19MA701

OPTIMIZATION THEORY

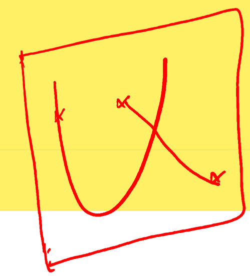
- Dr. Sarada Jayan
- Dr. Subramani R
Department of Mathematics

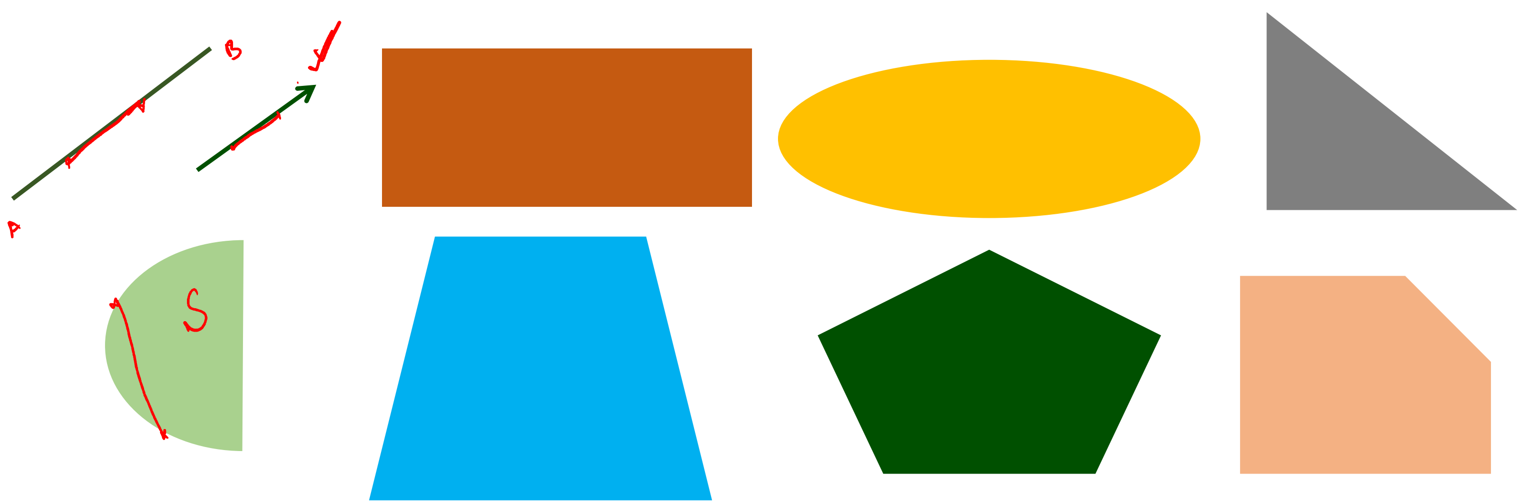
Convex Sets

- Let S be a set and let $\mathbf{x_1}$ and $\mathbf{x_2}$ be elements of the set. If the line segment joining $\mathbf{x_1}$ and $\mathbf{x_2}$ is also an element of S, then we say that S is convex
- Mathematically:

  Let $\boldsymbol{x_1}, \boldsymbol{x_2} \in S$

  If $\theta \boldsymbol{x_1} + (1-\theta)\boldsymbol{x_2} \in S$ for $0 < \theta < 1$, then S is a Convex set.



