Pricing Assets Using Jump-Diffusion Models

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Abstract

This working paper presents a detailed examination of asset pricing models that incorporate both continuous fluctuations and jumps. The jump-diffusion models are essential for accurately modeling asset prices, as they account for sudden, unpredictable market movements alongside regular change. We discuss three distinct cases: asset pricing with Brownian motion, asset pricing with Compound Poisson processes, and asset pricing combining both Brownian motion and Compound Poisson processes.

Pricing of an Asset Using Brownian Motion

We begin by defining the probability space $(\Omega, \mathbb{P}, \mathbb{F})$, where the price of the asset evolves over time. In this context, $\mathbb{P}$ represents the physical probability measure, and $\mathbb{F}$ denotes the filtration, which captures the flow of information over time.

$(\Omega, \mathbb{P}, \mathbb{F})$

The asset price, denoted $S(t)$, is assumed to follow an Itô process. The infinitesimal change in the asset price, $dS(t)$, is composed of two components: a drift term and a diffusion term. The drift term $\alpha S(t)dt$ reflects the deterministic growth of the asset, while the diffusion term $\sigma S(t)dW(t)$ represents the stochastic fluctuation in the price, driven by the Brownian motion $W(t)$.

$\forall t : S(t)$ such that

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

subject to $\forall t : W(t)$ is a Brownian motion.

To transition towards pricing in a risk-neutral framework, we utilize Girsanov's Theorem. This theorem enables a change of measure from the physical probability measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$. Under the new measure $\mathbb{Q}$, the asset price dynamics are modified, and the drift term becomes $r - \frac{1}{2}\sigma^2$, where r is the risk-free rate.

$\mathbb{P} \to \mathbb{Q}$ such that

$$dS^{\mathbb{Q}}(t) = (r - \frac{1}{2}\sigma^2)dt + \sigma S(t)dW^{\mathbb{Q}}(t)$$

subject to $\forall t : W^{\mathbb{Q}}(t)$ is a Brownian motion.

By applying Itô's Lemma to this process, we derive the explicit solution for the asset price under the risk-neutral measure. The solution for $S^{\mathbb{Q}}(T)$ at time T depends on the initial price $S(t)$, the time to maturity $T - t$, and the stochastic component represented by a standard normal random variable Y.

Thus, $\forall t \leq T$:

$$S^{\mathbb{Q}}(T) = S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y}$$

subject to

$$Y = \frac{dW^{\mathbb{Q}}(t)}{\sqrt{T-t}} \sim \Phi.$$

We now proceed to price an asset or derivative using the numeraire pricing approach. The value of the derivative at maturity, $V(T)$, is determined by the payoff function, which may include an indicator function, for instance, to determine whether the asset price at maturity S_T exceeds the strike price K.

Numeraire Pricing:

$$V(T) = \mathbb{I}_{(S_T \geq K)}(S_T - K)$$

such that $\forall t \leq T$:

$$\begin{aligned} &V(t) \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\mathbb{I}_{(S_T \geq K)} \mid \mathbb{F}(t)\right] \\ &= f(t, S_t) \end{aligned}$$

The function $f(t, x)$, representing the price of the option, can be expressed as an expectation under the risk-neutral measure, incorporating the evolution of the asset price over time.

such that

$$\begin{aligned} &f(t, x) \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(x \geq K)}\left(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y} - K\right) \mid \mathbb{F}(t)\right] \end{aligned}$$

subject to

$$xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y} \geq K$$

implying

$$Y \leq d^-(T-t, x).$$

By solving the integral for the expected payoff, we arrive at the Black-Scholes-Merton formula, which provides a closed-form solution for pricing European options. This formula expresses the option price in terms of cumulative normal distribution functions.

$$\begin{aligned}
&= e^{-r(T-t)} \int_{-\infty}^{d^-(T-t,x)} \left[xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y} - K \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= x \int_{-\infty}^{d^-(T-t,x)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\sigma\sqrt{T-t})^2} dy - Ke^{-r(T-t)}\Phi(d^-(T-t,x)) \\
&= x \int_{-\infty}^{d^+(T-t,x)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - Ke^{-r(T-t)}\Phi(d^-(T-t,x)) \\
&= x\Phi(d^+(T-t,x)) - Ke^{-r(T-t)}\Phi(d^-(T-t,x)).
\end{aligned}$$

The closed-form solution is given by:

$$V(t) = S_t\Phi(d^+(T-t, S_t)) - Ke^{-r(T-t)}\Phi(d^-(T-t, S_t)).$$

Finally, the option price can also be derived using the partial differential equation (PDE) approach. The PDE for the option price is obtained by applying stochastic calculus and Itô's Lemma to the asset price process under the risk-neutral measure.

