A Quadrotomy for Partial Orders

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Abstract

This paper uses a calculational system to formally prove a quadrotomy for partial orders $\preceq$, which states that any element x of the partial order must be related to any other element y by exactly one of (a) $x \prec y$, (b) $x = y$, (c) $y \prec x$, or (d) $\text{incomp}(x,y)$, where $\prec$ is the reflexive reduction of $\preceq$ and $\text{incomp}(x,y)$ is defined as $\text{incomp}(x,y) \equiv \neg(x \preceq y) \wedge \neg(y \preceq x)$. The calculational system, developed by Dijkstra and Scholten and extended by Gries and Schneider in their text *A Logical Approach to Discrete Math*, is based on only four inference rules – Substitution, Leibniz, Equanimity, and Transitivity. Inference rules in the older Hilbert-style systems, notably modus ponens, appear as theorems in this calculational deductive system, which is used to prove algorithm correctness in computer science. The theorem presented in this paper is a generalization of the trichotomy theorem for integers proved by Gries and Schneider, which states that any integer n must be related to any other integer m by exactly one of (a) $n < m$, (b) $n = m$, or (c) $m < n$.

1 Introduction

1.1 Background

A proof calculus is a system for proving propositions through the use of axioms and inference rules. Two of the most popular proof calculi are the Hilbert system and the natural deduction system. The former is sometimes used in the high-school environment to aid in the teaching of geometry; the latter is typically taught in undergraduate symbolic logic courses. There is a stark distinction between the two systems, due to their different approaches to axiomatization and their use of inference rules. Natural deduction attempts to mimic how people naturally reason, and contains few or no axioms and numerous inference rules. The Hilbert system makes no such attempt to mimic natural language – it contains only one inference rule, modus ponens, and a hefty amount of axiom schemes.

A third proof calculus, which will be used in this paper, is the equational system. This system is widely applicable in the realm of computer science, incorporating the usual propositional and predicate calculus of the other proof systems and seamlessly extending them to reason about sets, sequences, functions, programs, and graphs. The equational system was developed in the late 1980's by Dijkstra and Scholten and was then largely expanded by Gries and Schneider. A distinctive feature of it is its use of only four inference rules - Substitution, Leibniz, Equanimity, and Transitivity - and its parsimonious use of axioms. Its axiomatization strikes a pleasant balance between the extensive use of axioms in the Hilbert style and the lack of them in natural deduction.

The equational system's cumulative structure and its resemblance to high-school algebra make it the optimal proof calculus to expound the case analysis proof technique for which this paper presents a new metatheorem. Case analysis is sometimes referred to as proof by exhaustion, for it attempts to prove that an exhaustive set of cases leads to a specific outcome. It is instrumental in natural deduction, where it is used as an inference rule called disjunction exploitation. The technique has two stages: the first involves proving that there is an exhaustive set of cases that altogether represents all possibilities; the second presents a proof for each individual case.

1.2 Case analysis

Describe Gries and Schneider's treatment of case analysis including Shannon as justification of the metatheorem.

Include the metatheorem and show how Gries justifies it using Shannon.

Present the new theorem and show how it is a more direct justification of the case analysis metatheorem.

2 Results

Lemma 1: $b \prec c \vee b = c \equiv b \preceq c$, where $\prec$ is the reflexive reduction of $\preceq$.

Proof:

$$
\begin{array}{ll}
 & b \prec c \vee b = c \\
= & \langle \text{(14.15.4) Notation, } \langle b,c\rangle \in \rho \text{ and } b\rho c \text{ are interchangeable notations.} \rangle \\
 & \langle b,c\rangle \in \prec \vee b = c \\
= & \langle \text{(14.15.3) Identity lemma, } \langle x,y\rangle \in i_B \equiv x = y \rangle \\
 & \langle b,c\rangle \in \prec \vee \langle b,c\rangle \in i_B \\
= & \langle \text{(11.20) Axiom, Union, } v \in S \bigcup T \equiv v \in S \vee v \in T \rangle \\
 & \langle b,c\rangle \in \prec \bigcup i_B \\
= & \langle \text{(14.49b) If } \rho \text{ is a quasi order over a set B, then } \rho \bigcup i_B \text{ is a partial order} \rangle \\
 & \langle b,c\rangle \in \preceq \\
= & \langle \text{(14.15.4) Notation} \rangle \\
 & b \preceq c \quad \blacksquare
\end{array}
$$

