A Quadrotomy for Partial Orders

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Abstract

This paper uses a calculational system to formally prove a quadrotomy for partial orders $\preceq$, which states that any element x of the partial order must be related to any other element y by exactly one of (a) $x \prec y$, (b) $x = y$, (c) $y \prec x$, or (d) incomp(x, y), where $\prec$ is the reflexive reduction of $\preceq$ and incomp(x, y) is defined as incomp$(x, y) \equiv \neg(x \preceq y) \wedge \neg(y \preceq x)$. The calculational system, developed by Dijkstra and Scholten and extended by Gries and Schneider in their text *A Logical Approach to Discrete Math*, is based on only four inference rules – Substitution, Leibniz, Equanimity, and Transitivity. Inference rules in the older Hilbert-style systems, notably modus ponens, appear as theorems in this calculational deductive system, which is used to prove algorithm correctness in computer science. The theorem presented in this paper is a generalization of the trichotomy theorem for integers proved by Gries and Schneider, which states that any integer n must be related to any other integer m by exactly one of (a) $n < m$, (b) $n = m$, or (c) $m < n$.

1 Introduction

1.1 Background

Propositional calculus is a formal system of logic based on the unary operator negation $\neg$, the binary operators conjunction $\wedge$, disjunction $\vee$, implies $\Rightarrow$ (also written $\rightarrow$), and equivalence $\equiv$ (also written $\leftrightarrow$), variables (lowercase letters p, q, $\ldots$), and the constants *true* and *false*. Hilbert-style logic systems, $\mathcal{H}$, are the deductive logic systems traditionally used in mathematics to describe the propositional calculus. A key feature of such systems is their multiplicity of inference rules and the importance of modus ponens as one of them.

In the late 1980's, Dijkstra and Scholten [1], and Feijen [2] developed a method of proving program correctness with a new logic based on an equational style. In contrast to $\mathcal{H}$ systems, $\mathcal{E}$ has only four inference rules – Substitution, Leibniz, Equanimity, and Transitivity. In $\mathcal{E}$, modus ponens plays a secondary role. It is not an inference rule, nor is it assumed as an axiom, but instead is proved as a theorem from the axioms using the inference rules.

Gries and Schneider [3,5] show that $\mathcal{E}$, also known as a *calculational* system, has several advantages over traditional logic systems. The primary advantage of $\mathcal{E}$ over $\mathcal{H}$ systems is that the calculational system has only four proof rules, with inference rule Leibniz as the primary one. Roughly speaking, Leibniz is "substituting equals for equals," hence the moniker *equational* deductive system. In contrast, $\mathcal{H}$ systems rely on a more extensive set of inference rules.

In 1994, Gries and Schneider published *A Logical Approach to Discrete Math* (LADM) [4], in which they first develop $\mathcal{E}$ for propositional and predicate calculus, and then extend it to a theory of sets, a theory of sequences, relations and functions, a theory of integers, recurrence relations, modern algebra, and a theory of graphs. Using calculational logic as a tool, LADM brings all the advantages of $\mathcal{E}$ to these additional knowledge domains. The treatment is in marked contrast to the traditional one exemplified by the classic undergraduate text by Rosen [6].

1.2 Partial Orders

Paragraph or section on this application of $\mathcal{E}$ to the problem. Reference to LADM trichotomy.

2 Results

2.1 Quadrotomy

Lemma 1: $b \prec c \lor b = c \;\equiv\; b \preceq c$, where $\prec$ is the reflexive reduction of $\preceq$.

Proof:

$$
\begin{array}{ll}
 & b \prec c \lor b = c \\
= & \langle \text{(14.15.4) Notation, } \langle b,c\rangle \in \rho \text{ and } b\,\rho\,c \text{ are interchangeable notations.} \rangle \\
 & \langle b,c\rangle \in \prec\; \lor b = c \\
= & \langle \text{(14.15.3) Identity lemma, } \langle x,y\rangle \in i_B \;\equiv\; x = y \rangle \\
 & \langle b,c\rangle \in \prec\; \lor \langle b,c\rangle \in i_B \\
= & \langle \text{(11.20) Axiom, Union, } v \in S \cup T \;\equiv\; v \in S \lor v \in T \rangle \\
 & \langle b,c\rangle \in \prec \cup\, i_B \\
= & \langle \text{(14.49b) If } \rho \text{ is a quasi order over a set B, then } \rho \cup i_B \text{ is a partial order} \rangle \\
 & \langle b,c\rangle \in \preceq \\
= & \langle \text{(14.15.4) Notation} \rangle \\
 & b \preceq c \quad \blacksquare
\end{array}
$$

