Stochastic Portfolio Theory

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1 Introduction

Stochastic portfolio theory provides a mathematical framework for analyzing and constructing portfolios in financial markets. Unlike traditional portfolio theories, which often rely on static assumptions and deterministic optimization, stochastic portfolio theory embraces the inherent uncertainty of financial markets by modeling asset prices and portfolio behavior as stochastic processes.

This approach is particularly powerful in understanding long-term growth and relative performance of portfolios, as it accounts for dynamic interactions between asset prices and portfolio weights. Applications of stochastic portfolio theory extend beyond academic research, offering practical tools for portfolio construction, risk management, and performance benchmarking in real-world investment scenarios.

In this paper, we review the foundational aspects of this theory, including the modeling of asset price dynamics, portfolio construction, and optimization. We aim to make these concepts accessible to a broad audience by providing detailed explanations of the mathematical framework and exploring practical implications for modern finance.

2 Mathematical Framework

2.1 Probability Space

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, Ω represents the sample space. The set $\mathcal{F}$ is a σ-algebra of events, which is a collection of subsets of Ω. The probability measure $\mathbb{P}$ assigns probabilities to the events in $\mathcal{F}$, allowing us to quantify uncertainty.

2.2 Brownian Motion and Filtration

To model the uncertainty inherent in financial markets, we introduce a vector of independent Brownian Motions, denoted by $\{W_m(t)\}_{m=1}^{M}$. Each $W_m(t)$ is a continuous stochastic process that models the random fluctuations in the market. Which satisfy the following properties:

1. At time $t = 0 : W_m(0) = 0$.
2. For any $0 \leq s \leq t$, the increments $W_m(t) - W_m(s) \sim \Phi(0, t - s)$.

The collection of all information available up to time t is represented by the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, where each $\mathcal{F}_t$ is a σ-algebra generated by the Brownian Motions. This filtration models the flow of information in the market.

Proposition 2.1. *Each $W_m(t)$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}$, satisfying:*

1. *$W_m(t)$ is adapted to the filtration $\mathcal{F}_t$, so its value at time t depends only on information available up to time t.*
2. *The expected absolute value is finite: $\mathbb{E}^{\mathbb{P}}[|W_m(t)|] < \infty$.*
3. *For any $0 \leq s \leq t$, the conditional expectation satisfies:*

$$\mathbb{E}^{\mathbb{P}}[W_m(t)|\mathcal{F}_s] = W_m(s).$$

This proposition establishes that the future expected value of $W_m(t)$, given all past information up to time s, is simply its current value at time s.

2.3 Asset Price Dynamics

Consider a financial market consisting of n assets, each with a price process denoted by $\{X_i(t)\}_{t\geq 0}$ for $i = 1, 2, \ldots, n$. We model the logarithm price using a stochastic differential equation (SDE):

$$d \ln X_i(t) = \gamma_i(t)\, dt + \sum_{m=1}^{M} \epsilon_{i,m}(t)\, dW_m(t). \tag{1}$$

Such that

- $\gamma_i(t)$ represents the drift term, which is the expected instantaneous growth rate of asset i at time t.
- $\epsilon_{i,m}(t)$ represents the sensitivity of asset i to the m-th Brownian Motion at time t.

Assume

1. The drift term $\gamma_i(t)$ is adapted to the filtration $\mathcal{F}_t$ and satisfies the integrability condition:

   $$\int_0^t |\gamma_i(s)|\, ds < \infty.$$

   This condition ensures that the cumulative drift over any finite interval is finite.

2. The volatility coefficients $\epsilon_{i,m}(t)$ are adapted to $\mathcal{F}_t$ and satisfy:

   $$\int_0^t \|\epsilon_i(s)\|\, ds < \infty,$$

   This condition ensures that the cumulative volatility is finite over any finite interval.

These assumptions guarantee that the stochastic integrals involved in the model are well-defined in the sense that the asset prices do not exhibit explosive behavior.

