A Revised Axiomatic System for Linear Temporal Logic (Complete)

John Wiegley
DFINITY
West Sacramento, CA 95691

Scott M. Staley
Ford Motor Company Research Labs (retired)
Seattle, WA 98198

J. Stanley Warford
Computer Science Department
Pepperdine University
Malibu, CA 90263

Abstract

Calculational deductive systems, developed by Dijkstra and Scholten and extended by Gries and Schneider, are based on only four inference rules – Substitution, Leibniz, Equanimity, and Transitivity. The paper, A Calculational Deductive System for Linear Temporal Logic (CDS4LTL), is a survey of the linear temporal logic (LTL) literature and presents all the LTL theorems from the survey, plus many new ones, in a calculational deductive system. This paper improves on the CDS4LTL system by decreasing the number of axioms while at the same time using two newly-discovered axioms as its basis. It presents additional theorems not in CDS4LTL and also extends the presentation of duality. All the theorems have been proved with calculation logic without using the temporal deduction metatheorem.

1 Introduction

Propositional calculus is a formal system of logic based on the unary operator negation $\neg$, the binary operators conjunction $\wedge$, disjunction $\vee$, implies $\Rightarrow$, and equivalence $\equiv$, variables (lowercase letters p, q, $\ldots$), and the constants *true* and *false*. Hilbert-style logic systems, $\mathcal{H}$, are the deductive logic systems traditionally used in mathematics to describe the propositional calculus. Typical of such descriptions with applications to computer science is the text by Ben-Ari [2]. A key feature of such systems is their multiplicity of inference rules and the importance of modus ponens as one of them.

In the late 1980's, Dijkstra and Scholten [5], and Feijen [9] developed a method of proving program correctness with a new logic based on an equational style. This equational deductive system, $\mathcal{E}$, is the basis of books by Kaldewaij [17] and Cohen [4]. In contrast to $\mathcal{H}$ systems, $\mathcal{E}$ has only four inference rules – Substitution, Leibniz, Equanimity, and Transitivity. In $\mathcal{E}$, modus ponens plays a secondary role. It is not an inference rule, nor is it assumed as an axiom, but instead is proved as a theorem from the axioms using the inference rules.

Gries and Schneider [10, 14] show that $\mathcal{E}$, also known as *calculation logic*, has several advantages over traditional logic systems. The primary advantage of $\mathcal{E}$ over $\mathcal{H}$ systems is that calculation logic has only four proof rules, with inference rule Leibniz as the primary one. Roughly speaking, Leibniz is "substituting equals for equals," hence the moniker *equational* deductive system. In contrast, $\mathcal{H}$ systems rely on a more extensive set of inference rules.

In 1994, Gries and Schneider published *A Logical Approach to Discrete Math* (LADM) [11], in which they first develop $\mathcal{E}$ for propositional and predicate calculus, and then extend it to a theory of sets, a theory of sequences, relations and functions, a theory of integers, recurrence relations, modern algebra, and a theory of graphs. Using calculation logic as a tool, LADM brings all the advantages of $\mathcal{E}$ to these additional knowledge domains.

The paper, A Calculational Deductive System for Linear Temporal Logic (CDS4LTL) [23], extends the calculation logic of LADM to linear temporal logic (LTL). It surveys the LTL literature and presents all the theorems from the survey, plus many new ones, in a calculation logic system. This paper improves on the CDS4LTL system by decreasing the number of axioms while at the same time using two newly-discovered axioms as its basis. It presents additional theorems not in CDS4LTL and also extends the presentation of duality.

The following sections duplicate CDS4LTL to a large extent but omit details of the survey. Section 2 describes the deductive axioms and the proof rules for $\mathcal{E}$. It also defines the syntax and semantics of linear temporal logic. Section 3 presents the revised calculation logic system for linear temporal logic. Section 4 summarizes previous linear temporal logic axiomatization systems and compares them with the current work. [*Revise this sentence*]

2 Background

The first section below summarizes the calculational system $\mathcal{E}$ from LADM [11]. The summary is minimal, and assumes the reader is familiar with the propositional and predicate calculus. The second section introduces temporal logic and assumes no prior familiarity with it. The paper can serve as an introduction to linear temporal logic.

$[x := e]$ (textual substitution)	Highest precedence
$\neg \quad \bigcirc \quad \Diamond \quad \Box$	
$\mathcal{U} \quad \mathcal{W}$	
$=$ (conjunctional)	
$\vee \quad \wedge$	
$\Rightarrow \quad \Leftarrow$	
$\equiv$ (associative)	Lowest precedence

Figure 1: Precedence of the propositional and temporal logic operators.

2.1 Calculational Deductive Systems

2.1.1 Propositional and temporal operators

Expressions are the basis of propositional calculus in the calculational system. Propositional theorems are simply boolean expressions that are true in all states. The definition of an expression has four parts:

- A constant or variable is an expression.
- If E is an expression, then (E) is an expression.
- If $\triangleright$ is a unary prefix operator and E is an expression, then $\triangleright E$ is an expression with operand E.
- If $\star$ is a binary infix operator and D and E are expressions, then $D \star E$ is an expression with operands D and E.

By convention, upper-case letters (*e.g.* X, Y, ...) represent expressions and lower-case letters (*e.g.* x, y, ...) represent variables. In the propositional calculus, the constants are *true* and *false*.

Figure 1 is the table of precedences. Textual substitution has the highest precedence. All the unary operators have the next highest precedence. They are necessarily right associative. For example, $\neg\bigcirc\neg p$ means $\neg(\bigcirc(\neg p))$. In this system, two binary operators that have the same precedence require parentheses to disambiguate. As in LADM, conjunction $\wedge$ and disjunction $\vee$ have the same precedence so that $p \wedge q \vee r$ must be disambiguated as either $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$. This contrasts with many systems in which conjunction has higher precedence than disjunction.

Also consistent with the calculational system of LADM but different from most other deductive logic systems is the difference between operators equals $=$ and equivales $\equiv$. Equals applies to any mathematical type including, *e.g.*, boolean, natural number, and set. Equivales applies only to boolean and is commonly denoted $\leftrightarrow$ in other systems. Another difference is that equals is conjunctive, while equivales is associative. For example, the expression $p = q = r$ has conjunctive meaning $(p = q) \wedge (q = r)$, while the expression

$p \equiv q \equiv r$ can be taken as either $(p \equiv q) \equiv r$ or $p \equiv (q \equiv r)$. Associativity of equivales is the first axiom in the calculational deductive system of LADM.

2.1.2 Inference rules

The inference rules for calculation logic are Substitution, Leibniz, Equanimity, and Transitivity.

(I1) **Substitution:** $\dfrac{E}{E[z := F]}$

(I2) **Leibniz:** $\dfrac{X = Y}{E[z := X] = E[z := Y]}$

(I3) **Equanimity:** $\dfrac{X, \quad X = Y}{Y}$

(I4) **Transitivity:** $\dfrac{X = Y, \quad Y = Z}{X = Z}$

where the square bracket in $E[z := F]$ indicates textual substitution of expression F for variable z everywhere z occurs in expression E. In a typical proof, Substitution and Leibniz are explicit, while Equanimity and Transitivity are implicit.

Substitution allows the generalization of a single theorem to represent an infinite number of theorems. For example, because $p \Rightarrow \mathit{false} \equiv \neg p$ is a theorem, then, with $p := p \wedge q$, the expression $(p \wedge q) \Rightarrow \mathit{false} \equiv \neg(p \wedge q)$ is also a theorem.

Roughly speaking, Leibniz allows for the substitution of equals for equals in a proof step. The general form of a proof step is

$$\begin{array}{ll} & E[z := X] \\ = & \quad \langle X = Y \rangle \\ & E[z := Y] \end{array}$$

where the expression enclosed in angle brackets $\langle\ \rangle$, called the "hint", is the justification for the step.

An example of a proof step from the proof of theorem (166) in Section 3.8 is

$$\begin{array}{ll} & \Box(\Box p \wedge \Diamond q) \Rightarrow \Box\Diamond(p \wedge q) \\ = & \quad \langle \text{(99) Distributivity of } \Box \text{ over } \wedge \rangle \\ & \Box\Box p \wedge \Box\Diamond q \Rightarrow \Box\Diamond(p \wedge q) \end{array}$$

This proof step uses the previously proved theorem (99) Distributivity of $\Box$ over $\wedge$, which is $\Box(p \wedge q) \equiv \Box p \wedge \Box q$. The justification in the hint $X = Y$ comes from inference rule Substitution, with the textual substitution of $\Box p$ for p and $\Diamond q$ for q in (99) as follows

$$(\Box(p \wedge q) \equiv \Box p \wedge \Box q)[p, q := \Box p, \Diamond q] : \quad \Box(\Box p \wedge \Diamond q) \equiv \Box\Box p \wedge \Box\Diamond q$$

The expressions in Leibniz for the step are

$$\begin{array}{ll} E: & z \Rightarrow \Box \Diamond (p \wedge q) \\ X: & \Box(\Box p \wedge \Diamond q) \\ Y: & \Box \Box p \wedge \Box \Diamond q \end{array}$$

The textual substitutions are

$$\begin{array}{ll} E[z := X]: & \Box(\Box p \wedge \Diamond q) \Rightarrow \Box \Diamond (p \wedge q) \\ E[z := Y]: & \Box \Box p \wedge \Box \Diamond q \Rightarrow \Box \Diamond (p \wedge q) \end{array}$$

The proof of a theorem consists of showing the equivalence of that theorem to a previously proved theorem through a sequence of the proof steps. For example, here is a one-step proof of (3) $\bigcirc p \equiv \neg \bigcirc \neg p$ in Section 3.1.

Proof:

$$\begin{array}{ll} & \bigcirc p \equiv \neg \bigcirc \neg p \\ = & \langle (3.11)\ \neg p \equiv q \equiv p \equiv \neg q \text{ with } p, q := \bigcirc \neg p, \bigcirc p, \\ & \quad \neg \bigcirc \neg p \equiv \bigcirc p \equiv \bigcirc \neg p \equiv \neg \bigcirc p \rangle \\ & \neg \bigcirc p \equiv \bigcirc \neg p \quad \text{—(1) Self-dual} \quad \blacksquare \end{array}$$

In a proof hint, numeric references that contain a decimal point, such as (3.11) above, refer to a theorem in $\mathcal{E}$ from LADM. Equanimity is implicit in the proof. Because $\neg \bigcirc p \equiv \bigcirc \neg p$ (*i.e.* X) is a previous theorem, and $\neg \bigcirc p \equiv \bigcirc \neg p$ is equivalent to $\bigcirc p \equiv \neg \bigcirc \neg p$ (*i.e.* $X = Y$), by equanimity $\bigcirc p \equiv \neg \bigcirc \neg p$ (*i.e.* Y) is proved.

Transitivity of equality allows a derivation to be given as a sequence of equivalent expressions, which, at the end, proves the equivalence of the first expression in the sequence with the last expression in the sequence. For example, here is a two-step proof of (22) Idempotency of $\mathcal{U}$, $p \, \mathcal{U} \, p \equiv p$ in Section 3.2.

Proof:

$$\begin{array}{ll} & p \, \mathcal{U} \, p \equiv p \\ = & \langle (10) \text{ Expansion of } \mathcal{U} \rangle \\ & p \vee (p \wedge \bigcirc (p \, \mathcal{U} \, p)) \equiv p \\ = & \langle (3.43b) \text{ Absorption, } p \vee (p \wedge q) \equiv p \text{ with } q := \bigcirc (p \, \mathcal{U} \, p) \rangle \\ & p \equiv p \quad \text{—(3.5) Reflexivity of } \equiv \quad \blacksquare \end{array}$$

Transitivity of equality is implicit in the proof. Because $p \, \mathcal{U} \, p \equiv p$ is equivalent to $p \vee (p \wedge \bigcirc (p \, \mathcal{U} \, p)) \equiv p$ (*i.e.* $X = Y$), and $p \vee (p \wedge \bigcirc (p \, \mathcal{U} \, p)) \equiv p$ is equivalent to $p \equiv p$ (*i.e.* $Y = Z$), by transitivity $p \, \mathcal{U} \, p \equiv p$ is equivalent to $p \equiv p$ (*i.e.* $X = Z$).

2.1.3 Proof technique metatheorems

The logic system $\mathcal{E}$ of LADM [11] has 13 axioms for the propositional calculus from which theorems are deduced with the above inference rules in the calculational style. The system also contains a number of metatheorems based on properties of equivalence and implication, which allow the proof style to be extended. Here are four of the proof technique metatheorems.

(4.4) **Deduction (assume conjuncts of antecedent):**
To prove $P_1 \wedge P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q.
You cannot use textual substitution in P_1 or P_2.

(4.7) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.

(4.7.1) **Truth implication:** To prove P, prove $true \Rightarrow P$.

(4.12) **Contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$.

The validity of each metatheorem is established from the theorems of the propositional calculus and the inference rules. Deduction is established by showing that any deductive proof has an equivalent calculational proof. Mutual implication is based on (3.80), Truth implication is based on (3.73), and Contrapositive is based on (3.61).

(3.80) **Mutual implication:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \ \equiv\ (p \equiv q)$

(3.73) **Left identity of $\Rightarrow$:** $true \Rightarrow p \ \equiv\ p$

(3.61) **Contrapositive:** $p \Rightarrow q \ \equiv\ \neg q \Rightarrow \neg p$

The proof format is extended to the case where the theorem to be proved is of the form $P \equiv Q$. For theorems of this form, the proof may begin with the left hand side and show equivalence to the right hand side through a sequence of proof steps. This proof style is established by showing that such a proof is equivalent to a calculational proof and is based on (3.5).

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

For example, the following proof is the preferred style for the previous proof of (22).

Proof:

$$
\begin{array}{ll}
 & p \,\mathcal{U}\, p \\
= & \langle (10) \text{ Expansion of } \mathcal{U} \rangle \\
 & p \vee (p \wedge \circ\, (p \,\mathcal{U}\, p)) \\
= & \langle (3.43b) \text{ Absorption, } p \vee (p \wedge q) \equiv p \text{ with } q := \circ\, (p \,\mathcal{U}\, p) \rangle \\
 & p \quad \blacksquare
\end{array}
$$

Gries and Schneider also extend the proof format to incorporate implication using its transitive properties with itself and with equivales. Instead of proving a theorem of the form $P \Rightarrow Q$ to be equivalent to a previously proved theorem, P can be shown to imply Q, or Q can be shown to follow from P. The following mutual transitivity theorems justify this extension.

(3.82) **Transitivity:**
(a) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

An example is a proof of (46) $p \Rightarrow \Diamond p$ in Section 3.3.

Proof:

$\Diamond p$
$=$ ⟨(45) Expansion of $\Diamond$⟩
$p \vee \circ \Diamond p$
$\Leftarrow$ ⟨(3.76a) Weakening $p \Rightarrow p \vee q$ with $q := \circ \Diamond p$⟩
p ∎

Because $\Diamond p$ equivales $p \vee \circ \Diamond p$, and $p \vee \circ \Diamond p$ follows from p, it follows by mutual transitivity that $\Diamond p$ follows from p.

The following two theorems from LADM provide a further extension to proof steps with implication.

(4.2) **Monotonicity of $\vee$:** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$
(4.3) **Monotonicity of $\wedge$:** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

They are required to justify an implication when the antecedent of the implication is a conjunct or disjunct. For example, here is a proof step where the antecedent $p \,\mathcal{U}\, q$ is a disjunct in the expression $\Box p \vee p \,\mathcal{U}\, q$.

$\Box p \vee p \,\mathcal{U}\, q$
$\Rightarrow$ ⟨(42) Eventuality and (4.2) Monotonicity of $\vee$⟩
$\Box p \vee \Diamond q$

Previously proved theorem (42) Eventuality is $p \,\mathcal{U}\, q \Rightarrow \Diamond q$. The application of (4.2) is with the following textual substitution.

$((p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r))[p,q,r := p \,\mathcal{U}\, q, \Diamond q, \Box p]$
$=$ ⟨Textual substitution⟩
$(p \,\mathcal{U}\, q \Rightarrow \Diamond q) \Rightarrow (p \,\mathcal{U}\, q \vee \Box p \Rightarrow \Diamond q \vee \Box p)$

In other words, because $p \,\mathcal{U}\, q$ implies $\Diamond q$, $p \,\mathcal{U}\, q \vee \Box p$ implies $\Diamond q \vee \Box p$.

2.1.4 Predicate calculus

The predicate calculus in calculation logic has a consistent quantification notation that applies to Abelian monoids in both mathematics and logic. Denoting a general Abelian monoid as the infix operator $\star$, the form of a quantification is

$(\star\ \mathit{dummies} \mid \mathit{range} : \mathit{body})$

All quantifications have explicit scope for the dummy variable denoted by the outer parentheses. Within the parentheses the quantification consists of three parts:

- the infix operator $\star$ and dummy variable(s),
- the range, which is a boolean expression, and
- the body, which is an expression that is type compatible with the operator $\star$.

A vertical bar separates the operator and dummy variable from the range, and a colon separates the range from the body. An abbreviation is to omit the range when it is *true*. For example, $(\forall i \mid : P)$ is an abbreviation for $(\forall i \mid \mathit{true} : P)$.

For example, the standard mathematical notation for writing the sum of the squares of the first n positive integers is

$$\sum_{i=1}^{n} i^2$$

where $\star$ is Abelian monoid $+$, and Σ is the quantified symbol for addition. The calculational notation for the same expression is

$$(\Sigma i \mid 1 \leq i \leq n : i^2)$$

Similarly, the standard logic notation for writing that there exists a number between 10 and 20 inclusive that divides n is

$$\exists i (10 \leq i \leq 20 \wedge \mathrm{divides}(i,n))$$

where $\star$ is Abelian monoid $\vee$, $\exists$ is the quantified symbol for disjunction, and divides is a predicate that is true when i divides n. The calculational notation for the same expression is

$$(\exists i \mid 10 \leq i \leq 20 : \mathrm{divides}(i,n))$$

Predicate calculus in the calculational system of LADM begins with nine general axioms that apply to all Abelian monoids. For example, here are the first two axioms.

For symmetric and associative binary operator $\star$ with identity u.

