A Quadrotomy for Partial Orders

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Abstract

This paper uses a calculational system to formally prove a quadrotomy for partial orders $\preceq$, which states that any element x of the partial order must be related to any other element y by exactly one of (a) $x \prec y$, (b) $x = y$, (c) $y \prec x$, or (d) incomp(x, y), where $\prec$ is the reflexive reduction of $\preceq$ and incomp(x, y) is defined as incomp$(x, y) \equiv \neg(x \preceq y) \wedge \neg(y \preceq x)$. The calculational system, developed by Dijkstra and Scholten and extended by Gries and Schneider in their text *A Logical Approach to Discrete Math*, is based on only four inference rules – Substitution, Leibniz, Equanimity, and Transitivity. Inference rules in the older Hilbert-style systems, notably modus ponens, appear as theorems in this calculational deductive system, which is used to prove algorithm correctness in computer science. The theorem presented in this paper is a generalization of the trichotomy theorem for integers proved by Gries and Schneider, which states that any integer n must be related to any other integer m by exactly one of (a) $n < m$, (b) $n = m$, or (c) $m < n$.

1 Introduction

1.1 Background

Propositional calculus is a formal system of logic based on the unary operator negation $\neg$, the binary operators conjunction $\wedge$, disjunction $\vee$, implies $\Rightarrow$ (also written $\rightarrow$), and equivalence $\equiv$ (also written $\leftrightarrow$), variables (lowercase letters p, q, $\ldots$), and the constants *true* and *false*. Hilbert-style logic systems, $\mathcal{H}$, are the deductive logic systems traditionally used in mathematics to describe the propositional calculus. A key feature of such systems is their multiplicity of inference rules and the importance of modus ponens as one of them.

In the late 1980's, Dijkstra and Scholten [**?**], and Feijen [**?**] developed a method of proving program correctness with a new logic based on an equational style. In contrast to $\mathcal{H}$ systems, $\mathcal{E}$ has only four inference rules – Substitution, Leibniz, Equanimity, and Transitivity. In $\mathcal{E}$, modus ponens plays a secondary role. It is not an inference rule, nor

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is it assumed as an axiom, but instead is proved as a theorem from the axioms using the inference rules.

Gries and Schneider [**?**, **?**] show that $\mathcal{E}$, also known as a *calculational* system, has several advantages over traditional logic systems. The primary advantage of $\mathcal{E}$ over $\mathcal{H}$ systems is that the calculational system has only four proof rules, with inference rule Leibniz as the primary one. Roughly speaking, Leibniz is "substituting equals for equals," hence the moniker *equational* deductive system. In contrast, $\mathcal{H}$ systems rely on a more extensive set of inference rules.

In 1994, Gries and Schneider published *A Logical Approach to Discrete Math* (LADM) [**?**], in which they first develop $\mathcal{E}$ for propositional and predicate calculus, and then extend it to a theory of sets, a theory of sequences, relations and functions, a theory of integers, recurrence relations, modern algebra, and a theory of graphs. Using calculational logic as a tool, LADM brings all the advantages of $\mathcal{E}$ to these additional knowledge domains. The treatment is in marked contrast to the traditional one exemplified by the classic undergraduate text by Rosen [**?**].

1.2 Partial Orders

This paper applies $\mathcal{E}$ to the properties of partial and total orders. A partial order, denoted $\preceq$, is a binary relation which is reflexive, antisymmetric, and transitive. A total order is a partial order which for all elements b and c satisfy $b \preceq c$, $c \preceq b$, or both.

The subset relation $\subseteq$ over sets is an example of a partial order that is not a total order. It is reflexive, because every set is a subset of itself; it is antisymmetric, because if $S \subseteq T$, then $T \subseteq S$ only if $S = T$; and it is transitive, because if $S \subseteq T$ and $T \subseteq U$, then $S \subseteq U$. The subset relation is not a total order, because there can be sets S and T such that $S \nsubseteq T$ and $T \nsubseteq S$.

The at most relation $\leq$ over integers is an example of a partial order that is also a total order. It is reflexive, because every integer is at most itself; it is antisymmetric, because if $a \leq b$, then $b \leq a$ only if $a = b$; it is transitive, because if $a \leq b$ and $b \leq c$, then $a \leq c$. The at most relation is a total order because at least one of $a \leq b$ and $b \leq a$ must be true.

