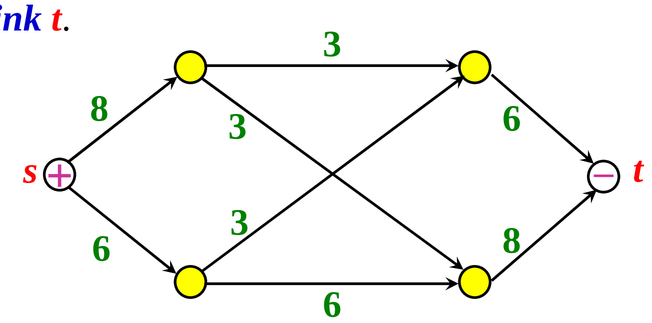
# Unit 10
# Maximum Flow

**T.H. Cormen et al., "Introduction to Algorithms", 3rd ed., Chapters 26**



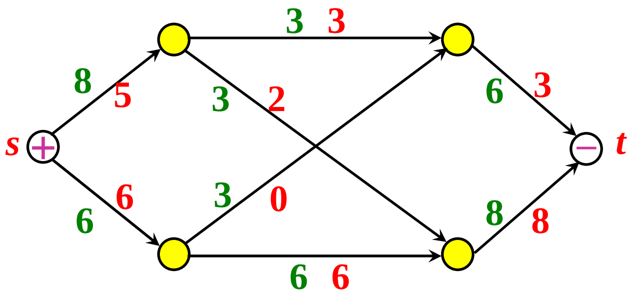
## Flow networks

- A digraph $G=(V, E)$ is a ***flow network*** if :
  - $\forall\ e \in E$, it has a ***capacity*** $c(e) \geq 0$.
  - Two distinguished vertices: a ***source*** $s$ and a ***sink*** $t$.



## A Flow on a Network

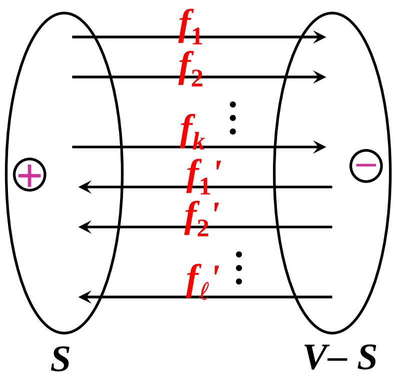
- A ***flow*** is a function $f : E \to \Re^*$ s.t.
  - $f(e) \le c(e)$, $\forall\ e \in E$.
  - $\Sigma_{u \in IN(v)} f(u,v) = \Sigma_{w \in OUT(v)} f(v,w)$, $\forall v \in V-\{s, t\}$.
- The ***value*** of a flow: $|f| = \Sigma_{w \in OUT(s)} f(s,w) - \Sigma_{w \in IN(s)} f(w,s)$.






## Flow and Cut

- For any flow $f$,
  $|f| = \Sigma_{w \in IN(t)} f(w,t) - \Sigma_{w \in OUT(t)} f(t,w)$.
- A set $S \subseteq V$ is a ***cut*** if $s \in S$ and $t \in V - S$.
- For any cut $S$,
  $f(S, V-S)$
  $= f_1 + f_2 + \ldots + f_k$
  $\quad - f_1' - f_2' - \ldots - f_\ell'$
  $= |f|$



## The maximum-flow problem

- In the ***maximum-flow problem***, we are given a flow network with source ***s*** and sink ***t***, and we wish to find a flow of maximum value.
- Many problems can be reduced to this problem such as: ***vertex connectivity***, ***edge connectivity***, ***maximum bipartite matching***.

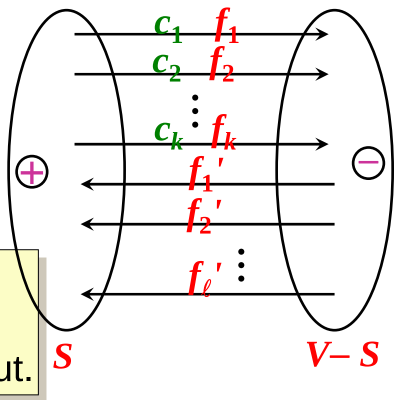


## Max Flow & Min Cut

- For any cut $S$, $c(S) = \Sigma_{u \in S,\ v \in V-S}\, c(u, v)$
- **Thm.** For any flow $f$, and any cut $S$, $|f| \le c(S)$