Proposition 2.2. *Under the assumptions, the logarithm price, $\ln X_i(t)$, is a continuous semi-martingale. Due that the stochastic integral $L_i(t)$ is a local-martingale with respect to the filtration $\{\mathcal{F}_t\}$.*

$$L_i(t) = \sum_{m=1}^{M} \int_0^t \epsilon_{i,m}(s)\, dW_m(s) \tag{2}$$

There exists a sequence of stopping times $\{\tau_k\}_{k=1}^{\infty}$ such that:

1. *Each stopping time satisfies $\tau_k \leq \tau_{k+1}$ almost surely, ensuring that the sequence is non-decreasing.*
2. *The sequence converges to infinity almost surely: $\lim_{k\to\infty} \tau_k = \infty$.*
3. *For each k, the stopped process $L_i^{\tau_k}(t) = L_i(t \wedge \tau_k)$ is a martingale.*

This proposition assures us that despite $L_i(t)$ not being a martingale over the entire time horizon, it behaves like a martingale up to certain stopping times. This property is essential for applying martingale techniques in the analysis of asset prices.

2.4 Itô's Lemma and Asset Process

To analyze the behavior of functions of stochastic processes, we use Itô's lemma. Let $f(t,x) \in C^{1,2}$ be a function that is continuously differentiable in t and twice continuously differentiable in x. Then, applying Itô's lemma to $f(t, X_i(t))$, we obtain:

$$df(t, X_i(t)) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma_i^2(t) \right) dt + \frac{\partial f}{\partial x}\, dX_i(t), \tag{3}$$

where $\sigma_i^2(t)$ is the variance of asset i, given by:

$$\sigma_i^2(t) = \sum_{m=1}^{M} \epsilon_{i,m}^2(t).$$

We can derive the SDE for the asset price $X_i(t)$ itself. Since $\ln X_i(t)$ follows an SDE, we can express $dX_i(t)$ as:

$$dX_i(t) = X_i(t) \left(\alpha_i(t)\, dt + \sum_{m=1}^{M} \epsilon_{i,m}(t)\, dW_m(t) \right), \tag{4}$$

where the adjusted drift $\alpha_i(t)$ is defined as:

$$\alpha_i(t) = \gamma_i(t) + \frac{1}{2}\sigma_i^2(t). \tag{5}$$

The term $\alpha_i(t)$ represents the expected return of asset i, incorporating both the drift and the volatility adjustment.

2.5 Covariance Structure

Understanding the relationships between different assets is crucial for portfolio construction. We define the covariance between assets i and j as:

$$\sigma_{i,j}(t) = \sum_{m=1}^{M} \epsilon_{i,m}(t)\epsilon_{j,m}(t). \tag{6}$$

This expression captures how the returns of assets i and j co-move due to their sensitivities to the underlying Brownian Motions. We assume that the covariances are integrable over any finite time interval:

$$\int_0^t |\sigma_{i,j}(s)|\, ds < \infty. \tag{7}$$

This condition ensures that the cumulative covariance between any pair of assets remains finite.

2.6 Portfolio Construction

Consider a market $\mathcal{M} = \{X_i(t)\}_{i=1}^n$ consisting of n assets. An investor allocates wealth among these assets according to portfolio weights $\pi(t) = \{\pi_i(t)\}_{i=1}^n$, where each $\pi_i(t)$ represents the proportion of wealth invested in asset i at time t. We impose the following conditions on the portfolio weights:

1. Each weight $\pi_i(t)$ is adapted to the filtration $\mathcal{F}_t$, meaning it depends only on information available up to time t.

2. The weights sum to one:

   $$\sum_{i=1}^{n} \pi_i(t) = 1.$$

   This condition ensures that the entire wealth is invested in the assets.

