Brouwer's Cambridge lectures on intuitionism

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CAMBRIDGE UNIVERSITY PRESS
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LONDON NEW YORK NEW ROCHELLE
MELBOURNE SYDNEY

Luitzen Egbertus Jan Brouwer founded a school of thought whose aim was to include mathematics within the framework of intuitionistic philosophy; mathematics was to be regarded as an essentially free development of the human mind. What emerged diverged considerably at some points from tradition, but intuitionism has survived well the struggle between contending schools in the foundations of mathematics to influence the new constructive mathematics and exact philosophy.

This monograph contains a series of lectures dealing with most of the fundamental topics such as choice sequences, the continuum, the fan theorem, order and well-order.

Brouwer's own powerful style is evident throughout the work. Graduate students and professionals working in mathematics, logic and philosophy will welcome this masterful account.

Cover design by Michael and Shirley Tucker

CAMBRIDGE UNIVERSITY PRESS

0 521 23441 7

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
32 East 57th Street, New York, NY 10022, USA
296 Beaconsfield Parade, Middle Park, Melbourne 3206, Australia.

First published 1981

Printed in Great Britain at the University Press, Cambridge

British Library cataloguing in publication data

Brouwer, Luitzen Egbertus Jan
Brouwer's Cambridge lectures on intuitionism
1. Intuitionistic mathematics – Addresses,
essays, lectures
I. Dalen, D Van
511'.2 QA9.47 80-41239

ISBN 0 521 23441 7

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Editorial preface

In the years after the Second World War, from 1946 to 1951, Brouwer gave various series of lectures on intuitionism at the University of Cambridge. He decided to collect the material together in a monograph, which would be the first systematic exposition of intuitionistic mathematics in book form. As a matter of fact it was not his first such enterprise; in a way Brouwer meant his dissertation to be a comprehensive exposition of his philosophical and mathematical views. In 'The rejected parts of Brouwer's dissertation on the foundations of mathematics',† van Stigt tells how the intervention of Brouwer's thesis adviser prevented the insertion of the philosophical parts. Although he never got beyond the stage of intentions, Brouwer seriously considered the publication of a revised version of his thesis. In 1929 the Noordhoff publishing company invited him to carry out these plans, but Brouwer was at that time already involved in the preparation of a German monograph. From the late twenties onwards Brouwer gave expository and propagandistic lectures, mainly in Germany but also in other countries. According to eye-witnesses Brouwer, with his lectures, made quite an impression. The intuitionist 'putsch' was the topic of many a heated discussion.

In particular his *Gastvorlesungen* in Berlin in 1927 raised high hopes for the establishment of intuitionism as an accepted part of mathematical practice. The barely veiled rivalry between Berlin and Göttingen most probably played an important role in the enthusiastic reception of intuitionism in Berlin; the spirited

† van Stigt, W. P. (1979) The rejected parts of Brouwer's dissertation on the foundations of mathematics. *Historia Mathematica* **6**, 385–404.

challenge of Hilbert's complacent supremacy was welcomed by more than one mathematician, especially in Berlin. The *Berliner Tageblatt* invited Brouwer to contribute a series of articles on intuitionism in a public debate with Hilbert. At the same time the publishing house of Walter de Gruyter asked Brouwer if it could publish his Berlin lectures. In March 1927, when a first draft of the text was ready, de Gruyter proposed to Brouwer that he should expand the text into a book 'which the public no doubt will tear out of your hands'. Unfortunately nothing came of it; Brouwer dropped the Berlin lectures and started to work on the book, but never finished it.

Apart from his course on intuitionism in Amsterdam (1927/28), Brouwer also lectured in Groningen (1933), Geneva (1934), Cape Town (1952), the USA and Canada (1953).

These courses were all modelled after the Berlin lectures. The Vienna guest lecture, 'Die Struktur des Kontinuums',† the Cape Town lectures,‡ and his Canadian and American lectures§ are in a sense selections of Brouwer's projected *magnum opus*.

Eventually none of the projected books or monographs appeared. When van Stigt and I undertook the collecting of Brouwer's estate in 1976 we found, through the kind assistance of Professor Dijkman, various manuscripts, among which was one of the Berlin lectures and one of the Cambridge lectures. The Berlin lectures had not been rewritten or greatly changed, but the Cambridge lectures showed signs of being extensively reworked for a longer period. The manuscript we found carried the handwritten note (in Brouwer's handwriting) *Laatste herziening tekst van Cambridge-boek* (Final revision text of Cambridge book), so there is some reason to believe that it represents Brouwer's views (of the fifties) in a reasonably adequate way.

It is an open question why Brouwer never published his

† Brouwer, L. E. J. (1930) *Die Struktur des Kontinuums.* Gistel, Vienna. 14 pp.

‡ Brouwer, L. E. J. (1952) Historical background, principles and methods of intuitionism. *South African Journal of Science* **49**, 139–46.

§ Brouwer, L. E. J. (1954) Points and spaces. *Canadian Journal of Mathematics* **6**, 1–17.

lectures. In 1951, when Brouwer lectured for the last time in Cambridge, five of the six planned chapters were finished (the sixth chapter would probably have dealt with the theory of functions). Professor S. W. P. Steen and Dr N. Routledge, who were involved in the preparation of the Cambridge lectures, had the impression that Brouwer never intended to publish the lectures, and that he used the proposed book as a pretext to return to Cambridge where he loved to lecture. Nevertheless, Brouwer kept revising his manuscript long after he stopped lecturing in Cambridge.