(8.13) **Axiom, Empty range:** $(\star x \mid \mathit{false} : P) = u$

(8.14) **Axiom, One-point rule:** Provided $\neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}E\text{’})$,
$(\star x \mid x = E : P) = P[x := E]$

It has two axioms for universal quantification.

(9.2) **Axiom, Trading:** $(\forall x \mid R : P) \equiv (\forall x \mid : R \Rightarrow P)$

(9.5) **Axiom, Distributivity of $\vee$ over $\forall$:** Provided $\neg \mathit{occurs}(\text{‘}x\text{’}, \text{‘}P\text{’})$,
$P \vee (\forall x \mid R : Q) \equiv (\forall x \mid R : P \vee Q)$

And it has one axiom for existential quantification.

(9.17) **Axiom, Generalized De Morgan:** $(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$

2.2 Linear Temporal Logic

The operators of propositional calculus, $\neg$, $=$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftarrow$, and $\equiv$ are static. That is, they apply at a single point in time. Each operator has a truth table that dictates how to evaluate the truth value of an expression. A state is an assignment of a truth value to each variable in the expression. A given boolean expression may be false in all states, true in some states and false in others, or true in all states, in which case the expression is known as a theorem or validity or tautology.

The operators of temporal logic, $\bigcirc$, $\Diamond$, $\Box$, $\mathcal{U}$, and $\mathcal{W}$ are dynamic. That is, they do not apply at a single point in time, but apply over an infinite sequence of states. Each state corresponds to a discrete point in time that represents one point in the execution of a program, possibly having several threads running concurrently but whose instruction executions have been serialized. As one instruction in the program is executed, the state changes, and hence the truth value of an expression may change as well.

2.2.1 Models and Anchored Sequences

A program consists of, among other things, a set of variables and constants. Using state expressions, with operations provided by the programming language, the program changes the values of the variables as it executes. When the value of a variable changes, a property associated with that variable might also change. Properties are described by state formulas (assertions). V is the set of variables that are combined with the operations of the programming language to form state expressions and with the boolean operations of logic to describe properties.

A model σ over V is an infinite sequence of the form

$$\sigma : s_0, s_1, s_2, \ldots$$

where s_0 is the initial state of a computation and each state $s_i, 0 \leq i$ is the state at time i. [19]

For example, suppose x is an integer variable whose value varies at each step of the computation. Then, x and the property $x \geq 10$ might evolve as follows.

σ	s_0	s_1	s_2	s_3	s_4	$\ldots$
x	8	9	10	11	12	$\ldots$
$x \geq 10$	F	F	T	T	T	$\ldots$

The bottom row shows the evaluation of the state formula for each state in the sequence.

An anchored sequence is a pair (σ, j) where j is a natural number (the anchor) that specifies a state in model σ. [21] The anchor point j partitions the states s_i of σ into the past $0 \leq i < j$, present $i = j$, and future $i > j$. The notation

$$(\sigma, j) \models p$$

means that the property p holds at position j in a sequence σ. In this example,

$(\sigma, 3) \models x \geq 10.$

The symbol $\models$ means "satisfies", so the above expression is read as "State 3 of sequence σ satisfies $x \geq 10$." Or, using "holds", the same expression is read as, "$x \geq 10$ holds in state 3 of sequence σ."

In the above example, evaluation of the property $x \geq 10$ in the anchored sequence (σ, j) depends only on the value of variable x at the anchor point j. In general, the truth of a temporal assertion at j may depend on future states as well. For example, an informal English assertion is, "p is now, and always will be from this point on, true." The temporal notation for this assertion is $\Box p$. If you assume that x in the above sequence keeps increasing by one, then $\Box p$ holds in state 3 of sequence σ. However, the truth of this assertion depends not only on the fact that p holds at s_3 but that p also holds at $s_4, s_5, \ldots$. A more precise formulation is that p holds at s_3 and that $\Box p$ also holds at each subsequent state. See Schneider [21] for a formal treatment, which depends on the formulation of prefix and suffix anchored sequences.

There is a distinction between the constant *true* and the truth value of an expression T in a given state. The constant *true* is an expression that evaluates to T in every state. Similarly, there is a distinction between the constant *false* and the truth value of an expression F in a given state. The constant *false* is an expression that evaluates to F in every state.

σ	s_0	s_1	s_2	s_3	s_4	$\ldots$
true	T	T	T	T	T	$\ldots$
false	F	F	F	F	F	$\ldots$

The propositional logic system of LADM [11] describes a case analysis metatheorem as follows: If $E[z := true]$ and $E[z := false]$ are theorems, then so is $E[z := p]$. This metatheorem does *not* hold in LTL because the two cases, $z := true$ and $z := false$, account for only two out of an infinite number of possible sequences of T's and F's in σ.

2.2.2 The temporal operators

This section uses the anchored sequence (σ, j) and $\models$ to formalize the interpretation of each temporal operator.

The *next* operator $\circ$

The semantics of the unary prefix operator $\circ$ are

$$(\sigma, j) \models \circ p \quad \text{iff} \quad (\sigma, j+1) \models p$$

That is, $\circ p$ holds at position j iff p holds at position $j+1$.

For example, in the following sequence $\circ\, 10 \leq x < 13$ holds at state s_1 because $10 \leq x < 13$ holds at state s_2.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\dots$
x	8	9	10	11	12	13	14	$\dots$
$10 \le x < 13$	F	F	T	T	T	F	F	$\dots$
$\circ\, 10 \le x < 13$	F	T	T	T	F	F	F	$\dots$

In other words,

$$(\sigma, 1) \models \circ\, 10 \le x < 13 \quad \text{because} \quad (\sigma, 2) \models 10 \le x < 13$$

Furthermore, $\circ\, 10 \le x < 13$ does not hold at state s_4 even though $10 \le x < 13$ does hold in that state, because $10 \le x < 13$ does not hold in state s_5.

This definition of $\circ$ assumes an infinite sequence of states. Emerson [8] shows variations of the *next* operator that would apply to a finite sequence of states suitable for modeling a program that terminates.

The *until* operator $\mathcal{U}$

The semantics of the binary infix operator $\mathcal{U}$ are

$$(\sigma, j) \models p\, \mathcal{U}\, q \quad \text{iff} \quad (\exists k \mid k \ge j : (\sigma, k) \models q \wedge (\forall i \mid j \le i < k : (\sigma, i) \models p))$$

If $p\, \mathcal{U}\, q$ holds at state s_j, then p holds at state s_j and continues to hold at every state after s_j until q holds at some future state. $p\, \mathcal{U}\, q$ guarantees that q will eventually hold at some future state and that p will continue to hold until then. After the state in which q holds for the first time, there are no restrictions on either p or q.

For example, suppose x and y evolve in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\dots$
x	-1	0	1	2	3	4	5	6	7	8	$\dots$
y	9	8	7	6	5	4	3	2	1	0	$\dots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\dots$
$2 \le y < 5$	F	F	F	F	F	T	T	T	F	F	$\dots$
$(0 < x < y)\, \mathcal{U}\, (2 \le y < 5)$	F	F	T	T	T	T	T	T	F	F	$\dots$

The bottom row shows the evaluation of the expression $p\, \mathcal{U}\, q$ where $p \equiv 0 < x < y$ and $q \equiv 2 \le y < 5$. In states s_0 and s_1, $p\, \mathcal{U}\, q$ is false because both p and q are false. Starting at state s_2, $p\, \mathcal{U}\, q$ is true because in that state p is true and will remain true until q eventually becomes true in state s_5.

From the semantics of $p\, \mathcal{U}\, q$, if q is true in any state, then $p\, \mathcal{U}\, q$ is true in that state regardless of p. For example, not only is $p\, \mathcal{U}\, q$ true in state s_5, before which p was true in several preceding states, it is also true in states s_6 and s_7, because in those states q is true. This behavior of $p\, \mathcal{U}\, q$ comes from the empty range and one-point rules [11] of the predicate calculus in the case that q holds in state s_j and $k = j$.

Proof:

$(\exists k \mid k \geq j : (\sigma,k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma,i) \models p))$
$= \quad \langle \text{Case } k = j \rangle$
$(\exists k \mid k = j : (\sigma,k) \models q \wedge (\forall i \mid j \leq i < j : (\sigma,i) \models p))$
$= \quad \langle j \leq i < j \equiv \mathit{false} \rangle$
$(\exists k \mid k = j : (\sigma,k) \models q \wedge (\forall i \mid \mathit{false} : (\sigma,i) \models p))$
$= \quad \langle \text{(8.13) Empty range rule } (\star x \mid \mathit{false} : P) = u \text{ with } \mathit{true} \text{ the identity of } \wedge \rangle$
$(\exists k \mid k = j : (\sigma,k) \models q \wedge \mathit{true})$
$= \quad \langle \text{(3.39) Identity of } \wedge,\ p \wedge \mathit{true} \equiv p \rangle$
$(\exists k \mid k = j : (\sigma,k) \models q)$
$= \quad \langle \text{(8.14) One-point rule } (\star x \mid x = E : P) = P[x := E] \rangle$
$((\sigma,k) \models q)[k := j]$
$= \quad \langle \text{Textual substitution} \rangle$
$(\sigma, j) \models q$
$= \quad \langle \text{Case } q \text{ holds in state } s_j \rangle$
true ∎

This result is theorem (20) $p \,\mathcal{U}\, \mathit{true} \equiv \mathit{true}$ listed in Section 3.2. *true* is the right zero of the *until* operator.

The *until* operator $\mathcal{U}$ is not associative as shown by the following sequence.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	...
p	F	F	T	T	T	T	F	F	...
q	F	T	F	T	F	F	F	F	...
r	F	F	F	T	T	F	T	F	...
$p \,\mathcal{U}\, q$	F	T	T	T	F	F	F	F	...
$q \,\mathcal{U}\, r$	F	F	F	T	T	F	T	F	...
$p \,\mathcal{U}\, (q \,\mathcal{U}\, r)$	F	F	T	T	T	T	T	F	...
$(p \,\mathcal{U}\, q) \,\mathcal{U}\, r$	F	T	T	T	T	F	T	F	...

State s_1 in the last two rows of the above table shows that $(p \,\mathcal{U}\, q) \,\mathcal{U}\, r$ does not imply $p \,\mathcal{U}\, (q \,\mathcal{U}\, r)$, and state s_5 shows that $p \,\mathcal{U}\, (q \,\mathcal{U}\, r)$ does not imply $(p \,\mathcal{U}\, q) \,\mathcal{U}\, r$.

The *eventually* operator $\diamond$

The semantics of the unary prefix operator $\diamond$ are

$$(\sigma, j) \models \diamond p \quad \text{iff} \quad (\exists k \mid k \geq j : (\sigma,k) \models p)$$

So, $\diamond p$ holds in state s_j if p holds in state s_j or in any other state s_k where $k \geq j$, that is, if p holds in the current state or in any other future state.

For example, suppose x evolves in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\dots$
x	1	2	3	4	5	6	7	$\dots$
$3 \le x < 6$	F	F	T	T	T	F	F	$\dots$
$\Diamond(3 \le x < 6)$	T	T	T	T	T	F	F	$\dots$

The bottom row shows the evaluation of the expression $\Diamond p$ where $p \equiv 3 \le x < 6$. In states s_0 and s_1, $\Diamond p$ is true because there is a state, either now or in the future, in which p will hold.

If $\Diamond p$ is ever false in any state s_i in a sequence σ, it must be false in all subsequent states s_j, $j \ge i$. If $\Diamond p$ is ever true in any state s_i in a sequence σ, it must be true in all preceding states s_j, $j \le i$. For example, suppose p and q evolve in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\dots$
p	F	F	T	F	F	T	F	F	F	F	$\dots$
q	F	F	T	T	F	F	T	T	F	F	$\dots$
$\Diamond p$	T	T	T	T	T	T	F	F	F	F	$\dots$
$\Diamond q$	T	T	T	T	T	T	T	T	T	T	$\dots$

The bottom two rows show the evaluation of the expressions $\Diamond p$ and $\Diamond q$ assuming that p remains false indefinitely and q continues to switch between true and false indefinitely.

The *eventually* operator is a special case of the *until* operator. Namely, *true* $\mathcal{U}$ q is equivalent to $\Diamond q$.

Proof:

$$
\begin{array}{ll}
 & (\sigma, j) \models \mathit{true}\ \mathcal{U}\ q \\
= & \quad \langle \text{Semantics of } p\ \mathcal{U}\ q \text{ with } p := \mathit{true} \rangle \\
 & (\exists k \mid k \ge j : (\sigma,k) \models q \wedge (\forall i \mid j \le i < k : (\sigma,i) \models \mathit{true})) \\
= & \quad \langle \mathit{true} \text{ holds in all states} \rangle \\
 & (\exists k \mid k \ge j : (\sigma,k) \models q \wedge (\forall i \mid j \le i < k : \mathit{true})) \\
= & \quad \langle (9.8)\ (\forall x \mid R : \mathit{true}) \equiv \mathit{true} \rangle \\
 & (\exists k \mid k \ge j : (\sigma,k) \models q \wedge \mathit{true}) \\
= & \quad \langle (3.39) \text{ Identity of } \wedge,\ p \wedge \mathit{true} \equiv p \rangle \\
 & (\exists k \mid k \ge j : (\sigma,k) \models q) \\
= & \quad \langle \text{Semantics of } \Diamond q \rangle \\
 & (\sigma, j) \models \Diamond q \quad \blacksquare
\end{array}
$$

This relationship is the basis of the definition of $\Diamond q$ in (38) $\Diamond q \equiv \mathit{true}\ \mathcal{U}\ q$ assumed in Section 3.3.

The *always* operator $\Box$

The semantics of the unary prefix operator $\Box$ are

$$(\sigma, j) \models \Box p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma,k) \models p)$$

So, $\Box p$ holds in state s_j if p holds in state s_j and in all other states s_k where $k \geq j$, that is, if p holds in the current state and in all other future states.

For example, suppose x evolves in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	$\dots$
x	1	2	3	4	5	6	7	8	$\dots$
$x < 4 \lor x \geq 6$	T	T	T	F	F	T	T	T	$\dots$
$\Box (x < 4 \lor x \geq 6)$	F	F	F	F	F	T	T	T	$\dots$

The bottom row shows the evaluation of the expression $\Box p$ where $p \equiv x < 4 \lor x \geq 6$. In states s_3 and s_4, $\Box p$ is false because p does not hold in those states. In states s_0, s_1, and s_2, p is true. However, $\Box p$ is false in those states because p does no hold in all future states. In states s_5, s_6, s_7, and subsequent states, $\Box p$ is true because p holds in in those states and in all future states as well.

If $\Box p$ is ever true in any state s_i in a sequence σ, it must be true in all subsequent states s_j, $j \geq i$. If $\Box p$ is ever false in any state s_i in a sequence σ, it must be false in all preceding states s_j, $j \leq i$.

For example, suppose p and q evolve in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\dots$
p	T	T	F	T	T	F	T	T	T	T	$\dots$
q	T	T	F	F	T	T	F	F	T	T	$\dots$
$\Box p$	F	F	F	F	F	F	T	T	T	T	$\dots$
$\Box q$	F	F	F	F	F	F	F	F	F	F	$\dots$

The bottom two rows show the evaluation of the expressions $\Box p$ and $\Box q$ assuming that p remains true indefinitely and q continues to switch between true and false indefinitely.

Note that $\Box p$ is a universal operator, while $\Diamond p$ is an existential operator. In the same way that $(\forall x \mid R : P)$ is equivalent to $\neg(\exists x \mid R : \neg P)$ through the generalized De Morgan theorem, $\Box p$ is equivalent to $\neg \Diamond \neg p$.

Proof:

$$
\begin{array}{ll}
 & (\sigma, j) \models \Box p \\
= & \langle \text{Semantics of } \Box p \rangle \\
 & (\forall k \mid k \geq j : (\sigma,k) \models p) \\
= & \langle \text{(9.18a) Generalized De Morgan } \neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P) \rangle \\
 & \neg(\exists k \mid k \geq j : \neg((\sigma,k) \models p)) \\
= & \langle p \text{ does not hold in a state iff } \neg p \text{ holds in that state} \rangle \\
 & \neg(\exists k \mid k \geq j : (\sigma,k) \models \neg p) \\
= & \langle \text{Semantics of } \Diamond p \rangle \\
 & \neg((\sigma, j) \models \Diamond \neg p)
\end{array}
$$

$$= \quad \langle p \text{ does not hold in a state iff } \neg p \text{ holds in that state} \rangle$$
$$(\sigma, j) \models \neg \Diamond \neg p \quad \blacksquare$$

This relationship is the basis of the definition of $\Box p$ in equation (54) $\Box p \equiv \neg\Diamond\neg p$ assumed in Section 3.4.

The above calculational proof illustrates a common advantage of $\mathcal{E}$ over traditional logic systems. The same equivalence is proved in [2] but only by resorting to proof by mutual implication using proof by contradiction within each case. Gries and Schneider [11] point out that many equivalence proofs are shorter in $\mathcal{E}$, in which equality is central, than in traditional systems, in which implication is central.

The above demonstration that $(\sigma, j) \models \Box p \;\equiv\; (\sigma, j) \models \neg\Diamond\neg p$ depends on the rule, "p does not hold in a state iff $\neg p$ holds in that state", written formally as

$$\neg((\sigma, j) \models p) \quad \text{iff} \quad (\sigma, j) \models \neg p$$

The corresponding rules for the binary operators are

$$\begin{array}{lll}
((\sigma, j) \models p) \wedge ((\sigma, j) \models q) & \text{iff} & (\sigma, j) \models p \wedge q \\
((\sigma, j) \models p) \vee ((\sigma, j) \models q) & \text{iff} & (\sigma, j) \models p \vee q \\
((\sigma, j) \models p) \Rightarrow ((\sigma, j) \models q) & \text{iff} & (\sigma, j) \models p \Rightarrow q \\
((\sigma, j) \models p) \equiv ((\sigma, j) \models q) & \text{iff} & (\sigma, j) \models p \equiv q
\end{array}$$

The *wait* operator $\mathcal{W}$

The semantics of the binary infix operator $\mathcal{W}$ in terms of $\mathcal{U}$ and $\Box$ are

$$(\sigma, j) \models p \,\mathcal{W}\, q \quad \text{iff} \quad (\sigma, j) \models p \,\mathcal{U}\, q \vee (\sigma, j) \models \Box p$$

The *wait* operator $\mathcal{W}$ is weaker than the *until* operator $\mathcal{U}$, because $p \,\mathcal{W}\, q$ does not require q to ever be true, while $p \,\mathcal{U}\, q$ does. Furthermore, theorem (174) shows that $p \,\mathcal{U}\, q \Rightarrow p \,\mathcal{W}\, q$.

