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NATURAL DEDUCTION

A Proof-Theoretical Study

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PREFACE

Systems of natural deduction, invented by Jáskowski and by Gentzen in the early 1930's, constitute a form for the development of logic that is natural in many respects. In the first place, there is a similarity between natural deduction and intuitive, informal reasoning. The inference rules of the systems of natural deduction correspond closely to procedures common in intuitive reasoning, and when informal proofs—such as are encountered in mathematics for example—are formalized within these systems, the main structure of the informal proofs can often be preserved. This in itself gives the systems of natural deduction an interest as an explication of the informal concept of logical deduction.

Gentzen's variant of natural deduction is natural also in a deeper sense. His inference rules show a noteworthy systematization, which, among other things, is closely related to the interpretation of the logical signs. Futhermore, as will be shown in this study, his rules allow the deduction to proceed in a certain direct fashion, affording an interesting normal form for deductions. The result that every natural deduction can be transformed into this normal form is equivalent to what is known as the *Hauptsatz* or the *normal form theorem*, a basic result in proof theory, which was established by Gentzen for the calculi of sequents. The proof of this result for systems of natural deduction is in many ways simpler and more illuminating.

Gentzen's systems of natural deduction are developed in detail in Chapter I. Their characteristic features can be described roughly as follows. With one exception, the inference rules are of two kinds, viz., *introduction rules* and *elimination rules*. The properties of each sentential connective and quantifier are expressed by two rules, one of each kind. The introduction rule for a logical constant allows an inference *to* a formula that has the constant as principal sign. For instance, the introduction rule for & (and) allows the inference to A & B given the two premisses A and B, and the introduction rule for

$\supset$ (if—, then—) allows the inference to $A \supset B$ given a deduction of B from A. The elimination rule for a constant, on the other hand, allows an inference *from* a formula that has the constant as principal sign. For instance, the elimination rule for & allows an inference from $A \,\&\, B$ to A (or to B), and the elimination rule for $\supset$ allows the inference from the premisses A and $A \supset B$ to B.

For inferring certain formulas, the introduction rule gives thus a sufficient condition that is formulated in terms of subformulas of these formulas. The elimination rule, on the other hand, is related to the corresponding introduction rule according to a certain *inversion principle*: the elimination rule is in a sense only the inverse of the corresponding introduction rule.

This inversion principle will be described in more detail in Chapter II, and by its use the result mentioned above that every deduction can be brought to a certain normal form is easily obtained. A deduction in normal form proceeds from the assumptions of the deduction to the conclusion in a direct and rather perspicuous way without detours; roughly speaking, the assumptions are first broken down into their parts by successive applications of the elimination rules, and these parts are then combined to form the conclusion by successive applications of the introduction rules. This is established for classical logic in Chapter III and for intuitionistic and minimal logic in Chapter IV. Applications of the result are exemplified by some corollaries.

The systems are extended in Chapter V to incorporate second order logic also. The main results for first order logic are then extended to ramified second order logic.

In Chapter VI the results are extended in another direction to the modal systems known as S4 and S5, and in Chapter VII, to some systems based upon concepts of implication that differ from the usual ones. Among the concepts considered is one introduced by Church and another one introduced by Ackerman (studied especially by Anderson and Belnap).

Some related systems are taken up in three appendices. The relation between the systems of natural deduction and the calculi of sequents is described in Appendix A, and Gentzen's Hauptsatz is then obtained as a corollary. A demonstrably consistent set-theory developed by Fitch is discussed in Appendix B. The final appendix is devoted to some other systems of natural deduction that have been described in the literature. Among them are the first systems developed by Jaskowski, some variants of Gentzen's systems of natural deduction, and systems containing rules for existential instantiation.

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The main results concerning a normal form for natural deduction was presented at joint colloquiums at the Universities of Uppsala and Stockholm in 1961 and—in a more complete form—at two colloquiums at the University of Münster (Westfalen) in 1962. An abstract of it was presented at a meeting of the Association for Symbolic Logic in New York 1964. The material in Chapters II and IV is presented essentially as at these colloquiums. Some of the corollaries, however, are of a somewhat later date, and so is the present treatment of classical logic in Chapter III. Most of the content in Chapters VI and VII was included in seminars that I held at UCLA during the spring semester of 1964.

I am indebted to participants in these colloquiums and seminars for criticism and discussion. I am especially grateful to Professors Anders Wedberg, Stig Kanger, and Christer Lech, who read parts of the manuscript and made valuable suggestions. I also want to thank Fil.lic. Göran Enger for help with proof-reading, and Mrs. Muriel Bengtsson, Mr. John Swaffield, and Mr. Rolf Schock for checking parts of the manuscript with respect to the English.

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CONTENTS

Some Notational Conventions. The letters i, j, k, m, and n are used as numerical indices. They are usually supposed to range over the positive integers. Sometimes the range is to include o also, which will then—if not obvious from the context—be explicitely stated.

I sometimes give two definitions in one sentence by using square brackets to indicate two readings of the sentence.

A name followed by a numeral within square brackets is a reference to the bibliography at the end of the monograph.

The number of a theorem refers to its order within the chapter; the same holds for the corollaries.

I. NATURAL DEDUCTION OF GENTZEN-TYPE

A system of natural deduction can be thought of as a set of rules (of the "natural" kind suggested in the Introduction) that determines the concept of deduction for some language or set of languages. Together with a language such a system can thus be said to constitute a logical calculus.[1]

In this chapter, I shall describe systems of natural deduction for classical, intuitionistic, and minimal predicate logic (of first order). The systems, which essentially are of a kind introduced by Gentzen [3], will be called *Gentzen-type systems of natural deduction*. I begin by describing the languages that the systems are intended for (§ 1), and the inference rules and the deductions of the systems (§ 2).