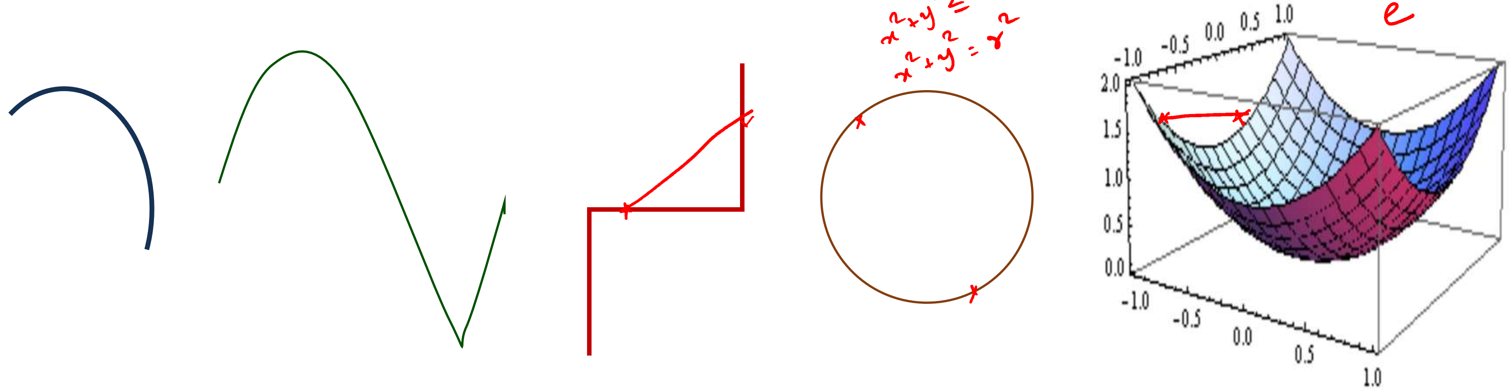


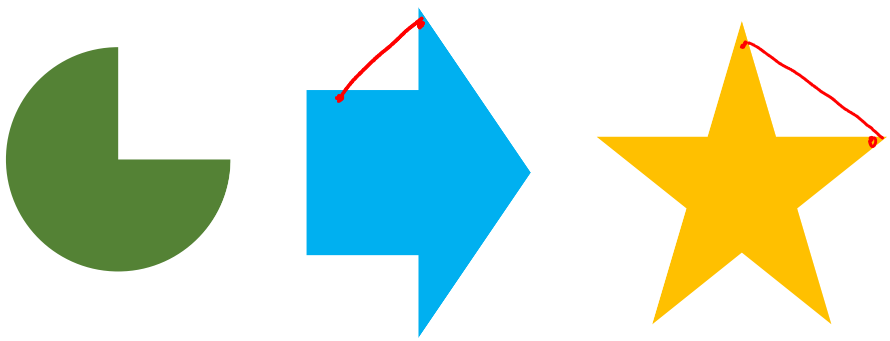
Examples : Non-Convex Sets

$x^2+y^2 \le r^2 \Rightarrow$ Region

$x^2+y^2 = r^2$

$e^{(-x^2-y^2)}$



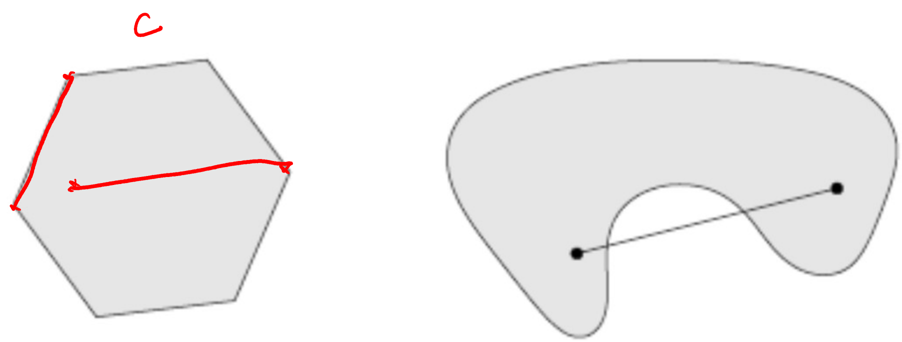


Any curve or curved surface is non-convex

Convex Sets

<u>Definition</u>: A set C is convex if a line segment between any two points in C lies in C.

