A Quadrotomy for Partial Orders

Alex Prieger
J. Stanley Warford
Pepperdine University
Malibu, CA 90263

Abstract

This paper uses a calculational system to formally prove a quadrotomy for partial orders $\preceq$, which states that any element x of the partial order must be related to any other element y by exactly one of (a) $x \prec y$, (b) $x = y$, (c) $y \prec x$, or (d) $\mathsf{incomp}(x,y)$, where $\prec$ is the reflexive reduction of $\preceq$ and $\mathsf{incomp}(x,y)$ is defined as $\mathsf{incomp}(x,y) \equiv \neg(x \preceq y) \wedge \neg(y \preceq x)$. The calculational system, developed by Dijkstra and Scholten and extended by Gries and Schneider in their text A Logical Approach to Discrete Math, is based on only four inference rules – Substitution, Leibniz, Equanimity, and Transitivity. Inference rules in the older Hilbert-style systems, notably modus ponens, appear as theorems in this calculational deductive system, which is used to prove algorithm correctness in computer science. The theorem presented in this paper is a generalization of the trichotomy theorem for integers proved by Gries and Schneider, which states that any integer n must be related to any other integer m by exactly one of (a) $n < m$, (b) $n = m$, or (c) $m < n$.

1 Introduction

1.1 Background

Propositional calculus is a formal system of logic based on the unary operator negation $\neg$, the binary operators conjunction $\wedge$, disjunction $\vee$, implies $\Rightarrow$ (also written $\rightarrow$), and equivalence $\equiv$ (also written $\leftrightarrow$), variables (lowercase letters p, q, $\ldots$), and the constants true and false. Hilbert-style logic systems, $\mathcal{H}$, are the deductive logic systems traditionally used in mathematics to describe the propositional calculus. A key feature of such systems is their multiplicity of inference rules and the importance of modus ponens as one of them.

In the late 1980's, Dijkstra and Scholten [?], and Feijen [?] developed a method of proving program correctness with a new logic based on an equational style. In contrast to $\mathcal{H}$ systems, $\mathcal{E}$ has only four inference rules – Substitution, Leibniz, Equanimity, and Transitivity. In $\mathcal{E}$, modus ponens plays a secondary role. It is not an inference rule, nor is it assumed as an axiom, but instead is proved as a theorem from the axioms using the inference rules.

Gries and Schneider [?, ?] show that $\mathcal{E}$, also known as a calculational system, has several advantages over traditional logic systems. The primary advantage of $\mathcal{E}$ over $\mathcal{H}$ systems is that the calculational system has only four proof rules, with inference rule Leibniz as the primary one. Roughly speaking, Leibniz is "substituting equals for equals," hence the moniker equational deductive system. In contrast, $\mathcal{H}$ systems rely on a more extensive set of inference rules.

In 1994, Gries and Schneider published A Logical Approach to Discrete Math (LADM) [?], in which they first develop $\mathcal{E}$ for propositional and predicate calculus, and then extend it to a theory of sets, a theory of sequences, relations and functions, a theory of integers, recurrence relations, modern algebra, and a theory of graphs. Using calculational logic as a tool, LADM brings all the advantages of $\mathcal{E}$ to these additional knowledge domains. The treatment is in marked contrast to the traditional one exemplified by the classic undergraduate text by Rosen [?].

1.2 Partial Orders

Paragraph or section on this application of $\mathcal{E}$ to the problem. Reference to LADM trichotomy.

2 Results

Section 2.1 formally proves a general quadrotomy for partial orders. Section 2.2 applies the general quadrotomy to prove the special case of a trichotomy for all total orders. In the solutions manual for LADM, Gries and Schneider prove using the properties of integers a trychotomy for $\leq$, which is an example of a total order. Section 2.3 applies our trichotomy for all total orders to prove Gries and Schneider's trichotomy for the total order $\leq$ without using the properties of integers.

