GOTTLOB FREGE

THE BASIC LAWS OF ARITHMETIC

Exposition of the System

Translated and Edited,
with an Introduction,
by MONTGOMERY FURTH

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INTRODUCTION

v In this book there are to be found theorems upon which arithmetic is based, proved by the use of symbols, which collectively I call Begriffsschrift. The most important of these propositions are collected at the end, some with translation appended. It will be seen that negative, fractional, irrational, and complex numbers have still been left out of account, as have addition, multiplication, and so on. Even the propositions concerning Numbers are still not present with the completeness originally planned; in particular, the proposition is still lacking that the Number of objects falling under a concept is finite if the Number of objects that fall under a superordinated concept is finite. External circumstances have caused me to reserve this, as well as the treatment of other numbers and of the arithmetical operations, for a later installment whose appearance will depend upon the reception accorded this first volume. What I have offered here may be sufficient to give an idea of my method. It might be thought that the propositions concerning the Number 'Infinite'[1] could have been omitted; to be sure, they are not necessary for the foundation of arithmetic in its traditional compass, but their derivation is for the most part simpler than that of the corresponding propositions for finite Numbers and can serve as preparation for it. Other propositions occur which do not treat of Numbers, but which are needed for the proofs; they treat, for example, of following in a series, of the many-oneness of relations, of relative products and of 'coupled' relations, of mapping by means of relations, and the like. Perhaps one might assign these propositions to an enlarged theory of combination.

The proofs are entirely contained in the paragraphs entitled "Construction", whereas the paragraphs entitled "Analysis" are meant to facilitate understanding by providing rough preliminary *outlines* of the proofs that follow them. The proofs

[1] Number of a denumerably infinite set.

themselves contain no words but are carried out entirely in my symbols; they appear as sequences of formulas separated by solid or broken lines or other signs. Each of these formulas is vi
a complete proposition including all of the conditions necessary to its validity. This completeness, not permitting the tacit attachment of presuppositions in thought, seems to me indispensable for the rigor of the conduct of proof.

The advance from one proposition to the next takes place according to the Rules summarized in §48, and no transition occurs that is not in accordance with these Rules. How, and by what Rule, an inference is made is indicated by the sign between the formulas, while the sign ——•—— terminates a chain of inference. Here there have to be propositions that are not derived from other propositions; such are in part the Basic Laws summarized in §47, and in part the Definitions collected in a table at the end of the volume with indication of the points at which they first occur. The need of definitions never ceases to be apparent in any attempt of this sort. The principles that must govern the giving of definitions are set out in §33. The definitions do not really create anything, and in my opinion may not do so; they merely introduce abbreviated notations (names), which could be dispensed with were it not that lengthiness would then make for insuperable external difficulties.

The ideal of a strictly scientific method in mathematics, which I have here attempted to realize, and which might indeed be named after Euclid, I should like to describe as follows. It cannot be demanded that everything be proved, because that is impossible; but we can require that all propositions used without proof be expressly declared as such, so that we can see distinctly what the whole structure rests upon. After that we must try to diminish the number of these primitive laws as far as possible, by proving everything that can be proved. Furthermore, I demand—and in this I go beyond Euclid—that all methods of inference employed be specified in advance; otherwise we cannot be certain of satisfying the first requirement. This ideal I believe I have now essentially attained. Only in a few points could one be any more exacting. So as to secure more flexibility and avoid extravagant length I have allowed myself to make tacit use of the interchangeability of subcomponents (conditions) and of the *possibility* of amalgamating identical subcomponents, and have not reduced the methods of inference to the smallest possible number. Readers of my *Begriffsschrift* will be able to gather from it the way in which here too the strictest requirements could be satisfied, but likewise, that this would entail a considerable increase in volume.

Apart from this, I believe, the only criticisms that can justly
vii be made against this book concern not the rigor but merely the choice of the course of proof and of intermediate steps. Frequently several routes for a proof are open; I have not tried to travel them all, and thus it is possible—even probable—that I have not invariably chosen the shortest. Let him who finds fault in this respect do better himself. Other matters will be disputable. Some might perhaps have preferred me to draw the limits of the permissible methods of inference more widely, thus attaining greater flexibility and brevity. But we must call a halt at some point, if the ideal which I have set is approved of at all—and wherever we do so, people can always say: it would have been better if more methods of inference had been permitted.

Because there are no gaps in the chains of inference, every 'axiom', every 'assumption', 'hypothesis', or whatever you wish to call it, upon which a proof is based is brought to light; and in this way we gain a basis upon which to judge the epistemological nature of the law that is proved. Of course the pronouncement is often made that arithmetic is merely a more highly developed logic; yet that remains disputable so long as transitions occur in the proofs that are not made according to acknowledged laws of logic, but seem rather to be based upon something known by intuition. Only if these transitions are split up into logically simple steps can we be persuaded that the root of the matter is logic alone. I have drawn together everything that can facilitate a judgment as to whether the chains of inference are cohesive and the buttresses solid. If anyone should find anything defective, he must be able to state precisely where, according to him, the error lies: in the Basic Laws, in the Definitions, in the Rules, or in the application of the Rules at a definite point. If we find everything in order, then we have accurate knowledge of the grounds upon which each individual theorem is based. A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values

(V), which logicians perhaps have not yet expressly enunciated, and yet is what people have in mind, for example, where they speak of the extensions of concepts. I hold that it is a law of pure logic. In any event the place is pointed out where the decision must be made.

My purpose necessitates many departures from what is customary in mathematics. The requirements upon the rigor of proof inevitably entail greater length; anyone not bearing this in mind will indeed be surprised at how laboriously a proposition is often proved here that he believes he can grasp in one single act of understanding. This will strike us particularly if we compare Herr Dedekind's work *Was sind und was sollen die Zahlen?*, the most thoroughgoing work on the foundations of arithmetic that has lately come to my notice. In much less space it pur- viii
sues the laws of arithmetic much farther than is done here. To be sure, this brevity is attained only because a great deal is really not proved at all. Frequently Herr Dedekind merely says that the proof follows from such and such propositions; he makes use of dots, as in the expression "$\mathfrak{M}(A, B, C, \cdots)$"; an inventory of the logical or other laws taken by him as basic is nowhere to be found, and even if it were, there would be no way of telling whether no others were actually used; for that to be possible the proofs would have to be not merely indicated but carried out, without gaps. Herr Dedekind, like myself, is of the opinion that the theory of numbers is a part of logic; but his work hardly contributes to its confirmation, because the expressions "system" and "a thing belongs to a thing", which he uses, are not usual in logic and are not reduced to acknowledged logical notions. I do not say this as a reproach, for his procedure may have been the most appropriate for his purpose; I say it only to set my intention in a clearer light by contrast. The length of a proof ought not to be measured by the yard. It is easy to make a proof look short on paper by skipping over many intermediate links in the chain of inference and merely indicating large parts of it. Generally people are satisfied if every step in the proof is evidently correct, and this is permissible if one merely wishes to be persuaded that the proposition to be proved is true. But if it is a matter of gaining an insight into the nature of this 'being evident', this procedure does not suffice; we must put down all of the intermediate steps, that the

full light of consciousness may fall upon them. Mathematicians generally are indeed only concerned with the content of a proposition and with the fact that it is to be proved. What is new in this book is not the content of the proposition, but the way in which the proof is carried out and the foundations on which it rests. That this essentially different viewpoint calls for a different method of treatment should not surprise us. If one of our propositions is derived in the customary way some proposition will easily be overlooked which does not seem necessary to the proof; yet careful pondering of my proof will, I believe, show the proposition to be indispensable unless some quite different route is taken. Then again, in one or another of our propositions conditions may be found that first strike one as unnecessary, but that turn out either to be necessary after all, or to be dispensable only by means of a proposition that must be especially proved for the purpose.

With this book I carry out a design that I had in view as early as my *Begriffsschrift* of 1879 and announced in my *Grundlagen der Arithmetik* of 1884[2]. I wish here to substantiate in actual
ix practice the view of Number that I expounded in the latter book. The most fundamental of my results I expressed there, in §46, by saying that a statement of number expresses an assertion about a concept; and the present account rests upon this. If anyone is of another view, let him try, using symbols, to base upon his view an account both consistent and practicable, and he will see that it does not work. In ordinary language, to be sure, the situation is not so transparent; but if we attend with sufficient care we find that here, too, in a statement of number there is invariably mention of a concept—not a group, or an aggregate, or the like—or that, if a group or aggregate is mentioned, it is invariably determined by a concept, that is, by the properties an object must have in order to belong to the group; while what makes the group into a group, or a system into a system—the relations of the members to one another—is for the Number wholly irrelevant.

One reason why the execution appears so long after the announcement is to be found in internal changes in my Begriffsschrift, which forced me to discard an almost completed

[2]Cf. the Introduction and §§90 and 91 of my *Grundlagen der Arithmetik*, Breslau, Verlag Wilhelm Koebner, 1884.

manuscript. These improvements may be mentioned here briefly. The primitive signs used in *Begriffsschrift* occur here also, with one exception. Instead of the three parallel lines I have adopted the ordinary sign of equality, since I have persuaded myself that it has in arithmetic precisely the meaning that I wish to symbolize. That is, I use the word "equal"* to mean the same as "coinciding with" or "identical with"; and the sign of equality is actually used in arithmetic in this way. The opposition that may arise against this will very likely rest on an inadequate distinction between sign and thing signified. Of course in the equation "$2^2 = 2+2$" the sign on the left is different from that on the right; but both designate or denote the same number[3]. To the old primitive signs two more have now been added: the smooth breathing, for the notation for the course-of-values of a function, and a sign meant to do the work of the definite article of everyday language. The introduction of the courses-of-values of functions is a vital advance, thanks to which we gain far greater flexibility. The former derivative signs can now be replaced by other, simpler ones, although the definitions of the many-oneness of a relation, of following in a series, and of a mapping, are essentially the same as those which I gave in part in *Begriffsschrift* and in part in *Grundlagen der Arithmetik.* But the courses-of-values are also extremely x
important in principle; in fact, I define Number itself as the extension of a concept, and extensions of concepts are by my definitions courses-of-values. Thus we just cannot get on without them. The old signs that appear here outwardly unchanged, and whose algorithm has also hardly changed, are nonetheless provided with different explanations. The former 'content-stroke' reappears as the 'horizontal'. These are consequences of a thoroughgoing development of my logical views. Formerly I distinguished two components in that whose external form is a declarative sentence: (1) the acknowledgment of truth, (2) the content that is acknowledged to be true. The content I called a 'possible content of judgment'. This last has now split for

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**gleich.*

[3] I also say: the sense of the sign on the right side is different from that of the sign on the left; but the denotation is the same. Cf. my essay on sense and denotation in the *Zeitschrift für Philosophie und philosophische Kritik*, vol. 100, p. 25.

me into what I call 'thought' and 'truth-value', as a consequence of distinguishing between sense and denotation of a sign. In this case the sense of a sentence is a thought, and its denotation a truth-value. Over and above this is the acknowledgment that the truth-value is the True. That is, I distinguish two truth-values: the True and the False. I have justified this more thoroughly in my essay on sense and denotation, mentioned above; here it may merely be mentioned that only in this way can indirect discourse be correctly understood. That is, the thought, which otherwise is the sense of a sentence, in indirect discourse becomes its denotation. How much simpler and sharper everything becomes by the introduction of truth-values, only detailed acquaintance with this book can show. These advantages alone put a great weight in the balance in favor of my own conception, which indeed may seem strange at first sight. Also the nature of the function, as distinguished from the object, is characterized more sharply here than in *Begriffsschrift*. From this results further the distinction between first- and second-level functions. As I have explained in my lecture *Function und Begriff*[4], concepts and relations are 'functions' in my extended meaning of the term; and so we have to distinguish first- and second-level concepts, equal-leveled and unequal-leveled relations.

It will be seen that the years have not passed in vain since the appearance of my *Begriffsschrift* and *Grundlagen*: they have brought the work to maturity. But just that which I recognize as a vital advance stands, as I cannot conceal from myself, as a great obstacle in the way of the dissemination and the effectiveness of my book. And that which is not its least value, the rigorous avoidance of gaps in the chains of inference, will, I fear, win it little thanks. I have moved farther away from the
xi accepted conceptions, and have thereby stamped my views with an impress of paradox. An expression cropping up here or there, as one leafs through these pages, may easily appear strange and create prejudice. I myself can estimate to some extent the resistance with which my innovations will be met, because I had first to overcome something similar in myself in order to make them. For I have not arrived at them haphazardly or out of a craving for novelty, but was driven by the nature of the

[4]Jena, Verlag Hermann Pohle, 1891.

case.

With this I arrive at the second reason for my delay: the discouragement that overcame me at times because of the cool reception—or more accurately the lack of reception—accorded by mathematicians to the writings of mine that I have mentioned[5], and because of the unpropitious currents in scientific thought against which my book will have to struggle. Even the first impression must frighten people off: unfamiliar signs, pages of nothing but alien-looking formulas. And so at times I turned to other subjects. But I could not keep the results of my thinking, which seemed valuable to me myself, locked up in my desk for long, and the labor already expended kept requiring new labor so as not to be in vain. So the subject did not let me go. In a case of this kind, where the value of a book cannot be recognized by a quick reading, criticism ought to step in with assistance. But criticism is in general too poorly remunerated. A critic can never hope to be repaid in cash for the toil that a deep study of this book sets before him. My only remaining hope is that someone may have enough confidence in the matter beforehand to expect in the intellectual profit a sufficient reward, and that he will make public the outcome of his careful examination. Not that only a laudatory review could satisfy me; on the contrary, I should far prefer an attack that is thoroughly well-informed than a commendation in general terms not touching the root of the matter. I should like to facilitate the work of the reader who approaches the book with such purposes by means of a few hints.

So as to gain at the outset a rough idea of the way in which I express thoughts with my signs, it will be useful to consider more closely some of the simpler theorems in the Table of More Important Theorems to which a translation is appended. One will then be able to conjecture what other theorems, similar to these but not followed by translation, are intended to assert. After that one may begin with paragraph §0 and set a b o u t mastering the Exposition of the Begriffsschrift. But my advice is to begin by acquainting oneself with it rapidly and not to xii

[5]In vain do we seek [notice of] my *Grundlagen der Arithmetik* in the *Jahrbuch über die Fortschritte der Mathematik*. Researchers in the same area, Dedekind, Otto Stolz, von Helmholtz, seem not to be aware of my works. And Kronecker fails to mention them in his essay on the concept of number.

delay too long over the discussions of details. Some matters had to be taken up in order to be able to meet all objections, but are nevertheless inessential to an understanding of the propositions of Begriffsschrift. I count among these the second half of §8, beginning on p. 42 with the words "If we now set up"; also the second half of §9, beginning on p. 44 with the words "If I say generally"; and the whole of §10. These portions may be skipped entirely on a first reading. The same holds for §§26 and 28 through 31. On the other hand I should like to stress as especially important for comprehension the first half of §8, and also §§12 and 13. A more exact reading may begin with §34 and continue to the end; one will have to return occasionally to the §§ previously only skimmed, and this will be made easier by the index at the end and the Table of Contents. The derivations in §§49 to 52 can serve as preparation for understanding the proofs themselves. All of the methods of inference and practically all of the applications of our Basic Laws already occur there. After one has reached the end in this way, he may reread the Exposition of the Begriffsschrift as a connected whole, keeping in mind that the stipulations that are not made use of later and hence seem superfluous serve to carry out the basic principle that every correctly-formed name is to denote something, a principle that is essential for full rigor. In this way, I believe, the suspicion that may at first be aroused by my innovations will gradually be dispelled. The reader will recognize that my basic principles at no point lead to consequences that he is not himself forced to acknowledge as correct. Perhaps then he will also grant that at the outset he overrated the labor involved, that my gapless procedure even facilitates understanding, once the obstacle of the novelty of the signs is overcome. May I be so fortunate as to find such a reader and judge! for a notice based on superficial perusal can easily do more harm than good.

Otherwise the prospects of my book are of course slight. In any event I must relinquish as readers all those mathematicians who, if they bump into logical expressions such as "concept", "relation", "judgment", think: *metaphysica sunt, non leguntur,* and likewise those philosophers who at the sight of a formula cry: *mathematica sunt, non leguntur*; and the number of such persons is surely not small. Perhaps the number of

mathematicians who trouble themselves over the foundation of their science is not great, and even these frequently seem to be in a great hurry until they have got the fundamental principles behind them. And I scarcely dare hope that my reasons for painstaking rigor and its inevitable lengthiness will persuade many of
them; what once becomes established indeed has great power xiii
over men's minds. If I compare arithmetic with a tree that unfolds upward into a multitude of techniques and theorems while its root drives into the depths, then it seems to me that the impetus of the root, at least in Germany, is rather weak. Even in a work that we should like to count as tending in the latter direction, the *Algebra der Logik* of E. Schröder, the top-growth quickly gains the upper hand even before any great depth has been reached, the effect being a bending upward and an unfolding into techniques and theorems.

Also unpropitious for my book is the widespread inclination to acknowledge as existing only what can be perceived by the senses. That which cannot, people try to deny or else to ignore. Now the objects of arithmetic, i.e., numbers, cannot be perceived by the senses. How do we come to terms with them? Simplicity itself! We pronounce the numerical signs to be the numbers. Then in the signs we have something visible, and that is naturally the chief thing. Of course the signs have totally different properties from the numbers themselves, but what does that matter? We simply invest them with the properties we wish by so-called 'definitions'. Of course it is a puzzle how there can be a definition where no question is raised about the connection between sign and thing signified. So far as possible we knead sign and thing signified indistinguishably together; and then we can make assertions of existence on the basis of tangibility[6], or then again bring to the fore the true properties of numbers, as the occasion requires. Sometimes, it seems, the numerical signs are regarded as chess pieces and the so-called 'definitions' as rules of the game. The sign then does not designate anything: it is the subject matter itself. To be sure, in all this one trifling detail is overlooked: namely, that with

[6] Cf. E. Heine, *Die Elemente der Functionslehre*, in Crelle's *Journal*, vol. 74, p. 173: "As for definition, I adopt the purely formalistic standpoint; what I call numbers are certain tangible signs, so that the existence of these numbers is thus unquestionable."

"$3^2 + 4^2 = 5^2$" we express a thought, whereas a configuration of chess pieces asserts nothing. Where people are satisfied with such superficialities, of course there is no basis for any deeper understanding.

It is important that we make clear at this point what definition is and what can be attained by means of it. It seems frequently to be credited with a creative power; but all it accomplishes is that something is marked out in sharp relief and designated by a name. Just as the geographer does not create a sea when he draws boundary lines and says: the part of the ocean's surface bounded by these lines I am going to call the Yellow Sea, so too the mathematician cannot really create anything by his defining. Nor can one by pure definition magically conjure into a thing a property that in fact it does not possess—save that of now being called by the name with which one has named it.
xiv But that an oval figure produced on paper with ink should by a definition acquire the property of yielding one when added to one, I can only regard as a scientific superstition. One could just as well by a pure definition make a lazy pupil diligent. It is easy for unclarity to arise here if we do not distinguish sufficiently between concept and object. If we say: 'a square is a rectangle in which the adjacent sides are equal', we define the concept *square* by specifying what properties a thing must have in order to fall under this concept. These properties I call 'characteristic marks' of the concept. But these characteristic marks of a concept, properly understood, are not the same as its properties. The concept *square* is not a rectangle; only such objects as may fall under this concept are rectangles, just as the concept *black cloth* is neither black nor a cloth. Whether there are any such objects is not known immediately from the definition. Now suppose one defines, for instance, the number zero, by saying: it is something which yields one when added to one. In so doing one has defined a concept, by specifying what property an object must have in order to fall under the concept. But this property is not a property of the concept defined. People frequently seem to fancy that by the definition something has been created that yields one when added to one. A great delusion! The concept defined does not possess this property, nor is the definition any guarantee that the concept is realized—a matter requiring separate investigation. Only when

we have proved that there exists at least and at most one object with the required property are we in a position to invest this object with the proper name "zero". To create zero is consequently impossible. I have already repeatedly explained these things, but apparently without effect[7].

From the prevailing logic, too, I cannot hope for any understanding of my distinction between a characteristic mark of a concept and a property of an object[8]; for it seems to be infected through and through with psychology. If people consider, instead of things themselves, only their subjective *simulacra*, their ideas of them, then naturally all the more delicate distinctions within the subject matter are lost, and others appear in their place that are logically completely worthless. And this brings me to what stands in the way of the influence of my book among logicians: namely, the corrupting incursion of psychology into logic. Our conception of the laws of logic is necessarily decisive for our treatment of the science of logic, and that conception in turn is connected with our understanding of the word xv "true". It will be granted by all at the outset that the laws of logic ought to be guiding principles for thought in the attainment of truth, yet this is only too easily forgotten, and here what is fatal is the double meaning of the word "law". In one sense a law asserts what is; in the other it prescribes what ought to be. Only in the latter sense can the laws of logic be called 'laws of thought': so far as they stipulate the way in which one ought to think. Any law asserting what is, can be conceived as prescribing that one ought to think in conformity with it, and is thus in that sense a law of thought. This holds for laws of geometry and physics no less than for laws of logic. The latter have a special title to the name "laws of thought" only if we mean to assert that they are the most general laws, which prescribe universally the way in which one ought to think if one is to think at all. But the expression "law of thought" seduces us into supposing that these laws govern thinking in the same way as laws of nature govern events in the external world. In that case they can be nothing but laws of psychology:

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[7]Mathematicians reluctant to venture into the labyrinths of philosophy are requested to leave off reading the Introduction at this point.