Lemma 2: $(p \lor q) \land \neg(q \lor r) \;\equiv\; p \land \neg q \land \neg r$

Proof:

$$
\begin{array}{ll}
 & (p \lor q) \land \neg(q \lor r) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \lor q) \;\equiv\; \neg p \land \neg q \rangle
\end{array}
$$

$(p \vee q) \wedge \neg q \wedge \neg r$
$=$ $\langle$(3.44a) Absorption, $p \wedge (\neg p \vee q) \equiv (p \wedge q)$, with $p,\ q := \neg q,\ p$ and with (3.12) Double negation$\rangle$
$p \wedge \neg q \wedge \neg r$ ∎

Lemma 3: $(p \Rightarrow q) \Rightarrow (p \wedge q \equiv p)$
Proof:

$p \wedge q \equiv p$
$=$ $\langle$(3.60) Implication, $p \Rightarrow q \equiv p \wedge q \equiv p\rangle$
$p \Rightarrow q$
$\Leftarrow$ $\langle$(3.71) Reflexivity of $\Rightarrow$, $p \Rightarrow p\rangle$
$p \Rightarrow q$ ∎

Lemma 4: ρ is irreflexive $\Rightarrow (b\rho c \Rightarrow \neg(b = c))$
Proof: The proof is by (4.4) Deduction.

$b\rho c \Rightarrow \neg(b = c)$
$=$ $\langle$(3.61) Contrapositive, $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$, with (3.12) Double negation$\rangle$
$b = c \Rightarrow \neg(b\rho c)$
$=$ $\langle$(3.84b) Substitution, $(e = f) \Rightarrow E^z_e \equiv (e = f) \Rightarrow E^z_f\rangle$
$b = c \Rightarrow \neg(b\rho b)$
$=$ $\langle$Assume the antecedent, ρ is irreflexive, or $(\forall b \mid\ : \neg(b\rho b))\rangle$
$b = c \Rightarrow true$
$=$ $\langle$(3.72) Right Zero of $\Rightarrow$, $p \Rightarrow true \equiv true\rangle$
$true$ ∎

Lemma 5: ρ is antisymmetric $\wedge$ ρ is reflexive $\Rightarrow (b\rho c \wedge c\rho b \equiv b = c)$
Proof: The proof is by (4.4) Deduction, *i.e.*, prove the consequent, $b\rho c \wedge c\rho b \equiv b = c$, assuming the conjuncts of the antecedent.
Using (4.7) Mutual implication, the proof of $b\rho c \wedge c\rho b \Rightarrow b = c$ follows.

$b\ \rho\ c \wedge c\ \rho\ b$
$\Rightarrow$ $\langle$Assume the conjunct of the antecedent ρ is antisymmetric, or $(\forall b, c \mid: b\ \rho\ c \wedge c\ \rho\ b \Rightarrow b = c)\rangle$
$b = c$ ∎

The proof of $b = c \Rightarrow b\rho c \wedge c\rho b$ is by (4.4) Deduction.
Proof:

$$
\begin{aligned}
& b\,\rho\,c \;\wedge\; c\,\rho\,b \\
= \quad & \langle \text{Assume the antecedent, } b = c \rangle \\
& b\,\rho\,b \;\wedge\; b\,\rho\,b \\
= \quad & \langle \text{(3.38) Idempotency of } \wedge,\ p \wedge p \;\equiv\; p \rangle \\
& b\,\rho\,b \\
= \quad & \langle \text{Assume the conjunct of the antecedent } \rho \text{ is reflexive, or } (\forall b \,|\!: b\,\rho\,b) \rangle \\
& \mathit{true} \quad \blacksquare
\end{aligned}
$$

Theorem, Quadrotomy (a): $b \prec c \;\equiv\; \neg(c \prec b \vee b = c \vee \mathrm{incomp}(b,c))$