The total value of the portfolio at time t is given by:

$$Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) X_i(t). \tag{8}$$

To understand how the portfolio evolves over time, we derive the SDE for its logarithmic return.

$$d \ln Z_\pi(t) = \gamma_\pi(t)\, dt + \sum_{m=1}^{M} \left(\sum_{i=1}^{n} \pi_i(t) \epsilon_{i,m}(t) \right) dW_m(t), \tag{9}$$

where the portfolio drift $\gamma_\pi(t)$ is given by:

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t) \gamma_i(t) + \gamma_\pi^*(t). \tag{10}$$

The term $\gamma_\pi^*(t)$ represents the portfolio's *excess growth rate*, capturing the additional expected return due to diversification and is calculated as:

$$2\gamma_\pi^*(t) = \sum_{i=1}^{n} \pi_i(t) \sigma_i^2(t) - \sum_{i,j=1}^{n} \pi_i(t) \pi_j(t) \sigma_{i,j}(t). \tag{11}$$

This excess growth rate arises because, in a diversified portfolio, the variance of the portfolio is less than the weighted sum of individual variances due to the covariances between assets. Consequently, the portfolio may achieve a higher growth rate than the weighted average of individual asset growth rates. By rearranging terms, we can express the portfolio's logarithmic return as:

$$d \ln Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) d \ln X_i(t) + \gamma_\pi^*(t)\, dt. \tag{12}$$

This equation shows that the portfolio's return is the weighted sum of individual asset returns plus the excess growth rate.

2.7 Relative Portfolio Performance

Investors often assess portfolio performance relative to a benchmark. Let $\mu(t) = \{\mu_i(t)\}_{i=1}^n$ denote the weights of a benchmark portfolio, which generates the value process $Z_\mu(t)$. The relative performance of asset i with respect to the benchmark is given by the logarithm of the ratio of their values:

$$\ln\left(\frac{X_i(t)}{Z_\mu(t)}\right). \tag{13}$$

The dynamics of the portfolio's performance relative to the benchmark are:

$$d\ln\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \sum_{i=1}^{n} \pi_i(t)d\ln\left(\frac{X_i(t)}{Z_\mu(t)}\right) + \gamma_\pi^*(t)\,dt. \tag{14}$$

This equation shows that the relative performance depends on the weighted sum of individual assets' relative performances and the portfolio's excess growth rate.

To analyze the variability of the portfolio's relative performance, we define the *relative covariance matrix* $\tau^\mu(t) = [\tau^\mu_{i,j}(t)]$, where:

$$\tau^\mu_{i,j}(t) = \sigma_{i,j}(t) - \sigma_{i,\mu}(t) - \sigma_{j,\mu}(t) + \sigma_{\mu,\mu}(t). \tag{15}$$

Here, $\sigma_{i,\mu}(t)$ represents the covariance between asset i and the benchmark portfolio, calculated as:

$$\sigma_{i,\mu}(t) = \sum_{k=1}^{n} \mu_k(t)\sigma_{i,k}(t), \tag{16}$$

and $\sigma_{\mu,\mu}(t)$ is the variance of the benchmark portfolio:

$$\sigma_{\mu,\mu}(t) = \sum_{i,j=1}^{n} \mu_i(t)\mu_j(t)\sigma_{i,j}(t). \tag{17}$$

The relative covariance matrix $\tau^\mu(t)$ measures the variance of the assets relative to the benchmark. The covariance matrix is positive semi-definite implying that the variance of the relative returns is non-negative.

3 Portfolio Optimization

In this section, we explore strategies for constructing portfolios that optimize certain criteria, such as minimizing variance.

3.1 Classic Minimum Variance Portfolio

One traditional approach to portfolio optimization is to minimize the portfolio's variance while achieving a desired expected return. The optimization problem is formulated as:

$$\begin{aligned} \min_{\pi(t)} \quad & \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{i,j}(t) \\ \text{subject to} \quad & \gamma_0 \leq \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t), \\ & \sum_{i=1}^{n} \pi_i(t) = 1. \end{aligned} \tag{18}$$

In this formulation:

- The objective is to minimize the portfolio variance, which is a function of the portfolio weights and the covariances between assets.
- The constraint ensures that the portfolio's expected return is at least γ_0, a target minimum return specified by the investor.
- The excess growth rate $\gamma_\pi^*(t)$ depends on the portfolio weights in a non-linear way.

