Material Summary: Statistics

1. Basic Concepts

1.1 Descriptive Statistics

- Numbers which are used to **summarize** and **describe** data
  - We work with **all items of interest** – **statistical population**
  - We don't try to make predictions, just describe what we're seeing
- Not very useful on their own
  - But an important part of other methods
- Example: pet shop sales
  - 100 pets in one month: 40 dogs, 30 cats, 30 other
- What percent of all pets are dogs?
- What's the mean number of cats sold per month?
- We can also represent the information graphically
  - What does the distribution of dog sales per day look like?
  - What does the cumulative distribution of sales look like?
  - How do sales compare?

1.2 Inferential Statistics

- In many cases the **population** is too large (or even infinite)
  - We represent the population by a subset – **sample**
  - **The population characteristics can be estimated by using the sample**
    - We have to be extremely careful how to choose the sample
  - In most cases we need **random sampling** of the population
- Examples
  - Voting predictions
    - We ask a small number of people and we draw **inferences** about the entire country
  - Mean salary by age
    - We divide people into age groups (e.g. $< 20, 20-25, 25-30, 30-35, \ldots$) and ask several people within each age group
    - This also makes the continuous variable "age" easier to work with

1.3 Sampling

- The process of selecting a sample from the population
- Steps in the sampling process
  - Define the population
  - Specify the **sampling frame** – a set of items from the population
  - Specify the **sampling method** – how to select items from the frame

SoftUni

Follow us:

        - Determine the sample size
        - Implement the sampling and collect data
    - A badly done sampling can induce **biases** and **errors**
        - **Selection bias** – selecting a non-random sample
            - E.g., asking only CEOs of companies when sampling data for salaries by age
        - **Random sampling error** – random variations in the results

1.4 Sampling Methods

- Non-random sampling
    - Can be biased
    - **Not representative** of the population
- Random sampling
    - Every member of the population has equal chance of being chosen
    - Example: insect population in trees
        - Trees are numbered 1-200, 10 trees are chosen at random
        - All insects are counted on the 10 random trees
- Stratified sampling
    - Divide the population into categories (subpopulations)
    - For each category, sample at random
    - Example: foot measurement study → male / female; age groups
        - Select samples for each combination { gender; age }

2. Properties of Distributions

2.1 Summarizing Distributions

- A **histogram** is a **complete description** of the sample distribution
- We often summarize it using a few descriptive statistics
    - **Central tendency**
        - Do the values tend to cluster around a center?
    - **Modes**
        - How many clusters are there? Where are they?
    - **Variance**
        - How much variability is there (how "spread out" is the distribution)?
    - **Tails**
        - How quickly do probabilities drop off as we move away from the center(s)?

        - **Outliers**
            - Are there extreme values, far from the center(s)?
- These are also called **summary statistics**

2.2 Measures of Central Tendency

- **Average** – a number which describes a typical data point
    - Can be calculated in many ways
- **Arithmetic mean**
    - The sum of all measurements divided by the number of observations

$$\bar{x} = \frac{1}{n}\sum_{x=1}^{n} x_i$$

- **Median**
    - The middle value of the distribution
    - To calculate it, the numbers must be sorted in ascending order
    - Examples:
        - $\text{Me}(\{1,\ 2,\ 2,\ 3,\ 4\}) = 2;\ \text{Me}(\{1,\ 2,\ 2,\ 3,\ 4,\ 10\}) = 2{,}5$
- **Mode**
    - The most frequent item
        - $\text{Mo}(\{1,\ 3,\ 2,\ 3,\ 4,\ 3\}) = 3$
    - Many "most frequent items" ⇒ multimodal distribution

2.3 Variance

- Describes how far away a sample is from the sample mean
    - All differences from the mean $x_i - \bar{x}$ can be positive or negative
    - They all sum up to 0 (that's the definition of the mean)
    - So we square them to make them positive

$$S^2(x) = \frac{1}{n-1}\sum_{x=1}^{n}(x_i - \bar{x})^2$$

    - Standard deviation: $S(x) = \sqrt{S^2(x)}$
- In the sample variance formula, there is $n - 1$ in the denominator
    - It refers to "degrees of freedom" – how many items we can remove
        - The number of parameters that can vary
    - Because all distances sum up to 0, if we know $n - 1$ of them, we can find the last one
    - Gives us an unbiased estimator (more on that *here*)
- Why bother to take the standard deviation?

