I realized upon reflection that I had not pointed out in the derivation in class the change from a constant time of 1 unit per segment to distance-based time. Below is a complete derivation for the first two problems we discussed in class, items (a) and (b) below. Accompanying this pdf is a C++ program which implements all of these equations and draws the graphs for the first two problems. You should become very comfortable with everything in both of these documents before class on Tuesday and then begin work on continuing the derivations as additional constraints are added.

(a) p1, p2 for each segment: 2 constraints per segment; 3 segments = 6 constraints or 3 linear polynomials (2 c's each)

(b) problem (a) + segment-to-segment velocity equality; 6 + 2 v=v = 8 constraints; 2 quadratic (3 c's each) & 1 linear (2 c's)

(c) problem (b) + 2 terminal velocities = 0; 8 + 2 v=0 = 10 constraints; 2 quadratic (3 c's each) & 1 cubic (4 c's)

(d) problem (b) + segment-to-segment acceleration equality; 8 + 2 a=a = 10 constraints; 2 quadratic & 1 cubic (as in 1c)

(e) problems (c) & (d): 12 constraints; 3 cubic (4 c's each)

(f) problem (e) + 2 terminal accelerations = 0; 12 + 2 = 14 constraints; 2 quartic (5 c's each) + 1 cubic (4 c's)

## 0.1 constraints: segment end points

A function mapping trajectory between two end points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ is to be created. In order to satisfy two constraints, such as end points, a model with two coefficients is required.

$$S(t) = C_1\ t + C_0 \tag{1}$$

Function $S(t)$ will yield trajectory position as a function of a parameter $t$, which may be thought of as time. Assuming start and end times of 0 and 1 respectively, the constraints are then denoted as follows.

$$S(0) = P_0 \tag{2}$$
$$S(1) = P_1 \tag{3}$$

The constraints may then be expressed using Equation 1.

$$S(0) = C_1 \cdot 0 + C_0 = P_0 \tag{4}$$
$$S(1) = C_1 \cdot 1 + C_0 = P_1 \tag{5}$$
$$C_0 = P_0 \tag{6}$$
$$C_1 + C_0 = P_1 \tag{7}$$

Equation 6 is then substituted into Equation 7.

$$C_1 + P_0 = P_1 \tag{8}$$
$$C_1 = P_1 - P_0 \tag{9}$$

Equation 1 may then be rewritten using Equations 6 and 9.

$$S(t) = (P_1 - P_0)\ t + P_0 \tag{10}$$

Now assume a series of perhaps four fixed points, $P_0$, $P_1$, $P_2$ and $P_3$. A simple trajectory through these points could be defined using a set of three trajectory functions of the form of Equation 1.

$$S_0(t) = C_{01}\ t + C_{00} \tag{11}$$
$$S_1(t) = C_{11}\ t + C_{10} \tag{12}$$
$$S_2(t) = C_{21}\ t + C_{20} \tag{13}$$

Applying the same substitutions to each of the three trajectory functions produces results analagous to Equation 10.

$$\begin{aligned} S_0(t) &= (P_1 - P_0)\ t + P_0 && (14) \\ S_1(t) &= (P_2 - P_1)\ t + P_1 && (15) \\ S_2(t) &= (P_3 - P_2)\ t + P_2 && (16) \end{aligned}$$

Since the four points may be at different successive distances from one another however, treating time as running from 0 to 1 on each segment could result in traversing each segment at widely different speeds. An alternative might be to treat time as proportional to distance.

$$\begin{aligned} t_0 &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} && (17) \\ t_1 &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && (18) \\ t_2 &= \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} && (19) \end{aligned}$$

For the first segment trajectory, the constraints of Equations 2 and 3 are restated as follows.

$$\begin{aligned} S_0(0) &= P_0 && (20) \\ S_0(t_0) &= P_1 && (21) \end{aligned}$$

Equation 11 is used to express the constraints and the coefficients are found as in Equations 4 through 9.

$$\begin{aligned} S_0(0) = C_{01} \cdot 0 + C_{00} &= P_0 && (22) \\ S_0(t_0) = C_{01} \cdot t_0 + C_{00} &= P_1 && (23) \\ C_{00} &= P_0 && (24) \\ C_{01}\ t_0 + P_0 &= P_1 && (25) \\ C_{01}\ t_0 &= P_1 - P_0 && (26) \\ C_{01} &= \frac{P_1 - P_0}{t_0} && (27) \end{aligned}$$

Results from Equations 24 and 27 are then substituted into Equation 11; parallel derivations result in completed forms for Equations 12 and 13.