For example, suppose p and q evolve in the computation as follows.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	$\ldots$
p	F	F	T	T	F	F	F	F	T	T	T	$\ldots$
q	F	F	F	F	T	T	F	F	F	F	F	$\ldots$
$\Box p$	F	F	F	F	F	F	F	F	T	T	T	$\ldots$
$p \,\mathcal{U}\, q$	F	F	T	T	T	T	F	F	F	F	F	$\ldots$
$p \,\mathcal{W}\, q$	F	F	T	T	T	T	F	F	T	T	T	$\ldots$

The bottom two rows show the evaluation of the expressions $p \,\mathcal{U}\, q$ and $p \,\mathcal{W}\, q$ assuming that p remains true indefinitely and q remains false indefinitely. From s_0 to s_7, $p \,\mathcal{U}\, q$ and $p \,\mathcal{W}\, q$ hold in the same states. From s_8 on, however, $p \,\mathcal{U}\, q$ does not hold because q never holds thereafter, while $p \,\mathcal{W}\, q$ does hold because p always holds thereafter.

2.2.3 Duality

LADM [11] defines the dual P_D of a boolean expression P to be the expression constructed from P by interchanging occurrences of

true and *false*
$\wedge$ and $\vee$
$\equiv$ and $\not\equiv$
$\Rightarrow$ and $\not\Leftarrow$
$\Leftarrow$ and $\not\Rightarrow$

This definition of P_D gives rise to the following metatheorem from LADM.

(2.3) **Metatheorem Duality**
(a) P is valid iff $\neg P_D$ is valid
(b) $P \equiv Q$ is valid iff $P_D \equiv Q_D$ is valid

Linear temporal logic extends the definition of P_D for the temporal operators to include interchanging occurrences of $\circ$ and $\circ$ (self dual), and $\Box$ and $\Diamond$. It is not the case that $\mathcal{W}$ is the dual of $\mathcal{U}$. Instead, each is the dual of one of the following two operators.

Definition of Weak Release $\mathcal{R}$: $p \,\mathcal{R}\, q \equiv q \,\mathcal{W}\, (p \wedge q)$

Definition of Strong Release $\mathcal{M}$: $p \,\mathcal{M}\, q \equiv q \,\mathcal{U}\, (p \wedge q)$

With these definitions, the *dual* P_D of a LTL expression P is constructed from P by interchanging occurrences of:

true and *false*
$\wedge$ and $\vee$
$\equiv$ and $\not\equiv$
$\Rightarrow$ and $\not\Leftarrow$
$\Leftarrow$ and $\not\Rightarrow$
$\circ$ and $\circ$ (self dual)
$\Box$ and $\Diamond$
$\mathcal{U}$ and $\mathcal{R}$
$\mathcal{W}$ and $\mathcal{M}$

3 The Revised Temporal System

This section presents the revised axiomatic deductive system of temporal logic and proves its theorems with the calculation logic $\mathcal{E}$ of Gries and Schneider's *A Logical Approach to*

Discrete Math (LADM). [11] Theorems cited in a proof hint take two forms. A numbered reference enclosed in parentheses *without* a period is a reference to an axiom or a previously-proved theorem in this paper. A numbered reference enclosed in parentheses *with* a period is a reference to an axiom or a theorem from the propositional calculus in LADM. The numbering is consistent with that text with the chapter number followed by the equation number separated by the period. Additional theorems, either not included in LADM or included but not numbered, are indicated by a three-part number with two period separators. The terms "definition" and "axiom" are synonymous. The following exposition includes the theorems from LADM in the proof hints, except that theorems are omitted for (4.2) and (4.3) Monotonicity, as they are described in Section 2.1.

3.1 Next

The following two axioms define the *next* operator $\circ$.

(1) **Axiom, Self-dual:** $\circ \neg p \equiv \neg \circ p$

(2) **Axiom, Distributivity of $\circ$ over $\Rightarrow$:** $\circ (p \Rightarrow q) \equiv \circ p \Rightarrow \circ q$

Self duality states that p not holding in the next state is equivalent to *next* p not holding in the current state. Distributivity states that p implies q in the next state is equivalent to *next* p implies *next* q in the current state. From this axiom, subsequent theorems prove that the *next* operator distributes over all the propositional binary operators.

Linearity follows from self-dual.

(3) **Linearity:** $\circ p \equiv \neg \circ \neg p$

Proof:

$$
\begin{array}{ll}
 & \circ p \equiv \neg \circ \neg p \\
= & \langle (3.11)\ \neg p \equiv q \equiv p \equiv \neg q \text{ with } p, q := \circ \neg p, \circ p \rangle \\
 & \neg \circ p \equiv \circ \neg p \quad \text{—(1) Self-dual} \quad \blacksquare
\end{array}
$$

The proof that $\circ$ distributes over $\vee$ uses the distributivity of $\circ$ over $\Rightarrow$. The proofs that it also distributes over $\wedge$ and $\equiv$ are similar.

(4) **Distributivity of $\circ$ over $\vee$:** $\circ (p \vee q) \equiv \circ p \vee \circ q$

Proof:

$$
\begin{array}{ll}
 & \circ (p \vee q) \\
= & \langle (3.59) \text{ Implication } p \Rightarrow q \equiv \neg p \vee q \rangle \\
 & \circ (\neg p \Rightarrow q) \\
= & \langle (2) \text{ Distributivity of } \circ \text{ over } \Rightarrow \rangle
\end{array}
$$

$$
\begin{array}{ll}
& \bigcirc \neg p \Rightarrow \bigcirc q \\
= & \langle (3.59)\ \text{Implication } p \Rightarrow q \ \equiv\ \neg p \vee q \text{ with } p, q := \bigcirc \neg p, \bigcirc q \rangle \\
& \neg\bigcirc \neg p \vee \bigcirc q \\
= & \langle (3)\ \text{Linearity} \rangle \\
& \bigcirc p \vee \bigcirc q \quad \blacksquare
\end{array}
$$

(5) **Distributivity of $\bigcirc$ over $\wedge$:** $\bigcirc (p \wedge q) \equiv \bigcirc p \wedge \bigcirc q$

Proof:

$$
\begin{array}{ll}
& \bigcirc (p \wedge q) \\
= & \langle (3.12)\ \text{Double negation}, \neg\neg p \equiv p, \text{twice} \rangle \\
& \bigcirc (\neg\neg p \wedge \neg\neg q) \\
= & \langle (3.47b)\ \text{De Morgan}, \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
& \bigcirc \neg(\neg p \vee \neg q) \\
= & \langle (1)\ \text{Self-dual with } p := (\neg p \vee \neg q) \rangle \\
& \neg\bigcirc (\neg p \vee \neg q) \\
= & \langle (4)\ \text{Distributivity of } \bigcirc \text{ over } \vee \text{ with } p, q := \neg p, \neg q \rangle \\
& \neg(\bigcirc \neg p \vee \bigcirc \neg q) \\
= & \langle (3.47b)\ \text{De Morgan}, \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
& \neg\bigcirc \neg p \wedge \neg\bigcirc \neg q \\
= & \langle (3)\ \text{Linearity, twice} \rangle \\
& \bigcirc p \wedge \bigcirc q \quad \blacksquare
\end{array}
$$

(6) **Distributivity of $\bigcirc$ over $\equiv$:** $\bigcirc (p \equiv q) \equiv \bigcirc p \equiv \bigcirc q$

Proof:

$$
\begin{array}{ll}
& \bigcirc (p \equiv q) \\
= & \langle (3.80)\ \text{Mutual implication } (p \Rightarrow q) \wedge (q \Rightarrow p) \ \equiv\ (p \equiv q) \rangle \\
& \bigcirc ((p \Rightarrow q) \wedge (q \Rightarrow p)) \\
= & \langle (5)\ \text{Distributivity of } \bigcirc \text{ over } \wedge \rangle \\
& \bigcirc (p \Rightarrow q) \wedge \bigcirc (p \Rightarrow q) \\
= & \langle (2)\ \text{Distributivity of } \bigcirc \text{ over } \Rightarrow \rangle \\
& (\bigcirc p \Rightarrow \bigcirc q) \wedge (\bigcirc q \Rightarrow \bigcirc p) \\
= & \langle (3.80)\ \text{Mutual Implication } (p \Rightarrow q) \wedge (q \Rightarrow p) \ \equiv\ (p \equiv q) \rangle \\
& \bigcirc p \equiv \bigcirc q \quad \blacksquare
\end{array}
$$

Now, *true* holds in the next state, and *false* does not hold in the next state. In the calculational logic of LADM, *true* is theorem (3.4) and is equivalent to all other theorems.

Theorem (7) shows that all propositional logic theorems hold at the next state and, by induction, hold in all states. The proof of (7) uses (3.28) Excluded middle. The proof of (8) uses (3.8) Definition of *false*, $false \equiv \neg true$.

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$$
\begin{aligned}
& \bigcirc\, true \\
= \quad & \langle (3.28) \text{ Excluded middle } p \vee \neg p \rangle \\
& \bigcirc\, (p \vee \neg p) \\
= \quad & \langle (4) \text{ Distributivty of } \bigcirc \text{ over } \vee \rangle \\
& \bigcirc\, p \vee \bigcirc\, \neg p \\
= \quad & \langle (1) \text{ Self-dual} \rangle \\
& \bigcirc\, p \vee \neg \bigcirc\, p \\
= \quad & \langle (3.28) \text{ Excluded middle } p \vee \neg p \text{ with } p := \bigcirc\, p \rangle \\
& true \quad \blacksquare
\end{aligned}
$$

(8) **Falsehood of $\bigcirc$:** $\bigcirc\, false \equiv false$

Proof:

$$
\begin{aligned}
& \bigcirc\, false \equiv false \\
= \quad & \langle (3.8) \text{ Definition of } false, false \equiv \neg true \rangle \\
& \bigcirc\, \neg true \equiv \neg true \\
= \quad & \langle (3.11)\ \neg p \equiv q \equiv p \equiv \neg q \text{ with } p, q := true, \bigcirc\, \neg true \rangle \\
& \neg \bigcirc\, \neg true \equiv true \\
= \quad & \langle (3) \text{ Linearity} \rangle \\
& \bigcirc\, true \equiv true \quad \text{—(7) Truth of } \bigcirc\,. \quad \blacksquare
\end{aligned}
$$

3.2 Until

This system defines the *until* operator $\mathcal{U}$ with the following ten axioms. The first axiom, distributivity of $\bigcirc$ over $\mathcal{U}$, implies the distributivity of $\bigcirc$ over $\mathcal{W}$ as Section 3.9 shows. Thus, the *next* operator distributes over all binary operators, both propositional and temporal.

(9) **Axiom, Distributivity of $\bigcirc$ over $\mathcal{U}$:** $\bigcirc\, (p\, \mathcal{U}\, q) \equiv \bigcirc\, p\, \mathcal{U} \bigcirc\, q$

The second axiom, expansion of $\mathcal{U}$, makes the *until* operator different from most propositional binary operators. Its right operand has an existential characteristic and its left operand has a universal characteristic. Expansion states that $p\, \mathcal{U}\, q$ is true iff q is true in the current state, or p is true in the current state and $p\, \mathcal{U}\, q$ is true in the next state. Thus,

q relates to the definition through disjunction, which is existential, while p relates through conjunction, which is universal. Consequently, the *until* operator is neither symmetric (*i.e.* commutative) nor associative.

(10) **Axiom, Expansion of $\mathcal{U}$:** $p \,\mathcal{U}\, q \equiv q \vee (p \wedge \circ\,(p \,\mathcal{U}\, q))$

The third axiom states that *false* is the right zero of $\mathcal{U}$ and is not noted in other LTL deductive systems.

(11) **Axiom, Right zero of $\mathcal{U}$:** $p \,\mathcal{U}\, \mathit{false} \equiv \mathit{false}$

The next four axioms describe how the *until* operator distributes over conjunction and disjunction. Because $\mathcal{U}$ is not symmetric, this system requires separate axioms for left and right distributivity.

(12) **Axiom, Left distributivity of $\mathcal{U}$ over $\vee$:** $p \,\mathcal{U}\, (q \vee r) \equiv p \,\mathcal{U}\, q \vee p \,\mathcal{U}\, r$

(13) **Axiom, Right distributivity of $\mathcal{U}$ over $\vee$:** $p \,\mathcal{U}\, r \vee q \,\mathcal{U}\, r \Rightarrow (p \vee q) \,\mathcal{U}\, r$

(14) **Axiom, Left distributivity of $\mathcal{U}$ over $\wedge$:** $p \,\mathcal{U}\, (q \wedge r) \Rightarrow p \,\mathcal{U}\, q \wedge p \,\mathcal{U}\, r$

(15) **Axiom, Right distributivity of $\mathcal{U}$ over $\wedge$:** $(p \wedge q) \,\mathcal{U}\, r \equiv p \,\mathcal{U}\, r \wedge q \,\mathcal{U}\, r$

The *until* operator is not associative. The last three axioms describe the ordering property, the right ordering property under disjunction, and the right ordering property under conjunction, of $\mathcal{U}$. These theorems do not appear in other LTL systems. Other systems do, however, list their $\mathcal{W}$ versions, which in this system are (253), (249), and (250).

(16) **Axiom, $\mathcal{U}$ implication ordering:** $p \,\mathcal{U}\, q \wedge \neg q \,\mathcal{U}\, r \Rightarrow p \,\mathcal{U}\, r$

(17) **Axiom, Right $\mathcal{U} \vee$ ordering:** $p \,\mathcal{U}\, (q \,\mathcal{U}\, r) \Rightarrow (p \vee q) \,\mathcal{U}\, r$

(18) **Axiom, Right $\wedge \mathcal{U}$ ordering:** $p \,\mathcal{U}\, (q \wedge r) \Rightarrow (p \,\mathcal{U}\, q) \,\mathcal{U}\, r$

Theorem (19) shows how $\mathcal{U}$ distributes over $\Rightarrow$ and is not listed in other deductive systems.

(19) **Right distributivity of $\mathcal{U}$ over $\Rightarrow$:** $(p \Rightarrow q) \,\mathcal{U}\, r \Rightarrow (p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r)$

Proof:

$$
\begin{aligned}
& (p \Rightarrow q) \,\mathcal{U}\, r \Rightarrow (p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r) \\
= \quad & \langle \text{(3.65) Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
& (p \Rightarrow q) \,\mathcal{U}\, r \wedge p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r
\end{aligned}
$$

And now,

$$
\begin{array}{ll}
& (p \Rightarrow q) \,\mathcal{U}\, r \wedge p \,\mathcal{U}\, r \\
= & \langle (15) \text{ Right distributivity of } \mathcal{U} \text{ over } \wedge \rangle \\
& (p \wedge (p \Rightarrow q)) \,\mathcal{U}\, r \\
= & \langle (3.66)\ p \wedge (p \Rightarrow q) \ \equiv\ p \wedge q \rangle \\
& (p \wedge q) \,\mathcal{U}\, r \\
= & \langle (15) \text{ Right distributivity of } \mathcal{U} \text{ over } \wedge \rangle \\
& p \,\mathcal{U}\, r \wedge q \,\mathcal{U}\, r \\
\Rightarrow & \langle (3.76\text{b}) \text{ Strengthening, } p \wedge q \Rightarrow p \rangle \\
& q \,\mathcal{U}\, r \quad \blacksquare
\end{array}
$$

Theorem (20) shows that *true* is a right zero of $\mathcal{U}$, which is unusual because axiom (11) shows that *false* is also a right zero of $\mathcal{U}$. Theorem (21) shows that *false* is the left identity of $\mathcal{U}$. Proofs of both use (10) Expansion of $\mathcal{U}$. Theorems (11), (20), and (21) cover three of the possibilities of constants *true* and *false* on either side of $\mathcal{U}$. None of these three theorems seem to appear in the temporal logic literature. The fourth possibility with *true* as the left argument is the basis of the definition of the *eventually* operator $\Diamond$ in Section 3.3.

(20) **Right zero of $\mathcal{U}$:** $p \,\mathcal{U}\, \mathit{true} \equiv \mathit{true}$

Proof:

$$
\begin{array}{ll}
& p \,\mathcal{U}\, \mathit{true} \\
= & \langle (10) \text{ Expansion of } \mathcal{U} \rangle \\
& \mathit{true} \vee (p \wedge \circ\, (p \,\mathcal{U}\, \mathit{true})) \\
= & \langle (3.29) \text{ Zero of } \vee,\ p \vee \mathit{true} \equiv \mathit{true} \rangle \\
& \mathit{true} \quad \blacksquare
\end{array}
$$

(21) **Left identity of $\mathcal{U}$:** $\mathit{false} \,\mathcal{U}\, q \equiv q$

Proof:

$$
\begin{array}{ll}
& \mathit{false} \,\mathcal{U}\, q \\
= & \langle (10) \text{ Expansion of } \mathcal{U} \rangle \\
& q \vee (\mathit{false} \wedge \circ\, (\mathit{false} \,\mathcal{U}\, q)) \\
= & \langle (3.40) \text{ Zero of } \wedge,\ p \wedge \mathit{false} \equiv \mathit{false} \rangle \\
& q \vee \mathit{false} \\
= & \langle (3.30) \text{ Identity of } \vee,\ p \vee \mathit{false} \equiv p \rangle \\
& q \quad \blacksquare
\end{array}
$$

Theorem (22) shows that the *until* operator is idempotent. Its proof uses (10) Expansion of $\mathcal{U}$ followed by (3.43b) Absorption. Theorem (23) is the *until* version of excluded middle, which is proved from (12) Left distributivity of $\mathcal{U}$ over $\vee$. The proof of (24) illustrates proof by Truth implication. The proof of (25) is similar. The proofs of (26) and (27) use (16) $\mathcal{U}$ implication ordering. The proofs of (28), (29), and (30) require only two steps.