        - Instead of using variance directly
    - It's all about units
    - Example:
        - Let's say we're measuring length in m
        - By definition, the variance will have units of m^2
        - We want to see how far is a certain point from the center and the units don't match
            - Compare $d = 2m,\ S^2 = 0{,}25m^2$ to $d = (2 \pm 0{,}5)\ m$
        - In order to make units match, we take the square root
        - Example
            - If we measure $2{,}75m$, we can say "This measurement is located at **1,5 standard deviations above the mean**"
            - Comparisons like these are very useful in statistics

2.4 Population vs. Sample: Measures

- There are differences between a population and samples from that population ⇒ we have different statistics
    - Notation
        - Sample statistics – sample mean, sample variance, etc. – Latin letters
        - Population statistics – Greek letters
- Population mean μ
    - Also called **expected value**

$$\mu(x) = E[x] = \frac{1}{N}\sum_{i=1}^{N} x_i$$

    - N – population size
- Population variance σ^2
    - Note how since we know the entire population, there is **no estimation** going on

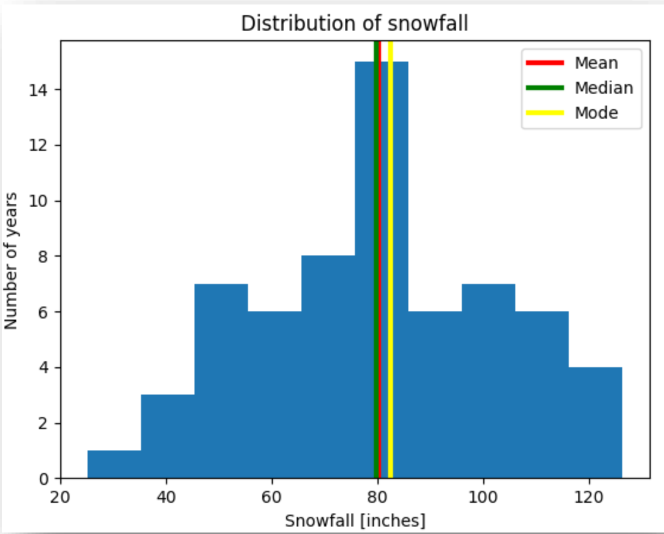
$$\sigma^2(x) = E[(x_i - \mu)^2] =$$

$$= \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

    - So, there is N in the denominator
- Population standard error
    - $\sigma(x) = \sqrt{\sigma^2(x)}$

2.5 Example: Snowfall Data

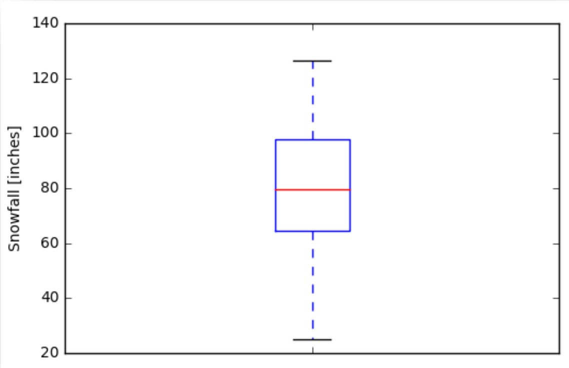
- You are given data of snowfall in Buffalo, NY in inches for years 1910 – 1972 (**snowfall.csv**)
- Plot a histogram
- Print the mean, standard deviation and modes
- Print the standard deviation
  - Note: If you're using **numpy**, it returns the biased estimator of standard deviation. Pass a parameter **ddof = 1** (difference in degrees of freedom) to calculate the unbiased estimator
- Overlay the mean, median and first mode on the histogram



2.6 Five-Number Summary

- Conveys similar information to a histogram
  - How many percent of the data are less than or equal to a specified number
    - Minimum (0%); first quartile (25%); median (50%); third quartile (75%); maximum (100%)
      - Generalization: quantiles – divide the frequency distribution into equal groups
      - 100 groups = percentiles
- Visualization: boxplot
  - Middle line – median
  - Box – quartiles
  - Whiskers – largest "non-outliers" – 1.5 times the interquartile range