Proof:

$$
\begin{aligned}
|f| &= f_1 + f_2 + \ldots + f_k \\
&\quad - f_1' - f_2' - \ldots - f_\ell' \\
&\le c_1 + c_2 + \ldots + c_k \\
&= c(S)
\end{aligned}
$$

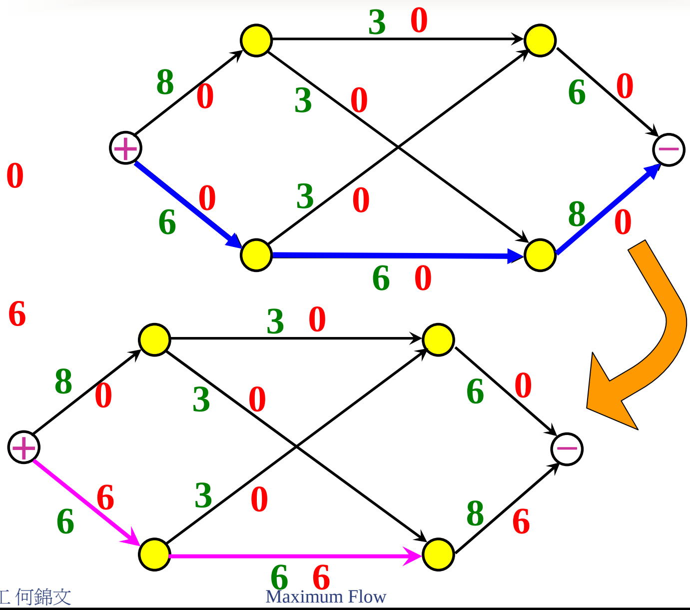



$\therefore$ If we can find a $f$ and $S$ s.t. $|f| = c(S)$, then $f$ is a max flow and $S$ is a min cut.