[8]In the *Logik* of Herr B. Erdmann I find no trace of this important distinction.

for thinking is a mental process. And if logic were concerned with these psychological laws it would be a part of psychology; it is in fact viewed in just this way. These laws of thought can in that case be regarded as guiding principles in the sense that they give an average, like statements about 'how it is that good digestion occurs in man', or 'how one speaks grammatically', or 'how one dresses fashionably'. Then one can only say: men's taking something to be true conforms on the average to these laws, at present and relative to our knowledge of men; thus if one wishes to correspond with the average one will conform to these. But just as what is fashionable in dress at the moment will shortly be fashionable no longer and among the Chinese is not fashionable now, so these psychological laws of thought can be laid down only with restrictions on their authority. Of course—if logic has to do with something's being taken to be true, rather than with its being true! And these are what the psychological logicians confuse. Thus Herr B. Erdmann in the first volume of his *Logik*[9] (pp. 272-275) equates truth with 'general validity', and bases this upon 'general certainty regarding the object of judgment', and bases this in turn upon 'general agreement among the subjects who judge'. Thus in the end truth is reduced to individuals' taking something to be true. All I have to say to this is: being true is different from being taken to be true, whether by one or many or everybody, and in no case is to be reduced to it. There is no contradiction in something's
xvi being true which everybody takes to be false. I understand by 'laws of logic' not psychological laws of takings-to-be-true, but laws of truth. If it is true that I am writing this in my chamber on the 13th of July, 1893, while the wind howls out-of-doors, then it remains true even if all men should subsequently take it to be false. If being true is thus independent of being acknowledged by somebody or other, then the laws of truth are not psychological laws: they are boundary stones set in an eternal foundation, which our thought can overflow, but never displace. It is because of this that they have authority for our thought if it would attain to truth. They do not bear the relation to thought that the laws of grammar bear to language; they do not make explicit the nature of our human thinking and change as it changes. Of course, Herr Erdmann's conception of the laws of logic is

[9] Halle a. S., Max Niemeyer, 1892.

quite different. He doubts their unconditional and eternal validity and would restrict them to our thought as it is now (pp. 375 ff.). "Our thought" surely can only mean the thought of human beings up to the present. Accordingly the possibility remains of men or other beings being discovered who were capable of bringing off judgments contradicting our laws of logic. If this were to happen? Herr Erdmann would say: here we see that these principles do not hold generally. Certainly!—if these are psychological laws, their verbal expression must single out the family of beings whose thought is empirically governed by them. I should say: thus there exist beings that recognize certain truths not as we do, immediately, but perhaps led by some lengthier route of induction. But what if beings were even found whose laws of thought flatly contradicted ours and therefore frequently led to contrary results even in practice? The psychological logician could only acknowledge the fact and say simply: those laws hold for them, these laws hold for us. I should say: we have here a hitherto unknown type of madness. Anyone who understands laws of logic to be laws that prescribe the way in which one ought to think—to be laws of truth, and not natural laws of human beings' taking a thing to be true—will ask, who is right? Whose laws of taking-to-be-true are in accord with the laws of truth? The psychological logician cannot ask this question; if he did he would be recognizing laws of truth that were not laws of psychology. One could scarcely falsify the sense of the word "true" more mischievously than by including in it a reference to the subjects who judge. Someone will now no doubt object that the sentence "I am hungry" can be true for one person and false for another. The sentence, certainly—but not the thought; for the word "I" in the mouth of the other person denotes a different man, and hence the sentence xvii
uttered by the other person expresses a different thought. All determinations of the place, the time, and the like, belong to the thought whose truth is in point; its truth itself is independent of place or time. How, then, is the Principle of Identity really to be read? Like this, for instance: "It is impossible for people in the year 1893 to acknowledge an object as being different from itself"? Or like this: "Every object is identical with itself"? The former law concerns human beings and contains a temporal reference; in the latter there is no talk either

of human beings or of time. The latter is a law of truth, the former a law of people's taking-to-be-true. The content of the two is wholly different and they are independent of one another; neither can be inferred from the other. Hence it is extremely confusing to designate both by the same name, "Principle of Identity". These mixings-together of wholly different things are to blame for the frightful unclarity that we encounter among the psychological logicians.

The question why and with what right we acknowledge a law of logic to be true, logic can answer only by reducing it to another law of logic. Where that is not possible, logic can give no answer. If we step away from logic, we may say: we are compelled to make judgments by our own nature and by external circumstances; and if we do so, we cannot reject this law—of Identity, for example; we must acknowledge it unless we wish to reduce our thought to confusion and finally renounce all judgment whatever. I shall neither dispute nor support this view; I shall merely remark that what we have here is not a logical consequence. What is given is not a reason for something's being true, but for our taking it to be true. Not only that: this impossibility of our rejecting the law in question hinders us not at all in supposing beings who do reject it; where it hinders us is in supposing that these beings are right in so doing, it hinders us in having doubts whether we or they are right. At least this is true of myself. If other persons presume to acknowledge and doubt a law in the same breath, it seems to me an attempt to jump out of one's own skin against which I can do no more than urgently warn them. Anyone who has once acknowledged a law of truth has by the same token acknowledged a law that prescribes the way in which one ought to judge, no matter where, or when, or by whom the judgment is made.

Surveying the whole question, it seems to me that the source of the dispute lies in a difference in our conceptions of what is true. For me, what is true is something objective and independent of the judging subject; for psychological logicians it
xviii is not. What Herr B. Erdmann calls 'objective certainty' is merely a general acknowledgment on the part of the subjects who judge, which is thus not independent of them but susceptible to alteration with the constitution of their minds.

We can generalize this still further: for me there is a domain of what is objective, which is distinct from that of what is actual, whereas the psychological logicians without ado take what is not actual to be subjective. And yet it is quite impossible to understand why something that has a status independent of the judging subject has to be actual, i.e., has to be capable of acting directly or indirectly on the senses*. No such connection is to be found between the concepts [of being objective and being actual], and we can even adduce examples pointing in the opposite direction. The number one, for instance, is not easily taken to be actual, unless we are disciples of John Stuart Mill. On the other hand, it is impossible to ascribe to every person his own number one; for in that case we should first have to investigate the extent to which the properties of these ones agreed, and if one person said "one times one is one" and the next said "one times one is two", we could only register the difference and say: your one has one property, mine has another. There could be no question of any argument as to who was right, or of any attempt to correct anyone; for they would not be speaking of the same object. Obviously this is totally contrary to the sense of the word "one" and the sense of the sentence "one times one is one". Since the number one, being the same for everyone, stands apart from everyone in the same way**, it can no more be researched by making psychological observations than can the moon. Whatever ideas there may be of the number one in individual souls, they are still to be as carefully distinguished from the number one, as ideas of the moon are to be distinguished from the moon itself. Because the psychological logicians fail to recognize the possibility of there being something objective that is not actual, they take concepts to be ideas and thereby consign them to psychology. But the true state of affairs makes itself felt too forcibly for this to be easily carried through. From this there stems an equivocation on the word "idea": at some times it seems to mean something that belongs to the mental life of an individual and that merges with other ideas with which it is associated, according to the laws of psychology; at other times it seems to mean something standing apart from everyone in the same way, where a possessor of the

*The translation reproduces a play upon the expressions "*wirklich*" and "*auf die Sinne wirken*".

**It is not possible to reproduce in English the implied play upon the expressions "*Gegenstand*" and "*gegenüberstehen*".

'idea' is neither mentioned nor even tacitly presupposed. These two uses of the word cannot be reconciled; for those associations and mergings occur only within the individual mind whose ideas are involved, and have only to do with something belonging to that mind as idiosyncratically as its pleasure or its pain. We must never forget that different people's ideas, however similar they may be (which, incidentally, we cannot accurately determine), nevertheless do not coincide but have to be distinguished. Every man has his own ideas, which are not those of any other. Here of course I am understanding "idea" in the
xix psychological sense. The equivocation on this word obscures the issue and helps the psychological logicians to conceal their weakness. When will a stop be put to this? In the end everything is drawn into the sphere of psychology; the boundary that separates objective and subjective fades away more and more, and even actual objects themselves are treated psychologically, as ideas. For what else is *actual* but a predicate? and what else are logical predicates but ideas? Thus everything drifts into idealism and from that point with perfect consistency into solipsism. If every man designated something different by the name "moon", namely one of his own ideas, much as he expresses his own pain by the cry "Ouch", then of course the psychological point of view would be justified; but an argument about the properties of the moon would be pointless: one person could perfectly well assert of his moon the opposite of what the other person, with equal right, said of his. If we could not grasp anything but what was within our own selves, then a conflict of opinions [based on] a mutual understanding would be impossible, because a common ground would be lacking, and no idea in the psychological sense can afford us such a ground. There would be no logic to be appointed arbiter in the conflict of opinions.

However, so as to dispel any notion that I am tilting at windmills, I shall demonstrate this helpless foundering in idealism in the case of a definite book. I choose for this the *Logik* of Herr B. Erdmann already mentioned, as one of the most recent works of the psychological school and not likely to be denied all importance. First let us look at the following sentence:

> "Thus psychology teaches with certainty that the objects of memory and imagination, as well as those of morbid hallucinatory and delusive ideation, are ideal in their

> nature.... Ideal as well is the whole realm of mathematical ideas properly so called, from the number-series down to the objects of mechanics (Vol. I, p. 85)."

What an assemblage! The number ten shall thus stand on a level with hallucinations! Obviously what is objective and not actual is being mixed up here with what is subjective. Some objective things are actual, others are not. *Actual* is merely one predicate out of many and has no more special relevance to logic than, e.g., the predicate *algebraic* as asserted of a curve. Of course by this confusion Herr Erdmann becomes entangled in metaphysics, however much he struggles to keep clear of it. I take it as a sure sign of a mistake if logic has need of metaphysics and psychology—sciences that require their own logical first principles. In this case then, where is the ultimate basis upon which everything rests? Or is it like Münchhausen, who pulled himself out of the bog by his own hair? I am pessimistic about that possibility and conjecture that Herr Erdmann is stuck xx
in his psychologico-metaphysical bog.

For Herr Erdmann there is no real objectivity; everything is idea. We may be convinced of this by his own statements; thus we read,

> "As a relation between what is ideated, the judgment presupposes at least two points of reference between which the relation holds. As an *assertion* about what is ideated it requires that one of these points of reference be determined as the object about which something is asserted, as the subject..., and the second as the object that is asserted, as the predicate... (Vol. I, p. 187)."

We see here first of all that both the subject about which something is asserted and the predicate are marked as 'objects' or 'things ideated'. Instead of "object" he could just as well have said "thing ideated"; for we read,

> "For objects are things ideated (Vol. I, p. 81)."

But conversely too, everything ideated is an object:

> "According to its origin, the ideated divides on the one hand into objects of sense-perception and of self-consciousness, and on the other hand into original and derived (Vol. I, p. 38)."

What arises from sense-perception and from self-consciousness is certainly mental in nature. The objects, the ideated, and hence subject and predicate as well, are thereby assigned to psychology. This is confirmed by the following passage:

> "In general it is either the ideated or the idea, for the two are one and the same: what is ideated is the idea, and the idea is what is ideated (Vol. I, pp. 147-148)."

The word "idea" nowadays is as a rule taken in the psychological sense; that this is Herr Erdmann's use as well, we see from the following passages:

> "Consciousness accordingly is the general case of feeling, ideating, willing (Vol. I, p. 35);"
>
> "Ideating is a compound of the *ideas* [in which objects are given us] and *passages of ideas* [by which these ideas are remembered, combined, or predicatively analyzed according to their associative relations] (Vol. I, p. 36)."*

After this, we ought not to be surprised that an object comes into existence in a psychological way:

> "To the extent that a mass of perceptions [e.g., the noise of a passing carriage], presents the same thing as have previous stimuli and the excitations triggered by them, it *reproduces* the memory-traces that stem from the sameness of previous stimuli and *amalgamates* with them to form the object of the apperceived idea (Vol. I, p. 42)."

Then on p. 43 it is shown as an example how a steel engraving of Raphael's Sistine Madonna comes into existence in a purely psychological way without steel plate, ink, press, or
xxi paper! After all this there cannot be any doubt that the object about which an assertion is made, the subject, is according to Herr Erdmann's opinion supposed to be an idea in the psychological sense of the word, and likewise for the predicate, the object that is asserted. If this were right it could not be truly asserted of any subject that it was green; for there are no green ideas. Nor could I assert of any subject that it was independent of being ideated, or of myself, who ideates it, any more than my decisions are independent of my willing, or of myself who wills; to the contrary, if I were destroyed they would be destroyed along with me. Thus there is no real objectivity for Herr Erdmann, as follows also from his representing what is ideated or the idea in general, the object in the most general

*In this and the following two extracts I have translated some phrases from Erdmann omitted by Frege in citing him.

sense of the word, as *summum genus* (γενικώτατον) (p. 147). And so he is an idealist. If the idealists were consistent, they would put down the sentence "Charlemagne conquered the Saxons" as neither true nor false, but as fiction, just as we are accustomed to regard, for example, the sentence "Nessus carried Deianeira across the river Evenus"; for even the sentence "Nessus did not carry Deianeira across the river Evenus" could be true only if the name "Nessus" had a bearer. It would not be easy to dislodge the idealists from this point of view. But we do not need to put up with their falsification of the sense of the sentence, as if I meant to assert something about my idea when I speak of Charlemagne; I simply mean to designate a man, independent of me and my ideating, and to assert something about him. We may grant the idealists that the attainment of this intention is not completely sure and that, without wishing to, I may perhaps lapse from truth into fiction; but this can change nothing in the sense. With the sentence "This blade of grass is green" I assert nothing about my idea; I am not designating any of my ideas with the words "this blade of grass", and if I were, then the sentence would be false. Here there enters a second falsification, namely that my idea of green is asserted of my idea of this blade of grass. I repeat: in this sentence there is no talk whatever of my ideas; it is the idealists who foist that sense upon us. By the way, I entirely fail to understand how an idea can be asserted of anything. It would be just as much a falsification to say that in the sentence "the moon is independent of myself and my ideating", my idea of independence-of-myself-and-my-ideating was asserted of my idea of the moon; this would be to surrender all objectivity in the proper sense of the word and push something wholly different into its place. Certainly it is possible that in making a judgment such a play of ideas occurs; but that is not xxii
the sense of the sentence. It may also be observed that with the same sentence and the same sense of that sentence the play of ideas can be wholly different. And it is this logically irrelevant accompanying phenomenon that our logicians take for the proper object of their study.

As may be seen, the nature of the situation opposes this foundering in idealism, and Herr Erdmann would not like to admit that for him there is no real objectivity; but equally plain is the futility of trying to deny it. For if all subjects and all predicates are ideas and if all thinking is nothing but the creating, connecting, and altering of ideas, then it is impossible to understand how anything objective is ever to be reached. A symptom of this futile struggle is the use of the words "thing ideated" and "object", which at first sight seem intended to designate something objective, as opposed to an idea, though as it turns out they only *seem* so intended and in fact denote that very thing. Well then, why this superfluity of expressions? That is not hard to unriddle. We observe that what is being spoken of is an object of the idea, although the object is supposed to be itself an idea. Thus it would be an idea of an idea. What relation between ideas are we supposed to be designating here? Obscure as this is, it is also readily understandable how, from the opposed workings of the nature of the situation and idealism, such giddy whirlpools can arise. Everywhere here we see the object of which I make an idea for myself being mixed up with this idea, and then the difference between them moving into prominence again. We recognize this conflict also in the following sentence:

> "For an idea whose object is general is on that account, as such, as a conscious event, no more general itself than an idea is real because its object is posited as real, or than an object that we perceive as sweet, [brown, warm, triangular, distant], is given by ideas that are themselves sweet, [brown, warm, triangular, distant] (Vol. I, p. 86)."

Here the true situation is forcibly making itself felt. I could almost agree with it. But if we observe that on Erdmann's principles the object of an idea and the object that is given by ideas are themselves ideas, then we see that all struggle is in vain. I request too that the words "as such" not be forgotten; they also occur similarly in this passage:

> "Where actuality is asserted of an object, the real subject of this judgment is not the object or the thing ideated as such, but rather the *Transcendent*, which is presupposed as the ground of being of this thing that is ideated and represented in it. The Transcendent is not to be taken
> xxiii here as the unknowable . . . rather its transcendence is to consist only in its independence of being ideated (Vol. I, p. 83)."

Another futile attempt to work himself out of the bog! If we take the words seriously, what is said is that in this case the subject is not an idea. But if this is possible then it is not clear why for other predicates, which indicate particular modes of activity or actuality, the real subject always has to be an idea, e.g., in the judgment "the earth is magnetic". Thus we should arrive at the view that only in a few judgments was the real subject an idea. But if it is once admitted that it is not essential for either the subject or the predicate to be an idea, then the ground is pulled from beneath the feet of the whole psychological logic. All psychological considerations, with which our logic-books of today are swollen, then prove to be irrelevant.

But probably we ought not to take Transcendence in Herr Erdmann's case so seriously. I need only remind him of his own pronouncement:

> "... the *metaphysical* limit of our ideation, the Transcendent, is also subordinate to the *summum genus* (Vol. I, p. 148),"

and with that he founders; for this *summum genus* (γενικώτατον) is according to him just the ideated or the idea in general. Or was the sense of the word "Transcendent" as used above, supposed to differ from its sense here? One would have thought that in *every* case the Transcendent had to be subordinate to the *summum genus*.

Let us dwell a moment longer on the expression "as such". I shall suppose that somebody wishes me to imagine that all objects are nothing but pictures on my retina. Very well; so far I have no objections. But now he declares further that the tower is bigger than the window through which I suppose that I am seeing it. Now to this I should say: "Either it is not the case that both the tower and the window are pictures on my retina, and then the tower may be bigger than the window; or else the tower and the window are, as you say, pictures on my retina, and then the tower is not bigger, but smaller than the window." It is at this point that he tries to extricate himself from the dilemma with "as such", and he says: "To be sure, the retinal picture of the tower is *as such* not bigger than the retinal picture of the window." At this I almost feel like losing my temper entirely and shouting at him: "Well then, the retinal picture of the tower is not bigger than the retinal picture of the window at all, and if the tower were the retinal picture of the tower and the window were the

retinal picture of the window, then the tower would not be bigger than the window either, and if your logic teaches you differently it is absolutely worthless!" This "as such" is a splendid discovery for hazy writers reluctant to say either yes or no. But I
xxiv will not put up with this hovering between the two; I ask: If actuality is asserted of an object, then is the real subject of the judgment the idea? Yes or no? If it is not, then presumably the subject is the Transcendent, which is presupposed as the ground of being of this idea. But this Transcendent is itself a thing ideated or an idea. Thus we are driven on to the supposition that the subject of the judgment is not the ideated Transcendent, but the Transcendent that is presupposed as the ground of being of this ideated Transcendent. Thus we should have to go on forever; but however far we went we should never emerge from the subjective. Moreover, we could begin the same game with the predicate too, and not merely with the predicate *actual*; we could do it just as well with, say, *sweet*. We should then start off by saying: If actuality (or sweetness) is asserted of an object, then the real predicate is not the object's ideated actuality (or sweetness), but the Transcendent that is presupposed as the ground of being of this thing that is ideated. But we could not relax there; we should be driven ever further without end. What can we learn from this? That psychological logic is on the wrong track entirely if it conceives subject and predicate of a judgment as ideas in the psychological sense, that psychological considerations have no more place in logic than they do in astronomy or geology. If we want to emerge from the subjective at all, we must conceive of knowledge as an activity that does not create what is known but grasps what is already there. The picture of grasping is very well suited to elucidate the matter. If I grasp a pencil, many different events take place in my body: nerves are stimulated, changes occur in the tension and pressure of muscles, tendons, and bones, the circulation of the blood is altered. But the totality of these events neither is the pencil nor creates the pencil; the pencil exists independently of them. And it is essential for grasping that something be there which is grasped; the internal changes alone are not the grasping. In the same way, that which we grasp with the mind also exists independently of this activity, independently of the ideas and their alterations that are a part of this grasping or accompany it; and it is neither

identical with the totality of these events nor created by it as a part of our own mental life.

Now let us see how for the psychological logicians, more delicate distinctions within the subject matter are blotted out. For the case of characteristic mark and property this has already been mentioned; a related case is the distinction between object and concept, which I stress, and also that between concepts of first and second level. Naturally these distinctions are indiscernible to psychological logicians; for them everything is just idea. With this goes their wrong conception of those judgments xxv
that in everyday language we express by using "there is". This existence Herr Erdmann jumbles up with actuality (Vol. I, p. 311), which, as we saw, also is not clearly distinguished from objectivity. Of what thing are we really asserting that it is actual if we say that there are square roots of four? Is it 2 or –2? But neither the one nor the other is named here in any way at all. And if I wished to say that the number 2 acts or is active or actual, this would be false and wholly different from what I mean by the sentence, "There are square roots of four." The confusion before us is just about the grossest possible; for it is not between concepts of the same level, but rather between a concept of first level and a concept of second level. This is characteristic of the obtuseness of psychological logic. When we have gained a less obstructed viewpoint we may be astonished that such a blunder could be committed by a professional logician, but of course before we can gauge the magnitude of the blunder we must ourselves have grasped the distinction between first- and second-level concepts, and psychological logic will be quite incapable of that. The chief impediment is that its exponents take such fantastic pride in psychological profundity, which is after all nothing but psychological falsification of logic. And that is how our thick logic books come into being; they are bloated with unhealthy psychological fat that conceals all more delicate forms. Thus a fruitful collaboration between mathematicians and logicians is made impossible. While the mathematician defines objects, concepts, and relations, the psychological logician is spying upon the origin and evolution of ideas, and to him at bottom the mathematician's defining can only appear foolish because it does not reproduce the essence of ideation. He looks into his psychological peep-show and

tells the mathematician: "I see nothing at all of what you are defining." And the mathematician can only reply: "No wonder, for it is not where you are looking for it."

This may be enough to place my logical standpoint in a clearer light by the contrast. The distance between my view and the psychological logicians' seems to me so enormous that there is no prospect of my book's having any effect on them at present. It seems to me as if the tree that I have planted would have to lift a colossal weight of stone in order to gain space and light. And yet I should not like to abandon entirely the hope that my book might later help to overthrow psychological logic. To that end there surely will be some notice of the book by the mathematicians, notice which will compel psychological logic to come to terms with it. And I believe that I may expect some support
xxvi from mathematicians; they have at bottom a common cause with me against the psychological logicians. I believe that as soon as the latter so much as condescend to occupy themselves seriously with my book, if only to refute it, then I have won. For the whole of the second part [the Proofs of the Basic Laws of Number], is really a test of my logical convictions. It is *prima facie* improbable that such a structure could be erected on a base that was uncertain or defective. Anyone who holds other convictions has only to try to erect a similar structure upon them, and I think he will perceive that it does not work, or at least does not work so well. As a refutation in this I can only recognize someone's actually demonstrating either that a better, more durable edifice can be erected upon other fundamental convictions, or else that my principles lead to manifestly false conclusions. But no one will be able to do that. May my book, then, even if belatedly, contribute to a renewal of logic.

Jena, July, 1893

G. Frege

EXPOSITION OF THE BEGRIFFSSCHRIFT

EXPOSITION OF THE BEGRIFFSSCHRIFT

1 §0. Our task. Demands on the conduct of proof. Dedekind's system. Schröder's class.

In my *Grundlagen der Arithmetik,*[1] I sought to make it plausible that arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition. In the present book this shall now be confirmed, by the derivation of the simplest laws of Numbers by logical means alone. But for this to be convincing, considerably higher demands must be placed on the conduct of proof than is customary in arithmetic.[2] A few methods of inference must be marked out in advance, and no step may be taken that is not in accordance with one of these. Thus in passing on to a new judgment one must not be satisfied, as the mathematicians have nearly always been hitherto, with the transition's being evidently correct; rather one must split it into the logically simple steps of which it is composed—and of which there are frequently not a few. In this way no presupposition can pass unnoticed; every axiom required must be uncovered. It is indeed precisely the presuppositions made tacitly and without clear awareness that obstruct our insight into the epistemological nature of a law.