Lemma 2: $(p \vee q) \wedge \neg(q \vee r) \equiv p \wedge \neg q \wedge \neg r$

Proof:

$$
\begin{array}{ll}
 & (p \vee q) \wedge \neg(q \vee r) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & (p \vee q) \wedge \neg q \wedge \neg r \\
= & \langle \text{(3.44a) Absorption, } p \wedge (\neg p \vee q) \equiv (p \wedge q) \text{, with } p, q := \neg q, p \text{ and with (3.12)} \\
 & \ \text{Double negation} \rangle \\
 & p \wedge \neg q \wedge \neg r \quad \blacksquare
\end{array}
$$

Lemma 3: $(p \Rightarrow q) \Rightarrow (p \wedge q \equiv p)$
Proof:

$$
\begin{array}{ll}
 & p \wedge q \equiv p \\
= & \langle \text{(3.60) Implication, } p \Rightarrow q \equiv p \wedge q \equiv p \rangle \\
 & p \Rightarrow q \\
\Leftarrow & \langle \text{(3.71) Reflexivity of } \Rightarrow, p \Rightarrow p \rangle \\
 & p \Rightarrow q \quad \blacksquare
\end{array}
$$

Lemma 4: ρ is irreflexive $\Rightarrow (b\rho c \Rightarrow \neg(b = c))$
Proof by (4.4) Deduction:

$$
\begin{array}{ll}
 & b\rho c \Rightarrow \neg(b = c) \\
= & \langle \text{(3.61) Contrapositive, } p \Rightarrow q \equiv \neg q \Rightarrow \neg p \text{, with (3.12) Double negation} \rangle \\
 & b = c \Rightarrow \neg(b\rho c) \\
= & \langle \text{(3.84b) Substitution, } (e = f) \Rightarrow E^z_e \equiv (e = f) \Rightarrow E^z_f \rangle \\
 & b = c \Rightarrow \neg(b\rho b) \\
= & \langle \text{Assume the antecedent, } \rho \text{ is irreflexive, or } (\forall b \mid\ : \neg(b\rho b)) \text{, with (9.16)} \\
 & \ \text{Metatheorem, P is a theorem iff } (\forall x \mid\ : P) \text{ is a theorem} \rangle \\
 & b = c \Rightarrow true \\
= & \langle \text{(3.72) Right Zero of } \Rightarrow, p \Rightarrow true \equiv true \rangle \\
 & true \quad \blacksquare
\end{array}
$$

Lemma 5: ρ is antisymmetric $\wedge$ ρ is reflexive $\Rightarrow (b\rho c \wedge c\rho b \equiv b = c)$
Proof by (4.4) Deduction and (4.7) Mutual Implication:
a. $b\rho c \wedge c\rho b \Rightarrow b = c$

$$
\begin{array}{ll}
 & b\rho c \wedge c\rho b \\
= & \langle \text{Assume the conjunct of the antecedent } \rho \text{ is antisymmetric, or} \\
 & \ (\forall b, c \mid\ : b\rho c \wedge c\rho b \Rightarrow b = c) \text{, with (9.16) Metatheorem, P is a theorem iff} \\
 & \ (\forall x \mid\ : P) \text{ is a theorem} \rangle \\
 & b = c \\
\Rightarrow & \langle \text{(3.71) Reflexivity of } \Rightarrow, p \Rightarrow p \rangle \\
 & b = c \quad \blacksquare
\end{array}
$$