Comparing the Cambridge lectures with earlier expositions, e.g. the Berlin lectures, it will strike the reader that Brouwer did not strive for new revolutionary breakthroughs, but rather for a consolidation of the fundamental material of intuitionism. I think that it is safe to agree that Brouwer's two most spectacular performances in his foundational work, given the basic notions (e.g. natural number, choice sequence), are *the continuity theorem* (involving continuity, bar induction, etc.) and *the strong counterexamples* (involving the creative subject). Although the ultimate exploitation of the notion of the creative subject is achieved only in 1949,† the notion already occurs implicitly in *Die Struktur des Kontinuums* and in the Berlin lectures (1927). In the Cambridge lectures extensive use is made of the creative subject, but *not* in the optimal way (most counterexamples are weak, i.e. in the form 'as long as non-tested statements are known we cannot say that', and not strong, i.e. of the form '$\neg A$ holds' or, in Brouwer's terminology, 'A is absurd'). It remains a baffling puzzle why Brouwer did not fully exploit the strength of the creative subject. Possibly he arrived at the stronger results (what he calls the contradictority of, for example, the classical theory of functions) only after he had designed his course, so that he did not choose to embark on a large-scale overhaul of his lecture notes. It is

† Brouwer, L. E. J. (1949a) De non-aequivalentie van de constructieve en de negatieve orderelatie in het continuum. *Indagationes Mathematicae* **11**, 37–9. Translated as 'The non-equivalence of the constructive and negative order relation on the continuum' in Brouwer, L. E. J. (1975) *Collected works*, Vol. I, ed. A. Heyting. North-Holland Publ. Co., Amsterdam. pp. 495–6.

also conceivable that Brouwer had second thoughts on the matter. It is remarkable that he published his first strong application of the creative subject in 1949 in Dutch,† whereas the first such application in English appeared only in 1954.‡ Notes written in pencil in the margin of the manuscript indicate that Brouwer entertained the idea of strengthening the presentation. He never did so, however.

It will be convenient to keep the notion of the creative subject, e.g. in Kreisel's presentation,§ in mind when reading the text. It will be helpful to sort out the weak from the strong negations. Posy has pointed out that, in particular where virtual and inextensible order are concerned, an analysis in terms of $\vdash_n$ is useful.‖

On the one hand Brouwer tries to obtain maximal generality; in particular in the theory of spreads he adds a great number of distinctions which result from a generous generalization of the basic notions as presented in earlier expositions.¶ On the other hand he wishes to restrict the notion of choice sequence by abolishing higher order restrictions (i.e. restrictions on restrictions, etc.), which were explicitly introduced in Brouwer (1942a).* In the Cambridge lectures Brouwer is vaguely suspicious of higher order restrictions: 'But at present the author is inclined to think this admission superfluous and perhaps leading to needless complications'. In Brouwer (1952)** this has been strengthened to: 'However, this admission is not justified by

† Brouwer, L. E. J. (1949a) De non-aequivalentie van de constructieve en de negatieve orderelatie in het continuum. *Indagationes Mathematicae* **11**, 37–9.

‡ Brouwer, L. E. J. (1954) An example of contradictority in classical theory of functions. *Indagationes Mathematicae* **16**, 204–5.

§ Kreisel, G. (1967) Informal rigour and completeness proofs. In *Problems in the philosophy of mathematics*, ed. I. Lakatos, pp. 138–86. North-Holland Publ. Co., Amsterdam.

‖ Posy, C. J. A note on Brouwer's definitions of unextendable order (to appear).

¶ Brouwer, L. E. J. (1925) Zur Begründung der intuitionistischen Mathematik I. *Mathematische Annalen* **93**, 244–57.

* Brouwer, L. E. J. (1942a) Zum freien Werden von Mengen und Funktionen. *Indagationes Mathematicae* **4**, 107–8.

Brouwer, L. E. J. (1952) Historical background, principles and methods of intuitionism. *South African Journal of Science* **49, 139–46.

close introspection and moreover would endanger the simplicity and rigour of further developments'. Unfortunately Brouwer has not elaborated the point, and so far no convincing arguments have come forward to decide the issue for or against higher order restrictions. The matter is of some interest as it has been argued that, according to Brouwer (1952), lawless sequences are not intuitionistically acceptable. It is curious that Brouwer never mentions this simplest of all notions of choice sequence in print; he mentions it, however, explicitly in a letter to Heyting in 1924. Freudenthal recalls that when, during the Berlin lectures, the notion of a sequence based on throwing a die was suggested Brouwer firmly rejected it (probably because it was based on the physical world).

A feature that might irritate a modern reader, but which typically belongs to Brouwer, is the consistent refusal to use symbolic notation. It seems tempting to ascribe this to Brouwer's aversion for formalization with its Hilbertian undertones. I think, however, that it was simply a characteristic of Brouwer's style. Even in his topological papers he writes in a leisurely style, avoiding the economic use of handy formalisms. In the present monograph, for example, Brouwer consistently uses such expressions as 'the absurdity of the absurdity of α', 'α is contradictory', where $\neg\neg\alpha$ and $\neg\alpha$ would be infinitely more readable. Still in 1951 Brouwer uses the old terminology of Schoenfliess for union and intersection, $\mathfrak{P}(M,N)$ and $\mathfrak{D}(M,N)$, but even in this instance Brouwer does not make an effective use of the notation. Almost always he prefers 'the union of M and N' to '$\mathfrak{P}(M,N)$'.

Taking into account Brouwer's views on communication it seems more reasonable to state that his style was a logical consequence of his personal experience and technique of transferring knowledge and insight. That is to say, in particular in the case of intuitionism that cannot be taught as if it were, say, linear algebra, there was a strong aspect of convincing, conversion, especially in the eyes of Brouwer who considered intuitionistic mathematics as the one and only correct mathematics. As a result a persuasive, personal, non-formal style is exactly what one would expect. The only change in notation I have made is to use the standard union and intersection symbols.

