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A weighted Kendall's tau statistic [1]

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Abstract

A weighted Kendall's tau statistic (τ_w) is proposed to measure weighted correlation. It can place more emphasis on items having low rankings than those have high rankings, or vice versa. The null limiting distribution is derived by the theory of U-statistics. An application, power comparison, and some critical values of τ_w are presented. © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

A weighted correlation is one which emphasizes items having low rankings and de-emphasizes those having high rankings, or vice versa. For example, 1234 is regarded more highly correlated with 1243 than 2134 (see Quade and Salama, 1992). Applications of weighted correlation abound in real life. For instance, *similarity* of every pre-season forecast from various sources and the post-season ranking can be compared to determine which forecast is most accurate. The similarity of two techniques, in terms of agreement in the top ranks (1, 2, 3, etc.), in sensitivity analysis (Iman and Conover, 1987), and the similarity of two methods to select gifted students, all can be measured by a weighted correlation.

Salama and Quade (1981, 1982) first studied the weighted correlation of two sets of rankings. Quade and Salama (1992) reviewed and introduced a few statistics for weighted correlation. These are sensitive to agreements in the top rankings and ignore disagreements on the rest variables in certain degree. Nevertheless, power has not been studied under any alternative which addresses the weighted correlation of the two sets of rankings.

In Section 2 of this note, we propose a class of weighted Kendall's tau statistics (τ_w). A novel alternative setting for weighted correlation is proposed in Section 4. For a certain choice of weights, we compare the power of τ_w with that of r_T in Iman and Conover (1987). The latter is the locally most powerful rank (LMPR) test under one alternative to independence. In Section 3, the null limiting distribution of τ_w is derived. We close with some remarks in Section 5. Some critical values of τ_w are tabulated in Appendix B.

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1.1. Motivating example

We assume that there are no ties among the variables being ranked. The fidelity evaluation of software packages for structural engineering, (Butler, 1986; Iman; 1987), motivated the study of τ_w. Suppose that two sets of twenty rankings by technique A and B are as follows:

A :	1	2	3	4	5	6	7	8	9	10
	11	12	12	14	15	16	17	18	19	20
B :	1	2	4	3	6	5	20	13	15	19
	17	18	14	16	12	9	11	8	10	7.

Although the two sets of rankings agree on the top six important variables (not necessarily on the exact order), there is an apparent disagreement in the rankings of the remaining variables. In this case, we may wish to emphasize agreement of the low rankings (for example, 1, 2, 3, etc.)

2. Weighted Kendall's tau

Let (X_i, Y_i), $i=1,\dots,n$ be independent and identically distributed (iid) random vectors. Further, let (i, R_i), $i=1,\dots,n$, be paired rankings, where R_i is the rank of Y whose corresponding X has rank i among $\{X_j\}$. Throughout this paper $\sum_i$, $\sum_j$ and $\sum_{i,j}$ will denote summations in which i,j and both i and j run through 1 to n, respectively. Let $w(i,j)$ be a weight function which is bounded and symmetric and $w:\mathbf{N}^2 \to \mathbf{R}$. For simplicity, we shall use w_{ij} to denote $w(i,j)$ henceforth. A weighted Kendall's tau is defined as

$$\tau_w = 1 \Bigg/ \left[\sum_{i,j} w_{ij} - \sum_i w_{ii}\right] \sum_{i \neq j} w_{ij}\,\mathrm{sgn}(i-j)\mathrm{sgn}(R_i - R_j), \tag{1}$$

where $\mathrm{sgn}(x) = -1, 0$ or 1, if $x <, =$ or > 0.

We note that τ_w is a rank test. Further, it is a weighted U-statistic of degree 2 (Shapiro and Hubert, 1979; O'Neil and Redner, 1993). Let $Z_1, Z_2, \dots$ be iid random variables (r.v.'s) or random vectors with distribution function F. Weighted U-statistics of degree 2 are defined as

$$U_w = 2/[n(n-1)] \sum_{i>j} w_{ij}\, h(Z_i, Z_j),$$

where h is a real symmetric function with mean zero and variance finite. For developments of weighted U-statistics, we refer to O'Neil and Redner (1993) and Shieh (1996). The kernel function of τ_w, h, is equal to $\mathrm{sgn}(X_i - X_j)\mathrm{sgn}(Y_i - Y_j)$.