b. $b = c \Rightarrow b\rho c \wedge c\rho b$
Proof by (4.4) Deduction:

$$\begin{array}{ll}
& b\rho c \wedge c\rho b \\
= & \langle \text{Assume the antecedent, } b = c \rangle \\
& b\rho b \wedge b\rho b \\
= & \langle \text{(3.38) Idempotency of } \wedge,\ p \wedge p \equiv p \rangle \\
& b\rho b \\
= & \langle \text{Assume the conjunct of the antecedent } \rho \text{ is reflexive, or } (\forall b \mid : b\rho b)\text{, with} \\
& \quad \text{(9.16) Metatheorem, P is a theorem iff } (\forall x \mid : P) \text{ is a theorem} \rangle \\
& true \quad \blacksquare
\end{array}$$

Theorem (a): $b \prec c \equiv \neg(c \prec b \vee b = c \vee \text{incomp}(b,c))$
Proof by (4.4) Deduction:

$$\begin{array}{ll}
& \neg(c \prec b \vee b = c \vee \text{incomp}(b,c)) \\
= & \langle \text{Lemma 1, } b \prec c \vee b = c \equiv b \preceq c \rangle \\
& \neg(c \preceq b \vee \text{incomp}(b,c)) \\
= & \langle \text{(14.47.1) Definition, Incomparable, incomp}(b,c) \equiv \neg(b \preceq c) \wedge \neg(c \preceq b) \rangle \\
& \neg(c \preceq b \vee (\neg(b \preceq c) \wedge \neg(c \preceq b))) \\
= & \langle \text{(3.44b) Absorption, } p \vee (\neg p \wedge q) \equiv p \vee q \rangle \\
& \neg(c \preceq b \vee \neg(b \preceq c)) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q\text{, with (3.12) Double negation} \rangle \\
& b \preceq c \wedge \neg(c \preceq b) \\
= & \langle \text{Lemma 1, twice} \rangle \\
& (b \prec c \vee b = c) \wedge \neg(c \prec b \vee b = c) \\
= & \langle \text{Lemma 2, } (p \vee q) \wedge \neg(q \vee r) \equiv p \wedge \neg q \wedge \neg r \rangle \\
& b \prec c \wedge \neg(b = c) \wedge \neg(c \prec b) \\
= & \langle \text{Lemma 3, } (p \Rightarrow q) \Rightarrow (p \wedge q \equiv p)\text{, with } p,q := b \prec c, \neg(c \prec b)\text{. The antecedent,} \\
& \quad b \prec c \Rightarrow \neg(c \prec b)\text{, is true by the definition of asymmetry and the fact that strict} \\
& \quad \text{orders are asymmetric.} \rangle \\
& b \prec c \wedge \neg(b = c) \\
= & \langle \text{Lemma 3, with } p,q := b \prec c, \neg(b = c)\text{. The antecedent, } b \prec c \Rightarrow \neg(b = c)\text{, is} \\
& \quad \text{true by Lemma 4, } \rho \text{ is irreflexive} \Rightarrow (b\rho c \Rightarrow \neg(b = c))\text{, and the fact that strict} \\
& \quad \text{orders are irreflexive.} \rangle \\
& b \prec c \quad \blacksquare
\end{array}$$

Theorem (b): $c \prec b \equiv \neg(b \prec c \vee b = c \vee \text{incomp}(b,c))$
Since $=$ and *incomp* are symmetric, the proof of b is identical to that of a with $b,c := c,b$.