Together with the manuscript a number of scraps and private notes for use during the lectures were filed. I have collected some of them into an appendix, which also contains parts of an address he gave in 1951. Furthermore I have added a number of notes of Brouwer that are relevant to the matter and that may be clarifying. Notes by the editor in the text are contained in square brackets [ ]. In addition to Brouwer's own footnotes, indicated by symbols, there are numbered endnotes provided by the editor.

Thanks are due to Professor Dijkman, who donated all the material to the Brouwer Archive, and also to the Mathematics Department of the Rijksuniversiteit Utrecht for its generous secretarial and material assistance.

I am particularly indebted to W. P. van Stigt, who with great enthusiasm and ingenuity joined in the researches connected with Brouwer's estate, and also to the Netherlands Organization for the Advancement of Pure Research (Z.W.O.) that enabled van Stigt to spend the academic year 1976/77 in Utrecht.

In the matter of the collecting of the biographical and scientific material the Brouwer family, in the person of the executor of the will, Ir. L. E. J. Brouwer, has been very helpful. I am grateful for his generous cooperation in the founding of the Brouwer Archive and for his approval of the publication of the material.

I want to express my gratitude to the Cambridge University Press for its willingness to legitimize the present, rather aged, child and for the helpfulness of its staff, in particular Mrs Jane Holland, in the preparation of the manuscript.

D. van Dalen Utrecht, June 1980

1

Historical introduction and fundamental notions[1]

The gradual transformation of the mechanism of mathematical thought is a consequence of the modifications which, in the course of history, have come about in the prevailing philosophical ideas, firstly concerning the origin of mathematical certainty, secondly concerning the delimitation of the object of mathematical science. In this respect we can remark that in spite of the continual trend from object to subject of the place ascribed by philosophers to time and space in the subject–object medium, the belief in the existence of immutable properties of time and space, properties independent of experience and of language, remained well-nigh intact far into the nineteenth century. To obtain exact knowledge of these properties, called mathematics, the following means were usually tried: some very familiar regularities of outer or inner experience of time and space were postulated to be invariable, either exactly, or at any rate with any attainable degree of approximation. They were called axioms and put into language. Thereupon systems of more complicated properties were developed from the linguistic substratum of the axioms by means of reasoning guided by experience, but linguistically following and using the principles of classical logic. We will call the standpoint governing this mode of thinking and working the *observational* standpoint, and the long period characterized by this standpoint the observational period. It considered logic as autonomous, and mathematics as (if not existentially, yet functionally) dependent on logic.

For space the observational standpoint became untenable when, in the course of the nineteenth and the beginning of the twentieth centuries, at the hand of a series of discoveries with which the names of Lobatchefsky, Bolyai, Riemann, Cayley, Klein, Hilbert, Einstein, Levi-Cività and Hahn are associated, mathematics was gradually transformed into a mere science of numbers; and when besides observational space a great number of other spaces, sometimes exclusively originating from logical speculations, with properties distinct from the traditional, but no less beautiful, had found their arithmetical realization. Consequently the science of classical (Euclidean, three-dimensional) space had to continue its existence as a chapter without priority, on the one hand of the aforesaid (exact) science of numbers, on the other hand (as applied mathematics) of (naturally approximative) descriptive natural science.

In this process of extending the domain of geometry, an important part had been played by the *logico-linguistic method*, which operated on words by means of logical rules, sometimes without any guidance from experience and sometimes even starting from axioms framed independently of experience. Encouraged by this the *Old Formalist School* (Dedekind, Cantor, Peano, Hilbert, Russell, Zermelo, Couturat), for the purpose of a rigorous treatment of mathematics *and logic* (though not for the purpose of furnishing objects of investigation to these sciences), finally rejected any elements extraneous to language, thus divesting logic and mathematics of their essential difference in character, as well as of their autonomy. However, the hope originally fostered by this school that mathematical science erected according to these principles would be crowned one day with a proof of its non-contradictority was never fulfilled, and nowadays, after the logical investigations performed in the last few decades, we may assume that this hope has been relinquished universally.

Of a totally different orientation was the *Pre-intuitionist School*, mainly led by Poincaré, Borel and Lebesgue. These thinkers seem to have maintained a modified observational standpoint for the introduction of natural numbers, for the principle of complete induction, and for all mathematical entities springing from this source without the intervention of axioms of existence, hence for what might be called the 'separable' parts of arithmetic and of algebra. For these, even for such theorems as were deduced by means of classical logic, they postulated an existence and exactness independent of language and logic and regarded its non-contradictority as certain, even without logical proof. For the continuum, however, they seem not to have sought an origin strictly extraneous to language and logic. On some occasions they seem to have contented themselves with an ever-unfinished and ever-denumerable species of 'real numbers' generated by an ever-unfinished and ever-denumerable species of laws defining convergent infinite sequences of rational numbers. However, such an ever-unfinished and ever-denumerable species of 'real numbers' is incapable of fulfilling the mathematical function of the continuum for the simple reason that it cannot have a positive measure. On other occasions they seem to have introduced the continuum by having recourse to some logical axiom of existence, such as the 'axiom of ordinal connectedness', or the 'axiom of completeness', without either sensory or epistemological evidence. In both cases in their further development of mathematics they continued to apply classical logic, including the *principium tertii exclusi*, without reserve and independently of experience. This was done regardless of the fact that the non-contradictority of systems thus constructed had become doubtful by the discovery of the well-known logico-mathematical antonomies.

In point of fact, pre-intuitionism seems to have maintained on the one hand the essential difference in character between logic and mathematics, and on the other hand the autonomy of logic, and of a part of mathematics. The rest of mathematics became dependent on these two.

