Logic, First Course, Winter 2020. Week 6, Lecture 2. Back to course website

Natural deduction for disjunction

In this section, we continue the study of natural deduction, focusing on disjunction.

Introduction rule for disjunction

The rule is: if you have ϕ on line ℓ_1, then you may write $\phi \vee \psi$ on any subsequent line $\ell > \ell_1$. Likewise, if you have ϕ on line ℓ_1, then you may write $\psi \vee \phi$ on any subsequent line $\ell > \ell_1$.

This rule is abbreviated as $I\vee$, where the 'I' is for *introduction*.

In terms of a picture, the rule is either of the following:

$l_1. \quad \varphi$

$l. \quad \varphi \vee \psi \quad : I\vee \, l_1$

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$l_1. \quad \varphi$

$l. \quad \psi \vee \varphi \quad : I\vee \, l_1$

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Note that the rule does **not** require that ψ appear on any previous line. In many ways, this is what gives $I\vee$ its strength.

Example of disjunction introduction

Example 1: Show that $p, (p \vee q) \to r \vdash r$.

Here is the proof written out by hand:

Show: $p, (p \vee q) \to r \vdash r.$

1. p : assumption
2. $(p \vee q) \to r$: assumption
3. $p \vee q$: $I\vee$ 1
4. r : $E\to$ 2, 3

You can try to input it this proof here, or come back later for practice:

exercise

```
p, ((p ∨ q) → r) ⊢ r

1.
```

Another example of disjunction introduction

Example 2: Show that $a, c, ((a \vee b) \wedge (b \vee c)) \rightarrow d \vdash d$.

Show: $a, c, ((a \vee b) \wedge (b \vee c)) \rightarrow d \vdash d$

1.	a	: assumption
2.	c	: assumption
3.	$((a \vee b) \wedge (b \vee c)) \rightarrow d$	: assumption
4.	$a \vee b$	: I∨ 1
5.	$b \vee c$	: I∨ 2
6.	$(a \vee b) \wedge (b \vee c)$	: I∧ 4, 5
7.	d	: E→ 3, 6

You can try to input it this proof here, or come back later for practice:

exercise

a, c, (((a ∨ b) ∧ (b ∨ c)) → d) ⊢ d

1.