(22) **Idempotency of $\mathcal{U}$:** $p \,\mathcal{U}\, p \equiv p$

Proof:

$$
\begin{aligned}
& p \,\mathcal{U}\, p \\
= \quad & \langle (10) \text{ Expansion of } \mathcal{U} \rangle \\
& p \vee (p \wedge \circ (p \,\mathcal{U}\, p)) \\
= \quad & \langle (3.43b) \text{ Absorption, } p \vee (p \wedge q) \equiv p \text{ with } q := \circ (p \,\mathcal{U}\, p) \rangle \\
& p \quad \blacksquare
\end{aligned}
$$

(23) $\mathcal{U}$ **excluded middle:** $p \,\mathcal{U}\, q \vee p \,\mathcal{U}\, \neg q$

Proof: (Ravi Mohan)

$$
\begin{aligned}
& p \,\mathcal{U}\, q \vee p \,\mathcal{U}\, \neg q \\
= \quad & \langle (12) \text{ Left distributivity of } \mathcal{U} \text{ over } \vee \rangle \\
& p \,\mathcal{U}\, (q \vee \neg q) \\
= \quad & \langle (3.28) \text{ Excluded middle, } p \vee \neg p \rangle \\
& p \,\mathcal{U}\, \mathit{true} \\
= \quad & \langle (20) \text{ Right zero of } \mathcal{U} \rangle \\
& \mathit{true} \quad \blacksquare
\end{aligned}
$$

(24) $\neg p \,\mathcal{U}\, (q \,\mathcal{U}\, r) \wedge p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r$

Proof: The proof is by (4.7.1) Truth implication.

$$
\begin{aligned}
& \mathit{true} \\
\Rightarrow \quad & \langle (17) \text{ Right } \mathcal{U} \vee \text{ ordering with } p := \neg p \rangle \\
& \neg p \,\mathcal{U}\, (q \,\mathcal{U}\, r) \Rightarrow (\neg p \vee q) \,\mathcal{U}\, r \\
= \quad & \langle (3.59) \text{ Implication, } p \Rightarrow q \ \equiv \ \neg p \vee q \rangle \\
& \neg p \,\mathcal{U}\, (q \,\mathcal{U}\, r) \Rightarrow (p \Rightarrow q) \,\mathcal{U}\, r \\
\Rightarrow \quad & \langle (19) \text{ Right distributivity of } \mathcal{U} \text{ over } \Rightarrow \text{ and (3.82a) Transitivity} \rangle \\
& \neg p \,\mathcal{U}\, (q \,\mathcal{U}\, r) \Rightarrow (p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r) \\
= \quad & \langle (3.65) \text{ Shunting, } p \wedge q \Rightarrow r \ \equiv \ p \Rightarrow (q \Rightarrow r) \rangle \\
& \neg p \,\mathcal{U}\, (q \,\mathcal{U}\, r) \wedge p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r \quad \blacksquare
\end{aligned}
$$

(25) $p \,\mathcal{U}\, (\neg q \,\mathcal{U}\, r) \wedge q \,\mathcal{U}\, r \Rightarrow p \,\mathcal{U}\, r$

Proof: The proof is by (4.7.1) Truth implication.

$true$
$\Rightarrow \quad \langle(17)$ Right $\mathcal{U} \vee$ ordering with $q := \neg q\rangle$
$p\,\mathcal{U}\,(\neg q\,\mathcal{U}\,r) \Rightarrow (p \vee \neg q)\,\mathcal{U}\,r$
$= \quad \langle(3.59)$ Implication, $p \Rightarrow q \ \equiv\ \neg p \vee q\rangle$
$p\,\mathcal{U}\,(\neg q\,\mathcal{U}\,r) \Rightarrow (q \Rightarrow p)\,\mathcal{U}\,r$
$\Rightarrow \quad \langle(19)$ Right distributivity of $\mathcal{U}$ over $\Rightarrow$ and (3.82a) Transitivity$\rangle$
$p\,\mathcal{U}\,(\neg q\,\mathcal{U}\,r) \Rightarrow (q\,\mathcal{U}\,r \Rightarrow p\,\mathcal{U}\,r)$
$= \quad \langle(3.65)$ Shunting, $p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r)\rangle$
$p\,\mathcal{U}\,(\neg q\,\mathcal{U}\,r) \wedge q\,\mathcal{U}\,r \Rightarrow p\,\mathcal{U}\,r$ ∎

(26) $p\,\mathcal{U}\,q \wedge \neg q\,\mathcal{U}\,p \Rightarrow p$

Proof:

$p\,\mathcal{U}\,q \wedge \neg q\,\mathcal{U}\,p$
$\Rightarrow \quad \langle(16)$ $\mathcal{U}$ implication ordering with $r := p\rangle$
$p\,\mathcal{U}\,p$
$= \quad \langle(22)$ Idempotency of $\mathcal{U}\rangle$
p ∎

(27) $p \wedge \neg p\,\mathcal{U}\,q \Rightarrow q$

Proof: The proof is by (4.7.1) Truth implication.

$true$
$\Rightarrow \quad \langle(16)$ $\mathcal{U}$ implication ordering with $p, q, r := false, p, q\rangle$
$false\,\mathcal{U}\,p \wedge \neg p\,\mathcal{U}\,q \Rightarrow false\,\mathcal{U}\,q$
$= \quad \langle(21)$ Left identity of $\mathcal{U}$, twice$\rangle$
$p \wedge \neg p\,\mathcal{U}\,q \Rightarrow q$ ∎

(28) $p\,\mathcal{U}\,q \Rightarrow p \vee q$

Proof:

$p\,\mathcal{U}\,q$
$= \quad \langle(10)$ Expansion of $\mathcal{U}\rangle$
$q \vee (p \wedge \circ\,(p\,\mathcal{U}\,q))$
$\Rightarrow \quad \langle(3.76\text{d})$ $p \vee (q \wedge r) \Rightarrow p \vee q$ with $p, q, r := q, p, \circ\,(p\,\mathcal{U}\,q)\rangle$
$p \vee q$ ∎

(29) $\mathcal{U}$ **Insertion:** $q \Rightarrow p\,\mathcal{U}\,q$

Proof:

$$
\begin{array}{ll}
 & p \,\mathcal{U}\, q \\
= & \langle (10) \text{ Expansion of } \mathcal{U} \rangle \\
 & q \vee (p \wedge \circ\, (p \,\mathcal{U}\, q)) \\
\Leftarrow & \langle (3.76\text{a}) \text{ Weakening, } p \Rightarrow p \vee q \rangle \\
 & q \quad \blacksquare
\end{array}
$$

(30) $p \wedge q \Rightarrow p \,\mathcal{U}\, q$

Proof:

$$
\begin{array}{ll}
 & p \wedge q \\
\Rightarrow & \langle (3.76\text{b}) \text{ Strengthening, } p \wedge q \Rightarrow p \rangle \\
 & q \\
\Rightarrow & \langle (29) \ \mathcal{U} \text{ insertion} \rangle \\
 & p \,\mathcal{U}\, q \quad \blacksquare
\end{array}
$$

This system has the following five absorption properties that do not seem to appear in the temporal logic literature. Most can be proved in one step.

(31) **Absorption:** $p \vee p \,\mathcal{U}\, q \equiv p \vee q$

Proof:

$$
\begin{array}{ll}
 & p \vee p \,\mathcal{U}\, q \\
= & \langle (10) \text{ Expansion of } \mathcal{U} \rangle \\
 & p \vee q \vee (p \wedge \circ\, (p \,\mathcal{U}\, q)) \\
= & \langle (3.43\text{b}) \text{ Absorption } p \vee (p \wedge q) \equiv p \rangle \\
 & p \vee q \quad \blacksquare
\end{array}
$$

(32) **Absorption:** $p \,\mathcal{U}\, q \vee q \equiv p \,\mathcal{U}\, q$

Proof:

$$
\begin{array}{ll}
 & p \,\mathcal{U}\, q \vee q \equiv p \,\mathcal{U}\, q \\
= & \langle (3.57) \text{ Definition of implication, } p \Rightarrow q \ \equiv \ p \vee q \ \equiv \ q \rangle \\
 & q \Rightarrow p \,\mathcal{U}\, q \quad \text{—(29) } \mathcal{U} \text{ Insertion} \quad \blacksquare
\end{array}
$$

(33) **Absorption:** $p \,\mathcal{U}\, q \wedge q \equiv q$

Proof:

$$
\begin{array}{ll}
 & p \,\mathcal{U}\, q \wedge q \equiv q \\
= & \langle (3.60) \text{ Implication, } p \Rightarrow q \ \equiv \ p \wedge q \ \equiv \ p \rangle \\
 & q \Rightarrow p \,\mathcal{U}\, q \quad \text{—(29) } \mathcal{U} \text{ Insertion} \quad \blacksquare
\end{array}
$$

(34) **Absorption:** $p \,\mathcal{U}\, q \vee (p \wedge q) \equiv p \,\mathcal{U}\, q$

Proof:

$$\begin{array}{ll} & p \,\mathcal{U}\, q \vee (p \wedge q) \equiv p \,\mathcal{U}\, q \\ = & \langle \text{(3.57) Definition of implication, } p \Rightarrow q \;\equiv\; p \vee q \;\equiv\; q \rangle \\ & p \wedge q \Rightarrow p \,\mathcal{U}\, q \quad \text{—(30).} \quad \blacksquare \end{array}$$

(35) **Absorption:** $p \,\mathcal{U}\, q \wedge (p \vee q) \equiv p \,\mathcal{U}\, q$

Proof: (Ravi Mohan)

$$\begin{array}{ll} & p \,\mathcal{U}\, q \wedge (p \vee q) \equiv p \,\mathcal{U}\, q \\ = & \langle \text{(3.60) Implication, } p \Rightarrow q \;\equiv\; p \wedge q \;\equiv\; p \rangle \\ & p \,\mathcal{U}\, q \Rightarrow p \vee q \quad \text{—(28)} \quad \blacksquare \end{array}$$

All systems have the following two absorption theorems. Manna and Pnueli [19] refer to these as idempotence properties. This paper follows Schneider [21], which refers to them as absorption properties. The proof of each uses mutual implication.

(36) **Left absorption of $\mathcal{U}$:** $p \,\mathcal{U}\, (p \,\mathcal{U}\, q) \equiv p \,\mathcal{U}\, q$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$$\begin{array}{ll} & p \,\mathcal{U}\, (p \,\mathcal{U}\, q) \\ \Rightarrow & \langle \text{(17) Right } \mathcal{U} \vee \text{ ordering} \rangle \\ & (p \vee p) \,\mathcal{U}\, q \\ = & \langle \text{(3.26) Idempotency of } \vee, \; p \vee p \equiv p \rangle \\ & p \,\mathcal{U}\, q \end{array}$$

The proof in the second direction follows.

$$\begin{array}{ll} & p \,\mathcal{U}\, q \\ \Rightarrow & \langle \text{(29) } \mathcal{U} \text{ Insertion with } q := p \,\mathcal{U}\, q \rangle \\ & p \,\mathcal{U}\, (p \,\mathcal{U}\, q) \quad \blacksquare \end{array}$$

(37) **Right absorption of $\mathcal{U}$:** $(p \,\mathcal{U}\, q) \,\mathcal{U}\, q \equiv p \,\mathcal{U}\, q$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$$\begin{array}{ll} & (p \,\mathcal{U}\, q) \,\mathcal{U}\, q \\ \Rightarrow & \langle \text{(28) with } p := p \,\mathcal{U}\, q \rangle \\ & p \,\mathcal{U}\, q \vee q \\ = & \langle \text{(32) Absorption} \rangle \\ & p \,\mathcal{U}\, q \end{array}$$

The proof in the second direction follows.

$$
\begin{array}{ll}
 & (p \,\mathcal{U}\, q) \,\mathcal{U}\, q \\
\Leftarrow & \langle \text{(18) Right } \wedge \,\mathcal{U}\, \text{ ordering} \rangle \\
 & p \,\mathcal{U}\, (q \wedge q) \\
= & \langle \text{(3.38) Idempotency of } \wedge,\ p \wedge p \equiv p \rangle \\
 & p \,\mathcal{U}\, q \quad \blacksquare
\end{array}
$$

3.3 Eventually

Eventually $\Diamond$ is a special case of $\mathcal{U}$ when the left hand side is *true*. Equation (38) is its only defining axiom.

(38) **Definition of $\Diamond$:** $\Diamond q \equiv true \,\mathcal{U}\, q$

Theorem (39) shows how the unary operator *eventually* absorbs into the binary operator *until*. Its proof uses (15) Right distributivity of $\mathcal{U}$ over $\wedge$. Theorems (40) and (41) show also that the binary operator *until* absorbs into the unary operator *eventually*. They are proved by mutual implication. Theorem (42) shows that $p \,\mathcal{U}\, q$ guarantees that q will eventually be *true*. Its proof uses (39). Theorems (43) and (44), Truth and Falsehood of $\Diamond$, do not appear in other LTL systems. Their proofs are simple.

(39) **Absorption of $\Diamond$ into $\mathcal{U}$:** $p \,\mathcal{U}\, q \wedge \Diamond q \equiv p \,\mathcal{U}\, q$

Proof:

$$
\begin{array}{ll}
 & p \,\mathcal{U}\, q \wedge \Diamond q \\
= & \langle \text{(38) Definition of } \Diamond \rangle \\
 & p \,\mathcal{U}\, q \wedge true \,\mathcal{U}\, q \\
= & \langle \text{(15) Right distributivity of } \mathcal{U} \text{ over } \wedge \rangle \\
 & (p \wedge true) \,\mathcal{U}\, q \\
= & \langle \text{(3.39) Identity of } \wedge,\ p \wedge true \equiv p \rangle \\
 & p \,\mathcal{U}\, q \quad \blacksquare
\end{array}
$$

(40) **Absorption of $\mathcal{U}$ into $\Diamond$:** $p \,\mathcal{U}\, q \vee \Diamond q \equiv \Diamond q$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$$
\begin{array}{ll}
 & p \,\mathcal{U}\, \Diamond q \vee \Diamond q \\
= & \langle \text{(38) Definition of } \Diamond \rangle \\
 & p \,\mathcal{U}\, q \vee true \,\mathcal{U}\, q \\
\Rightarrow & \langle \text{(13) Right distributivity of } \mathcal{U} \text{ over } \vee \rangle
\end{array}
$$

$$\begin{array}{ll} & (p \vee true)\; \mathcal{U}\; q \\ = & \langle (3.29)\text{ Zero of } \vee,\ p \vee true \equiv true \rangle \\ & true\; \mathcal{U}\; q \\ = & \langle (38)\text{ Definition of } \Diamond \rangle \\ & \Diamond q \end{array}$$

The proof in the second direction follows.

$$\begin{array}{ll} & \Diamond q \\ \Rightarrow & \langle (3.76a)\text{ Weakening the consequent, } p \Rightarrow p \vee q \rangle \\ & p\; \mathcal{U}\; q \vee \Diamond q \quad \blacksquare \end{array}$$

(41) **Absorption of $\mathcal{U}$ into $\Diamond$:** $p\; \mathcal{U} \Diamond q \equiv \Diamond q$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$$\begin{array}{ll} & p\; \mathcal{U} \Diamond q \\ = & \langle (38)\text{ Definition of } \Diamond \rangle \\ & p\; \mathcal{U}\; (true\; \mathcal{U}\; q) \\ \Rightarrow & \langle (17)\text{ Right } \mathcal{U} \vee \text{ ordering with } q, r := true, q \rangle \\ & (p \vee true)\; \mathcal{U}\; q \\ = & \langle (3.29)\text{ Zero of } \vee,\ p \vee true \equiv true \rangle \\ & true\; \mathcal{U}\; q \\ = & \langle (38)\text{ Definition of } \Diamond \rangle \\ & \Diamond q \end{array}$$

The proof in the second direction follows.

$$\begin{array}{ll} & \Diamond q \\ \Rightarrow & \langle (29)\ \mathcal{U}\text{ Insertion with } q := \Diamond q \rangle \\ & p\; \mathcal{U} \Diamond q \quad \blacksquare \end{array}$$

(42) **Eventuality:** $p\; \mathcal{U}\; q \Rightarrow \Diamond q$

Proof:

$$\begin{array}{ll} & p\; \mathcal{U}\; q \\ = & \langle (39)\text{Absorption of } \Diamond \text{ into } \mathcal{U} \rangle \\ & p\; \mathcal{U}\; q \wedge \Diamond q \\ \Rightarrow & \langle (3.76b)\text{ Strengthening the antecedent, } p \wedge q \Rightarrow p \rangle \\ & \Diamond q \quad \blacksquare \end{array}$$

(43) **Truth of $\Diamond$:** $\Diamond\, true \equiv true$

Proof:

$$
\begin{aligned}
& \Diamond\, true \\
= \quad & \langle (38) \text{ Definition of } \Diamond \rangle \\
& true\ \mathcal{U}\ true \\
= \quad & \langle (22) \text{ Idempotency of } \mathcal{U} \rangle \\
& true \quad \blacksquare
\end{aligned}
$$

(44) **Falsehood of $\Diamond$:** $\Diamond\, false \equiv false$

Proof:

$$
\begin{aligned}
& \Diamond\, false \\
= \quad & \langle (38) \text{ Definition of } \Diamond \rangle \\
& true\ \mathcal{U}\ false \\
= \quad & \langle (11) \text{ Right zero of } \mathcal{U} \rangle \\
& false \quad \blacksquare
\end{aligned}
$$

Expansion of $\Diamond$, like expansion of $\mathcal{U}$, has two disjuncts. The first describes the current state and the second contains the operation in the next state. The expansion of $\Diamond$ follows directly from the expansion of $\mathcal{U}$. The two weakening theorems (46) and (47) follow directly from expansion of $\Diamond$.