Meanwhile, under the pressure of well-founded criticism exerted upon old formalism, Hilbert founded the *New Formalist School*, which postulated existence and exactness independent of language not for proper mathematics but for meta-mathematics, which is the scientific consideration of the symbols occurring in perfected mathematical language, and of the rules of manipulation of these symbols. On this basis new formalism, in contrast to old formalism, *in confesso* made primordial practical use of the intuition of natural numbers and of complete induction. It is true that only for a small part of mathematics (much smaller than in pre-intuitionism) was autonomy postulated in this way. New formalism was not deterred from its procedure by the objection that between the perfection of mathematical language and the perfection of mathematics itself no clear connection could be seen.

So the situation left by formalism and pre-intuitionism can be summarized as follows: for the elementary theory of natural numbers, the principle of complete induction and more or less considerable parts of arithmetic and of algebra, exact existence, absolute reliability and non-contradictority were universally acknowledged, independently of language and without proof. As for the continuum, the question of its languageless existence was neglected, its establishment as a set of real numbers with positive measure was attempted by logical means and no proof of its non-contradictory existence appeared. For the whole of mathematics the four principles of classical logic were accepted as means of deducing exact truths.

In this situation intuitionism intervened with two acts, of which the first seems to lead to destructive and sterilizing consequences, but then the second yields ample possibilities for new developments.

FIRST ACT OF INTUITIONISM *Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is* *divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics.*

Inner experience reveals how, by unlimited unfolding of the basic intuition, much of 'separable' mathematics can be rebuilt in a suitably modified form. In the edifice of mathematical thought thus erected, language plays no part other than that of an efficient, but never infallible or exact, technique for memorizing mathematical constructions, and for communicating them to others, so that mathematical language by itself can never create new mathematical systems. But because of the highly logical character of this mathematical language the following question naturally presents itself. *Suppose that, in mathematical language, trying to deal with an intuitionist mathematical operation, the figure of an application of one of the principles of classical logic is, for once, blindly formulated. Does this figure of language then accompany an actual languageless mathematical procedure in the actual mathematical system concerned?*

A careful examination reveals that, briefly expressed, the answer is *in the affirmative, as far as the principles of contradiction and syllogism are concerned,*[2] *if one allows for the inevitable inadequacy of language as a mode of description and communication. But with regard to the principle of the excluded third, except in special cases, the answer is in the negative, so that this principle cannot in general serve as an instrument for discovering new mathematical truths.*

Indeed, if each application of the *principium tertii exclusi* in mathematics accompanied some actual mathematical procedure, this would mean that each mathematical assertion (i.e. an assignment of a property to a mathematical entity) could be *judged*, that is to say could either be proved or be reduced to absurdity.

Now every construction of a bounded finite nature in a finite mathematical system can only be attempted in a finite number of ways, and each attempt proves to be successful or abortive in a finite number of steps. We conclude that every assertion of possibility of a construction of a bounded finite nature in a finite mathematical system can be judged, so that in these circumstances applications of the *principium tertii exclusi* are legitimate.

But now let us pass to infinite systems and ask for instance if there exists a natural number n such that in the decimal expansion of π the nth, $(n+1)$th, ..., $(n+8)$th and $(n+9)$th digits form a sequence 0123456789. This question, relating as it does to a so far not judgeable assertion, can be answered neither affirmatively nor negatively. But then, from the intuitionist point of view, because outside human thought there *are* no mathematical truths, the assertion that in the decimal expansion of π a sequence 0123456789 either does or does not occur is devoid of sense.

The aforesaid property, suppositionally assigned to the number n, is an example of a *fleeing property*,[3] by which we understand a property f, which satisfies the following three requirements:

(i) for each natural number n it can be decided whether or not n possesses the property f;
(ii) no way of calculating a natural number n possessing f is known;
(iii) the assumption that at least one natural number possesses f is not known to be an absurdity.

Obviously the fleeing nature of a property is not necessarily permanent, for a natural number possessing f might at some time be found, or the absurdity of the existence of such a natural number might at some time be proved.

By the *critical number* κ_f of the fleeing property f we understand the (hypothetical) smallest natural number possessing f.[4] A natural number will be called an *up-number* of f if it is not smaller than κ_f, and a *down-number* if it is smaller than κ_f. Of course, f would cease to be fleeing if an up-number of f were found.

A fleeing property is called *two-sided with regard to parity* if neither of an odd nor of an even κ_f the absurdity of existence has been demonstrated. Let s_f be the real number which is the limit of the infinite sequence $a_1, a_2, \ldots$, where $a_\nu = (-2)^{-\nu}$ if ν is a down-number and $a_\nu = (-2)^{-\kappa_f}$ if ν is an up-number of f. This real number violates the principle of the excluded third, for neither is it equal to zero nor is it different from zero and, although its irrationality is absurd, it is not a rational number. Moreover if f is two-sided with regard to parity then s_f is neither $\geqq 0$ nor $\leqq 0$.