Proof:

$$
\begin{aligned}
& \neg(c \prec b \vee b = c \vee \mathrm{incomp}(b,c)) \\
= \quad & \langle \text{Lemma 1, } b \prec c \vee b = c \;\equiv\; b \preceq c \rangle \\
& \neg(c \preceq b \vee \mathrm{incomp}(b,c)) \\
= \quad & \langle \text{(14.47.1) Definition, Incomparable, } \mathrm{incomp}(b,c) \;\equiv\; \neg(b \preceq c) \wedge \neg(c \preceq b) \rangle \\
& \neg(c \preceq b \vee (\neg(b \preceq c) \wedge \neg(c \preceq b))) \\
= \quad & \langle \text{(3.44b) Absorption, } p \vee (\neg p \wedge q) \;\equiv\; p \vee q \rangle \\
& \neg(c \preceq b \vee \neg(b \preceq c)) \\
= \quad & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \text{, with (3.12) Double negation} \rangle \\
& b \preceq c \wedge \neg(c \preceq b) \\
= \quad & \langle \text{Lemma 1, twice} \rangle \\
& (b \prec c \vee b = c) \;\wedge\; \neg(c \prec b \vee b = c) \\
= \quad & \langle \text{Lemma 2, } (p \vee q) \wedge \neg(q \vee r) \;\equiv\; p \wedge \neg q \wedge \neg r \rangle \\
& b \prec c \wedge \neg(b = c) \wedge \neg(c \prec b) \\
= \quad & \langle \text{Lemma 3, } (p \Rightarrow q) \Rightarrow (p \wedge q \equiv p) \text{, with } p,\, q := b \prec c,\, \neg(c \prec b)\text{. The antecedent} \\
& \quad \text{is true because strict orders are asymmetric.} \rangle \\
& b \prec c \wedge \neg(b = c) \\
= \quad & \langle \text{Lemma 3, with } p,\, q := b \prec c,\, \neg(b = c)\text{. The antecedent, } b \prec c \Rightarrow \neg(b = c)\text{, is} \\
& \quad \text{true by Lemma 4, } \rho \text{ is irreflexive} \Rightarrow (b\,\rho\,c \Rightarrow \neg(b = c))\text{, and the fact that strict} \\
& \quad \text{orders are irreflexive.} \rangle \\
& b \prec c \quad \blacksquare
\end{aligned}
$$

Theorem, Quadrotomy (b): $c \prec b \;\equiv\; \neg(b \prec c \vee b = c \vee \mathrm{incomp}(b,c))$

Because = and *incomp* are symmetric, the proof of (b) is identical to that of (a) with $b,\, c := c,\, b$.

Theorem, Quadrotomy (c): $b = c \;\equiv\; \neg(b \prec c \vee c \prec b \vee \mathrm{incomp}(b,c))$

Proof:

$$
\begin{aligned}
& \neg(b \prec c \vee c \prec b \vee \mathrm{incomp}(b,c)) \\
= \quad & \langle \text{(3.26) Idempotency of } \vee,\ p \;\equiv\; p \vee p \rangle \\
& \neg(b \prec c \vee c \prec b \vee \mathrm{incomp}(b,c) \vee \mathrm{incomp}(b,c))
\end{aligned}
$$

$$= \quad \langle (14.48.2), \neg(b \preceq c) \;\equiv\; c \prec b \vee \text{incomp}(b,c), \text{twice}\rangle$$
$$\neg(\neg(b \preceq c) \vee \neg(c \preceq b))$$
$$= \quad \langle (3.47a) \text{ De Morgan}, \neg(p \wedge q) \;\equiv\; \neg p \vee \neg q, \text{with (3.12) Double negation}\rangle$$
$$b \preceq c \wedge c \preceq b$$
$$= \quad \langle \text{Lemma 5}, \rho \text{ is antisymmetric } \wedge\ \rho \text{ is reflexive} \Rightarrow (b\,\rho\,c \wedge c\,\rho\,b \;\equiv\; b = c),$$
$$\text{with the fact that partial orders are antisymmetric and reflexive}\rangle$$
$$b = c \quad \blacksquare$$

Theorem, Quadrotomy (d): $\text{incomp}(b,c) \;\equiv\; \neg(b \prec c \vee c \prec b \vee b = c)$
Proof:

$$\neg(b \prec c \vee c \prec b \vee b = c)$$
$$= \quad \langle (3.26) \text{ Idempotency of } \vee, p \;\equiv\; p \vee p\rangle$$
$$\neg(b \prec c \vee c \prec b \vee b = c \vee b = c)$$
$$= \quad \langle \text{Lemma 1}, b \prec c \vee b = c \;\equiv\; b \preceq c \text{ twice}\rangle$$
$$\neg(b \preceq c \vee c \preceq b)$$
$$= \quad \langle (3.47b) \text{ De Morgan}, \neg(p \vee q) \;\equiv\; \neg p \wedge \neg q\rangle$$
$$\neg(b \preceq c) \wedge \neg(c \preceq b)$$
$$= \quad \langle (14.47.1) \text{ Definition, Incomparable: incomp}(b,c) \;\equiv\; \neg(b \preceq c) \wedge \neg(c \preceq b)\rangle$$
$$\text{incomp}(b,c) \quad \blacksquare$$

2.2 Generalized Trichotomy

Corollary to (14.50): $\text{incomp}(b,c) \;\equiv\; false$, where $\text{incomp}(b,c)$ refers to b and c being incomparable under a total order $\leq$.

