Pricing Assets Using Jump-Diffusion Models

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Abstract

This working paper presents a detailed examination of asset pricing models that incorporate both continuous fluctuations and jumps. The jump-diffusion models are essential for accurately modeling asset prices, as they account for sudden, unpredictable market movements alongside regular changes. We discuss three distinct cases: asset pricing with Brownian motion, asset pricing with compound Poisson processes, and asset pricing combining both Brownian motion and compound Poisson processes.

Contents

1 Pricing of an Asset Using Brownian Motion

In this section, we explore the classical Black-Scholes-Merton framework, where the asset price dynamics are driven solely by continuous fluctuations modeled through Brownian motion. This model forms the foundation for understanding more complex models that incorporate jumps.

1.1 Model Setup

We begin by defining the probability space in which our stochastic processes are defined:

$$(\Omega, \mathbb{P}, \mathbb{F}),$$

where Ω is the sample space representing all possible outcomes, $\mathbb{P}$ is the physical probability measure reflecting real-world probabilities, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the filtration representing the evolution of information over time.

1.2 Asset Price Dynamics

The asset price, denoted $S(t)$, is assumed to follow an Itô process, which captures both deterministic trends and random shocks:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t),$$

subject to

$$W(t) \text{ is a standard Brownian motion.}$$

In this equation:

- α is the drift coefficient, representing the expected rate of return of the asset.
- σ is the volatility coefficient, representing the magnitude of random fluctuations.
- $dW(t)$ is the increment of a standard Brownian motion, capturing the continuous stochastic shocks to the asset price.

This model assumes that asset prices evolve continuously over time, with no sudden jumps or discontinuities.

1.3 Risk-Neutral Measure and Pricing

To price derivatives in a risk-neutral world, we transition from the physical probability measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$. Under $\mathbb{Q}$, investors are indifferent to risk, and all assets are expected to grow at the risk-free rate r, adjusted for risk. This change of measure is facilitated by Girsanov's Theorem, which allows us to adjust the drift of the Brownian motion:

$$dS^{\mathbb{Q}}(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t),$$

subject to

$$W^{\mathbb{Q}}(t) \text{ is a Brownian motion under } \mathbb{Q}.$$

Under the risk-neutral measure, the drift term becomes r, reflecting that in expectation, all assets earn the risk-free rate. The volatility remains unchanged, as it represents the inherent uncertainty in the asset price.

1.4 Solving the Stochastic Differential Equation

Applying Itô's Lemma to solve the stochastic differential equation, we find the explicit solution for the asset price at time T:

$$\begin{aligned} S^{\mathbb{Q}}(T) &= S^{\mathbb{Q}}(t) + \int_t^T rS^{\mathbb{Q}}(u)du + \int_t^T \sigma S^{\mathbb{Q}}(u)dW^{\mathbb{Q}}(u) \\ &= S^{\mathbb{Q}}(t) \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)\right)\right). \end{aligned}$$

To derive this, we integrate the SDE using the property that for geometric Brownian motion, the solution is known to be exponential of the integral form.

We can simplify the expression by introducing a standard normal variable Y:

$$Y = \frac{W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)}{\sqrt{T-t}} \sim \mathcal{N}(0,1),$$

so the asset price at time T becomes:

$$S^{\mathbb{Q}}(T) = S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y}.$$

Here, we used the fact that the increment of the Brownian motion over $[t,T]$ is normally distributed with mean zero and variance $T-t$.

1.5 Pricing a European Call Option

Next, we aim to price a European call option with strike price K and maturity T. The option gives the holder the right, but not the obligation, to purchase the asset at price K at time T. The payoff at maturity is:

$$V(T) = \max(S^{\mathbb{Q}}(T) - K, 0) = \mathbb{I}_{(S^{\mathbb{Q}}(T) \geq K)}(S^{\mathbb{Q}}(T) - K),$$

where $\mathbb{I}_{(A)}$ is the indicator function, which is 1 if event A occurs and 0 otherwise.