## The Ford-Fulkerson Method (例 1/5)

|*f*| = 0

|*f*| = 6

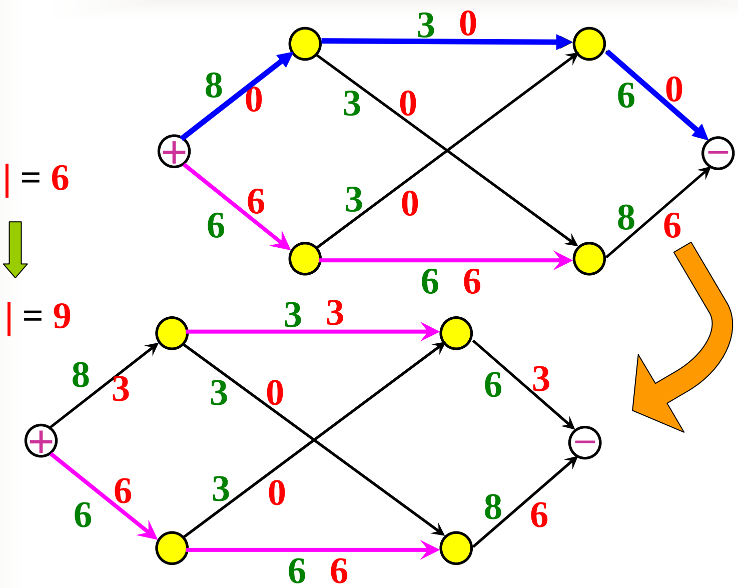





## The Ford-Fulkerson Method (例 2/5)

|*f*| = 6

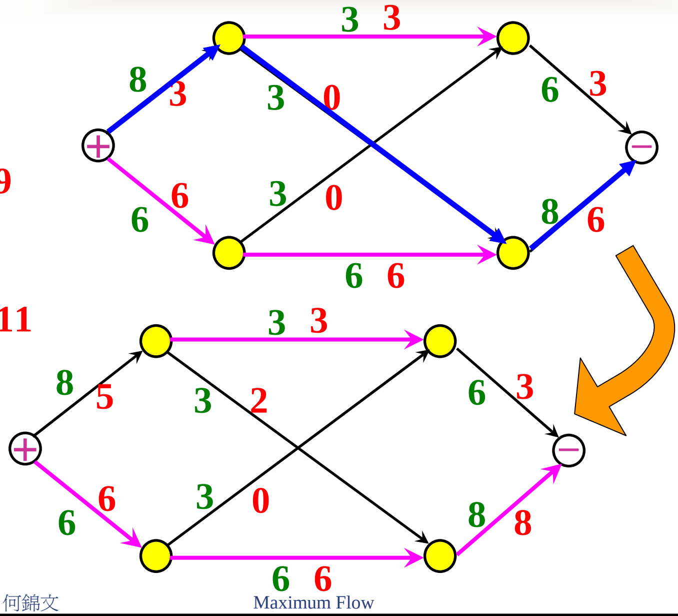
|*f*| = 9



# The Ford-Fulkerson Method (例 3/5)

| *f* | = 9

| *f* | = 11

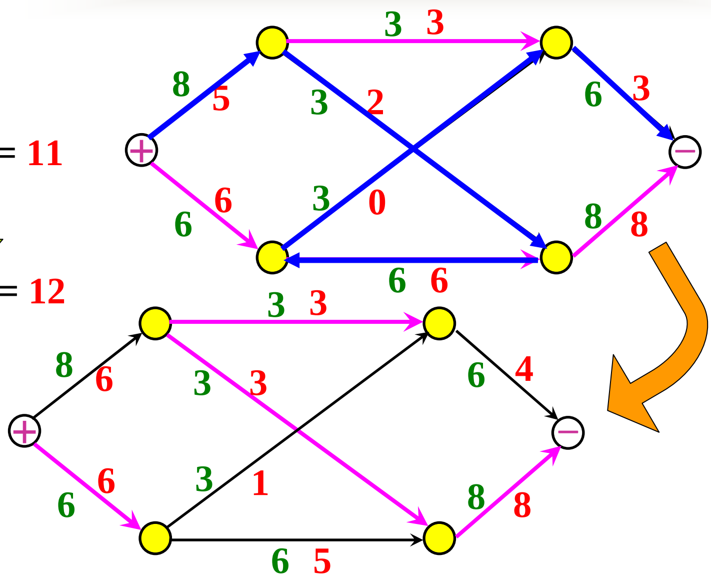




# The Ford-Fulkerson Method (例 4/5)

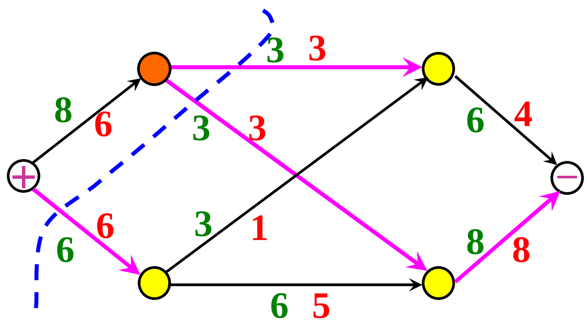
| *f* | = 11

| *f* | = 12



## The Ford-Fulkerson Method (例 5/5)

$|f| = 12$




There is a cut $S$ with its capacity $c(S) = |f| = 12$.
So, $f$ is a max flow and $S$ is a min cut.

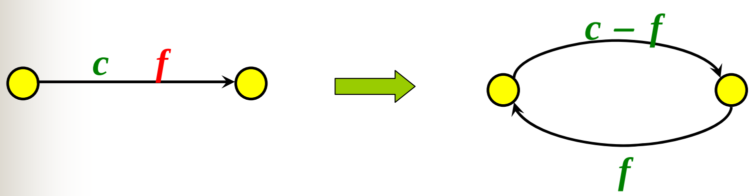


## The Ford-Fulkerson Method (pseudo-code)

S1: Star with zero flow everywhere.

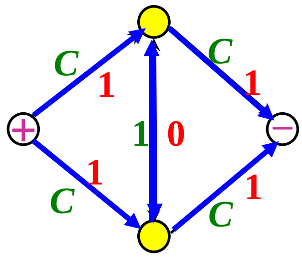
S2: Increase the flow along any path from source to sink ***with no full forward edges or empty backward edges***.

S3: Repeat S2 until no such paths can be found.