Proof:

$$\text{incomp}(b,c)$$
$$= \quad \langle (14.47.1) \text{ Definition, Incomparable, incomp}(b,c) \;\equiv\; \neg(b \preceq c) \wedge \neg(c \preceq b),$$
$$\text{with } \preceq := \leq\rangle$$
$$\neg(b \leq c) \wedge \neg(c \leq b)$$
$$= \quad \langle (3.47b) \text{ De Morgan}, \neg(p \vee q) \;\equiv\; \neg p \wedge \neg q\rangle$$
$$\neg(b \leq c \vee c \leq b)$$
$$= \quad \langle (14.50) \text{ Definition, Total Order: A partial order } \preceq \text{ over } B \text{ is called a total or}$$
$$\text{linear order if } (\forall b,c \mid : b \preceq c \vee c \preceq b)\rangle$$
$$\neg true$$
$$= \quad \langle (3.8) \text{ Definition of false}\rangle$$
$$false \quad \blacksquare$$

Theorem, Trichotomy (a): $b < c \;\equiv\; \neg(c < b \vee b = c)$, where < is a strict total order.
Proof:

$$
\begin{aligned}
& b < c \\
= \quad & \langle \text{(Theorem, Quadrotomy (a)}, b \prec c \;\equiv\; \neg(c \prec b \vee b = c \vee \text{incomp}(b,c)), \text{with} \\
& \quad \prec := < \rangle \\
& \neg(c < b \vee b = c \vee \text{incomp}(b,c)) \\
= \quad & \langle \text{Corollary to (14.50)}, \text{incomp}(b,c) \;\equiv\; false \text{ for a total order} \rangle \\
& \neg(c < b \vee b = c \vee false) \\
= \quad & \langle \text{(3.30) Identity of } \vee, p \vee false \;\equiv\; false \rangle \\
& \neg(c < b \vee b = c) \quad \blacksquare
\end{aligned}
$$

Theorem, Trichotomy (b): $c < b \;\equiv\; \neg(b < c \vee b = c)$, where < is a strict total order.
Proof:

$$
\begin{aligned}
& c < b \\
= \quad & \langle \text{(Theorem, Quadrotomy (b)}, c \prec b \;\equiv\; \neg(b \prec c \vee b = c \vee \text{incomp}(b,c)), \text{with} \\
& \quad \prec := < \rangle \\
& \neg(b < c \vee b = c \vee \text{incomp}(b,c)) \\
= \quad & \langle \text{Corollary to (14.50)}, \text{incomp}(b,c) \;\equiv\; false \text{ for a total order} \rangle \\
& \neg(b < c \vee b = c \vee false) \\
= \quad & \langle \text{(3.30) Identity of } \vee, p \vee false \;\equiv\; false \rangle \\
& \neg(b < c \vee b = c) \quad \blacksquare
\end{aligned}
$$

Theorem, Trichotomy (c): $b = c \;\equiv\; \neg(b < c \vee c < b)$, where < is a strict total order.
Proof:

$$
\begin{aligned}
& b = c \\
= \quad & \langle \text{(Theorem, Quadrotomy (c)}, b = c \;\equiv\; \neg(b \prec c \vee c \prec b \vee \text{incomp}(b,c)), \text{with} \\
& \quad \prec := < \rangle \\
& \neg(b < c \vee c < b \vee \text{incomp}(b,c)) \\
= \quad & \langle \text{Corollary to (14.50)}, \text{incomp}(b,c) \;\equiv\; false \text{ for a total order} \rangle \\
& \neg(b < c \vee c < b \vee false) \\
= \quad & \langle \text{(3.30) Identity of } \vee, p \vee false \;\equiv\; false \rangle \\
& \neg(b < c \vee c < b) \quad \blacksquare
\end{aligned}
$$

2.3 Gries and Schneider's Trichotomy

Lemma 6:
$(p \equiv q \equiv r) \wedge \neg(p \wedge q \wedge r) \;\equiv\; (p \;\equiv\; \neg(q \vee r)) \wedge (q \;\equiv\; \neg(p \vee r)) \wedge (r \;\equiv\; \neg(p \vee q))$

Proof: The proof is by (4.5) Case Analysis on p, q, and r. There are eight cases: p, q, and r can be either true or false.