Using the risk-neutral valuation formula, the price of the option at time t is the discounted expected payoff under the risk-neutral measure:

$$\begin{aligned} V(t) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[V(T) \mid \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(S^{\mathbb{Q}}(T) \geq K)}(S^{\mathbb{Q}}(T) - K) \mid S^{\mathbb{Q}}(t)\right]. \end{aligned}$$

1.6 Deriving the Black-Scholes-Merton Formula

Substituting the expression for $S^{\mathbb{Q}}(T)$, we have:

$$V(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\left(S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y} \geq K\right)}\left(S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y} - K\right)\right].$$

We define:

$$\ln S^{\mathbb{Q}}(T) = \ln S(t) + \left(r - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}Y.$$

Then, the condition $S^{\mathbb{Q}}(T) \geq K$ is equivalent to:

$$\ln S^{\mathbb{Q}}(T) \geq \ln K,$$

which simplifies to:

$$\ln S(t) + \left(r - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}Y \geq \ln K.$$

Rearranging, we get:

$$Y \geq \frac{\ln\left(\dfrac{S(t)}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d^-.$$

Similarly, we define:

$$d^+ = d^- + \sigma\sqrt{T-t} = \frac{\ln\left(\dfrac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Now, we can express the option price as:

$$V(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(Y \geq d^-)}\left(S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y} - K\right)\right].$$

We split the expectation into two parts:

$$V(t) = e^{-r(T-t)}\left(S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(Y\geq d^-)}e^{\sigma\sqrt{T-t}Y}\right] - K\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(Y\geq d^-)}\right]\right).$$

Using the properties of the log-normal distribution and the standard normal cumulative distribution function $\Phi(\cdot)$, we have:

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(Y\geq d^-)}\right] = 1 - \Phi(d^-),$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(Y\geq d^-)}e^{\sigma\sqrt{T-t}Y}\right] = e^{\frac{1}{2}\sigma^2(T-t)}\left(1 - \Phi(d^+)\right).$$

Substituting back, we obtain:

$$\begin{aligned} V(t) &= e^{-r(T-t)}\left(S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)}e^{\frac{1}{2}\sigma^2(T-t)}\left(1-\Phi(d^+)\right) - K\left(1-\Phi(d^-)\right)\right) \\ &= S(t)\left(1-\Phi(d^+)\right) - Ke^{-r(T-t)}\left(1-\Phi(d^-)\right). \end{aligned}$$

Noting that $\Phi(-d) = 1 - \Phi(d)$, we have:

$$V(t) = S(t)\Phi(d^+) - Ke^{-r(T-t)}\Phi(d^-).$$

This is the Black-Scholes-Merton formula, providing a closed-form solution for the price of a European call option.

1.7 Alternative Derivation via PDE

Alternatively, we can derive the option price by solving the associated partial differential equation (PDE). The PDE represents the dynamics of the option price over time and is given by:

$$\frac{\partial f}{\partial t} + (rx)\frac{\partial f}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - rf = 0,$$

with the terminal condition:

$$f(T, x) = V(T) = \max(x - K, 0).$$

This is the famous Black-Scholes PDE. Solving this PDE with the given boundary condition yields the same option price as obtained from the Black-Scholes-Merton formula.

2 Pricing of an Asset Using a Compound Poisson Process

In this section, we extend the asset price model to include jumps, capturing sudden and significant movements in asset prices that cannot be explained by continuous Brownian motion alone. Such jumps are modeled using a compound Poisson process.

2.1 Model Setup

We define the probability space as before:

$$(\Omega, \mathbb{P}, \mathbb{F}).$$

2.2 Asset Price Dynamics with Jumps

The asset price $S(t)$ now follows a jump process, where the price can change discontinuously due to jumps at random times:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dM(t),$$

subject to

$$M(t) = C(t) - \lambda\gamma,$$

where:

- $C(t)$ is a compound Poisson process with intensity λ, representing the cumulative effect of jumps up to time t.
- $\gamma = \mathbb{E}[Y]$ is the expected jump size.
- σ scales the impact of the jump sizes on the asset price.

The compensated process $M(t)$ adjusts the compound Poisson process to ensure that it is a martingale under $\mathbb{P}$, meaning it has an expected change of zero, making it suitable for modeling in finance.

2.3 Transition to the Risk-Neutral Measure

To price derivatives under this model, we switch to the risk-neutral measure $\mathbb{Q}$, adjusting the drift to reflect risk-neutral probabilities:

$$dS^{\mathbb{Q}}(t) = (r - \lambda^{\mathbb{Q}}\gamma)S(t)dt + \sigma S(t)dC^{\mathbb{Q}}(t),$$

subject to

$$C^{\mathbb{Q}}(t) \text{ is a compound Poisson process with intensity } \lambda^{\mathbb{Q}}.$$

Under $\mathbb{Q}$, the drift term becomes $r - \lambda^{\mathbb{Q}}\gamma$, accounting for the expected loss due to jumps, ensuring that the expected return of the asset is the risk-free rate when adjusted for jumps.