2.1 Quadrotomy

To prove that two elements a and b are related by exactly one of (a) $x \prec y$, (b) $x = y$, (c) $y \prec x$, or (d) $\mathsf{incomp}(x,y)$, we prove the four-part theorem:

Theorem, Quadrotomy (a): $b \prec c \ \equiv\ \neg(c \prec b \vee b = c \vee \mathsf{incomp}(b,c))$

Theorem, Quadrotomy (b): $c \prec b \ \equiv\ \neg(b \prec c \vee b = c \vee \mathsf{incomp}(b,c))$

Theorem, Quadrotomy (c): $b = c \ \equiv\ \neg(b \prec c \vee c \prec b \vee \mathsf{incomp}(b,c))$

Theorem, Quadrotomy (d): $\mathsf{incomp}(b,c) \ \equiv\ \neg(b \prec c \vee c \prec b \vee b = c)$

By mutual implication $p \equiv q$ is equivalent to $p \Rightarrow q$ and $q \Rightarrow p$. The left-to-right implication for Quadrotomy (a) asserts that if $b \prec c$ holds, then none of the other three parts of the quadrotomy holds, and similarly for theorems (b), (c), and (d). Consequently, at most one of the four parts of the quadrotomy is true. The right-to-left implication for Quadrotomy (a) asserts that if none of the other three parts of the quadrotomy holds, then $b \prec c$ holds, and similarly for theorems (b), (c), and (d). Consequently, at least one of the four parts of the quadrotomy is true. Therefore, exactly one of the four parts of the quadrotomy is true.

2.1.1 Lemmas

To prove this four-part theorem, we first prove several lemmas. The first lemma,

$$b \prec c \ \vee\ b = c \ \equiv\ b \preceq c$$

where $\prec$ is the reflexive reduction of $\preceq$, states that comparing two elements under a partial order is equivalent to comparing them under a strict order or equality. This intuitive fact is used in several points during the proofs, and follows closely from the definition of a strict order.

The second lemma,

$$(p \vee q) \wedge \neg(q \vee r) \ \equiv\ p \wedge \neg q \wedge \neg r$$

is based solely on propositional calculus. It is used only in the proof of Quadrotomy (a), but is introduced as a lemma to avoid cluttering that proof with simple boolean steps.

The third lemma,

$$(p \Rightarrow q) \Rightarrow (p \wedge q \ \equiv\ p)$$

is an implication. In fact, the consequent is equivalent to the antecedent, a step which is used in the proof, but we state it here in the weaker form of an implication. This is because the calculational system allows us to use the consequent of an implication (here, $p \wedge q \equiv p$) as a theorem as long as the antecedent (here, $p \Rightarrow q$) is true.

The fourth lemma,

$$\rho \text{ is irreflexive} \Rightarrow (b \rho c \Rightarrow \neg(b = c))$$

follows closely from the definition of irreflexivity, and is used much like the third lemma: when the antecedent is true, the consequent can be used as a theorem.

The fifth lemma,

$$\rho \text{ is antisymmetric } \wedge\ \rho \text{ is reflexive} \Rightarrow (b \rho c \wedge c \rho b \ \equiv\ b = c)$$

is an important property of partial orders: if $b \preceq c$ and $c \preceq b$, then $b = c$, because no element can both precede and follow any other element. Like the third and fourth lemmas, when the conjuncts of the antecedent of this theorem are true, the consequent can be used as a theorem.

Lemma 1: $b \prec c \ \vee\ b = c \ \equiv\ b \preceq c$, where $\prec$ is the reflexive reduction of $\preceq$.
Proof:

$$
\begin{array}{ll}
 & b \prec c \lor b = c \\
= & \langle (14.15.4) \text{ Notation, } \langle b,c\rangle \in \rho \text{ and } b\,\rho\,c \text{ are interchangeable notations.}\rangle \\
 & \langle b,c\rangle \in \prec \lor b = c \\
= & \langle (14.15.3) \text{ Identity lemma, } \langle x,y\rangle \in i_B \equiv x = y\rangle \\
 & \langle b,c\rangle \in \prec \lor \langle b,c\rangle \in i_B \\
= & \langle (11.20) \text{ Axiom, Union, } v \in S \cup T \equiv v \in S \lor v \in T\rangle \\
 & \langle b,c\rangle \in \prec \cup\, i_B \\
= & \langle (14.49b) \text{ If } \rho \text{ is a quasi order over a set B, then } \rho \cup i_B \text{ is a partial order}\rangle \\
 & \langle b,c\rangle \in \preceq \\
= & \langle (14.15.4) \text{ Notation}\rangle \\
 & b \preceq c \quad \blacksquare
\end{array}
$$

Lemma 2: $(p \lor q) \land \neg(q \lor r) \equiv p \land \neg q \land \neg r$
Proof:

$$
\begin{array}{ll}
 & (p \lor q) \land \neg(q \lor r) \\
= & \langle (3.47b) \text{ De Morgan, } \neg(p \lor q) \equiv \neg p \land \neg q\rangle \\
 & (p \lor q) \land \neg q \land \neg r \\
= & \langle (3.44a) \text{ Absorption, } p \land (\neg p \lor q) \equiv (p \land q) \text{, with } p, q := \neg q, p \text{ and with} \\
 & \quad (3.12) \text{ Double negation}\rangle \\
 & p \land \neg q \land \neg r \quad \blacksquare
\end{array}
$$

Lemma 3: $(p \Rightarrow q) \Rightarrow (p \land q \equiv p)$
Proof:

$$
\begin{array}{ll}
 & p \land q \equiv p \\
= & \langle (3.60) \text{ Implication, } p \Rightarrow q \equiv p \land q \equiv p\rangle \\
 & p \Rightarrow q \\
\Leftarrow & \langle (3.71) \text{ Reflexivity of } \Rightarrow, p \Rightarrow p\rangle \\
 & p \Rightarrow q \quad \blacksquare
\end{array}
$$

Lemma 4: ρ is irreflexive $\Rightarrow (b\rho c \Rightarrow \neg(b = c))$
Proof: The proof is by (4.4) Deduction.

$$
\begin{array}{ll}
 & b\rho c \Rightarrow \neg(b = c) \\
= & \langle (3.61) \text{ Contrapositive, } p \Rightarrow q \equiv \neg q \Rightarrow \neg p \text{, with (3.12) Double negation}\rangle \\
 & b = c \Rightarrow \neg(b\rho c) \\
= & \langle (3.84b) \text{ Substitution, } (e = f) \Rightarrow E^z_e \equiv (e = f) \Rightarrow E^z_f\rangle \\
 & b = c \Rightarrow \neg(b\rho b) \\
= & \langle \text{Assume the antecedent, } \rho \text{ is irreflexive, or } (\forall b \mid : \neg(b\rho b))\rangle \\
 & b = c \Rightarrow \mathit{true}
\end{array}
$$

$= \quad \langle (3.72) \text{ Right Zero of } \Rightarrow,\ p \Rightarrow \mathit{true} \equiv \mathit{true} \rangle$
$\quad \mathit{true} \quad \blacksquare$

Lemma 5: ρ is antisymmetric $\wedge$ ρ is reflexive $\Rightarrow (b\rho c \wedge c\rho b \equiv b = c)$
Proof: The proof is by (4.4) Deduction, i.e., prove the consequent, $b\rho c \wedge c\rho b \equiv b = c$, assuming the conjuncts of the antecedent.
Using (4.7) Mutual implication, the proof of $b\rho c \wedge c\rho b \Rightarrow b = c$ follows.