(45) **Expansion of $\Diamond$:** $\Diamond\, p \equiv p \vee \bigcirc \Diamond\, p$

Proof:

$$
\begin{aligned}
& \Diamond\, p \\
= \quad & \langle (38) \text{ Definition of } \Diamond \rangle \\
& true\ \mathcal{U}\ p \\
= \quad & \langle (10) \text{ Expansion of } \mathcal{U} \rangle \\
& p \vee (true \wedge \bigcirc (true\ \mathcal{U}\ p)) \\
= \quad & \langle (38) \text{ Definition of } \Diamond \rangle \\
& p \vee (true \wedge \bigcirc \Diamond\, p) \\
= \quad & \langle (3.39) \text{ Identity of } \wedge,\ p \wedge true \equiv p \rangle \\
& p \vee \bigcirc \Diamond\, p \quad \blacksquare
\end{aligned}
$$

(46) **Weakening of $\Diamond$:** $p \Rightarrow \Diamond\, p$

Proof:

$$
\begin{array}{ll}
 & \Diamond p \\
= & \langle (45) \text{ Expansion of } \Diamond \rangle \\
 & p \vee \bigcirc \Diamond p \\
\Leftarrow & \langle (3.76\text{a}) \text{ Weakening the consequent}, \ p \Rightarrow p \vee q \rangle \\
 & p \quad \blacksquare
\end{array}
$$

(47) **Weakening of $\Diamond$:** $\bigcirc p \Rightarrow \Diamond p$

Proof:

$$
\begin{array}{ll}
 & \Diamond p \\
= & \langle (45) \text{ Expansion of } \Diamond \rangle \\
 & p \vee \bigcirc \Diamond p \\
= & \langle (45) \text{ Expansion of } \Diamond \rangle \\
 & p \vee \bigcirc (p \vee \bigcirc \Diamond p) \\
= & \langle (4) \text{ Distributivity of } \bigcirc \text{ over } \vee \rangle \\
 & p \vee \bigcirc p \vee \bigcirc \bigcirc \Diamond p \\
\Leftarrow & \langle (3.76\text{a}) \text{ Weakening the consequent}, \ p \Rightarrow p \vee q \rangle \\
 & \bigcirc p \quad \blacksquare
\end{array}
$$

The two absorption theorems (48) and (49) do not seem to appear in the temporal logic literature. The following four theorems (50), (51), (52), and (53) are common to all temporal logic systems.

(48) **Absorption of $\vee$ into $\Diamond$:** $p \vee \Diamond p \equiv \Diamond p$

Proof: (Ravi Mohan)

$$
\begin{array}{ll}
 & p \vee \Diamond p \equiv \Diamond p \\
= & \langle (3.57) \text{ Definition of Implication}, \ p \Rightarrow q \ \equiv \ p \vee q \ \equiv \ q \rangle \\
 & p \Rightarrow \Diamond p \quad \text{—(46) Weakening of } \Diamond \quad \blacksquare
\end{array}
$$

(49) **Absorption of $\Diamond$ into $\wedge$:** $\Diamond p \wedge p \equiv p$

Proof: (Ravi Mohan)

$$
\begin{array}{ll}
 & \Diamond p \wedge p \equiv p \\
= & \langle (3.60) \text{ Implication}, \ p \Rightarrow q \ \equiv \ p \wedge q \ \equiv \ p \rangle \\
 & p \Rightarrow \Diamond p \quad \text{—(46) Weakening of } \Diamond \quad \blacksquare
\end{array}
$$

(50) **Absorption of $\Diamond$:** $\Diamond \Diamond p \equiv \Diamond p$

Proof:

$\Diamond \Diamond p$
$= \quad \langle$(38) Definition of $\Diamond$, with $p := \Diamond p\rangle$
$true\, \mathcal{U} \Diamond p$
$= \quad \langle$(41) Absorption of $\mathcal{U}$ into $\Diamond$, with $p, q := true, p\rangle$
$\Diamond p$ ∎

(51) **Exchange of $\circ$ and $\Diamond$:** $\circ \Diamond p \equiv \Diamond \circ p$

Proof:

$\circ \Diamond p$
$= \quad \langle$(38) Definition of $\Diamond\rangle$
$\circ (true\, \mathcal{U}\, p)$
$= \quad \langle$(9) Distributivity of $\circ$ over $\mathcal{U}\rangle$
$\circ\, true\, \mathcal{U} \circ p$
$= \quad \langle$(7) Truth of $\circ\rangle$
$true\, \mathcal{U} \circ p$
$= \quad \langle$(38) Definition of $\Diamond\rangle$
$\Diamond \circ p$ ∎

(52) **Distributivity of $\Diamond$ over $\vee$:** $\Diamond (p \vee q) \equiv \Diamond p \vee \Diamond q$

Proof:

$\Diamond (p \vee q)$
$= \quad \langle$(38) Definition of $\Diamond\rangle$
$true\, \mathcal{U}\, (p \vee q)$
$= \quad \langle$(12) Left distributivity of $\mathcal{U}$ over $\vee\rangle$
$true\, \mathcal{U}\, p \vee true\, \mathcal{U}\, q$
$= \quad \langle$(38) Definition of $\Diamond$, twice$\rangle$
$\Diamond p \vee \Diamond q$ ∎

(53) **Distributivity of $\Diamond$ over $\wedge$:** $\Diamond (p \wedge q) \Rightarrow \Diamond p \wedge \Diamond q$

Proof:

$\Diamond (p \wedge q)$
$= \quad \langle$(38) Definition of $\Diamond\rangle$
$true\, \mathcal{U}\, (p \wedge q)$
$\Rightarrow \quad \langle$(14) Left distributivity of $\mathcal{U}$ over $\wedge\rangle$
$true\, \mathcal{U}\, p \wedge true\, \mathcal{U}\, q$
$= \quad \langle$(38) Definition of $\Diamond$, twice$\rangle$
$\Diamond p \wedge \Diamond q$ ∎

3.4 Always

This system defines the *always* operator $\Box$ in terms of the *eventually* operator $\Diamond$. $\Box p$ is true when p is true in the current state and in all future states. The defining equation (54) states that p is always true iff it is not the case that $\neg p$ is eventually true. The two induction axioms do not appear in other LTL systems, either as axioms or theorems.

(54) **Definition of $\Box$:** $\Box p \equiv \neg\Diamond\neg p$

(55) **Axiom, $\mathcal{U}$ Induction:** $\Box(p \Rightarrow (\circ p \wedge q) \vee r) \Rightarrow (p \Rightarrow \Box q \vee q\,\mathcal{U}\,r)$

(56) **Axiom, $\mathcal{U}$ Induction:** $\Box(p \Rightarrow \circ(p \vee q)) \Rightarrow (p \Rightarrow \Box p \vee p\,\mathcal{U}\,q)$

Induction theorem (57) is common to many systems. It follows from (56) with $q := \mathit{false}$. The negation of the dual of theorem (58) is equivalent to theorem (57). Theorem (59) expresses $\Diamond p$ in terms of $\Box p$ and is the dual of the defining equation (54).

(57) **$\Box$ Induction:** $\Box(p \Rightarrow \circ p) \Rightarrow (p \Rightarrow \Box p)$

Proof:

$$
\begin{array}{ll}
 & \mathit{true} \\
= & \langle (56) \text{ with } q := \mathit{false} \rangle \\
 & \Box(p \Rightarrow \circ(p \vee \mathit{false})) \Rightarrow (p \Rightarrow \Box p \vee p\,\mathcal{U}\,\mathit{false}) \\
= & \langle (11) \text{ Right zero of } \mathcal{U} \rangle \\
 & \Box(p \Rightarrow \circ(p \vee \mathit{false})) \Rightarrow (p \Rightarrow \Box p \vee \mathit{false}) \\
= & \langle (3.30) \text{ Identity of } \vee \quad p \vee \mathit{false} \equiv p, \text{twice} \rangle \\
 & \Box(p \Rightarrow \circ p) \Rightarrow (p \Rightarrow \Box p) \quad \blacksquare
\end{array}
$$

(58) **$\Diamond$ Induction:** $\Box(\circ p \Rightarrow p) \Rightarrow (\Diamond p \Rightarrow p)$

Proof: The proof is by (2.3a) Metatheorem Duality. The negation of the dual of (58) is

$$
\begin{array}{ll}
 & \neg(\Diamond(\circ p \not\Leftarrow p) \not\Leftarrow (\Box p \not\Leftarrow p)) \\
= & \langle \text{Definition of } / \text{ and (3.58) Consequence } p \Leftarrow q \equiv q \Rightarrow p \rangle \\
 & \neg\neg((\Box p \not\Leftarrow p) \Rightarrow \Diamond(\circ p \not\Leftarrow p)) \\
= & \langle (3.12) \text{ Double negation, } \neg\neg p \equiv p \rangle \\
 & (\Box p \not\Leftarrow p) \Rightarrow \Diamond(\circ p \not\Leftarrow p) \\
= & \langle \text{Definition of } / \text{ and (3.58) Consequence } p \Leftarrow q \equiv q \Rightarrow p, \text{twice} \rangle \\
 & \neg(p \Rightarrow \Box p) \Rightarrow \Diamond\neg(p \Rightarrow \circ p) \\
= & \langle (3.61) \text{ Contrapositive } p \Rightarrow q \ \equiv\ \neg q \Rightarrow \neg p \rangle \\
 & \neg\Diamond\neg(p \Rightarrow \circ p) \Rightarrow (p \Rightarrow \Box p) \\
= & \langle (54) \text{ Definition of } \Box \rangle \\
 & \Box(p \Rightarrow \circ p) \Rightarrow (p \Rightarrow \Box p) \quad \text{—(57) } \Box \text{ Induction} \quad \blacksquare
\end{array}
$$

(59) $\Diamond p \equiv \neg\Box\neg p$

Proof:

$$
\begin{array}{ll}
 & \neg\Box\neg p \\
= & \langle (54) \text{ Definition of } \Box \rangle \\
 & \neg\neg\Diamond\neg\neg p \\
= & \langle (3.12) \text{ Double negation, } \neg\neg p \equiv p, \text{twice} \rangle \\
 & \Diamond p \quad \blacksquare
\end{array}
$$

Whereas the *next* operator $\bigcirc$ is its own dual, the *eventually* operator $\Diamond$ and the *always* operator $\Box$ are mutually dual, as are $\Diamond\Box$ and $\Box\Diamond$. Each of the following four theorems can be proved directly without invoking (2.3) Metatheorem Duality. However, with P and Q defined as the expressions $P : \neg\Box p$ and $Q : \Diamond\neg p$, the dual expressions are $P_D : \neg\Diamond p$ and $Q_D : \Box\neg p$. Because theorem (60) is the expression $P \equiv Q$ and theorem (61) is the expression $P_D \equiv Q_D$, the validity of (61) can be asserted by invoking (2.3b) Metatheorem Duality with theorem (60). Similarly, the validity of (63) can be asserted by invoking duality with theorem (62).

(60) **Dual of $\Box$:** $\neg\Box p \equiv \Diamond\neg p$

Proof:

$$
\begin{array}{ll}
 & \neg\Box p \equiv \Diamond\neg p \\
= & \langle (3.11)\ \neg p \equiv q \equiv p \equiv \neg q \text{ with } p, q := \Box p, \Diamond\neg p \rangle \\
 & \Box p \equiv \neg\Diamond\neg p \quad \text{—(54) Definition of } \Box \quad \blacksquare
\end{array}
$$

(61) **Dual of $\Diamond$:** $\neg\Diamond p \equiv \Box\neg p$

Proof:

$$
\begin{array}{ll}
 & \Box\neg p \\
= & \langle (54) \text{ Definition of } \Box \rangle \\
 & \neg\Diamond\neg\neg p \\
= & \langle (3.12) \text{ Double negation, } \neg\neg p \equiv p \rangle \\
 & \neg\Diamond p \quad \blacksquare
\end{array}
$$

(62) **Dual of $\Diamond\Box$:** $\neg\Diamond\Box p \equiv \Box\Diamond\neg p$

Proof:

$$
\begin{array}{ll}
 & \neg\Diamond\Box p \\
= & \langle (61) \text{ Dual of } \Diamond, \text{ with } p := \Box p \rangle \\
 & \Box\neg\Box p \\
= & \langle (60) \text{ Dual of } \Box \rangle \\
 & \Box\Diamond\neg p \quad \blacksquare
\end{array}
$$

(63) **Dual of $\Box\Diamond$:** $\neg\Box\Diamond p \equiv \Diamond\Box\neg p$

Proof:

$$\begin{array}{ll} & \neg\Box\Diamond p \\ = & \langle (60)\ \text{Dual of } \Box,\ \text{with } p := \Diamond p \rangle \\ & \Diamond\neg\Diamond p \\ = & \langle (61)\ \text{Dual of } \Diamond \rangle \\ & \Diamond\Box\neg p \quad \blacksquare \end{array}$$

Theorems (64) and (65), Truth and Falsehood of $\Box$, do not appear in other LTL systems.

(64) **Truth of $\Box$:** $\Box\, true \equiv true$

Proof:

$$\begin{array}{ll} & \Box\, true \\ = & \langle (54)\ \text{Definition of } \Box \rangle \\ & \neg\Diamond\neg true \\ = & \langle (3.8)\ \text{Definition of } false,\ false \equiv \neg true \rangle \\ & \neg\Diamond false \\ = & \langle (44)\ \text{Falsehood of } \Diamond \rangle \\ & \neg false \\ = & \langle (3.13)\ \text{Negation of } false,\ \neg false \equiv true \rangle \\ & true \quad \blacksquare \end{array}$$

(65) **Falsehood of $\Box$:** $\Box false \equiv false$

Proof:

$$\begin{array}{ll} & \Box false \equiv false \\ = & \langle (3.8)\ \text{Definition of } false,\ false \equiv \neg true,\ \text{twice} \rangle \\ & \Box\neg true \equiv \neg true \\ = & \langle (3.11)\ \neg p \equiv q \equiv p \equiv \neg q \rangle \\ & \neg\Box\neg\ true \equiv true \\ = & \langle (59)\ \Diamond p \equiv \neg\Box\neg p \rangle \\ & \Diamond true \equiv true \quad \text{—(43) Truth of } \Diamond \quad \blacksquare \end{array}$$

While the expansions of $\mathcal{U}$ and $\Diamond$ have two disjuncts, the expansion of $\Box$ has two conjuncts. As usual, the first describes the current state and the second contains the operation in the next state. Theorem (66) is the dual of (45) which can be used in its direct proof.

(66) **Expansion of $\Box$:** $\Box p \equiv p \wedge \bigcirc\Box p$

Proof:

$\Box\, p$
$= \quad \langle (54)$ Definition of $\Box \rangle$
$\neg\Diamond\, \neg p$
$= \quad \langle (45)$ Expansion of $\Diamond$ with $p := \neg p \rangle$
$\neg(\neg p \vee \bigcirc \Diamond \neg p)$
$= \quad \langle (3.47b)$ De Morgan, $\neg(p \vee q) \equiv \neg p \wedge \neg q \rangle$
$\neg\neg p \wedge \neg\bigcirc \Diamond \neg p$
$= \quad \langle (3.12)$ Double negation, $\neg\neg p \equiv p \rangle$
$p \wedge \neg\bigcirc \Diamond \neg p$
$= \quad \langle (1)$ Self dual$\rangle$
$p \wedge \bigcirc \neg\Diamond \neg p$
$= \quad \langle (54)$ Definition of $\Box \rangle$
$p \wedge \bigcirc \Box\, p \quad \blacksquare$

(67) **Expansion of $\Box$:** $\quad \Box\, p \equiv p \wedge \bigcirc\, p \wedge \bigcirc \Box\, p$

Proof:

$p \wedge \bigcirc\, p \wedge \bigcirc \Box\, p$
$= \quad \langle (5)$ Distributivity of $\bigcirc$ over $\wedge \rangle$
$p \wedge \bigcirc (p \wedge \Box\, p)$
$= \quad \langle (66)$ Expansion of $\Box \rangle$
$p \wedge \bigcirc (p \wedge p \wedge \bigcirc \Box\, p)$
$= \quad \langle (3.38)$ Idempotency of $\wedge$, $p \wedge p \equiv p \rangle$
$p \wedge \bigcirc (p \wedge \bigcirc \Box\, p)$
$= \quad \langle (66)$ Expansion of $\Box \rangle$
$p \wedge \bigcirc \Box\, p$
$= \quad \langle (66)$ Expansion of $\Box \rangle$
$\Box\, p \quad \blacksquare$

Theorem (68) absorption of $\wedge$ into $\Box$, is the dual of (48) the absorption of $\vee$ into $\Diamond$, while (69) absorption of $\Box$ into $\vee$ is the dual of (49) the absorption of $\Diamond$ into $\wedge$. As with (48) and (49), neither seem to appear in the temporal logic literature.

Conjunction $\wedge$ and the *always* operator $\Box$ are both universal, while disjunction $\vee$ and the *eventually* operator $\Diamond$ are both existential. When the left side of the equivalence contains both existential (or both universal) operators as in (48) and (68), the right side retains the same type of unary operator. When existential and universal operators are mixed on the left side, as in (49) and (69), the equivalence is just a statement about p at the current time.

(68) **Absorption of $\wedge$ into $\Box$:** $\quad p \wedge \Box\, p \equiv \Box\, p$

Proof:

$$
\begin{array}{ll}
& p \wedge \Box p \\
= & \langle (66)\ \text{Expansion of } \Box \rangle \\
& p \wedge p \wedge \bigcirc \Box p \\
= & \langle (3.38)\ \text{Idempotency of } \wedge,\ p \wedge p \equiv p \rangle \\
& p \wedge \bigcirc \Box p \\
= & \langle (66)\ \text{Expansion of } \Box \rangle \\
& \Box p \quad \blacksquare
\end{array}
$$

(69) **Absorption of $\Box$ into $\vee$:** $\Box p \vee p \equiv p$

Proof:

$$
\begin{array}{ll}
& \Box p \vee p \\
= & \langle (66)\ \text{Expansion of } \Box \rangle \\
& (p \wedge \bigcirc \Box p) \vee p \\
= & \langle (3.43b)\ \text{Absorption},\ p \vee (p \wedge q) \equiv p \rangle \\
& p \quad \blacksquare
\end{array}
$$

The absorption of $\Diamond$ into $\Box$ and of $\Box$ into $\Diamond$ do not appear in the LTL systems we survey. Their proofs are straightforward applications of the previous absorption theorems.