2.4 Solving the Jump Process

By integrating the stochastic differential equation, we find the explicit solution for the asset price at time T:

$$S^{\mathbb{Q}}(T) = S(t)e^{(r-\lambda^{\mathbb{Q}}\gamma)(T-t)}\prod_{i=1}^{N}(1 + \sigma Y_i),$$

where:

- N is the number of jumps in the interval $(t, T]$, following a Poisson distribution with parameter $\lambda^{\mathbb{Q}}(T - t)$.
- Y_i are i.i.d. random variables representing the jump sizes.

Each jump at time t_i scales the asset price by a factor of $(1 + \sigma Y_i)$, reflecting the multiplicative effect of jumps on the asset price.

2.5 Pricing a European Call Option with Jumps

We aim to price a European call option with payoff at maturity:

$$V(T) = \max(S^{\mathbb{Q}}(T) - K, 0).$$

Using risk-neutral valuation, the option price at time t is:

$$V(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\max(S^{\mathbb{Q}}(T) - K, 0) \mid S^{\mathbb{Q}}(t)\right].$$

2.6 Computing the Expectation

To compute this expectation, we condition on the number of jumps N occurring between t and T:

$$V(t) = e^{-r(T-t)} \sum_{j=0}^{\infty} P(N=j)\mathbb{E}^{\mathbb{Q}} \left[\max(S_j - K, 0) \mid N = j, S^{\mathbb{Q}}(t) = x\right],$$

where:

- $P(N=j) = e^{-\lambda^{\mathbb{Q}}(T-t)} \dfrac{[\lambda^{\mathbb{Q}}(T-t)]^j}{j!}$ is the probability of j jumps.
- $S_j = xe^{(r-\lambda^{\mathbb{Q}}\gamma)(T-t)} \prod_{i=1}^{j}(1+\sigma Y_i)$ is the asset price at time T given j jumps.

By conditioning on N, we simplify the problem into computing expectations over scenarios with a fixed number of jumps. Without specifying the distribution of the jump sizes Y_i, we cannot compute this expectation explicitly.

2.7 Partial Integro-Differential Equation

Finally, we derive the partial integro-differential equation (PIDE) that the option price must satisfy. This PIDE incorporates the effect of jumps on the option price dynamics:

$$\frac{\partial f}{\partial t} + (rx)\frac{\partial f}{\partial x} - rf + \lambda^{\mathbb{Q}} \left[\int_{\mathbb{R}} f(t, x(1+\sigma y))\nu(dy) - f(t,x)\right] = 0,$$

with the terminal condition:

$$f(T,x) = V(T) = \max(x-K, 0),$$

where $\nu(dy)$ is the probability measure of the jump sizes Y under $\mathbb{Q}$.

The integral term represents the expected change in the option price due to jumps, integrating over all possible jump sizes and their impact on the asset price.

3 Pricing of an Asset Using Brownian Motion and a Compound Poisson Process

In this section, we consider a more comprehensive model that combines both continuous fluctuations and jumps, capturing a wider range of market behaviors. This jump-diffusion model reflects the reality that asset prices experience both small, continuous changes and sudden, significant movements.

3.1 Model Setup

We define the probability space as before:

$$(\Omega, \mathbb{P}, \mathbb{F}).$$

3.2 Asset Price Dynamics with Diffusion and Jumps

The asset price dynamics are modeled with both diffusion (Brownian motion) and jump components:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) + S(t)dM(t),$$

where:

- $W(t)$ is a standard Brownian motion, capturing continuous fluctuations.
- $M(t) = C(t) - \lambda\gamma$ is the compensated compound Poisson process, capturing jumps.

This model allows us to account for both types of market movements within a single framework.

3.3 Risk-Neutral Measure Adjustment

Under the risk-neutral measure $\mathbb{Q}$, the asset price dynamics adjust to reflect risk-neutral probabilities:

$$dS^{\mathbb{Q}}(t) = (r - \lambda^{\mathbb{Q}}\gamma)S(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t) + S(t)dC^{\mathbb{Q}}(t).$$

Here, the drift term $r - \lambda^{\mathbb{Q}}\gamma$ accounts for the risk-free rate and the expected loss due to jumps. The Brownian motion and jump components are adjusted to reflect the risk-neutral measure.