PDE Pricing:

$$
\begin{aligned}
&e^{-rt}f(t,x)\\
&= \int_0^t e^{-ru}\Bigg[-rf(u,S(u)) + \frac{\partial f(u,S(u))}{\partial t} + \frac{\partial f(u,S(u))}{\partial x}S(u)\\
&= +\frac{1}{2}\frac{\partial^2 f(u,S(u))}{\partial x^2}\sigma^2 S(u)^2\Bigg]du + \int_0^t e^{-ru}\frac{\partial f(u,S(u))}{\partial x}S(u)dW^{\mathbb{Q}}(u).
\end{aligned}
$$

Let $du \to 0$, implying

$$
-rf(u,S(u)) + \frac{\partial f(u,S(u))}{\partial t} + \frac{\partial f(u,S(u))}{\partial x}S(u) + \frac{1}{2}\frac{\partial^2 f(u,S(u))}{\partial x^2}\sigma^2 S(u)^2 = 0,
$$

subject to $f(T,x) = V(T)$.

Pricing of an Asset Using a Compound Poisson Process

We begin by defining the probability space $(\Omega, \mathbb{P}, \mathbb{F})$, where the asset price evolves. In this framework, $\mathbb{P}$ represents the physical probability measure, and $\mathbb{F}$ is the filtration of information available over time.

$(\Omega, \mathbb{P}, \mathbb{F})$

Next, we introduce the jump process. The asset price $S(t)$ follows a process with jumps, where the infinitesimal change in the asset price, $dS(t)$, consists of two components: a drift term $\alpha S(t)dt$ and a jump term $\sigma S(t)dM(t)$, where $M(t)$ represents the jump process. The process $M(t)$ is defined as the difference between a compound Poisson process $C(t)$ and the expected number of jumps $\lambda\gamma$, with λ representing the intensity of the jumps.

$\forall t : S(t)$ such that

$$dS(t) = \alpha S(t)dt + \sigma S(t)dM(t)$$

subject to $\forall t : M(t) = C(t) - \lambda\gamma$, Where $\forall t : C^(t)$ compound Poisson with intensity λ.

To transition to pricing under the risk-neutral measure, we apply Girsanov's Theorem, which facilitates the change of measure from the physical probability measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$. Under the risk-neutral measure, the asset price $S^{\mathbb{Q}}(t)$ follows a jump-diffusion process, where the drift is adjusted to account for the risk-neutral jump intensity $\lambda^{\mathbb{Q}}$.

$\mathbb{P} \to \mathbb{Q}$ such that

$$dS^{\mathbb{Q}}(t) = (r - \sigma\lambda^{\mathbb{Q}}\gamma)S(t)dt + \sigma S(t)dC^{\mathbb{Q}}(t)$$

subject to $\forall t : C^{\mathbb{Q}}(t)$ compound Poisson with intensity $\lambda^{\mathbb{Q}}$.

By applying Itô's Lemma to the jump-diffusion process, we derive the explicit solution for the asset price under the risk-neutral measure. The asset price $S^{\mathbb{Q}}(T)$ at maturity T depends on the initial price $S(t)$ and incorporates the effect of the jumps over time, where $N(t)$ represents the number of jumps occurring between t and T.

Thus, $\forall t \leq T$:

$$S^{\mathbb{Q}}(T) = S(t)e^{(r-\lambda^{\mathbb{Q}}\gamma)(T-t)} \prod_{i=N(t)+1}^{N(T)} (1 + \sigma Y_i),$$

where Y_i represents the size of the i-th jump.

We now apply the numeraire pricing approach to value a derivative. The value of the derivative at maturity, denoted by $V(T)$, is determined by the payoff function, which depends on whether the asset price at maturity exceeds a strike price K.

Numeraire Pricing:

$$V(T) = \mathbb{I}_{(S_T \geq K)}(S_T - K)$$

such that $\forall t \leq T$:

$$\begin{aligned} &V(t) \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\mathbb{I}_{(S_T \geq K)} \mid \mathbb{F}(t)\right] \\ &= f(t, S_t) \end{aligned}$$

We now derive the step-by-step pricing formula for the derivative. The function $f(t, x)$ represents the expectation under the risk-neutral measure, accounting for the jumps in the asset price.

such that

$$
\begin{aligned}
& f(t,x) \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(x\geq K)}\left(xe^{(r-\lambda^{\mathbb{Q}}\gamma)(T-t)}\prod_{i=N(t)+1}^{N(T)}(1+\sigma Y_i)-K\right)\mid \mathbb{F}(t)\right] \\
&= e^{-r(T-t)}\sum_{j=0}^{\infty} h(j,x)e^{-\lambda^{\mathbb{Q}}(T-t)}\frac{(\lambda^{\mathbb{Q}}(T-t))^j}{j!},
\end{aligned}
$$

subject to

$$
h(j,x) = \mathbb{E}^{\mathbb{Q}}\left[xe^{(r-\lambda^{\mathbb{Q}}\gamma)(T-t)}\prod_{i=1}^{j}(1+\sigma Y_i)\right].
$$

Finally, we approach the pricing of the derivative via a partial differential equation (PDE). The PDE is derived by applying stochastic calculus to the jump-diffusion process, accounting for both the continuous price changes and the jumps.