(70) **Absorption of $\Diamond$ into $\Box$:** $\Diamond p \wedge \Box p \equiv \Box p$

Proof:

$$
\begin{array}{ll}
& \Diamond p \wedge \Box p \\
= & \langle (68)\ \text{Absorption of } \wedge \text{ into } \Box \rangle \\
& \Diamond p \wedge p \wedge \Box p \\
= & \langle (49)\ \text{Absorption of } \Diamond \text{ into } \wedge \rangle \\
& p \wedge \Box p \\
= & \langle (68)\ \text{Absorption of } \wedge \text{ into } \Box \rangle \\
& \Box p \quad \blacksquare
\end{array}
$$

(71) **Absorption of $\Box$ into $\Diamond$:** $\Box p \vee \Diamond p \equiv \Diamond p$

Proof:

$$
\begin{array}{ll}
& \Box p \vee \Diamond p \\
= & \langle (48)\ \text{Absorption of } \vee \text{ into } \Diamond \rangle \\
& \Box p \vee p \vee \Diamond p \\
= & \langle (69)\ \text{Absorption of } \Box \text{ into } \vee \rangle \\
& p \vee \Diamond p \\
= & \langle (48)\ \text{Absorption of } \vee \text{ into } \Diamond \rangle \\
& \Diamond p \quad \blacksquare
\end{array}
$$

Theorem (72) absorption of $\Box$ is the dual of (50) the absorption of $\Diamond$, and (73) the exchange of $\bigcirc$ and $\Box$ is the dual of (51) the exchange of $\bigcirc$ and $\Diamond$.

(72) **Absorption of $\Box$:** $\Box\Box p \equiv \Box p$

Proof: (Kyle Sundman)

$$
\begin{array}{ll}
 & \Box\Box p \\
= & \langle(54)\text{ Definition of }\Box\text{ with }p := \Box p\rangle \\
 & \neg\Diamond\neg\Box p \\
= & \langle(60)\text{ Dual of }\Box\rangle \\
 & \neg\Diamond\Diamond\neg p \\
= & \langle(50)\text{ Absorption of }\Diamond\rangle \\
 & \neg\Diamond\neg p \\
= & \langle(54)\text{ Definition of }\Box\rangle \\
 & \Box p \quad \blacksquare
\end{array}
$$

(73) **Exchange of $\bigcirc$ and $\Box$:** $\bigcirc\Box p \equiv \Box\bigcirc p$

Proof:

$$
\begin{array}{ll}
 & \bigcirc\Box p \\
= & \langle(54)\text{ Definition of }\Box\rangle \\
 & \bigcirc\neg\Diamond\neg p \\
= & \langle(1)\text{ Self-dual}\rangle \\
 & \neg\bigcirc\Diamond\neg p \\
= & \langle(51)\text{ Exchange of }\bigcirc\text{ and }\Diamond\text{ with }p := \neg p\rangle \\
 & \neg\Diamond\bigcirc\neg p \\
= & \langle(1)\text{ Self-dual}\rangle \\
 & \neg\Diamond\neg\bigcirc p \\
= & \langle(54)\text{ Definition of }\Box\rangle \\
 & \Box\bigcirc p \quad \blacksquare
\end{array}
$$

Theorem (74) does not appear in other LTL systems. Theorem (75) states that if p holds and eventually $\neg p$ holds, there must eventually be a state where p holds in that state and it does not hold in the next state. Theorem (75) is the contrapositive of, and therefore equivalent to, (57) $\Box$ Induction.

(74) $p \Rightarrow \Box p \equiv p \Rightarrow \bigcirc\Box p$

Proof: (Ravi Mohan)

$$
\begin{array}{ll}
 & p \Rightarrow \Box\, p \\
= & \langle (66) \text{ Expansion of } \Box \rangle \\
 & p \Rightarrow p \wedge \bigcirc \Box\, p \\
= & \langle (3.63.1) \text{ Distributivity of } \Rightarrow \text{ over } \wedge,\ p \Rightarrow q \wedge r \;\equiv\; (p \Rightarrow q) \wedge (p \Rightarrow r) \rangle \\
 & (p \Rightarrow p) \wedge (p \Rightarrow \bigcirc \Box\, p) \\
= & \langle (3.71) \text{ Reflexivity of } \Rightarrow,\ p \Rightarrow p \rangle \\
 & \mathit{true} \wedge (p \Rightarrow \bigcirc \Box\, p) \\
= & \langle (3.39) \text{ Identity of } \wedge,\ p \wedge \mathit{true} \equiv p \rangle \\
 & p \Rightarrow \bigcirc \Box\, p \quad \blacksquare
\end{array}
$$

(75) $p \wedge \Diamond \neg p \Rightarrow \Diamond (p \wedge \bigcirc \neg p)$

Proof: By contrapositive, $\neg \Diamond (p \wedge \bigcirc \neg p) \Rightarrow \neg (p \wedge \Diamond \neg p)$

$$
\begin{array}{ll}
 & \neg \Diamond (p \wedge \bigcirc \neg p) \\
= & \langle (61) \text{ Dual of } \Diamond \rangle \\
 & \Box \neg (p \wedge \bigcirc \neg p) \\
= & \langle (3.47a) \text{ De Morgan } \neg (p \wedge q) \equiv \neg p \vee \neg q \rangle \\
 & \Box (\neg p \vee \neg \bigcirc \neg p) \\
= & \langle (3) \text{ Linearity} \rangle \\
 & \Box (\neg p \vee \bigcirc\, p) \\
= & \langle (3.59) \text{ Implication},\ p \Rightarrow q \;\equiv\; \neg p \vee q \rangle \\
 & \Box (p \Rightarrow \bigcirc\, p) \\
\Rightarrow & \langle (57)\ \Box \text{ Induction} \rangle \\
 & p \Rightarrow \Box\, p \\
= & \langle (3.59) \text{ Implication},\ p \Rightarrow q \;\equiv\; \neg p \vee q \rangle \\
 & \neg p \vee \Box\, p \\
= & \langle (54) \text{ Definition of } \Box \rangle \\
 & \neg p \vee \neg \Diamond \neg p \\
= & \langle (3.47a) \text{ De Morgan } \neg (p \wedge q) \equiv \neg p \vee \neg q \rangle \\
 & \neg (p \wedge \Diamond \neg p) \quad \blacksquare
\end{array}
$$

The following four strengthening theorems for $\Box$ are common and contrast with the weakening theorems (46) and (47) for $\Diamond$. Theorem (80) is listed in one other system. Theorem (81) is unique to this one.

(76) **Strengthening of $\Box$:** $\Box\, p \Rightarrow p$

Proof:

$$\Box\, p$$

$= \quad \langle (66) \text{ Expansion of } \Box \rangle$
$\quad p \wedge \bigcirc \Box p$
$\Rightarrow \quad \langle (3.76b) \text{ Strengthening, } p \wedge q \Rightarrow p \rangle$
$\quad p$ ■

(77) **Strengthening of $\Box$:** $\Box p \Rightarrow \Diamond p$

Proof:

$\quad \Box p$
$\Rightarrow \quad \langle (76) \text{ Strengthening of } \Box \rangle$
$\quad p$
$\Rightarrow \quad \langle (46) \text{ Weakening of } \Diamond \rangle$
$\quad \Diamond p$ ■

(78) **Strengthening of $\Box$:** $\Box p \Rightarrow \bigcirc p$

Proof:

$\quad \Box p$
$= \quad \langle (67) \text{ Expansion of } \Box \rangle$
$\quad p \wedge \bigcirc p \wedge \bigcirc \Box \bigcirc p$
$\Rightarrow \quad \langle (3.76b) \text{ Strengthening, } p \wedge q \Rightarrow p \rangle$
$\quad \bigcirc p$ ■

(79) **Strengthening of $\Box$:** $\Box p \Rightarrow \bigcirc \Box p$

Proof:

$\quad \Box p$
$= \quad \langle (66) \text{ Expansion of } \Box \rangle$
$\quad p \wedge \bigcirc \Box p$
$\Rightarrow \quad \langle (3.76b) \text{ Strengthening, } p \wedge q \Rightarrow p \rangle$
$\quad \bigcirc \Box p$ ■

(80) $\bigcirc$ **generalization:** $\Box p \Rightarrow \Box \bigcirc p$

Proof:

$\quad \Box p \Rightarrow \Box \bigcirc p$
$= \quad \langle (73) \text{ Exchange of } \bigcirc \text{ and } \Box \rangle$
$\quad \Box p \Rightarrow \bigcirc \Box p \quad$ —(79) Strengthening of $\Box$ ■

(81) $\Box p \Rightarrow \neg(q \, \mathcal{U} \, \neg p)$

Proof:

$$
\begin{array}{ll}
 & \Box\, p \Rightarrow \neg(q\,\mathcal{U}\,\neg p) \\
= & \langle (3.61) \text{ Contrapositive } p \Rightarrow q \;\equiv\; \neg q \Rightarrow \neg p \rangle \\
 & q\,\mathcal{U}\,\neg p \Rightarrow \neg\Box\, p \\
= & \langle (60) \text{ Dual of } \Box \rangle \\
 & q\,\mathcal{U}\,\neg p \Rightarrow \Diamond\neg p \quad \text{—(42) Eventuality with } p,q := q,\neg p \quad \blacksquare
\end{array}
$$

3.5 Temporal deduction

The following deduction proof technique is a metatheorem in LADM.

(4.4) **Deduction (assume conjuncts of antecedent):**
To prove $P_1 \wedge P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q.
You cannot use textual substitution in P_1 or P_2.

Corresponding to the deduction metatheorem of the propositional calculus is the following temporal deduction metatheorem.

(82) **Temporal deduction:**

To prove $\Box P_1 \wedge \Box P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q.
You cannot use textual substitution in P_1 or P_2.

Temporal deduction is Theorem (2.1.6) of Kröger and Merz [18], who also give the justification. Note that if you assume P in a step of an LTL proof of Q, you have *not* proved that $P \Rightarrow Q$, but rather that $\Box P \Rightarrow Q$.

3.6 Always, continued

The following two theorems, (83) and (84), do not appear in other LTL systems. However, they are required for the proof of later theorems that are included in other systems. In particular, the proofs of (85) and (86) depend on (83) Distributivity of $\wedge$ over $\mathcal{U}$. The proof of (83) illustrates temporal deduction in a calculational proof. The proof of (84) is similar.

(83) **Distributivity of $\wedge$ over $\mathcal{U}$:** $\Box\, p \wedge q\,\mathcal{U}\,r \Rightarrow (p \wedge q)\,\mathcal{U}\,(p \wedge r)$

Proof: The proof is by (82) Temporal deduction.

$$
\begin{array}{ll}
 & \Box\, p \wedge q\,\mathcal{U}\,r \Rightarrow (p \wedge q)\,\mathcal{U}\,(p \wedge r) \\
= & \langle (3.65) \text{ Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
 & \Box\, p \Rightarrow (q\,\mathcal{U}\,r \Rightarrow (p \wedge q)\,\mathcal{U}\,(p \wedge r))
\end{array}
$$

And now,

$$
\begin{aligned}
& q\,\mathcal{U}\,r \Rightarrow (p \wedge q)\,\mathcal{U}\,(p \wedge r) \\
= \quad & \langle \text{Assume antecedent } p \rangle \\
& q\,\mathcal{U}\,r \Rightarrow (\mathit{true} \wedge q)\,\mathcal{U}\,(\mathit{true} \wedge r) \\
= \quad & \langle \text{(3.39) Identity of } \wedge,\ p \wedge \mathit{true} \equiv p \rangle \\
& q\,\mathcal{U}\,r \Rightarrow q\,\mathcal{U}\,r \quad \text{—(3.71) Reflexivity of } \Rightarrow,\ p \Rightarrow p. \quad \blacksquare
\end{aligned}
$$

(84) $\mathcal{U}$ **implication:** $\Box\, p \wedge \Diamond\, q \Rightarrow p\,\mathcal{U}\,q$

Proof: The proof is by (82) temporal deduction.

$$
\begin{aligned}
& \Box\, p \wedge \Diamond\, q \Rightarrow p\,\mathcal{U}\,q \\
= \quad & \langle \text{(3.65) Shunting, } p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box\, p \Rightarrow (\Diamond\, q \Rightarrow p\,\mathcal{U}\,q)
\end{aligned}
$$

And now,

$$
\begin{aligned}
& \Diamond\, q \Rightarrow p\,\mathcal{U}\,q \\
= \quad & \langle \text{Assume antecedent } p \rangle \\
& \Diamond\, q \Rightarrow \mathit{true}\,\mathcal{U}\,q \\
= \quad & \langle \text{(38) Definition of } \Diamond \rangle \\
& \Diamond\, q \Rightarrow \Diamond\, q \quad \text{—(3.71) Reflexivity of } \Rightarrow,\ p \Rightarrow p \quad \blacksquare
\end{aligned}
$$

(85) **Right monotonicity of $\mathcal{U}$:** $\Box\,(p \Rightarrow q) \Rightarrow (r\,\mathcal{U}\,p \Rightarrow r\,\mathcal{U}\,q)$

Proof:

$$
\begin{aligned}
& \Box\,(p \Rightarrow q) \Rightarrow (r\,\mathcal{U}\,p \Rightarrow r\,\mathcal{U}\,q) \\
= \quad & \langle \text{(3.65) Shunting, } p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box\,(p \Rightarrow q) \wedge r\,\mathcal{U}\,p \Rightarrow r\,\mathcal{U}\,q
\end{aligned}
$$

And now,

$$
\begin{aligned}
& \Box\,(p \Rightarrow q) \wedge r\,\mathcal{U}\,p \\
\Rightarrow \quad & \langle \text{(83) Distributivity of } \wedge \text{ over } \mathcal{U} \rangle \\
& ((p \Rightarrow q) \wedge r)\,\mathcal{U}\,((p \Rightarrow q) \wedge p) \\
= \quad & \langle \text{(3.66) } p \wedge (p \Rightarrow q) \ \equiv\ p \wedge q \rangle \\
& ((p \Rightarrow q) \wedge r)\,\mathcal{U}\,(p \wedge q) \\
= \quad & \langle \text{(15) Right Distributivity of } \mathcal{U} \text{ over } \wedge \rangle \\
& (p \Rightarrow q)\,\mathcal{U}\,(p \wedge q) \wedge r\,\mathcal{U}\,(p \wedge q) \\
\Rightarrow \quad & \langle \text{(14) Left Distributivity of } \mathcal{U} \text{ over } \wedge \text{ and (4.3) Monotonicity of } \wedge \rangle \\
& (p \Rightarrow q)\,\mathcal{U}\,(p \wedge q) \wedge r\,\mathcal{U}\,p \wedge r\,\mathcal{U}\,q \\
\Rightarrow \quad & \langle \text{(3.76b) Strengthening, } p \wedge q \Rightarrow p \rangle \\
& r\,\mathcal{U}\,q \quad \blacksquare
\end{aligned}
$$

(86) **Left monotonicity of $\mathcal{U}$:** $\square(p \Rightarrow q) \Rightarrow (p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r)$

Proof:

$$\begin{array}{ll} & \square(p \Rightarrow q) \Rightarrow (p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r) \\ = & \langle \text{(3.65) Shunting, } p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r) \rangle \\ & \square(p \Rightarrow q) \wedge p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r \end{array}$$

And now,

$$\begin{array}{ll} & \square(p \Rightarrow q) \wedge p \,\mathcal{U}\, r \\ \Rightarrow & \langle \text{(83) Distributivity of } \wedge \text{ over } \mathcal{U} \rangle \\ & ((p \Rightarrow q) \wedge p) \,\mathcal{U}\, ((p \Rightarrow q) \wedge r) \\ = & \langle \text{(3.66) } p \wedge (p \Rightarrow q) \equiv p \wedge q \rangle \\ & (p \wedge q) \,\mathcal{U}\, ((p \Rightarrow q) \wedge r) \\ = & \langle \text{(15) Right Distributivity of } \mathcal{U} \text{ over } \wedge \rangle \\ & p \,\mathcal{U}\, ((p \Rightarrow q) \wedge r) \wedge q \,\mathcal{U}\, ((p \Rightarrow q) \wedge r) \\ \Rightarrow & \langle \text{(14) Left Distributivity of } \mathcal{U} \text{ over } \wedge \text{ and (4.3) Monotonicity of } \wedge \rangle \\ & p \,\mathcal{U}\, ((p \Rightarrow q) \wedge r) \wedge q \,\mathcal{U}\, (p \Rightarrow q) \wedge q \,\mathcal{U}\, r) \\ \Rightarrow & \langle \text{(3.76b) Strengthening, } p \wedge q \Rightarrow p \rangle \\ & q \,\mathcal{U}\, r \quad \blacksquare \end{array}$$

Theorem (87) states that if it is always the case that p is false then it is not the case that p is always true, but not the converse. Suppose, for example, that p continually oscillates between true and false over time. Then, the consequent of (87) is true, but the antecedent is false. Theorem (88) shows how $\diamond$ distributes over $\wedge$.

(87) **Distributivity of $\neg$ over $\square$:** $\square \neg p \Rightarrow \neg \square p$

Proof:

$$\begin{array}{ll} & \square \neg p \\ \Rightarrow & \langle \text{(77) Strengthening of } \square \rangle \\ & \diamond \neg p \\ = & \langle \text{(60) Dual of } \square \rangle \\ & \neg \square p \quad \blacksquare \end{array}$$

(88) **Distributivity of $\diamond$ over $\wedge$:** $\square p \wedge \diamond q \Rightarrow \diamond (p \wedge q)$

Proof: (Ravi Mohan)

$$\begin{array}{ll} & \square p \wedge \diamond q \\ = & \langle \text{(38) Definition of } \diamond \rangle \\ & \square p \wedge \mathit{true} \,\mathcal{U}\, q \end{array}$$

$\Rightarrow$ $\langle$(83) Distributivity of $\wedge$ over $\mathcal{U}$ with $q, r := true, q\rangle$
$(p \wedge true)\ \mathcal{U}\ (p \wedge q)$
$\Rightarrow$ $\langle$(42) Eventuality with $p, q := p \wedge true, p \wedge q\rangle$
$\Diamond (p \wedge q)$ ∎

Theorems (89), (90), and (91) are the linear temporal versions of the excluded middle axiom of propositional logic, (3.28) $p \vee \neg p$. Theorems (92), (93), and (94) are the linear temporal versions of the contradiction theorem of propositional logic, (3.42) $p \wedge \neg p \equiv false$. Theorems (95), (96), (97), and (98) are variations. These theorems are obvious, and their proofs are simple, which is perhaps why they do not appear in other LTL systems.