We note that weighted Kendall's tau ranges from -1 to 1. Further, the weight function for τ_w can be adjusted easily so that it can measure not only top–down correlation but also down–top correlation. Namely, when high-ranked variables are more important than low-ranked ones. Likewise, τ_w can measure weighted correlation which emphasizes on "the middle part" of two sets of rankings.

A useful case of weighted Kendall's tau, which has weight function $w_{ij} = v_i v_j$, will be demonstrated in later sections. τ_w with product weights $w_{ij} = v_i v_j$ has the form

$$\tau_w = 2 \Bigg/ \left[\left(\sum_i v_i\right)^2 - \sum_i v_i^2\right] \sum_{i>j} v_i v_j\,\mathrm{sgn}(R_i - R_j).$$

If we take $v_i = I[i \leqslant [(n+1)p]]$, where $p = m/n$. Then τ_w can be further simplified to

$$\tau_w = 2/[m(m-1)] \sum_{1 \leqslant j < i \leqslant m} \mathrm{sgn}(R_i - R_j), \tag{2}$$

where $m = [(n+1)p]$. We note that m can be determined either by the questions or by the data set. We shall demonstrate how to determine m in Section 2.1.

2.1. Example 1 continued

In Example 1, the two sets of rankings of the top six variables are in "agreement", thus we can take $m = 6$, and $p = 6/20 = 0.3$. Substituting $m = 6$ into Eq. (2) and carrying out some computation, we obtain that $\tau_w = 0.733$. Compared to the critical value at $\alpha = 0.05$ in Table 5, it is significant. However, Spearman's rho equals 0.367 and it is smaller than the critical value 0.380.

3. Null limiting distribution

In this section, we derive the limiting distribution of a weighted Kendall's tau statistic under the null hypothesis of independence between the X- and Y-variable. Our proof is based on the theory of U-statistics, but it can also be derived by a theorem in O'Neil and Redner (1993). Let $w_{i.} = \sum_j w_{ij}$.

Theorem 1. *Suppose that* H_0 *holds,* $\sum_{i,j} w_{ij}^2 / \sum_i w_{i.}^2 \to 0$ *and* $\max_i w_{i.}^2 / \sum_i w_{i.}^2 \to 0$ *as* $n \to \infty$, *then*

$$\sqrt{n}\tau_w 3\bar{w}/(4\bar{w}^2)^{1/2} \to_{\mathrm{D}} \mathrm{N}(0,1),$$

where $\bar{w} = \lim_{n\to\infty} n^{-2} \sum_{i,j} w_{ij}$ *and* $\bar{w}^2 = \lim_{n\to\infty} n^{-1} \sum_i (n^{-1} \sum_j w_{ij})^2$.

Proof. See Appendix A.

Remark. *For random* w_{ij}, *which is independent of* $\{Y_i\}$, *under* H_0 *the asymptotic normality of* τ_w *still holds. For* τ_w *with product weights* $w_{ij} = v_i v_j$, *it has the following limiting distribution.*

Corollary 1. *For* τ_w *with* $w_{ij} = v_i v_j$, *suppose that* H_0 *holds,* $\sum_i v_i^2 / (\sum_i v_i)^2 \to 0$ *and* $\max_i v_i^2 / (\sum_i v_i^2) \to 0$ *as* $n \to \infty$, *then*

$$\sqrt{n}\tau_w 3\bar{v}/(4\bar{v}^2)^{1/2} \to_{\mathrm{D}} \mathrm{N}(0,1),$$

where $\bar{v} = \lim_{n\to\infty} n^{-1} \sum_i v_i$ *and* $\bar{v}^2 = \lim_{n\to\infty} n^{-1} \sum_i v_i^2$.

Proof. Substituting w_{ij} by $v_i v_j$, we obtain the result from Theorem 1.

4. Simulation results

In this section, we shall compare the power of $\tau_w = 2/[m(m-1)] \sum_{1 \leqslant j < i \leqslant m} \mathrm{sgn}(R_i - R_j)$ and $m = [(n+1)p]$, to that of the top–down statistic r_T in Iman and Conover (1987). Recall that r_T is the Pearson correlation