This problem is a non-quadratic programming problem due to the non-linear constraint involving $\gamma_\pi^*(t)$ as $\gamma_\pi^*(t) = \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\sigma_{i,i}(t) - \frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{i,j}(t)$

3.2 Risk-Controlled Portfolio

Alternatively, an investor may wish to minimize the variance of the portfolio's returns relative to a benchmark portfolio $\mu(t)$. This approach focuses on controlling the tracking error, which is the deviation of the portfolio's performance from that of the benchmark. The optimization problem is:

$$\begin{aligned}
\min_{\pi(t)} \quad & \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\tau_{i,j}^{\mu}(t) \\
\text{subject to} \quad & \gamma_0 \leq \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t), \\
& \sum_{i=1}^{n} \pi_i(t) = 1.
\end{aligned} \tag{19}$$

In this problem:

- The objective is to minimize the relative variance, measured by the relative covariance matrix $\tau^{\mu}(t)$.
- The constraint ensures that the portfolio's expected return is at least γ_0, a target minimum return specified by the investor.

Assuming that the tracking error is small, we can simplification the optimization problem into a quadratic programming problem, which is more tractable computationally. As $\gamma_\pi^*(t) \approx \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\sigma_{i,i}(t)$.

4 Generating Portfolio Functions

A essential concept in stochastic portfolio theory is the use of generating functions to construct portfolios.

4.1 Generating Functions

A positive, continuous function $\boldsymbol{S}(\mu(t))$ is said to generate the portfolio weights $\pi(t)$ if the following relationship holds:

$$\ln\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \ln\left(\boldsymbol{S}(\mu(t))\right) + \Theta(t), \tag{20}$$

where $\Theta(t)$ is a finite variation process, meaning it has paths of bounded variation over finite intervals. The expression states that the logarithm portfolio's value relative to the benchmark can be expressed in terms of the generating function and an adjustment term. Assuming that $\boldsymbol{S} \in C^{1,2}$ (i.e., it is continuously differentiable in μ_i and twice continuously differentiable in μ_i), we can apply Itô's lemma to $\ln(\boldsymbol{S}(\mu(t)))$. The dynamics are given by:

$$\begin{aligned} d\ln(\boldsymbol{S}(\mu(t))) &= \sum_{i=1}^{n} \frac{\partial \ln(\boldsymbol{S})}{\partial \mu_i}\, d\mu_i(t) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \ln(\boldsymbol{S})}{\partial \mu_i \partial \mu_j}\, d\mu_i(t) d\mu_j(t). \end{aligned} \tag{21}$$

Since the benchmark weights $\mu_i(t)$ are proportions of the benchmark portfolio, their dynamics are influenced by the asset returns and the benchmark return. Similarly, the dynamics of the portfolio's performance relative to the benchmark are:

$$d\ln\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \sum_{i=1}^{n} \pi_i(t) d\ln\left(\frac{X_i(t)}{Z_\mu(t)}\right) + \gamma_\pi^*(t)\, dt. \tag{22}$$

To ensure that the portfolio generated by $\boldsymbol{S}$ behaves as desired, we define the portfolio weights as:

$$\pi_i(t) = \frac{\partial \ln(\boldsymbol{S}(\mu(t)))}{\partial \mu_i} + \lambda(t)\mu_i(t), \tag{23}$$

where $\lambda(t)$ is a Lagrange multiplier chosen to enforce the budget constraint $\sum_{i=1}^{n} \pi_i(t) = 1$.

Substituting the definition of $\pi_i(t)$ into the budget constraint, we obtain:

$$\sum_{i=1}^{n} \pi_i(t) = \sum_{i=1}^{n} \left(\frac{\partial \ln(\boldsymbol{S}(\mu(t)))}{\partial \mu_i} + \lambda(t)\mu_i(t) \right) = 1. \tag{24}$$

Expanding the summation:

$$\sum_{i=1}^{n} \frac{\partial \ln(\boldsymbol{S}(\mu(t)))}{\partial \mu_i} + \lambda(t) \sum_{i=1}^{n} \mu_i(t) = 1. \tag{25}$$