The belief in the universal validity of the principle of the excluded third in mathematics is considered by the intuitionists as a phenomenon of the history of civilization of the same kind as the former belief in the rationality of π, or in the rotation of the firmament about the earth. The intuitionist tries to explain the long duration of the reign of this dogma by two facts: firstly that within an arbitrarily given domain of mathematical entities the non-contradictority of the principle for a single assertion is easily recognized; secondly that in studying an extensive group of simple every-day phenomena of the exterior world, careful application of the whole of classical logic was never found to lead to error.†

The mathematical activity made possible by the first act of intuitionism seems at first sight, because mathematical creation by means of logical axioms is rejected, to be confined to 'separable' mathematics, mentioned above; while, because also the principle of the excluded third is rejected, it would seem that even within 'separable' mathematics the field of activity would have to be considerably curtailed. In particular, since the continuum appears to remain outside its scope, one might fear at this stage that in intuitionism there would be no place for analysis. But this fear would have assumed that infinite se-

† This means *de facto* that common objects and mechanisms subjected to familiar manipulations behave as if the system of states they can assume formed part of a finite discrete set, whose elements are connected by a finite number of relations.

quences generated by the intuitionistic unfolding of the basic intuition would have to be fundamental sequences,[5] i.e. predeterminate infinite sequences proceeding, like classical ones, in such a way that from the beginning the nth term is fixed for each n. Such however is not the case; on the contrary, a much wider field of development, including analysis and often exceeding the frontiers of classical mathematics, is opened by the second act of intuitionism.

SECOND ACT OF INTUITIONISM *Admitting two ways of creating new mathematical entities: firstly in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired* (so that, for example, infinite decimal fractions having neither exact values[6], nor any guarantee of ever getting exact values are admitted); *secondly in the shape of mathematical species, i.e. properties supposable for mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be 'equal' to it, definitions of equality having to satisfy the conditions of symmetry, reflexivity and transitivity.*

By the *elements* of a species we understand the mathematical entities previously acquired† for which the property in question holds.‡ These elements will also be said to *belong* to the species.

Two mathematical entities will be called *different* if their equality proves to be absurd. The notations for equality and difference will be $=$ and $\neq$ respectively.

Two infinite sequences of mathematical entities $a_1,a_2,\ldots$ and $b_1,b_2,\ldots$ will be said to be *equal*, or *identical*, if $a_\nu = b_\nu$ for each ν and *distinct* if a natural number n can be indicated (or calculated) such that a_n and b_n are different.

† It follows that during the development of intuitionist mathematics some species may have to be considered as being tacitly defined again and again in the same way.

‡ A species can be an element of another species, but never an element of itself!

A species is called *discrete* if any two of its elements can be proved either to be equal or to be different.

If the species M possesses an element which cannot possibly belong to the species N, or, what is the same, is different from each element of N, we shall say that M *deviates* from N.

The species M will be called a *subspecies* of the species N, and we shall write $M \subset N$, if every element of M can be proved to belong to N. If in addition N deviates from M, then M is called a *proper subspecies* of N. If each element of N either belongs to M or cannot possibly belong to M, then M is called a *removable subspecies* of N.

Two species are said to be *equal*, or *identical*, if for each element of either of them an element of the other equal to it can be indicated. They are called *different* if their equality is absurd, and *congruent* if neither can deviate from the other.[7] For instance the following three species are congruent to one another: the species P of the infinite sequences of 0s and 1s; the species M of those elements of P which either consist only of 0s or begin with a 1 or begin with a finite number of 0s followed by a 1; and the species N of those elements of P which either consist of 1s or begin with a 0, or begin with a finite number of 1s followed by a 0.

Obviously each property that is absurd for all elements of one of two congruent species is also absurd for those of the other. Suppose, conversely, that every property that is absurd for the elements of the one species is also absurd for the elements of the other. Then this holds in particular for the property of being different from each element of one of the two species, from which we immediately deduce the congruence of the two species. Note that two species can be congruent and different at the same time.

Let S be a species of species s. The property of being an element of all these species s will be called the *intersection* $\cap S$,[8] and the property of being an element of at least one of the species s will be called the *union* $\cup S$ of the species s.[9] If S has only a finite number of elements $s_1, \ldots, s_n$, or a fundamental sequence $s_1, s_2, s_3, \ldots$, we shall also write $s_1 \cap s_2 \cap \ldots \cap s_n, s_1 \cup s_2 \cup \ldots \cup s_n$, and $\cap_{i \in N} s_i$, $\cup_{i \in N} s_i$ respectively.[10]

A species which cannot possess an element is said to be *empty*. Two different species whose intersection is empty are called *disjoint*.

The above-mentioned well-nigh evident non-contradictority within an arbitrarily given domain of mathematical entities of the *simple principle of judgeability* or the *simple principle of the excluded third*, i.e. of the principle of the excluded third enunciated for an arbitrary single assertion, holds for the whole of intuitionistic mathematics. It even holds for the simultaneous enunciation of the principle of the excluded third for an arbitrary finite number of assertions. For, the enunciation of the principle of the excluded third for an assertion is itself an assertion, and *finite additivity of non-contradictority* in an arbitrarily given domain of mathematical assertions is easily established in the following way.

Let the assertions ρ and σ be non-contradictory, and let us start for a moment from the supposition ω that the conjunction τ of ρ and σ is contradictory. Then the truth of ρ would entail the contradictority of σ. Since the contradictority of σ clashes with the data, the truth of ρ is absurd, i.e. ρ is absurd. Thus a consequence of the supposition ω clashes with the data, and so this supposition is contradictory, i.e. τ is non-contradictory.

This finite additivity of the non-contradictority of the principle of judgeability cannot be extended to universal additivity; in particular the contradictority can be proved of the following *complete principle of judgeability*.

If a, b and c are species of mathematical entities, and both a and b are subspecies of c, and b consists of those elements of c which cannot belong to a, then c is identical with the union of a and b.

We formulate the following pair of corollaries of the simple and the complete principles of judgeability respectively, of which the former is non-contradictory and the latter contradictory:

(1) if of two assertions a and b, a is equivalent to the absurdity of b, then b is equivalent to the absurdity of a (*simple principle of reciprocity of absurdity* or *simple principle of truth by non-contradiction*);

(2) if a, b and c are species of mathematical entities, and both a and b are subspecies of c, and b consists of those elements of c which cannot belong to a, then a consists of those elements of c which cannot belong to b (*complete principle of reciprocity of absurdity* or *principle of reciprocity of complementarity* or *complete principle of truth by non-contradictority*).