- Points – outliers

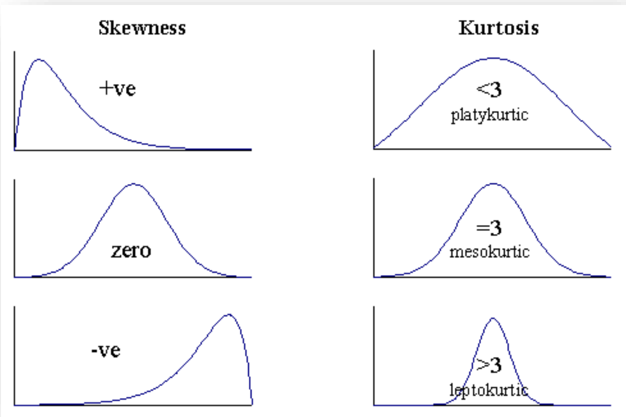


2.7 Moments of Distributions

- r^{th} central moment:

$$\mu_r(x) = \frac{\sum(x-\mu)^r}{N}$$

    - Defined for discrete and continuous variables
    - Measure the shape of the probability distribution
- Zeroth moment: 1 (**total probability**)
- First moment: **arithmetic mean** μ
- Second moment: **variance** σ^2
- Third moment: **skewness** γ
    - Asymmetry in the distribution
- Fourth moment: **kurtosis** β
    - Heaviness of the "tails"
    - "Normal": $\beta = 3$
        - Excess kurtosis: $\beta - 3$



2.8 Moments of the Gaussian Distribution

- Generalization of the binomial distribution
- Probability density function

$$N(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Mean: μ
- Median: μ
- Mode: μ
- Variance: σ^2
- Skewness: 0
- Excess kurtosis: 0
- *"And that, kids, is why I love the Gaussian distribution."*

2.9 Standard Score

- In order to compare different Gaussian distributions, we can "normalize" them
  - Change their parameters to get a "standard" Gaussian distribution with $\mu = 0$ and $\sigma = 1$
  - We need to "shift" the distribution left or right and "squish" or "stretch" to achieve the required standard deviation
  - The shift is denoted by the standard score (or z-score): $\mathrm{Z}(x) = \frac{x-\mu}{\sigma}$
- Example: 50 student scores
  - Normal distribution, mean 60 (out of 100) and standard deviation 15
  - How well did a student perform if they had 70 / 100?
    - Top 25% of the class
  - What marks do the top 10% of the class have?
    - 79 and up

3. Many Variables

3.1 Covariance

- Up to now, we've been looking at variables on their own
  - But in many cases, they interact with each other
- Covariance is a measure of the joint variability of two variables

$$\mathrm{cov}(x,y) = \frac{1}{n}\sum(x_i - \bar{x})(y_i - \bar{y})$$

  - Positive: as one variable increases, the other also increases
  - Negative: as one variable increases, the other decreases
  - Zero: the two variables don't vary together at all
- We can see that $\mathrm{cov}(X,X) = \sigma^2(X)$

- In higher dimensions, we calculate a covariance matrix
  - The same idea: element $(i,\ j)$ is equal to the covariance of the i^{th} and j^{th} dimensions: $A_{ij} = \text{cov}(x_i,\ x_j)$

3.2 Correlation

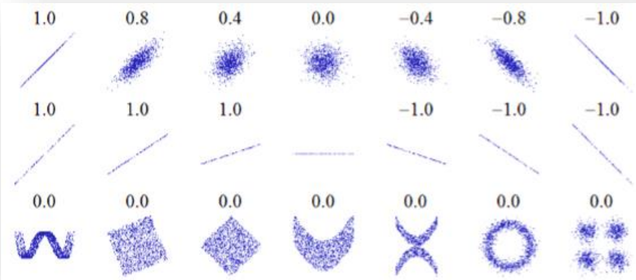
- Like the variance, covariance is in "weird" units
  - We divide by the standard deviations to normalize them ⇒ standard scores (similar to z-scores)

$$p_i = \frac{(x_i - \bar{x})}{s_x}\frac{(y_i - \bar{y})}{s_y}$$

  - The mean value can be calculated as

$$\rho = \frac{1}{n}\sum p_i = \frac{\text{cov}(x, y)}{s_x s_y}$$

    - This is called **Pearson's correlation coefficient**
- The correlation coefficient can be in $[-1;\ 1]$
  - **High absolute value ⇒ strong correlation**
  - Measures the linearity of a relationship between two variables
  - Cannot express other, more complex relationships