Case 1: p, q, and r are all false.

$$
\begin{array}{ll}
& (p \equiv q \equiv r) \wedge \neg(p \wedge q \wedge r) \;\equiv\; (p \;\equiv\; \neg(q \vee r)) \wedge (q \;\equiv\; \neg(p \vee r)) \wedge (r \;\equiv\; \neg(p \vee q)) \\
= & \langle \text{Case 1} \rangle \\
& (false \equiv false \equiv false) \wedge \neg(false \wedge false \wedge false) \;\equiv\; (false \;\equiv\; \neg(false \vee false)) \\
& \wedge (false \;\equiv\; \neg(false \vee false)) \wedge (false \;\equiv\; \neg(false \vee false)) \\
= & \langle (3.15), \neg p \;\equiv\; p \;\equiv\; false \rangle \\
& \neg(false \equiv false) \wedge \neg(false \wedge false \wedge false) \;\equiv\; \neg\neg(false \vee false) \\
& \wedge \neg\neg(false \vee false) \wedge \neg\neg(false \vee false) \\
= & \langle (3.12) \text{ Double negation}, \neg\neg p \;\equiv\; p \rangle \\
& \neg(false \equiv false) \wedge \neg(false \wedge false \wedge false) \;\equiv\; (false \vee false) \wedge (false \vee false) \\
& \wedge (false \vee false) \\
= & \langle (3.30) \text{ Identity of } \vee, p \vee false \;\equiv\; p \rangle \\
& \neg(false \equiv false) \wedge \neg(false \wedge false \wedge false) \;\equiv\; false \wedge false \wedge false \\
= & \langle (3.40) \text{ Identity of } \wedge, p \wedge false \;\equiv\; false \rangle \\
& \neg(false \equiv false) \wedge \neg false \;\equiv\; false \\
= & \langle (3.3) \text{ Identity of } \equiv, true \;\equiv\; q \vee q \rangle \\
& \neg true \wedge \neg false \;\equiv\; false \\
= & \langle (3.8) \text{ Definition of false}, \neg true \;\equiv\; false \rangle \\
& false \wedge true \;\equiv\; false \\
= & \langle (3.40) \text{ Zero of } \wedge, p \wedge false \;\equiv\; false \rangle \\
& true \quad \blacksquare
\end{array}
$$

Cases 2-8?

(15.44) Trichotomy: $(a < b \vee a = b \vee a > b) \wedge \neg(a < b \vee a = b \vee a > b)$

Proof

$$
\begin{array}{ll}
& (a < b \vee a = b \vee a > b) \wedge \neg(a < b \vee a = b \vee a > b) \\
= & \langle \text{Lemma, MECE-3} \rangle \\
& (a < b \;\equiv\; \neg(b < a \vee b = c)) \wedge (b < a \;\equiv\; \neg(a < b \vee b = c)) \\
& \wedge (a = b \;\equiv\; \neg(a < b \vee b < a)) \\
= & \langle \text{Theorem, Trichotomy (a)}, b < c \;\equiv\; \neg(c < b \vee b = c), \text{with} < \text{(generic strict} \\
& \quad \text{total order) replaced by} < \text{(integers)} \rangle \\
& true \wedge (b < a \;\equiv\; \neg(a < b \vee b = c)) \wedge (a = b \;\equiv\; \neg(a < b \vee b < a)) \\
= & \langle \text{Theorem, Trichotomy (b): } c < b \;\equiv\; \neg(b < c \vee b = c), \text{with} <:=< \rangle \\
& true \wedge true \wedge (a = b \;\equiv\; \neg(a < b \vee b < a)) \\
= & \langle \text{Theorem, Trichotomy (c): } b = c \;\equiv\; \neg(b < c \vee c < b), \text{with} <:=< \rangle \\
& true \wedge true \wedge true \\
= & \langle (3.39) \text{ Identity of } \wedge, p \wedge true \;\equiv\; p, \text{twice} \rangle \\
& true \quad \blacksquare
\end{array}
$$

3 Conclusion

References

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