Table 1
Powers of weighted Kendall's tau with $p=0.1$ and top-down statistic

		N(0,1)		Logistic(0, −1)		Extreme[a]	
n	ρ	τ_w	r_T	τ_w	r_T	τ_w	r_T
10	0.0	—	0.050	—	0.050	—	0.050
	0.1	—	0.064	—	0.064	—	0.063
	0.2	—	0.079	—	0.073	—	0.073
	0.3	—	0.086	—	0.085	—	0.090
20	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.079	0.092	0.076	0.086	0.073	0.080
	0.2	0.090	0.114	0.087	0.105	0.085	0.104
	0.3	0.098	0.125	0.096	0.124	0.096	0.130
30	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.146	0.110	0.141	0.106	0.131	0.098
	0.2	0.209	0.148	0.200	0.146	0.189	0.146
	0.3	0.272	0.167	0.269	0.175	0.263	0.185

[a] $f(x)=g(x)\sim\exp\{x-e^x\}$, where f and g are p.d.f. of $\{X_i\}$ and $\{Y_i\}$.

coefficient computed on Savage scores (Savage, 1956) and has the form

$$r_T=\left(\sum_{i=1}^{n} S_iS_{R_i}-n\right)\Big/(n-S_1),$$

where Savage scores $S_i=\sum_{j=i}^{n}1/j$, and i is the rank assigned to the ith order statistic in a sample of size n.

The alternative used is the following:

$$X_i=X_i^*+w(X_i^*)\Delta Z_i \quad\text{and}\quad Y_i=Y_i^*+\Delta Z_i, \tag{3}$$

where X_i^*, Y_i^* and Z_i are mutually independent r.v.'s, $w(X^*)=I[X^*\leqslant X_{(m)}^*]$ with $m=[(n+1)p]$ and $X_{(m)}^*$ being the mth order statistic among $\{X_i^*\}$, and Δ is a nonnegative constant. We note that this alternative setting is novel and it addresses the weighted correlation between X and Y.

Let ρ denote the correlation between X_i and Y_i. In this study, $\rho=0.0(0.1$ or $0.2)\rho_{\max}$, where $\rho_{\max}$ is $\sqrt{p}$ rounded to first decimal place; and $p=0.1, 0.2, 0.3$ and 0.5.

The power of τ_w and r_T under the alternative in Eq. (3) with X_i, Y_i and $Z_i\sim N(0,1)$, X_i, Y_i and $Z_i\sim$ Logistic$(0,-1)$ and X_i, Y_i and $Z_i\sim$ Extreme distribution, i.e., $f(x)\sim\exp\{x-e^x\}$, are summarized in Tables 1–4, respectively. In each simulation, the number of replications used was 5000 which yields a standard error about 0.0071. The sample sizes studied are 10, 20 and 30.

From Tables 1–4, we found that for all sample sizes and p we studied, τ_w is much more powerful than r_T in every case when $m=[(n+1)p]\geqslant 4$. τ_w is compatible to r_T when $m=3$ and is slightly less powerful than r_T when $m=1$ or 2. This is due to the fact that simplified τ_w in Eq. (2) with $w_i=I_{[i\leqslant m]}$ is the usual Kendall's tau computed on m rankings.

Table 5, in Appendix B, summarizes the tail critical values of distributions of simplified weighted Kendall's tau in (2) with $p=0.3$. Other tables of critical values of simplified τ_w with $p=0.1, 0.2$ and 0.5 can be obtained from the author on request.

Table 2
Powers of weighted Kendall's tau with $p = 0.2$ and top–down statistic

		N(0,1)		Logistic(0, −1)		Extreme	
n	ρ	τ_W	r_T	τ_W	r_T	τ_W	r_T
10	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.067	0.081	0.066	0.082	0.065	0.075
	0.2	0.078	0.103	0.076	0.102	0.073	0.099
	0.3	0.085	0.130	0.082	0.117	0.080	0.118
	0.4	0.092	0.141	0.090	0.134	0.089	0.142
20	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.169	0.110	0.157	0.111	0.137	0.101
	0.2	0.302	0.159	0.281	0.164	0.245	0.149
	0.3	0.454	0.197	0.415	0.197	0.378	0.191
	0.4	0.681	0.253	0.648	0.248	0.609	0.261
30	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.283	0.147	0.252	0.129	0.221	0.114
	0.2	0.533	0.220	0.478	0.205	0.416	0.189
	0.3	0.764	0.278	0.714	0.276	0.654	0.272
	0.4	0.948	0.348	0.921	0.337	0.897	0.352