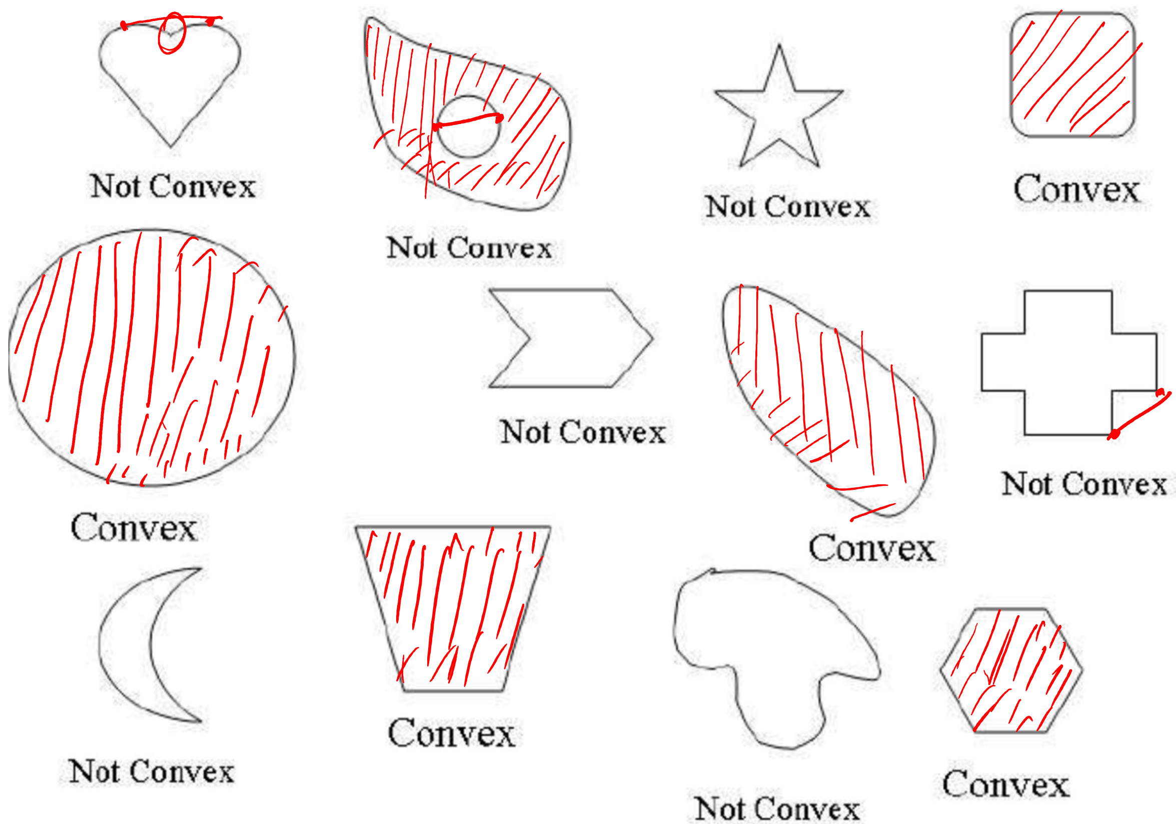
<u>Ex</u>: Which of the below are convex sets?



The set on the left is convex. The set on the right is not.

Convex Sets

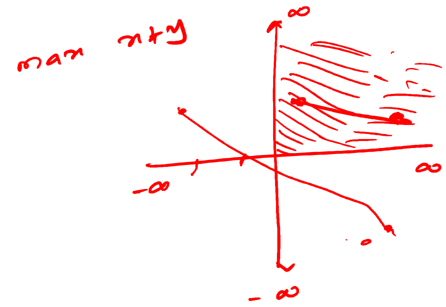
<u>Definition</u>: A set C is convex if a line segment between any two points in C lies in C.

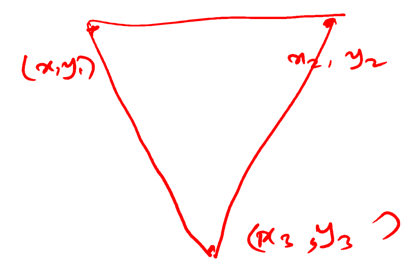


Examples of Convex Sets

- The empty set Φ is a convex set. $\{ \quad \}$
- A single point (singleton), i.e., $\{x_0\}$ is a convex set. $\{\cdot\}$
- **R**, **R**^{2}, **R**^{3}, …, **R**^{n} are convex sets.
- **X axis**, **Y axis** and any other **line** is convex.
- Any line segment and any ray is convex.
- Any vector space and subspace is convex.
- A **hyperplane** is convex.
  - Hyperplane is a solution set of a single linear equation.
  - In **R**^{2} it is a line, in **R**^{3} it is a plane.
- Solution set of linear equations, $C = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b}\}$ is convex (point/ line/ plane/ hyperplane)

