The Reflection Method for the Numerical Solution of Linear Systems

Numerical Methods (MA2003)

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Introduction

- This paper discusses Cimmino's iterative method for the solution of the system $A\mathbf{x} = \mathbf{b}$, of n linear equations on $\mathbb{R}^n$, where A is a real $n \times n$ sparse matrix that is initially supposed to be invertible and $\mathbf{x}$, $\mathbf{b}$ belong to $\mathbb{R}^n$ are column vectors.
- Cimmino's observation states that *'Given two lines on a plane intersecting at **Z** and a point **P** $\neq$ **Z**, the mirror points of P with respect to the lines lie on the circle of center **Z** and radius dist(P, Z).'*

The Problem Definition of the Paper

The problem definition presented in the paper focuses on Cimmino's Reflection Method for solving linear systems numerically.

- **Initial Point and Reflections:** The method begins with an arbitrary point in $\mathbb{R}^n$ and reflects it across the hyperplanes defined by the linear system.
- **Geometrical Approach:** It uses geometric reflections to converge to the solution, which is the centroid of points on a spherical surface centered at the solution point. The centroid of the reflected points is used as the starting point for the next iteration.
- **Convergence and Error Estimates:** The paper provides error estimates for convergence at each iteration and shows that the distance to the solution decreases with each iteration.
- **Applications and Extensions:** The reflection method applies to large linear systems and has been adapted for parallel computing and image reconstruction in tomography.

Motivation of the Paper

- The main motivation of the paper is to solve a system of linear equations using various approaches such as the geometrical approach, numerical approach, and iterative method.
- The paper demonstrates the interaction of linear algebra in mathematical fields and how it is widely used in the geometric field as well.
- The study aims to offer a theoretical framework for understanding and analyzing error bounds associated with Cimmino's method.

Consider, in the plane $\mathbb{R}^2$, the straight line r of equation

$$\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle = b_i, \quad i = 1, 2, \tag{1}$$

Let's fix $P^{(0)} = P^{(0)}(\boldsymbol{x}^{(0)}) = P^{(0)}(x_1^{(0)}, x_2^{(0)})$ such that $P^{(0)} \notin r$.

$$\begin{aligned} \text{Orthogonal projection of } P^{(0)} \text{ onto } r, \text{ R} &= \boldsymbol{x}^{(0)} + \frac{b - \langle \boldsymbol{a}, \boldsymbol{x}^{(0)} \rangle}{\|\boldsymbol{a}\|^2} \boldsymbol{a}, \\ \text{Symmetric point of } P^{(0)} \text{ w.r.to } R, \text{ Q} &= \boldsymbol{x}^{(0)} + 2 \frac{b - \langle \boldsymbol{a}, \boldsymbol{x}^{(0)} \rangle}{\|\boldsymbol{a}\|^2} \boldsymbol{a}. \end{aligned} \tag{2}$$

Brief Overview Of The Mathematical Methods

Cimmino's Method for $\mathbb{R}^2$ (n =2)

Given the initial approximation $P^{(0)} = P^{(0)}(\boldsymbol{x}^{(0)})$, then $P^{(1)} = P^{(1)}(\boldsymbol{x}^{(1)})$ is the centroid of unit masses placed at $Q^{(i)}$, $i = 1, 2$, of $P^{(0)}$ with respect to the straight line $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle = b_i$:

$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \sum_{i=1}^{2} \frac{b_i - \langle \boldsymbol{a}_i, \boldsymbol{x}^{(0)} \rangle}{\|\boldsymbol{a}_i\|^2} \boldsymbol{a}_i. \tag{3}$$

This will be taken as the starting point of the next iteration, and so on. At step $\nu + 1$,

$$\boldsymbol{x}^{(\nu+1)} = \boldsymbol{x}^{(\nu)} + \sum_{i=1}^{2} \frac{b_i - \langle \boldsymbol{a}_i, \boldsymbol{x}^{(\nu)} \rangle}{\|\boldsymbol{a}_i\|^2} \boldsymbol{a}_i. \tag{4}$$