§ 1. The languages of first order

The languages considered are languages of first order predicate logic formulated in the usual way. However, I follow the convenient practice (which is not always adopted) of using two kinds of signs that range over the individuals; one kind, called *variables*, is only used bound, and the other, called *parameters*, is only used free. The languages are to contain (primitive) signs for conjunction (&), disjunction (v), implication ($\supset$), universal quantification ($\forall$), existential quantification ($\exists$), and falsehood (or absurdity) ($\curlywedge$). In more detail, these languages (referred to by the letter L) are described as follows.

SYMBOLS. Every language L is to contain non-descriptive symbols divided into categories as follows:

[1] Cf. note 1 on p. 23.

1. *Individual variables.* There are to be denumerably many of this kind. As a syntactic notation to refer to arbitrary symbols of this kind, I use the letters: x, y, z.
2. *Individual parameters.* Denumerably many. Syntactic notation: a, b, c.
3. *n-place predicate parameters,* $n = 0, 1, 2, \ldots$. Denumerably many for each n. Syntactic notation: P^n, Q^n; or, when the number of places is arbitrary or clear from the context: P, Q.
4. *Logical constants.*
   (a) A sentential constant for falsehood (absurdity): $\curlywedge$.
   (b) Sentential connectives: &, v, and $\supset$.
   (c) Quantifiers: $\forall$ and $\exists$.
5. *Auxiliary signs:* Parantheses.

In addition a language L may contain a set of descriptive constants divided into the following categories:

6. *Individual constants.*
7. *n-place operation constants,* $n = 1, 2, \ldots$
8. *n-place predicate constants,* $n = 0, 1, \ldots$

All the different categories are to be disjoint. Certain strings of symbols are singled out as terms and formulas in the usual way.

TERMS. t is an *individual term* in a language L if and only if t is a (finite) symbol string and either

1) t is an individual parameter or an individual constant in L, or

2) $t = f(t_1\, t_2 \ldots t_n)$, where $t_1, t_2, \ldots, t_n$ are individual terms in L and f is an n-place operation constant in L.

FORMULAS. A is an *atomic formula* in L if and only if either (1) A is $\curlywedge$, or (2) A consists of an n-place predicate parameter or constant in L followed by n individual terms in L.

The notion of *formula* in L is defined inductively by:[1]

1) An atomic formula in L is a formula in L.

2) If A and B are formulas in L, then so are $(A \mathbin{\&} B)$, $(A \vee B)$, and $(A \supset B)$.

[1] For perspicuity, I shall usually give inductive definitions by clauses as above. It will always be possible to state them more accurately in the form of equivalences (as in the definition of terms above).

3) If A is a formula in L, then so are $\forall x A^*$ and $\exists x A^*$, where A^* is A or is obtained from A by replacing occurrences of an individual parameter with the variable x.

I sometimes omit outer parentheses around a formula and parentheses in repeated conjunctions and disjunctions. Parantheses may also be omitted under the convention that $\supset$ (and $\equiv$ introduced below) makes a greater break than & and v.

The ***degree*** of a formula A is defined as the number of occurrences of logical constants in A except $\curlywedge$.

Occasionally there are reasons to consider also symbol strings that are like terms or formulas except for containing variables at places where a term or formula has parameters; they will be called ***pseudo-terms*** and ***pseudo-formulas***. Formulas and terms are considered to be special cases of pseudo-formulas and pseudo-terms.

Let A be a pseudo-formula that is not atomic. Then, A has exactly one of the forms $(B \& C)$, $(B \vee C)$, $(B \supset C)$, $\forall x B$, and $\exists x B$; the symbol &, v, $\supset$, $\forall$, or $\exists$, respectively, is said to be the ***principal sign*** of A.

The ***scope*** of a certain occurrence of a logical constant in a pseudo-formula A is the part of A that has this occurrence as principal sign.

An occurrence of a variable x in a pseudo-formula A is ***bound*** or ***free*** according as the occurrence belongs or does not belong to the scope of a quantifier that is immediately followed by x. In a formula, all occurrences of variables are obviously bound.[1]

A ***universal*** [***existential***] ***closure*** of a formula A with respect to the parameters $a_1, a_2, \ldots, a_n$ is to be a formula of the form $\forall x_1 \forall x_2 \ldots \forall x_n A^*$ [$\exists x \exists x_2 \ldots \exists x_n A^*$], where the variables $x_1, x_2, \ldots, x_n$ are all different from each other and from variables occurring in A, and A^* is obtained from A by replacing each occurrence of a_i by x_i ($i \leqslant n$). A formula in which there are no parameters is a ***sentence***.

SOME NOTATIONS. Notations like A^t_x and A^t_u, where the letter A represents a formula and the letters t and u terms, are used to denote the result of substituting x or u, respectively, for all occurrences of t in A (if any). In contexts where a notation like A^t_x is used, it is always to be assumed that t does not occur in A within the scope of a quantifier that is immediately followed by x. A notation like A^x_t, where A

[1] The distinction between free and bound occurrences of variables can be left out entirely if we required in clause 3 in the definition of formula that x not occur in A.

represents a pseudo-formula and t a term, is used to denote the result of substituting t for all free occurrences of x in A.

Notations like $A^{a\,x}_{x\,t}$ are used to denote the result of first substituting x for a and then t for x. To denote the result of carrying out these two substitutions simultaneously, I use the notation $\mathrm{S}^{a\,x}_{x\,t}A$.