$\quad b\ \rho\ c \wedge c\ \rho\ b$
$\Rightarrow \quad \langle$Assume the conjunct of the antecedent ρ is antisymmetric, or $(\forall b, c \mid : b\ \rho\ c \wedge c\ \rho\ b \Rightarrow b = c)\rangle$
$\quad b = c \quad \blacksquare$

The proof of $b = c \Rightarrow b\rho c \wedge c\rho b$ is by (4.4) Deduction.
Proof:

$\quad b\ \rho\ c \wedge c\ \rho\ b$
$= \quad \langle$Assume the antecedent, $b = c\rangle$
$\quad b\ \rho\ b \wedge b\ \rho\ b$
$= \quad \langle (3.38) \text{ Idempotency of } \wedge,\ p \wedge p \equiv p \rangle$
$\quad b\ \rho\ b$
$= \quad \langle$Assume the conjunct of the antecedent ρ is reflexive, or $(\forall b \mid : b\ \rho\ b)\rangle$
$\quad \mathit{true} \quad \blacksquare$

2.1.2 quadrotomy theorems

Proof of the four quadrotomy theorems using the five lemmas follows.

Theorem, Quadrotomy (a): $b \prec c \equiv \neg(c \prec b \vee b = c \vee \mathsf{incomp}(b,c))$
Proof:

$\quad \neg(c \prec b \vee b = c \vee \mathsf{incomp}(b,c))$
$= \quad \langle \text{Lemma 1, } b \prec c \vee b = c \equiv b \preceq c \rangle$
$\quad \neg(c \preceq b \vee \mathsf{incomp}(b,c))$
$= \quad \langle (14.47.1) \text{ Definition, Incomparable, } \mathsf{incomp}(b,c) \equiv \neg(b \preceq c) \wedge \neg(c \preceq b) \rangle$
$\quad \neg(c \preceq b \vee (\neg(b \preceq c) \wedge \neg(c \preceq b)))$
$= \quad \langle (3.44b) \text{ Absorption, } p \vee (\neg p \wedge q) \equiv p \vee q \rangle$
$\quad \neg(c \preceq b \vee \neg(b \preceq c))$
$= \quad \langle (3.47b) \text{ De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q, \text{ with (3.12) Double negation} \rangle$
$\quad b \preceq c \wedge \neg(c \preceq b)$
$= \quad \langle \text{Lemma 1, } b \prec c \vee b = c \equiv b \preceq c, \text{ twice} \rangle$

$$
\begin{array}{ll}
& (b \prec c \,\vee\, b = c) \;\wedge\; \neg(c \prec b \,\vee\, b = c) \\
= & \quad \langle \text{Lemma 2, } (p \vee q) \wedge \neg(q \vee r) \;\equiv\; p \wedge \neg q \wedge \neg r \rangle \\
& b \prec c \,\wedge\, \neg(b = c) \,\wedge\, \neg(c \prec b) \\
= & \quad \langle \text{Lemma 3, } (p \Rightarrow q) \Rightarrow (p \wedge q \equiv p) \text{, with } p, q := b \prec c, \neg(c \prec b) \text{. The antecedent} \\
& \quad\;\; \text{is true because strict orders are asymmetric.} \rangle \\
& b \prec c \,\wedge\, \neg(b = c) \\
= & \quad \langle \text{Lemma 3, with } p, q := b \prec c, \neg(b = c) \text{. The antecedent, } b \prec c \Rightarrow \neg(b = c) \text{, is} \\
& \quad\;\; \text{true by Lemma 4, } \rho \text{ is irreflexive} \Rightarrow (b \,\rho\, c \Rightarrow \neg(b = c)) \text{, and the fact that strict} \\
& \quad\;\; \text{orders are irreflexive.} \rangle \\
& b \prec c \quad \blacksquare
\end{array}
$$

Theorem, Quadrotomy (b): $c \prec b \;\equiv\; \neg(b \prec c \vee b = c \vee \mathsf{incomp}(b,c))$

Because $=$ and incomp are symmetric, Quadrotomy (b) is simply Quadrotomy (a) with $b, c := c, b$.