Theorem (c): $b = c \equiv \neg(b \prec c \vee c \prec b \vee \text{incomp}(b,c))$
Proof by (4.4) Deduction:

$$
\begin{array}{cl}
& \neg(b \prec c \vee c \prec b \vee \mathrm{incomp}(b,c)) \\
= & \langle (3.26)\ \text{Idempotency of } \vee,\ p \equiv p \vee p \rangle \\
& \neg(b \prec c \vee c \prec b \vee \mathrm{incomp}(b,c) \vee \mathrm{incomp}(b,c)) \\
= & \langle (14.48.2),\ \neg(b \preceq c) \equiv c \prec b \vee \mathrm{incomp}(b,c),\ \text{twice} \rangle \\
& \neg(\neg(b \preceq c) \vee \neg(c \preceq b)) \\
= & \langle (3.47b)\ \text{De Morgan},\ \neg(p \vee q) \equiv \neg p \wedge \neg q,\ \text{with (3.12) Double negation twice} \rangle \\
& b \preceq c \wedge c \preceq b \\
= & \langle \text{Lemma 5, } \rho \text{ is antisymmetric} \wedge \rho \text{ is reflexive} \Rightarrow (b\rho c \wedge c\rho b \equiv b = c),\ \text{with the} \\
& \quad \text{fact that partial orders are antisymmetric and reflexive} \rangle \\
& b = c \quad \blacksquare
\end{array}
$$

Theorem (d): $\mathrm{incomp}(b,c) \equiv \neg(b \prec c \vee c \prec b \vee b = c)$
Proof by (4.4) Deduction:

$$
\begin{array}{cl}
& \neg(b \prec c \vee c \prec b \vee b = c) \\
= & \langle (3.26)\ \text{Idempotency of } \vee,\ p \equiv p \vee p \rangle \\
& \neg(b \prec c \vee c \prec b \vee b = c \vee b = c) \\
= & \langle \text{Lemma 1, } b \prec c \vee b = c \equiv b \preceq c \text{ twice} \rangle \\
& \neg(b \preceq c \vee c \preceq b) \\
= & \langle (3.47b)\ \text{De Morgan},\ \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
& \neg(b \preceq c) \wedge \neg(c \preceq b) \\
= & \langle (14.47.1)\ \text{Definition, Incomparable: } \mathrm{incomp}(b,c) \equiv \neg(b \preceq c) \wedge \neg(c \preceq b) \rangle \\
& \mathrm{incomp}(b,c) \quad \blacksquare
\end{array}
$$

Four abbreviations for expressions will be introduced here to make the following proofs more concise. Note that replacing A, B, C, and D with $b \prec c, c \prec b, b = c$, and $\mathrm{incomp}(b,c)$, in any order, will make the expressions identical to Theorems (a)-(d). However, the expressions in general suffice to describe a situtation in which exactly one of A, B, C, and D is true.

α. $A \equiv \neg(B \vee C \vee D)$
β. $B \equiv \neg(A \vee C \vee D)$
γ. $C \equiv \neg(A \vee B \vee D)$
δ. $D \equiv \neg(A \vee B \vee C)$
Theorem (e): $\alpha \wedge \beta \wedge \gamma \wedge \delta \Rightarrow A \vee B \vee C \vee D$
Proof by (4.4) Deduction:

$$
\begin{array}{cl}
& A \vee B \vee C \vee D \\
= & \langle \text{Assume conjunct } \alpha \text{ of the antecedent, } A \equiv \neg(B \vee C \vee D) \rangle \\
& \neg(B \vee C \vee D) \vee B \vee C \vee D \\
= & \langle (3.28)\ \text{Excluded middle},\ p \vee \neg p \rangle \\
& \mathit{true} \quad \blacksquare
\end{array}
$$

Lemma 6: $\alpha \wedge \beta \wedge \gamma \wedge \delta \Rightarrow \neg(A \wedge B \wedge C)$
Proof by (4.4) Deduction:

$$
\begin{array}{ll}
 & \neg(A \wedge B \wedge C) \\
= & \langle \text{(3.47a) De Morgan, } \neg(p \wedge q) \equiv \neg p \vee \neg q\text{, twice} \rangle \\
 & \neg A \vee \neg B \vee \neg C \\
= & \langle \text{Assume conjunct } \alpha \text{ of the antecedent, } A \equiv \neg(B \vee C \vee D)\text{, with (3.12) Double negation} \rangle \\
 & B \vee C \vee D \vee \neg B \vee \neg C \\
= & \langle \text{(3.28) Excluded middle, } p \vee \neg p\text{, twice} \rangle \\
 & true \vee true \vee D \\
= & \langle \text{(3.29) Identity of } \vee\text{, } p \vee true \equiv true \rangle \\
 & true \quad \blacksquare
\end{array}
$$