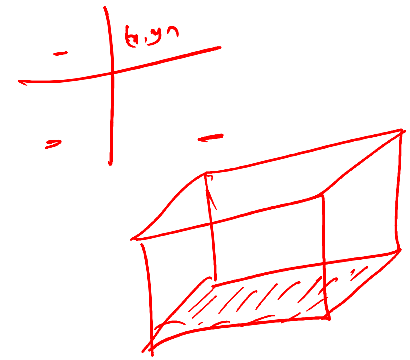
$3x - y = 7$
$y + 5x = 3$
$x = 3/5$





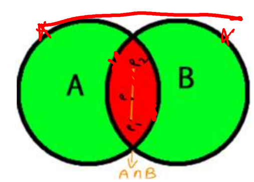
More Examples of Convex Sets

$a x \le b$



- A **halfspace** is convex.
  - Halfspace is the solution set of a single linear inequality.
    $H = \{\boldsymbol{x} \mid \boldsymbol{a}^T \boldsymbol{x} \le b\},\ \boldsymbol{a} \neq \mathbf{0}$
  - Hyperplane divides the domain into 2 halfspaces.
- Every **quadrant**(n=2), **octant**(n=3) and **orthant**(n) is convex.
- A **polyhedron** and a **polytope** is convex.
  - Polyhedron is a solution set of finitely many inequalities.
  - A bounded polyhedron is often called a polytope.
- **Feasible region** of a linear optimization problem is always convex.

Operations that preserve convexity



- Intersection of convex sets is convex

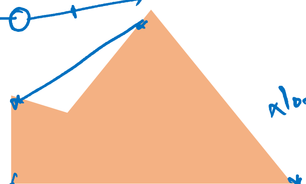
  (Union of convex sets need not be convex)

Problems

1. Which of the following sets are convex?

(a) $\mathbf{R} - \{0\}$ Not Convex

(b)



Not Convex

Answers1. Only (c) and (d) are convex

(c) All the points in an interval, [a, b] ⇒ Convex

(d) All the points in an interval, (a, b) ⇒ Convex

(e) All points in the XY plane except the origin ⇒ Not Convex

(f) All points in the parabola, $y = x^2$ ⇒ Not Convex

2. Find the domain of the following functions and mention if it is convex or not

(a) $f(x) = \frac{1}{x-1}$ (b) $g(x) = \sqrt{4 - x^2}$ (c) $h(x) = \frac{\sqrt{x-1}}{x}$ $(1, \infty)$

Answers2. (a) Domain of f is $\mathbf{R} - \{1\}$, which is not convex

(b) Domain of g is [-2,2], which is convex

(c) Domain of h is [1,∞), which is convex

Convex functions

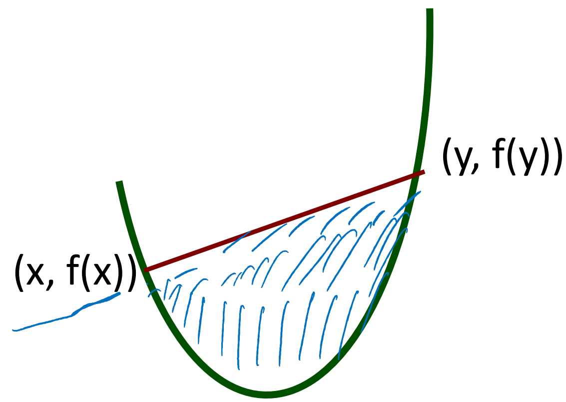
- A function $f: \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ is a convex function if:

  (i) the domain of f is convex and $< \rightarrow$ strictly Convex

  (ii) $f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}) \leq \theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y})$

  for all $x, y \in$ domain f and $0 \leq \theta \leq 1$

- Graphically, a function y=f(x) is convex if the curve y = f(x) lies below the line segment joining any two points on the curve.