Theorem, Quadrotomy (c): $b = c \;\equiv\; \neg(b \prec c \vee c \prec b \vee \mathsf{incomp}(b,c))$

Proof:

$$
\begin{array}{ll}
& \neg(b \prec c \,\vee\, c \prec b \,\vee\, \mathsf{incomp}(b,c)) \\
= & \quad \langle \text{(3.26) Idempotency of } \vee,\; p \;\equiv\; p \vee p \rangle \\
& \neg(b \prec c \,\vee\, c \prec b \,\vee\, \mathsf{incomp}(b,c) \,\vee\, \mathsf{incomp}(b,c)) \\
= & \quad \langle \text{(14.48.2), } \neg(b \preceq c) \;\equiv\; c \prec b \vee \mathsf{incomp}(b,c) \text{, twice} \rangle \\
& \neg(\neg(b \preceq c) \,\vee\, \neg(c \preceq b)) \\
= & \quad \langle \text{(3.47a) De Morgan, } \neg(p \wedge q) \;\equiv\; \neg p \vee \neg q \text{, with (3.12) Double negation} \rangle \\
& b \preceq c \,\wedge\, c \preceq b \\
= & \quad \langle \text{Lemma 5, } \rho \text{ is antisymmetric } \wedge\; \rho \text{ is reflexive} \Rightarrow (b \,\rho\, c \wedge c \,\rho\, b \;\equiv\; b = c), \\
& \quad\;\; \text{with the fact that partial orders are antisymmetric and reflexive} \rangle \\
& b = c \quad \blacksquare
\end{array}
$$

Theorem, Quadrotomy (d): $\mathsf{incomp}(b,c) \;\equiv\; \neg(b \prec c \vee c \prec b \vee b = c)$

Proof:

$$
\begin{array}{ll}
& \neg(b \prec c \,\vee\, c \prec b \,\vee\, b = c) \\
= & \quad \langle \text{(3.26) Idempotency of } \vee,\; p \;\equiv\; p \vee p \rangle \\
& \neg(b \prec c \,\vee\, c \prec b \,\vee\, b = c \,\vee\, b = c) \\
= & \quad \langle \text{Lemma 1, } b \prec c \vee b = c \;\equiv\; b \preceq c \text{ twice} \rangle \\
& \neg(b \preceq c \,\vee\, c \preceq b) \\
= & \quad \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
& \neg(b \preceq c) \,\wedge\, \neg(c \preceq b) \\
= & \quad \langle \text{(14.47.1) Definition, Incomparable: } \mathsf{incomp}(b,c) \;\equiv\; \neg(b \preceq c) \wedge \neg(c \preceq b) \rangle \\
& \mathsf{incomp}(b,c) \quad \blacksquare
\end{array}
$$

2.2 Generalized Trichotomy

This subsection presents a three-part proof of the trichotomy of total orders:

Theorem, Trichotomy (a): $b < c \ \equiv\ \neg(c < b \lor b = c)$, where $<$ is a strict total order.

Theorem, Trichotomy (b): $c < b \ \equiv\ \neg(b < c \lor b = c)$, where $<$ is a strict total order.

Theorem, Trichotomy (c): $b = c \ \equiv\ \neg(b < c \lor c < b)$, where $<$ is a strict total order.

It will be noticed that these theorems closely mirror the four theorems of Quadrotomy, but with the disjunct $\mathsf{incomp}(b,c)$ removed. The reason for this is Lemma 6, $\mathsf{incomp}(b,c) \equiv \mathit{false}$, where $\mathsf{incomp}(b,c)$ refers to b and c being incomparable under a total order $\leq$. Lemma 6 follows closely from the definition of a total order, and the proofs of the three parts of Trichotomy follow closely from Lemma 6 and Quadrotomy.

Lemma 6: $\mathsf{incomp}(b,c) \ \equiv\ \mathit{false}$, where $\mathsf{incomp}(b,c)$ refers to b and c being incomparable under a total order $\leq$.