In the sequel, I shall usually simply speak about formulas and related notions, tacitly understanding a reference to a language L. The letters A, B, C, D, E, and F when standing alone will always represent formulas unless otherwise stated. As a part of another notation, such as $\forall xA$, they will also represent pseudo-formulas, but then the whole notation will always represent a formula.

The letters Γ and Δ will represent sets of formulas.

The letters t and u will represent individual terms unless otherwise stated.

I shall at no occasion write down the symbols mentioned above, but shall only speak about them, using the syntactical notations indicated in the preceding paragraphs.[1] How the symbols actually look, we may thus leave indeterminate.

Defined signs. There are no symbols for negation and equivalence in the languages L but "$\sim$" and "$\equiv$" are used syntactically to abbreviate names of formulas. Thus, $\sim A$ is defined to be the formula $A \supset \curlywedge$ and $A \equiv B$ to be the formula $(A \supset B)$ & $(B \supset A)$.

Subformulas. The notion of *subformula* is defined inductively by:

1) A is a subformula of A.

2) If B & C, $B \vee C$, or $B \supset C$ is a subformula of A, then so are B and C.

3) If $\forall xB$ or $\exists xB$ is a subformula of A, then so is B^x_t.

§ 2. Inference rules and deductions

A. Informal account

Preliminary explanations. I first give an informal account of how the deductions are obtained in the systems under consideration. All the notions that are introduced informally in this account will later be defined in part B of this section and in § 3.

[1] On the syntactical level, I use signs as names of themselves if there is no risk of confusion.

We start the deduction by inferring a consequence from some assumptions by means of one of the inference rules listed below. We indicate this by writing the formulas assumed on a horizontal line and the formula inferred immediately below this line. From the formula inferred—possibly together with some other formulas obtained in a similar manner or some other assumptions that we want to make—we now infer a new consequence in accordance with one of the inference rules. We again indicate this by arranging all the formulas involved in such a way that the premisses in the last inference come on a horizontal line and the consequence comes immediately below this line. Continuing in this way, we obtain successively larger configurations that have the form of a tree as in the example below, where we want to derive $A \supset (B \,\&\, C)$ from $(A \supset B) \,\&\, (A \supset C)$:

$$\dfrac{\dfrac{A \qquad \dfrac{(A \supset B) \,\&\, (A \supset C)}{A \supset B}}{B} \qquad\qquad \dfrac{A \qquad \dfrac{(A \supset B) \,\&\, (A \supset C)}{A \supset C}}{C}}{B \,\&\, C}$$

We note that, if a formula is used as a premiss in two different inferences, then it also occurs twice in the configuration.

At each step so far, the configuration is a deduction of the undermost formula from the set of formulas that occur as assumptions. The assumptions are the uppermost formula occurrences, and we say that the undermost formula depends on these assumptions. Thus, the example above is a deduction of $B \,\&\, C$ from the set $\{(A \supset B) \,\&\, (A \supset C), A\}$, and, in this deduction, $B \,\&\, C$ is said to depend on the top-occurrences of these two formulas.

As a result of certain inferences, however, the formula inferred becomes independent of some or all assumptions, and we then say that we discharge the assumptions in question. There will be four ways to discharge assumptions, namely:

1) given a deduction of B from $\{A\} \cup \Gamma$, we may infer $A \supset B$ and discharge the assumptions of the form A;

2) given a deduction of $\wedge$ from $\{\sim A\} \cup \Gamma$, we may infer A and discharge the assumptions of the form $\sim A$;

3) given three deductions, one of $A \vee B$, one of C from $\{A\} \cup \Gamma_1$, and one of C from $\{B\} \cup \Gamma_2$, we may infer C and discharge the assumptions of the form A and B that occur in the second and third deduction

respectively, i.e., below the end-formulas of the three deductions, we write C and then obtain a new deduction of C that is independent of the mentioned assumptions;

4) given one deduction of $\exists xA$ and one of B from $\{A_a^x\} \cup \Gamma$, we may infer B and discharge assumptions of the form A_a^x, provided that a does not occur in $\exists xA$, in B, or in any assumption—other than those of the form A_a^x—on which B depends in the given deduction.

To facilitate the reading of a deduction, one may mark the assumptions that are discharged by numerals and write the same numeral at the inference by which the assumption is discharged (see e.g. p. 21).

To continue the deduction in the example above, we may write $A \supset (B \,\&\, C)$ below $B \,\&\, C$ and obtain then a deduction of $A \supset (B \,\&\, C)$ from $\{(A \supset B) \,\&\, (A \supset C)\}$.

AN EXAMPLE. Before I list the inference rules, I give one further example of a deduction and illustrate how it may correspond to informal reasoning. An informal derivation of $\forall x \exists y(Pxy \,\&\, Pyx)$ from the two assumptions

(1) $$\forall x \exists y Pxy$$

(2) $$\forall x \forall y(Pxy \supset Pyx)$$

may run somewhat as follows:

From (1), we obtain

(3) $$\exists y Pay.$$

Let us therefore assume

(4) $$Pab.$$

From (2), we have

(5) $$Pab \supset Pba,$$

and from (4) and (5)

(6) $$Pba.$$

Hence, from (4) and (6), we obtain

(7) $$Pab \,\&\, Pba$$

and from (7) we get

(8) $$\exists y(Pay \,\&\, Pya)$$

Now, (8) is obtained from assumption (4), but the argument is independent of the particular value of the parameter b that satisfies (4). In view of (3), we therefore have:

(9) (8) is independent of the assumption (4).

