ACTA UNIVERSITATIS STOCKHOLMIENSIS
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NATURAL DEDUCTION

A Proof-Theoretical Study

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APPENDIX C

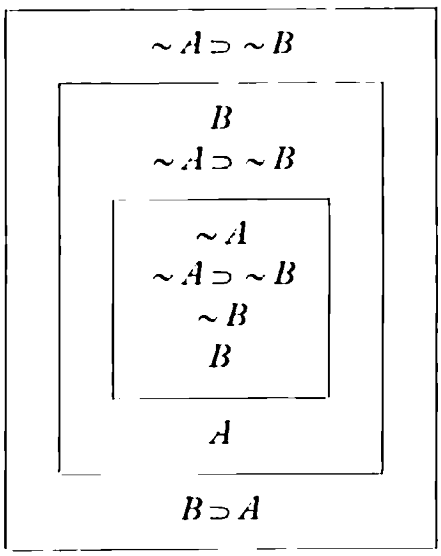
NOTES ON SOME OTHER VARIANTS OF NATURAL DEDUCTION

§ 1. The origin of natural deduction

The first person to express the idea of constructing a system of natural deduction seems to have been Łukasiewics in seminars in 1926. He called attention to the fact that in informal mathematical reasoning, one does not proceed according to the principles of the then common logical systems of Frege, Russell, and Hilbert among others, drawing inferences from axioms (or theorems) with the help of (proper) inference rules. Instead, one uses most frequently the method of drawing inferences from assumptions. Łukasiewics suggested that one should try to formalize this kind of reasoning, and the first results in this direction was obtained by *Jáskowski* in these seminars. He presented the results also at the First Polish Mathematical Congress in Lwow 1929.

Deductions in this system of Jáskowski's consist of formulas written in a number of boxes, some of which could appear within others. A new assumption is marked by the introduction of a new box. The assumption is written as the first formula in this box, and below one writes the formulas that are inferred from this assumption. If the box appears inside another box, formulas that stand in the larger box may be repeated in the smaller box. When an assumption is discharged and one obtains a consequence that is independent of the assumption, one writes this consequence outside and immediately below the box. The technique is exemplified by the deduction at the top of the next page.

In 1934, Jáskowski published a revised version of this system as well as some other systems under the title "On the rules of suppositions in formal logic" (Jáskowski [1]). He now develops systems for (1) classical sentential logic, (2) the sentential logic of Kolmogoroff, (3)



$$(\sim A \supset \sim B) \supset (B \supset A)$$

extended propositional logic (allowing quantifiers over propositional variables), and (4) a predicate logic in which the provable formulas are those classically valid in all domains including the empty one. I shall briefly describe systems (1), (2) and (4).

The languages of these systems are like those in Chapter I with some differences, the main ones of which are as follows: There are no individual parameters, and a formula is thus allowed to contain free variables. The logical constants are $\sim$, $\supset$, $\forall$. There are no descriptive symbols.

Instead of using the device with boxes explained above, Jáskowski now prefixes the formula occurrences in a deduction by strings of numerals; the numerals indicate the assumptions that the formula occurrences depend on. Let us call a string of numerals separated by commas—including the empty string—a *prefix*, and let p and q (sometimes with subscripts) refer to prefixes. Let us write $p \leq q$ to denote that p agrees with an initial part of q which is to be understood as including the case where $p = q$ and the case where p is the empty prefix; let us say that *p immediately precedes q* if either p is the empty prefix and q is a numeral, or $p = q,\nu$ where ν is a numeral.

In a sequence p_1A_1, p_2A_2, ..., p_nA_n, we say that p_iA_i is an *assumption* if p_i is not empty and does not occur earlier in the sequence.

Somewhat changing Jáskowski's terminology, we can make the following definition:

$\mathcal{D}$ is a *deduction* in *Jáskowski's system for classical sentential logic* if and only if $\mathcal{D}$ is a sequence $p_1A_1, p_2A_2, \ldots, p_nA_n$ such that for each $i \leq n$ either

1) p_iA_i is an assumption in $\mathcal{D}$; or

2) A_i is obtained from A_j and A_k by $\supset$E for some $j, k < i$ such that $p_j \lesssim p_i$ and $p_k \lesssim p_i$; or

3) $A_i = A_j \supset A_k$ for some $j, k < i$ such that p_jA_j is an assumption in $\mathcal{D}$, $p_k \lesssim p_j$, and p_i immediately precedes p_j; or

4) $\sim A_i = A_j$ for some $j < i$ such that p_jA_j is an assumption in $\mathcal{D}$, p_i immediately precedes p_j, and for some $k, m < i$ it holds that $A_k = \sim A_m$, $p_k \lesssim p_j$, and $p_m \lesssim p_j$.

