Material Summary: Basic Algebra

1. Polynomials

- We already looked at **linear** and **quadratic polynomials**
- **Term (monomial):** $2x^2$
  - Coefficient (number), variable, power (number ≥ 0)
- **Polynomial**: sum of monomials
  - $2x^4 + 3x^2 - 0{,}5x + 2{,}72$
  - Degree: the highest degree of the variable (with coefficient $\neq 0$)
- **Operations**
  - Defined the same way as with numbers
  - **Addition** and **subtraction**
    - $(2x^2 + 5x - 8) + (3x^4 - 2) = 3x^4 + 2x^2 + 5x - 10$
  - **Multiplication** and **division**
    - $(2x^2 + 5x - 8)(3x^4 - 2) = 6x^6 + 15x^5 - 24x^4 - 4x^2 - 10x + 16$

2. Polynomials in Python

- **numpy** has a module for working with polynomials
  - Includes the "general" polynomials, as well as a few special cases
    - Chebyshev, Legandre, Hermit
- **Storing polynomials**
  - As arrays (index $\Rightarrow$ power, value $\Rightarrow$ coefficient)
  - Keep in mind this will look "reversed" relative to the way we write

```
import numpy.polynomial.polynomial as p
p.polyadd([-8, 5, 2], [-2, 0, 0, 0, 3])
p.polymul([-8, 5, 2], [-2, 0, 0, 0, 3])
# array([-10., 5., 2., 0., 3.])
# array([ 16., -10., -4., 0., -24., 15., 6.])
```

- **Pretty printing**
  - Use **sympy** to print the polynomial
    - If it's a list, use it **directly**
    - If it's a Polynomial object, call the **coef property**
  - Reverse the order of coefficients (sympy expects them from highest to lowest)

```
import sympy
from sympy.abc import x
polynomial = p.Polynomial([-2, 0, 0, 0, 3])
sympy.init_printing()
print(sympy.Poly(reversed(polynomial.coef), x).as_expr())
# Output: 3.0*x**4 - 2.0
```

3. Set

- An **unordered collection** of things
  - Usually, numbers
  - No repetitions
- <u>**Set notation**</u>:
  - "**The set of numbers x, which are a subset of the real numbers, which are greater than or equal to zero**"
  - **Left**: example element
  - **Right**: conditions to satisfy
- Python **set comprehensions**
  - Very similar to what we already wrote
  - Also very similar to list comprehensions (but with curly braces)

```
positive_x = {x for x in range(-5, 5) if x >= 0}
# {0, 1, 2, 3, 4}
```

- <u>**Cardinality**</u>: number of elements
- Checking whether an **element is in the set**: $x \in S$
- Checking whether a **set is subset of another set**: $S_1 \subseteq S_2$
- **Union**: $S_1 \cup S_2$
- **Intersection**: $S_1 \cap S_2$
- **Difference**: $S_1 \setminus S_2$

4. Functions

- A relation between set of inputs *X* (**domain**) and a set of outputs *Y* (**codomain**)
- **One input produces exactly one output**
- The inputs don't need to be numbers
- Functions don't know how to compute the output, they're just mappings
- In programming, we write **procedures**
- Math notation:
  - Commonly abbreviated as: $f : X \to Y$
- Some more definitions:
  - **Injective** (one-to-one): unique inputs => unique outputs
  - **Surjective** (onto): every element in the codomain is mapped
  - **Bijective** (one-to-one correspondence): injective and surjective
  - Here is [a graphical view]

5. Function Composition

- Also called <u>**pipelining**</u> in most languages
- Takes two functions and applies them in order
  - **Innermost to outermost**
  - <u>**Math notation**</u>: $f \circ g = f(g(x))$
  - Can be generalized to more functions

- Note that the order matters

$$f(x) = 2x + 3,\ \ g(x) = x^2$$
$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 3$$
$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2$$

- This kind of notation can be confusing sometimes
  - x is only a placeholder for the input
  - We've used the same letter x for different inputs
  - **Tip**: When working with complicated functions, be very careful what the inputs and outputs are, and how variables depend on other variables
- Functions and composition are the basis of [functional programming]