$$\begin{aligned} S_0(t) &= \frac{P_1 - P_0}{t_0}\ t + P_0 && (28) \\ S_1(t) &= \frac{P_2 - P_1}{t_1}\ t + P_1 && (29) \\ S_2(t) &= \frac{P_3 - P_2}{t_2}\ t + P_2 && (30) \end{aligned}$$

## 0.2 additional constraints: segment-to-segment velocity equality

Velocity at the end of the first segment is to be made equal to that at the beginning of the second; similarly, velocities at the end of the second and beginning of the third segments are to be equal.

$$\begin{aligned} S_0'(t_0) &= S_1'(0) && (31) \\ S_1'(t_1) &= S_2'(0) && (32) \end{aligned}$$

Two additional constraints will require two additional coefficients; one possible arrangement is an increase in degree by one for the first and last segment polynomial models.

$$\begin{aligned} S_0(t) &= C_{02}\ t^2 + C_{01}\ t + C_{00} && (33) \\ S_1(t) &= C_{11}\ t + C_{10} && (34) \\ S_2(t) &= C_{22}\ t^2 + C_{21}\ t + C_{20} && (35) \end{aligned}$$

The position constraints are then restated in terms of Equations 33 through 35.

$$\begin{aligned}
S_0(0) &= C_{02} \cdot 0^2 + C_{01} \cdot 0 + C_{00} &=& \quad P_0 && (36)\\
S_0(t_0) &= C_{02}\ t_0^2 + C_{01}\ t_0 + C_{00} &=& \quad P_1 && (37)\\
S_1(0) &= C_{11} \cdot 0 + C_{10} &=& \quad P_1 && (38)\\
S_1(t_1) &= C_{11}\ t_1 + C_{10} &=& \quad P_2 && (39)\\
S_2(0) &= C_{22} \cdot 0^2 + C_{21} \cdot 0 + C_{20} &=& \quad P_2 && (40)\\
S_2(t_2) &= C_{22}\ t_2^2 + C_{21}\ t_2 + C_{20} &=& \quad P_3 && (41)
\end{aligned}$$

The zero terms are removed to reveal solutions for the constant term coefficient of each polynomial.

$$\begin{aligned}
C_{00} &= P_0 && (42)\\
C_{02}\ t_0^2 + C_{01}\ t_0 + C_{00} &= P_1 && (43)\\
C_{10} &= P_1 && (44)\\
C_{11}\ t_1 + C_{10} &= P_2 && (45)\\
C_{20} &= P_2 && (46)\\
C_{22}\ t_2^2 + C_{21}\ t_2 + C_{20} &= P_3 && (47)
\end{aligned}$$

The constant term coefficients of Equations 42, 44 and 46 are then substituted into segment end position Equations 43, 45 and 47, respectively.

$$\begin{aligned}
C_{02}\ t_0^2 + C_{01}\ t_0 + P_0 &= P_1 && (48)\\
C_{02}\ t_0^2 + C_{01}\ t_0 &= P_1 - P_0 && (49)\\
C_{11}\ t_1 + P_1 &= P_2 && (50)\\
C_{11}\ t_1 &= P_2 - P_1 && (51)\\
C_{11} &= \frac{P_2 - P_1}{t_1} && (52)\\
C_{22}\ t_2^2 + C_{21}\ t_2 + P_2 &= P_3 && (53)\\
C_{22}\ t_2^2 + C_{21}\ t_2 &= P_3 - P_2 && (54)
\end{aligned}$$

Note that Equations 49 and 54 still have four unknowns between them.

Velocity polynomials may be obtained by differentiating position Equations 33 through 35 and substituting from Equation 52 for the single known coefficient involved.

$$\begin{aligned}
S_0'(t) &= 2\ C_{02}\ t + C_{01} && (55)\\
S_1'(t) &= C_{11} && (56)\\
&= \frac{P_2 - P_1}{t_1} && (57)\\
S_2'(t) &= 2\ C_{22} t + C_{21} && (58)
\end{aligned}$$

The velocity constraints of Equations 31 and 32 are now restated in terms of Equations 55, 57 and 58.

$$\begin{aligned}
2\ C_{02}\ t_0 + C_{01} &= \frac{P_2 - P_1}{t_1} && (59)\\
\frac{P_2 - P_1}{t_1} &= 2\ C_{22} \cdot 0 + C_{21} && (60)\\
&= C_{21} && (61)
\end{aligned}$$

This result for $C_{21}$ may be substituted into Equation 54 in order to obtain $C_{22}$.