- The function is strictly convex if the inequality is strictly less (<)

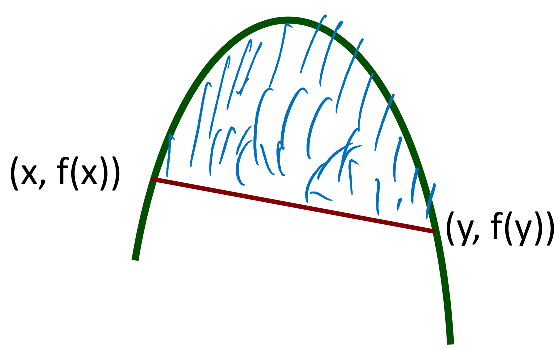
Concave functions

- A function $f: \boldsymbol{R}^n \to \boldsymbol{R}$ is a concave function if $-f$ is convex, i.e., when,

$$f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}) \geq \theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y})$$

for all $x, y \in$ domain f and $0 \leq \theta \leq 1$

- Graphically, a function y=f(x) is concave if the curve y = f(x) lies above the line segment joining any two points on the curve.

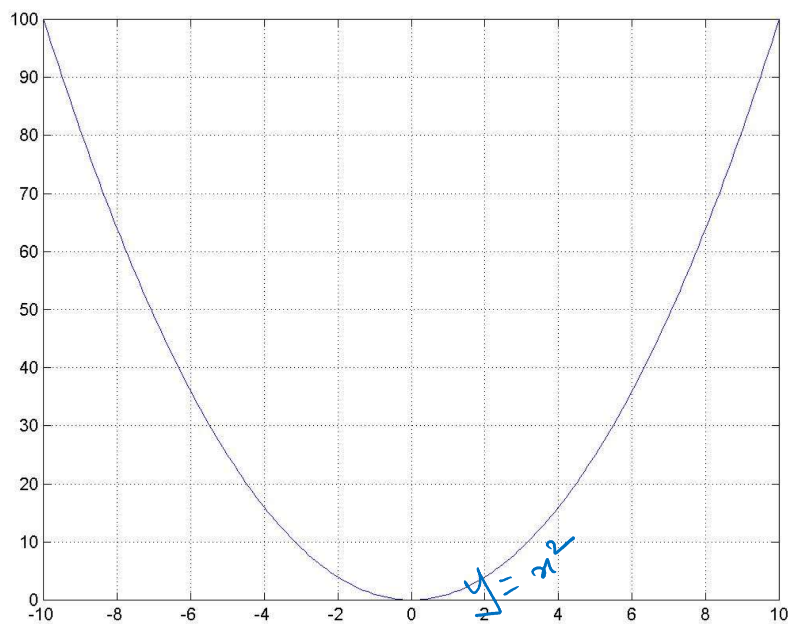


- The function is strictly concave if the inequality is strictly greater (>)

Convex functions

<u>Definition</u>: A function f(x) is convex in an interval if its second derivative is positive on that interval.