Lemma 7: $\alpha \wedge \beta \wedge \gamma \wedge \delta \wedge \neg D \Rightarrow (A \wedge \neg B \wedge \neg C \equiv A)$
Proof by (4.4) Deduction:

$$
\begin{array}{ll}
 & A \wedge \neg B \wedge \neg C \\
= & \langle \text{(3.39) Identity of } \wedge\text{, } p \wedge true \equiv p \rangle \\
 & A \wedge \neg B \wedge \neg C \wedge true \\
= & \langle \text{Assume conjunct } \neg D \text{ of the antecedent} \rangle \\
 & A \wedge \neg B \wedge \neg C \wedge \neg D \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q\text{, twice} \rangle \\
 & A \wedge \neg(B \vee C \vee D) \\
= & \langle \text{Assume conjunct } \alpha \text{ of the antecedent, } A \equiv \neg(B \vee C \vee D) \rangle \\
 & A \wedge A \\
= & \langle \text{(3.40) Idempotency of } \wedge\text{, } p \wedge p \equiv p \rangle \\
 & A \quad \blacksquare
\end{array}
$$

Lemma 8: $(p \equiv q \equiv r) \equiv (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$
Proof:

$$
\begin{array}{ll}
 & p \equiv q \equiv r \\
= & \langle \text{(3.52) Equivalence, } p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q) \rangle \\
 & (p \wedge q) \vee (\neg p \wedge \neg q) \equiv r \\
= & \langle \text{(3.52) Equivalence} \rangle \\
 & (((p \wedge q) \vee (\neg p \wedge \neg q)) \wedge r) \vee (\neg((p \wedge q) \vee (\neg p \wedge \neg q)) \wedge \neg r) \\
= & \langle \text{(3.46) Distributivity of } \wedge \text{ over } \vee\text{, } p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \rangle \\
 & (p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg((p \wedge q) \vee (\neg p \wedge \neg q)) \wedge \neg r) \\
= & \langle \text{(3.52) Equivalence} \rangle \\
 & (p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg(p \equiv q) \wedge \neg r) \\
= & \langle \text{(3.10) Definition of } \not\equiv\text{, } (p \not\equiv q) \equiv \neg(p \equiv q) \rangle \\
 & (p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee ((p \not\equiv q) \wedge \neg r)
\end{array}
$$

$=$ $\langle$(3.46) Excusive or, $p \not\equiv q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$$\rangle$
$(p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (((\neg p \wedge q) \vee (p \wedge \neg q)) \wedge \neg r)$
$=$ $\langle$(3.46) Distributivity of $\wedge$ over $\vee$$\rangle$
$(p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)$
$=$ $\langle$(3.24) Symmetry of $\vee$$\rangle$
$(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$ ■

Lemma 9: $\text{incomp}(b,c) \equiv false$, where $\text{incomp}(b,c)$ refers to b and c being incomparable under $\leq$.

Proof by (4.4) Deduction:

$\text{incomp}(b,c)$
$=$ $\langle$(14.47.1) Definition, Incomparable, $\text{incomp}(b,c) \equiv \neg(b \preceq c) \wedge \neg(c \preceq b)$, where the partial order is $\leq$$\rangle$
$\neg(b \leq c) \wedge \neg(c \leq b)$
$=$ $\langle$(3.47b) De Morgan, $\neg(p \vee q) \equiv \neg p \wedge \neg q$$\rangle$
$\neg(b \leq c \vee c \leq b)$
$=$ $\langle$(14.50) Definition, Total Order: A partial order over B is called a total or linear order if $(\forall b,c \mid\, : b \preceq c \vee c \preceq b)$, with (9.16) Metatheorem, P is a theorem iff $(\forall x \mid\, : P)$ is a theorem, and the fact that $\leq$ is a total order$\rangle$
$\neg true$
$=$ $\langle$(3.8) Definition of false$\rangle$
$false$ ■