Two elements b, c of a partial order are comparable if $b \preceq c$ or $c \preceq b$. We formally define the predicate $\text{incomp}(b,c)$ as the negation of comparable.

$$\text{incomp}(b,c) \equiv \neg(b \preceq c) \wedge \neg(c \preceq b)$$

In addition to partial and total orders, there are strict partial and strict total orders, denoted $\prec$. A strict partial or strict total order is computed by removing the reflexive pairs from a partial order or total order. This strict partial order is called the reflexive reduction of the corresponding partial order. For example, $\subset$ is the reflexive reduction of $\subseteq$, and $<$ is the reflexive reduction of $\leq$.

2 Results

This paper proves that exactly one of (a) $x \prec y$, (b) $x = y$, (c) $y \prec x$, or (d) $\text{incomp}(x, y)$ holds for any given partial order $\preceq$ and its reflexive reduction $\prec$. Section 2.1 formally proves a general quadrotomy for partial orders. Section 2.2 applies the general quadrotomy to prove the special case of a trichotomy for all total orders. In the solutions manual for LADM, Gries and Schneider prove using the properties of integers a trichotomy for $\leq$, which is an example of a total order. Section 2.3 shows that Gries and Schneider's trichotomy for integers is a special case of our more general trichotomy for all total orders.

2.1 Quadrotomy

To prove that two elements a and b are related by exactly one of (a) $x \prec y$, (b) $x = y$, (c) $y \prec x$, or (d) $\text{incomp}(x, y)$, we prove the following four-part theorem:

Theorem, Quadrotomy (a): $b \prec c \;\equiv\; \neg(c \prec b \lor b = c \lor \text{incomp}(b, c))$
Theorem, Quadrotomy (b): $c \prec b \;\equiv\; \neg(b \prec c \lor b = c \lor \text{incomp}(b, c))$
Theorem, Quadrotomy (c): $b = c \;\equiv\; \neg(b \prec c \lor c \prec b \lor \text{incomp}(b, c))$
Theorem, Quadrotomy (d): $\text{incomp}(b, c) \;\equiv\; \neg(b \prec c \lor c \prec b \lor b = c)$

By mutual implication $p \equiv q$ is equivalent to $p \Rightarrow q$ and $q \Rightarrow p$. The left-to-right implication for Quadrotomy (a) asserts that if $b \prec c$ holds, then none of the other three parts of the quadrotomy holds, and similarly for theorems (b), (c), and (d). Consequently, at most one of the four parts of the quadrotomy is true. The right-to-left implication for Quadrotomy (a) asserts that if none of the other three parts of the quadrotomy holds, then $b \prec c$ holds, and similarly for theorems (b), (c), and (d). Consequently, at least one of the four parts of the quadrotomy is true. Therefore, exactly one of the four parts of the quadrotomy is true.

2.1.1 Lemmas

To prove this four-part theorem, we first prove several lemmas. The first lemma,

$$b \prec c \lor b = c \;\equiv\; b \preceq c$$

where $\prec$ is the reflexive reduction of $\preceq$, states that comparing two elements under a partial order is equivalent to comparing them under a strict order or equality. This lemma follows closely from the definition of a strict order.

The second lemma,

$$(p \lor q) \land \neg(q \lor r) \;\equiv\; p \land \neg q \land \neg r$$

is based solely on propositional calculus. It is used only in the proof of Quadrotomy (a), but is introduced as a lemma to avoid cluttering that proof with simple steps.

The third lemma,

$$(p \Rightarrow q) \Rightarrow (p \land q \;\equiv\; p)$$

is an implication. In fact, the consequent is equivalent to the antecedent, a step which is used in the proof, but we state it here in the weaker form of an implication. This is because the calculational system allows us to use the consequent of an implication (here, $p \wedge q \equiv p$) as a theorem as long as the antecedent (here, $p \Rightarrow q$) is true.

The fourth lemma,

$$\rho \text{ is irreflexive} \Rightarrow (b\rho c \Rightarrow \neg(b = c))$$

follows closely from the definition of irreflexivity, and is used like the third lemma: when the antecedent is true, the consequent can be used as a theorem.