Proof:

$\mathsf{incomp}(b,c)$
$=$ ⟨(14.47.1) Definition, Incomparable, $\mathsf{incomp}(b,c) \ \equiv\ \neg(b \preceq c) \land \neg(c \preceq b)$, with $\preceq := \leq$⟩
$\neg(b \leq c) \land \neg(c \leq b)$
$=$ ⟨(3.47b) De Morgan, $\neg(p \lor q) \equiv \neg p \land \neg q$⟩
$\neg(b \leq c \lor c \leq b)$
$=$ ⟨(14.50) Definition, Total Order: A partial order $\preceq$ over B is called a total or linear order if $(\forall b,c \mid : b \preceq c \lor c \preceq b)$⟩
$\neg \mathit{true}$
$=$ ⟨(3.8) Definition of false⟩
false ∎

Theorem, Trichotomy (a): $b < c \ \equiv\ \neg(c < b \lor b = c)$, where $<$ is a strict total order.

Proof:

$b < c$
$=$ ⟨(Theorem, Quadrotomy (a), $b \prec c \ \equiv\ \neg(c \prec b \lor b = c \lor \mathsf{incomp}(b,c))$, with $\prec := <$⟩
$\neg(c < b \lor b = c \lor \mathsf{incomp}(b,c))$
$=$ ⟨Lemma 6, $\mathsf{incomp}(b,c) \equiv \mathit{false}$ for a total order⟩
$\neg(c < b \lor b = c \lor \mathit{false})$
$=$ ⟨(3.30) Identity of $\lor$, $p \lor \mathit{false} \equiv \mathit{false}$⟩
$\neg(c < b \lor b = c)$ ∎

Theorem, Trichotomy (b): $c < b \;\equiv\; \neg(b < c \vee b = c)$, where $<$ is a strict total order.

Trichotomy (b) is simply Trichotomy (a) with $b, c := c, b$.

Theorem, Trichotomy (c): $b = c \;\equiv\; \neg(b < c \vee c < b)$, where $<$ is a strict total order.

Proof:

$$\begin{array}{ll}
 & b = c \\
= & \langle \text{(Theorem, Quadrotomy (c), } b = c \;\equiv\; \neg(b \prec c \vee c \prec b \vee \mathbf{incomp}(b,c)) \text{), with } \\
 & \quad \prec := < \rangle \\
 & \neg(b < c \vee c < b \vee \mathbf{incomp}(b,c)) \\
= & \langle \text{Lemma 6, } \mathbf{incomp}(b,c) \equiv \mathit{false} \text{ for a total order} \rangle \\
 & \neg(b < c \vee c < b \vee \mathit{false}) \\
= & \langle \text{(3.30) Identity of } \vee,\ p \vee \mathit{false} \equiv \mathit{false} \rangle \\
 & \neg(b < c \vee c < b) \quad \blacksquare
\end{array}$$

2.3 Gries and Schneider's Trichotomy

It is not easy to concisely describe a stituation in which several propositions are both mutually exclusive (that is, at most one can be true) and collectively exhaustive (that is, at least one must be true) using only binary boolean operators. There are several different ways to do so, and differences in the ways chosen account for most of the difficulty in showing that Gries and Schneider's Trichotomy is a special case of our Quadrotomy.

Figure 1 below shows three different ways to do so. The first we name "Double Implication," since it can be intuitively read as a series of two-way implications, "if and only if" statements. This is the format we use for the Quadrotomy and the Generalized Trichotomy. We do so because with larger numbers of propositions, it has fewer conjuncts than Equivalence String. In addition, it has conjuncts rather than disjuncts as Truth Table does, and therefore can be broken up into several theorems rather than one long, unwieldy theorem.

The second we name "Equivalence String," since the core element is a string of equivalences, followed by a string of excluded specific cases. This is the format Gries and Schneider use for their Trichotomy.

The third we name "Truth Table" because of its strong resemblance to a truth table; each disjunct says that one particular row of the truth table is true. This is used in none of the theorems, but is an important step in the proof that they are equivalent.