<u>Example</u>: $f(x)=x^2$ is convex since $f'(x)=2x$, $f''(x)=2>0$



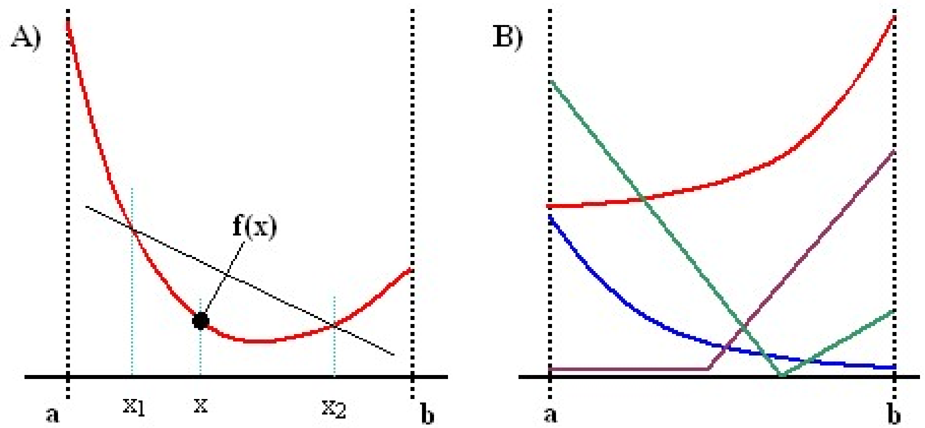
$y = x^2$

$f'(y) = 2x \rightarrow 0 \Rightarrow x = 0$

$f''(y) = 2$

Convex functions

The second derivative test is sufficient but not necessary.



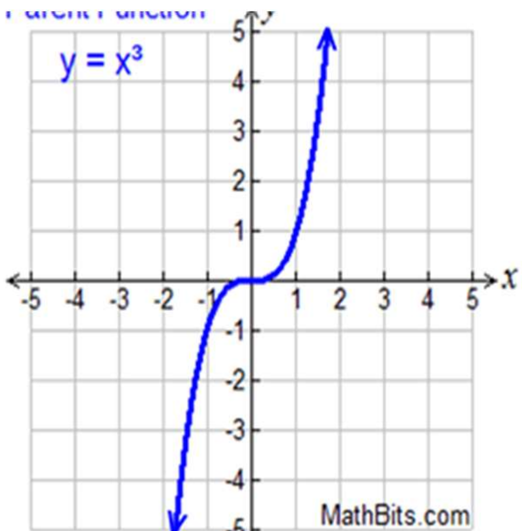
Definition : A function f(x) is convex if a line drawn between any two points on the function remains on or above the function in the interval between the two points.

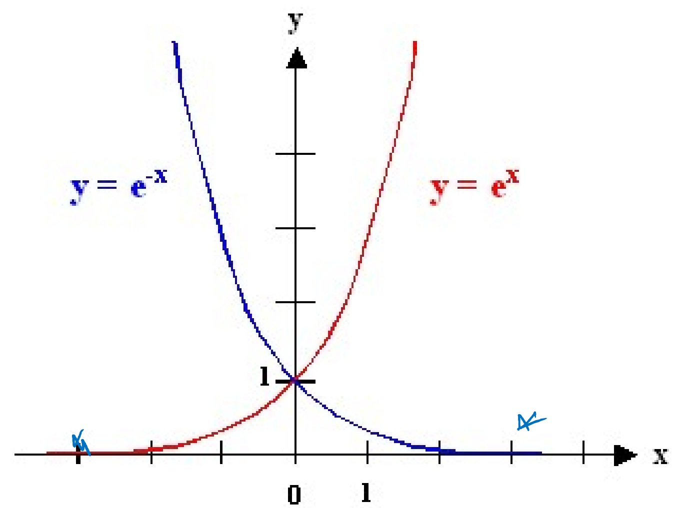
Examples

- Linear functions and affine functions are taken to be convex as well as concave as they don't have a curvature.

$f(x) = x + 3y$

- $y = x^2$ is a convex function
- $y = x^4, y = x^6, y = x^8, \ldots$ are convex functions for all values of x.
- $y = x^3, y = x^5, y = x^7, \ldots$ are convex functions if $x > 0$ (in the first quadrant) and concave in the third quadrant (x<0).
- Exponential function, $y = e^{ax}$ for any $a \in \mathbf{R}$ is a convex function as it always curves up.





Second derivative test for convexity/concavity

$f''(\ ) = 0$ → critical pt

- A twice differentiable function f(x) is convex if its Hessian $\nabla^2 f(x)$ is at least positive definite.

$f'(x) = 0 \rightarrow p$

- A twice differentiable function f(x) is concave if its Hessian $\nabla^2 f(x)$ is at least negative definite.