The fifth lemma,

$$\rho \text{ is antisymmetric } \wedge \ \rho \text{ is reflexive} \Rightarrow (b\rho c \ \wedge \ c\rho b \ \equiv \ b = c)$$

is proved by deduction and mutual implication. Partial orders satisfy the antecedent, and the consequent therefore applies: no element can both precede and follow any other element, unless the two are identical. Like the third and fourth lemmas, when the conjuncts of the antecedent of this theorem are true, the consequent can be used as a theorem.

Lemma 1: $b \prec c \vee b = c \ \equiv \ b \preceq c$, where $\prec$ is the reflexive reduction of $\preceq$.
Proof:

$$
\begin{array}{ll}
 & b \prec c \vee b = c \\
= & \langle (14.15.4) \text{ Notation, } \langle b,c\rangle \in \rho \text{ and } b\ \rho\ c \text{ are interchangeable notations.} \rangle \\
 & \langle b,c\rangle \in \prec\ \vee b = c \\
= & \langle (14.15.3) \text{ Identity lemma, } \langle x,y\rangle \in\ i_B \ \equiv\ x = y \rangle \\
 & \langle b,c\rangle \in \prec\ \vee \langle b,c\rangle \in i_B \\
= & \langle (11.20) \text{ Axiom, Union, } v \in\ S \cup T \ \ \equiv\ \ v \in S \vee v \in T \rangle \\
 & \langle b,c\rangle \in \prec \cup\, i_B \\
= & \langle (14.49\text{b}) \text{ If } \rho \text{ is a quasi order over a set B, then } \rho \cup i_B \text{ is a partial order} \rangle \\
 & \langle b,c\rangle \in \preceq \\
= & \langle (14.15.4) \text{ Notation} \rangle \\
 & b \preceq c \quad \blacksquare
\end{array}
$$

Lemma 2: $(p \vee q) \wedge \neg(q \vee r) \ \equiv \ p \wedge \neg q \wedge \neg r$
Proof:

$$
\begin{array}{ll}
 & (p \vee q) \wedge \neg(q \vee r) \\
= & \langle (3.47\text{b}) \text{ De Morgan, } \neg(p \vee q) \ \equiv\ \neg p \wedge \neg q \rangle \\
 & (p \vee q) \wedge \neg q \wedge \neg r \\
= & \langle (3.44\text{a}) \text{ Absorption, } p \wedge (\neg p \vee q) \ \equiv\ (p \wedge q) \text{, with } p,\, q := \neg q,\, p \text{ and with} \\
 & \ \ (3.12) \text{ Double negation} \rangle \\
 & p \wedge \neg q \wedge \neg r \quad \blacksquare
\end{array}
$$

Lemma 3: $(p \Rightarrow q) \Rightarrow (p \wedge q \equiv p)$

Proof:

$$\begin{array}{ll} & p \wedge q \equiv p \\ = & \langle (3.60)\ \text{Implication},\ p \Rightarrow q \ \equiv\ p \wedge q \equiv p \rangle \\ & p \Rightarrow q \\ \Leftarrow & \langle (3.71)\ \text{Reflexivity of} \Rightarrow,\ p \Rightarrow p \rangle \\ & p \Rightarrow q \quad \blacksquare \end{array}$$

Lemma 4: ρ is irreflexive $\Rightarrow (b\rho c \Rightarrow \neg(b = c))$

Proof: The proof is by (4.4) Deduction.

$$\begin{array}{ll} & b\rho c \Rightarrow \neg(b = c) \\ = & \langle (3.61)\ \text{Contrapositive},\ p \Rightarrow q \ \equiv\ \neg q \Rightarrow \neg p,\ \text{with (3.12) Double negation} \rangle \\ & b = c \Rightarrow \neg(b\rho c) \\ = & \langle (3.84\text{b})\ \text{Substitution},\ (e = f) \Rightarrow E^z_e \ \equiv\ (e = f) \Rightarrow E^z_f \rangle \\ & b = c \Rightarrow \neg(b\rho b) \\ = & \langle \text{Assume the antecedent, } \rho \text{ is irreflexive, or } (\forall b \mid\ : \neg(b\rho b)) \rangle \\ & b = c \Rightarrow true \\ = & \langle (3.72)\ \text{Right Zero of} \Rightarrow,\ p \Rightarrow true \ \equiv\ true \rangle \\ & true \quad \blacksquare \end{array}$$

Lemma 5: ρ is antisymmetric $\wedge$ ρ is reflexive $\Rightarrow (b\rho c \wedge c\rho b \equiv b = c)$

Proof: The proof is by (4.4) Deduction, *i.e.*, prove the consequent, $b\rho c \wedge c\rho b \equiv b = c$, assuming the conjuncts of the antecedent.