3.4 Solving the Jump-Diffusion SDE

Integrating the SDE, considering both continuous and jump parts, we obtain the asset price at time T:

$$S^{\mathbb{Q}}(T) = S^{\mathbb{Q}}(t)\exp\left(\left(r - \lambda^{\mathbb{Q}}\gamma - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)\right)\right)\prod_{i=1}^{N}(1+\sigma Y_i),$$

where:

- N is the number of jumps in $(t, T]$, following a Poisson distribution.
- Y_i are the jump sizes.

3.4.1 Derivation Steps

The solution is obtained by integrating the SDE in segments between jumps and accounting for the jumps multiplicatively.

Between jumps, the asset price evolves according to the SDE:

$$dS^{\mathbb{Q}}(t) = (r - \lambda^{\mathbb{Q}}\gamma - \tfrac{1}{2}\sigma^2)S^{\mathbb{Q}}(t)dt + \sigma S^{\mathbb{Q}}(t)dW^{\mathbb{Q}}(t).$$

Integrating this between jumps, we have:

$$S^{\mathbb{Q}}(t_{i+1}^-) = S^{\mathbb{Q}}(t_i^+)\exp\left(\left(r - \lambda^{\mathbb{Q}}\gamma - \tfrac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\left(W^{\mathbb{Q}}(t_{i+1}) - W^{\mathbb{Q}}(t_i)\right)\right).$$

At each jump time t_i, the asset price experiences a jump:

$$S^{\mathbb{Q}}(t_i^+) = S^{\mathbb{Q}}(t_i^-)(1 + \sigma Y_i).$$

By iterating this process over all jumps in $[t, T]$, we obtain the solution.

3.5 Pricing a European Call Option in the Jump-Diffusion Model

We aim to price a European call option with payoff:

$$V(T) = \max(S^{\mathbb{Q}}(T) - K, 0).$$

Using risk-neutral valuation, the option price at time t is:

$$V(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\max(S^{\mathbb{Q}}(T) - K, 0) \mid S^{\mathbb{Q}}(t)\right].$$

3.6 Computing the Expectation with Diffusion and Jumps

Conditioning on the number of jumps N, we have:

$$V(t) = e^{-r(T-t)}\sum_{j=0}^{\infty} P(N=j)\mathbb{E}^{\mathbb{Q}}\left[\max(S_j - K, 0) \mid N=j, S^{\mathbb{Q}}(t)=x\right],$$

where:

- $P(N=j) = e^{-\lambda^{\mathbb{Q}}(T-t)}\dfrac{[\lambda^{\mathbb{Q}}(T-t)]^j}{j!}$.
- $S_j = xe^{(r-\lambda^{\mathbb{Q}}\gamma-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y}\prod_{i=1}^{j}(1+\sigma Y_i)$.
- $Y \sim \mathcal{N}(0,1)$ represents the standard normal variable from the Brownian motion.

Again, without specifying the distribution of Y_i, we cannot compute the expectation explicitly.

3.7 Deriving the Partial Integro-Differential Equation

The option price $f(t,x)$ satisfies the PIDE:

$$\frac{\partial f}{\partial t} + (rx)\frac{\partial f}{\partial x} + \tfrac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2} - rf + \lambda^{\mathbb{Q}}\left[\int_{\mathbb{R}} f(t, x(1+\sigma y))\nu(dy) - f(t,x)\right] = 0,$$

with the terminal condition:

$$f(T, x) = V(T) = \max(x - K, 0).$$

This PIDE captures the combined effects of continuous price changes due to Brownian motion and discrete jumps due to the compound Poisson process, making it a powerful tool for option pricing in more realistic market conditions.

4 Pricing of an Asset Using Merton's Jump-Diffusion Model

In this section, we consider the special case where the jump sizes in the asset price model are normally distributed. This is known as Merton's Jump-Diffusion Model. By specifying the distribution of the jumps, we can derive a closed-form solution for option pricing, extending the Black-Scholes framework to accommodate jumps.

4.1 Model Setup

We define the probability space:

$$(\Omega, \mathbb{P}, \mathbb{F}),$$

and model the asset price dynamics under the risk-neutral measure $\mathbb{Q}$ as:

$$dS^{\mathbb{Q}}(t) = (r - \lambda\kappa)S^{\mathbb{Q}}(t)dt + \sigma S^{\mathbb{Q}}(t)dW^{\mathbb{Q}}(t) + S^{\mathbb{Q}}(t)dJ(t),$$

where:

- $W^{\mathbb{Q}}(t)$ is a standard Brownian motion under $\mathbb{Q}$.
- $J(t)$ is a compound Poisson process representing the cumulative jumps up to time t.
- The jump sizes Y_i are i.i.d. random variables such that $\ln Y_i \sim \mathcal{N}(m, \delta^2)$.
- λ is the intensity (mean arrival rate) of the Poisson process.
- $\kappa = \mathbb{E}^{\mathbb{Q}}[Y_i - 1] = e^{m+\frac{1}{2}\delta^2} - 1$ is the expected percentage change due to a jump.