Another pair of corollaries of the simple and of the complete principles of judgeability respectively follows:

(1) every mathematical assertion can be *tested*, i.e. can either be proved to be non-contradictory or to be absurd (*simple principle of testability*, which is non-contradictory);

(2) if a, b, d and c are species of mathematical entities, and a, b and d are subspecies of c, and b consists of those elements of c which cannot belong to a, and d of those elements of c which cannot belong to b, then c is identical with the union of b and d (*complete principle of testability*, which is contradictory).

The assertion mentioned above of the existence of a sequence 0123456789 in the decimal expansion of π so far neither satisfies the simple principle of judgeability nor the simple principle of testability.

Let us consider a 'subordinating' sequence of n predicates of absurdity: *absurdity of absurdity* ... *of absurdity*, or in a shortened form, abs abs ... abs. The classical point of view admits the principle of reciprocity of complementarity, and thus allows this sequence to be reduced, by repeated cancellation of pairs of successive predicates, either to truth or to absurdity. One might expect that from the intuitionistic point of view such cancellations are strictly excluded, so that unequal sequences of this kind would have to be treated as inequivalent. But, surprisingly, this is not the case; cancellations of the kind mentioned are admissable, provided that they leave the first predicate of the sequence untouched, as follows from the following intuitionistic

THEOREM *Absurdity of absurdity of absurdity is equivalent to absurdity.*[11]

PROOF *Firstly,* since implication of the assertion y by the assertion x implies implication of absurdity of x by absurdity of y, the implication of *absurdity of absurdity* by *truth* (which is an established fact) implies the implication of *absurdity of truth,* that is to say of *absurdity, by absurdity of absurdity of absurdity. Secondly,* since truth of an assertion implies absurdity of its absurdity, in particular truth of absurdity implies absurdity of absurdity of absurdity.

From the theorem thus proved it follows that in intuitionistic mathematics every subordinating sequence of $n>2$ predicates of absurdity can be reduced either to absurdity or to absurdity of absurdity.

Another corollary of the same theorem is the intuitionistic validity of the principle of reciprocity of complementarity for negative assertions, which proves that this principle has a larger domain of validity than the principle of the excluded third.

By a *node of order n* we understand a sequence of n natural numbers ($n \geqq 1$) called the *indices* of the node.

A node p' of order $n+m$ ($m \geqq 1$) will be called an *mth descendant* of the node p of order n, and p will be called the *mth predecessor* of p', if the sequence of indices of p is an initial segment of the sequence of indices of p'.

The union of the node p and the species of its descendants will be called a *pyramid*, of which p will be called the top.

If $m=1$, p' will also be called an *immediate descendant* of p and p the *immediate predecessor* of p'.

The immediate descendants of a node p of order n in their natural order (i.e. ordered according to their last index) constitute a species Q of nodes. This species will be called a *row of nodes* of order $n+1$ and the *ramifying row* of p, whilst p will be called the *dominant* of Q.

The species of the nodes of order 1, in their natural order, will be called the *row of nodes of order* 1.

A finite sequence of nodes consisting of a node p_1 of order 1, an immediate descendant p_2 of p_1, an immediate descendant p_3 of $p_2, \ldots$ up to an immediate descendant p_n of p_{n-1}, will be called a *stick of order n*.

An infinite (but not necessarily predeterminate) sequence of nodes consisting of a node p_1 of order 1, an immediate descendant p_2 of p_1, an immediate descendant p_3 of p_2, etc., *ad infinitum*, will be called an *arrow*.

The arrow may proceed throughout with complete freedom, i.e. in the passage from p_ν to $p_{\nu+1}$ the choice of a new index to be joined to those of p_ν may be completely free for each ν as long as the creating subject likes. But this freedom of proceeding may at any stage be completely abolished at the beginning, or at any p_ν, by means of a law fixing all further nodes in advance. From this moment the arrow concerned will be called a *sharp arrow*. Finally the freedom of proceeding, without being completely abolished, may at some p_ν undergo some restriction, and later on further restrictions.†

All these intervening acts, as well as the choices of the p_ν themselves, may (at the beginning or at any later stage) be made

† In some former publications of the author restrictions of freedom of future restrictions of freedom, restrictions of freedom of future restrictions of freedom of future restrictions of freedom, and so on, were also admitted. But at present the author is inclined to think this admission superfluous and perhaps leading to needless complications. [A stronger statement can be found in Brouwer (1952).]

to depend on the influence of possible future occurrences in the world of mathematical thought of the creating subject.

Let ρ be a fundamental sequence $a_1, a_2, \ldots$ into which the species of nodes has been arranged in such a way that each node comes before its descendants, and before the nodes it precedes in its row of nodes. Then even if no details of this arrangement are known, the sequence of indices of each a_ν can be reconstructed as soon as for each a_ν its ramifying row can be indicated [including the row of nodes of order 1]. The arrangement can be effected, for example, in the following way.

Let G_n be the species of the nodes of order $\leqq n$ and indices $\leqq n$, $G_{n\nu}$ the species of the nodes of G_n of order ν, and A_n $(n \geqq 2)$ the species of the nodes of G_n not belonging to G_{n-1}. Each $G_{n\nu}$ is enumerated in such a way that p precedes q if the first index in which they differ is smaller for p than for q. Then each G_n is enumerated by making each $G_{n\nu}$ precede $G_{n,\nu+1}$. This implies an enumeration for each A_n. Finally the species of the nodes is enumerated by making G_1 precede each A_ν, and each A_ν precede $A_{\nu+1}$.