Using (4.7) Mutual implication, the proof of $b\rho c \wedge c\rho b \Rightarrow b = c$ follows.

$$\begin{array}{ll} & b\ \rho\ c \wedge c\ \rho\ b \\ \Rightarrow & \langle \text{Assume the conjunct of the antecedent } \rho \text{ is antisymmetric, or} \\ & \quad (\forall b, c \mid : b\ \rho\ c \wedge c\ \rho\ b \Rightarrow b = c) \rangle \\ & b = c \quad \blacksquare \end{array}$$

The proof of $b = c \Rightarrow b\rho c \wedge c\rho b$ follows.

Proof:

$$\begin{array}{ll} & b = c \Rightarrow b\rho c \wedge c\rho b \\ = & \langle (3.84\text{b})\ \text{Substitution},\ (e = f) \Rightarrow E^z_e \equiv (e = f) \Rightarrow E^z_f \rangle \\ & b = c \Rightarrow b\ \rho\ b \wedge b\ \rho\ b \\ = & \langle (3.38)\ \text{Idempotency of} \wedge,\ p \wedge p \ \equiv\ p \rangle \\ & b = c \Rightarrow b\ \rho\ b \\ = & \langle \text{Assume the conjunct of the antecedent } \rho \text{ is reflexive, or } (\forall b \mid : b\ \rho\ b) \rangle \end{array}$$

$$
\begin{array}{ll}
& b = c \Rightarrow true \\
= & \langle \text{(3.72) Right zero of } \Rightarrow\text{: } p \Rightarrow true \equiv true \rangle \\
& true \quad \blacksquare
\end{array}
$$

2.1.2 Quadrotomy Theorems

Proof of the four quadrotomy theorems using the five lemmas follows.

Theorem, Quadrotomy (a): $b \prec c \;\equiv\; \neg(c \prec b \vee b = c \vee \text{incomp}(b,c))$
Proof:

$$
\begin{array}{ll}
& \neg(c \prec b \vee b = c \vee \text{incomp}(b,c)) \\
= & \langle \text{Lemma 1, } b \prec c \vee b = c \;\equiv\; b \preceq c \rangle \\
& \neg(c \preceq b \vee \text{incomp}(b,c)) \\
= & \langle \text{(14.47.1) Definition, Incomparable, incomp}(b,c) \;\equiv\; \neg(b \preceq c) \wedge \neg(c \preceq b) \rangle \\
& \neg(c \preceq b \vee (\neg(b \preceq c) \wedge \neg(c \preceq b))) \\
= & \langle \text{(3.44b) Absorption, } p \vee (\neg p \wedge q) \;\equiv\; p \vee q \rangle \\
& \neg(c \preceq b \vee \neg(b \preceq c)) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \;\equiv\; \neg p \wedge \neg q\text{, with (3.12) Double negation} \rangle \\
& b \preceq c \wedge \neg(c \preceq b) \\
= & \langle \text{Lemma 1, } b \prec c \vee b = c \;\equiv\; b \preceq c\text{, twice} \rangle \\
& (b \prec c \vee b = c) \;\wedge\; \neg(c \prec b \vee b = c) \\
= & \langle \text{Lemma 2, } (p \vee q) \wedge \neg(q \vee r) \;\equiv\; p \wedge \neg q \wedge \neg r \rangle \\
& b \prec c \wedge \neg(b = c) \wedge \neg(c \prec b) \\
= & \langle \text{Lemma 3, } (p \Rightarrow q) \Rightarrow (p \wedge q \equiv p)\text{, with } p,\, q := b \prec c,\, \neg(c \prec b)\text{. The antecedent} \\
& \quad \text{is true because strict orders are asymmetric.} \rangle \\
& b \prec c \wedge \neg(b = c) \\
= & \langle \text{Lemma 3, with } p,\, q := b \prec c,\, \neg(b = c)\text{. The antecedent, } b \prec c \Rightarrow \neg(b = c)\text{, is} \\
& \quad \text{true by Lemma 4, } \rho \text{ is irreflexive} \Rightarrow (b \,\rho\, c \Rightarrow \neg(b = c))\text{, and the fact that strict} \\
& \quad \text{orders are irreflexive.} \rangle \\
& b \prec c \quad \blacksquare
\end{array}
$$

Theorem, Quadrotomy (b): $c \prec b \;\equiv\; \neg(b \prec c \vee b = c \vee \text{incomp}(b,c))$
Because = and *incomp* are symmetric, Quadrotomy (b) is simply Quadrotomy (a) with $b,\, c := c,\, b$.

