Logic, First Course, Winter 2020. Week 9, Lecture 2, Handout.

Assigning probabilities to rows of the truth table

f = the coin flips heads on the *first* flip

s = the coin flips heads on the *second* flip

(f ∧ ¬s), (¬f ∨ s)

f	s	(	f	∧	¬	s	),(	¬	f	∨	s	)
T	T		T			T			T		T	
T	F		T			F			T		F	
F	T		F			T			F		T	
F	F		F			F			F		F	

Check

It is natural to assign probabilities to each of these outcomes. If our coin was a fair coin, we would assign probabilities as follows:

Probability assignment for fair coin, for propositions *f* and *s*:

1/4 -- TT = Heads on first flip, heads on second flip.
1/4 -- TF = Heads on first flip, tails on second flip.
1/4 -- FT = Tails on first flip, heads on second flip.
1/4 -- FF = Tails on first flip, tails on second flip.

What is the probability of *f* ∧ ¬*s*?

What is the probability of ¬*f* ∨ *s*?

(f ∧ ¬s), (¬f ∨ s)

f	s	(	f	∧	¬	s	),(	¬	f	∨	s	)
T	T		T			T			T		T	
T	F		T			F			T		F	
F	T		F			T			F		T	
F	F		F			F			F		F	

Check

Probability assignment for biased coin, for propositions *f* and *s*:

1/9 -- TT = Heads on first flip and heads on second flip.
2/9 -- TF = Heads on first flip and tails on second flip.

2/9 -- FT = Tails on first flip and heads on second flip.
4/9 -- FF = Tails on first flip and tails on second flip.

What is the probability of $f \wedge \neg s$?

What is the probability of $\neg f \vee s$?

In general, we can summarize as follows:

- A *probability assignment* assigns numbers between 0 and 1 to each row of the truth-table, so that the numbers all add up to 1.
- One then assigns probabilities $Pr(\phi)$ to propositions ϕ by adding up the probabilities of all those rows at which the proposition ϕ is true.

To illustrate, let's note that it is possible to assign a row of a truth-table probability zero. For instance, suppose we use the key

> *p* = it is sunny on Monday
>
> *q* = it is sunny on Tuesday

Then we might assign probabilities as follows, since we live in a part of the world where it is very unlikely to have two days in a row that are not sunny:

Probability assignment for an agent's degrees of belief about the weather, for propositions *p* and *q*:

3/5 -- TT = It is sunny on Monday and it is sunny on Tuesday.
1/5 -- TF = It is sunny on Monday and it is not sunny on Tuesday.
1/5 -- FT = It is not sunny on Monday and it is sunny on Tuesday.
0 -- FF = It is not sunny on Monday and it is not sunny on Tuesday.

(p → q), (p ∧ ¬q)

p	q	(p	→	q),	(p	∧	¬	q)
T	T	T		T	T			T
T	F	T		F	T			F
F	T	F		T	F			T
F	F	F		F	F			F

Check

What is Pr(p->q), relative to the agent's degrees of belief about the weather?

- 1/5
- 2/5
- 3/5
- 4/5

Check

What is Pr(p/\~q), relative to the agent's degrees of belief about the weather?

- 1/5
- 2/5
- 3/5
- 4/5

Check

The rules of probability

- First rule: $0 \leq Pr(\phi) \leq 1$
- Second rule: if ϕ is a tautology, then $Pr(\phi) = 1$.
- Third rule: $Pr(\neg\phi) = 1 - Pr(\phi)$.
- Fourth rule: $Pr(\phi \vee \psi) = Pr(\phi) + Pr(\psi) - Pr(\phi \wedge \psi)$

Example 1. Suppose that $Pr(p) = .4$ and $Pr(q) = .5$ and $Pr(p \wedge q) = .2$. What is $Pr(p \vee q)$?

Example 2. Suppose that $Pr(p) = .3$ and $Pr(q) = .2$ and $Pr(p \vee q) = .45$. What is $Pr(p \wedge q)$?

Example 3. Suppose that $Pr(p) = .3$ and $Pr(q) = .2$ and $Pr(\neg p \wedge \neg q) = .5$. What is $Pr(\neg p \vee \neg q)$?

Connections between probability and logic

- If $\phi \to \psi$ is a tautology, then $Pr(\phi) \leq Pr(\psi)$.
- If ϕ and ψ are equivalent, then $Pr(\phi) = Pr(\psi)$.

Example 4. Show that $Pr(\phi \wedge (\psi \vee \xi)) = Pr(\phi \wedge \psi) + Pr(\phi \wedge \xi) - Pr(\phi \wedge (\psi \wedge \xi))$.

One can check the equivalence mentioned at the end of this video by doing a truth-table like follows:

((p ∧ q) ∧ (p ∧ r)), (p ∧ (q ∧ r))

p q r	((p	∧	q)	∧	(p	∧	r)),	(p	∧	(q	∧	r))
T T T	T		T		T		T	T		T		T
T T F	T		T		T		F	T		T		F
T F T	T		F		T		T	T		F		T
T F F	T		F		T		F	T		F		F
F T T	F		T		F		T	F		T		T
F T F	F		T		F		F	F		T		F
F F T	F		F		F		T	F		F		T
F F F	F		F		F		F	F		F		F

Check

One can calso check the equivalence by doing two simple proofs:

exercise

((p ∧ q) ∧ (p ∧ r)) ⊢ (p ∧ (q ∧ r))

```
1. (p∧q)∧(p∧r) :assumption
```

exercise

(p ∧ (q ∧ r)) ⊢ ((p ∧ q) ∧ (p ∧ r))

```
1. p∧(q∧r) :assumption
```

Example 5. Show that $Pr(\phi \vee (\psi \vee \xi)) = Pr(\phi) + Pr(\psi) + Pr(\xi) - Pr(\phi \wedge \psi) - Pr(\psi \wedge \xi) - Pr(\phi \wedge \xi) + Pr(\phi \wedge (\psi \wedge \xi))$.

This is a special case of the so-called 'Inclusion-Exclusion' formula.

Example 6. Let us verify this formula, in a specific case. Consider the following truth-table, which we complete just like in Week 1 (the commas separating the formulas is just a way for us to write many truth-tables in one box):

(p ∨ (q ∨ r)), (p ∧ q), (q ∧ r), (p ∧ r), (p ∧ (q ∧ r))

pqr	(p	∨	(q	∨	r)),(	p	∧	q),(	q	∧	r),(	p	∧	r),(	p	∧	(q	∧	r))
TTT	T		T		T	T		T	T		T	T		T	T		T		T
TTF	T		T		F	T		T	T		F	T		F	T		T		F
TFT	T		F		T	T		F	F		T	T		T	T		F		T
TFF	T		F		F	T		F	F		F	T		F	T		F		F
FTT	F		T		T	F		T	T		T	F		T	F		T		T
FTF	F		T		F	F		T	T		F	F		F	F		T		F
FFT	F		F		T	F		F	F		T	F		T	F		F		T
FFF	F		F		F	F		F	F		F	F		F	F		F		F

Check

Suppose that each of the rows is weighted equally, that is each has equal chance of happening, namely a 1/8 chance of happening.

These are lecture notes written for this course.[1]

1. It is run on the Carnap software, which is↩