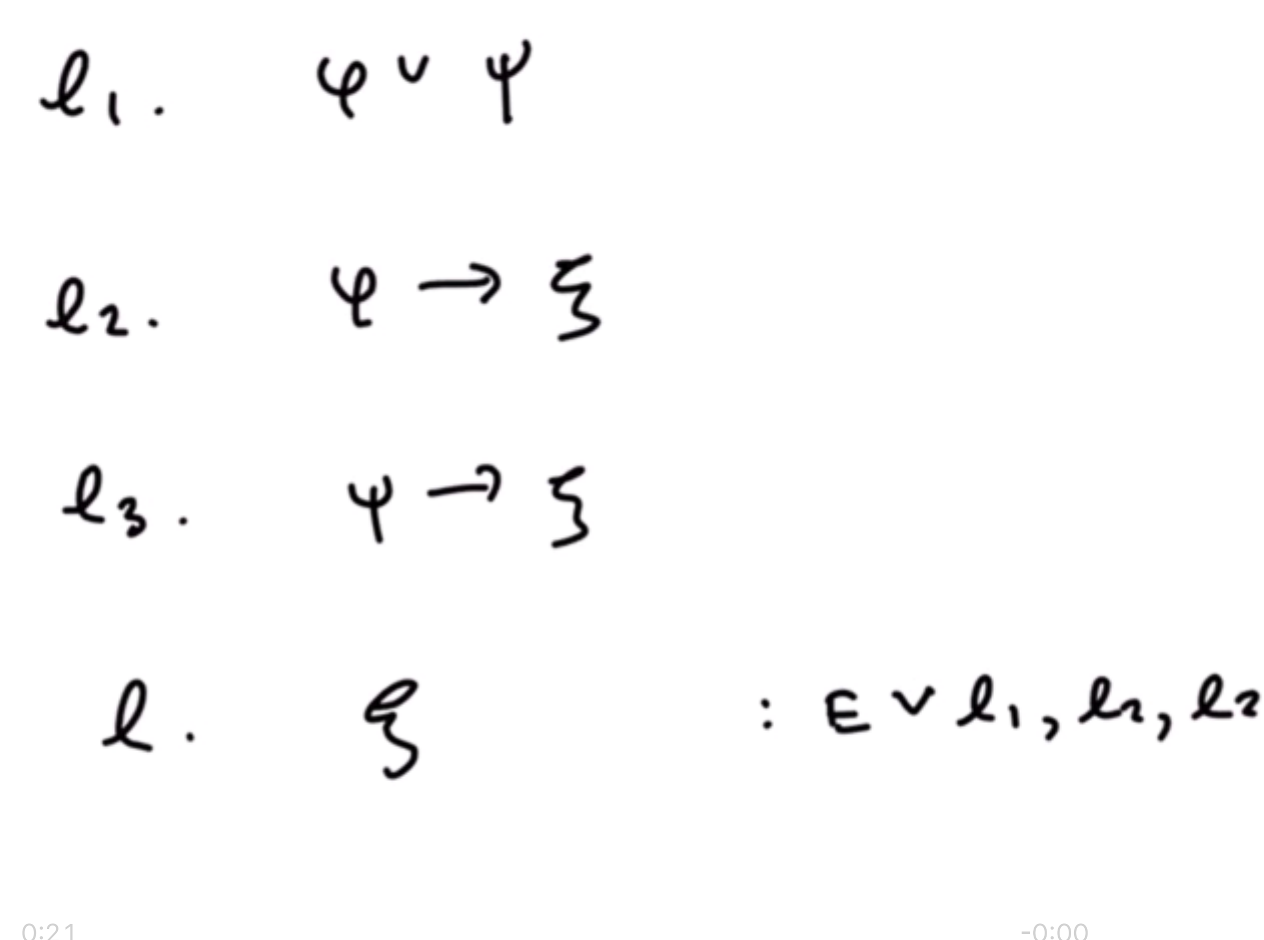
Elimination rule for disjunction

The rule is: if you have $\phi \vee \psi$ on line ℓ_1, and you have $\phi \to \xi$ on line ℓ_2, and you have $\psi \to \xi$ on line ℓ_3, then you may write ξ on any subsequent line $\ell > \ell_1, \ell_2, \ell_3$.

Again, the order in which ℓ_1, ℓ_2, ℓ_3 occurs does not matter. All that matters is the all of three of these come before the ℓ, where we apply the rule.

This rule is abbreviated as $E\vee$, where the 'E' is for *elimination*.

In terms of a picture, the rule is the following:



Note that the rule just says that "reasoning by cases" is valid.

Example of disjunction elimination

Example 3: Show that $(c \vee d) \wedge a, a \wedge (c \rightarrow e), (d \rightarrow e) \wedge b \vdash e$

Here is the proof written out by hand, where the idea is to use arrow elimination to get lines with $c \vee d$ and $c \rightarrow e$ and $d \rightarrow e$ and then to apply disjunction elimination:

Show: $(c \vee d) \wedge a \;,\; a \wedge (c \rightarrow e), (d \rightarrow e) \wedge b \;\vdash\; e$

1.	$(c \vee d) \wedge a$	: assumption
2.	$a \wedge (c \rightarrow e)$	: assumption
3.	$(d \rightarrow e) \wedge b$	: assumption
4.	$c \vee d$	: $E\wedge$ 1
5.	$c \rightarrow e$	: $E\wedge$ 2
6.	$d \rightarrow e$	: $E\wedge$ 3
7.	e	: $E\vee$ 4, 5, 6

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You can try to input it this proof here, or come back later for practice:

exercise

```
((c ∨ d) ∧ a), (a ∧ (c → e)), ((d → e) ∧ b)
⊢ e
```

1.

Another example of disjunction elimination

Example 4: Show that $(a \wedge b) \vee (a \wedge c) \vdash a$

This proof is typical in that often, to apply disjunction elimination, one first has to prove the one or both of the two arrow statements. In this case, we have to prove $(a \wedge b) \rightarrow a$ and $(a \wedge c) \rightarrow a$. Each of these proofs is itself an easy instance arrow introduction preceded by conjunction elimination.

Here is the proof written out by hand:

Show: $((a \wedge b) \vee (a \wedge c)) \vdash a$

1.	$(a \wedge b) \vee (a \wedge c)$	: assumption
2.	$a \wedge b$	: assumption
3.	a	: $E\wedge$ 2
4.	$(a \wedge b) \rightarrow a$	: $I\rightarrow$ 2-3
5.	$a \wedge c$	: assumption
6.	a	: $E\wedge$ 5
7.	$(a \wedge c) \rightarrow a$	: $I\rightarrow$ 5, 6
8.	a	: $E\vee$ 1, 4, 7

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You can try to input it this proof here, or come back later for practice:

exercise

```
((a ∧ b) ∨ (a ∧ c)) ⊢ a
```

```
1.
```

Deriving commutativity of disjunction

Example 5: Show that $\vdash (p \vee q) \rightarrow (q \vee p)$

To prove this, we think about using arrow introduction at the end to go from the disjunction $p \vee q$ to the conclusion $q \vee p$. To do this, we need to get $p \rightarrow (q \vee p)$ and $p \rightarrow (q \vee p)$. But both of these follow via simple applications of arrow introduction preceded by disjunction introduction.

Here is the proof written out by hand:

Show: $\vdash (p \vee q) \to (q \vee p)$

1.	$p \vee q$	: assumption
2.	p	: assumption
3.	$q \vee p$	: $I\vee$ 2
4.	$p \to (q \vee p)$	: $I\to$ 2–3
5.	q	: assumption
6.	$q \vee p$	: $I\vee$ 5
7.	$q \to (q \vee p)$	: $I\to$ 5–6
8.	$q \vee p$	: $E\vee$ 1, 4, 7
9.	$(p \vee q) \to (q \vee p)$	: $I\to$ 1–8

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You can try to input it this proof here, or come back later for practice:

exercise

$\top \vdash ((p \vee q) \to (q \vee p))$

1.

These are lecture notes written for this course.[1]

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1. It is run on the Carnap software, which is↩