$$\begin{aligned}
C_{22}\ t_2^2 + C_{21}\ t_2 &= P_3 - P_2 && (62)\\
C_{22}\ t_2^2 + \frac{P_2 - P_1}{t_1}\ t_2 &= P_3 - P_2 && (63)\\
C_{22}\ t_2^2 &= P_3 - P_2 - \frac{P_2 - P_1}{t_1}\ t_2 && (64)
\end{aligned}$$

$$
\begin{aligned}
C_{22}\ t_2^2 &= \frac{P_3 - P_2}{t_1}\ t_1 - \frac{P_2 - P_1}{t_1}\ t_2 && (65)\\
C_{22} &= \frac{(P_3 - P_2)\ t_1 - (P_2 - P_1)\ t_2}{t_1\ t_2^2} && (66)
\end{aligned}
$$

Equations 49 and 59 amount to two equations in the two remaining unknowns, $C_{02}$ and $C_{01}$, and are restated here.

$$
\begin{aligned}
C_{02}\ t_0^2 + C_{01}\ t_0 &= P_1 - P_0 && (67)\\
2C_{02}\ t_0 + C_{01} &= \frac{P_2 - P_1}{t_1} && (68)
\end{aligned}
$$

A new equation with only $C_{02}$ unknown can be obtained by multiplying Equation 68 by $-t_0$ and adding the resulting equation to Equation 67.

$$
\begin{aligned}
-2\ C_{02}\ t_0^2 - C_{01}\ t_0 &= -\frac{P_2 - P_1}{t_1}\ t_0 && (69)\\
-C_{02}\ t_0^2 &= P_1 - P_0 - \frac{P_2 - P_1}{t_1}\ t_0 && (70)\\
-C_{02}\ t_0^2 &= \frac{P_1 - P_0}{t_1}\ t_1 - \frac{P_2 - P_1}{t_1}\ t_0 && (71)\\
&= \frac{(P_1 - P_0)\ t_1 - (P_2 - P_1)\ t_0}{t_1} && (72)\\
C_{02} &= \frac{(P_2 - P_1)\ t_0 - (P_1 - P_0)\ t_1}{t_1\ t_0^2} && (73)
\end{aligned}
$$

A new equation with only $C_{01}$ unknown can similarly be obtained by multiplying Equation 67 by -2, multiplying Equation 68 by $t_0$, adding these new equations together and then isolating $C_{01}$.

$$
\begin{aligned}
-2\ C_{02}\ t_0^2 - 2\ C_{01}\ t_0 &= -2\ (P_1 - P_0) && (74)\\
2\ C_{02}\ t_0^2 + C_{01}\ t_0 &= \frac{P_2 - P_1}{t_1}\ t_0 && (75)\\
-C_{01}\ t_0 &= \frac{P_2 - P_1}{t_1}\ t_0 - 2\ (P_1 - P_0) && (76)\\
-C_{01}\ t_0 &= \frac{P_2 - P_1}{t_1}\ t_0 - 2\ \frac{P_1 - P_0}{t_1}\ t_1 && (77)\\
-C_{01}\ t_0 &= \frac{(P_2 - P_1)\ t_0 - 2\ (P_1 - P_0)\ t_1}{t_1} && (78)\\
C_{01} &= \frac{2\ (P_1 - P_0)\ t_1 - (P_2 - P_1)\ t_0}{t_0\ t_1} && (79)
\end{aligned}
$$

The coefficients for the polynomial models put forth in Equations 33 through 35 are now all known; the models and their coefficients from Equations 73, 79, 42, 52, 44, 66, 61, and 46 are all summarized here.

$$
\begin{aligned}
S_0(t) &= C_{02}\ t^2 + C_{01}\ t + C_{00} && (80)\\
C_{02} &= \frac{(P_2 - P_1)\ t_0 - (P_1 - P_0)\ t_1}{t_1\ t_0^2} && (81)\\
C_{01} &= \frac{2\ (P_1 - P_0)\ t_1 - (P_2 - P_1)\ t_0}{t_0\ t_1} && (82)\\
C_{00} &= P_0 && (83)
\end{aligned}
$$

$$
\begin{aligned}
S_1(t) &= C_{11}\ t + C_{10} && (84)\\
C_{11} &= \frac{P_2 - P_1}{t_1} && (85)\\
C_{10} &= P_1 && (86)
\end{aligned}
$$

$$
S_2(t) = C_{22}\ t^2 + C_{21}\ t + C_{20} \qquad (87)
$$

$$C_{22} = \frac{(P_3 - P_2)\ t_1 - (P_2 - P_1)\ t_2}{t_1\ t_2^2} \tag{88}$$

$$C_{21} = \frac{P_2 - P_1}{t_1} \tag{89}$$

$$C_{20} = P_2 \tag{90}$$