Theorem, Quadrotomy (c): $b = c \;\equiv\; \neg(b \prec c \vee c \prec b \vee \text{incomp}(b,c))$
Proof:

$$
\begin{array}{ll}
& \neg(b \prec c \vee c \prec b \vee \text{incomp}(b,c)) \\
= & \langle \text{(3.26) Idempotency of } \vee\text{, } p \;\equiv\; p \vee p \rangle \\
& \neg(b \prec c \vee c \prec b \vee \text{incomp}(b,c) \vee \text{incomp}(b,c))
\end{array}
$$

$=\quad \langle(14.48.2), \neg(b \preceq c) \;\equiv\; c \prec b \vee \text{incomp}(b,c), \text{twice}\rangle$

$\neg(\neg(b \preceq c) \vee \neg(c \preceq b))$

$=\quad \langle(3.47\text{a}) \text{ De Morgan}, \neg(p \wedge q) \;\equiv\; \neg p \vee \neg q, \text{with } (3.12) \text{ Double negation}\rangle$

$b \preceq c \wedge c \preceq b$

$=\quad \langle\text{Lemma 5}, \rho \text{ is antisymmetric } \wedge\; \rho \text{ is reflexive} \Rightarrow (b\,\rho\,c \wedge c\,\rho\,b \;\equiv\; b = c),$ with the fact that partial orders are antisymmetric and reflexive$\rangle$

$b = c$ ∎

Theorem, Quadrotomy (d): $\text{incomp}(b,c) \;\equiv\; \neg(b \prec c \vee c \prec b \vee b = c)$

Proof:

$\neg(b \prec c \vee c \prec b \vee b = c)$

$=\quad \langle(3.26) \text{ Idempotency of } \vee, p \;\equiv\; p \vee p\rangle$

$\neg(b \prec c \vee c \prec b \vee b = c \vee b = c)$

$=\quad \langle\text{Lemma 1}, b \prec c \vee b = c \;\equiv\; b \preceq c \text{ twice}\rangle$

$\neg(b \preceq c \vee c \preceq b)$

$=\quad \langle(3.47\text{b}) \text{ De Morgan}, \neg(p \vee q) \;\equiv\; \neg p \wedge \neg q\rangle$

$\neg(b \preceq c) \wedge \neg(c \preceq b)$

$=\quad \langle(14.47.1) \text{ Definition, Incomparable: incomp}(b,c) \;\equiv\; \neg(b \preceq c) \wedge \neg(c \preceq b)\rangle$

$\text{incomp}(b,c)$ ∎

2.2 Generalized Trichotomy

This subsection presents a three-part proof of the trichotomy of total orders:

Theorem, Trichotomy (a): $b < c \;\equiv\; \neg(c < b \vee b = c)$, where < is a strict total order.

Theorem, Trichotomy (b): $c < b \;\equiv\; \neg(b < c \vee b = c)$, where < is a strict total order.

Theorem, Trichotomy (c): $b = c \;\equiv\; \neg(b < c \vee c < b)$, where < is a strict total order.

These theorems closely mirror the four theorems of Quadrotomy, but with the disjunct $\text{incomp}(b,c)$ removed. The reason for this is Lemma 6, $\text{incomp}(b,c) \;\equiv\; \mathit{false}$, where $\text{incomp}(b,c)$ refers to b and c being incomparable under a total order $\leq$. Lemma 6 follows closely from the definition of a total order, and the proofs of the three parts of Trichotomy follow closely from Lemma 6 and Quadrotomy.

Lemma 6: $\text{incomp}(b,c) \;\equiv\; \mathit{false}$, where $\text{incomp}(b,c)$ refers to b and c being incomparable under a total order $\leq$.