Since $\sum_{i=1}^{n} \mu_i(t) = 1$ (by the definition of $\mu(t)$), the equation simplifies to:

$$\sum_{i=1}^{n} \frac{\partial \ln(\boldsymbol{S}(\mu(t)))}{\partial \mu_i} + \lambda(t) = 1. \tag{26}$$

Rearranging to solve for $\lambda(t)$, we find:

$$\lambda(t) = 1 - \sum_{i=1}^{n} \frac{\partial \ln(\boldsymbol{S}(\mu(t)))}{\partial \mu_i}. \tag{27}$$

This ensures that the portfolio weights satisfy the budget constraint:

$$\sum_{i=1}^{n} \pi_i(t) = 1. \tag{28}$$

Thus, the final expression for $\pi_i(t)$ is:

$$\pi_i(t) = \frac{\partial \ln(\boldsymbol{S}(\mu(t)))}{\partial \mu_i} + \mu_i(t) \left(1 - \sum_{j=1}^{n} \frac{\partial \ln(\boldsymbol{S}(\mu(t)))}{\partial \mu_j} \right). \tag{29}$$

By substituting the defined weights into the dynamics of the portfolio's relative performance, we can show that:

$$d \ln \left(\frac{Z_\pi(t)}{Z_\mu(t)} \right) = d \ln(\boldsymbol{S}(\mu(t))) - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \ln(\boldsymbol{S})}{\partial \mu_i \partial \mu_j} \mu_i(t) \mu_j(t) \tau^{\mu}_{i,j}(t)\, dt. \tag{30}$$

The difference between the two dynamics is captured by the finite variation process $\Theta(t)$. This relationship establishes that the portfolio generated by $\boldsymbol{S}$ behaves similarly to the generating function, adjusted for the curvature of $\boldsymbol{S}$ and the covariance structure of the assets.

4.2 Example: Entropy Function

Consider the generating function defined by the negative entropy of the benchmark weights:

$$\boldsymbol{S}(\mu(t)) = - \sum_{i=1}^{n} \mu_i(t) \ln(\mu_i(t)). \tag{31}$$

This function is positive and continuously differentiable, satisfying the requirements for a generating function. The first and second derivatives of $\boldsymbol{S}$ are:

$$\frac{\partial \boldsymbol{S}}{\partial \mu_i} = - \ln(\mu_i(t)) - 1, \quad \frac{\partial^2 \boldsymbol{S}}{\partial \mu_i \partial \mu_j} = - \frac{\delta_{ij}}{\mu_i(t)}, \tag{32}$$

where δ_{ij} is the Kronecker delta (1 if $i = j$, 0 otherwise).

Using the definition of the portfolio weights, we have:

$$\pi_i(t) = \frac{-\ln(\mu_i(t)) - 1}{\sum_{j=1}^{n} \mu_j(t) \left(-\ln(\mu_j(t)) - 1\right)} \mu_i(t). \tag{33}$$

This weighting scheme allocates more weight to assets with smaller benchmark weights, promoting diversification.

The dynamics of the portfolio's performance relative to the benchmark become:

$$d \ln \left(\frac{Z_\pi(t)}{Z_\mu(t)} \right) = d \ln(\boldsymbol{S}(\mu(t))) + \frac{1}{2} \sum_{i=1}^{n} \mu_i(t) \tau_{i,i}^{\mu}(t) \, dt. \tag{34}$$

This expression shows that the portfolio may outperform the benchmark due to the positive contribution of the covariance terms.

4.3 Performance Analysis

We can analyze the portfolio's potential for outperformance by examining the finite variation process $\Theta(t)$.

Proposition 4.1. *Assuming that the portfolio weights satisfy $\pi_i(t) \geq 0$ for all i (i.e., no short selling), we have:*

$$d\Theta(t) = -\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \ln(\boldsymbol{S})}{\partial \mu_i \partial \mu_j} \mu_i(t) \mu_j(t) \tau_{i,j}^{\mu}(t) \, dt. \tag{35}$$

If the Hessian matrix H is negative semi-definite and the relative covariance matrix $\tau^\mu(t)$ is positive semi-definite, then $d\Theta(t) \geq 0$. This condition implies that the portfolio may have an expected outperformance relative to the benchmark.