It is important to recognize that while the expressions as given may look significantly different, all of the rows are equivalent because they describe the same truth table, which we have checked with an automated truth-table generator. This can also be proven with the calculational system, but the proof is tedious, and will not be

Format	2 Propositions	3 Propositions	4 Propositions
Double Implication	$(p \equiv \neg q) \wedge$ $(q \equiv \neg p)$	$(p \equiv \neg(q \vee r)) \wedge$ $(q \equiv \neg(p \vee r)) \wedge$ $(r \equiv \neg(p \vee q))$	$(p \equiv \neg(q \vee r \vee s)) \wedge$ $(q \equiv \neg(p \vee r \vee s)) \wedge$ $(r \equiv \neg(p \vee q \vee s)) \wedge$ $s \equiv \neg(p \vee q \vee r)$
Equivalence String	$p \not\equiv q$	$(p \equiv q \equiv r) \wedge$ $\neg(p \wedge q \wedge r)$	$(p \not\equiv q \not\equiv r \not\equiv s) \wedge$ $\neg(\neg p \wedge q \wedge r \wedge s) \wedge$ $\neg(p \wedge \neg q \wedge r \wedge s) \wedge$ $\neg(p \wedge q \wedge \neg r \wedge s) \wedge$ $\neg(p \wedge q \wedge r \wedge \neg s)$
Truth Table	$(p \wedge \neg q) \vee$ $(\neg p \wedge q)$	$(p \wedge \neg q \wedge \neg r) \vee$ $(\neg p \wedge q \wedge \neg r) \vee$ $(\neg p \wedge \neg q \wedge r)$	$(p \wedge \neg q \wedge \neg r \wedge \neg s) \vee$ $(\neg p \wedge q \wedge \neg r \wedge \neg s) \vee$ $(\neg p \wedge \neg q \wedge r \wedge \neg s) \vee$ $(\neg p \wedge \neg q \wedge \neg r \wedge s)$

Table 1: Different ways to express mutually exclusive, collectively exhaustive propositions

presented here. Lemma 7 follows from this table—in particular, from the equivalence of the second and third elements of the third column.

Lemma 7:
$(p \equiv q \equiv r) \wedge \neg(p \wedge q \wedge r) \;\equiv\; (p \equiv \neg(q \vee r)) \wedge (q \equiv \neg(p \vee r)) \wedge (r \equiv \neg(p \vee q))$
Lemma 7 is stated without proof, since as mentioned the proof is tedious.

(15.44) Trichotomy: $(a < b \vee a = b \vee a > b) \wedge \neg(a < b \vee a = b \vee a > b)$
Proof

$$
\begin{array}{ll}
 & (a < b \vee a = b \vee a > b) \wedge \neg(a < b \vee a = b \vee a > b) \\
= & \langle \text{Lemma 7} \rangle \\
 & (a < b \equiv \neg(b < a \vee a = b)) \wedge (b < a \equiv \neg(a < b \vee a = b)) \\
 & \wedge\, (a = b \equiv \neg(a < b \vee b < a)) \\
= & \langle \text{Theorem, Trichotomy (a), } b < c \equiv \neg(c < b \vee b = c)\text{, with } < \text{ (generic strict} \\
 & \quad \text{total order) replaced by } < \text{ (integers)} \rangle \\
 & \mathit{true} \wedge (b < a \equiv \neg(a < b \vee a = b)) \wedge (a = b \equiv \neg(a < b \vee b < a)) \\
= & \langle \text{Theorem, Trichotomy (b): } c < b \equiv \neg(b < c \vee b = c)\text{, with } {<}{:=}{<} \rangle \\
 & \mathit{true} \wedge \mathit{true} \wedge (a = b \equiv \neg(a < b \vee b < a)) \\
= & \langle \text{Theorem, Trichotomy (c): } b = c \equiv \neg(b < c \vee c < b)\text{, with } {<}{:=}{<} \rangle \\
 & \mathit{true} \wedge \mathit{true} \wedge \mathit{true} \\
= & \langle \text{(3.39) Identity of } \wedge, \; p \wedge \mathit{true} \equiv p\text{, twice} \rangle \\
 & \mathit{true} \quad \blacksquare
\end{array}
$$

3 Conclusion

Something