For an undertaking of this kind to succeed, it is of course necessary that we grasp precisely the concepts required. This applies particularly to what the mathematicians would like to designate by the word "set". Dedekind[3] uses the word "system" with very much the same purpose. But despite the explanations in my *Grundlagen* four years earlier, a clear insight into the essence of the matter is not to be found in Dedekind, although he occasionally comes near the mark, as here (p. 2): "Such a system S . . . is completely determined if for every thing it is

[1] Breslau, 1884.
[2] Cf. my *Grundlagen,* §90.
[3] *Was sind und was sollen die Zahlen?* Braunschweig, 1888.

determined whether it is an element of S or not. Hence a system S is the same as a system T (in symbols, $S = T$) if every element of S is also an element of T and every element [of] T is 2
also an element of S." But other passages wander off again, for example the following (pp. 1-2): "It very frequently occurs that different things $a, b, c, \ldots$, regarded for some reason from a common point of view, are put together in the mind; and we say then that they form a *system* S." Here a presentiment of the truth is indeed contained in the 'common point of view'; but this 'regarding', this 'putting together in the mind', is not an objective characteristic. I ask, in whose mind? If they are put together in one mind but not in another, do they form a system then? What is supposed to be put together in my mind, no doubt must be in my mind: then do the things outside myself not form systems? Is a system a subjective figure in the individual soul? In that case is the constellation Orion a system? And what are its elements? The stars, or the molecules, or the atoms? The following passage is worthy of note (p. 2): "For uniformity of expression it is advantageous to admit also the special case in which a system S consists of a *single* (one and only one) element a, i.e., in which the thing a is an element of S but every thing different from a is not an element of S." Subsequently (p. 3) this is so understood that every element s of a system S can itself be regarded as a system. Since in this case element and system coincide, it is here particularly clear that according to Dedekind it is the elements that really make up the 'system'. E. Schröder in his *Vorlesungen über die Algebra der Logik*[4] progresses a step beyond Dedekind in calling attention to the connection of his systems with concepts—which Dedekind seems to have overlooked. In fact what Dedekind really means when he calls a system part of a system (p. 2) is the subordination of a concept under a concept or an object's falling under a concept: cases that he distinguishes no better than Schröder, owing to an error of conception shared by them both; for Schröder too at bottom regards the elements as what constitute his *class*. With him an empty class may really no more occur than may an empty system with Dedekind; yet the need of it that arises from the nature of the situation makes itself felt, in different ways, with both

[4]Leipzig, 1890, p. 253.

authors. Dedekind continues the above passage: "On the other hand, for certain reasons we will here wholly exclude the empty system, which contains no element, although for other investigations it can prove convenient to invent such a system." So according to this, such an invention is allowed; only 'for certain
3 reasons' it is waived. Schröder ventures to invent an empty class. Thus as it seems, both are in agreement with many mathematicians in the view that we may invent anything we please that is not there—and even anything that is unthinkable; for if the elements constitute the system, then where the elements are abolished the system goes with them. As to where the limits of this inventive caprice may lie, or whether there are any limits at all, there is to be found little clarity or agreement; yet the correctness of a proof may depend upon it. I believe this question to have been settled, for all reasonable persons, in my *Grundlagen der Arithmetik* (§92 ff.) and in my lecture "Über formale Theorien der Arithmetik."[5] Schröder does 'invent' his 'Null', and thereby entangles himself in great difficulties.[6] Accordingly, while a clear insight is lacking in Schröder as in Dedekind, the true situation nevertheless makes itself felt whereever a system is to be specified. Dedekind then cites properties that a thing must have in order to belong to the system; i.e., he defines a concept by means of its characteristic marks.[7] Now if it is the characteristic marks that make up the concept, and not the objects falling under it, then there are no difficulties or objections against an empty concept. Of course then an object can never be at the same time a concept; and a concept under which falls only one object must not be confused with it. In this way, then, it will finally be acknowledged that a statement of number contains an assertion about a concept.[8] I have reduced Number to the relation of equinumeracy, and reduced the latter to many-one correspondence. Much the same holds for the word "correspondence" as for the word "set"; both are today

[5] *Sitzungsberichte der Jenaischen Gesellschaft für Medicin und Naturwissenschaft, Jahrgang 1885*, session of 17th July.

[6] Cf. E. G. Husserl in the *Göttinger gelehrte Anzeigen*, 1891, no. 7, p. 272, where, however, the problems are not solved.

[7] On *concept, object, property, characteristic mark*, cf. my *Grundlagen*, §§38, 47, 53, and my essay "Über Begriff und Gegenstand", in the *Vierteljahrsschrift für wissenschaftliche Philosophie*, Vol. XVI, no. 2 [1892].

[8] §46 of my *Grundlagen*.

used frequently in mathematics, and for the most part there is lacking any deeper insight into what they are really intended to mean. If I am right in thinking that arithmetic is a branch of pure logic, then a purely logical expression must be selected for "correspondence". I choose "relation" for this purpose. Concept and relation are the foundation-stones upon which I erect my structure.

But even when the concepts have been grasped precisely, it would be difficult–in fact, almost impossible–to satisfy without special aid the demands we must here place on the conduct of proof. Such an aid is my Begriffsschrift, and my first task will be to expound it. The following observation may be made before we proceed. It will not always be possible to give a 4
regular definition of everything, precisely because our endeavor must be to trace our way back to what is logically simple, which as such is not properly definable. I must then be satisfied with indicating what I intend by means of hints. Above all I must strive to be understood, and on this account I shall try to unfold the subject gradually and not attempt full generality or a final expression at the very outset. The reader may be surprised at the frequent use made of quotation marks; by their use I distinguish between the cases in which I am speaking of the sign itself, and those in which I am speaking of its denotation. Pedantic as this may appear, I nevertheless hold it to be necessary. It is remarkable how an inaccurate manner of speaking or writing, perhaps originally employed for convenience or brevity but with full consciousness of its inaccuracy, can end in a confusion of thought when once that consciousness has disappeared. But people have succeeded in mistaking numerals for numbers, the name for what is named, the mere auxiliary devices of arithmetic for its real subject matter. Such experiences teach us how necessary it is to place the highest demands on the accuracy of a manner of speaking and writing. And I have taken pains to do justice to these demands, at all events wherever it seemed to me to be of importance.

5 1. PRIMITIVE SIGNS

i. INTRODUCTION: FUNCTION, CONCEPT, RELATION[9]

§1. The function is unsaturated.

If we are asked to state the original meaning of the word "function" as used in mathematics, it is easy to fall into calling function of x an expression, formed from "x" and particular numbers by use of the notation for sum, product, power, difference, and so on. This is incorrect, because a function is here represented as an *expression*, as a concatenation of signs, not as what is designated thereby. Hence one will attempt to say, in place of "expression", rather "denotation of an expression". But now, there occurs in the expression the letter "x", which does not denote a number as the sign "2" does, for example, but only indeterminately indicates one. For different numerals that we put in the place of "x" we obtain in general different denotations. For example, if for "x" in the expression

$$\text{“}(2 + 3x^2)x\text{”}$$

we substitute the numerals "0", "1", "2", "3" in order, then we obtain as corresponding denotations the numbers 0, 5, 28, 87. None of these denotations can claim to be our function. The essence of the function manifests itself rather in the connection it establishes between the numbers whose signs we put for "x" and the numbers that then appear as denotations of our expression—a connection intuitively represented in the course of the

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[9] Cf. my lecture *Über Function und Begriff* (Jena, 1891) and my essay "Über Begriff und Gegenstand", in the *Vierteljahrsschrift für wissenschaftliche Philosophie*, Vol. XVI, no. 2 (1892). My *Begriffsschrift* (Halle, 1879) no longer fully corresponds to my present standpoint, and hence should be used only with caution to elucidate that set forth here.

curve whose equation in rectangular coördinates is

$$``y = (2 + 3x^2)x"$$

Accordingly the essence of the *function* lies in that part of the expression which is there over and above the "x". The expression for a *function* is *in need of completion, unsaturated.* The letter "x" serves only to hold places open for a numeral that is 6
to complete the expression, and in this way renders recognizable the particular type of need for completion that constitutes the specific nature of the function designated above. Hereafter, the letter "ξ" will be used for this purpose instead of "x".[10] This holding-open is to be understood as follows: all places at which "ξ" stands must be filled always by the same sign, never by different ones. I call these places *argument-places*, and that whose sign (name) occupies these places in a given case, I call the *argument* of the function for this case. The function is completed by the argument; what it becomes on completion I call the *value* of the function for the argument. Thus we obtain a name of the value of a function for an argument, if we fill the argument-places in the name of the function with the name of the argument. In this way, for example, "$(2+3\cdot 1^2)\cdot 1$" is a name of the number 5, composed of the function-name "$(2+3\xi^2)\xi$" and "1". Thus the argument is not to be counted a part of the *function,* but serves to complete the function, which in itself is *unsaturated.* In the sequel, where use is made of an expression like "the function $\Phi(\xi)$", it is always to be observed that "ξ" contributes to the designation of the function only so far as it renders recognizable the argument-places, but not in such a way that the essence of the function is altered if some sign is substituted for "ξ".

§2. Truth-values. Denotation and sense. Thought. Object.

To the fundamental arithmetical operations mathematicians have added, as constituting functions, the process of proceeding to a limit in its various forms, as infinite series, differential quotients, and integrals; and finally have understood the word "function" so widely that in some cases the connection between argument and value of the function can no longer be designated

[10] However, nothing is here stipulated for the Begriffsschrift. Rather, the "ξ" will not occur at all in the developments of the Begriffsschrift itself; I shall use it only in the exposition of it, and in elucidations.

by the signs of mathematical analysis, but only by words. Another extension has been to admit complex numbers as arguments and consequently as values of functions. In both directions I have gone still farther. That is, while on the one hand the signs of analysis have not hitherto always been sufficient, on the other hand not all of them have been employed in forming function-names, in that "$\xi^2 = 4$" and "$\xi > 2$", for example, were not allowed to count as names of functions—as I allow them to do. But this is also to say that the domain of the values of functions cannot remain restricted to numbers; for if I take as arguments of the function $\xi^2 = 4$ the numbers 0, 1, 2, 3 in order, I do not obtain numbers [as values]. The expressions

"$0^2 = 4$", "$1^2 = 4$", "$2^2 = 4$", "$3^2 = 4$"

are expressions some of true, some of false thoughts. I put this
7 as follows: the value of the function $\xi^2 = 4$ is either the *truth-value* of what is true or that of what is false.[11] It can be seen from this that I do not mean to assert anything if I merely write down an equation, but that I merely *designate* a truth-value, just as I do not assert anything if I merely write down "2^2", but merely *designate* a number. I say: the *names* "$2^2 = 4$" and "$3 > 2$" *denote* the same truth-value, which I call for short *the True*. Likewise, for me "$3^2 = 4$" and "$1 > 2$" *denote* the same truth-value, which I call for short *the False*, precisely as the name "2^2" *denotes* the number four. Accordingly I call the number four the *denotation* of "4" and of "2^2", and I call the True the denotation of "$3 > 2$". However, I distinguish from the *denotation* of a name its *sense*. "2^2" and "$2 + 2$" do not have the same *sense*, nor do "$2^2 = 4$" and "$2 + 2 = 4$" have the same *sense*. The sense of a name of a truth-value I call a *thought*. I further say a name *expresses* its sense and *denotes* its denotation. I *designate* with the name that which it denotes.

Thus the function $\xi^2 = 4$ can have only two values, namely the True for the arguments 2 and -2, and the False for all other arguments.

The domain of what is admitted as argument must also be extended to objects in general. *Objects* stand opposed to functions. Accordingly I count as *objects* everything that is not a

[11] I have justified this more thoroughly in my essay "Über Sinn und Bedeutung" in the *Zeitschrift für Philosophie und philosophische Kritik*, 100 (1892).

function, for example, numbers, truth-values, and the courses-of-values to be introduced below. The names of objects—the *proper names*—therefore carry no argument-places; they are saturated, like the objects themselves.

§3. Course-of-values of a function. Concept. Extension of a concept.

I use the words

"the function $\Phi(\xi)$ has the same *course-of-values* as the function $\Psi(\xi)$"

generally to denote the same as the words

"the functions $\Phi(\xi)$ and $\Psi(\xi)$ have always the same value for the same argument".

We have this circumstance with the functions $\xi^2 = 4$ and $3\xi^2 = 12$, at least if numbers are taken as arguments. However, we can imagine the signs for squaring and multiplication to be so defined that the function

$$(\xi^2 = 4) = (3\xi^2 = 12)$$

has the True as value for every argument whatever. At this point we may also use an expression from logic: "the concept *square root of 4* has the same extension as the concept *some-* 8
thing whose square trebled is 12". With such functions, whose value is always a truth-value, one may accordingly say, instead of "course-of-values of the function", rather "extension of the concept"; and it seems appropriate to call directly a *concept* a function whose value is always a truth-value.

§4. Functions of two arguments.

Hitherto I have spoken only of functions of a single argument; but we can easily pass on to *functions of two arguments*. These are *doubly in need of completion*, in the sense that a function of one argument is obtained once a completion by means of one argument has been effected. Only by means of yet another completion do we attain an object, and this is then called the *value* of the function for the two arguments. Just as the letter "ξ" served us with functions of one argument, so here we make use of the letters "ξ" and "ζ" to indicate the twofold unsaturatedness of functions of two arguments, as in

$$\text{“}(\xi + \zeta)^2 + \zeta\text{”}.$$

By substituting (for example) "1" for "ζ", we saturate the function in such a way that in $(\xi + 1)^2 + 1$ we still have a function, but of one argument. This way of using the letters "ξ" and "ζ" must always be kept in mind if an expression occurs like "the function $\Phi(\xi,\zeta)$" (cf. n. 10, above). I call the places at which "ξ" stands *ξ-argument-places*, and those at which "ζ" stands *ζ-argument-places.* I say that the ξ-argument-places are *related* to one another, and likewise for the ζ-argument-places; while I call a ξ-argument-place not *related* to a ζ-argument-place.

The functions of two arguments $\xi = \zeta$ and $\xi > \zeta$ always have a truth-value as value (at least if the signs "=" and ">" are appropriately defined). Such functions it will be appropriate to call *relations.* In the first relation, for example, 1 stands to 1, and in general every object to itself; in the second, for example, 2 stands to 1. We say that the object Γ *stands to* the object Δ *in the relation* $\Psi(\xi,\zeta)$ if $\Psi(\Gamma,\Delta)$ is the True. Likewise we say that the object Δ *falls under* the concept $\Phi(\xi)$ if $\Phi(\Delta)$ is the True. Of course it is presupposed in this that the functions $\Phi(\xi)$ and $\Psi(\xi,\zeta)$ always have as value a truth-value.[12]

9 *ii. SIGNS FOR FUNCTIONS*

§5. Judgment and thought. Judgment-stroke and horizontal.

We have already said that in a mere equation there is as yet no assertion; "2 + 3 = 5" only designates a truth-value, without its being said which of the two it is. Again, if I wrote

"$(2 + 3 = 5) = (2 = 2)$"

and presupposed that we knew $2 = 2$ to be the True, I still should not have asserted thereby that the sum of 2 and 3 is 5; rather I should only have designated the truth-value of *"2 + 3 = 5"'s denoting the same as "2 = 2".* We therefore require another special sign to be able to assert something as true. For this

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[12] There is a difficulty here which can easily obscure the true state of affairs and hence arouse suspicion as to the correctness of my view. If we compare the expression "the truth-value of *Δ's falling under the concept* $\Phi(\xi)$" with "$\Phi(\Delta)$", we see that what really corresponds to the expression "$\Phi(\ )$" is

"the truth-value of *( )'s falling under the concept* $\Phi(\xi)$",

and not

"the concept $\Phi(\xi)$".

These last words therefore do not really designate a concept (in our sense), even though by their linguistic form it appears as if they do. As to the awkward position in which language here finds itself, cf. my essay "Über Begriff und Gegenstand".

purpose I let the sign "$\vdash$" precede the name of the truth-value, so that for example in

$$\text{"}\vdash 2^2 = 4\text{"},^{13}$$

it is asserted that the square of 2 is 4. I distinguish the *judgment* from the *thought* in this way: by a *judgment* I understand the acknowledgement of the truth of a *thought*. The presentation in Begriffsschrift of a judgment by use of the sign "$\vdash$" I call a *proposition of Begriffsschrift* or briefly a *proposition*. I regard this "$\vdash$" as composed of the vertical line, which I call the *judgment-stroke*, and the horizontal line, which I will now simply call the *horizontal*.[14] The horizontal will mostly occur fused with other signs, as here with the judgment-stroke, and thereby will be protected against confusion with the *minus sign*. Where it does occur apart, for purposes of distinction it must be made somewhat longer than the minus sign. I regard it as a function-name, as follows:

$$—\Delta$$

is the True if Δ is the True; on the other hand it is the False if Δ is not the True.[15] Accordingly,

$$—\xi$$

is a function whose value is always a truth-value—or by our stipulation, a concept. Under this concept there falls the True and only the True. Thus,

$$\text{"}—2^2 = 4\text{"}$$

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[13] I frequently make use here, in a provisional way, of the notations for the sum, product, power, although these signs have here not yet been defined, to enable me to form examples more easily and to facilitate understanding by means of hints. But we must keep it in mind that nothing is made to rest on the denotations of these notations.

[14] I used to call it the *content-stroke*, when I still combined under the expression "possible content of judgment" what I have now learned to distinguish as truth-value and thought. Cf. my essay "Über Sinn und Bedeutung."

[15] Obviously the sign "Δ" may not be denotationless, but must denote an object. Denotationless names must not occur in the Begriffsschrift. The stipulation above is made in such a way that "$—\Delta$" denotes something under all circumstances so long merely as "Δ" denotes something. Otherwise $—\xi$ would not be a concept having sharp boundaries, thus in our sense not a concept at all. I here use *capital Greek letters* as if they were names denoting something, although I do not specify their denotation. In the developments of the Begriffsschrift itself they will occur no more than will "ξ" and "ζ".

denotes the same thing as "$2^2 = 4$", namely the True. In order to dispense with brackets I specify that everything standing to the right of the horizontal is to be regarded as a whole that occupies the argument-place of the function-name "$—\xi$", except as *brackets* prohibit this.

$$\text{"}—2^2 = 5\text{"}$$

denotes the False, thus the same thing as does "$2^2 = 5$"; as against this,

$$\text{"}—2\text{"}$$

denotes the False, thus something different from the number 2. If Δ is a truth-value, then $—\Delta$ is the same truth-value, and consequently

$$\Delta = (—\Delta)$$

is the True. But this is the False if Δ is not a truth-value. We can therefore say that

$$\Delta = (—\Delta)$$

is the truth-value of *Δ's being a truth-value*.

Accordingly the function $—\Phi(\xi)$ is a concept and the function $—\Psi(\xi,\zeta)$ is a relation, regardless of whether $\Phi(\xi)$ is a concept or $\Psi(\xi,\zeta)$ a relation.

Of the two signs of which "$\vdash$" is composed, only the judgment-stroke contains the act of assertion.

§ 6. Negation-stroke. Amalgamation of horizontals.

We need no special sign to declare a truth-value to be the False, so long as we possess a sign by which either truth-value is changed into the other; it is also indispensable on other grounds. I now stipulate:

The value of the function

$$\top\xi$$

shall be the False for every argument for which the value of the function

$$—\xi$$

is the True; and shall be the True for all other arguments.

Accordingly we possess in

$$\top\xi$$

a function whose value is always a truth-value; it is a concept, under which falls every object with the sole exception of the True. From this it follows that "$\top\Delta$" always denotes the same thing as "$\top(—\Delta)$", and as "$—\top\Delta$", and as "$—\top(—\Delta)$". Hence we regard "$\top$" as composed of the small vertical stroke, the *negation-stroke*, and the two portions of the horizontal stroke, each of which may be regarded as *horizontals* in our sense. The transition from "—┬—(——Δ)" or "——┬—Δ" to "—┬—Δ", as well as that from "————Δ" to "——Δ", I call *amalgamation* of horizontals.

By our stipulation —┬—$2^2 = 5$ is the True; thus: 11

$$\vdash\!\!\top\!\!-2^2 = 5,$$

in words: $2^2 = 5$ is not the True; or: the square of 2 is not 5.

So also: $\vdash\!\!\top\!\!-2$.

§7. The identity-sign.

We have been using the identity-sign as we went along, to form examples; but it is necessary to stipulate something more precise regarding it.

$$\text{“}\Gamma = \Delta\text{”}$$

shall denote the True if Γ is the same as Δ; in all other cases it shall denote the False.

In order to dispense with brackets I specify that everything standing to the left of the identity-sign as far as the nearest horizontal, as a whole denotes the ξ-argument of the function $\xi = \zeta$, except as *brackets* prohibit this; and that everything standing to the right of the identity-sign as far as the nearest identity-sign, as a whole denotes the ζ-argument of that function, except as *brackets* prohibit this (cf. p. 39, above).

§8. Generality. Gothic letters. Their scope. Amalgamation of horizontals.

We considered in §3 the case in which an equation such as

$$\text{“}\Phi(x) = \Psi(x)\text{”}$$

always yields a name of the True, whatever proper name we may substitute for "x", provided only that this name actually denotes an object. We then have the generality of an identity, whereas in "$2^2 = 4$" we have only an identity. This difference manifests itself in the fact that in the former case we have a letter "x" that *indicates* only indeterminately, whereas in "$2^2 = 4$" every sign has a determinate denotation. To obtain an expression for the generality, we might think of a definition of this sort:

> 'By "$\Phi(x)$" is to be understood the True, if the value of the function $\Phi(\xi)$ is the True for every argument; otherwise it denotes the False.'

It would be assumed here, as in all our considerations of this sort, that "$\Phi(\xi)$" always acquires a denotation if in it we replace "ξ" by a name that denotes an object. Otherwise I should not call $\Phi(\xi)$ a *function*. Accordingly then,

$$``x \cdot (x - 1) = x^2 - x"$$

would denote the True, at least if the notations for multiplication, subtraction and squaring were so defined also for objects that are not numbers that the equation did hold universally. On the other hand "$x \cdot (x - 1) = x^2$" would denote the False, because we obtain the False as denotation if we substitute "1" for "x", although we obtain the True if we substitute "0". But by this stipulation the scope of the generality would not be well enough demarcated. A doubt could arise; e.g., whether

$$``\neg\, 2 + 3x = 5x"$$

was to be taken as the negation of a generality or as the gener-
12 ality of a negation; or, more precisely, whether it was supposed to denote the truth-value of

> *the value of the function $2 + 3\xi = 5\xi$'s not being the True for every argument,*

or the truth-value of

> *the value of the function $\neg\, 2 + 3\xi = 5\xi$'s being the True for every argument.*

In the first case, "$\neg\, 2 + 3x = 5x$" would denote the True; in the second, the False. But the negation of the generality must be expressible as well as the generality of the negation. I express the latter thus:

$$``\overset{\mathfrak{a}}{\smile}\neg\, 2 + 3\mathfrak{a} = 5\mathfrak{a}";$$

and the negation of the generality thus:

$$``\neg\overset{\mathfrak{a}}{\smile}\, 2 + 3\mathfrak{a} = 5\mathfrak{a}";$$

and the generality itself thus:

$$``\overset{\mathfrak{a}}{\smile}\, 2 + 3\mathfrak{a} = 5\mathfrak{a}".$$

This last denotes the True if for every argument the value of the function $2 + 3\xi = 5\xi$ is the True. Since this is not the case, then

$$\overset{\mathfrak{a}}{\smile}\, 2 + 3\mathfrak{a} = 5\mathfrak{a}$$

is the False, hence

$$\neg\overset{\mathfrak{a}}{\smile}\, 2 + 3\mathfrak{a} = 5\mathfrak{a}$$

is the True.

$$\overset{\mathfrak{a}}{\smile}\neg\, 2 + 3\mathfrak{a} = 5\mathfrak{a}$$

is the False, because the value of the function $\neg\, 2 + 3\xi = 5\xi$

is not the True for every argument; for it is the False for the argument 1. Consequently

$$\neg\overset{\mathfrak{a}}{\smile}\neg\; 2 + 3\mathfrak{a} = 5\mathfrak{a}$$

is the True, and

$$\text{“}\vdash\neg\overset{\mathfrak{a}}{\smile}\neg\; 2 + 3\mathfrak{a} = 5\mathfrak{a}\text{”}$$

asserts: *there is* at least one solution of the equation

$$\text{“}2 + 3x = 5x\text{”}.$$

In the same way,

$$\vdash\neg\overset{\mathfrak{a}}{\smile}\neg\; \mathfrak{a}^2 = 1;$$

in words: *there is* at least one square root of 1. We see from this how "*there is*" is to be rendered in the Begriffsschrift.