Table 3
Powers of weighted Kendall's tau with $p = 0.3$ and top–down statistic

		N(0,1)		Logistic(0, −1)		Extreme	
n	ρ	τ_W	r_T	τ_W	r_T	τ_W	r_T
10	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.086	0.083	0.091	0.088	0.085	0.082
	0.2	0.119	0.122	0.120	0.121	0.112	0.112
	0.3	0.153	0.160	0.152	0.161	0.141	0.152
	0.4	0.193	0.202	0.184	0.187	0.175	0.185
	0.5	0.237	0.239	0.236	0.240	0.228	0.248
20	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.189	0.127	0.186	0.125	0.165	0.111
	0.2	0.353	0.197	0.347	0.203	0.305	0.190
	0.3	0.543	0.252	0.526	0.258	0.466	0.247
	0.4	0.750	0.341	0.719	0.325	0.665	0.324
	0.5	0.932	0.400	0.918	0.400	0.888	0.413
30	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.293	0.150	0.272	0.148	0.233	0.130
	0.2	0.571	0.256	0.527	0.245	0.456	0.225
	0.3	0.811	0.374	0.764	0.343	0.692	0.322
	0.4	0.949	0.461	0.924	0.437	0.885	0.435
	0.5	0.997	0.553	0.996	0.545	0.992	0.568

5. Conclusion

Weighted Kendall's tau statistics are proposed for measuring weighted correlation. The proposed measure is based on ranks and it is also a weighted U-statistic. Parallel to τ_w proposed here, a weighted Spearman's rho

Table 4
Powers of weighted Kendall's tau with $p = 0.5$ and top–down statistic

		N(0,1)		Logistic(0, −1)		Extreme	
n	ρ	τ_{W}	r_{T}	τ_{W}	r_{T}	τ_{W}	r_{T}
10	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.106	0.090	0.108	0.097	0.102	0.090
	0.3	0.253	0.182	0.270	0.187	0.243	0.178
	0.5	0.507	0.323	0.487	0.303	0.443	0.301
	0.7	0.967	0.495	0.961	0.502	0.951	0.513
20	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.142	0.121	0.183	0.118	0.164	0.104
	0.3	0.505	0.297	0.559	0.321	0.485	0.304
	0.5	0.884	0.517	0.886	0.529	0.832	0.535
	0.7	1.000	0.778	1.000	0.777	1.000	0.806
30	0.0	0.050	0.050	0.050	0.050	0.050	0.050
	0.1	0.249	0.147	0.255	0.148	0.225	0.133
	0.3	0.771	0.420	0.750	0.404	0.672	0.379
	0.5	0.987	0.677	0.978	0.678	0.955	0.679
	0.7	1.000	0.926	1.000	0.916	1.000	0.939

r_{w} and weighted top–down statistic T_{w} are studied in Bai et al. (1997). They show that under the alternative setting in (3) r_{w} and T_{w} are LMPR tests under certain alternative distribution families.

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Appendix A. Proof of Theorem 1

Let $t_w = 2/[n(n-1)]\sum_{i>j} w_{ij}\mathrm{sgn}(i-j)\mathrm{sgn}(R_i - R_j)$. We write t_{w} as

$$t_{\mathrm{w}} = 2/[n(n-1)]\sum_{i>j} w_{ij}\,\mathrm{sgn}(X_i - X_j)\,\mathrm{sgn}(Y_i - Y_j),$$

which is a weighted U-statistics. Since $\mathrm{sgn}(X_i - X_i) = 0$, for $1 \leqslant i \leqslant n$, without loss of generality, we can assume that $w_{ii} = 0$. Suppose that $\{X_i\}$ and $\{Y_i\}$ follow continuous F and G, respectively. Note that

$$E(\mathrm{sgn}(X_i - X_j)\,|\,X_i) = 2F(X_i) - 1 \quad \text{and} \quad E(\mathrm{sgn}(Y_i - Y_j)\,|\,Y_i) = 2F(Y_i) - 1.$$

Let $g_{ij} = \mathrm{sgn}(X_i - X_j)\mathrm{sgn}(Y_i - Y_j) - (2F_i - 1)(2G_i - 1) - (2F_j - 1)(2G_j - 1)$, where $F_i = F(X_i)$ and $G_i = G(Y_i)$, with $1 \leqslant i \leqslant n$. Then

$$t_{\mathrm{w}} = 2/[n(n-1)]\left\{\sum_{i>j} w_{ij}[(2F_i - 1)(2G_i - 1) + (2F_j - 1)(2G_j - 1)] + \sum_{i>j} w_{ij}g_{ij}\right\} = U_1 + U_2.$$

Recall that $w_{i.} = \sum_j w_{ij}$. After some calculation, we have that $\mathrm{Var}(U_1) = (4/9)[n(n-1)]^{-2} \sum_i w_{i.}^2$ and $\mathrm{Var}(U_2) = 4E(g_{12}^2)[n(n-1)]^{-2} \sum_{i,j} w_{ij}^2$. Assume that

$$\sum_{i,j} w_{ij}^2 \Big/ \sum_i w_{i.}^2 \to 0\,, \tag{A.1}$$

then $n\,\mathrm{Var}(t_w) = (4/9)n^{-1} \sum_i (n^{-1} \sum_j w_{ij})^2 \to (4/9)\;\bar{w}^2$, as $n \to \infty$.