6. Function Graphs (Plots)

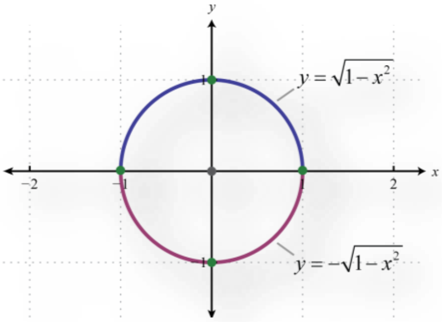
- One very intuitive way to get to **know functions is to plot them**
  - Generate values in the **domain** (independent variable)
  - For each value **compute the output** (dependent variable)
  - Create a **graph**
  - Plot all computed points and connect them with tiny straight lines
- **lambda** in Python is a short syntax for a function
  - We can define it outside as well (it's just shorter and simpler to use it inline)

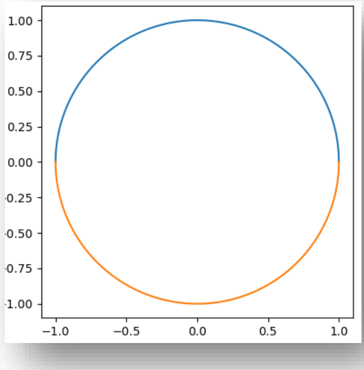
```python
import numpy as np
import matplotlib.pyplot as plt
def plot_function(f, x_min = -10, x_max = 10, n_values = 2000):
  x = np.linspace(x_min, x_max, n_values)
  y = f(x)
  plt.plot(x, y)
  plt.show()
plot_function(lambda x: np.sin(x))
```

7. Graphing a Circle

- Let's try to graph the unit circle
  - Equation: $x^2 + y^2 = 1$
- This cannot be represented as one function
  - We have multiple values of y

    $x = 0 \rightarrow y = \{-1,\ 1\}$
- We can try two functions:
  - But we want to represent the circle as one object






```python
def plot_function(f, x_min = -10, x_max = 10, n_values = 2000):
  plt.gca().set_aspect("equal")
  x = np.linspace(x_min, x_max, n_values)
  y = f(x)
  plt.plot(x, y)

plot_function(lambda x: np.sqrt(1 - x**2), -1, 1)
plot_function(lambda x: -np.sqrt(1 - x**2), -1, 1)
plt.show()
```

- In math and science, many problems can be solved by **changing our viewpoint**
- We can use another type of **reference system**
  - One which incorporates **angles naturally**
  - **Polar coordinate system** $(\boldsymbol{r}, \boldsymbol{\varphi})$:
    - (r: distance from origin ($r \geq 0$); φ: angle to x-axis)
  - We can easily convert **Cartesian to polar coordinates**

$$x^2 + y^2 = 1$$
$$(r\cos\varphi)^2 + (r\sin\varphi)^2 = 1$$
$$r^2\cos^2\varphi + r^2\sin^2\varphi = 1$$
$$r^2(\cos^2\varphi + \sin^2\varphi) = 1$$
$$r^2 = 1,\ r \geq 0 \Rightarrow r = 1$$

  - Now we can see the equation is very, very simple
  - Doesn't even depend on φ
  - This is why we needed the change of viewpoint (coordinates)

- Graphing a function in **polar coordinates**
  - This applies to any function, circles in particular
  - Generate initial values of r and φ
  - Convert them to rectangular coordinates
  - Plot the rectangular coordinates

```
import numpy as np
import matplotlib.pyplot as plt
r = 1 # Radius
phi = np.linspace(0, 2 * np.pi, 1000) # Angle (full circle)
x = r * np.cos(phi)
y = r * np.sin(phi)
plt.plot(x, y)
plt.gca().set_aspect("equal")
plt.show()
```

```
plt.polar(phi, r)
```