If we now set up the definition as follows:

> "$\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a})$" is to denote the True if for every argument the value of the function $\Phi(\xi)$ is the True, and otherwise is to denote the False,

a supplementation is required: namely, a more exact statement as to what this function $\Phi(\xi)$ is in every case. We will call it the *corresponding* function. For uncertainties can arise; $\Delta = \Delta$ is as much the value of the function $\Delta = \xi$ as it is of the function $\xi = \xi$, in both cases for the argument Δ. Thus someone starting from $\overset{\mathfrak{a}}{\smile}\mathfrak{a} = \mathfrak{a}$ might take either $\xi = \mathfrak{a}$, or $\mathfrak{a} = \xi$, or $\xi = \xi$ to be the corresponding function. However, by our use of Gothic 13
letters we should have in the first two cases no *function* at all, because "$\xi = \mathfrak{a}$" and "$\mathfrak{a} = \xi$" never denote anything, whatever one may substitute for the "ξ": the Gothic letter "$\mathfrak{a}$" may not occur without "$\overset{\mathfrak{a}}{\smile}$" prefixed, save in "$\overset{\mathfrak{a}}{\smile}$" itself. Therefore only "$\xi = \xi$" may here be considered our corresponding function. The case is less simple with an expression like

$$\text{“}\overset{\mathfrak{a}}{\smile}[(\mathfrak{a} + \mathfrak{a} = 2\mathfrak{a}) = (\overset{\mathfrak{a}}{\smile}\mathfrak{a} = \mathfrak{a})]\text{”}.$$

If we went ahead blindly, we could think we had the corresponding function in

$$\text{“}(\xi + \xi = 2\xi) = (\overset{\xi}{\smile}\xi = \xi)\text{”}.$$

I will now say, "$\mathfrak{a}$" stands in "$\overset{\mathfrak{a}}{\smile}$" over the *concavity*. The place over the concavity is never an *argument-place*; therefore at least the "$\mathfrak{a}$" standing over the second concavity is to be preserved. But since the "$\overset{\mathfrak{a}}{\smile}$" must always be followed by a combination of signs that includes "$\mathfrak{a}$", it follows that "$\mathfrak{a}$" must remain intact in at least one of the two places in "$\mathfrak{a} = \mathfrak{a}$". Accordingly we might incline toward these functions as corresponding:

$$(\xi + \xi = 2\xi) = (\overset{\mathfrak{a}}{\smile}\,\xi = \mathfrak{a}),$$
$$(\xi + \xi = 2\xi) = (\overset{\mathfrak{a}}{\smile}\,\mathfrak{a} = \xi),$$
$$(\xi + \xi = 2\xi) = (\overset{\mathfrak{a}}{\smile}\,\mathfrak{a} = \mathfrak{a});$$

but against the first two notions is the fact that the denotation of the "$\overset{\mathfrak{a}}{\smile}\,\mathfrak{a} = \mathfrak{a}$" occurring in

$$\text{"}\overset{\mathfrak{a}}{\smile}[(\mathfrak{a} + \mathfrak{a} = 2\mathfrak{a}) = (\overset{\mathfrak{a}}{\smile}\,\mathfrak{a} = \mathfrak{a})]\text{"}$$

is already established, and may not be called back into question.

Let us call that which follows a concavity containing a *Gothic letter* and which together with this concavity forms the name of the truth-value of *the value of the corresponding function's being the True for every argument,* the *scope* of the Gothic letter standing over the concavity. The *corresponding* function is now determined by the following rule:

1. All places at which there occurs a Gothic letter within its scope, yet not within an enclosed scope of the same letter, and not over a concavity, are related argument-places, namely those of the corresponding function.

But if we want to designate the truth-value of *the function*

$$(\xi + \xi = 2\xi) = (\overset{\mathfrak{a}}{\smile}\,\xi = \mathfrak{a})\text{'}s$$

having the True as value for every argument, then we shall choose a different Gothic letter:

$$\overset{\mathfrak{e}}{\smile}(\mathfrak{e} + \mathfrak{e} = 2\mathfrak{e}) = (\overset{\mathfrak{a}}{\smile}\,\mathfrak{e} = \mathfrak{a}).$$

14 I comprise this in the following rule:

2. If in the name of a function Gothic letters already occur, within whose scope lie argument-places of this function, then to form the corresponding expression of generality a Gothic letter different from these must be chosen.

According to our stipulations, one Gothic letter is in general as good as another, with the restriction however that distinctness among these letters can be essential. For certain Gothic letters we shall later lay down a somewhat different type of use.

"$\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a})$" denotes the same thing as "$\overset{\mathfrak{a}}{\smile}(-\!\!-\!\!-\,\Phi(\mathfrak{a}))$" and "$-\!\!-\!\!-(\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}))$". Hence I regard the horizontal strokes to the right and left of the concavity in "$\overset{\mathfrak{a}}{\smile}$" as *horizontals* in our special sense of the word, so that by *amalgamation* of horizontals we can pass over immediately from the forms "$-\!\!-\!\!-(\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}))$" and "$\overset{\mathfrak{a}}{\smile}(-\!\!-\!\!-\,\Phi(\mathfrak{a}))$" to "$\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a})$".

§9. Notation for the course-of-values. Small Greek vowels. Their scope.

If $\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$ is the True, then by our earlier stipulation

(§3) we can also say that the function $\Phi(\xi)$ has the same course-of-values as the function $\Psi(\xi)$; i.e., we can transform the generality of an identity into an identity of courses-of-values and vice versa. This possibility must be regarded as a law of logic, a law that is invariably employed, even if tacitly, whenever discourse is carried on about the extensions of concepts. The whole Leibniz-Boole calculus of logic rests upon it. One might perhaps regard this transformation as unimportant or even as dispensable. As against this, I recall the fact that in my *Grundlagen der Arithmetik* I defined a Number as the extension of a concept, and indicated then that negative, irrational, in short all numbers were to be defined as extensions of concepts. We can set down a simple sign for a course-of-values, and in that way for example the name of the Number Nought will be introduced. On the other hand, in

$$\text{“}\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})\text{”}$$

we cannot put a simple sign for “$\Phi(\mathfrak{a})$”, because the letter “$\mathfrak{a}$” must always occur in whatever is substituted for “$\Phi(\mathfrak{a})$”.

The conversion of the generality of an identity into an identity of courses-of-values has to be capable of being carried out in our symbolism. Therefore, e.g., for

$$\text{“}\overset{\mathfrak{a}}{\smile}\mathfrak{a}^2 - \mathfrak{a} = \mathfrak{a}\cdot(\mathfrak{a} - 1)\text{”},$$

I write

$$\text{“}\acute{\epsilon}(\epsilon^2 - \epsilon) = \acute{\alpha}(\alpha\cdot(\alpha - 1))\text{”},$$

in which by “$\acute{\epsilon}(\epsilon^2 - \epsilon)$” I understand the course-of-values of the function $\xi^2 - \xi$, and by “$\acute{\alpha}(\alpha\cdot(\alpha - 1))$” the course-of-values of the function $\xi\cdot(\xi - 1)$. Similarly, $\acute{\epsilon}(\epsilon^2 = 4)$ is the course-of-values of the function $\xi^2 = 4$, or, as we can also say, the extension of the concept *square root of 4*.

If I say generally that

“$\acute{\epsilon}\Phi(\epsilon)$” denotes the course-of-values of the function $\Phi(\xi)$,

this requires a supplementation like that in §8, above, in our explanation of “$\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a})$”; i.e., the question is which function in each case is to be regarded as the *corresponding* function $\Phi(\xi)$. It is obvious that $\acute{\epsilon}(\epsilon^2 - \epsilon)$ is the course-of-values of the function $\xi^2 - \xi$, and not of $\xi^2 - \epsilon$ or $\epsilon^2 - \xi$, because by our way of using *small Greek vowels* neither “$\xi^2 - \epsilon$” nor “$\epsilon^2 - \xi$” would acquire a denotation for any object whose name was substituted for “ξ”, or as we can also put it, because these combinations of signs do not denote any functions but rather, separated from the "$\grave{\epsilon}$", are devoid of denotation. A combination of signs like

$$\text{“}\grave{\epsilon}\Psi(\epsilon,\ \grave{\epsilon}X(\epsilon))\text{”}$$

must be adjudicated as in §8 with "$\overset{\mathfrak{a}}{\smile}\Psi(\mathfrak{a},\ \overset{\mathfrak{a}}{\smile}X(\mathfrak{a}))$". The place under the smooth breathing is no more an *argument-place* than that over the concavity. Let us call what follows a *small Greek vowel* with the smooth breathing, and together with this forms the name of the course-of-values of the *corresponding* function, the *scope* of this Greek vowel. Then we can set up the following rule:

1. All places at which there occurs a small Greek vowel within its scope, yet not within an enclosed scope of the same letter, and not with a smooth breathing, are related argument-places, namely those of the corresponding function.

This is hereby stipulated. Accordingly $\grave{\epsilon}(\epsilon = \grave{\epsilon}(\epsilon^2 - \epsilon))$ is the course-of-values of the function $\xi = \grave{\epsilon}(\epsilon^2 - \epsilon)$, and $\grave{\alpha}(\alpha = \grave{\epsilon}(\epsilon = \alpha)$ the course-of-values of the function $\xi = \grave{\epsilon}(\epsilon = \xi)$. For the formation of a name of a course-of-values the following rule also holds:

2. If in the name of a function small Greek vowels already occur, within whose scope lie argument-places of this function, then to form the name of the course-of-values of this function a Greek vowel different from these must be chosen.

According to our stipulations, one small Greek vowel is in general as good as another, with the restriction however that distinctness among these letters can be essential.

The introduction of a notation for courses-of-values seems to
16 me to be one of the most important supplementations that I have made of my Begriffsschrift since my first publication on this subject. By introducing it we also extend the domain of arguments of any function. For example,

$$\grave{\epsilon}(\epsilon^2 - \epsilon) = \grave{\alpha}(\alpha \cdot (\alpha - 1))$$

is the value of the function

$$\xi = \grave{\alpha}(\alpha \cdot (\alpha - 1))$$

for the argument $\grave{\epsilon}(\epsilon^2 - \epsilon)$.

§10. The course-of-values of a function more exactly specified.

Although we have laid it down that the combination of signs

$$\text{“}\grave{\epsilon}\Phi(\epsilon) = \grave{\alpha}\Psi(\alpha)\text{”}$$

has the same denotation as

$$\text{“}\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})\text{”},$$

this by no means fixes completely the denotation of a name like "$\grave{\epsilon}\Phi(\epsilon)$". We have only a means of always recognizing a course-of-values if it is designated by a name like "$\grave{\epsilon}\Phi(\epsilon)$", by which it is already recognizable as a course-of-values. But we can neither decide, so far, whether an object is a course-of-values that is not given us as such, and to what function it may correspond, nor decide in general whether a given course-of-values has a given property unless we know that this property is connected with a property of the corresponding function. If we assume that

$$X(\xi)$$

is a function that never takes on the same value for different arguments, then for objects whose names are of the form

$$\text{"}X(\grave{\epsilon}\Phi(\epsilon))\text{"}$$

just the same distinguishing mark for recognition holds, as for objects signs for which are of the form "$\grave{\epsilon}\Phi(\epsilon)$". To wit,

$$\text{"}X(\grave{\epsilon}\Phi(\epsilon)) = X(\grave{\alpha}\Psi(\alpha))\text{"}$$

then also has the same denotation as "$\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$".[16] From this it follows that by identifying the denotation of "$\grave{\epsilon}\Phi(\epsilon) = \grave{\alpha}\Psi(\alpha)$" with that of "$\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$", we have by no means fully determined the denotation of a name like "$\grave{\epsilon}\Phi(\epsilon)$"–at least if there does exist such a function $X(\xi)$ whose value for a course-of-values as argument is not always the same as the course-of-values itself. How may this indefiniteness be overcome? By its being determined for every function when it is introduced, what values it takes on for courses-of-values as arguments, just as for all other arguments. Let us do this for the functions considered up to this point. There are the following:

$$\xi = \zeta, \quad -\!\!-\!\!-\,\xi, \quad -\!\!\!\top\!\!-\,\xi.$$

We can leave the last out of account, since it can be considered always to take a truth-value as argument. With this function it makes no difference whether one takes as argument an object, or the value of the function $-\!\!-\!\!-\,\xi$ for this object as argument. Now we can still reduce the function $-\!\!-\!\!-\,\xi$ to the function $\xi = \zeta$. That is, by our stipulations the function $\xi = (\xi = \xi)$ has for every argument the same value as the function $-\!\!-\!\!-\,\xi$; for the value of the function $\xi = \xi$ is the True for every argument. From this it follows that the value of the function $\xi = (\xi = \xi)$ is the True 17

[16] This is not to say that the sense is the same.

only for the True as argument, and that it is the False for all other arguments, exactly as with the function $——\xi$. Since in this way everything reduces to consideration of the function $\xi = \zeta$, we ask what value this has if a course-of-values occurs as argument. Since up to now we have introduced only the truth-values and courses-of-values as objects, it can only be a question of whether one of the truth-values can perhaps be a course-of-values. If not, then it is thereby also decided that the value of the function $\xi = \zeta$ is always the False if a truth-value is taken as one of its arguments and a course-of-values as the other. If on the other hand the True is at the same time the course-of-values of some function $\Phi(\xi)$, then it is thereby also decided what the value of the function $\xi = \zeta$ is in all cases in which the True is taken as one of the arguments, and likewise if the False is at the same time the course-of-values of a certain function. Now the question whether one of the truth-values is a course-of-values cannot be decided from the fact that

$$``\grave{\epsilon}\Phi(\epsilon) = \grave{\alpha}\Psi(\alpha)"$$

is to have the same denotation as

$$``\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})".$$

It is possible to stipulate generally that

$$``\bar{\eta}\Phi(\eta) = \bar{\alpha}\Psi(\alpha)"$$

shall denote the same thing as

$$``\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})"$$

without the identity of $\grave{\epsilon}\Phi(\epsilon)$ and $\bar{\eta}\Phi(\eta)$ being derivable from this. We should then have a class of objects with names of the form "$\bar{\eta}\Phi(\eta)$", and for whose differentiation and recognition the same distinguishing mark held good as for courses-of-values. We could now determine the function $X(\xi)$ by saying that its value shall be the True for $\bar{\eta}\Lambda(\eta)$ as argument, and shall be $\bar{\eta}\Lambda(\eta)$ for the True as argument; further the value of the function $X(\xi)$ shall be the False for the argument $\bar{\eta}M(\eta)$, and shall be $\bar{\eta}M(\eta)$ for the False as argument; for every other argument the value of the function $X(\xi)$ is to coincide with the argument itself. If now the functions $\Lambda(\xi)$ and $M(\xi)$ do not always have the same value for the same argument, then our function $X(\xi)$ never has the same value for different arguments, hence

$$``X(\bar{\eta}\Phi(\eta)) = X(\bar{\alpha}\Psi(\alpha))"$$

also always has the same denotation as

$$``\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})".$$

The objects whose names were of the form "$X(\acute{\eta}\Phi(\eta))$" would then be recognized by the same means as the courses-of-values, and $X(\acute{\eta}\Lambda(\eta))$ would be the True and $X(\acute{\eta}M(\eta))$ the False. Thus without contradicting our setting

$$\text{"}\acute{\epsilon}\Phi(\epsilon) = \acute{\epsilon}\Psi(\epsilon)\text{"}$$

equal to

$$\text{"}\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})\text{"}$$

it is always possible to stipulate that an arbitrary course-of-values is to be the True and another the False. Accordingly let us lay it down that $\acute{\epsilon}(—\,\epsilon)$ is to be the True and $\acute{\epsilon}(\epsilon = (\top\!\overset{\mathfrak{a}}{\smile}\,\mathfrak{a} = \mathfrak{a}))$ is to be the False. $\acute{\epsilon}(—\,\epsilon)$ is the course-of-values of the function $—\,\xi$, whose value is the True only if the argument is the True, and whose value for all other arguments is the False. All functions for which this holds, have the same course-of-values, 18
and this is by our stipulation the True. Accordingly $—\,\acute{\epsilon}\Phi(\epsilon)$ is the True only if the function $\Phi(\xi)$ is a concept under which falls only the True; in all other cases $—\,\acute{\epsilon}\Phi(\epsilon)$ is the False. Further, $\acute{\epsilon}(\epsilon = (\top\!\overset{\mathfrak{a}}{\smile}\,\mathfrak{a} = \mathfrak{a}))$ is the course-of-values of the function $\xi = (\top\!\overset{\mathfrak{a}}{\smile}\,\mathfrak{a} = \mathfrak{a})$, whose value is the True only if the argument is the False, and whose value for all other arguments is the False. All functions for which this holds have the same course-of-values, and this is by our stipulation the False. Thus every concept under which falls the False and only the False, has as its extension the False.[17]

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[17] A natural suggestion is to generalize our stipulation so that every object is regarded as a course-of-values, viz., as the extension of a concept under which it and it alone falls. A concept under which the object Δ and Δ alone falls is $\Delta = \xi$. Suppose we attempt the stipulation: let $\acute{\epsilon}(\Delta = \epsilon)$ be the same as Δ. Such a stipulation is possible for every object that is given us independent of courses-of-values on the same basis as we have observed with the truth-values. But before it may be generalized, the question arises whether it may not contradict our notation for recognizing courses-of-values if we take for Δ an object that is already given us as a course-of-values. In particular it is intolerable to allow it to hold only for such objects as are not given us as courses-of-values; the way in which an object is given must not be regarded as an immutable property of it, since the same object can be given in a different way. Thus if we substitute "$\acute{\alpha}\Phi(\alpha)$" for "$\Delta$", then we obtain

$$\text{"}\acute{\epsilon}(\acute{\alpha}\Phi(\alpha) = \epsilon) = \acute{\alpha}\Phi(\alpha)\text{"},$$

and this would denote the same as

$$\text{"}\overset{\mathfrak{a}}{\smile}(\acute{\alpha}\Phi(\alpha) = \mathfrak{a}) = \Phi(\mathfrak{a})\text{"},$$

With this we have determined the courses-of-values so far as is here possible. As soon as there is a further question of introducing a function that is not completely reducible to functions known already, we can stipulate what value it is to have for courses-of-values as arguments; and this can then be regarded as much as a further determination of the courses-of-values as of that function.

§11. Substitute for the definite article: the function $\backslash\xi$.

In fact, we do require such functions still. If the equating of "$\dot{\varepsilon}(\Delta = \varepsilon)$" with "$\Delta$" were allowed generally to stand,[18] then we
19 should have a substitute in the form $\dot{\varepsilon}\Phi(\varepsilon)$ for the definite article of ordinary language. That is, assuming $\Phi(\xi)$ to be a concept under which fell the object Δ and only Δ, then $\smile^{\mathfrak{a}}\Phi(\mathfrak{a}) = (\Delta = \mathfrak{a})$ would be the True, and consequently $\dot{\varepsilon}\Phi(\varepsilon) = \dot{\varepsilon}(\Delta = \varepsilon)$ would also be the True, and by virtue of our equating of "$\dot{\varepsilon}(\Delta = \varepsilon)$" with "$\Delta$", $\dot{\varepsilon}\Phi(\varepsilon)$ would be the same as Δ; i.e., in the case in which $\Phi(\xi)$ was a concept under which one and only one object fell, "$\dot{\varepsilon}\Phi(\varepsilon)$" would designate this object. Now of course this is not possible, because that equation could not be sustained in its general form, but we can serve our purpose by introducing the function

$$\backslash\xi$$

with the stipulation that two cases are to be distinguished:

1. If to the argument there corresponds an object Δ such that the argument is $\dot{\varepsilon}(\Delta = \varepsilon)$, then let the value of the function $\backslash\xi$ be Δ itself;

---

which however denotes the True only if $\Phi(\xi)$ is a concept under which one and only one object falls, namely $\dot{\alpha}\Phi(\alpha)$. Since this last is not necessary, our stipulation cannot remain intact in its general form.

The identity "$\dot{\varepsilon}(\Delta = \varepsilon) = \Delta$" with which we have tested this stipulation is a special case of "$\dot{\varepsilon}\Omega(\varepsilon, \Delta) = \Delta$", and one may ask how the function $\Omega(\xi, \zeta)$ would have to be constituted so that it might be generally determined that Δ was to be the same as $\dot{\varepsilon}\Omega(\varepsilon, \Delta)$. Then

$$\dot{\varepsilon}\Omega(\varepsilon, \dot{\alpha}\Phi(\alpha)) = \dot{\alpha}\Phi(\alpha)$$

must be the True, consequently

$$\smile^{\mathfrak{a}}\Omega(\mathfrak{a}, \dot{\alpha}\Phi(\alpha)) = \Phi(\mathfrak{a})$$

must be the True as well, whatever function $\Phi(\xi)$ may be. We shall later become acquainted with a function having this property in the function $\xi \frown \zeta$; but we shall define this with the aid of the course-of-values, so that we can make no use of it here.

[18] See n. 17, above.

2. if to the argument there does not correspond an object Δ such that the argument is $\grave{\epsilon}(\Delta = \epsilon)$, then let the value of the function be the argument itself.