Because of (9), (8) depends only on (1) and (2) and thus holds on these assumptions for any arbitrary value of a. Hence, the desired result:

(10) $$\forall x \exists y(Pxy \,\&\, Pyx).$$

The corresponding natural deduction is given below; the numerals refer to the steps in the informal argument above (rather than to the way the assumptions are discharged).

$$
\dfrac{\dfrac{{}^{(1)}\ \forall x\exists yPxy}{{}^{(3)}\ \exists yPay} \qquad \dfrac{\dfrac{{}^{(4)}\ Pab \qquad \dfrac{{}^{(4)}\ Pab \qquad \dfrac{\dfrac{\forall x\forall y(Pxy\supset Pyx)\ {}^{(2)}}{\forall y(Pay\supset Pya)}}{Pab\supset Pba\ {}^{(5)}}}{Pba\ {}^{(6)}}}{Pab\,\&\,Pba\ {}^{(7)}}}{\exists y(Pay\,\&\,Pya)\ {}^{(8)}}}{\dfrac{\exists y(Pay\,\&\,Pya)\ {}^{(9)}}{\forall x\exists y(Pxy\,\&\,Pyx)\ {}^{(10)}}}
$$

INFERENCE RULES. The inference rules consist of an introduction (I) rule and an elimination (E) rule for each logical constant except $\curlywedge$. We have thus a &I-rule, a &E-rule, a $\vee$I-rule etc. In addition, there are two rules for the sentential constant $\curlywedge$, one used in intuitionistic logic called the $\curlywedge_I$-rule and one used in classical logic called the $\curlywedge_C$-rule. The rules are indicated by the figures below. Letters within parentheses indicate that the inference rule discharges assumptions as explained above.

&I) $\dfrac{A \quad B}{A \& B}$ &E) $\dfrac{A \& B}{A} \quad \dfrac{A \& B}{B}$

vI) $\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$ vE) $\dfrac{A \vee B \quad \begin{matrix}(A)\\ C\end{matrix} \quad \begin{matrix}(B)\\ C\end{matrix}}{C}$

$\supset$I) $\dfrac{\begin{matrix}(A)\\ B\end{matrix}}{A \supset B}$ $\supset$E) $\dfrac{A \quad A \supset B}{B}$

$\forall$I) $\dfrac{A}{\forall x A^a_x}$ $\forall$E) $\dfrac{\forall x A}{A^x_t}$

$\exists$I) $\dfrac{A^x_t}{\exists x A}$ $\exists$E) $\dfrac{\exists x A \quad \begin{matrix}(A^x_a)\\ B\end{matrix}}{B}$

λ_I) $\dfrac{\lambda}{A}$ λ_C) $\dfrac{\begin{matrix}(\sim A)\\ \lambda\end{matrix}}{A}$

Restriction on the $\forall$I*-rule*: *a* must not occur in any assumption on which A depends.

Restriction on the $\exists$E*-rule*: *a* must not occur in $\exists x A$, in B, or in any assumption on which the upper occurrence of B depends other than A^x_a.

Restriction on the λ*-rules*: A is to be different from λ. This restriction is, of course, not essential, but saves us from considering certain trivial cases in the sequel.

Restriction on the λ_C*-rule*: A is not to have the form $B \supset \lambda$. This restriction is also only a matter of convenience. That nothing is lost by this restriction can easily be seen. Suppose we have an application of the λ_C-rule that does not satisfy the restriction. Then the assumption $(B \supset \lambda) \supset \lambda$ discharged by this application can be replaced with the following deduction of $(B \supset \lambda) \supset \lambda$ from $\{B\}$:

$$\dfrac{B \qquad \dfrac{\overset{(1)}{B \supset \curlywedge}}{}}{\dfrac{\curlywedge}{(B \supset \curlywedge) \supset \curlywedge}\ (1)}$$

By this replacement, the given application of the $\curlywedge_C$-rule is turned into an application of the $\supset$I-rule discharging the assumption B.

REMARK 1. Gentzen stated also two rules for negation, namely:

$$\sim\mathrm{I})\ \ \dfrac{\begin{matrix}(A)\\ \curlywedge\end{matrix}}{\sim A} \qquad\qquad \sim\mathrm{E})\ \ \dfrac{A \quad \sim A}{\curlywedge}$$

When we define negation as above, these rules become special cases of the $\supset$I-rule and the $\supset$E-rule respectively.

CLASSICAL, INTUITIONISTIC, AND MINIMAL LOGIC. The differences between classical, intuitionistic, and minimal logic have a simple and illuminating characterization in the systems of natural deduction. For deductions in *minimal logic*, we may use only the introduction and elimination rules for sentential connectives and quantifiers. For deductions in *intuitionistic logic*, we may use all these rules and in addition the $\curlywedge_I$-rule. For deductions in *classical logic*, we may use all the inference rules including the $\curlywedge_C$-rule (note that the $\curlywedge_I$-rule is a special case of the $\curlywedge_C$-rule).

REMARK 2. To get classical logic from intuitionistic logic, we can, instead of adding the inference rule $\curlywedge_C$, add all formulas of the form $A \vee \sim A$ as axioms. This is the course followed by Gentzen, who also considers the following inference rule for the elimination of double negation:

$$\frac{\sim\sim A}{A}$$

(By this addition, the $\curlywedge_I$-rule becomes redundant.)

This concludes the informal explanation of the Gentzen-type systems of natural deduction. I shall now give a more precise definition of the notion of a deduction and some related notions. (For another variant of Gentzen's systems, see Appendix C, § 2.)