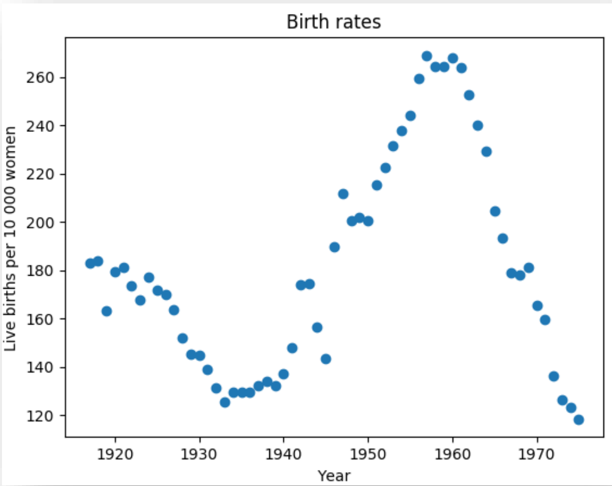
3.3 Scatter Plots

- The easiest way to see how two variables are correlated
- Two versions:
  - "Independent" variable – x-axis, "dependent" variable – y-axis
  - Two correlated variables (we can't say which is "independent")
- Besides, outliers usually become easily visible
- Best practices
  - Label your axes
  - If needed, include a legend
  - Scale / transform the variables if needed
    - Simplifies the relationship

- Add trendlines if needed
  - You can also plot line charts if that's what your data suggests

3.4 Example: Birth Rates

- You are given the number of live births per 10 000 23-year-old women in the US between 1917 and 1975
  - File: birth_rates.csv
- Plot a scatter plot of the birth rates per year
  - What conclusions can you make?
  - This is called "time series analysis" – we are analyzing a process as it evolves with time
- Additionally, you can still inspect the variables one by one
  - Plot a histogram of the birth rates,disregarding the years
  - Are there any "typical" birth rates?
    - Are they distributed normally?



3.5 Example: Brain and Body Weights

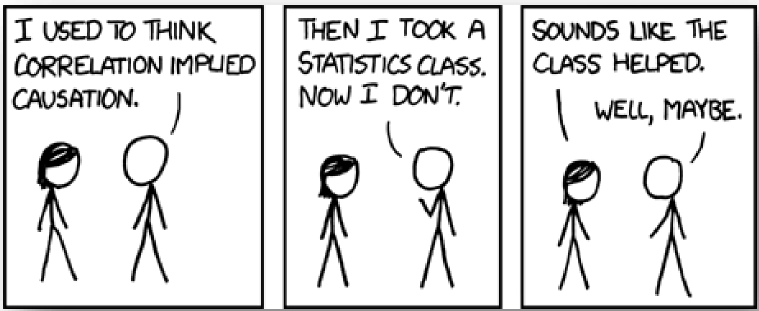
- File: brain_weight.csv
- Inspect the two variables: body weight [kg], brain weight [g]
  - Plot histograms, even boxplots if needed
- Create a scatterplot
  - The distribution is highly skewed, almost nothing is visible
- Transform the data
  - Take logarithms of both the body weight and the brain weight

- Plot histograms of the logarithms
- Create another (log-log) scatterplot
- Is there any significant relationship?
  - If so, what is the real relationship (between the untransformed variables)?
  - To find it, you have to "reverse" the transformation

4. Common Pitfalls

4.1 Correlation Does Not Imply Causation!

- If two variables are correlated, this does not mean that necessarily the first causes the second
- Example: height and weight
  - Does a greater weight cause a greater height?
- We can still describe them
- **We can predict** height from weight and vice versa
- But that still does not say anything about one causing the other



