Material Summary: Linear Algebra

1. Vectors

Vector Definitions

- "Physics definition"
  - A pointed segment in space
- "Computer science definition"
  - A list of objects (usually numbers)
  - Dimensions = length
- Math definition
  - Encompasses both, and allows even more abstraction: _v_
  - Vectors can be added and multiplied
    - By numbers and other vectors
  - Similar to how we defined a field
- Another perspective
  - Transformations
  - Actually, things a just a little more complicated…
    - You can look up "tensors" if you're interested
    - We'll talk a little about tensors later

Vector Components

- The distances to all coordinate axes: _v_x_, _v_y_
- Equivalent to $\begin{bmatrix} v_x \\ v_y \end{bmatrix}$
- Polar coordinates: $v = |\vec{v}|,\ \theta$
- Finding components: Pythagoras $v_x = v\cos(\theta),\ \ v_y = v\sin(\theta)$
- Finding the polar form (magnitude, direction):

$$v = \sqrt{v_x^2 + v_y^2},\ \ \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

- All these operations generalize to more than 2 dimensions
- We usually denote vectors by _v_ or with bold type: $\boldsymbol{v}$
  - Another notation: Latin letters for vectors, Greek letters for numbers
  - Reason: The vector $\boldsymbol{v}$ and its length v can be easily confused

Vector Operations

- **Addition**
  - Result: length = distance from start to end, direction: start → end
  - In component form: sum all components for every direction

- **Multiplication by a number (scalar)**
  - Result: length = scaled length, direction: same (if scalar ≥*0*), opposite otherwise
  - In component form: multiply each component by the number

- **Scalar product of two vectors**
  - Also called dot product or inner product
  - Result: scalar
  - Definition: $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$
  - Using the vector components: $\vec{a}.\vec{b} = \sum_{i=1}^{n} a_i b_i$

- **Vector product of two vectors**
  - Also called cross product
  - Result: vector, perpendicular to both initial vectors
  - Definition: $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin(\theta)\vec{n}$
    - $\vec{n}$ – normal vector
    - Magnitude: $|\vec{a}||\vec{b}|\sin(\theta)$ = volume of parallelogram between $\vec{a}$ and $\vec{b}$
    - Direction: coincides with the direction of $\vec{n}$

Vector Spaces

- A field (usually R or C): F
- A set of elements (vectors): V
- Operations
  - Addition of two vectors: $w=u+v$
  - Multiplication by an element of the field: $w=\lambda u$
- A "checklist" of eight axioms
- We read this as "vector space (or linear space) V over the field F"
- Examples of vector spaces
  - Coordinate space, e.g., real coordinate space
    - n-dimensional vectors
  - Infinite coordinate space $\mathbb{R}^\infty$
    - Vectors with infinitely many components
  - Polynomial space
    - All polynomials of variable x with real coefficients $\mathbb{R}[x]$
  - Function space

Linear Combinations

- Vectors: $v_1, v_2, \dots, v_n$
- Numbers (scalars): $\lambda_1, \lambda_2, \dots, \lambda_n$
- **Linear combination:** The sum of each vector multiplied by a scalar coefficient

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = \sum \lambda_i v_i$$

- Why linear? No fancy functions, no vector multiplications

- **Span** (linear hull) of vectors: the set of all their linear combinations
- Linear (in)dependence
  - The vectors $v_1, \dots, v_n$ are **linearly independent** if the only solution to the equation: $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = \vec{0}$ is $\lambda_1 = 0,\ \lambda_2 = 0,\ \dots, \lambda_n = 0$
  - Conversely, they are **linearly dependent** if there is a non-trivial linear combination which is equal to zero
- Example:

$$u = (2, -1, 1),\ \ v = (3, -4, -2),\ \ w = (5, -10, -8)$$

$$w = -2u + 3v \Rightarrow 2u - 3v + 1w = 0$$

- Consider:

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

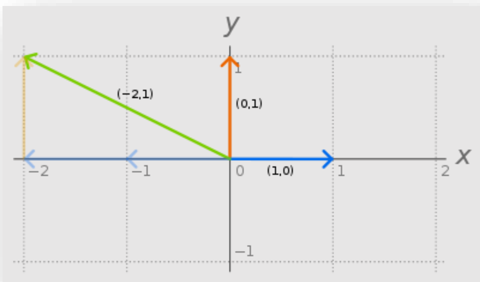
- Now consider the vector:

$$a = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- We can see that we can express a as the linear combination:

$$a = -2\hat{i} + 1\hat{j}$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



- **Linearly independent**
- Every other vector in the space can be represented as their linear combination
- This linear combination is **unique**
- **Each vector space has a basis**
- Each pair of **two** LI vectors forms a basis in **2D** coordinate space
- Each set of n **LI vectors forms a basis in n-dimensional vector space**

2. Matrices

Definition

- A **rectangular table of numbers**
- Dimensions: **rows** × **columns**

- Examples:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 4.2 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 7 & 12 \\ 0 & 5 & -3 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