Rewrite $U_1 = 3/[n(n-1)(4\sum_i w_{i.}^2)^{1/2}] \sum_i w_{i.}\eta_i$, where $\eta_i = (2F_i - 1)(2G_i - 1)$. Suppose that $\{w_{ij}\}$ satisfy

$$\max_i w_{i.}^2 \Big/ \sum_i w_{i.}^2 \to 0 \text{ as } n \to \infty\,, \tag{A.2}$$

which is derived from Lindeberg condition:

$$\left(9 \Big/ \sum_i w_{i.}^2\right) \sum_i w_{i.}^2 E(\eta_i^2) I \left[(w_{i.}\eta_i) \Big/ \left(\sum_i w_{i.}^2\right)^{1/2} \geqslant \varepsilon \right] \to 0,$$

for all $\varepsilon > 0$. Thus $\sqrt{n} t_w 3/(4\bar{w}^2)^{1/2} \to_{\mathrm{D}} \mathrm{N}(0,1)$.

By Slusky's theorem, we have that $\mathscr{L}(\tau_w) \to_{\mathrm{D}} \mathscr{L}(t_w)/\bar{w}$, where $\bar{w} = \lim_{n\to\infty} n^{-2} \sum_{i,j} w_{ij} > 0$. Thus Theorem 1 follows.

We note that the conditions (A.1) and (A.2) are the same as those given in Theorem 3 of O'Neil and Redner (1993).

Appendix B

Table 5
Quantiles of weighted Kendall's tau with $p = 0.3$

n	$m = [(n+1)p]$	0.900	0.950	0.975	0.990	0.995
6	2	1.0000	1.0000	1.0000	1.0000	1.0000
9	3	1.0000	1.0000	1.0000	1.0000	1.0000
13	4	0.6667	0.6667	1.0000	1.0000	1.0000
16	5	0.6000	0.6000	0.8000	0.8000	1.0000
19	6	0.4667	0.6000	0.7333	0.7333	0.8667
23	7	0.4286	0.5238	0.6190	0.7143	0.7143
26	8	0.3571	0.5000	0.5714	0.6429	0.7143
29	9	0.3333	0.4444	0.5000	0.6111	0.6667
33	10	0.3333	0.4222	0.4667	0.5556	0.6000
36	11	0.3091	0.3818	0.4545	0.5273	0.6000
39	12	0.3030	0.3636	0.4242	0.5152	0.5455
43	13	0.2821	0.3590	0.4103	0.4615	0.5128
46	14	0.2747	0.3407	0.3846	0.4505	0.4945
49	15	0.2571	0.3143	0.3714	0.4476	0.4667
53	16	0.2333	0.3000	0.3500	0.4167	0.4667
56	17	0.2353	0.2941	0.3529	0.4118	0.4559
59	18	0.2288	0.2810	0.3464	0.3987	0.4379
63	19	0.2164	0.2749	0.3333	0.3918	0.4269
66	20	0.2105	0.2632	0.3158	0.3684	0.4000
69	21	0.2095	0.2571	0.3143	0.3619	0.4000

Table 5
Continued

n	$m = [(n+1)p]$	0.900	0.950	0.975	0.990	0.995
73	22	0.2035	0.2554	0.3074	0.3506	0.3939
76	23	0.1937	0.2490	0.2964	0.3439	0.3834
79	24	0.1884	0.2464	0.2899	0.3406	0.3768
83	25	0.1867	0.2333	0.2867	0.3400	0.3667
86	26	0.1754	0.2246	0.2738	0.3231	0.3662
89	27	0.1795	0.2308	0.2707	0.3162	0.3561
93	28	0.1693	0.2222	0.2646	0.3175	0.3439
96	29	0.1675	0.2217	0.2611	0.3153	0.3399
99	30	0.1678	0.2138	0.2598	0.3057	0.3333

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