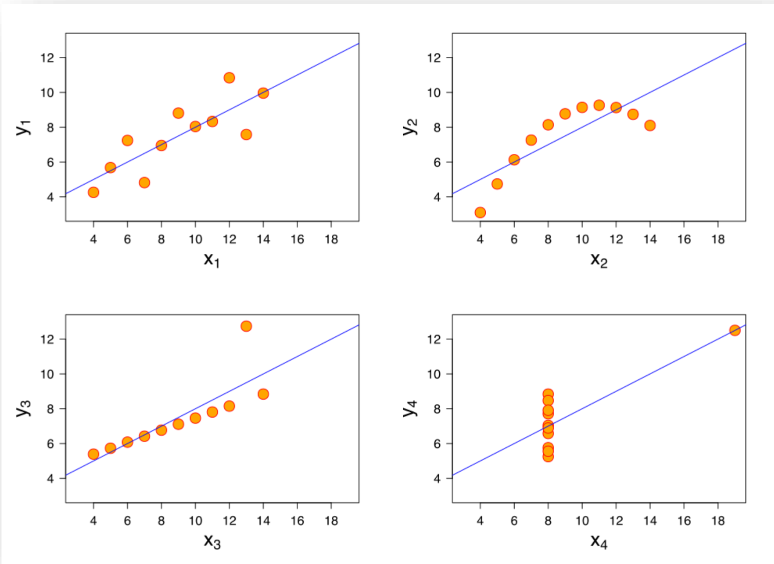
4.2 Correlation vs. Causation

- Reverse causation
  - The faster the windmills rotate, the more wind there is ⇒ Windmills cause wind
- Lurking variable
  - The more firefighters there are to put out a fire, the greater the damage caused ⇒ Firefighters being present at fires, cause more damage
- Bidirectional relationship
  - Predator numbers affect prey numbers, but prey numbers (amount of food) also affect predator numbers
- Coincidence
  - http://tylervigen.com/spurious-correlations

- More information about causal relationships: [minutephysics (YouTube)]

4.3 Anscombe's Quartet

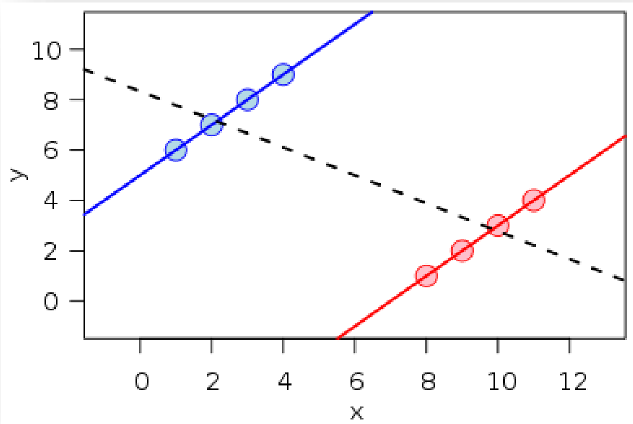
- Four datasets with similar descriptive statistics which look completely different when plotted
  - More information: [Wikipedia]
- Takeaways
  - Plot the data
    - In general, it's important to get to know your data
  - List as many assumptionsand simplifications as possible
  - **Do not rely** simply on a bunch of numbers
    - Even worse, a single number



4.4 Simpson's Paradox

- 1973, University of California – Berkeley was sued for sex discrimination
  - Accepted 44% male applicants but 35% female applicants
  - When researches dug in, they found it was not so
  - ***"If the data are properly pooled...there is a small but statistically significant bias in favor of women."***
- Simpson's paradox
  - A case of **omitted variable** bias
  - Observed explanatory variable → explained variable
  - **Lurking variable**
  - **Uneven sample sizes (in most cases)**

        - The effect of the observed explanatory variable reverses when we take the lurking variable into account
- When we consider both samples together, it appears that x has a negative effect on y
    - When we take color into account, the relationship reverses
- Other example: kidney stone treatment
    - One treatment is better for large stones, and better for small stones; but the other one is better overall
        - Confounders – the severity of the illness + different sample sizes
- An article with *more info* on the topic



4.5 UCB Admissions – Explanation

- The *research paper* concluded that 6 departments were significantly biased towards men and 4 – towards women
    - The other 75 weren't (significantly) biased at all
- Actually, the overall bias was (slightly) in favor of women
- Women tended to apply to competitive departments with low admission rates
- Men tended to apply to less competitive departments with high admission rates
    - We cannot observe that directly from our dataset
    - **Lurking variable** – competitiveness
        - Students didn't have the same motivations to apply