B. DEFINITIONS

FORMULA-TREES. As explained above, the deductions are certain trees of formulas, or shorter, *formula-trees*. If $\Pi_1, \Pi_2, \ldots, \Pi_n$ is a sequence[1] of formula-trees, then

(1) $$(\Pi_1, \Pi_2, \ldots, \Pi_n/A)$$

is to be the tree obtained by arranging the configurations Π's and A so that the Π's end on a horizontal line immediately above A.[2] I shall write (1) also in the more graphical notation:

$$\frac{\Pi_1 \; \Pi_2 \ldots \Pi_n}{A}$$

NOTIONS CONCERNING INFERENCE RULES. The inference rules were indicated by certain figures above. I shall say that $(A_1, A_2, \ldots, A_n/A)$ is an *instance of a certain inference rule* if it has the form indicated by the corresponding figure. Thus, $(A_1, A_2, \ldots, A_n/B)$ is said to be an *instance of*:

the &*I-rule* if $n = 2$ and B is $A_1 \& A_2$;
the &E*-rule* if $n = 1$ and A_1 is $(B \& C)$ or $(C \& B)$ for some C;
the vI*-rule* if $n = 1$ and B is $(A_1 \vee C)$ or $(C \vee A_1)$ for some C;
the v*E-rule* if $n = 3$, A_1 has the form $(C \vee D)$ and $A_2 = A_3 = B$; etc.

If $(A_1, A_2, \ldots, A_n/B)$ is an instance of an inference rule, we call $A_1, A_2, \ldots, A_n$ the *premisses* and B the *consequence* of this instance. In an instance $(A, A \supset B/B)$ of the $\supset$E-rule, an instance $(B \vee C, A, A/A)$ of the vE-rule, or an instance $(\exists xB, A/A)$ of the $\exists$E-rule, A is said to be *minor premiss*. The first [second] occurrence of A in the instance of the vE-rule is the first [second] minor premiss of this instance. A premiss that is not minor is a *major premiss*.

DEDUCTION RULES. The inference rules do not characterize a system of natural deduction completely, since it is not stated in them how assumptions are discharged, and since the use of certain inference

[1] By a sequence, I understand simply a list of elements (rather than a function as in the set-theoretical sense). In particular, a sequence of one term is identified with the term itself, and I shall say that $\alpha_i (i \leq n)$ belongs to or is an element in the sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$. Empty sequences are not considered except when explicitly admitted.

[2] We have: Π is a formula-tree if and only if either (i) Π is a formula or (ii) Π is $(\Pi_1, \Pi_2, \ldots, \Pi_n/A)$, where $\Pi_1, \Pi_2, \ldots,$ and Π_n are formula-trees. Abstractly, we may understand (1) as just the ordered $(n+1)$-tuple $\langle \Pi_1, \Pi_2, \ldots, \Pi_n, A \rangle$, but I shall usually have the more concrete interpretation in mind.

rules are circumscribed by restrictions which are formulated in terms of what assumptions the premisses depend on. In particular, to characterize the roles of $\vee$E, $\supset$I, $\forall$I, $\exists$E, and the principle for indirect proof, I shall state a number of rules that I shall call *deduction rules*. The inference rules stated above for $\vee$E, $\supset$I, $\forall$I, $\exists$E, and $\curlywedge_C$ will be called *improper inference rules* while the others will be called *proper inference rules*.

A deduction rule can be thought of as a rule that allows us to infer a formula from a whole deduction given as a "premiss" and to determine what set of formulas the inferred formula depends on. For instance, the deduction rule for $\vee$E will allow us to infer C given three "premisses" consisting of a deduction of $A \vee B$ from Γ_1, a deduction of C from Γ_2, and a deduction of C from Γ_3. Further, it allows us to conclude that the inferred formula depends on the union of Γ_1, $\Gamma_2 - \{A\}$, and $\Gamma_3 - \{B\}$. When applying a deduction rule in the systems considered here, however, the only things we need to know about a given deduction of A from Γ are the Γ and A. A deduction rule can therefore be taken as a relation between pairs of the form $\langle \Gamma, A \rangle$. The form of an *instance of* the different *deduction rules* is to be as specified below:

$\vee$E: $\langle\langle \Gamma_1, A \vee B\rangle, \langle \Gamma_2, C\rangle, \langle \Gamma_3, C\rangle, \langle \Delta, C\rangle\rangle$ where $\Delta = \Gamma_1 \cup (\Gamma_2 - \{A\}) \cup (\Gamma_3 - \{B\})$.

$\supset$I: $\langle\langle \Gamma, B\rangle, \langle \Delta, A \supset B\rangle\rangle$ where $\Delta = \Gamma - \{A\}$.

$\forall$I: $\langle\langle \Gamma, A\rangle, \langle \Gamma, \forall x A^a_x\rangle\rangle$ where a does not occur in any formula of Γ.

$\exists$E: $\langle\langle \Gamma_1, \exists x A\rangle, \langle \Gamma_2, B\rangle, \langle \Delta, B\rangle\rangle$ where $\Delta = \Gamma_1 \cup (\Gamma_2 - \Gamma_3)$, Γ_3 being the set of all formulas A^x_a such that a does not occur in $\exists x A$, in B, or in any formula of Γ_2 except A^x_a.

$\curlywedge_C$: $\langle\langle \Gamma, \curlywedge\rangle, \langle \Delta, A\rangle\rangle$ where $\Delta = \Gamma - \{\sim A\}$ and A is different from $\curlywedge$ and is not of the form $\sim B$.