Accordingly $\backslash\grave{\epsilon}(\Delta = \epsilon) = \Delta$ is the True, and "$\backslash\grave{\epsilon}\Phi(\epsilon)$" denotes the object falling under the concept $\Phi(\xi)$ if $\Phi(\xi)$ is a concept under which falls one and only one object; in all other cases "$\backslash\grave{\epsilon}\Phi(\epsilon)$" denotes the same as "$\grave{\epsilon}\Phi(\epsilon)$". Thus for example, $2 = \backslash\grave{\epsilon}(\epsilon + 3 = 5)$ is the True, because 2 is the one and only object that falls under the concept

what when increased by 3 yields 5

—a proper definition of the plus sign being presupposed. $\grave{\epsilon}(\epsilon^2 = 1) = \backslash\grave{\epsilon}(\epsilon^2 = 1)$ is the True, because more than one single object falls under the concept *square root of 1*.

$$\grave{\epsilon}(\neg\, \epsilon = \epsilon) = \backslash\grave{\epsilon}(\neg\, \epsilon = \epsilon)$$

is the True, because no object falls under the concept *not identical with itself*. $\grave{\epsilon}(\epsilon + 3) = \backslash\grave{\epsilon}(\epsilon + 3)$ [is the True], because the function $\xi + 3$ is not a concept.

We have here a substitute for the definite article of ordinary language, which serves to form proper names out of concept-words. For example, we form from the words

"positive square root of 2",

which denote a concept, the proper name

"the positive square root of 2".

Here there is a logical danger. For if we wanted to form from the words "square root of 2" the proper name "the square root of 2" we should commit a logical error, because this proper name, in the absence of further stipulation, would be ambiguous,[19] hence even devoid of denotation. If there were no irrational numbers—as has indeed been maintained—then even the proper name "the positive square root of 2" would be without a denotation, at least by the straightforward sense of the words, 20
without special stipulation. And if we were to give this proper name a denotation expressly, the object denoted would have no connection with the formation of the name, and we should not be entitled to infer that it was a positive square root of 2, while yet we should be only too inclined to conclude just that. This danger about the definite article is here completely circumvented, since "$\backslash\grave{\epsilon}\Phi(\epsilon)$" always has a denotation, whether the function $\Phi(\xi)$ be not a concept, or a concept under which falls

[19] I am taking for granted here that there exist negative and irrational numbers.

no object or more than one, or a concept under which falls exactly one object.

§12. Condition-stroke. And. Neither-nor. Or. Subcomponents. Main component.

In order to enable us to designate the subordination of a concept under a concept, and other important relations, I introduce the function of two arguments

$$\begin{array}{l}\top\, \xi \\ \llcorner\, \zeta\end{array}$$

by stipulating that its value shall be the False if the True be taken as ζ-argument and any object other than the True be taken as ξ-argument, and that in all other cases the value of the function shall be the True. By these and earlier stipulations the value of this function is specified also for courses-of-values as arguments. It follows that

$$\begin{array}{l}\top\, \Gamma \\ \llcorner\, \Delta\end{array}$$

is the same as

$$\text{—}\left(\begin{array}{l}\top\, (\text{—}\, \Gamma) \\ \llcorner\, (\text{—}\, \Delta)\end{array}\right),$$

hence in

$$\text{“}\begin{array}{l}\top\, \Gamma \\ \llcorner\, \Delta\end{array}\text{”}$$

we can regard the horizontal stroke in front of "Δ", as well as the two parts into which the upper horizontal stroke is divided by the vertical, as *horizontals* in our specific sense. We speak here, as earlier, of the *amalgamation of horizontals*. The vertical stroke I call the *condition-stroke*. It may be lengthened as required.

The following propositions hold:

$$\text{“}\begin{array}{l}\vdash\!\top\, 3^2 > 2 \\ \quad\llcorner\, 3 > 2\end{array}\text{”},\qquad \text{“}\begin{array}{l}\vdash\!\top\, 2^2 > 2 \\ \quad\llcorner\, 2 > 2\end{array}\text{”},\qquad \text{“}\begin{array}{l}\vdash\!\top\, 1^2 > 2 \\ \quad\llcorner\, 1 > 2\end{array}\text{”}.$$

The function $\begin{array}{l}\top\!\!\!\top\, \xi \\ \ \llcorner\, \zeta\end{array}$ or $\begin{array}{l}\text{—}\!\top\!\text{—}\top\, \xi \\ \qquad\ \llcorner\, \zeta\end{array}$ has as value always the True if the function $\begin{array}{l}\top\, \xi \\ \llcorner\, \zeta\end{array}$ has as value the False, and conversely. Therefore

21 $\begin{array}{l}\top\!\!\!\top\, \Gamma \\ \ \llcorner\, \Delta\end{array}$ is the True if and only if Δ is the True and Γ is not the True. Consequently

$$\begin{array}{l}\vdash\!\!\top\!\!\!\top\, 2 > 3 \\ \qquad\llcorner\, 2 + 3 = 5;\end{array}$$

in words: 2 is not greater than 3 *and* the sum of 2 and 3 is 5.

$$\begin{array}{l}\vdash\!\top\!\!\top\!\!\top\; 3 > 2\\ \qquad\llcorner\; 2 + 3 = 5;\end{array}$$

in words: 3 is greater than 2 *and* the sum of 2 and 3 is 5. That is,

$$\begin{array}{l}-\!\top\!\!\top\; 3 > 2\\ \qquad\llcorner\; 2 + 3 = 5\end{array}$$

is the value of the function $\begin{array}{l}\top\!\!\top\;\xi\\ \quad\llcorner\;\zeta\end{array}$ for the ξ-argument $-\!\top\!-\,3 > 2$ and the ζ-argument $2 + 3 = 5$.

$$\begin{array}{l}\vdash\!\top\!\!\top\; 2^3 = 3^2\\ \qquad\llcorner\!\top\; 1^2 = 2^1;\end{array}$$

in words: *neither* is the third power of 2 the second power of 3, *nor* is the second power of 1 the first power of 2.

In place of the propositions

$$\text{“}\begin{array}{l}\vdash\!\top\; 3^2 > 3\\ \quad\llcorner\!\top\; 3 < 3\end{array}\text{”},\qquad \text{“}\begin{array}{l}\vdash\!\top\; 2^2 > 3\\ \quad\llcorner\!\top\; 2 < 3\end{array}\text{”},\qquad \text{“}\begin{array}{l}\vdash\!\top\; 1^2 > 3\\ \quad\llcorner\!\top\; 1 < 3\end{array}\text{”},$$

we have the following:

$$\text{“}\begin{array}{l}\vdash\!\top\!\!\top\; 3^2 > 3\\ \qquad\llcorner\!\top\; 3 < 3\end{array}\text{”},\qquad \text{“}\begin{array}{l}\vdash\!\top\!\!\top\; 2^2 > 3\\ \qquad\llcorner\!\top\; 2 < 3\end{array}\text{”},\qquad \text{“}\begin{array}{l}\vdash\!\top\!\!\top\; 1^2 > 3\\ \qquad\llcorner\!\top\; 1 < 3\end{array}\text{”}.$$

Now since $\begin{array}{l}\top\; 1^2 > 3\\ \llcorner\!\top\; 1 < 3\end{array}$ is the truth-value of *neither the square of 1's being greater than 3 nor 1's being smaller than 3,* this is negated by our last proposition above; thus it is affirmed that of the following two at least one is true: either that the square of 1 is greater than 3, *or* that 1 is smaller than 3. We see from these examples how the "*and*" of ordinary language (used as a sentence-connective), the "*neither-nor*", and the "*or*" between sentences, are to be rendered.

In "$\begin{array}{l}\top\;\xi\\ \llcorner\;\Delta\end{array}$", any proper name may be substituted for "ξ", thus also, for example, the proper name "$\begin{array}{l}\top\;\Theta\\ \llcorner\;\Lambda\end{array}$". In this way we obtain

$$\text{“}\begin{array}{l}\top\!\left(\begin{array}{l}\top\;\Theta\\ \llcorner\;\Lambda\end{array}\right)\\ \llcorner\!\!-\!\!-\;\Delta\end{array}\text{”},$$

in which we may now *amalgamate* the horizontals:

$$\text{“}\begin{array}{l}-\!\top\!\!\top\;\Theta\\ \qquad\llcorner\;\Lambda\\ \quad\llcorner\!-\;\Delta\end{array}\text{”}.$$

This denotes the False if Δ is the True and $\begin{array}{l}\top\;\Theta\\ \llcorner\;\Lambda\end{array}$ is not the True, 22 i.e., in this case if $\begin{array}{l}\top\, \Theta \\ \llcorner\, \Lambda\end{array}$ is the False. But this is the case if and only if Λ is the True and Θ is not the True. Accordingly,

$$\begin{array}{l}\top\!\!\top\, \Theta \\ \;\;\llcorner\, \Lambda \\ \llcorner\, \Delta\end{array}$$

is the False if Δ and Λ are the True while Θ is not the True; in all other cases it is the True. From this there follows the interchangeability of Λ and Δ;

$$\begin{array}{l}\top\!\!\top\, \Theta \\ \;\;\llcorner\, \Lambda \\ \llcorner\, \Delta\end{array}$$

is the same truth-value as

$$\begin{array}{l}\top\!\!\top\, \Theta \\ \;\;\llcorner\, \Delta \\ \llcorner\, \Lambda .\end{array}$$

In

$$\begin{array}{l}``\top\!\!\top\, \Theta" \\ \;\;\llcorner\, \Delta \\ \llcorner\, \Lambda \;,\end{array}$$

we may call "$—\,\Theta$" the *main component* and "$—\,\Delta$" and "$—\,\Lambda$" *subcomponents*; however, we may also regard "$\begin{array}{l}\top\, \Theta \\ \llcorner\, \Delta\end{array}$" as the *main component* and "$—\,\Lambda$" alone as *subcomponent*. The subcomponents are accordingly *interchangeable*. In the same way we see that

$$\begin{array}{l}\top\!\!\top\!\!\top\, \Theta \\ \;\;\;\;\llcorner\, \Lambda \\ \;\;\llcorner\, \Delta \\ \llcorner\, \Xi\end{array}$$

is the False if and only if both Λ and Δ and Ξ are the True while Θ is not the True; in all other cases it is the True. Here too we have *interchangeability of the subcomponents* "$—\,\Lambda$", "$—\,\Delta$" and "$—\,\Xi$". This interchangeability must properly be proved for every case that arises, and I have done this in my booklet *Begriffsschrift* for certain cases in such a way that it is easy to treat every case accordingly. So as not to become tied up in excessive complexity, I here wish to assume this interchangeability generally granted, and to make use of it in future without further explicit mention.

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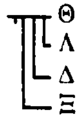
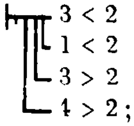
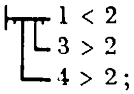


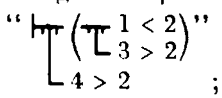
is the True if and only if both Λ and Δ and Ξ are the True while Θ is not the True. Accordingly,



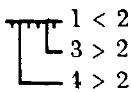
in words: 3 is not smaller than 2 and 1 is smaller than 2 and 3 is greater than 2 and 4 is greater than 2;



in words: 1 is smaller than 2 and 3 is greater than 2 and 4 is greater than 2. We may imagine it split up as follows:



in which the two negation-strokes between the condition-strokes may be canceled and the horizontals amalgamated. In



we have the value of the function $\begin{smallmatrix}\xi\\ \zeta\end{smallmatrix}$ for the ξ-argument $\begin{smallmatrix}1 < 2\\ 3 > 2\end{smallmatrix}$ and the ζ-argument $4 > 2$, where $\begin{smallmatrix}1 < 2\\ 3 > 2\end{smallmatrix}$ is the value of the same function for the ξ-argument $1 < 2$ and the ζ-argument $3 > 2$.

§13. If. All. Every. Subordination. Particular affirmative proposition. Some.

To justify the nomenclature "condition-stroke", let me point out that the names "$\begin{smallmatrix}3^2 > 2\\ 3 > 2\end{smallmatrix}$", "$\begin{smallmatrix}2^2 > 2\\ 2 > 2\end{smallmatrix}$", "$\begin{smallmatrix}1^2 > 2\\ 1 > 2\end{smallmatrix}$", result from "$\begin{smallmatrix}\xi^2 > 2\\ \xi > 2\end{smallmatrix}$" by "3", "2", "1" being substituted for "ξ". Let us now employ the sign ">" in such a way that "$\Gamma > \Delta$" denotes the True if Γ and Δ are real numbers and Γ is greater than Δ, and that "$\Gamma > \Delta$" denotes the False in all other cases; let us further assume that the notation "Γ^2" is so defined that it always has a denotation if Γ is an object; then the value of the function

$$\begin{array}{l}\top\ \xi^2 > 2\\ \llcorner\ \xi > 2\end{array}$$

24 is the True for every argument. Therefore,

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\ \mathfrak{a}^2 > 2\\ \llcorner\ \mathfrak{a} > 2\ ;\end{array}$$

in words: *if* something is greater than 2, *then* its square is also greater than 2. Thus too,

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\ \mathfrak{a}^4 = 1\\ \llcorner\ \mathfrak{a}^2 = 1\ ;\end{array}$$

in words: *if* the square of something is 1, *then* its fourth power also is 1. But one can also say: *every* square root of 1 is also fourth root of 1; or: *all* square roots of 1 are fourth roots of 1.[20] Here we have the *subordination* of a concept under a concept, a *universal* affirmative proposition. We have called a concept a function of one argument whose value is always a truth-value. Here $\xi^4 = 1$ and $\xi^2 = 1$ are functions of this kind; the latter is the *subordinated*, the former the *superordinated* concept. The concept $\begin{array}{l}\top\ \xi^4 = 1\\ \llcorner\ \xi^2 = 1\end{array}$ is composed out of these concepts as its characteristic marks; under it falls, for example, the number -1:

$$\vdash\!\begin{array}{l}\top\ (-1)^4 = 1\\ \llcorner\ (-1)^2 = 1\ ;\end{array}$$

in words: -1 is square root of 1 and fourth root of 1. We saw in §8 how the "there is" of natural language is to be rendered; this may be applied so as to express that there is something which is square root of 1 and fourth root of 1: $\vdash\!\neg\overset{\mathfrak{a}}{\smile}\!\neg\!\begin{array}{l}\top\!\neg\ \mathfrak{a}^4 = 1.\\ \llcorner\ \mathfrak{a}^2 = 1\end{array}$ Plainly two negation-strokes here cancel each other: $\vdash\!\neg\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\!\neg\ \mathfrak{a}^4 = 1.\\ \llcorner\ \mathfrak{a}^2 = 1\end{array}$

Now let us consider this from yet another side. $\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\!\neg\ \mathfrak{a}^4 = 1\\ \llcorner\ \mathfrak{a}^2 = 1\end{array}$ is the truth-value of *if something is square root of 1 then its not being fourth root of 1*; or, as we can also say, of *no square root*

[20] It is easy to link this with a related thought, that there exists something which is square root of 1; but we must here hold entirely aloof from this. Likewise we must resist another related thought, that there exists more than one square root of 1.

of 1's being fourth root of 1. This truth-value is the False, and consequently $\vdash\!\!\!\top\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!\begin{array}{l} \mathfrak{a}^4 = 1. \\ \llcorner\, \mathfrak{a}^2 = 1 \end{array}$ Here we have the negation of a *universal* negative proposition; i.e., we have a *particular* affirmative proposition,[21] for which we can also say: "*some* square roots of 1 are fourth roots of 1", in which, however, the plural 25
form must not be so understood that there has to be more than one.

$$\vdash\!\!\!\top\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!\begin{array}{l} \mathfrak{a}^4 = 1 \\ \llcorner\, \mathfrak{a}^3 = 1\,; \end{array}$$

in words: there is at least one cube root of 1 that is also fourth root of 1, or: *some* (at least one) cube root of 1 is fourth root of 1.

In our symbolism the sentence-connective "and" appears less simple than the function-name "$\top\!\!\begin{array}{l}\xi \\ \llcorner\,\zeta\end{array}$", for which a simple expression in words is lacking. The relation that obtains in ordinary language seems easily the more natural and appropriate, because we are used to it. What is simpler from a logical standpoint is hard to say: one can define our "$\top\!\!\begin{array}{l}\xi \\ \llcorner\,\zeta\end{array}$" by means of "and" and negation; but then conversely, one can define "and" by means of the function-name "$\top\!\!\begin{array}{l}\xi \\ \llcorner\,\zeta\end{array}$" and the negation-stroke. Obviously "$\vdash\!\!\begin{array}{l} 2 + 3 = 5 \\ \llcorner\, 2 + 2 = 4 \end{array}$", for example, asserts less than "$\vdash\!\!\!\top\!\!\begin{array}{l} 2 + 3 = 5 \\ \llcorner\!\top 2 + 2 = 4 \end{array}$", hence could be held to be simpler. The real reason for introducing "$\top\!\!\begin{array}{l}\xi \\ \llcorner\,\zeta\end{array}$" is the ease and perspicuity with which one can by its use represent deductive inference. To this we now proceed.

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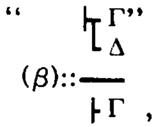
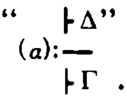
[21]The *particular* affirmative proposition on the one hand indeed says less than does the *universal* affirmative, but on the other hand (what is easily overlooked) also says more, since it asserts the realization of concepts, whereas subordination occurs also in the case of empty concepts—with the latter, even occurs invariably. Many logicians seem to assume without ado that concepts are realized, and to overlook entirely the very important case of empty concepts, perhaps because they quite wrongly do not recognize empty concepts as justified. This is why I use the expressions "subordination", "universal affirmative", "particular affirmative", in a sense somewhat different from theirs, and arrive at statements that they will be impelled to hold (wrongly) to be false.

§14. First method of inference.

From the propositions “$\vdash\!\begin{array}{l}\Gamma\\ \Delta\end{array}$” and “$\vdash \Delta$” we may infer “$\vdash\Gamma$”; for if Γ were not the True, then since Δ is the True $-\!\begin{array}{l}\Gamma\\ \Delta\end{array}$ would be the False. To every proposition set up in signs of Begriffsschrift that is to be used in a subsequent proof I shall give an *index* for purposes of citation. If the proposition “$\vdash\!\begin{array}{l}\Gamma\\ \Delta\end{array}$” has thus received the index “a” and “$\vdash \Delta$” the index “β”, then I write the inference either in this way, with a double colon:

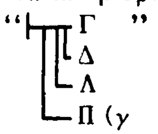


or in this way, with a single colon:



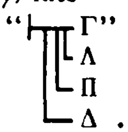
26 This is the sole method of inference used in my book *Begriffsschrift,* and one can actually manage with it alone. The dictates of scientific economy would properly require that we do so; yet in this book, where I wish to set up lengthy chains of inference, I must make some concessions to practical considerations. In fact, if I were not willing to admit some additional methods of inference the result would be exorbitant lengthiness—a point already anticipated in the Foreword to *Begriffsschrift.*

If we are given the propositions



and “$\vdash \Delta$ (β”,

then we cannot immediately make the inference described above, but only after we have made use of the interchangeability of subcomponents, transforming (γ) into



But in order to avoid excessive length I do not write it all out explicitly, but infer directly

$$
\begin{array}{ll}
\begin{array}{l}
\text{“}\vdash\!\top\!\top\!\top\,\Gamma\text{”} \\
\qquad\llcorner\,\Delta \\
\qquad\llcorner\,\Lambda \\
\qquad\llcorner\,\Pi \\
(\beta)::\overline{\qquad\qquad} \\
\vdash\!\top\!\top\,\Gamma \\
\quad\llcorner\,\Lambda \\
\quad\llcorner\,\Pi
\end{array}
& \quad\text{or}\quad
\begin{array}{l}
\text{“}\ \vdash\Delta\ \text{”} \\
(\gamma):\overline{\qquad} \\
\vdash\!\top\!\top\,\Gamma \\
\quad\llcorner\,\Lambda \\
\quad\llcorner\,\Pi\ ,
\end{array}
\end{array}
$$

where in the conclusion the subcomponents could also be differently ordered.

If a subcomponent of a proposition differs from a second proposition only in lacking the judgment-stroke, then a proposition may be inferred that results from the first proposition by suppressing that subcomponent.

We also combine two such inferences in a way to be gathered from the following. Let there be given the further proposition "$\vdash \Lambda$ $(\rho$". Then we write the double inference in this way:

$$
\begin{array}{l}
\text{“}\vdash\!\top\!\top\!\top\,\Gamma\text{”} \\
\qquad\llcorner\,\Delta \\
\qquad\llcorner\,\Lambda \\
\qquad\llcorner\,\Pi \\
(\beta,\,\rho)::\overline{\overline{\qquad\qquad}} \\
\vdash\!\top\,\Gamma \\
\quad\llcorner\,\Pi
\end{array}
$$

15. Second method of inference. Contraposition.

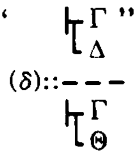
The following method of inference is a little more complicated. From the two propositions

$$
\begin{array}{l}
\text{“}\vdash\!\top\,\Gamma\ \text{”} \\
\quad\llcorner\,\Delta\ (\alpha
\end{array}
\qquad\text{and}\qquad
\begin{array}{l}
\text{“}\vdash\!\top\,\Delta\ \text{”} \\
\quad\llcorner\,\Theta\ (\delta
\end{array}
$$

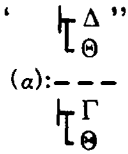
we may infer the proposition "$\begin{array}{l}\vdash\!\top\,\Gamma \\ \ \llcorner\,\Theta\end{array}$". For $\begin{array}{l}\top\,\Gamma \\ \llcorner\,\Theta\end{array}$ is the False only if Θ is the True and Γ is not the True. But if Θ is the True then Δ too must be the True, for otherwise $\begin{array}{l}\top\,\Delta \\ \llcorner\,\Theta\end{array}$ would be the False. But if Δ is the True then if Γ were not the True then $\begin{array}{l}\top\,\Gamma \\ \llcorner\,\Delta\end{array}$ would 27
be the False. Hence the case of $\begin{array}{l}\top\,\Gamma \\ \llcorner\,\Theta\end{array}$'s being the False cannot arise; and $\begin{array}{l}\top\,\Gamma \\ \llcorner\,\Theta\end{array}$ is the True.

