deduction begins is discharged (as it must be when the deduction is a proof), then the last formula of the path contains an occurrence of $\supset$.)

Independence of $\forall$. Let P be a 1-place predicate parameter, and let $(\forall x Px)^* = A$. Let a be a parameter that does not occur in A. Since

$\vdash \forall xPx \supset Pa$, also $\vdash A \supset Pa$, and hence $A \vdash \forall xPx$. The following lemma holds in general:

If A does not contain $\forall$, then $A \vdash \forall xPx$ implies $A \vdash \curlywedge$.

The independence of $\forall$ then easily follows. (For instance, since $\forall xPx \supset Q$ is not provable, $A \supset Q$ is not provable, contradicting the result that $A \vdash \curlywedge$.)

To prove the lemma, let Π be a normal deduction of $\forall xPx$ from A that has only pure parameters, and let π_1 be a main path in Π. The minimum segment σ of π_1 consists either of occurrences of $\curlywedge$ or of occurrences of Pa for some a. In the latter case, σ is the premiss of an application α of $\forall$I. However, at which point in π can a enter? It holds in general that an individual parameter can enter into a path only (i) at the first element of the path, (ii) at the consequence of an I-rule, the $\curlywedge_I$-rule or the $\forall$E-rule, or (iii) at an assumption discharged by $\exists$E. Case (iii) is excluded. Since Π has only pure parameters, the assumption in question would have to be discharged at some formula occurrence below σ, and the premiss of α would then depend on an assumption containing a, violating the requirements in the deduction rule for $\forall$I. That case (ii) is excluded follows from Theorem 2 and the fact that A lacks occurrences of $\forall$: The minimum segment σ cannot be preceded by a consequence of an I-rule or the $\curlywedge_I$-rule, and if σ is preceded by a premiss of the $\forall$E-rule, then $\forall$ would occur in the first formula occurrence in π i.e., in A. But also case (i) is impossible, because the premiss of α would then again have to depend on a formula containing a, violating the requirements in the deduction rule for $\forall$I. It follows that every main path has a minimum segment consisting of occurrences of $\curlywedge$, and the lemma then follows as in the &-case above.

Independence of $\exists$. Let P be a 1-place parameter, and let $(\exists xPx)^* = A$. Let Q be a 0-place parameter not occurring in A. Since

$$\vdash \exists xPx \supset (\forall x(Px \supset Q) \supset Q)$$

it follows that $A \vdash \forall x(Px \supset Q) \supset Q$, and hence that

$$(1) \qquad A \vdash \exists xPx$$

(substitute $\exists xPx$ for Q). Corollary 7 applied to (1) gives that

$$(2) \qquad A \vdash Pt_1 \vee Pt_2 \vee \ldots \vee Pt_n$$

for certain terms t_i. Let a be different from all the t_i's ($i \leqslant n$). Since $\vdash Pa \supset \exists x Px$, it follows from (2) that

$$Pa \vdash Pt_1 \vee Pt_2 \vee \ldots \vee Pt_n,$$

which is absurd (apply Corollary 6 and then Theorem 2).

Weak independence of $\curlywedge$. $A \vdash \curlywedge$ cannot hold if A lacks occurrences of $\curlywedge$. Because, by Theorem 2, normal deductions contain occurrences of $\curlywedge$ in minimum segments only, and, by Corollary 2, the formula occurring in the minimum segment in a normal deduction of $\curlywedge$ from A is an assumption-part of $\langle\{A\}, \curlywedge\rangle$ and is hence a subformula of A.

Strong independence of $\curlywedge$ *in intuitionistic logic*. Suppose that $\curlywedge$ could be replaced by A without changing provability, and let P be a 0-place parameter not occurring in A. Since $\vdash_{\mathrm{I}} \curlywedge \supset P$, it follows that $A \vdash_{\mathrm{I}} P$, which is impossible just as above.