(89) $\Diamond$ **excluded middle:** $\Diamond p \vee \Box \neg p$

Proof:

$\Diamond p \vee \Box \neg p$
$=$ $\langle$(61) Dual of $\Diamond\rangle$
$\Diamond p \vee \neg \Diamond p$ —(3.28) Excluded middle, $p \vee \neg p$ with $p := \Diamond p$ ∎

(90) $\Box$ **excluded middle:** $\Box p \vee \Diamond \neg p$

Proof: (Ray McIntyre)

$\Box p \vee \Diamond \neg p$
$=$ $\langle$(60) Dual of $\Box\rangle$
$\Box p \vee \neg \Box p$ —(3.28) Excluded middle, $p \vee \neg p$ with $p := \Box p$ ∎

(91) **Temporal excluded middle:** $\Diamond p \vee \Diamond \neg p$

Proof:

$\Diamond p \vee \Diamond \neg p$
$=$ $\langle$(60) Dual of $\Box\rangle$
$\Diamond p \vee \neg \Box p$
$=$ $\langle$(3.59) Implication, $p \Rightarrow q \ \equiv\ \neg p \vee q\rangle$
$\Box p \Rightarrow \Diamond p$ —(77) Strengthening of $\Box$ ∎

(92) $\Diamond$ **contradiction:** $\Diamond p \wedge \Box \neg p \ \equiv\ false$

Proof: (Ray McIntyre)

$\Diamond p \wedge \Box \neg p \ \equiv\ false$
$=$ $\langle$(61) Dual of $\Diamond\rangle$
$\Diamond p \wedge \neg \Diamond p \ \equiv\ false$ —(3.42) Contradiction, $p \wedge \neg p \ \equiv\ false$ with $p := \Diamond p$ ∎

(93) $\Box$ **contradiction:** $\Box p \wedge \Diamond \neg p \;\equiv\; \mathit{false}$

Proof: (Ray McIntyre)

$$\begin{array}{ll} & \Box p \wedge \Diamond \neg p \;\equiv\; \mathit{false} \\ = & \langle (60) \text{ Dual of } \Box \rangle \\ & \Box p \wedge \neg \Box p \;\equiv\; \mathit{false} \quad \text{—(3.42) Contradiction, } p \wedge \neg p \;\equiv\; \mathit{false} \text{ with } p := \Box p \quad \blacksquare \end{array}$$

(94) **Temporal contradiction:** $\Box p \wedge \Box \neg p \;\equiv\; \mathit{false}$

Proof:

$$\begin{array}{ll} & \Box p \wedge \Box \neg p \;\equiv\; \mathit{false} \\ = & \langle (3.15)\ \neg p \equiv p \equiv \mathit{false} \rangle \\ & \neg(\Box p \wedge \Box \neg p) \\ = & \langle (3.47a) \text{ De Morgan } \neg(p \wedge q) \equiv \neg p \vee \neg q \rangle \\ & \neg \Box p \vee \neg \Box \neg p \\ = & \langle (60) \text{ Dual of } \Box \text{ and } (59) \rangle \\ & \Diamond \neg p \vee \Diamond p \quad \text{—(91) Temporal excluded middle} \quad \blacksquare \end{array}$$

(95) $\Box\Diamond$ **excluded middle:** $\Box \Diamond p \vee \Diamond \Box \neg p$

Proof:

$$\begin{array}{ll} & \Box \Diamond p \vee \Diamond \Box \neg p \\ = & \langle (63) \text{ Dual of } \Box \Diamond \rangle \\ & \Box \Diamond p \vee \neg \Box \Diamond p \quad \text{—(3.28) Excluded middle, } p \vee \neg p \text{ with } p := \Box \Diamond p \quad \blacksquare \end{array}$$

(96) $\Diamond\Box$ **excluded middle:** $\Diamond \Box p \vee \Box \Diamond \neg p$

Proof:

$$\begin{array}{ll} & \Diamond \Box p \vee \Box \Diamond \neg p \\ = & \langle (62) \text{ Dual of } \Diamond \Box \rangle \\ & \Diamond \Box p \vee \neg \Diamond \Box p \quad \text{—(3.28) Excluded middle, } p \vee \neg p \text{ with } p := \Diamond \Box p \quad \blacksquare \end{array}$$

(97) $\Box\Diamond$ **contradiction:** $\Box \Diamond p \wedge \Diamond \Box \neg p \;\equiv\; \mathit{false}$

Proof:

$$\begin{array}{ll} & \Box \Diamond p \wedge \Diamond \Box \neg p \;\equiv\; \mathit{false} \\ = & \langle (63) \text{ Dual of } \Box \Diamond \rangle \\ & \Box \Diamond p \wedge \neg \Box \Diamond p \;\equiv\; \mathit{false} \quad \text{—(3.42) Contradiction, } p \wedge \neg p \;\equiv\; \mathit{false} \text{ with } p := \Box \Diamond p \quad \blacksquare \end{array}$$

(98) $\Diamond\Box$ **contradiction:** $\Diamond \Box p \wedge \Box \Diamond \neg p \;\equiv\; \mathit{false}$

Proof:

$$
\begin{array}{ll}
& \Diamond \Box p \wedge \Box \Diamond \neg p \ \equiv \ \textit{false} \\
= & \langle (62) \text{ Dual of } \Diamond \Box \rangle \\
& \Diamond \Box p \wedge \neg \Diamond \Box p \ \equiv \ \textit{false} \quad \text{—(3.42) Contradiction, } p \wedge \neg p \ \equiv \ \textit{false} \text{ with } p := \Diamond \Box p \quad \blacksquare
\end{array}
$$

Theorem (99) shows that $\Box$, a universal operator, distributes over conjunction. Because disjunction is existential, (100) shows that $\Box$ distributes over conjunction in only one direction. Theorems (101), (102), and (103) reflect the concept of logical equivalence $\cong$ in [18]. Theorems (104) and (105) show how $\Diamond$ distributes over $\Rightarrow$.

(99) **Distributivity of $\Box$ over $\wedge$:** $\Box (p \wedge q) \equiv \Box p \wedge \Box q$

Proof:

$$
\begin{array}{ll}
& \Box (p \wedge q) \\
= & \langle (54) \text{ Definition of } \Box \rangle \\
& \neg \Diamond \neg (p \wedge q) \\
= & \langle (3.47\text{a}) \text{ De Morgan } \neg(p \wedge q) \equiv \neg p \vee \neg q \rangle \\
& \neg \Diamond (\neg p \vee \neg q) \\
= & \langle (52) \text{ Distributivity of } \Diamond \text{ over } \vee \rangle \\
& \neg (\Diamond \neg p \vee \Diamond \neg q) \\
= & \langle (3.47\text{b}) \text{ De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
& \neg \Diamond \neg p \wedge \neg \Diamond \neg q \\
= & \langle (54) \text{ Definition of } \Box \text{, twice} \rangle \\
& \Box p \wedge \Box q \quad \blacksquare
\end{array}
$$

(100) **Distributivity of $\Box$ over $\vee$:** $\Box p \vee \Box q \Rightarrow \Box (p \vee q)$

Proof:

$$
\begin{array}{ll}
& \Box p \vee \Box q \Rightarrow \Box (p \vee q) \\
= & \langle (3.60) \text{ Implication, } p \Rightarrow q \ \equiv \ p \wedge q \ \equiv \ p \rangle \\
& (\Box p \vee \Box q) \wedge \Box (p \vee q) \equiv \Box p \vee \Box q \\
= & \langle (3.46) \text{ Distributivity of } \wedge \text{ over } \vee \text{, } p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \rangle \\
& (\Box p \wedge \Box (p \vee q)) \vee (\Box q \wedge \Box (p \vee q)) \equiv \Box p \vee \Box q \\
= & \langle (99) \text{ Distributivity of } \Box \text{ over } \wedge \text{, twice} \rangle \\
& \Box (p \wedge (p \vee q)) \vee \Box (q \wedge (p \vee q)) \equiv \Box p \vee \Box q \\
= & \langle (3.43\text{a}) \text{ Absorption, } p \wedge (p \vee q) \equiv p \text{, twice} \rangle \\
& \Box p \vee \Box q \equiv \Box p \vee \Box q \quad \text{—(3.5) Reflexivity of } \equiv \text{, } p \equiv p \text{ with } p := \Box p \vee \Box q \quad \blacksquare
\end{array}
$$

(101) **Logical equivalence law of $\circ$:** $\Box (p \equiv q) \Rightarrow (\circ p \equiv \circ q)$

Proof:

$$\begin{array}{ll} & \Box(p \equiv q) \\ \Rightarrow & \langle (78) \text{ Strengthening of } \Box \text{ with } p := (p \equiv q)\ \rangle \\ & \bigcirc(p \equiv q) \\ = & \langle (6) \text{ Distributivity of } \bigcirc \text{ over } \equiv \rangle \\ & \bigcirc p \equiv \bigcirc q \quad \blacksquare \end{array}$$

(102) **Logical equivalence law of $\Diamond$:** $\Box(p \equiv q) \Rightarrow (\Diamond p \equiv \Diamond q)$

Proof: The proof is by (82) Temporal deduction.

$$\begin{array}{ll} & \Diamond p \equiv \Diamond q \\ = & \langle \text{Assume antecedent } p \equiv q, \text{twice} \rangle \\ & \Diamond(p \wedge (p \equiv q)) \equiv \Diamond(q \wedge (p \equiv q)) \\ = & \langle (3.50)\ p \wedge (q \equiv p) \ \equiv\ p \wedge q, \text{twice} \rangle \\ & \Diamond(p \wedge q) \equiv \Diamond(p \wedge q) \quad \text{—(3.5) Reflexivity of } \equiv,\ p \equiv p \text{ with } p := \Diamond(p \wedge q). \quad \blacksquare \end{array}$$

(103) **Logical equivalence law of $\Box$:** $\Box(p \equiv q) \Rightarrow (\Box p \equiv \Box q)$

Proof:

$$\begin{array}{ll} & \Box(p \equiv q) \Rightarrow (\Box p \equiv \Box q) \\ = & \langle (3.62)\ p \Rightarrow (q \equiv r) \ \equiv\ p \wedge q \ \equiv\ p \wedge r \rangle \\ & \Box(p \equiv q) \wedge \Box p \equiv \Box(p \equiv q) \wedge \Box q \\ = & \langle (99) \text{ Distributivity of } \Box \text{ over } \wedge, \text{twice} \rangle \\ & \Box((p \equiv q) \wedge p) \equiv \Box((p \equiv q) \wedge q) \\ = & \langle (3.50)\ p \wedge (q \equiv p) \ \equiv\ p \wedge q, \text{twice} \rangle \\ & \Box(p \wedge q) \equiv \Box(p \wedge q) \quad \text{—(3.5) Reflexivity of } \equiv,\ p \equiv p \text{ with } p := \Box(p \wedge q) \quad \blacksquare \end{array}$$

(104) **Distributivity of $\Diamond$ over $\Rightarrow$:** $\Diamond(p \Rightarrow q) \equiv (\Box p \Rightarrow \Diamond q)$

Proof:

$$\begin{array}{ll} & \Diamond(p \Rightarrow q) \\ = & \langle (3.59) \text{ Implication, } p \Rightarrow q \ \equiv\ \neg p \vee q \rangle \\ & \Diamond(\neg p \vee q) \\ = & \langle (52) \text{ Distributivity of } \Diamond \text{ over } \vee \rangle \\ & \Diamond \neg p \vee \Diamond q \\ = & \langle (60) \text{ Dual of } \Box \rangle \\ & \neg \Box p \vee \Diamond q \\ = & \langle (3.59) \text{ Implication, } p \Rightarrow q \ \equiv\ \neg p \vee q \rangle \\ & \Box p \Rightarrow \Diamond q \quad \blacksquare \end{array}$$

(105) **Distributivity of $\Diamond$ over $\Rightarrow$:** $(\Diamond p \Rightarrow \Diamond q) \Rightarrow \Diamond(p \Rightarrow q)$

Proof:

$$
\begin{array}{ll}
& (\Diamond p \Rightarrow \Diamond q) \Rightarrow \Diamond (p \Rightarrow q) \\
= & \langle (104) \text{ Distributivity of } \Diamond \text{ over } \Rightarrow \rangle \\
& (\Diamond p \Rightarrow \Diamond q) \Rightarrow (\Box p \Rightarrow \Diamond q) \\
= & \langle (3.65) \text{ Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
& (\Diamond p \Rightarrow \Diamond q) \wedge \Box p \Rightarrow \Diamond q
\end{array}
$$

And now,

$$
\begin{array}{ll}
& (\Diamond p \Rightarrow \Diamond q) \wedge \Box p \\
\Rightarrow & \langle (77) \text{ Strengthening of } \Box \text{ and (4.3) Monotonicity of } \wedge \rangle \\
& (\Diamond p \Rightarrow \Diamond q) \wedge \Diamond p \\
\Rightarrow & \langle (3.77) \text{ Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \rangle \\
& \Diamond q \quad \blacksquare
\end{array}
$$

The next three frame laws (106), (107), and (108) state that if $\Box p$ holds then p may be "added" by conjunction under each temporal operator. [18] For completeness, we show that they may be added under disjunction, implication, and equivalence as well. Theorems (148) to (150) extend the frame laws to $\mathcal{U}$ and theorems (210) to (212) extend them to $\mathcal{W}$.

(106) $\wedge$ **frame law of** $\circ$**:** $\Box p \Rightarrow (\circ q \Rightarrow \circ (p \wedge q))$

Proof:

$$
\begin{array}{ll}
& \Box p \Rightarrow (\circ q \Rightarrow \circ (p \wedge q)) \\
= & \langle (3.65) \text{ Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box p \wedge \circ q \Rightarrow \circ (p \wedge q)
\end{array}
$$

And now,

$$
\begin{array}{ll}
& \Box p \wedge \circ q \\
\Rightarrow & \langle (78) \text{ Strengthening of } \Box \text{ and (4.3) Monotonicity of } \wedge \rangle \\
& \circ p \wedge \circ q \\
= & \langle (5) \text{ Distributivity of } \circ \text{ over } \wedge \rangle \\
& \circ (p \wedge q) \quad \blacksquare
\end{array}
$$

(107) $\wedge$ **frame law of** $\Diamond$**:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond (p \wedge q))$

Proof:

$$
\begin{array}{ll}
& \Box p \Rightarrow (\Diamond q \Rightarrow \Diamond (p \wedge q)) \\
= & \langle (3.65) \text{ Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box p \wedge \Diamond q \Rightarrow \Diamond (p \wedge q) \quad \text{—(88) Distributivity of } \Diamond \text{ over } \wedge \quad \blacksquare
\end{array}
$$

(108) $\wedge$ **frame law of** $\Box$**:** $\Box p \Rightarrow (\Box q \Rightarrow \Box (p \wedge q))$

Proof:

$$\begin{array}{ll}
& \Box\, p \Rightarrow (\Box\, q \Rightarrow \Box\,(p \wedge q)) \\
= & \langle (3.65)\ \text{Shunting},\ p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box\, p \wedge \Box\, q \Rightarrow \Box\,(p \wedge q) \\
= & \langle (99)\ \text{Distributivity of } \Box \text{ over } \wedge \rangle \\
& \Box\,(p \wedge q) \Rightarrow \Box\,(p \wedge q) \quad \text{—(3.71) Reflexivity of } \Rightarrow,\ p \Rightarrow p \quad \blacksquare
\end{array}$$

(109) $\vee$ **frame law of** $\circ$**:** $\Box\, p \Rightarrow (\circ\, q \Rightarrow \circ\,(p \vee q))$

Proof:

$$\begin{array}{ll}
& \Box\, p \Rightarrow (\circ\, q \Rightarrow \circ\,(p \vee q)) \\
= & \langle (3.65)\ \text{Shunting},\ p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box\, p \wedge \circ\, q \Rightarrow \circ\,(p \vee q)
\end{array}$$

And now,

$$\begin{array}{ll}
& \Box\, p \wedge \circ\, q \\
\Rightarrow & \langle (78)\ \text{Strengthening of } \Box \text{ and (4.3) Monotonicity of } \wedge \rangle \\
& \circ\, p \wedge \circ\, q \\
\Rightarrow & \langle (3.76\text{c})\ (\text{Weakening/strengthening}),\ p \wedge q \Rightarrow p \vee q \rangle \\
& \circ\, p \vee \circ\, q \\
= & \langle (4)\ \text{Distributivity of } \circ \text{ over } \vee \rangle \\
& \circ\,(p \vee q) \quad \blacksquare
\end{array}$$

(110) $\vee$ **frame law of** $\Diamond$**:** $\Box\, p \Rightarrow (\Diamond\, q \Rightarrow \Diamond\,(p \vee q))$

Proof:

$$\begin{array}{ll}
& \Box\, p \Rightarrow (\Diamond\, q \Rightarrow \Diamond\,(p \vee q)) \\
= & \langle (3.65)\ \text{Shunting},\ p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box\, p \wedge \Diamond\, q \Rightarrow \Diamond\,(p \vee q)
\end{array}$$

And now,

$$\begin{array}{ll}
& \Box\, p \wedge \Diamond\, q \\
\Rightarrow & \langle (77)\ \text{Strengthening of } \Box \text{ and (4.3) Monotonicity of } \wedge \rangle \\
& \Diamond\, p \wedge \Diamond\, q \\
\Rightarrow & \langle (3.76\text{c})\ (\text{Weakening/strengthening}),\ p \wedge q \Rightarrow p \vee q \rangle \\
& \Diamond\, p \vee \Diamond\, q \\
= & \langle (52)\ \text{Distributivity of } \Diamond \text{ over } \vee \rangle \\
& \Diamond\,(p \vee q) \quad \blacksquare
\end{array}$$

(111) $\vee$ **frame law of** $\Box$**:** $\Box\, p \Rightarrow (\Box\, q \Rightarrow \Box\,(p \vee q))$

Proof:

$$
\begin{array}{ll}
& \Box p \Rightarrow (\Box q \Rightarrow \Box(p \vee q)) \\
= & \langle (3.65) \text{ Shunting}, p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box p \wedge \Box q \Rightarrow \Box(p \vee q)
\end{array}
$$

And now,

$$
\begin{array}{ll}
& \Box p \wedge \Box q \\
\Rightarrow & \langle (3.76\text{c}) \text{ (Weakening/strengthening)}, p \wedge q \Rightarrow p \vee q \rangle \\
& \Box p \vee \Box q \\
\Rightarrow & \langle (100) \text{ Distributivity of } \Box \text{ over } \vee \rangle \\
& \Box(p \vee q) \quad \blacksquare
\end{array}
$$