Proof:

$\text{incomp}(b,c)$

$=\quad \langle(14.47.1) \text{ Definition, Incomparable, incomp}(b,c) \;\equiv\; \neg(b \preceq c) \wedge \neg(c \preceq b),$ with $\preceq := \leq\rangle$

$\neg(b \leq c) \wedge \neg(c \leq b)$

$=\quad \langle(3.47\text{b}) \text{ De Morgan}, \neg(p \vee q) \;\equiv\; \neg p \wedge \neg q\rangle$

$\neg(b \leq c \ \vee \ c \leq b)$
$= \quad \langle$(14.50) Definition, Total Order: A partial order $\preceq$ over B is called a total or linear order if $(\forall b, c\,|\!: b \preceq c \ \vee \ c \preceq b)\rangle$
$\neg true$
$= \quad \langle$(3.8) Definition of false$\rangle$
$false$ ∎

Theorem, Trichotomy (a): $b < c \ \equiv \ \neg(c < b \ \vee \ b = c)$, where < is a strict total order.
Proof:

$b < c$
$= \quad \langle$(Theorem, Quadrotomy (a), $b \prec c \ \equiv \ \neg(c \prec b \ \vee \ b = c \ \vee \ \text{incomp}(b,c))$, with $\prec := <\rangle$
$\neg(c < b \ \vee \ b = c \ \vee \ \text{incomp}(b,c))$
$= \quad \langle$Lemma 6, $\text{incomp}(b,c) \ \equiv \ false$ for a total order$\rangle$
$\neg(c < b \ \vee \ b = c \ \vee \ false)$
$= \quad \langle$(3.30) Identity of $\vee$, $p \ \vee \ false \ \equiv \ false\rangle$
$\neg(c < b \ \vee \ b = c)$ ∎

Theorem, Trichotomy (b): $c < b \ \equiv \ \neg(b < c \ \vee \ b = c)$, where < is a strict total order.
Trichotomy (b) is simply Trichotomy (a) with $b, c := c, b$.

Theorem, Trichotomy (c): $b = c \ \equiv \ \neg(b < c \ \vee \ c < b)$, where < is a strict total order.
Proof:

$b = c$
$= \quad \langle$(Theorem, Quadrotomy (c), $b = c \ \equiv \ \neg(b \prec c \ \vee \ c \prec b \ \vee \ \text{incomp}(b,c))$, with $\prec := <\rangle$
$\neg(b < c \ \vee \ c < b \ \vee \ \text{incomp}(b,c))$
$= \quad \langle$Lemma 6, $\text{incomp}(b,c) \ \equiv \ false$ for a total order$\rangle$
$\neg(b < c \ \vee \ c < b \ \vee \ false)$
$= \quad \langle$(3.30) Identity of $\vee$, $p \ \vee \ false \ \equiv \ false\rangle$
$\neg(b < c \ \vee \ c < b)$ ∎

2.3 Gries and Schneider's Trichotomy

There are several different ways to describe a stituation in which several propositions are both mutually exclusive (that is, at most one can be true) and collectively exhaustive (that is, at least one must be true).

Table 1 below shows three different formulations. The first we name "Double Implication" because it can be intuitively read as a series of two-way implications, "if and only

Format	Two Propositions	Three Propositions	Four Propositions
Double Implication	$(p \equiv \neg q) \wedge$ $(q \equiv \neg p)$	$(p \equiv \neg(q \vee r)) \wedge$ $(q \equiv \neg(p \vee r)) \wedge$ $(r \equiv \neg(p \vee q))$	$(p \equiv \neg(q \vee r \vee s)) \wedge$ $(q \equiv \neg(p \vee r \vee s)) \wedge$ $(r \equiv \neg(p \vee q \vee s)) \wedge$ $s \equiv \neg(p \vee q \vee r)$
Equivalence String	$p \not\equiv q$	$(p \equiv q \equiv r) \wedge$ $\neg(p \wedge q \wedge r)$	$(p \not\equiv q \not\equiv r \not\equiv s) \wedge$ $\neg(\neg p \wedge q \wedge r \wedge s) \wedge$ $\neg(p \wedge \neg q \wedge r \wedge s) \wedge$ $\neg(p \wedge q \wedge \neg r \wedge s) \wedge$ $\neg(p \wedge q \wedge r \wedge \neg s)$
Truth Table	$(p \wedge \neg q) \vee$ $(\neg p \wedge q)$	$(p \wedge \neg q \wedge \neg r) \vee$ $(\neg p \wedge q \wedge \neg r) \vee$ $(\neg p \wedge \neg q \wedge r)$	$(p \wedge \neg q \wedge \neg r \wedge \neg s) \vee$ $(\neg p \wedge q \wedge \neg r \wedge \neg s) \vee$ $(\neg p \wedge \neg q \wedge r \wedge \neg s) \vee$ $(\neg p \wedge \neg q \wedge \neg r \wedge s)$