PDE Pricing:

$$
\begin{aligned}
& e^{-rt}f(t,x) \\
&= \int_0^t e^{-ru}\left[-rf(u,S(u)) + \frac{\partial f(u,S(u))}{\partial t} + \frac{\partial f(u,S(u))}{\partial x}S(u)\right]du \\
&+ \sum_{0\leq u\leq t} e^{-ru}\left[f(u,S(u)) - f(u,S(u-))\right].
\end{aligned}
$$

We then refine this expression by focusing on the jump terms:

$$
\begin{aligned}
& \sum_{0\leq u\leq t} e^{-ru}\left[f(u,S(u)) - f(u,S(u-))\right] \\
&= \sum_{0\leq u\leq N(t)} e^{-ru}\left[f(u,S(u)(1+\sigma\delta C(u))) - f(u,S(u-))\right] \\
&= \sum_{m=1}^{M}\int_0^t e^{-ru}\left[f(u,S(u-)(1+\sigma y_m)) - f(u,S(u))\right]dN_m(u).
\end{aligned}
$$

Letting $du \to 0$, we derive the final PDE that describes the price of the derivative under the compound Poisson process, which captures both the drift and the jumps in the asset price.

Thus,

$$
\begin{aligned}
&e^{-rt} f(t,x) \\
&= \int_0^t e^{-ru} \Bigg[-rf(u,S(u)) + \frac{\partial f(u,S(u))}{\partial t} + \frac{\partial f(u,S(u))}{\partial x} S(u) \\
&+ \sum_{m=1}^{M} \left(f(u,S(u)(1+\sigma y_m)) - f(u,S(u))\lambda_m^{\mathbb{Q}} \right) \Bigg] du \\
&+ \int_0^t e^{-ru} \left[f(u,S(u)) - f(u,S(u-)) \right] dM(u).
\end{aligned}
$$

Let $du \to 0$, implying the PDE:

$$
\begin{aligned}
&- rf(u,S(u)) + \frac{\partial f(u,S(u))}{\partial t} + \frac{\partial f(u,S(u))}{\partial x} S(u) \\
&+ \sum_{m=1}^{M} \left(f(u,S(u)(1+\sigma y_m)) - f(u,S(u))\lambda_m^{\mathbb{Q}} \right) = 0,
\end{aligned}
$$

subject to the boundary condition $f(T,x) = V(T)$.

Pricing of an Asset Using Brownian Motion and a Compound Poisson Process

We begin by considering an asset price model that incorporates both continuous fluctuations, represented by Brownian motion, and discontinuous jumps, modeled by a compound Poisson process. The probability space $(\Omega, \mathbb{P}, \mathbb{F})$ defines the framework in which asset prices evolve. Here, $\mathbb{P}$ is the physical probability measure, and $\mathbb{F}$ represents the filtration, capturing the information available over time.

$(\Omega, \mathbb{P}, \mathbb{F})$

The asset price $S(t)$ follows a jump-diffusion process, where the infinitesimal changes in the asset price $dS(t)$ are influenced by both a continuous Brownian motion term and a jump process. Specifically, $dS(t)$ consists of a drift term $\alpha S(t)dt$, a diffusion term $\sigma S(t)dW(t)$ driven by Brownian motion $W(t)$, and a jump term $S(t)dM(t)$ representing the jumps.

$\forall t : S(t)$ such that

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) + S(t)dM(t)$$

Next, we move to the risk-neutral pricing measure $\mathbb{Q}$ using Girsanov's Theorem. Under the risk-neutral measure, the dynamics of the asset price change to reflect the risk-neutral jump intensity $\lambda^{\mathbb{Q}}$. The drift term is now adjusted to reflect the risk-free rate r, accounting for the expected jump magnitude γ under $\mathbb{Q}$.

$\mathbb{P} \to \mathbb{Q}$ such that

$$dS^{\mathbb{Q}}(t) = (r - \lambda^{\mathbb{Q}}\gamma)S(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t) + S(t)dC(t)$$

By applying Itô's Lemma to the jump-diffusion process, we derive the explicit solution for the asset price under the risk-neutral measure. The asset price $S^{\mathbb{Q}}(T)$ at time T includes contributions from both the continuous Brownian motion and the discrete jumps occurring at random times.