Let us suppose that in ρ an assignment is made to each node successively of either a 'figure', i.e. no thing or a mathematical entity previously acquired, or the predicate of being 'sterilized', in such a way that each descendant of a sterilized node is sterilized likewise, that the figures assigned in the case of non-sterilization are predeterminate (but not the decisions between sterilization and non-sterilization) and that for each non-sterilized node a non-sterilized immediate descendant can be indicated. Such a sequence of assignments will be called a *spread direction.*

If, instead of requiring that for each non-sterilized node a non-sterilized immediate descendant can be indicated, we impose the condition that for each n at least a non-sterilized node of order n is available, the sequence of assignments will be called a *spread haze direction.*

If the decisions of sterilization or non-sterilization are predeterminate, a spread direction will be called a *spread law*, and a spread haze direction a *spread haze law*.

A spread direction for which a non-sterilized node of order 1 can be indicated will be called *substantial*. If, however, all nodes of order 1 are sterilized, the spread direction will be called *empty*.

The part $\psi(\sigma,\chi)$ of a spread direction σ which concerns the pyramid χ will be called a *pyramid direction*, and a *pyramidal subdirection of* σ.

Each 'free arrow' of a spread direction σ, i.e. each arrow which avoids the nodes sterilized by σ, yields an infinite sequence of figures. These infinite sequences of figures, by virtue of their genesis, together with all infinite sequences equal to any one of them, may be considered as the elements of a species $\eta(\sigma)$. This species is called a *spread*.† The species of the infinite sequences derived in the same way from a spread haze direction is called a *spread haze*.

The existence of an element is guaranteed neither for a spread nor for a spread haze.

According to σ each pyramid χ yields a subspread $\eta(\sigma,\chi)$ of $\eta(\sigma)$. It is easily proved that the union of a finite number or a fundamental sequence of spreads is again a spread.

[In our discussion of the material of the Cambridge lectures we will use conventions and notations as presented in, for example, Kleene & Vesley (1965) or Troelstra (1977). These conventions deviate inessentially from Brouwer's in that zero is counted as a natural number, and that the empty sequence, $\langle\ \rangle$, is taken into consideration. The reader should keep these points

† Viewing the creation of the elements of the corresponding spread we can say that a *spread law* yields an *instruction* according to which, if again and again an arbitrary natural number is chosen as 'index', each of these choices has as its predeterminate effect, depending also on the earlier choices, that either a certain 'figure' (viz. either no thing or a mathematical entity previously acquired) is generated, or that the choice is 'sterilized', in which case the figures already generated are destroyed and generation of any further figures is prevented, so that all further choices are sterilized likewise. It was from this definition that intuitionist analysis started originally.

in mind when translating Brouwer's notions into the standard notation of contemporary practice.

We list the relevant notions:

(i) $x,y,z,\ldots$ vary over natural numbers;

(ii) $f,g,h,\ldots$ vary over lawlike functions from N to N (sharp arrows);

(iii) $\xi,\eta,\zeta,\ldots$ vary over choice sequences (arrows);

(iv) $n,m,\ldots$ vary over (codes of) finite sequences of natural numbers (nodes);

(v) the empty sequence $\langle\ \rangle$ (or λ) has code 0;

(vi) finite sequences will be written as $\langle n_0,\ldots, n_{k-1}\rangle$;

(vii) the concatenation function $*$ satisfies $\langle n_0,\ldots, n_{k-1}\rangle * \langle n_k,\ldots, n_p\rangle = \langle n_0,\ldots, n_p\rangle$ and $n * \langle\ \rangle = n = \langle\ \rangle * n$;

(viii) initial segments of functions or sequences are given by $\bar{f}x = \langle f0,\ldots, f(x-1)\rangle$, $\bar{\xi}x = \langle \xi 0,\ldots, \xi(x-1)\rangle$ and $\bar{f}0 = \langle\ \rangle = \bar{\xi}0$;

(ix) the length of a sequence is given by $l(\langle n_0,\ldots, n_{k-1}\rangle) = k$ and $l(\langle\ \rangle) = 0$;

(x) $n \preceq m$ if $\exists n'(n * n' = m)$, $n \prec m$ if $n \preceq m \wedge n \neq m$, $n \prec_1 m$ if $\exists x(n * \langle x\rangle = m)$, where $\preceq$ ($\prec$) is the predecessor, or successor, relation (strict predecessor, or strict successor, relation) and $\prec_1$ is the immediate predecessor, or successor, relation.

The species of all (codes of) finite sequences of natural numbers is denoted by SEQ.

In the Cambridge lectures Brouwer considered spreads in a more general setting. The natural denumeration ρ of all nodes is assumed, and by following it step by step one assigns figures to nodes, or sterilizes nodes. The sterilization is subject to the condition that the descendant of a sterilized node is itself sterilized. The figures assigned to nodes are predeterminate, i.e. no entities depending on choice sequences are allowed (higher order spreads were considered in Brouwer (1942b) and shown to be superfluous).

One can view the effect of sterilization as the specifying of a subtree (where trees are always considered to be closed under predecessors). Since the sterilization is, in general, not a predeterminate process, the resulting subtree of the universal tree may depend on choice parameters.

In a spread direction we have the strong condition $\forall n(\neg\mathrm{St}(n) \rightarrow \exists m \succ_1 n(\neg\mathrm{St}(n)))$, where $\mathrm{St}(n)$ stands for 'n is sterilized'. We use $\mathrm{St}(n)$ instead of the more precise $\mathrm{St}_\sigma(n)$.

In a *spread haze* direction, however, we have the much weaker condition $\forall x \exists n(l(n) = x \wedge \neg\mathrm{St}(n))$. A spread haze direction thus can be, for example, well-founded, which is impossible for spread directions with at least one non-sterilized node.