4.3.1 Eigenvalue Decomposition

To further understand the behavior of $d\Theta(t)$, we can perform an eigenvalue decomposition of the Hessian matrix:

$$H = Q \Lambda Q^\top, \tag{36}$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ contains the eigenvalues, and Q is the matrix of corresponding orthonormal eigenvectors. We can express $d\Theta(t)$ in terms of the eigenvalues and eigenvectors:

$$d\Theta(t) = -\frac{1}{2}\sum_{k=1}^{n} \lambda_k \left(\sum_{i=1}^{n} q_{ik}\mu_i(t) \right)^2 \tau_k(t)\, dt, \tag{37}$$

where $\tau_k(t)$ represents the projection of the covariance structure onto the eigenvector directions. If any eigenvalue λ_k is negative, and since $\tau_k(t) \geq 0$, the corresponding term in $d\Theta(t)$ is positive, contributing to potential outperformance.

4.3.2 Example: Entropy Function Revisited

For the entropy function, the Hessian matrix is:

$$\frac{\partial^2 \ln(\boldsymbol{S})}{\partial \mu_i \partial \mu_j} = -\frac{\delta_{ij}}{\mu_i(t)}. \tag{38}$$

This matrix is diagonal with negative entries, implying that all eigenvalues are negative and equal to $-\frac{1}{\mu_i(t)}$. Since the relative covariance matrix $\tau^{\mu}(t)$ is positive semi-definite, we have $d\Theta(t) \geq 0$. This result suggests that the portfolio generated by the entropy function may consistently outperform the benchmark over time.

5 Admissible

In this section, we introduce key definitions and conditions that underpin the regulatory aspects of portfolio construction within the stochastic portfolio theory framework. These conditions ensure the robustness, diversity, and admissibility of portfolios while addressing arbitrage opportunities.

5.1 Non-Degeneracy

A portfolio $\pi(t)$ is considered **non-degenerate** if the diagonal elements of its covariance matrix $\tau^{\pi}_{i,i}(t)$ satisfy the following inequality:

$$\tau^{\pi}_{i,i}(t) \geq \epsilon\big(1 - \max_{1\leq i\leq n} \pi_i(t)\big)^2,$$

where $\epsilon > 0$ is a constant. This condition ensures that no single asset dominates the portfolio in terms of weight, thereby promoting stability and diversification. Furthermore, non-degeneracy implies:

$$2\gamma_\pi^*(t) = \sum_{i=1}^{n} \pi_i(t)\tau_{i,i}^{\pi}(t) \geq \epsilon\big(1 - \max_{1\leq i\leq n} \pi_i(t)\big)^2.$$

This property is essential for achieving a positive excess growth rate $\gamma_\pi^*(t)$, which is indicative of the portfolio's capacity to outperform.

5.2 Diversity

A benchmark portfolio $\mu(t)$ exhibits **diversity** if no single asset comprises more than a specified fraction of the portfolio, as expressed by:

$$\max_{1\leq i\leq n} \mu_i(t) \leq 1 - \phi,$$

where $\phi > 0$ is a constant representing the minimum level of diversification. This condition enforces a balanced allocation across assets, mitigating the risk associated with concentrated portfolios.

5.3 Admissibility

A portfolio $\pi(t)$ is deemed **admissible** if it satisfies the following criteria:

1. **Non-negativity:**
$$\pi_i(t) \geq 0 \quad \forall i,\ \forall t.$$

2. **Relative Performance:**
$$\exists\, c > 0 : \frac{Z_\pi(t)}{Z_\pi(0)} > c\frac{Z_\mu(t)}{Z_\mu(0)} \quad \text{for all } t.$$

3. **Weight Bound:**
$$\exists\, d > 0 : \frac{\pi_i(t)}{\mu_i(t)} \leq d \quad \forall i,\ \forall t.$$

These conditions ensure that admissible portfolios remain well-defined, avoid short selling, and consistently outperform a benchmark over time.