Thus, $\forall t \leq T$:

$$S^{\mathbb{Q}}(T) = S(t)e^{(r-\lambda^{\mathbb{Q}}\gamma-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y} \prod_{i=N(t)+1}^{N(T)} (1+\sigma Y_i)$$

We now apply the numeraire pricing approach to value a derivative security. The value of the derivative at maturity, denoted $V(T)$, depends on whether the asset price at maturity S_T exceeds the strike price K.

Numeraire Pricing:

$$V(T) = \mathbb{I}_{(S_T \geq K)}(S_T - K)$$

such that $\forall t \leq T$:

$$\begin{aligned} &V(t) \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\mathbb{I}_{(S_T \geq K)} \mid \mathbb{F}(t)\right] \\ &= f(t, S_t) \end{aligned}$$

To compute the price of the option step-by-step, we express the price function $f(t, x)$ as the expectation under the risk-neutral measure, considering both the continuous diffusion and the jump components in the asset price dynamics.

such that

$$\begin{aligned} &f(t, x) \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(x \geq K)}\left(xe^{(r-\lambda^{\mathbb{Q}}\gamma-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Y} \prod_{i=N(t)+1}^{N(T)} (1+\sigma Y_i) - K\right) \mid \mathbb{F}(t)\right] \\ &= e^{-r(T-t)}\sum_{j=0}^{\infty}\mathbb{E}^{\mathbb{Q}}[h(j, x)]e^{-\lambda^{\mathbb{Q}}(T-t)}\frac{(\lambda^{\mathbb{Q}}(T-t))^j}{j!} \end{aligned}$$

subject to

$$h(j,x) = \mathbb{E}^{\mathbb{Q}}\left[k(T-t, xe^{-\gamma(T-t)} \prod_{i=N(t)+1}^{N(T)} (1+\sigma Y_i)\right]$$

such that

$$k(t,x) = e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{(x \geq K)} x e^{r-\frac{1}{2}\sigma^2 t+\sigma\sqrt{t}Y} - K\right]$$

The option pricing problem can also be formulated as a partial differential equation (PDE), which captures both the continuous and jump components of the asset price dynamics. The PDE describes the evolution of the option price $f(t,x)$ over time, incorporating the risk-free rate, the diffusion term, and the jump terms.

PDE Pricing:

$$\begin{aligned}
&e^{-rt} f(t,x) \\
&= \int_0^t e^{-ru}\left[-rf(u,S(u)) + \frac{\partial f(u,S(u))}{\partial t} + \frac{\partial f(u,S(u))}{\partial x}(r-\gamma)S(u)\right. \\
&\left.+ \frac{1}{2}\frac{\partial^2 f(u,S(u))}{\partial x^2}\sigma^2 S^2(u)\right] du + \int_0^t e^{-ru}\frac{\partial f(u,S(u))}{\partial x} S(u) dW^{\mathbb{Q}} \\
&+ \sum_{0 \leq u \leq t} e^{-ru}\left[f(u,S(u)) - f(u,S(u-))\right]
\end{aligned}$$

The PDE formulation allows us to handle the complexity of both diffusion and jumps. By letting $du \to 0$, we derive the final PDE for the option price, incorporating both the continuous fluctuations in the asset price and the discontinuous jumps.

Thus,

$$\int_0^t e^{-ru}\Big[-rf(u,S(u)) + \frac{\partial f(u,S(u))}{\partial t} + \frac{\partial f(u,S(u))}{\partial x}(r-\gamma)S(u) + \frac{1}{2}\frac{\partial^2 f(u,S(u))}{\partial x^2}\sigma^2 S^2(u)$$
$$+ \sum_{m=1}^{M}\big(f(u,S(u)(1+\sigma y_m)) - f(u,S(u))\lambda_m^{\mathbb{Q}}\big)\Big]du + \int_0^t e^{-ru}\frac{\partial f(u,S(u))}{\partial x}S(u)dW^{\mathbb{Q}}$$
$$+ \int_0^t e^{-ru}\left(f(u,S(u)) - f(u,S(u-))\right)dM(u)$$

Letting $du \to 0$, the PDE simplifies to

$$- rf(u,S(u)) + \frac{\partial f(u,S(u))}{\partial t} + \frac{\partial f(u,S(u))}{\partial x}(r-\gamma)S(u) + \frac{1}{2}\frac{\partial^2 f(u,S(u))}{\partial x^2}\sigma^2 S^2(u)$$
$$+ \sum_{m=1}^{M}(f(u,S(u)(1+\sigma y_m)) - f(u,S(u))\lambda_m^{\mathbb{Q}} = 0$$

subject to the boundary condition $f(T,x) = V(T)$.

References

[1] Steven E Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. 1st. Vol. 11. Stochastic Modelling and Applied Probability. New York: Springer Science & Business Media, 2004.