Systems of natural deduction. A system of natural deduction of the kind considered here is determined by a set of proper inference rules, a set of deduction rules, and a set of axioms.[1] We consider three

[1] In a logical calculus, as this notion is defined by Church [2] e.g., the deductions are determined by a set of inference rules and a set of axioms. Deduction rules of the kind considered above have, however, the effective character that is essential in the idea of a logical calculus.

such systems in this chapter, all without axioms. The *system for minimal logic* contains exactly all the rules for introduction and elimination of the sentential connectives and quantifiers as described above; i.e., the proper inference rules &I, &E, $\vee$ I, $\supset$ E, $\forall$E, and $\exists$I and the deduction rules $\vee$ E, $\supset$ I, $\forall$I, and $\exists$E. The *system for intuitionistic* [*classical*] *logic* contains all these rules and in addition the proper inference rule $\curlywedge_I$ [the deduction rule $\curlywedge_C$]. The systems for classical, intuionistic and minimal logic will be dencted by the letters C, I, and M. We get a system of natural deduction for a particular axiomatic theory that is expressed within a first order language by adding the set of axioms of the theory to one of these three systems.

Deductions. For systems of the general kind described above, we now single out certain formula-trees as deductions. We first define what is meant by Π being a *deduction in* a system S *of* a formula A *depending on* a set Γ of formulas:

1) If A is not an axiom in S, then A is a deduction in S of A depending on $\{A\}$.

2) If A is an axiom in S, then A is a deduction in S of A depending on the empty set.

3) If Π_i is a deduction in S of A_i depending on Γ_i for every $i \leq n$, then $(\Pi_1, \Pi_2, ..., \Pi_n/B)$ is a deduction in S of B depending on Δ provided that either

(i) $(A_1, A_2, ..., A_n/B)$ is a proper inference rule in S and Δ is the union of all Γ_i for $i \leq n$, or

(ii) $\langle\langle\Gamma_1, A_1\rangle, \langle\Gamma_2, A_2\rangle ..., \langle\Gamma_n, A_n\rangle, \langle\Delta, B\rangle\rangle$ is an instance of a deduction rule in S.

Π is a *deduction* in the system S *of A from* Γ if and only if Π is a deduction in S of A depending on Γ or some subset of Γ.

A is *deducible from* Γ in the system S—abbreviated as $\Gamma \vdash_S A$—if and only if there is a deduction in S of A from Γ.[1]

A deduction of A depending on the empty set is a *proof* of A; A is *provable* in the system S—abbreviated as $\vdash_S A$—if and only if there is a proof in S of A.

[1] When Γ is a unit set $\{B\}$, I shall sometimes simply say that Π is a deduction of A from B and shall write: $B \vdash_S A$. Sometimes I simply speak about deductions, tacitly understanding a reference to a system S.

§ 3. Notions concerning deductions

To be able to speak about deductions conveniently, I need several concepts, which I bring together in this section; most of them are probably clear from their wording or from the informal explanations in the first part of § 2.

NOTIONS CONCERNING TREES. The following notions, here stated for formula-trees, are to apply also to trees formed by elements other than formulas.

The letter Π will always stand for trees, and the letter Σ for sequences of trees including the empty one. If Σ is empty, we define (Σ/A) to be equal to A.

I take for granted the notion of an *occurrence of a formula* or (synonymously) a *formula occurrence* in a formula-tree. Two formula occurrences are said to be of the same form or shape if they are occurrences of the same formula; they are identical only if they also stand at the same place in the formula-tree. I use the letters A, B, ..., F [Γ and Δ] to stand for [sets of] formula occurrences as well.[1] Sometimes, when I want to distinguish between two occurrences of a formula A, I use superscript notation, like A^1, A^2, etc.

I also take for granted the meaning of saying that a formula occurrence A stands *immediately above* a formula occurrence B (or that B stands *immediately below* A) in a formula-tree Π.

A *top-formula* in a formula-tree Π is a formula-occurrence that does not stand immediately below any formula occurrence in Π. The *end-formula* of Π is the formula occurrence in Π that does not stand immediately above any formula occurrence in Π.

A sequence $A_1, A_2, ..., A_n$ of formula occurrences in a formula-tree Π is a *thread* in Π if (1) A_1 is a top-formula in Π, (2) A_i stands immediately above A_{i+1} in Π for each $i < n$, and (3) A_n is the end-formula of Π. When that is the case, A_i is said to stand *above* [*below*] A_j if $i < j$ [$i > j$].

If A is a formula occurrence in the tree Π, the *subtree of* Π *determined by* A is the tree obtained from Π by removing all formula occurrences except A and the ones above A.

Let A be a formula occurrence in Π, let $(\Pi_1, \Pi_2, ..., \Pi_n/A)$ be the

[1] Also in the same context, the letters A, B, ... may be used to represent both a certain formula and a particular occurrence of this formula, but in the text, it will then usually be stated in each case whether the shape or the occurrence is intended.

subtree of Π determined by A, and let $A_1, A_2, \ldots, A_n$ be the end-formulas of $\Pi_1, \Pi_2, \ldots, \Pi_n$ respectively. We then say that $A_1, A_2, \ldots, A_n$ are the formula occurrences immediately above A in Π in *their order from left to right*. We shall then also say that A_i is *side-connected* with A_j $(i, j \leq n)$.

If Γ is a set of top-formulas in Π, then $(\Sigma/\Gamma/\Pi)$ is to be the tree obtained as the result of writing Σ above each top-formula in Π that belongs to Γ (i.e. so that the end-formulas of the trees in the sequence Σ come on a horizontal line immediately above each such top-formula).[1] If Σ or Γ is empty, $(\Sigma/\Gamma/\Pi) = \Pi$.

When Γ is the unit set of a formula occurrence A in Π, I will write simply $(\Sigma/A/\Pi)$; and, when Γ is a set of formula occurrences of the shape A, I will sometimes denote this set by $[A]$ and write $(\Sigma/[A]/\Pi)$. In a more graphic notation, I write in the respective cases:

$$\begin{array}{c} \Sigma \\ \overline{(A)} \\ \Pi \end{array} \qquad\qquad \begin{array}{c} \Sigma \\ \overline{[A]} \\ \Pi \end{array}$$

The *length* of a formula-tree is the number of formula occurrences in the treee.