→ Second order partial Derivative matrix

- A function f(x) is non-convex if its Hessian $\nabla^2 f(x)$ is indefinite

Examples:

1. f(x) = x + 1/x is convex in x > 0

   (Since $\nabla^2 f(x) = \frac{d^2 f}{dx^2} = \frac{2}{x^3} > 0$ for $x > 0$)

2. f(x) = cosx is concave in $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

3. f(x,y) = $5x^2 + 6xy + 7y^2$ is convex
4. f(x,y) = $-2x^2 + xy - 7y^2$ is concave
5. f(x,y) = $5x^2 + 19xy + 7y^2$ is non-convex
6. f(x,y) = x^2/y is convex if y>0 and concave if y < 0

③ $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 10x + 6y \\ 6x + 14y \end{bmatrix}$

$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$

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Operations that preserve convexity

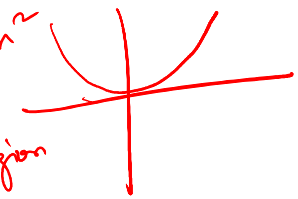
- If *f(x)* is a convex function, then $\alpha f(x)$ is also convex for $\alpha > 0$
  - $f(x) = 8e^x$ is convex
- The sum of convex functions is also convex
  - $f(x,y) = x^4 + y^4 + e^x$ is convex
- Non-negative weighted sum of convex functions is convex
  - $f(x,y,z) = 4x^2 + 5y^2 + 3e^z$ is convex

Practical methods for establishing convexity of a function

- Verify definition
- For twice differentiable functions, check whether hessian matrix is positive definite
- Show that f is obtained from simple convex functions by operations that preserve convexity

Relation between convex sets and convex functions

$y \le x^2$



Region

The set of all points in the region $g(x) \le 0$ is a convex set if the function $g(x)$ is a convex function.

Optimization Problem in standard form

Minimize $f(x)$

subject to $g_i(x) \leq 0, i = 1, 2, \dots, m$

$h_i(x) = 0, i = 1, 2, \dots, p$

x : decision variable

$f: \boldsymbol{R}^n \to \boldsymbol{R}$ is the objective function (or cost fn. if obj. is min.)

$g_i: \boldsymbol{R}^n \to \boldsymbol{R}$ are the inequality constraints

$h_i: \boldsymbol{R}^n \to \boldsymbol{R}$ are the equality constraints

Domain of the standard optimization problem:

$$\boldsymbol{D} = \cap_{i=1}^{m} dom\ g_i \ \cap \ \cap_{i=1}^{p} dom\ h_i$$

Convex Optimization Problem

An optimization problem in the standard form:

Minimize $f(x)$

subject to $g_i(x) \leq 0, i = 1, 2, \ldots, m$

$h_i(x) = 0, i = 1, 2, \ldots, p$

is said to be convex if

(i) the objective function is convex

(ii) the feasible region is convex

Convex Feasible regions

- Feasible region of the optimization problem is the intersection of each constraint $g_i(\boldsymbol{x}) \leq 0, i = 1, 2, \dots, m, \quad h_j(x) = 0, i = 1, 2, \dots, p$
- Since intersection of convex sets is convex, convexity of the feasible region can be checked by checking whether each constraint gives a convex set.
- $g_i(\boldsymbol{x}) \leq 0$ is a convex set if g_i is a convex function.
- $h_j(x) = 0$ will be a convex set only if $h_j(x)$ is linear. (Any linear equality constraint represents a line or a plane or a hyperplane which is a convex set. Any non-linear equality constraint represents a curve or a curved surface which cannot be a convex set.)

Convex Optimization Problem

An optimization problem in the standard form:

Minimize $f(x)$

subject to $g_i(x) \leq 0, i = 1, 2, \dots, m$

$h_j(x) = 0, i = 1, 2, \dots, p$

is said to be convex if

(i) the objective function is convex

(ii) the inequality constraint functions, $g_i(x)$ are convex

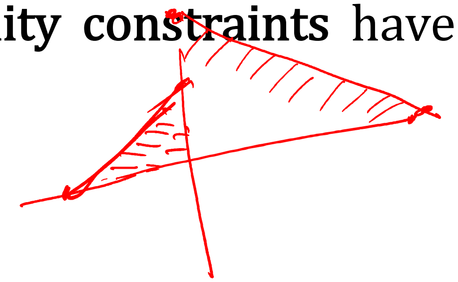
(iii)the equality constraint functions, $h_j(x)$ are linear

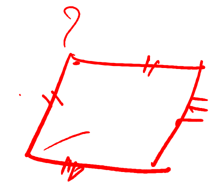
Optimization problems with convex Feasible regions

- Optimization problems involving **linear inequality constraints** have convex feasible region.