## The Ford-Fulkerson Method (討論 1)

- 解法很容易可以推廣至無向圖 (可用來解 connectivity 的問題).
- 若augmenting path 搜尋方式沒有限定, 則 #iterations 可能很大, 例：





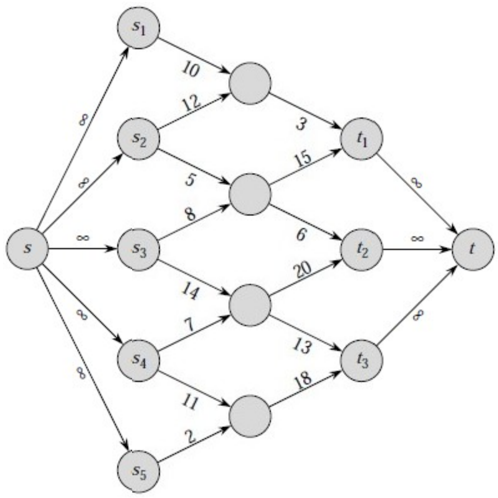

## The Ford-Fulkerson Method (討論 2)

- 若 augmenting path 搜尋方式用 BFS (稱為 the Edmonds-Karp algorithm), 則 **# *iterations*** $\leq$ ***nm*** (p.729) (Time = $O(nm^2)$ )
- 若 augmenting path 找使 flow 增加最大之 path, 則 **# *iterations*** $\leq 1 + \log_{M/(M-1)} f^*$

  相當於找一最長路徑 ***P*** 長度為 = $\mathbf{Min}_{e \in P}\, \mathbf{w}(e)$, 上式中 $f^*$ 為最大 flow 的值, $M$ 為 cut 中邊數最多的數值 (see Sedgewick’s book).
- Other efficient implementations see Section 26.4 & 26.5.

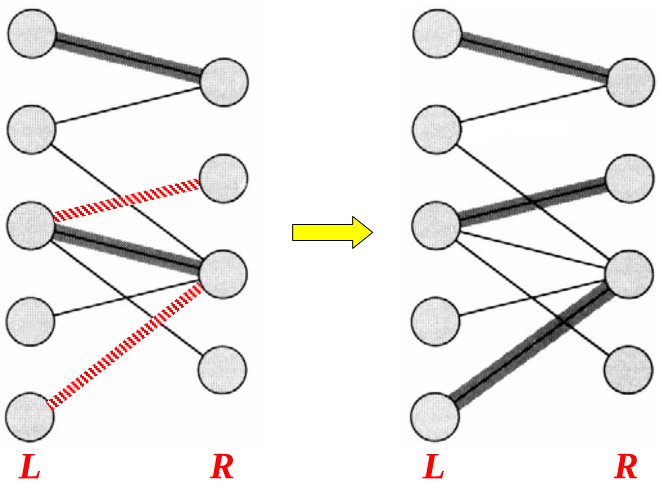
# Networks with multiple sources and sinks

- ☛ The network may have several sources and sinks:

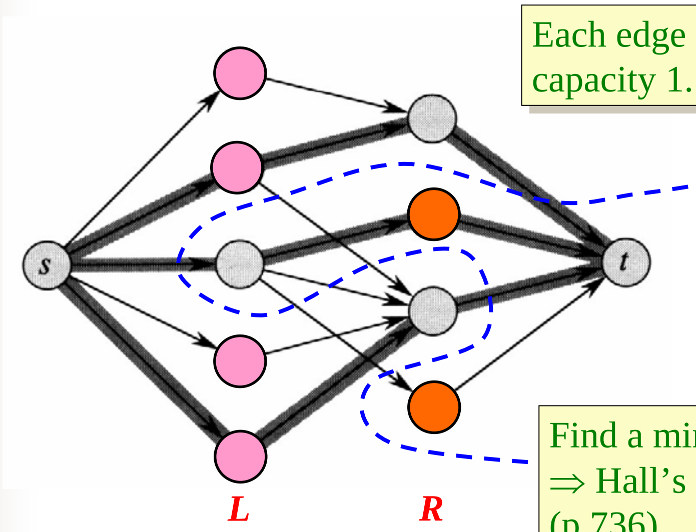





# Maximum bipartite matching



# Finding a maximum bipartite matching




Each edge has capacity 1.

Find a min cut
⇒ Hall’s Thm. (p.736)