The notation used for substitution in formulas is also used in connection with formula-trees and sequences of such trees. Thus, Σ_t^a denotes the result of carrying out the substitution in question in all formulas that occur in the trees that belong to the sequence Σ.

APPLICATIONS OF INFERENCE RULES. Let B be a formula occurrence in a deduction Π and let $A_1, A_2, \ldots, A_n$ be all the formula occurrences immediately above B in Π in their order from left to right. Then $\alpha = (A_1, A_2, \ldots, A_n/B)$ has the form of an instance of an inference rule R and I shall say that α is an *application of* R *in* Π. It can happen that α has the form of an instance of both the $\curlywedge$-rule and some other rule. I shall then follow the convention of considering α to be an application of only the $\curlywedge$-rule. It then holds that α is an application of at most one inference rule. In the case above, I shall say that A_i $(i \leq n)$ is a *premiss*

[1] More precisely, we may define this operation by the following recursion, where it is assumed that Γ is a non-empty set of formula occurrences in Π.

1) If $\Pi = A$, then $(\Sigma/\Gamma/\Pi) = (\Sigma/A)$.

2) If $\Pi = (\Pi_1, \Pi_2, \ldots, \Pi_n/A)$ and Γ_i $(1 \leq i \leq n)$ are the members of Γ that belong to Π_i, then $(\Sigma/\Gamma/\Pi) = ((\Sigma/\Gamma_1/\Pi_1), (\Sigma/\Gamma_2/\Pi_2), \ldots, (\Sigma/\Gamma_n/\Pi_n)/A)$.

and that B is the *consequence of the application* α *of* R (sometimes leaving out "the application of" for brevity).

ASSUMPTIONS. Let Π be a deduction in a system S and let A be a top-formula in Π. Then, the formula occurrence A is said to be an *axiom in* Π if it has the form of an axiom in S and is said to be an *assumption in* Π if it does not have such a form.

A formula occurrence A in a deduction Π is said to *depend* in Π *on the set* Γ *of formulas* if the subtree of Π determined by A is a deduction depending on Γ. If $B \in \Gamma$, then we say that A *depends* in Π *on the formula* B. We sometimes need to consider also the assumptions, i.e., the particular top-formulas, on which A depends. For the system for classical logic, we define:

Let A be an assumption in a deduction Π and let τ be the thread that begins with A. Then, A is *discharged in* Π *at* B *by the application* α of R if and only if B is the first formula occurrence C in τ such that one of the following four conditions holds:

1) R is vE, the major premiss of α has the form $A \vee D$ or $D \vee A$ (for some D), and C is the first or second minor premiss of α respectively.

2) R is $\supset$I, C is the premiss of α, and the consequence of α has the form $A \supset C$.

3) R is $\exists$E, the major premiss of α has the form $\exists x A^a_x$, C is the minor premiss of α, and a does not occur in C or in any formula—other than the one of the form A—on which C depends.

4) R is $\curlywedge_C$, C is the premiss of α, A has the form $\sim D$, where D in its turn is not a negation, and the consequence of α is D.

For the systems for intuitionistic and minimal logic we make the same definition but leave out clause 4).

A formula occurrence B is said to *depend* in the deduction Π *on the assumption* A if B belongs to the thread τ in Π that begins with A and A is not discharged in Π at a formula occurrence that precedes B in τ.

PROPER PARAMETERS. A parameter a is said to be the *proper parameter of an application* α *of* $\forall$I *in* Π if α has the form $(A/\forall x A^a_x)$ and a actually occurs in A. A parameter a is said to be the *proper parameter of an application* α *of* $\exists$E *in* Π if α has the form $(\exists x A^a_x, B/B)$, an assumption of the form A is discharged by α, and a actually occurs in A. A parameter a is said to be a proper parameter in a deduction Π if it is the proper parameter of some application of $\forall$I or $\exists$E in Π.

Pure parameters. In the rest of this section, I deal with certain details concerning proper parameters in order to facilitate operations on deductions that will be carried out in the sequel. In most cases, however, it is sufficient that the proper parameters are chosen so that they satisfy clauses (i)–(iii) in the lemma on parameters below. The reader may content himself with realizing that this is possible.

By a *connection* in a deduction Π between two formula occurrences A and B in Π, I understand a sequence $A_1, A_2, \ldots, A_n$ of formula occurrences in Π such that $A_1 = A$, $A_n = B$, and one of the following conditions holds for each $i < n$:

1) A_i is not major premiss of an application of $\vee$E and $\exists$E, and A_{i+1} stands immediately below A_i; or vice versa;

2) A_i is premiss of $\supset$E, and A_{i+1} is side connected with A_i;

3) A_i is major premiss of an application of $\vee$E or $\exists$E, and A_{i+1} is an assumption discharged by this application; or vice versa;

4) A_i is a consequence of an application of $\supset$I or the $\curlywedge_C$-rule, and A_{i+1} is an assumption discharged by this application; or vice versa.

I shall say that a formula occurrence B is *linked* in the deduction Π *to* a formula occurrence A *by* a parameter a if there is a connection between A and B in Π such that a occurs in every formula occurence in the connection.