5.4 Dominant

A portfolio $\pi(t)$ is defined as **dominant** if its value consistently surpasses the benchmark portfolio's value:

$$\frac{Z_\pi(t)}{Z_\pi(0)} \geq \frac{Z_\mu(t)}{Z_\mu(0)} \quad \text{for all } t.$$

Dominant portfolios guarantee superior performance, making them highly desirable under regulatory and practical considerations.

5.5 Arbitrage Opportunities

An **arbitrage opportunity** exists if a portfolio $\pi(t)$ satisfies the following:

1. **Probability of Outperformance:**

$$\mathcal{P}\left(\frac{Z_\pi(t)}{Z_\pi(0)} \geq \frac{Z_\mu(t)}{Z_\mu(0)}\right) = 1.$$

2. **Positive Likelihood of Strict Outperformance:**

$$\mathcal{P}\left(\frac{Z_\pi(t)}{Z_\pi(0)} > \frac{Z_\mu(t)}{Z_\mu(0)}\right) > 0.$$

5.5.1 Example: Arbitrage Opportunity

Such opportunities arise when specific portfolio constructions exploit inefficiencies in the market. For example, a generating function $\boldsymbol{S}(\mu(t))$ defined as:

$$\boldsymbol{S}(\mu(t)) = 1 - \frac{1}{2}\sum_{i=1}^{n} \mu_i^2(t),$$

can yield a portfolio $\pi(t)$ with weights:

$$\pi_i(t) = \left(\frac{2 - \mu_i(t)}{\boldsymbol{S}(\mu(t))} - 1\right)\mu_i(t).$$

This portfolio outperforms as long as $\boldsymbol{S}(\mu(t))$ satisfies:

$$0 \leq \boldsymbol{S}(\mu(t)) \leq 1,$$

and contributes to the finite variation process:

$$d\Theta(t) = \frac{1}{2\boldsymbol{S}(\mu(t))} \sum_{i=1}^{n} \mu_i^2(t)\tau_{i,i}(t).$$

Under non-degeneracy, we have:

$$d\Theta(t) = \frac{1}{2\boldsymbol{S}(\mu(t))} \sum_{i=1}^{n} \mu_i^2\tau_{i,i}(t) \geq \frac{1}{2} \sum_{i=1}^{n} \mu_i^2(t)\epsilon\big(1 - \max_{1\leq i\leq n} \pi_i(t)\big)^2.$$

due $0 \leq \boldsymbol{S} \geq 1$ By substituting the diversity condition:

$$\max_{1\leq i\leq n} \mu_i(t) \leq 1 - \phi,$$

we find:

$$d\Theta(t) \geq \frac{1}{2} \sum_{i=1}^{n} \mu_i^2(t)\epsilon\phi^2.$$

Assuming equal weights $\mu_i(t) = \frac{1}{n}$, the inequality simplifies to:

$$d\Theta(t) \geq \frac{\epsilon\phi^2}{2n}.$$

Integrating over time t, the total outperformance satisfies:

$$\Theta(t) \geq \frac{\epsilon\phi^2 t}{2n}.$$

Thus, the portfolio $\pi(t)$ outperforms the benchmark $\mu(t)$ for all times t satisfying:

$$t \geq \frac{2n\ln(2)}{\epsilon\phi^2}.$$

This result highlights the long-term superiority of portfolios adhering to regulatory standards while achieving sustained outperformance.

6 Conclusion

In this paper, we have reviewed a comprehensive framework for stochastic portfolio theory. By modeling asset price dynamics as stochastic differential equations, we captured the random nature of financial markets. We then examined how portfolios can be constructed and analyzed using these models, deriving key results for portfolio dynamics and optimization.

A major strength of stochastic portfolio theory lies in its ability to incorporate uncertainty, allowing for a dynamic analysis of portfolios over time. The introduction of generating functions offers a powerful tool for constructing portfolios that potentially outperform benchmarks, providing both theoretical insights and practical strategies for portfolio managers.

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