- R – row vector, C – column vector
- Elements $A = \{a_{ij}\}$
- Scalars have **no** dimensions: 2; 3; 18; -42; 0,5
- Vectors have **one** dimension: $v = \{v_i\}$
- Matrices have **two** dimensions: $A = \{a_{ij}\}$
- A generalization of this pattern to many dimensions is called a **tensor**
    - Tensors are quite more complicated than this
    - For almost all purposes it's OK to think about them as multidimensional matrices

Operation

- **Addition** (the dimensions must be the same)

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 8 & 9 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -4 & 1 \end{bmatrix} \Rightarrow A+B = \begin{bmatrix} 2+1 & 3-3 & 7+0 \\ 8+2 & 9-4 & 1+1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 7 \\ 10 & 5 & 2 \end{bmatrix}$$

- **Multiplication by a scalar**

$$\lambda = 2, A = \begin{bmatrix} 2 & 3 & 7 \\ 8 & 9 & 1 \end{bmatrix} \Rightarrow \lambda A = \begin{bmatrix} 2.2 & 2.3 & 2.7 \\ 2.8 & 2.9 & 2.1 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 14 \\ 16 & 18 & 2 \end{bmatrix}$$

- All $\boldsymbol{m} \times \boldsymbol{n}$ matrices **form a vector space**
    - You may check this
- Transposition:
    - Turning **rows into columns** and vice versa
    - The transpose of a matrix is denoted by an **upper index T**

$$A^T = (a_{ij})^T_{m \times n} = (a_{ji})_{n \times m} \qquad A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -3 & -4 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -4 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

Matrix Multiplication

- The dimensions **must match**:

$$A_{m \times p} B_{p \times n} = C_{m \times n}$$

- Definition:

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

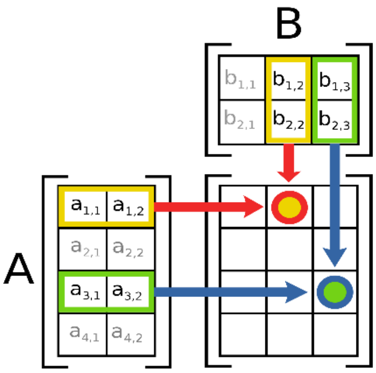
- Example:

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 8 & 9 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -3 & -4 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 7 & -8 & 10 & 18 \\ -17 & -20 & 10 & 36 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2.1 + 3.(-3) + 7.2 & 2.2 + 3.(-4) + 7.0 & 2.0 + 3.1 + 7.1 & 2.1 + 3.3 + 7.1 \\ 8.1 + 9.(-3) + 1.2 & 8.2 + 9.(-4) + 1.0 & 8.0 + 9.1 + 1.1 & 8.1 + 9.3 + 1.1 \end{bmatrix}$$

- Note that $\boldsymbol{AB} \neq \boldsymbol{BA}$
  - In this case, we can't even **multiply** $\boldsymbol{BA}$
  - We say that **matrix multiplication is not commutative**
    - Compare with numbers:

5.3 = 3.5 → commutative



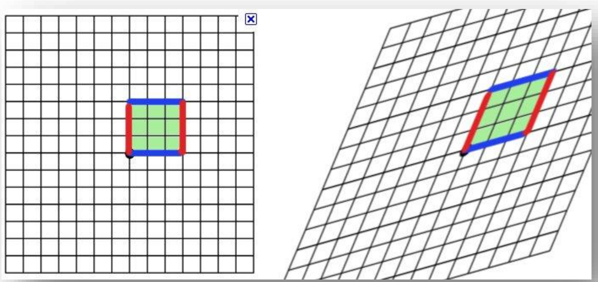
- We can use @ or dot() for both matrix multiplicationand dot products
- **Note:** Whenever possible, use **numpy** arrays instead of lists

```
A = np.array([
  [2, 3, 7],
  [8, 9, 1]
])
B = np.array([
  [1, -3, 0],
  [2, -4, 1]
])

print(A + B)
print(2 * A)
print(A * B) # Element-wise multiplication
print(A.dot(B)) # Error: shapes not aligned
print(A.dot(B.T)) # Matrix multiplication
```

Transformation

- A mapping (function) between two vector spaces: $V \to W$
- Special case: mapping a space onto itself: $V \to V$
  - This is called a linear operator
- Each vector of V gets mapped to a vector in W



Linear Transformations

- Only **linear combinations are allowed**
- The origin **remains fixed**
- All lines remain lines (not curves)
- All lines remain evenly spaced (equidistant)
- Each **space has a basis**
  - All other vectors can be expressed as **linear combinations of the basis vectors**
  - If we know how **basis vectors are transformed**, we can transform **every other vector**
- Consider the transformation

$$\hat{\imath}' = \begin{bmatrix}1\\-1\end{bmatrix}, \hat{\jmath}' = \begin{bmatrix}2\\1\end{bmatrix} \quad \Rightarrow \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