Remark. We may note in passing that the restrictions on the $\forall$I- and $\exists$E-rules can be liberalized. Thus, it is sufficient to require of an application α of the $\forall$I-rule that the premiss of α is not linked by the proper parameter of α to any assumption on which it depends; and to require of an application α of the $\exists$E-rule with major premiss $\exists xA$ and minor premiss B that, if A^x_a is an assumption discharged by α, then A^x_a is not linked by a to $\exists xA$, to B, or to any assumption on which B depends that differs in shape from that of A^x_a. If this more liberal restriction is adopted, the following is seen to hold: If Π_1 is a deduction of A from Γ and Π_2 is a deduction of B from $\{A\} \cup \Delta$ and $[A]$ is the set of assumptions of the form A on which the end-formula of Π_2 depends, then $(\Pi_1/[A]/\Pi_2)$ is a deduction of B from $\Gamma \cup \Delta$. — However, for our purpose, it will be more convenient to follow an opposite course, putting stronger restrictions on the proper parameters as follows.

I shall say that a proper parameter of an application α of $\forall$I [$\exists$E] in a deduction Π is *pure* in Π if every formula occurrence in Π in which a

occurs is linked by a to the premiss of α [an assumption discharged by α] and a is the only proper parameter of α (the latter being trivial in case of $\forall$I). I shall say that a deduction has *only pure parameters* when every proper parameter in Π is pure.

We can always transform a deduction Π so that all its proper parameters become pure. In addition, we can chose the proper parameters so that they come to belong to a given set K of individual parameters provided that K is sufficiently large (e.g., infinite). Namely, we take an uppermost application α of $\forall$I or $\exists$E that has a proper parameter that is not pure or does not belong to K (i.e., all applications of $\forall$I or $\exists$E with the consequence standing above the consequence of α are supposed to have proper parameters (if any) that are pure and belongs to K) and then we substitute a parameter b that belongs to K and does not occur in Π for all occurrences of every proper parameter a of α that belongs to formula occurrences linked by a to (i) the premiss of α in case α is an application of $\forall$I, (ii) an assumption discharged by α in case α is an application of $\exists$E. Then b is pure in the new deduction and pure parameters remain pure. We can obviously repeat this procedure until all proper parameters become pure and members of K, and we so have the following lemma. The three clauses (i)–(iii) are easy consequences of the fact that the parameters are pure.

LEMMA ON PARAMETERS. *If $\Gamma \vdash_S A$ and K is an infinite set of individual parameters, then there is a deduction* Π *in* S *of A from Γ such that all its proper parameters are pure and belong to K. In particular,* Π *then satisfies the following three clauses:*

(i) *The proper parameter of an application α of $\forall I$ in* Π *occurs in* Π *only in formula occurrences above the consequence of α.*

(ii) *The proper parameter of an appliation of* $\exists$E *in* Π *occurs in* Π *only in formula occurrences above the minor premiss of α.*

(iii) *Every proper parameter in* Π *is a proper parameter of exactly one application of the* $\forall$I*-rule or the* $\exists$E*-rule in* Π.

§ 4. An alternative definition of the deductions

With the help of the notions defined in § 3, a slightly different notion of deduction could have been defined. This alternative definition will, however, not be used before Chapter VI; the present section may thus be postponed until then.

Let all the inference rules (i.e., also the improper ones) be given with a system S of natural deduction. Let us say that Π is a *quasi-deduction* in a system S when Π is a formula-tree such that, if B is a formula occurrence in Π and $A_1, A_2, ..., A_n$ are all the formula occurrences immediately above B in Π in their order from left to right, then $(A_1, A_2, ..., A_n/B)$ is an application of an inference rule in S.

Axioms and assumptions in a quasi-deduction are defined as for deductions.

By a *discharge-function* $\mathfrak{J}$ *for a quasi-deduction* Π, we understand a function from a set of assumptions in Π that assigns to A either A itself or a formula occurrence in Π below A.

Let Π be a quasi-deduction and let $\mathfrak{J}$ be a discharge-function for Π. We say that an assumption A in Π is discharged with respect to $\mathfrak{J}$ at B if $\mathfrak{J}(A) = B$. B is said to depend with respect to $\mathfrak{J}$ on the assumption A if B belongs to the thread τ in Π that begins with A and A is not discharged with respect to $\mathfrak{J}$ at a formula occurrence that precedes B in τ. B is said to depend with respect to $\mathfrak{J}$ on the formula A if B depends on an assumption of the shape A.

For classical [intuitionistic or minimal] logic, we define $\mathfrak{J}$ as a *regular discharge-function* for a quasi-deduction Π if (i) the proper parameter in an application of $\forall$I does not occur in any assumption on which the premiss of this application depends and (ii) $\mathfrak{J}(A)$ is a premiss C in an application α of a rule R satisfying one of the conditions 1)–4) [1)–3)] in the definition on p. 27 of the phrase "A is discharged in Π at B by the application α of R".

To say that Π is a deduction of A from Γ as defined above is clearly equivalent to saying that Π is a quasi-deduction such that the end-formula has the form A and such that there exists a regular discharge-function $\mathfrak{J}$ for Π with the property that the end-formula of Π depends with respect to $\mathfrak{J}$ on formulas of Γ only.

However, note that, by the first definition, an assumption is discharged as early as possible whereas it is not required of a regular discharge-function $\mathfrak{J}$ that $\mathfrak{J}(A)$ be the uppermost formula occurrence in the thread starting with A that satisfies the stipulated conditions. In certain connections, it is an advantage not to require that an assumption be discharged as early as possible. We then use a *different notion of a deduction* defining a deduction of A depending on Γ as a pair $\langle\Pi, \mathfrak{J}\rangle$, such that (i) Π is a quasi-deduction with an end-formula of the shape A and (ii) $\mathfrak{J}$ is a regular discharge-function for Π such that the end-formula of Π depends with respect to $\mathfrak{J}$ exactly on the formulas of Γ. This way of defining the deductions is more in agreement with the way of Gentzen. Usually, however, it is more convenient to let the dependency on the assumptions be uniquely given with the formula-tree as first defined.