This inference I write,

either in this way: "



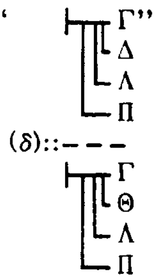
", or in this way: "



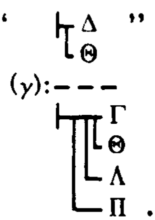
"

If instead of the proposition (α) we have as premiss the proposition given the index "γ" in §14, then properly we must first transform (γ) as we did there, before making the inference. But for brevity we do this tacitly, as above, writing

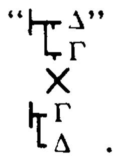
"



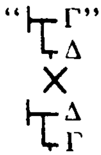
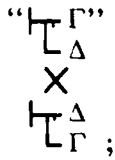
or "



$\begin{smallmatrix}\text{─┬─┬}\Delta\\ \text{└┬}\Gamma\end{smallmatrix}$ is the False if $\text{─┬}\Gamma$ is the True and $\text{─┬}\Delta$ is not the True; i.e., if — Γ is the False and Δ is the True. In all other cases $\begin{smallmatrix}\text{─┬─┬}\Delta\\ \text{└┬}\Gamma\end{smallmatrix}$ is the True. But the same holds for $\begin{smallmatrix}\text{─┬}\Gamma\\ \text{└}\Delta\end{smallmatrix}$; thus the functions $\begin{smallmatrix}\text{─┬─┬}\zeta\\ \text{└┬}\xi\end{smallmatrix}$ and $\begin{smallmatrix}\text{─┬}\xi\\ \text{└}\zeta\end{smallmatrix}$ always have the same value for the same arguments. In the same way the functions $\begin{smallmatrix}\text{─┬}\zeta\\ \text{└┬}\xi\end{smallmatrix}$ and $\begin{smallmatrix}\text{─┬}\xi\\ \text{└┬}\zeta\end{smallmatrix}$ always have the same value for the same arguments. We can reduce this case to the previous one by putting "$\text{─┬}\zeta$" for "ζ" and canceling juxtaposed negation-strokes. The functions $\begin{smallmatrix}\text{─┬─┬}\xi\\ \text{└}\zeta\end{smallmatrix}$ and $\begin{smallmatrix}\text{─┬─┬}\zeta\\ \text{└}\xi\end{smallmatrix}$ also always have the same value for the same arguments. Thus we may pass from the proposition "$\begin{smallmatrix}\text{├┬}\Gamma\\ \text{└}\Delta\end{smallmatrix}$" to the proposition "$\begin{smallmatrix}\text{├┬─┬}\Delta\\ \text{└┬}\Gamma\end{smallmatrix}$" and conversely. We write these transitions as follows:

 and 

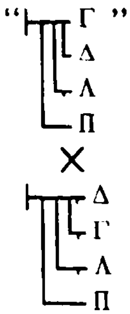
In the same way:

 and 

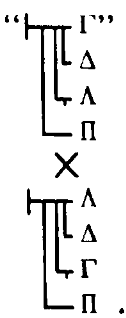
these cases reducing to the first by canceling negation-strokes. We may comprise this in a rule, as follows:

A subcomponent may be interchanged with the main component if the truth-value of each is simultaneously reversed.

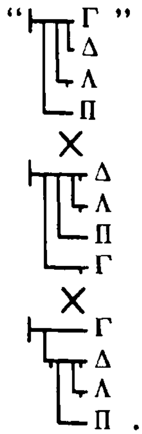
This transition we call *contraposition.* But further subcom- 28
ponents can be present as well; thus we have the transition



However, by making tacit use of the interchangeability of subcomponents we can also write



By double contraposition we achieve the concentration of all subcomponents into a single one, as follows:



That is, in the second contraposition we regard

as main component and “” as subcomponent. Suppose we give the truth-value

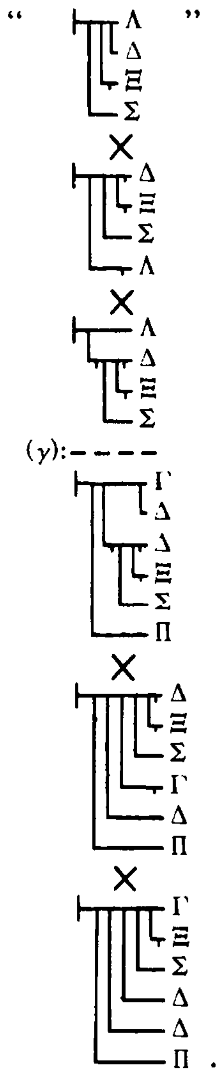
the abbreviated name “Θ”. Then the next-to-last proposition above becomes “”, from which follows “”. If we then replace “Θ” by the detailed expression again, we obtain the conclusion. As is to be seen from §12, in

we have the truth-value of Λ*'s being the True and* Λ*'s not being the True and* Π*'s being the True.*

If we assume the propositions

 (γ and  (ε

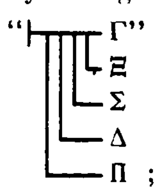
as given, then we may infer as follows: first we concentrate the subcomponents of (ε):



(We may now infer as at the beginning of this §, since this proposition is of the same form as (δ).) 29

(Now we unwind again the concentrated sub-component.)

This last we can simplify by writing "Δ" only once:

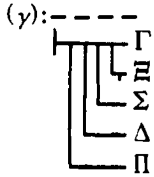


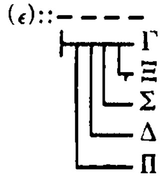
for

is always the same truth-value as .

A subcomponent occurring twice need be written only once. This we call the *amalgamation* of identical subcomponents.

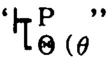
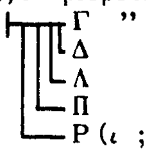
I now abbreviate the foregoing transition either in this way: "ε  ", or in this way: "γ  ", and for it I set



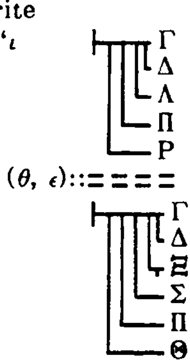
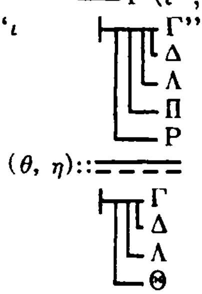


up the following rule:

If the same combination of signs occurs in one proposition as main component and in another as subcomponent, a proposition may be inferred in which the main component of the second is main component, and all subcomponents of either, save the one mentioned, are subcomponents. But subcomponents occurring in both need be written only once.

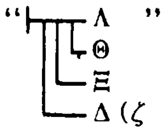
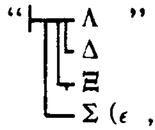
In a manner like that of §14 we may compress two inferences into one. Let there be given for example, besides (ε), the propositions "" and "$\vdash \Pi$ (η", and ""

then we may write

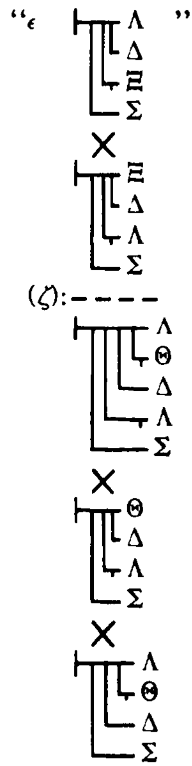
both "ι " and "ι "

§16. Third method of inference.

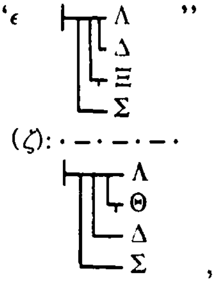
If we assume as given the propositions

 and 

then we can reduce this case to that just treated, namely in this way:



The purpose of the last two contrapositions is to eliminate one occurrence of "—┬— Λ" by amalgamation of identical subcomponents. That  is the same truth-value as —— Λ can also be understood directly; for  is the False if —┬— Λ is the True and Λ is not the True, and otherwise  is the True. The second condition implies the first. But —— Λ too is the False if Λ is not the True, and otherwise ——Λ is the True. We now abbreviate the foregoing transition in this way:



and express the rule as follows:

If two propositions agree in their main components, while a
subcomponent of one differs from a subcomponent of the other
only in a negation-stroke's being prefixed, then a proposition
31 *may be inferred in which the common main component is main*
component, and all subcomponents of either, save the two mentioned, are subcomponents. Here subcomponents occurring in both need be written only once (amalgamation of identical subcomponents).

§17. Roman letters. Transition from Roman to Gothic letters.

Now let us look to see how the inference called "Barbara" in logic fits into our scheme. From the two sentences,

"All square roots of 1 are fourth roots of 1"

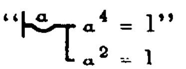
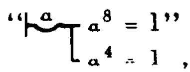
and

"All fourth roots of 1 are eighth roots of 1",

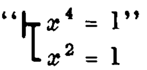
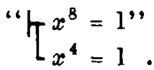
we can infer

"All square roots of 1 are eighth roots of 1".

Now if we write the premisses in this way:

 and  ,

then we cannot apply our methods of inference. We can, however, if we write them thus:

 and  .

Here we have the case of §15. Already earlier [in §8] we made an attempt to express generality in this way by the use of a *Roman letter*, but we left off again, because we observed that the scope of the generality was not well enough demarcated. We now meet this objection by stipulating that in the case of a

Roman letter the *scope* shall comprise everything that occurs in the proposition with the exception of the judgment-stroke.[22] Accordingly with a Roman letter we cannot ever express the negation of a generality, but we can express the generality of a negation. Thus no ambiguity is any longer present. However, we see that the expression of generality by Gothic letters and the concavity does not thereby become superfluous. Our stipulation regarding the *scope* of a *Roman letter* is to set only a lower bound upon the scope, not an upper bound. Thus it remains permissible to extend such a scope over several propositions, and this renders the Roman letters suitable to do duty in inferences, which the Gothic letters, with the strict closure of their scopes, cannot. If we have the premisses "$\begin{array}{l}\vdash\!\!\top x^4 = 1\\ \;\;\llcorner x^2 = 1\end{array}$" and "$\begin{array}{l}\vdash\!\!\top x^8 = 1\\ \;\;\llcorner x^4 = 1\end{array}$" and infer the proposition "$\begin{array}{l}\vdash\!\!\top x^8 = 1\\ \;\;\llcorner x^2 = 1\end{array}$", in making the transition we extend the scope of the "x" over both of the premisses and the conclusion, in order to perform the inference, although each of these propositions still holds good apart from this extension.

Of a Roman letter we say, not that it *denotes* an object, but rather that it *indicates* an object. In the same way we also say 32
that a Gothic letter *indicates* an object, where it does not stand over a concavity.

A proposition with a Roman letter can always be transformed into a proposition with a Gothic letter, whose concavity is separated from the judgment-stroke only by a horizontal. We write such a transition thus:

$$\text{“}\vdash\Phi(x)\text{”}$$

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!\Phi(\mathfrak{a})\ .$$

Here the second rule of §8 should be observed, as in the following example, where "$\mathfrak{e}$" may not be chosen as the newly introduced Gothic letter:

---

[22]Here the use of Roman letters is explained only for the case in which there occurs a judgment-stroke. This however is always the case in a development of pure Begriffsschrift; for therein we always proceed directly from one asserted proposition to another asserted proposition.

$$\vdash\!\begin{array}{l}\top\; 1 \geqq a\\ \llcorner\; a > 0\\ \llcorner\overset{\mathfrak{e}}{\smile}\!\begin{array}{l}\top\; a > \mathfrak{e}^{3}\\ \llcorner\; a > \mathfrak{e}\end{array}\end{array}$$

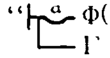
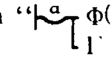
$$\smile$$

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\; 1 \geqq \mathfrak{a}\\ \llcorner\; \mathfrak{a} > 0\\ \llcorner\overset{\mathfrak{e}}{\smile}\!\begin{array}{l}\top\; \mathfrak{a} > \mathfrak{e}^{3}\\ \llcorner\; \mathfrak{a} > \mathfrak{e}\end{array}\end{array}.$$

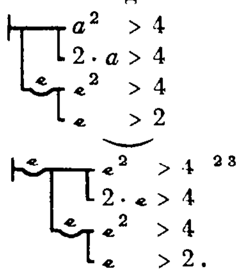
In connection with the transition from a Roman to a Gothic letter the following case should also be mentioned. Let us consider the proposition "$\vdash\!\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$", in which "$\Gamma$" is a proper name and "$\Phi(\xi)$" is a function-name. $\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$ is the False if for any argument the function $\begin{array}{l}\top\,\Phi(\xi)\\ \llcorner\,\Gamma\end{array}$ has the False as value. This in turn is the case if Γ is the True, and the value of the function $-\!\!-\,\Phi(\xi)$ is for any argument the False. In all other cases $\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$ is the True. Thus "$\vdash\!\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$" asserts that either Γ is not the True or the value of the function $\Phi(\xi)$ is the True for every argument. With this let us compare "$\begin{array}{l}\top\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$". This denotes the False if Γ is the True and $\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a})$ is the False. But the latter is the case if the value of the function $-\!\!-\,\Phi(\xi)$ is for any argument the False. In all other cases $\begin{array}{l}\top\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$ is the True. The proposition "$\vdash\!\begin{array}{l}\top\overset{\mathfrak{a}}{\smile}\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$" thus asserts the same as does "$\vdash\!\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$". If for "$\Gamma$" and "$\Phi(\xi)$", combinations of signs are substituted that do not denote an object and a function respectively, but only indicate, because they contain Roman letters, then the foregoing still holds generally if for each Roman letter a name is substituted, whatever this may be.

So as to express myself more accurately, I will introduce the following idiom. I call *names* only such signs and combinations of signs as are to denote something. Thus Roman letters, and combinations of signs in which they occur, are not *names*, because

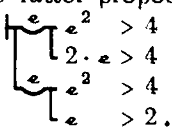
they only indicate. A combination of signs which contains Roman letters and which always results in a proper name if we re- 33
place every Roman letter by a name, I will call a *Roman object-mark*. And a combination of signs which contains Roman letters and which always results in a function-name if we replace every Roman letter by a name, I will call a *Roman function-mark*, or *Roman mark* of a function.

We can now say: the proposition "" always asserts the same as does the proposition "", not only if "$\Phi(\xi)$" is a function-name and "Γ" is a proper name, but also if "$\Phi(\xi)$" is a Roman function-mark and "Γ" is a Roman object-mark.

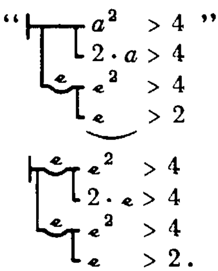
Let us apply this to the following case:



By the foregoing, for the latter proposition we can also write



Clearly, the only subcomponents which may be left out of the scope of the newly introduced Gothic letter are those which do not contain the Roman letter being replaced. I will write such transitions thus:



Rather than introducing several Gothic letters one after another,

[23] The second rule of §8 does not prohibit this iteration of "$\mathfrak{e}$", because "a" in the first proposition does not occur within the scope of "$\mathfrak{e}$".

we write the final result straightway under the sign "$\smile$".

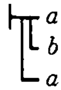
We comprise this in the following rule:

A Roman letter may be replaced at all of its occurrences in a proposition by one and the same Gothic letter. The Gothic letter must then at the same time be inserted over a concavity in front of a main component outside which the Roman letter does not occur.[24] *If within this main component is contained the scope of*
34 *a Gothic letter, and within this scope the Roman letter occurs, then the Gothic letter introduced for the Roman letter must be different from the Gothic letter already present* (second rule of §8).

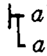
§18. Laws in symbols of Begriffsschrift (I. IV. VI.).

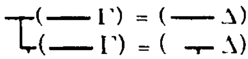
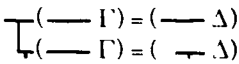
Now we shall set up, in Roman letters, some general laws that we shall require later. By §12,

could be the False only if both Γ and Δ were the True while Γ was not the True. This is impossible; therefore



(I

The "I" is assigned to this proposition as its index (§14), and indices will be attached to propositions in this way as we proceed. If we write "a" instead of "b", we can amalgamate the identical subcomponents, so that in "" we have a particular case of (I), which will be understood together with (I) without explicit notice.

$-\!\!-\,\Delta$ and $-\!\!\!\top\!\!\!-\,\Delta$ are always different, and always truth-values. Now since $-\!\!-\,\Gamma$ is in any case always a truth-value, it must coincide either with $-\!\!-\,\Delta$ or with $-\!\!\!\top\!\!\!-\,\Delta$. It follows from this that  is always the True; for it could be the False only if $-\!\!\!\top\!\!\!-(-\!\!-\,\Gamma) = (-\!\!\!\top\!\!\!-\,\Delta)$ were the True (i.e., if $(-\!\!-\,\Gamma) = (-\!\!\!\top\!\!\!-\,\Delta)$ were the False) and $(-\!\!-\,\Gamma) = (-\!\!-\,\Delta)$ were not the True (i.e., were the False). In other words,  could

[24] Thus if the Roman letter occurs in every subcomponent, then the whole proposition (excluding the judgment-stroke) must be regarded as the main component, and the concavity with the Gothic letter must then be placed separated from the judgment-stroke only by a horizontal.

be the False only if both $(\!-\!\!-\,\Gamma) = (\!-\!\!-\,\Delta)$ and $(\!-\!\!-\,\Gamma) = (-\!\!\!\top\!\!-\,\Delta)$ were the False, which, as we just saw, is not possible. Therefore

$$\vdash\!\!\begin{array}{l} \ulcorner (\!-\!\!-\, a) = (\!-\!\!-\, b) \\ \llcorner (\!-\!\!-\, a) = (-\!\!\!\top\!\!-\, b) \end{array} \qquad \text{(IV}$$

The brackets to the right of the identity-sign are dispensable.

From the denotation of the function-name "$\backslash\xi$" (§11), there follows:

$$\vdash a = \backslash\grave{\epsilon}(a = \epsilon) \qquad \text{(VI}$$

iv. EXTENSION OF THE NOTATION FOR GENERALITY

§19. Generality with respect to functions. Function-letters. Object-letters.

Up to this point, generality has been expressed only with respect to objects. In order to enable the same for functions, we distinguish as *function-letters* the letters "f", "g", "h", "F", "G", "H", and the corresponding Gothic letters: as opposed to the other letters, which we call *object-letters*,[25] so that the former shall indicate only functions, and never objects (as object-letters do). Among the object-letters we also reckon the small Greek vowels, since they occur without the smooth breathing only at places where proper names may also stand. In general after a function-letter there follows within its scope a *bracket* whose interior contains either one place, or two separated by a comma, according as the letter is to indicate a function of one argument or of two. Such a place serves to receive a simple or complex sign that denotes or indicates an argument, or occupies the argument-place in the manner of the small Greek vowels. Clearly, within its scope a function-letter must occur accompanied everywhere by one argument-place, or everywhere by two. The *scope* in the case of Roman function-letters comprises everything occurring in the proposition save the judgment-stroke, and in the case of Gothic letters is demarcated by means of a concavity with the Gothic letter standing alone. In this, the use of function-letters wholly agrees with that of object-letters. We may next consider some illustrative examples.

§20. Laws in symbols of Begriffsschrift (IIa. III. V.).

$\overset{\mathfrak{a}}{\smile}\!\!-\Phi(\mathfrak{a})$ is the True only if the value of the corresponding function $\Phi(\xi)$ is the True for every argument. Thus in such a case

[25] With the exception of "M", which is being reserved for a special purpose.

$\Phi(\Gamma)$ must likewise be the True. It follows that $\begin{array}{l}-\!\!\top\!\!-\Phi(\Gamma)\\ \;\;\llcorner\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a})\end{array}$ is always the True, whatever function of one argument $\Phi(\xi)$ may be. (Here we must observe the first rule of §8, so as to recognize the corresponding function $\Phi(\xi)$. For example, if one were to write "$\begin{array}{l}-\!\!\top\!\!-\Psi(\Gamma, \overset{\mathfrak{a}}{\smile}\!-X(\Gamma, \mathfrak{a}))\\ \;\;\llcorner\overset{\mathfrak{a}}{\smile}\!-\Psi(\mathfrak{a}, \overset{\mathfrak{a}}{\smile}\!-X(\mathfrak{a}, \mathfrak{a}))\end{array}$", then one could only appear to have names of the same function in the main component and subcomponent. In fact, the subcomponent would be formed by use of the function-name "$\Psi(\xi, \overset{\mathfrak{a}}{\smile}\!-X(\mathfrak{a}, \mathfrak{a}))$", and the main component by use of the function-name "$\Psi(\xi, \overset{\mathfrak{a}}{\smile}\!-X(\xi, \mathfrak{a}))$".) We now understand by "$\begin{array}{l}\overset{\mathfrak{f}}{\smile}\!\!\top\!\!-\mathfrak{f}(\Gamma)\\ \;\;\;\llcorner\overset{\mathfrak{a}}{\smile}\!-\mathfrak{f}(\mathfrak{a})\end{array}$" the truth-value of *one's always obtaining a name of the True, whatever function-name one may substitute in place of "$\mathfrak{f}$" in "$\begin{array}{l}-\!\!\top\!\!-\mathfrak{f}(\Gamma)\\ \;\;\llcorner\overset{\mathfrak{a}}{\smile}\!-\mathfrak{f}(\mathfrak{a})\end{array}$".* This truth-value is the True, whatever object "Γ" may denote: $\begin{array}{l}\vdash\!\overset{\mathfrak{f}}{\smile}\!\!\top\!\!-\mathfrak{f}(a)\\ \;\;\;\;\;\llcorner\overset{\mathfrak{a}}{\smile}\!-\mathfrak{f}(\mathfrak{a})\end{array}$. Since here the concavity with the "$\mathfrak{f}$" is separated from the judgment-stroke only by a horizontal, we may drop the concavity and write a Roman letter in place of the Gothic letter. Thus,

$$\begin{array}{l}\vdash\!\!\top\!\!-f(a)\\ \;\;\llcorner\overset{\mathfrak{a}}{\smile}\!-f(\mathfrak{a})\end{array} \qquad\qquad \text{(IIa}$$

Perhaps we might render this law in words in this way: What holds for all objects, holds also for any.

According to §7 the function of two arguments $\xi = \zeta$ always has as value a truth-value, viz., the True if and only if the ζ-argument coincides with the ξ-argument. If $\Gamma = \Delta$ is the True,
36 then $\begin{array}{l}\overset{\mathfrak{f}}{\smile}\!\!\top\!\!-\mathfrak{f}(\Gamma)\\ \;\;\;\;\llcorner\mathfrak{f}(\Delta)\end{array}$ is also the True; i.e., if Γ is the same as Δ, then Γ falls under every concept under which Δ falls; or, as we may also say: then every statement that holds for Δ holds also for Γ. But also conversely; if $\Gamma = \Delta$ is the False, then not every statement that holds for Δ also holds for Γ, i.e., then $\begin{array}{l}\overset{\mathfrak{f}}{\smile}\!\!\top\!\!-\mathfrak{f}(\Gamma)\\ \;\;\;\;\llcorner\mathfrak{f}(\Delta)\end{array}$ is the False. For example, Γ does not fall under the concept $\xi = \Delta$, under which Δ does fall. Thus, $\Gamma = \Delta$ is always the same truth-value as $\begin{array}{l}\overset{\mathfrak{f}}{\smile}\!\!\top\!\!-\mathfrak{f}(\Gamma)\\ \;\;\;\;\llcorner\mathfrak{f}(\Delta)\end{array}$. Consequently, [the truth-value] $\begin{array}{l}\overset{\mathfrak{f}}{\smile}\!\!\top\!\!-\mathfrak{f}(\Gamma)\\ \;\;\;\;\llcorner\mathfrak{f}(\Delta)\end{array}$ falls under every concept under which [the truth-value] $\Gamma = \Delta$ falls. Thus,

$$\begin{array}{l} \vdash\!\!\top g\Big(\overset{\mathfrak{f}}{\smile}\!\!\top \begin{array}{l} \mathfrak{f}(a) \\ \llcorner \mathfrak{f}(b) \end{array}\Big) \\ \llcorner\, g(a = b) \end{array} \qquad \text{(III}$$

We saw in §§3 and 9 that an identity of courses-of-values may always be transformed into the generality of an identity, and conversely:

$$\vdash (\grave{\varepsilon} f(\varepsilon) = \grave{\alpha} g(\alpha)) = (\overset{\mathfrak{a}}{\smile} f(\mathfrak{a}) = g(\mathfrak{a})) \qquad \text{(V}$$

Here the first rules of §§8 and 9 are to be observed.