(15.44) Trichotomy: $(a < b \equiv a = b \equiv a > b) \wedge \neg(a < b \wedge a = b \wedge a > b)$

Proof:

$(a < b \equiv a = b \equiv a > b) \wedge \neg(a < b \wedge a = b \wedge a > b)$
$=$ $\langle$Lemma 6, $\alpha \wedge \beta \wedge \gamma \wedge \delta \Rightarrow \neg(A \vee B \vee C)$ with $A,B,C,D := a < b, a = b, a > b, \text{incomp}(a,b)$. The antecedent is true by Theorems (a)-(d)$\rangle$
$(a < b \equiv a = b \equiv a > b) \wedge true$
$=$ $\langle$(3.39) Identity of $\wedge$, $p \wedge true \equiv p$$\rangle$
$(a < b \equiv a = b \equiv a > b)$
$=$ $\langle$Lemma 8, with $p,q,r := a < b, a = b, a > b$$\rangle$
$(a < b \wedge a = b \wedge a > b) \vee (a < b \wedge \neg(a = b) \wedge \neg(a > b)) \vee$
$(\neg(a < b) \wedge a = b \wedge \neg(a > b)) \vee (\neg(a < b) \wedge \neg(a = b) \wedge a > b)$
$=$ $\langle$Lemma 6, as in step 1, but with (3.12) Double negation$\rangle$
$false \vee (a < b \wedge \neg(a = b) \wedge \neg(a > b)) \vee (\neg(a < b) \wedge a = b \wedge \neg(a > b)) \vee$
$(\neg(a < b) \wedge \neg(a = b) \wedge a > b)$
$=$ $\langle$(3.30) Identity of $\vee$, $p \vee false \equiv p$$\rangle$
$(a < b \wedge \neg(a = b) \wedge \neg(a > b)) \vee (\neg(a < b) \wedge a = b \wedge \neg(a > b)) \vee$
$(\neg(a < b) \wedge \neg(a = b) \wedge a > b)$

$=$ $\langle$Lemma 7, $\alpha \wedge \beta \wedge \gamma \wedge \delta \wedge \neg D \Rightarrow (A \wedge \neg B \wedge \neg C \equiv A)$, with $A, B, C, D := a < b, a = b, a > b, \text{incomp}(a, b)$. The conjuncts α through δ of the antecedent are satisfied by Theorems (a)-(d), while the conjunct $\neg D$ is satisfied by Lemma 9, $\text{incomp}(b, c) \equiv false$ under $\leq\rangle$

$a < b \vee (\neg(a < b) \wedge a = b \wedge \neg(a > b)) \vee (\neg(a < b) \wedge \neg(a = b) \wedge a > b)$

$=$ $\langle$Lemma 7, with $A, B, C, D := a = b, a < b, a > b, \text{incomp}(a, b)$. The conjuncts of the antecedent are satisfied as in the previous step. Note that the expressions replacing A and B are swapped compared to the previous step$\rangle$

$a < b \vee a = b \vee (\neg(a < b) \wedge \neg(a = b) \wedge a > b)$

$=$ $\langle$Lemma 7, with $A, B, C, D := a < b, a = b, a > b, \text{incomp}(a, b)$. The conjuncts of the antecedent are satisfied as in the previous step. Note that the expressions are again swapped to make the lemma apply to the final disjunct$\rangle$

$a < b \vee a = b \vee a > b$

$=$ $\langle$Theorem d with $\preceq := \leq$, with (3.12) Double negation$\rangle$

$\neg\, \text{incomp}(a, b)$

$=$ $\langle$Lemma 9$\rangle$

$true$ ■

3 Conclusion