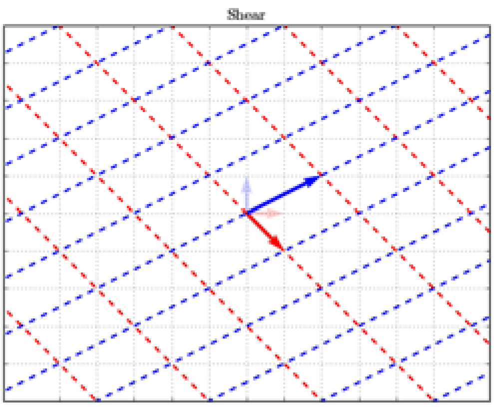
- Consider another vector
  - Old basis:

$$v = v_x\hat{\imath} + v_y\hat{\jmath} \quad \begin{bmatrix}v_x\\v_y\end{bmatrix} = v_x\begin{bmatrix}1\\0\end{bmatrix} + v_y\begin{bmatrix}0\\1\end{bmatrix}$$

  - New basis:

$$v' = v_x\hat{\imath}' + v_y\hat{\jmath}' \quad \begin{bmatrix}v'_x\\v'_y\end{bmatrix} = v_x\begin{bmatrix}1\\-1\end{bmatrix} + v_y\begin{bmatrix}2\\1\end{bmatrix}$$

    - **Same coefficients, new basis vectors**
- This operation is called applying the linear transformation



Multiple Transformations

- Consider the same transformation:

$$\hat{\imath}' = \begin{bmatrix}1\\-1\end{bmatrix}, \hat{\jmath}' = \begin{bmatrix}2\\1\end{bmatrix}$$

- We applied the linear transformation by taking dot products
  - Therefore, we can describe it in another way – using a matrix
  - This is called the **matrix of the linear transformation**
  - Its **columns** denote where the **basis vectors** go

$$T = \begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}$$

- Applying the transformation to a vector is the same as multiplying the matrix times the original vector:

$$v' = Tv$$

- Example:

$$T = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}, \ v = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \qquad v' = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 29 \\ -3 \end{bmatrix}$$

- We can apply many transformations, one right after the other
  - Result: composite transformation
  - We do this by multiplying **on the left** by the matrix of each transformation

  ⇒ matrix multiplication ≡ applying many transformations
- To visualize transformations, you can use the code in the **visualize_transformation.py** file
- Intuition
  - Apply each transformation in order
  - After the last one, record where the basis vectors land
  - The new matrix is the matrix of the composite transformation
- We can either apply all transformations one by one
  - Or **just** the **resulting** transformation ☺
- This is especially useful in computer graphics
- Rotation, then shearing:

  $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
- Apply rotation to a vector:

$$v' = Rv = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

- Apply shear to the resulting vector:

$$v'' = Sv' = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v'_x \\ v'_y \end{bmatrix}$$

- This is the same as:

$$v'' = SRv = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

- The new transformation matrix is:

$$T = SR = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

- Measure of how much the unit area (volume) changes
- Scalar value
- Defined only for square matrices
- For more than two dimensions: area → volume
- The determinant of a matrix A is denoted $\det(A)$
- The determinant has very useful [properties]
  - Notably, $\det(AB) = \det(A)\det(B)$

3. Linear Systems

Linear Systems in Matrix Form

- Consider the linear system

$$\left| \begin{array}{rcl} 2x - 5y + 3z & = & -3 \\ 4x + 0y + 8z & = & 0 \\ 1x + 3y + 0z & = & 2 \end{array} \right.$$

- Unknown variables x, y, z
- We can represent this as a matrix equation

$$\begin{bmatrix} 2 & -5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

  - Or more generally: $Ax = b$
  - Looks like a linear equation "on steroids"

Inverse Matrix

- Consider a **general**, "good" transformation
  - The inverse transformation will **"bring back" the basis vectors**
  - **90° clockwise rotation $\Rightarrow$ 90° counterclockwise rotation**
- The inverse transformation has its own matrix: T^{-1}
- If we apply the transformation and the inverse, we'll get our initial result
  - I.e., nothing will change
  - In math terms: $T^{-1}T = E$
- Let's now try to apply the inverse transformation to our linear system
  - Note that this means multiplying on the left

$$Ax = b \qquad Ex = x \Rightarrow x = A^{-1}b$$

$$A^{-1}A = E \Rightarrow Ex = A^{-1}b$$

- To find the **unknown vector x**:
  - We need to find the **inverse matrix of A**
  - There are many methods, the most popular of which is called **Gaussian elimination** (or Gauss – Jordan method)
- <u>**Basic idea**</u>: $A^{-1}A = E$
  - Apply some transformation to get **from A to E**
  - Apply the **same transformation to E**
  - What we get is **the inverse matrix**