4.2 Solving the Jump-Diffusion SDE with Normally Distributed Jumps

Integrating the stochastic differential equation, we obtain the asset price at time T:

$$S^{\mathbb{Q}}(T) = S^{\mathbb{Q}}(t) \exp\left(\left(r - \lambda\kappa - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)\right)\right) \prod_{i=1}^{N} Y_i,$$

where N is the number of jumps occurring in the interval $(t, T]$, which follows a Poisson distribution with parameter $\lambda(T-t)$.

Since the logarithm of each jump size is normally distributed, the cumulative effect of the jumps can be combined:

$$\sum_{i=1}^{N} \ln Y_i \sim \mathcal{N}\left(Nm, N\delta^2\right).$$

Therefore, conditional on $N = n$, the logarithm of the asset price at time T is normally distributed:

$$\ln S^{\mathbb{Q}}(T) \mid N = n \sim \mathcal{N}\left(\mu_n, \sigma_n^2\right),$$

where:

$$\mu_n = \ln S^{\mathbb{Q}}(t) + \left(r - \lambda\kappa - \tfrac{1}{2}\sigma^2\right)(T-t) + nm,$$

$$\sigma_n^2 = \sigma^2(T-t) + n\delta^2.$$

4.3 Pricing a European Call Option under Merton's Model

We aim to price a European call option with strike price K and maturity T. The option price at time t is given by:

$$V(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\max\left(S^{\mathbb{Q}}(T) - K, 0\right) \mid S^{\mathbb{Q}}(t)\right].$$

By conditioning on the number of jumps $N = n$, we have:

$$V(t) = e^{-r(T-t)} \sum_{n=0}^{\infty} P(N = n)\mathbb{E}^{\mathbb{Q}}\left[\max\left(S^{\mathbb{Q}}(T) - K, 0\right) \mid N = n, S^{\mathbb{Q}}(t)\right],$$

where:

$$P(N = n) = e^{-\lambda(T-t)} \frac{[\lambda(T-t)]^n}{n!}.$$

Since $\ln S^{\mathbb{Q}}(T) \mid N = n$ is normally distributed, we can use the Black-Scholes formula with adjusted parameters to compute the conditional expectation.

4.4 Adjusted Parameters and Closed-Form Solution

We define the adjusted volatility and drift:

$$\sigma_n = \sqrt{\sigma^2(T-t) + n\delta^2},$$

$$\mu_n = \ln S^{\mathbb{Q}}(t) + \left(r - \lambda\kappa - \tfrac{1}{2}\sigma^2\right)(T-t) + nm,$$

and

$$d_1^{(n)} = \frac{\mu_n - \ln K + \sigma_n^2}{\sigma_n},$$

$$d_2^{(n)} = d_1^{(n)} - \sigma_n.$$

The conditional expectation is then:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\max\left(S^{\mathbb{Q}}(T) - K, 0\right) \mid N = n, S^{\mathbb{Q}}(t)\right] &= e^{\mu_n + \frac{1}{2}\sigma_n^2} N\left(d_1^{(n)}\right) - KN\left(d_2^{(n)}\right) \\ &= S^{\mathbb{Q}}(t) e^{-\lambda\kappa(T-t)+nm} N\left(d_1^{(n)}\right) - KN\left(d_2^{(n)}\right).\end{aligned}$$

Substituting back into the option price formula, we obtain the closed-form solution for the option price under Merton's Jump-Diffusion Model:

$$V(t) = \sum_{n=0}^{\infty} e^{-\lambda(T-t)} \frac{[\lambda(T-t)]^n}{n!} \left[S^{\mathbb{Q}}(t) e^{nm-\lambda\kappa(T-t)} N\left(d_1^{(n)}\right) - Ke^{-r(T-t)} N\left(d_2^{(n)}\right)\right].$$

References

[1] Steven E Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. 1st. Vol. 11. Stochastic Modelling and Applied Probability. New York: Springer Science & Business Media, 2004.