Table 1: Different ways to express mutually exclusive, collectively exhaustive propositions

if" statements. The second we name "Equivalence String" because the core element is a string of equivalences, followed by a string of excluded specific cases. The third we name "Truth Table" because of its strong resemblance to a truth table; each disjunct says that one particular row of the truth table is true.

The first format we use for the Quadrotomy and the Generalized Trichotomy, because with larger numbers of propositions, it has fewer conjuncts than Equivalence String. Furthermore, it has conjuncts rather than disjuncts as Truth Table does, and therefore can be broken up into several theorems rather than one long, unwieldy theorem. The second is the format Gries and Schneider use for their Trichotomy. The third is used in none of the theorems, but is a step in the proof that they are equivalent.

While the expressions as given may look different, all the rows are equivalent because they describe the same truth table, which we have checked with an automated truth-table generator. One can prove that the rows are equivalent with the calculational system, but the proof is tedious, and is not presented here. Lemma 7 follows from Table 1 — in particular, from the equivalence of Double Implication / Three Propositions and Equivalence String / Three Propositions.

Lemma 7:
$(p \equiv q \equiv r) \wedge \neg(p \wedge q \wedge r) \;\equiv\; (p \equiv \neg(q \vee r)) \wedge (q \equiv \neg(p \vee r)) \wedge (r \equiv \neg(p \vee q))$

(15.44) Trichotomy: $(a < b \vee a = b \vee a > b) \wedge \neg(a < b \vee a = b \vee a > b)$
Proof

$$\begin{array}{ll} & (a < b \vee a = b \vee a > b) \wedge \neg(a < b \vee a = b \vee a > b) \\ = & \langle \text{Lemma 7} \rangle \end{array}$$

$$
\begin{aligned}
& (a < b \equiv \neg(b < a \vee a = b)) \wedge (b < a \equiv \neg(a < b \vee a = b)) \\
& \wedge (a = b \equiv \neg(a < b \vee b < a)) \\
= \quad & \langle \text{Theorem, Trichotomy (a), } b < c \;\equiv\; \neg(c < b \vee b = c) \text{, with } < \text{ (generic strict} \\
& \text{total order) replaced by } < \text{ (integers)} \rangle \\
& true \wedge (b < a \equiv \neg(a < b \vee a = b)) \wedge (a = b \equiv \neg(a < b \vee b < a)) \\
= \quad & \langle \text{Theorem, Trichotomy (b): } c < b \;\equiv\; \neg(b < c \vee b = c) \text{, with } <:=< \rangle \\
& true \wedge true \wedge (a = b \equiv \neg(a < b \vee b < a)) \\
= \quad & \langle \text{Theorem, Trichotomy (c): } b = c \;\equiv\; \neg(b < c \vee c < b) \text{, with } <:=< \rangle \\
& true \wedge true \wedge true \\
= \quad & \langle \text{(3.39) Identity of } \wedge,\ p \wedge true \;\equiv\; p \text{, twice} \rangle \\
& true \quad \blacksquare
\end{aligned}
$$

3 Conclusion

In general, elementsx and y of a partial order satisfy exactly one of (a) $x \prec y$, (b) $x = y$, (c) $y \prec x$, or (d) incomp(x,y), where incomp is defined as incomp$(x,y) \;\equiv\; \neg(x \preceq y) \wedge \neg(y \preceq x)$ and $\prec$ is the reflexive reduction of $\preceq$. This is a generalization of the trichotomy of the at most relation $\leq$, which states that integers a and b satisfy exactly one of (a) $a < b$, (b) $a = b$, or (c) $a > b$.

This paper demonstrates the power of $\mathcal{E}$ for proofs in discrete mathematics, in spite of its lack of wide adoption. The calculational system disambiguates every step of each proof while still allowing for easy conversion between intuitive understanding and formalization.