In the definition of a spread direction nothing is said about the nodes of order 1. They may all be sterilized; then the spread direction will not contain an arrow (infinite path) so it is called empty. If there is a non-sterilized node of order 1, the spread direction is called *substantial*. Note that it is not possible to decide if a spread direction is empty or substantial. A substantial spread direction can be characterized by $\neg\mathrm{St}(\langle\ \rangle)$.

A *spread law* is a spread direction in which the decision between sterilization and non-sterilization is predeterminate, i.e. $\exists f \forall n(\neg\mathrm{St}(n) \leftrightarrow f(n) = 0)$, so the subtree determined by the spread law has a lawlike characteristic function. This is the original notion of a spread (German: *Menge*), cf. Brouwer (1918), Kleene & Vesley (1965 p. 58), and Troelstra (1977 p. 127).

The notion of a spread was illustrated by Brouwer by the following example, taken from course notes of 1933. Consider two persons A and B. A calls out a natural number, which he has freely chosen. B has a stock of figures (signs), and according to a certain law B acts upon the number called by A in one of the following three ways.

(i) B writes down a figure and tells A to go on. A then calls out another natural number.

(ii) B writes down a figure and tells A that he may stop. Now A can stop or continue, but if A continues B does not write down any more signs.

(iii) B decides to destroy the results. A now stops and B destroys all the figures he has written down. We express this by saying that the choice sequence of A leads to sterilization (Dutch: *Stuiting*, German: *Hemmung*).

This is virtually the definition given in Brouwer (1918), where the notion was introduced. Note that the assignment of figures to sequences of numbers is supposed to be given by a law. The case presented under (ii) allows for the introduction of finite sequences. In the Cambridge lectures Brouwer replaced the halting by allowing B to write nothing (the empty figure). In (iii) the crucial notion of sterilization is introduced. One can think of this act of sterilization as a means of indicating subtrees of the universal tree of all finite sequences of natural numbers.

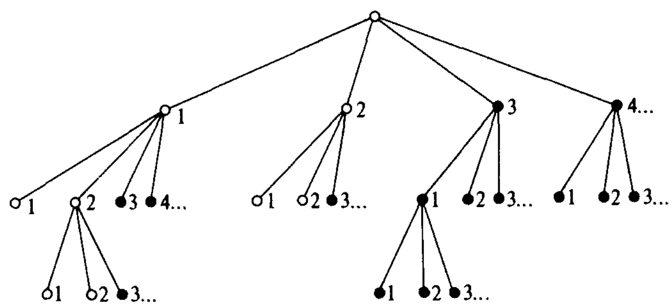
In 'Points and spaces' (Brouwer, 1954) Brouwer simplifies his presentation by dropping the sterilization clause. He achieves this either by allowing for each node all immediate descendants or by assigning a number m such that only immediate descendants with last index $\geqq m$ are allowed. Also in 'Intuitionism: an introduction' (Heyting, 1956) a notion of spread without sterilization is presented. This approach has been followed in all of the subsequent literature, e.g. Kleene & Vesley (1965), Troelstra (1969).

One might wonder why Brouwer chose to use the unwieldy notion of sterilization, whereas the 'subtree method' would do just as well. The reason is that Brouwer took the notion of mental, sequential process very seriously. For example, a construction is a mental process consisting of consecutive steps that might very well lead to an impossibility (for a geometric example see van Dalen, 1978), in which case the whole sequence is destroyed. Therefore the possibility of sterilization was built into his definition of spread. The intuitive picture is that of the chronicle of all construction attempts of a special kind (e.g. the construction of Cauchy sequences).

We proceed by exhibiting one of Brouwer's examples from his 1933 course.

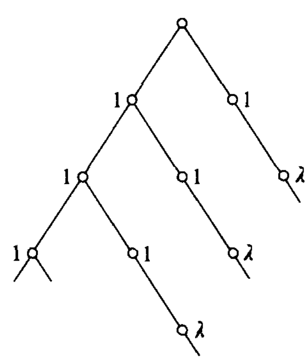
> Construction of a spread consisting of three elements, such that the corresponding sequences of figures will be three given figures $\square$, x, σ.
>
> Consider the following law. If A presents the first time 1, 2 or 3 then B writes down $\square$, x or σ with the comment 'you may stop'. All other first choices of A lead to sterilization, i.e. destruction.

It is customary and convenient to represent spreads geometrically by pictures of trees. Strictly speaking there should always be two trees: one for the choice sequences of natural numbers, one for the associated sequences of figures. One often only considers the latter tree if it contains all the relevant information. We have added an extra node at the top of the tree in this example (Brouwer does not do this).



EXAMPLE Each node $\langle n_1,\ldots,n_k\rangle$ with $\max\{n_i | i \leq k\} > 2$ is sterilized. The resulting subtree of non-sterilized nodes is the binary tree. The sterilized nodes are indicated in the figure as black ones. Note that all descendants of black nodes are black. We now introduce an assignment:

$$\langle n_1,\ldots,n_k\rangle \mapsto \begin{cases} 1 \text{ if } n_k \text{ is not preceded by any 2,} \\ \lambda \text{ else,} \end{cases}$$



where λ is the empty word (or, in Brouwer's terminology, 'no thing').

We cannot make a tree of the generated sequences. The associated sequences can conveniently be represented by a labelled tree (p. 19). The resulting set of sequences consists of all sequences which have only initial segments of the form $\langle 1,1,1,\ldots,1\rangle$. (N.B. We cannot say 'all finite sequences $\langle 1,1,\ldots,1\rangle$ plus one infinite sequence of 1s'.)]