8. Complex Numbers

- Field
  - A collection of values with operations "plus" and "times"
  - Algebra is so abstract we can redefine these operations
- History of number fields
  - **Natural numbers**: $\mathbb{N} = \{0, 1, 2, \dots\}$
  - **Integers:** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
  - **Rational numbers** $\mathbb{Q}$ **:** ratio of two integers
  - **Real numbers**: $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$
  - **Complex numbers**: $\mathbb{C}$
    - "Imaginary unit": i is the positive solution of $x^2 = -1$
    - Pairs of real numbers: $(a; b) : a, b \in \mathbb{R}$
    - Commonly written as: $a + bi$
    - **Real part**: $\mathrm{Re}(a + bi) = a$
    - **Imaginary part**: $\mathrm{Im}(a + bi) = b$
    - **In Python, we use `j` instead of i**

```
3j
```

```
1j
```

```
3 + 2j
```

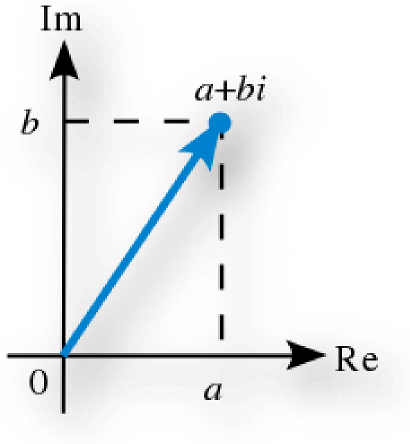
    - We can get the real and imaginary parts

```
z = 3 + 2j
print(z.real) # 3
print(z.imag) # 2
```

- Adding and multiplying complex numbers

```
print((3 + 2j) + (8 - 3j)) # (11-1j)
print((3 + 2j) * (8 - 3j)) # (30+7j)
```

- We can plot the coordinate **pairs on the plane**
- Each point in the **2D space represents one complex number**
- **Polar coordinates**: we can use the same transformation
  - $\rho = |z|$ – **module** of the complex number
  - $\varphi = \arg(z)$ – **argument** of the complex number
  - $a = \rho\cos(\varphi)$, $b = \rho\sin(\varphi)$
- Why do we do this?
  - Some operations (e.g., multiplication and division) are easier in polar coordinates
  - Powers of complex numbers become extremely easy
- Polar form: $z = a + bi = \rho(\cos(\varphi) + i\sin(\varphi))$



9. Euler's Formula

- Leonhard Euler proved that: $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$
  - Here's a summary of the proof
  - It involves series which **we haven't covered yet**
  - A very beautiful consequence: $e^{i\pi} + 1 = 0$
- Now we can write our complex number as: $z = |z|e^{i\varphi}$
- Why and how does multiplication work?
  - Multiplication by a real number
    - Scales the original vector
  - Multiplication by an imaginary number
    - Rotates the original vector
  - You can see a thorough explanation here
- **Main point:** Multiplication of complex numbers is the same as scaling and rotating 2D vectors

10. Fundamental Theorem of Algebra

- Theorem of Algebra: "**Every non-zero, single-variable, degree-n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.**"
- More simply said: **Every algebraic equation has as many roots as its power.**
- Back to quadratic equations
  - How do we get all roots?
  - Simply use the complex math Python module: **cmath**

```
import cmath
def solve_quadratic_equation(a, b, c):
  discriminant = cmath.sqrt(b * b - 4 * a * c)
  return [
    (-b + discriminant) / (2 * a),
    (-b - discriminant) / (2 * a)]

print(solve_quadratic_equation(1, -3, -4))
# [(4+0j), (-1+0j)]
print(solve_quadratic_equation(1, 0, -4)) # [(2+0j), (-2+0j)]
print(solve_quadratic_equation(1, 2, 1)) # [(-1+0j), (-1+0j)]
print(solve_quadratic_equation(1, 4, 5)) # [(-2+1j), (-2-1j)]
```

11. Galois Field

- In everyday algebra, we usually think about fields as those we already know
- But since algebra is abstract, we can define our own fields
- **Galois field: $GF(2)$**
  - **Elements $\{0,\ 1\}$**
  - **Addition: equivalent to XOR**
  - **Multiplication: as usual**
- **Usage: in cryptography**
- If you're interested, you can have a look at this paper

+	0	1
0	0	1
1	1	0

*	0	1
0	0	0
1	0	1