- This is where we can find Cimmino's geometrical observation, which is

$$\text{dist}(Q^{(i)}, Z) = \text{dist}(P^{(0)}, Z). \tag{5}$$

- Considering the triangles $\triangle ZP^{(0)}R^{(1)}$ and $\triangle ZR^{(0)}Q^{(1)}$ which follow ASA congruency rule, which gives $ZP^{(0)} = ZQ^{(1)}$. This can also be applied to the other reflective point $Q^{(2)}$.
- Therefore we get, $ZP^{(0)} = ZQ^{(1)} = ZQ^{(2)}$, which means the points $P^{(0)}$, $Q^{(1)}$, and $Q^{(2)}$ lie on a circle with the center as Z and a radius of $\text{dist}(Z, P^{(0)})$.

Brief Overview Of The Mathematical Methods

Cimmino's Method for $\mathbb{R}^2$ (n =2)

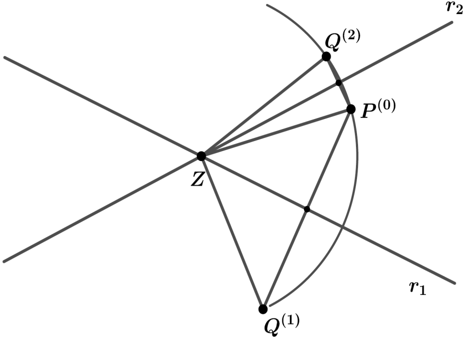

图: Reflection method for $n = 2$.

- And that was the proof of Cimmino's Observation.

Brief Overview Of The Mathematical Methods

Error bounds

Let's indicate $Z = Z(\xi_1, \xi_2)$ the point which solve our system Ax=b. And fix a point P$^{(0)} \neq Z. Using \langle a_i, \xi \rangle = b_i, i = 1, 2$, we define,

$$\eta_i = \langle a_i, x^{(0)} \rangle - b_i = \langle a_i, x^{(0)} - \xi \rangle \tag{6}$$

Now,

$$x^{(1)} = x^{(0)} - \sum_{i=1}^{2} \frac{\langle a_i, x^{(0)} - \xi \rangle}{||a_i||^2} a_i. \tag{7}$$

observe,

$$\|x^{(1)} - \xi\|^2 = \|x^{(0)} - \xi\|^2 - 2 \sum_{i=1}^{2} \frac{\eta_i^2}{|a_i||^2} + \left\| \sum_{i=1}^{2} \frac{1}{||a_i||^2} \cdot \eta_i \cdot a_i \right\|^2 \tag{8}$$

With,

$$\left\|\sum_{i=1}^{2} p_i \eta_i a_i\right\|^2 \leq \left(\sum_{i=1}^{2} p_i \eta_i^2\right) \cdot \left(\sum_{i=1}^{2} p_i \|a_i\|^2\right), \tag{9}$$

where, $p_i = \frac{1}{\|a_i\|^2}$ Our Eq (8) follows as,

$$\|x^{(1)} - \xi\| \leq \|x^{(0)} - \xi\| \tag{10}$$

Now, iterating, we obtain a sequence $P^{(\nu)}$ and corresponding estimates

$$\|x^{(\nu+1)} - \xi\| \leq \|x^{(\nu)} - \xi\| \tag{11}$$

there is equality in Eq (11) iff $x^{(\nu)}$ solves the given system, and hence $x^{(\nu)} = \xi$ and also $x^{(\nu+1)} = x^{(\nu)}$ for any $\nu \in \mathbb{N}$.

Objectives of This Project

- Here we use linear algebra in a well-versed manner for getting the reflections and orthogonal projections.
- Reflection Algorithm: Introduces Cimmino's reflection algorithm for solving linear systems using geometric reflections.
- In this paper, Cimmino's reflection method is used for solving the system of linear equations for N=2.
- We also discuss how linear algebra can interact fruitfully not only with algebra, geometry, and numerical analysis but also with probability theory.

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Thank You