§21. Functions and concepts of first and second level.

If we are to give a general explanation of the use of function-letters we require another technical term, which will be explained at this point.

If we consider the names

"$\overset{\mathfrak{a}}{\smile}\, \mathfrak{a}^2 = 4$", "$\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} > 0$", "$\overset{\mathfrak{a}}{\smile}\!\!\top \begin{array}{l} \mathfrak{a}^2 = 1 \\ \llcorner\, \mathfrak{a} > 0 \end{array}$",

then we easily see that we obtain them from "$\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})$"[26] by replacing the function-name "$\Phi(\xi)$" by names of the functions $\xi^2 = 4$, $\xi > 0$, and $\top \begin{array}{l} \xi^2 = 1 \\ \llcorner\, \xi > 0 \end{array}$. Clearly, only names of functions of one argument—not proper names, nor names of functions of two arguments—may be substituted, for the combinations of signs being substituted must always have open argument-places to receive the letter "$\mathfrak{a}$"; [27] and if on the other hand we wanted to substitute a name of a function of *two* arguments, then the ζ-argument-places would remain unfilled. For example, in order to substitute a name of the function $\Psi(\xi, \zeta)$, we might perhaps feel inclined to write "$\overset{\mathfrak{a}}{\smile}\, \Psi(\mathfrak{a}, \mathfrak{a})$"; but in that case we should in fact have substituted, not a name of the function $\Psi(\xi, \zeta)$, but rather a name of the function of one argument $\Psi(\xi, \xi)$ (first rule of §8). If we wanted to write "$\overset{\mathfrak{a}}{\smile}\, \Psi(\mathfrak{a}, 2)$", then we should again be substituting only a name of a function of one argument, the function $\Psi(\xi, 2)$. One might perhaps let the "ζ" stand: "$\overset{\mathfrak{a}}{\smile}\, \Psi(\mathfrak{a}, \zeta)$"; here we should have a function whose argument

[26] Cf. §13.

[27] That functions (such as $\xi = \xi$ or $\xi^2 - \xi \cdot \xi$) that have the same value for every argument—one might call them constant functions—are nevertheless to be distinguished from this value, this object, itself, I have shown in my essay *Über Function und Begriff*, p. 8.

was indicated by "ζ". We may combine our consideration of this case with the case in which the argument-sign in "$X(\xi)$" is replaced by "$\Phi(\xi)$": "$X(\Phi(\xi))$". We commonly speak here of
37 a 'function of a function', but inaccurately; for if we recall that functions are fundamentally different from objects, and further that the value of a function for an argument is to be distinguished from the function itself, then we see that a function-name can never occupy the place of a proper name, because it carries with it empty places that answer to the unsaturatedness of the function. If we say "the function $\Phi(\xi)$", then we must never forget that "ξ" belongs to the function-name only in the sense that it renders this unsaturatedness recognizable. Thus another function can never occur as argument of the function $X(\xi)$, though indeed the value of a function for an argument can do so: e.g., $\Phi(2)$, in which case the value is $X(\Phi(2))$. If we write "$X(\Phi(\xi))$", then by "$\Phi(\xi)$" we only indicate the argument in the way in which we indicate it in "$X(\xi)$" by "ξ". The function-name is really only a part of "$\Phi(\xi)$", so that the function does not occur here as argument of $X(\xi)$, because the function-name fills up only a part of the argument-place. So too, one c a n n o t say that in "$\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\ \Psi(\mathfrak{a}, \zeta)$" the function-name "$\Psi(\xi, \zeta)$" occupies the place which the function-name "$\Phi(\xi)$" occupies in "$\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\ \Phi(\mathfrak{a})$"; for it fills up only a part of it, whereas another part, namely the place of the "ζ", is still open for a proper name. *Functions of two arguments* are just as fundamentally different from *functions of one argument* as the latter are from *objects*. For whereas objects are wholly *saturated*, functions of two arguments are saturated to a lesser degree than functions of one argument, which too are already *unsaturated*.

In "$\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\ \Phi(\mathfrak{a})$", therefore, we have an expression in which we may replace the name of the function $\Phi(\xi)$ by names of functions of one argument, but not by names of objects, and not by names of functions of two arguments. This gives us cause to regard $\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\ \mathfrak{a}^2 = 4$, $\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\ \mathfrak{a} > 0$, and $\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l} \top\ \mathfrak{a}^2 = 1 \\ \llcorner\ \mathfrak{a} > 0 \end{array}$ as values of the same function

$$\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-\ \phi(\mathfrak{a})$$

for different *arguments*. Here, however, these arguments are themselves again functions, namely the functions of one argument $\xi^2 = 4$, $\xi > 0$, and $\begin{array}{l} \top\ \xi^2 = 1 \\ \llcorner\ \xi > 0 \end{array}$; and only functions of one argument

are capable of being arguments of our function $-\!\!\overset{\mathfrak{a}}{\smile}\!\!-\,\phi(\mathfrak{a})$. If we say, "the function $-\!\!\overset{\mathfrak{a}}{\smile}\!\!-\,\phi(\mathfrak{a})$", then "$\phi$" is a proxy for the sign of an argument, just as "ξ" in the expression "the function $\xi^2 - 4$" is a proxy for a proper name that could appear as sign of an argument. "ϕ" in our present case is not to be assigned to the function any more than "ξ" in the previous case. We now call those functions whose arguments are objects *first-level functions*; on the other hand, those functions whose arguments are first-level functions may be called *second-level functions*. The value of our function $-\!\!\overset{\mathfrak{a}}{\smile}\!\!-\,\phi(\mathfrak{a})$ is always a truth-value, whatever first-level function we may take as argument. To con- 38
form with earlier nomenclature, we shall accordingly call it a concept: namely a *second-level concept*, to distinguish it from *first-level concepts* which are first-level functions.

Our function $-\!\!\overset{\mathfrak{a}}{\smile}\!\!-\,\phi(\mathfrak{a})$ had, for the arguments taken previously, the True as value. If we take now as argument the function $\begin{array}{l} \top\!\!-\, \xi^3 = -1 \\ \llcorner\, \xi > 0 \end{array}$, then we obtain in $\begin{array}{l} -\!\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!-\, \mathfrak{a}^3 = -1 \\ \qquad\llcorner\, \mathfrak{a} > 0 \end{array}$ the False, because there is no positive cube root of -1. In the same way the value of our function for the argument $\xi + 3$ is the False; for we may always replace $-\!\!\overset{\mathfrak{a}}{\smile}\!\!-\,\mathfrak{a}+3$ by $-\!\!\overset{\mathfrak{a}}{\smile}\!\!-(\!-\!\!-\,\mathfrak{a}+3)$; and this is the False because the value of the function $-\!\!-\,\xi+3$ is always the False—if, that is, we assume the plus sign to be so defined that the value of the function $\xi+3$ is not the True for any argument.

§22. Examples of second-level functions. Unequal-leveled functions and relations.

We have another second-level function in

$$\begin{array}{l} -\!\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!\overset{\mathfrak{e}}{\smile}\!\!\top\!\!-\, \mathfrak{a} = \mathfrak{e} \\ \qquad\quad\llcorner\, \phi(\mathfrak{e}) \\ \qquad\llcorner\!\!-\!\!-\, \phi(\mathfrak{a}) \end{array} ,$$

where again "ϕ" is a proxy for the sign of the argument. Its value is the True for every first-level concept as argument, under which not more than a single object falls. Accordingly,

$$\begin{array}{l} \vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!\overset{\mathfrak{e}}{\smile}\!\!\top\!\!-\, \mathfrak{a} = \mathfrak{e} \\ \qquad\quad\llcorner\, \mathfrak{e} + 1 = 3 \\ \qquad\llcorner\!\!-\!\!-\, \mathfrak{a} + 1 = 3, \end{array} \qquad \begin{array}{l} \vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!\overset{\mathfrak{e}}{\smile}\!\!\top\!\!-\, \mathfrak{a} = \mathfrak{e} \\ \qquad\quad\llcorner\!\top\, \mathfrak{e} = \mathfrak{e} \\ \qquad\llcorner\!\!-\!\!\top\, \mathfrak{a} = \mathfrak{a}. \end{array}$$

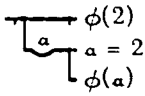
On the other hand,

$$\begin{array}{l} \vdash\!\!\top\!\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!\overset{\mathfrak{e}}{\smile}\!\!\top\!\!-\, \mathfrak{a} = \mathfrak{e} \\ \qquad\qquad\llcorner\, \mathfrak{e}^2 = 1 \\ \qquad\quad\llcorner\!\!-\!\!-\, \mathfrak{a}^2 = 1 . \end{array}$$

We also have a second-level function in $\phi(2)$. The values of this function are in part truth-values, as, for example, for the arguments $\xi + \xi = \xi \cdot \xi$, $\xi + 1 = 4$, to which there correspond the values $2 + 2 = 2 \cdot 2$ and $2 + 1 = 4$, and in part other objects, as, for example, the number 3 for the argument $\xi + 1$. This second-level function is distinct from the number 2 itself, since, like all functions, it is unsaturated.

The second-level function $\text{——}\,\phi(2)$ is distinguished from the previous function by the fact that its value is always a truth-value. It is thus a second-level concept, which we may call *property of the number 2*; because every concept under which falls the number 2, falls under this second-level concept, and no other first-level functions of one argument fall under this second-level concept.[28]

In



we also have a second-level concept, which we could call *property of the number 2 that belongs to it exclusively.*

$\overset{\mathfrak{a}}{\smile}\,\phi(\mathfrak{a})$ is also a second-level concept. We have a second-level function that is not a concept in $\grave{\epsilon}\phi(\epsilon)$.

39 In order to have an example from analysis, let us consider the first derivative of a function. We regard the function as argument. If we take a particular function, for example ξ^2, as argument, then we obtain first another first-level function $2 \cdot \xi$; and only if we take an object as argument of this function—for example, the number 3—do we obtain as value an object: the number 6. The first derivative is accordingly to be regarded as a function of two arguments, the first of which must be a first-level function of one argument, the second of which must be an object. On this account we may call it an *unequal-leveled* function of two arguments. From it we obtain a second-level function of one argument if we saturate it with an object-argument—for example, the number 3, i.e., if we determine that the first derivative is to be formed for the argument 3.[29]

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[28] Cf. n. 12, above.

[29] It must be presupposed here, as in all of the examples drawn from arithmetic, that the signs of addition, multiplication, and so on, as well as that of the first derivative, have been so defined that a name

We also have an unequal-leveled function of two arguments in $—\phi(\xi)$, where "ξ" occupies and renders recognizable the place of the object-argument and "$\phi(\ )$" that of the function-argument. Since the value of this function is always a truth-value, we can call it an unequal-leveled relation. It is the relation of an object to a concept under which it falls.

We have second-level, equal-leveled relations in:

$$\begin{array}{l} \overset{\mathfrak{a}}{\smile}\!\!\top\!\!\!-\ \phi(\mathfrak{a}) \\ \quad\ \llcorner\ \psi(\mathfrak{a}) \end{array} \qquad \text{and} \qquad \begin{array}{l} \top\!\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!\!-\ \phi(\mathfrak{a}) \\ \qquad\ \llcorner\ \psi(\mathfrak{a}), \end{array}$$

where "ϕ" and "ψ" render recognizable the argument-places. In the latter relation, for example, stand the concepts $\xi^3 = 1$ and $\xi^2 = 1$; for we have

$$\begin{array}{l} \vdash\!\!\top\!\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!\!-\ \mathfrak{a}^3 = 1 \\ \qquad\quad \llcorner\ \mathfrak{a}^2 = 1; \end{array}$$

in words: at least one square root of 1 is also cube root of 1.

§23. Types of arguments and argument-places. Second-level functions of arguments of type 2 and type 3.

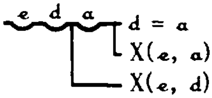
In the examples given hitherto, we have had as arguments functions of one argument. $\top\!\!\overset{\mathfrak{a}}{\smile}\!\!\overset{\mathfrak{e}}{\smile}\!\!\top\!\!-\ \phi(\mathfrak{a}, \mathfrak{e})$ is a second-level concept whose argument must be a function of two arguments. Under this concept there fall all relations such that there are objects which stand in these relations. We can, in fact, mention relations—one might call them empty relations—in which no objects stand to one another; for example, the relation $\begin{array}{l} \top\!\!\!-\ 2\cdot\xi = 2\cdot\zeta; \\ \llcorner\ \xi = \zeta \end{array}$

for, $\begin{array}{l} \vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!\overset{\mathfrak{e}}{\smile}\!\!\top\!\!\!-\ 2\cdot\mathfrak{a} = 2\cdot\mathfrak{e}. \\ \qquad\quad \llcorner\ \mathfrak{a} = \mathfrak{e} \end{array}$

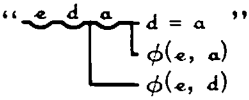
In order to have another example for this case, let us try to express the *many-oneness* of a relation. By this we understand that for every ξ-argument there exists not more than one ζ-argument such that the value of our function (relation) $X(\xi, \zeta)$ for these arguments is the True. We may also say: if from the fact that a stands to b in this relation, and that a stands to c in this 40
relation, it follows universally that b coincides with c, then we say that this relation is many-one. Or: if from the fact that $X(a, b)$ is the True and $X(a, c)$ is the True it follows universally

correctly formed out of these together with proper names always has a denotation—which, of course, the customary definitions do not achieve. In them, invariably, only numbers are taken into account, for the most part without saying what a number is.

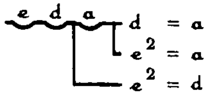
that $c = b$ is the True, then we call the function $X(\xi, \zeta)$ a many-one relation, if it is a relation at all.



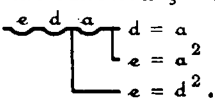
must be the True if the relation —— $X(\xi, \zeta)$ is to be many-one. If we put for "X" the "ϕ" that renders argument places recognizable, we then obtain in



the name of a second-level function that requires a first-level function of two arguments as argument. This second-level function is a second-level concept under which fall all many-one relations, but also functions $X(\xi, \zeta)$ such that —— $X(\xi, \zeta)$ is a many-one relation. (The many-oneness is here always intended in the direction from the ξ-argument to the ζ-argument.) If we take as a r g u m e n t of our second-level function the function $\xi^2 = \zeta$, then we obtain as value of the function,



that is, the True; whereas the False appears as function-value if we take as argument the function $\xi = \zeta^2$:



[30]

We see from these examples the great multiplicity of functions. We see, too, that there are basically different types of functions, since the various argument-places are basically different. Those argument-places, in fact, that are appropriate for admission of proper names, cannot admit names of functions, and vice versa. Further, those argument-places that may admit names of first-level functions of one argument, are unsuited to admit names of first-level functions of two arguments. Accordingly, we distinguish:

arguments of type 1: objects;
arguments of type 2: first-level functions of one argument;
arguments of type 3: first-level functions of two arguments.

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[30] With appropriate definition of ξ^2 for arguments that are not numbers.

In the same way we distinguish:

argument-places of type 1, which are appropriate to admit proper names;

argument-places of type 2, which are appropriate to admit names of first-level functions of one argument;

argument-places of type 3, which are appropriate tc admit names of first-level functions of two arguments. 41

Proper names and object-letters are *fitting* for the argument-places of type 1; names of first-level functions of one argument are *fitting* for the argument-places of type 2; names of first-level functions of two arguments are *fitting* for the argument-places of type 3. The objects and functions whose names are fitting for the argument-places of the name of a function, are *fitting* arguments for this function. Functions of one argument for which arguments of type 2 are fitting we call *second-level functions of one argument of type 2;* functions of one argument for which arguments of type 3 are fitting, we call *second-level functions of one argument of type 3.*

Just as in $\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$ we have the value of the second-level function $\overset{\mathfrak{a}}{\smile}\, \phi(\mathfrak{a})$ for the argument $\xi = \xi$, so too we may regard $\overset{\mathfrak{f}}{\smile}\!\begin{array}{l}\top\, \mathfrak{f}(1+1)\\ \llcorner\, \mathfrak{f}(2)\end{array}$ as value of a *third-level function* for the argument $\begin{array}{l}\top\, \phi(1+1)\\ \llcorner\, \phi(2)\end{array}$, which itself is a second-level function of one argument of type 2.

§24. General explanation of the use of function-letters.

It is now possible to explain generally the use of function-letters.

If after a concavity with a Gothic function-letter there follows a combination of signs composed of the name of a second-level function of one argument and this function-letter, which fills up the argument-places, then the whole denotes the True if the value of that second-level function is the True for every fitting argument; in all other cases, it denotes the False. Which places are argument-places of the *corresponding* second-level function, is to be decided according to the first rule of §8. Also the second rule of §8 has for function-letters all the validity it has for object-letters.

We have herewith introduced two third-level functions, whose names may be written thus:

$$\text{“}\overset{\mathfrak{f}}{\smile}\mu_{\beta}(\mathfrak{f}(\beta))\text{”} \quad \text{and} \quad \text{“}\overset{\mathfrak{f}}{\smile}\mu_{\beta\gamma}(\mathfrak{f}(\beta,\gamma))\text{”}$$

in which we render the argument-places recognizable here with "μ_{β}" and "$\mu_{\beta\gamma}$", just as we render the argument-places of types 2 and 3 recognizable with "ϕ" and "ψ", and of type 1 with "ξ" and "ζ". "μ_{β}" and "$\mu_{\beta\gamma}$" are, however, no more supposed to be signs of the Begriffsschrift than the former letters, but serve us only provisionally. If we take as arguments for the first-named third-level function the following second-level functions of one argument of type 2, in order:

$$\overset{\mathfrak{a}}{\smile}\phi(\mathfrak{a}), \qquad \phi(2), \qquad \overset{\mathfrak{a}}{\smile}\overset{\mathfrak{e}}{\smile}\begin{array}{l} \mathfrak{a}=\mathfrak{e} \\ \phi(\mathfrak{e}) \\ \phi(\mathfrak{a}) \end{array};$$

then we obtain as values:

$$\overset{\mathfrak{f}}{\smile}\overset{\mathfrak{a}}{\smile}\mathfrak{f}(\mathfrak{a}), \qquad \overset{\mathfrak{f}}{\smile}\mathfrak{f}(2), \qquad \overset{\mathfrak{f}}{\smile}\overset{\mathfrak{a}}{\smile}\overset{\mathfrak{e}}{\smile}\begin{array}{l} \mathfrak{a}=\mathfrak{e} \\ \mathfrak{f}(\mathfrak{e}) \\ \mathfrak{f}(\mathfrak{a}) \end{array}.$$

42 **§25. Generality with respect to second-level functions. Basic Law IIb.**

We still require a method of expressing generality with respect to second-level functions of one argument of type 2. One might suppose that this would not nearly suffice; but we shall see that we can make do with this, and that even this occurs only in a single proposition. It may be briefly observed here that this economy is made possible by the fact that second-level functions can be represented in a certain manner by first-level functions, whereby the functions that appear as arguments of the former are represented by their courses-of-values. But the notational device necessary for this does not belong to the primitive notations of the Begriffsschrift; we shall introduce it presently by the use of our primitive signs. Since our means of expression is used only in a single proposition, it is not necessary to explain it in full generality.

We indicate a second-level function of one argument of type 2 in this way:

$$\text{“}M_{\beta}(\phi(\beta))\text{”}$$

by using the *Roman function-letter "M"*,[31] as we indicate a

[31] Thus this letter is not an *object-letter*.

first-level function of one argument by "$f(\xi)$". "$\phi(\ )$" here renders recognizable the argument-place, just as "ξ" does in "$f(\xi)$". The letter "β" here in the brackets fills up the place of the argument of the function occurring as argument [of the whole]. The use of "$M_{\beta}(\phi(\beta))$" is to second-level functions precisely as the use of "$f(\xi)$" is to first-level functions. We avail ourselves of this expression of generality in the following Basic Law:

$$\begin{array}{l} \vdash\!\!\!\top\!\!\!-\!\!- M_{\beta}(f(\beta)) \\ \quad \llcorner\!\smile^{\mathfrak{f}}\!- M_{\beta}(\mathfrak{f}(\beta)) \end{array} \qquad \text{(IIb}$$

in words: What holds for all first-level functions of one argument holds also for any. Obviously this Basic Law is for our second-level functions what (IIa) is for first-level functions. "M_{β}" here corresponds to the letter "f" in (IIa); "f" here corresponds to the "a" in (IIa); and "$\mathfrak{f}$" here corresponds to "$\mathfrak{a}$" in (IIa). Let $\Omega_{\beta}(\phi(\beta))$ be a second-level function of one argument of type 2, whose place is rendered recognizable by "ϕ". Then $\smile^{\mathfrak{f}}\!\!- \Omega_{\beta}(\mathfrak{f}(\beta))$ is the True only if for every fitting argument the value of our second-level function is the True. Then $\Omega_{\beta}(\Phi(\beta))$ must also be the True. Consequently

$$\begin{array}{l} \top\!\!\!-\!\!- \Omega_{\beta}(\Phi(\beta)) \\ \llcorner\!\smile^{\mathfrak{f}}\!- \Omega_{\beta}(\mathfrak{f}(\beta)) \end{array}$$

is always the True, whatever first-level function of one argument $\Phi(\xi)$ may be, regardless whether $\smile^{\mathfrak{f}}\!\!- \Omega_{\beta}(\mathfrak{f}(\beta))$ is the True or the False, and our Basic Law (IIb) asserts this generally for every function of second level of one argument of type 2.

2. DEFINITIONS 43

i. GENERAL REMARKS

§26. Classification of signs. Names. Marks. Proposition of Begriffsschrift. Transition-sign.

The signs explained in the foregoing will now be used for the introduction of new names. But before I enter upon the rules to be observed in such an undertaking, it will aid understanding if I classify our signs and combinations of signs into types and adopt terminology for these.

The Gothic, Roman, and Greek letters occurring in the Begriffsschrift I will not call *names*, because they are not supposed to denote anything. On the other hand I call "$\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$", for example, a *name*, because it denotes the True; it is a *proper name*. Thus I call a *proper name*, or *name* of an object, a sign, simple or complex, that is supposed to denote an object, but not a sign that merely indicates an object.

If from a proper name we remove a proper name that forms a part of it or coincides with it, at some or all of the places where the constituent proper name occurs—but in such a way that these places remain recognizable as capable of being filled by one and the same arbitrary proper name (i.e., as being *argument-places of type 1*), then I call that which we obtain by this means a *name* of a first-level function of one argument. Such a name, combined with a proper name filling the argument-place, forms a proper name. Accordingly we have a function-name too in "ξ" itself, provided that the letter "ξ" is only to render recognizable the argument-place. The function named by it has the property that its value for every argument coincides with the argument itself.