A is said to be deducible in this system from Γ if there is a deduction $\mathcal{D}$ in the system ending with pA for which it holds that if $q \lesssim p$ and qB is an assumption in $\mathcal{D}$, then $B \in \Gamma$.

Below are two examples in this system of a proof of $(\sim A \supset \sim B) \supset (B \supset A)$ and of a deduction of $\sim\sim A \supset B$ from $A \supset \sim\sim B$:

1	$\sim A \supset \sim B$	1	$A \supset \sim\sim B$
1, 1	B	1, 1	$\sim\sim A$
1, 1, 1	$\sim A$	1, 1, 1	$\sim A$
1, 1, 1	$\sim B$	1, 1	A
1, 1	A	1, 1	$\sim\sim B$
1	$B \supset A$	1, 1, 2	$\sim B$
	$(\sim A \supset \sim B) \supset (B \supset A)$	1, 1	B
		1	$\sim\sim A \supset B$

Jaskowski's system for the sentential logic of Kolmogoroff is obtained from the system above by changing "$\sim A_i = A_j$" in clause 4) to "$A_i = \sim A_j$".

As was said above, *Jaskowski's system for predicate logic* allows proofs only of formulas valid in all domains including the empty one. A *deduction* in this system is a sequence $\mathcal{D} = p_1\alpha_1, p_2\alpha_2, \ldots, p_n\alpha_n$ such that for each $i \leq n$, α_i is a formula or a variable and either

(I) α_i is a variable and p_i is a non-empty prefix, neither of which occurs earlier in the sequence; or

(II) α_i is a formula for which one of the following clauses hold:

1) $p_i\alpha_i$ is an assumption in $\mathcal{D}$ (as defined before) and for every free variable x in α_i there is a $j < i$ such that $x = \alpha_j$ and $p_j \lesssim p_i$;

2)–4) the same as for sentential logic;

5) there are $j, k < i$ such that α_j has the form $\forall xA$, α_k is a variable y, x does not occur free in A within the scope of a quantifier with y, $\alpha_i = A^x_y$, $p_j \leq p_i$, and $p_k \leq p_i$;

6) $\alpha_i = \forall \alpha_j \alpha_k$ for some $j, k < i$ such that α_j is a variable, α_k is a formula, $p_k \leq p_j$, and p_i immediately precedes p_j.

A is *provable* in this system if A occurs without prefix as the last element of a deduction in the system.

Independently of this Polish development, *Gentzen* constructed in 1933 what he called "ein Kalkül des natürlichen Schliessens". It was published in 1934 in a paper with the title "Untersuchungen über das logische Schliessen" (Gentzen [3])[1], where he developed systems of natural deduction for classical and intuitionistic logic, essentially the ones described in Chapter I. (For some minor deviations see Remarks 1 and 2 in Chapter I, § 2)[2]

The minimal logic was introduced by *Johansson* [1] in 1937, and was stated by him in the form of a Gentzen-type system of natural deduction (as well as in the form of a calculus of sequents and in the form of a system of axiomatic type).[3]

§ 2. Variants of Gentzen-type systems

Various modifications of Gentzen's systems of natural deduction have been proposed. One variant, used in Gentzen [4], is to make

[1] A French translation of Gentzen's paper is given by Feys Ladrière [1], supplemented by comments relating Gentzen's systems to the systems of Jáskowski and Johansson among others. Accounts of Gentzen's systems can also be found in Feys [1] and [2] and in Curry [1].

[2] Gentzen's description is more like the one in § 4 of Chapter I. A deduction is, according to him, to be supplemented by some marks that indicate the places at which the assumptions are discharged. (But these marks are not theoretically necessary. As the ∃E-rule is stated by Gentzen, however, it allows the discharge of assumptions of the same shape only, and one can then not, without such marks, always decide uniquely what assumptions the end-formula in a deduction depends on.)

[3] However, the part of minimal logic containg no logical constants besides ⊃ and ~ coincides with the sentential logic of Kolmogoroff and was already developed in the form of a system of natural deduction by Jákowski as stated above.