  Eg: $-x + y \leq 2, 2x + 3y \leq 12, x \geq 0, y \geq 0$

  (Feasible region will be a null set, or a polyhedron)





- Optimization problems with **only linear equality constraints** have a convex feasible region.

  (Feasible region will be a null set, or a singleton or a line or a plane)

- Optimization problems with **linear equality and inequality constraints** have convex feasible region.

  Eg: $-x + 2y \leq 2, 2x + 3y = 12, x \geq 0, y \geq 0$

- Optimization problems involving **convex inequality constraints** have a convex feasible region.

  Eg: $x^2 + y^2 \leq 1, (x-1)^2 + (y-1)^2 \leq 1, x \geq 0, y \geq 0$

Note :

Feasible region for an optimization problem with **non-linear equality constraint(s) is never a convex set** regardless of whether the equality constraints is convex or not.

$$
\begin{aligned}
&x^2 + y^2 = 1,\\
&x \geq 0,\\
&y \geq 0
\end{aligned}
$$

Examples:

1. Minimize $5x + 9y^2$

subject to $x^2 - 4y + 9x \leq 7$

$2x^2 + 3xy + 81y^2 \leq 5$

Is a C.O.P. since $f(x)$, $g_1(x)$ and $g_2(x)$ are convex

2. Minimize $(x-1)^2 + (y-1)^2 + xy$

subject to $x + y \leq 4$

$2x + x^2 + y^2 = 16$

Is not a C.O.P. since the equality constraint is non-linear

3. Maximize $x^2 + y^2$

subject to $x + y \leq 4$

$2x + x^2 + y^2 \leq 16$

Is not a C.O.P. since the objective in the standard form is concave.

4. Maximize, $xy - (x-1)^2 - (y-1)^2$

subject to $2x + x^2 + y^2 \leq 16$

$3x - 7y = 9$

Is C.O.P. Since $f(x)$ (after writing in standard form), $g_1(x)$ is convex and $g_2(x)$ is linear

Fundamental Property of Convex Optimization Problems

Any locally optimal point of a convex optimization problem is globally optimal

Tutorial questions

1. Consider the function $f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_1x_2 - 4x_2 + 7x_1 + 15$. Is this function convex?
2. Is the given non-linear programming problem convex? Why or why not?

   Maximize $z = 9x_1^2 - 4x_1x_2 + 7x_2^2$

   Subject to $x_1^2 + 2x_1x_2 + 3x_2^2 = 40$; $2x_1^2 - x_2 \leq 80$; $x_1 \leq 60$.

3. Is the following statement true? Explain.

   "Maximize $(x-3)^2 + (y-8)^2$ is not a convex optimization problem but

   Minimize $(x-3)^2 + (y-8)^2$ is a convex optimization problem"

4. Determine if the following optimization problems are convex optimization problems. Use graphical methods to solve these problems.

   (a) Maximize $-6x + 9y$

   subject to $x - y \geq 2$

   $3x + y \geq 1$

   $2x - 3y \geq 3$

   (b) Minimize $x^2 + 2y^2$

   subject to $x + y \geq 1$; $x, y \geq 0$

Tutorial questions

5. Draw the feasible region and determine the convexity status:

Minimize $x^2+2y^2-24x-20y$

subject to $x+2y \geq 0;\ x+2y \leq 9;\ x+y \leq 8; x+y \geq 0$

6. Explain why the given optimization problem is not convex.

Maximize $(x-2)^2+(y-10)^2$

subject to $x^2+y^2=50$

$x^2+y^2\ +2xy-x-y+20 \geq 0; x, y\ \geq 0.$

7. Determine if the following optimization problem is convex optimization problems or not:

Minimize$\frac{100}{e^{2x+y}}$

subject to $e^x+e^y \leq 20;$

$x, y\ \geq 1.$