If, from a name of a first-level function of one argument, we remove a proper name that forms a part of it, at all or some of the places where it occurs—but in such a way that these places remain recognizable as capable of being filled by one and the same arbitrary proper name (i.e., as being argument-places of type 1), then I call that which we obtain by this means a *name* of a first-level function of two arguments.

If, from a proper name, we remove a name of a first-level function that forms a part of it, at all or some of the places where it occurs—but in such a way that these places remain recognizable as capable of being filled by one and the same arbitrary name of a first-level function (i.e., as being argument-places of type 2 or type 3), then I call that which we obtain by this means a *name*
44 of a second-level function of one argument—an argument of type 2 or type 3, according as the argument-places are of type 2 or type 3.

Names of functions I call for short *function-names*.

It is unnecessary to pursue further these explanations of the types of names.

If, in a proper name, we replace proper names that form a part of it or coincide with it by object-letters, and function-names by function-letters, then I call that which we obtain by this means an *object-mark* or *mark* of an object. If this replacement is by Roman letters only, then I call the mark so obtained a *Roman object-mark*. Thus the object-letters are also object-marks, and the Roman object-letters are Roman object-marks.

A sign (proper name or object-mark) consisting solely of the function-name "$\xi = \zeta$" and proper names or object-marks standing in the two argument-places, I call an *identity*.

If, in a function-name, we replace proper names by object-letters and function-names by function-letters, then I call that which we obtain by this means a *function-mark:* a *mark* of a function of the same kind as that from whose name the mark has been obtained. If this replacement is by Roman letters only, then I call the mark so obtained a *Roman* mark of a function. The function-letters are also function-marks, and the Roman function-letters are Roman function-marks.

The judgment-stroke I reckon neither among the *names* nor among the *marks*; it is a sign of its own special kind. A sign consisting of a judgment-stroke and a name of a truth-value with a horizontal prefixed, I call a *proposition of Begriffsschrift*, or where no doubt can arise, a *proposition*. In the same way, I call *proposition of Begriffsschrift* (or *proposition*) a sign consisting of a judgment-stroke and a Roman mark of a truth-value with a horizontal prefixed.

Signs such as

"(a):————", "(a, β)::= = = = = =", "(a)::- - - - - -", "X",

which stand between the propositions so as to indicate the way in which the one below is yielded by the one above, I call *transition-signs*.

§27. The double-stroke of definition.

In order now to introduce new signs in terms of those already familiar, we require the *double-stroke of definition,* which appears as an iterated judgment-stroke coupled with a horizontal:

"$\Vdash$",

and which is used in place of the judgment-stroke where something is to be, not judged, but abbreviated by definition. We introduce a new name by means of a *definition* by stipulating 45
that it is to have the same sense and the same denotation as some name composed of signs that are familiar. Thereby the

new sign becomes the same in meaning* as that being used to define it; and thus the definition goes over directly into a proposition. Hence we may cite a definition in the same way as a proposition, in the process replacing the stroke of definition by the judgment-stroke.

A definition is always presented here in the form of an identity with "$\Vdash$" prefixed. To the left of the identity-sign we will always write the definiens, and to the right the definiendum. The definiens will be composed of familiar signs.

§28. Correct formation of names.

For definitions I now set up the following leading principle:

Correctly-formed names must always denote something.

I call a name *correctly*-formed, if it consists only of signs introduced as primitive or by definition, and if these signs are used only as what they were introduced as being: thus, proper names as proper names, names of first-level functions of one argument as names of functions of this kind, and so on, so that the argument-places are always filled by fitting names or marks.

To *correct* formation it further appertains, that Gothic and small Greek letters are used only in a way suitable to their purpose.

Thus a Gothic letter may stand over a concavity only if there immediately succeeds this concavity a mark of a truth-value, composed of the name or the mark of a function of one argument together with the same Gothic letter in the argument-places. A function-letter must occur either everywhere in its scope with one argument-place or everywhere with two. A Gothic letter may stand in an argument-place only if there stands to its left a concavity with the same letter, demarcating its scope. Only a Gothic letter may stand over a concavity.

A small Greek vowel may stand under the smooth breathing only if there immediately follows an object-mark, composed of a name or a mark of a first-level function of one argument and of the same Greek letter filling the argument-places. A small Greek vowel may stand in an argument-place only if the same letter precedes it with a smooth breathing, demarcating its scope.

**wird gleichbedeutend.* In view of the previous sentence, it seems best to translate this in the manner of the ordinary German *gleich bedeuten* and not to restrict it to Frege's technical use.

Only a small Greek vowel may occur with the smooth breathing.

§29. When does a name denote something?

Now we answer the question, When does a name denote something? confining ourselves to the following cases.

A name of a first-level function of one argument has a *denotation* (*denotes* something, succeeds in *denoting*) if the proper 46
name that results from this function-name by its argument-places' being filled by a proper name always has a denotation if the name substituted denotes something.

A proper name has a *denotation* if the proper name that results from that proper name's filling the argument-places of a denoting name of a first-level function of one argument always has a denotation, and if the name of a first-level function of one argument that results from the proper name in question's filling the ξ-argument-places of a denoting name of a first-level function of two arguments always has a denotation, and if the same holds also for the ζ-argument-places.

A name of a first-level function of two arguments has a *denotation* if the proper name that results from this function-name by its ξ-argument-places' being filled by a denoting proper name and its ζ-argument-places' being filled by a denoting proper name always has a denotation.

A name of a second-level function of one argument of type 2 has a *denotation* if, from the fact that the name of a first-level function of one argument denotes something, it follows generally that the proper name that results from its being substituted in the argument-places of our [name of a] second-level function has a denotation.

It follows that every name of a first-level function of one argument which, combined with every denoting proper name, forms a denoting proper name, is also such that if combined with any denoting name of a second-level function of one argument of type 2, it forms a denoting [proper] name.

The name "$\overset{\mathfrak{f}}{\smile}\mu_{\beta}(\mathfrak{f}(\beta))$" of a third-level function succeeds in denoting, if, from the fact that a name of a second-level function of one argument of type 2 denotes something, it follows generally that the proper name that results from its being substituted in the argument-place of "$\overset{\mathfrak{f}}{\smile}\mu_{\beta}(\mathfrak{f}(\beta))$" has a denotation.

§30. Two ways to form a name.

The foregoing provisions are not to be regarded as definitions of the phrases "have a denotation" or "denote something", because their application always presupposes that we have already recognized some names as denoting. They can serve only in the extension step by step of the sphere of such names. From them it follows that every name formed out of denoting names does denote something. This formation is carried out in this way: a name fills the argument-places of another name that are fitting for it. Thus there arises

[A] a proper name

[1] from a proper name and a name of a first-level function of one argument,

47 or [2] from a name of a first-level function and a name of a second-level function of one argument,

or [3] from a name of a second-level function of one argument of type 2 and the name "$\overset{\mathfrak{f}}{\smile}\,\mu_{\beta}(\mathfrak{f}(\beta))$" of a third-level function;

[B] the name of a first-level function of one argument

[1] from a proper name and a name of a first-level function of two arguments.

The names so formed may be used in the same way for the formation of further names, and all names arising in this way succeed in denoting if the primitive simple names do so.

A proper name can be employed in the present process of formation only by its filling the argument-places of one of the simple or composite [names of] first-level functions. Composite names of first-level functions arise in the way provided above only from simple names of first-level functions of two arguments by a proper name's filling the ξ- or the ζ-argument-places. Thus the argument-places that remain open in a composite function-name are always also argument-places of a simple name of a function of two arguments. From this it follows that a proper name that is part of a name formed in this way, wherever it occurs, always stands at an argument-place of one of the simple names of first-level functions [of two arguments]. If now we replace this proper name at some or all places by another, then the function-name* so arising is likewise formed in the way

*The text has "*Eigenname*" ("proper name"), incorrectly, because the procedure being described does not in this case yield a proper stated above, and thus it also has a denotation, if all the simple names employed as well succeed in denoting.

Of course in this we are assuming that the simple names of first-level functions of one argument have only one argument-place, and that the simple names of first-level functions of two arguments have only one ξ- and one ζ-argument-place. Otherwise it could indeed occur in the case of the replacement just described that related argument-places of simple function-names were filled by different names, and an explanation of the denotation for this case would be lacking. But this can always be avoided: and must be avoided, so as to prevent the occurrence of denotationless names. And there would certainly be no point in introducing several ξ-argument-places or several ζ-argument-places into the simple function-names.

On this assumption, then, we see the possibility of a second procedure for forming names of first-level functions. To wit: we begin by forming a name in the first way, and we then exclude from it at all or some places, a proper name that is a part of it (or coincides with it entirely)—but in such a way that these places remain recognizable as argument-places of type 1. The function-name resulting from this likewise always has a denotation if the simple names from which it is formed denote something; and it may be used further to form denoting names in the first way or in the second.

Thus, for example, in the first way we can form, from the prop- 48
er name "Δ" and the function-name "$\xi = \zeta$", the function-name "$\Delta = \zeta$", and further from the latter name and "Δ", the proper name "$\Delta = \Delta$". In the second way we form from "$\Delta = \Delta$" the function-name "$\xi = \xi$", and then in the first way, from this and the function-name "$\overset{\mathfrak{a}}{\smile}\, \phi(\mathfrak{a})$", form the proper name "$\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$".

All correctly-formed names are formed in this manner.

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name. The point being introduced is this. If, in a denoting 'composite' expression "$\Phi(\Delta)$" or "$\Phi(\Delta, \zeta)$", containing a constituent proper name "Δ", replacement of that proper name by a proper name always results in an expression having denotation provided that the name substituted has a denotation, then the result of removing "Δ", i.e. "$\Phi(\xi)$" or "$\Phi(\xi, \zeta)$", may be called a name having denotation. This procedure of forming a name is the 'second way'. The special case at this point is of the type "$\Phi(\Delta, \zeta)$". Two paragraphs below, it is of the other type, "$\Phi(\Delta)$" (specifically, "$\Delta = \Delta$").

§31. Our simple names denote something.

Let us apply the foregoing in order to show that the proper names, and names of first-level functions, which we can form in this way out of our simple names introduced up to now, always have a denotation. By what has been said, it is necessary to this end only to demonstrate of our primitive names that they denote something. These are

1. names of first-level functions of one argument,

$$\text{“}\!-\!\!\!-\!\!\!-\,\xi\text{”}, \qquad \text{“}\!-\!\!\!\top\,\xi\text{”}, \qquad \text{“}\backslash\xi\text{”};$$

2. names of first-level functions of two arguments,

$$\text{“}\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}\text{”}, \qquad \text{“}\xi = \zeta\text{”};$$

3. names of second-level functions of one argument of type 2,

$$\text{“}\overset{\mathfrak{a}}{\smile}\,\phi(\mathfrak{a})\text{”}, \qquad \text{“}\grave{\epsilon}\phi(\epsilon)\text{”};$$

4. names of third-level functions,

$$\text{“}\overset{\mathfrak{f}}{\smile}\,\mu_{\beta}(\mathfrak{f}(\beta))\text{”}, \qquad \text{“}\overset{\mathfrak{f}}{\smile}\,\mu_{\beta\gamma}(\mathfrak{f}(\beta, \gamma))\text{”},$$

of which the last may be left out of account because it will not actually be used.

First let it be noted that there always occurs only one ξ- and only one ζ-argument-place. We start from the fact that the names of truth-values denote something, namely, either the True or the False. We then gradually widen the sphere of names to be recognized as succeeding in denoting by showing that those to be adopted, together with those already adopted, form denoting names by way of the one's appearing at fitting argument-places of the other.

In order now to show, first, that the function-names "$-\!\!\!-\!\!\!-\,\xi$" and "$-\!\!\!\top\,\xi$" denote something, we have only to show that those names succeed in denoting that result from our putting for "ξ" a name of a truth-value (we are not yet recognizing other objects). This follows immediately from our explanations. The names obtained are again names of truth-values.

If in the function-names "$\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}$" and "$\xi = \zeta$" we put names of truth-values for "ξ" and for "ζ", then we obtain names that denote truth-values. Consequently our names of first-level functions of two arguments have denotations.

49 To investigate whether the name "$\overset{\mathfrak{a}}{\smile}\,\phi(\mathfrak{a})$" of a second-level function denotes something, we ask whether it follows universally

from the fact that the function-name "$\Phi(\xi)$" denotes something, that "$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a})$" succeeds in denoting. Now "$\Phi(\xi)$" has a denotation if, for every denoting proper name "Δ", "$\Phi(\Delta)$" denotes something. If this is the case, then this denotation either always is the True (whatever "Δ" denotes), or not always. In the first case "$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a})$" denotes the True, in the second the False. Thus it follows universally from the fact that the substituted function-name "$\Phi(\xi)$" denotes something, that "$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a})$" denotes something. Consequently the function-name "$\overset{\mathfrak{a}}{\smile}\,\phi(\mathfrak{a})$" is to be admitted into the sphere of denoting names. The same follows similarly for "$\overset{\mathfrak{f}}{\smile}\,\mu_{\beta}(\mathfrak{f}(\beta))$".

The matter is less simple with "$\acute{\varepsilon}\phi(\varepsilon)$"; for with this we are introducing not merely a new function-name, but simultaneously answering to every name of a first-level function of one argument, a new proper name (course-of-values-name); in fact not just for those [function-names] known already, but in advance for all such that may be introduced in the future. To the inquiry whether a course-of-values-name denotes something, we need only subject such course-of-values-names as are formed from denoting names of first-level functions of one argument. We shall call these for short *fair** course-of-values-names. We must examine whether a fair course-of-values-name placed in the argument-places of "$—\,\xi$" and "$\top\,\xi$" yields a denoting proper name, and further whether, placed in the ξ-argument-places or in the ζ-argument-places of "$\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}$" and "$\xi = \zeta$", it always forms a denoting name of a first-level function of one argument. If we substitute the course-of-values-name "$\acute{\varepsilon}\Phi(\varepsilon)$" for "$\zeta$" in "$\xi = \zeta$", then the question is thus whether "$\xi = \acute{\varepsilon}\Phi(\varepsilon)$" is a denoting name of a first-level function of one argument, and to that end it is to be asked in turn whether all proper names denote something that result from our putting in the argument-place either a name of a truth-value or a fair course-of-values-name. By our stipulations, that "$\acute{\varepsilon}\Psi(\varepsilon) = \acute{\varepsilon}\Phi(\varepsilon)$" is always to have the same denotation as "$\overset{\mathfrak{a}}{\smile}\,\Psi(\mathfrak{a}) = \Phi(\mathfrak{a})$", that "$\acute{\varepsilon}(—\,\varepsilon)$" is to denote the True, and that "$\acute{\varepsilon}(\varepsilon = \top\overset{\mathfrak{a}}{\smile}\,\mathfrak{a} = \mathfrak{a})$" is to denote the False, a denotation is assured in every case for a proper name of the form "$\Gamma = \Delta$", if "Γ" and "Δ" are fair course-of-values-names or

*rechte.

names of truth-values. Thereby it is also known that we always obtain a denoting proper name from the function-name "$\xi = (\xi = \xi)$", if we put in the argument-places a fair course-of-values-name. Since now according to our stipulations the function $——\xi$ always has the same value for the same argument as the function $\xi = (\xi = \xi)$, it is also known of the function-name "$——\xi$" that a proper name of a truth-value always results from it by substitution of a fair course-of-values-name. By our stipulations the names "$—\!\!\top\!\!—\Delta$" and "$\begin{array}{l}\top\,\Gamma\\ \llcorner\,\Delta\end{array}$" always have denotations if the names "$——\Delta$" and "$——\Gamma$" denote something. Since this is now the case if "Γ" and "Δ" are fair course-of-values-names, we always obtain denoting proper names from the function-names "$—\!\!\top\!\!—\xi$" and "$\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}$" by placing fair course-of-values-names or names of truth-values in the argument-places. We have seen that each of our simple names of first-level functions "$——\xi$", "$—\!\!\top\!\!—\xi$", "$\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}$", "$\xi = \zeta$", up to now recognized as denoting, produces denoting names upon admission of fair course-of-values-names in the argument-places. Thus the fair course-of-values-names may be admitted into our sphere of denoting names. Thereby, however, the same thing is decided for our function-name "$\grave{\epsilon}\phi(\epsilon)$", since it now follows universally from the fact that a name of a first-level function of one argument denotes something, that the proper name resulting from its being substituted in "$\grave{\epsilon}\phi(\epsilon)$" denotes something.

From among our primitive names there now remains only "$\backslash\xi$". We have determined that "$\backslash\Delta$" shall denote Γ, if [there exists an object Γ such that] "Δ" is a name of the course-of-values $\grave{\epsilon}(\epsilon = \Gamma)$, and that on the other hand "$\backslash\Delta$" shall denote Δ, if there exists no object Γ such that "Δ" is a name of the course-of-values $\grave{\epsilon}(\epsilon = \Gamma)$. By this a denotation is assured for all cases for a proper name of the form "$\backslash\Delta$" and therewith for the function-name "$\backslash\xi$".

§ 32. Every proposition of Begriffsschrift expresses a thought.

In this way it is shown that our eight primitive names have denotation, and thereby that the same holds good for all names correctly compounded out of these. However, not only a denotation, but also a sense, appertains to all names correctly formed from our signs. Every such name of a truth-value *expresses* a

sense, a *thought*. Namely, by our stipulations it is determined under what conditions the name denotes the True. The sense of this name—the *thought*—is the thought that these conditions are fulfilled. Now a proposition of Begriffsschrift consists of the judgment-stroke and of a name or a Roman mark of a truth-value. (But such a mark is transformed into the name of a truth-value by the introduction of Gothic letters in place of Roman letters and the prefixing of concavities according to §17. If we imagine this carried out, then we have only the case in which the proposition is composed of the judgment-stroke and a name of a truth-value.) It is now asserted by such a proposition that this name does denote the True. Since at the same time it expresses a thought, we have in every correctly-formed proposition of Begriffsschrift a judgment that a thought is true; and here 51
a thought certainly cannot be lacking. It will be the reader's task to make clear to himself the thought of each proposition of Begriffsschrift, and I shall take pains to facilitate this as much as possible at the outset.

The names, whether simple or themselves composite, of which the name of a truth-value consists, contribute to the expression of the thought, and this contribution of the individual [component] is its *sense*. If a name is part of the name of a truth-value, then the sense of the former name is part of the thought expressed by the latter name.

§33. Principles of definition.

The following are our standard principles for definitions:

1. Every name correctly formed from the defined names must have a denotation. Thus it must always be possible to produce a name, compounded out of our eight primitive names, that is the same as it in meaning*, and the latter must be unambiguously determined by the definitions, up to inessential choices of particular Gothic and Greek letters.

2. It follows from this that the same thing may never be defined twice, because it would then remain in doubt whether these definitions were consistent with one another.

3. The name defined must be simple; that is, it may not be composed of any familiar names or names that are yet to be defined; for otherwise it would remain in doubt whether the

**Gleichbedeutend ist.* Cf. translator's note, p. 83, above.

definitions of the names were consistent with one another.

4. If on the left-hand side of the definitional identity we have a proper name formed from our primitive names or defined names, then this always has a denotation, and we shall place on the right-hand side a simple sign not previously employed, which is now introduced by the definition as a proper name having the same meaning*, so that we may in future replace this sign wherever it occurs by the name standing on the left. Of course this sign may never be used as a function-name, for to do so would be to cut off the route back to the primitive names.

5. A name introduced for a first-level function of one argument may contain only a single argument-place. With more argument-places it would be possible to fill these with different names, and then the name defined would be being used as the name of a function of more than one argument, whereas it would not be defined as such. If a name of a first-level function of one argument is defined, the argument-places on the left-hand side of the definitional identity must be filled with one Roman object-letter, which also renders recognizable the argument-place of the new function-name on the right-hand side. The def-
52 inition then asserts that the proper name that results on the right-hand side from substitution of a denoting proper name in the argument-place, shall always have the same meaning** as that which results on the left-hand side from substitution of the same proper name in all argument-places. The one argument-place of the name defined thus represents all those of the definiens. Wherever the defined name may occur subsequently, its argument-place must always be filled by a proper name or an object-mark.

6. A name introduced for a first-level function of two arguments must contain two and only two argument-places. The mutually related argument-places on the left-hand side must be occupied by one and the same Roman object-letter, which also renders recognizable one of the two argument-places on the right-hand side; argument-places that are not related must contain different Roman letters. The definition then asserts that the proper name which results on the right-hand side from substitution of denoting proper names in the argument-places, shall

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**Gleichbedeutend.* Cf. the previous note.

***Gleichbedeutend sein.*

always have the same meaning* as that which results on the left-hand side from substitution of the same proper names in the corresponding argument-places. One argument-place on the right-hand side thus represents all the ξ-argument-places on the left-hand side, and the other represents all the ζ-argument-places.

7. Thus there must never occur on one side of a definitional identity a Roman letter that does not occur on the other. If the object-mark on the left-hand side is transformed into a correctly-formed proper name, by the Roman letters' being replaced by proper names, then by our stipulations the function-name defined always has a denotation.

Cases other than those just mentioned will not occur in the sequel.

ii. PARTICULAR DEFINITIONS

§ 34. Definition of the function $\xi \frown \zeta$.

It has already been suggested, in §25, that in further developments, instead of second-level functions, we may employ first-level functions. This will now be shown. As was indicated then, this is made possible through the functions that appear as arguments of second-level functions being represented by their courses-of-values—though of course not in such a way that they simply give up their places to them, for that is impossible.

In the first instance it is a matter only of designating the value of the function $\Phi(\xi)$ for the argument Δ, i.e., $\Phi(\Delta)$, by means of "Δ" and "$\grave{\epsilon}\Phi(\epsilon)$". I do so in this way:

$$\text{“}\Delta \frown \grave{\epsilon}\Phi(\epsilon)\text{”},$$

which is to mean the same as* "$\Phi(\Delta)$". The object $\Phi(\Delta)$ thus appears as the value of the function of two arguments $\xi \frown \zeta$ for Δ as ξ-argument and $\grave{\epsilon}\Phi(\epsilon)$ as ζ-argument.

But $\xi \frown \zeta$ must now be defined for all possible objects as arguments. This may be done as follows: 53

$$\Vdash \backslash\grave{\alpha}\left(\neg\!\overset{\mathfrak{g}}{\smile}\!\neg\!\!\begin{array}{l} \mathfrak{g}(a) = \alpha \\ u = \grave{\epsilon}\,\mathfrak{g}(\epsilon) \end{array}\right) = a \frown u \qquad (A$$

Since here a function of two arguments is being defined, two Roman letters occur on both the left- and the right-hand side. Although the definiens contains only familiar notation, a few explanatory remarks are in order. We have on the left-hand side

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* *Gleichbedeutend sein.*