(112) $\Rightarrow$ **frame law of** $\circ$ **:** $\quad \Box p \Rightarrow (\circ q \Rightarrow \circ(p \Rightarrow q))$

Proof:

$$
\begin{array}{ll}
& \Box p \Rightarrow (\circ q \Rightarrow \circ(p \Rightarrow q)) \\
= & \langle (3.65) \text{ Shunting}, p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box p \wedge \circ q \Rightarrow \circ(p \Rightarrow q) \\
= & \langle (2) \text{ Axiom, Distributivity of } \circ \text{ over } \Rightarrow \rangle \\
& \Box p \wedge \circ q \Rightarrow (\circ p \Rightarrow \circ q) \\
= & \langle (3.65) \text{ Shunting}, p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box p \wedge \circ q \wedge \circ p \Rightarrow \circ q \\
& \quad \text{—(3.76b) Strengthening}, p \wedge q \Rightarrow p \text{ with } p, q := \circ q, \Box p \wedge \circ p \quad \blacksquare
\end{array}
$$

(113) $\Rightarrow$ **frame law of** $\Diamond$ **:** $\quad \Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p \Rightarrow q))$

Proof:

$$
\begin{array}{ll}
& \Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p \Rightarrow q)) \\
= & \langle (3.65) \text{ Shunting}, p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box p \wedge \Diamond q \Rightarrow \Diamond(p \Rightarrow q) \\
= & \langle (104) \text{ Distributivity of } \Diamond \text{ over } \Rightarrow \rangle \\
& \Box p \wedge \Diamond q \Rightarrow (\Box p \Rightarrow \Diamond q) \\
= & \langle (3.65) \text{ Shunting}, p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box p \wedge \Diamond q \wedge \Box p \Rightarrow \Diamond q \\
& \quad \text{—(3.76b) Strengthening}, p \wedge q \Rightarrow p \text{ with } p, q := \Diamond q, \Box p \wedge \Box p. \quad \blacksquare
\end{array}
$$

(114) $\Rightarrow$ **frame law of** $\Box$ **:** $\quad \Box p \Rightarrow (\Box q \Rightarrow \Box(p \Rightarrow q))$

Proof:

$$
\begin{array}{ll}
& \Box p \Rightarrow (\Box q \Rightarrow \Box(p \Rightarrow q)) \\
= & \langle (3.65)\ \text{Shunting},\ p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box p \wedge \Box q \Rightarrow \Box(p \Rightarrow q) \\
= & \langle (99)\ \text{Distributivity of } \Box \text{ over } \wedge \rangle \\
& \Box(p \wedge q) \Rightarrow \Box(p \Rightarrow q) \\
= & \langle (3.66)\ p \wedge (p \Rightarrow q) \ \equiv\ p \wedge q \rangle \\
& \Box(p \wedge (p \Rightarrow q)) \Rightarrow \Box(p \Rightarrow q) \\
= & \langle (99)\ \text{Distributivity of } \Box \text{ over } \wedge \rangle \\
& \Box p \wedge \Box(p \Rightarrow q) \Rightarrow \Box(p \Rightarrow q) \\
& \quad \text{—(3.76b) Strengthening, } p \wedge q \Rightarrow p \text{ with } p,q := \Box(p \Rightarrow q), \Box p. \quad \blacksquare
\end{array}
$$

(115) $\equiv$ **frame law of $\bigcirc$:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc(p \equiv q))$

Proof: The proof is by (82) Temporal deduction.

$$
\begin{array}{ll}
& \bigcirc q \Rightarrow \bigcirc(p \equiv q) \\
= & \langle \text{Assume antecedent } p \rangle \\
& \bigcirc q \Rightarrow \bigcirc(\mathit{true} \equiv q) \\
= & \langle (3.3)\ \text{Identity of } \equiv,\ \mathit{true} \equiv p \equiv p \rangle \\
& \bigcirc q \Rightarrow \bigcirc q \quad \text{—(3.71) Reflexivity of } \Rightarrow,\ p \Rightarrow p \quad \blacksquare
\end{array}
$$

(116) $\equiv$ **frame law of $\Diamond$:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p \equiv q))$

Proof:

$$
\begin{array}{ll}
& \Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p \equiv q)) \\
= & \langle (3.70)\ p \vee q \Rightarrow p \wedge q \ \equiv\ p \ \equiv\ q \rangle \\
& \Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p \vee q \Rightarrow p \wedge q)) \\
= & \langle (104)\ \text{Distributivity of } \Diamond \text{ over } \Rightarrow \rangle \\
& \Box p \Rightarrow (\Diamond q \Rightarrow (\Box(p \vee q) \Rightarrow \Diamond(p \wedge q))) \\
= & \langle (3.65)\ \text{Shunting},\ p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r), \text{twice} \rangle \\
& \Box p \wedge \Diamond q \wedge \Box(p \vee q) \Rightarrow \Diamond(p \wedge q) \\
= & \langle (99)\ \text{Distributivity of } \Box \text{ over } \wedge \rangle \\
& \Box(p \wedge (p \vee q) \wedge \Diamond q \Rightarrow \Diamond(p \wedge q) \\
= & \langle (3.43a)\ \text{Absorption},\ p \wedge (p \vee q) \equiv p \rangle \\
& \Box p \wedge \Diamond q \Rightarrow \Diamond(p \wedge q) \quad \text{—(88) Distributivity of } \Diamond \text{ over } \wedge \quad \blacksquare
\end{array}
$$

(117) $\equiv$ **frame law of $\Box$:** $\Box p \Rightarrow (\Box q \Rightarrow \Box(p \equiv q))$

Proof: The proof is by (82) Temporal deduction.

$$
\begin{array}{ll}
& \Box q \Rightarrow \Box(p \equiv q) \\
= & \langle \text{Assume antecedent } p \rangle
\end{array}
$$

$\quad \Box q \Rightarrow \Box(true \equiv q)$
$= \quad \langle (3.3) \text{ Identity of } \equiv, true \equiv p \equiv p \rangle$
$\quad \Box q \Rightarrow \Box q \quad$ —(3.71) Reflexivity of $\Rightarrow$, $p \Rightarrow p$ ∎

Theorems (118), (119), and (120) show that all unary temporal operators are monotonic. Theorem (120) can also be considered distributivity of $\Box$ over $\Rightarrow$. Proofs of the consequence rules (121), (122), and (123) use the monotonicity theorems as shown in the proof of (121). Proofs of the catenation rules (124), (125), and (126) are similar.

(118) **Monotonicity of $\bigcirc$:** $\quad \Box(p \Rightarrow q) \Rightarrow (\bigcirc p \Rightarrow \bigcirc q)$

Proof:

$\quad \Box(p \Rightarrow q)$
$\Rightarrow \quad \langle (78) \text{ Strengthening of } \Box \rangle$
$\quad \bigcirc(p \Rightarrow q)$
$= \quad \langle (2) \text{ Distributivity of } \bigcirc \text{ over } \Rightarrow \rangle$
$\quad \bigcirc p \Rightarrow \bigcirc q$ ∎

(119) **Monotonicity of $\Diamond$:** $\quad \Box(p \Rightarrow q) \Rightarrow (\Diamond p \Rightarrow \Diamond q)$

Proof:

$\quad \Box(p \Rightarrow q) \Rightarrow (\Diamond p \Rightarrow \Diamond q)$
$= \quad \langle (3.59) \text{ Implication}, p \Rightarrow q \;\equiv\; \neg p \vee q, \text{thrice} \rangle$
$\quad \neg\Box(\neg p \vee q) \vee \neg\Diamond p \vee \Diamond q$
$= \quad \langle (60) \text{ Dual of } \Box \rangle$
$\quad \Diamond \neg(\neg p \vee q) \vee \neg\Diamond p \vee \Diamond q$
$= \quad \langle (3.47b) \text{ De Morgan}, \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle$
$\quad \Diamond(p \wedge \neg q) \vee \neg\Diamond p \vee \Diamond q$
$= \quad \langle (52) \text{ Distributivity of } \Diamond \text{ over } \vee \rangle$
$\quad \Diamond((p \wedge \neg q) \vee q) \vee \neg\Diamond p$
$= \quad \langle (3.44b) \text{ Absorption}, p \vee (\neg p \wedge q) \equiv p \vee q \rangle$
$\quad \Diamond(p \vee q) \vee \neg\Diamond p$
$= \quad \langle (52) \text{ Distributivity of } \Diamond \text{ over } \vee \rangle$
$\quad \Diamond p \vee \Diamond q \vee \neg\Diamond p$
$= \quad \langle (3.28) \text{ Excluded middle}, p \vee \neg p \text{ with } p := \Diamond p \rangle$
$\quad \Diamond q \vee true$
$= \quad \langle (3.29) \text{ Zero of } \vee, p \vee true \equiv true \rangle$
$\quad true$ ∎

(120) **Monotonicity of $\Box$:** $\quad \Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$

Proof:

$\square(p \Rightarrow q)$
$= \quad \langle(3.60)$ Implication, $p \Rightarrow q \ \equiv\ p \wedge q \ \equiv\ p\rangle$
$\square(p \wedge q \equiv p)$
$\Rightarrow \quad \langle(103)$ Logical equivalence law of $\square\rangle$
$\square(p \wedge q) \equiv \square p$
$= \quad \langle(99)$ Distributivity of $\square$ over $\wedge\rangle$
$\square p \wedge \square q \equiv \square p$
$= \quad \langle(3.60)$ Implication, $p \Rightarrow q \ \equiv\ p \wedge q \ \equiv\ p\rangle$
$\square p \Rightarrow \square q$ ∎

(121) **Consequence rule of $\bigcirc$:** $\square((p \Rightarrow q) \wedge (q \Rightarrow \bigcirc r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow \bigcirc s)$

Proof:

$\square((p \Rightarrow q) \wedge (q \Rightarrow \bigcirc r) \wedge (r \Rightarrow s))$
$= \quad \langle(99)$ Distributivity of $\square$ over $\wedge\rangle$
$\square(p \Rightarrow q) \wedge \square(q \Rightarrow \bigcirc r) \wedge \square(r \Rightarrow s)$
$\Rightarrow \quad \langle(76)$ Strengthening of $\square$ and (4.3) Monotonicity of $\wedge$, twice$\rangle$
$(p \Rightarrow q) \wedge (q \Rightarrow \bigcirc r) \wedge \square(r \Rightarrow s)$
$\Rightarrow \quad \langle(3.82a)$ Transitivity and (4.3) Monotonicity of $\wedge\rangle$
$(p \Rightarrow \bigcirc r) \wedge \square(r \Rightarrow s)$
$\Rightarrow \quad \langle(118)$ Monotonicity of $\bigcirc$ and (4.3) Monotonicity of $\wedge\rangle$
$(p \Rightarrow \bigcirc r) \wedge (\bigcirc r \Rightarrow \bigcirc s)$
$\Rightarrow \quad \langle(3.82a)$ Transitivity$\rangle$
$p \Rightarrow \bigcirc s$ ∎

(122) **Consequence rule of $\Diamond$:** $\square((p \Rightarrow q) \wedge (q \Rightarrow \Diamond r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow \Diamond s)$

Proof:

$\square((p \Rightarrow q) \wedge (q \Rightarrow \Diamond r) \wedge (r \Rightarrow s))$
$= \quad \langle(99)$ Distributivity of $\square$ over $\wedge\rangle$
$\square(p \Rightarrow q) \wedge \square(q \Rightarrow \Diamond r) \wedge \square(r \Rightarrow s)$
$\Rightarrow \quad \langle(76)$ Strengthening of $\square$ and (4.3) Monotonicity of $\wedge$, twice$\rangle$
$(p \Rightarrow q) \wedge (q \Rightarrow \Diamond r) \wedge \square(r \Rightarrow s)$
$\Rightarrow \quad \langle(3.82a)$ Transitivity and (4.3) Monotonicity of $\wedge\rangle$
$(p \Rightarrow \Diamond r) \wedge \square(r \Rightarrow s)$
$\Rightarrow \quad \langle(119)$ Monotonicity of $\Diamond$ and (4.3) Monotonicity of $\wedge\rangle$
$(p \Rightarrow \Diamond r) \wedge (\Diamond r \Rightarrow \Diamond s)$
$\Rightarrow \quad \langle(3.82a)$ Transitivity$\rangle$
$p \Rightarrow \Diamond s$ ∎

(123) **Consequence rule of $\square$:** $\square((p \Rightarrow q) \wedge (q \Rightarrow \square r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow \square s)$

Proof:

$$
\begin{array}{ll}
& \square((p \Rightarrow q) \wedge (q \Rightarrow \square r) \wedge (r \Rightarrow s)) \\
= & \langle \text{(99) Distributivity of } \square \text{ over } \wedge \rangle \\
& \square(p \Rightarrow q) \wedge \square(q \Rightarrow \square r) \wedge \square(r \Rightarrow s) \\
\Rightarrow & \langle \text{(76) Strengthening of } \square \text{ and (4.3) Monotonicity of } \wedge \text{, twice} \rangle \\
& (p \Rightarrow q) \wedge (q \Rightarrow \square r) \wedge \square(r \Rightarrow s) \\
\Rightarrow & \langle \text{(3.82a) Transitivity and (4.3) Monotonicity of } \wedge \rangle \\
& (p \Rightarrow \square r) \wedge \square(r \Rightarrow s) \\
\Rightarrow & \langle \text{(120) Monotonicity of } \square \text{ and (4.3) Monotonicity of } \wedge \rangle \\
& (p \Rightarrow \square r) \wedge (\square r \Rightarrow \square s) \\
\Rightarrow & \langle \text{(3.82a) Transitivity} \rangle \\
& p \Rightarrow \square s \quad \blacksquare
\end{array}
$$

(124) **Catenation rule of $\diamond$:** $\square((p \Rightarrow \diamond q) \wedge (q \Rightarrow \diamond r)) \Rightarrow (p \Rightarrow \diamond r)$

Proof:

$$
\begin{array}{ll}
& \square((p \Rightarrow \diamond q) \wedge (q \Rightarrow \diamond r)) \\
= & \langle \text{(99) Distributivity of } \square \text{ over } \wedge \rangle \\
& \square(p \Rightarrow \diamond q) \wedge \square(q \Rightarrow \diamond r) \\
\Rightarrow & \langle \text{(76) Strengthening of } \square \text{ and (4.3) Monotonicity of } \wedge \rangle \\
& (p \Rightarrow \diamond q) \wedge \square(q \Rightarrow \diamond r) \\
\Rightarrow & \langle \text{(119) Monotonicity of } \diamond \text{ and (4.3) Monotonicity of } \wedge \rangle \\
& (p \Rightarrow \diamond q) \wedge (\diamond q \Rightarrow \diamond\diamond r) \\
= & \langle \text{(50) Absorption of } \diamond \rangle \\
& (p \Rightarrow \diamond q) \wedge (\diamond q \Rightarrow \diamond r) \\
\Rightarrow & \langle \text{(3.82a) Transitivity} \rangle \\
& p \Rightarrow \diamond r \quad \blacksquare
\end{array}
$$

(125) **Catenation rule of $\square$:** $\square((p \Rightarrow \square q) \wedge (q \Rightarrow \square r)) \Rightarrow (p \Rightarrow \square r)$

Proof:

$$
\begin{array}{ll}
& \square((p \Rightarrow \square q) \wedge (q \Rightarrow \square r)) \\
\Rightarrow & \langle \text{(76) Strengthening of } \square \rangle \\
& (p \Rightarrow \square q) \wedge (q \Rightarrow \square r) \\
= & \langle \text{(76) Strengthening of } \square \text{ and (3.39) Identity of } \wedge,\ p \wedge \mathit{true} \equiv p \rangle \\
& (p \Rightarrow \square q) \wedge (\square q \Rightarrow q) \wedge (q \Rightarrow \square r) \\
\Rightarrow & \langle \text{(3.82a) Transitivity and (4.3) Monotonicity of } \wedge \rangle
\end{array}
$$

$$
\begin{array}{ll}
 & (p \Rightarrow q) \wedge (q \Rightarrow \Box r) \\
\Rightarrow & \langle \text{(3.82a) Transitivity} \rangle \\
 & p \Rightarrow \Box r \quad \blacksquare
\end{array}
$$

(126) **Catenation rule of $\mathcal{U}$:** $\Box((p \Rightarrow q\,\mathcal{U}\,r) \wedge (r \Rightarrow q\,\mathcal{U}\,s)) \Rightarrow (p \Rightarrow q\,\mathcal{U}\,s)$

Proof:

$$
\begin{array}{ll}
 & \Box((p \Rightarrow q\,\mathcal{U}\,r) \wedge (r \Rightarrow q\,\mathcal{U}\,s)) \Rightarrow (p \Rightarrow q\,\mathcal{U}\,s) \\
= & \langle \text{(3.65) Shunting, } p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r) \rangle \\
 & \Box((p \Rightarrow q\,\mathcal{U}\,r) \wedge (r \Rightarrow q\,\mathcal{U}\,s)) \wedge p \Rightarrow q\,\mathcal{U}\,s
\end{array}
$$

And now,

$$
\begin{array}{ll}
 & \Box((p \Rightarrow q\,\mathcal{U}\,r) \wedge (r \Rightarrow q\,\mathcal{U}\,s)) \wedge p \\
= & \langle \text{(99) Distributivity of } \Box \text{ over } \wedge \rangle \\
 & \Box(p \Rightarrow q\,\mathcal{U}\,r) \wedge \Box(r \Rightarrow q\,\mathcal{U}\,s) \wedge p \\
\Rightarrow & \langle \text{(76) Strengthening of } \Box \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & (p \Rightarrow q\,\mathcal{U}\,r) \wedge \Box(r \Rightarrow q\,\mathcal{U}\,s) \wedge p \\
\Rightarrow & \langle \text{(3.77) Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & q\,\mathcal{U}\,r \wedge \Box(r \Rightarrow q\,\mathcal{U}\,s) \\
\Rightarrow & \langle \text{(85) Right monotonicity of } \mathcal{U} \text{ with } p,q,r := r, q\,\mathcal{U}\,s, q \\
 & \quad \text{and (4.3) Monotonicity of } \wedge \rangle \\
 & q\,\mathcal{U}\,r \wedge (q\,\mathcal{U}\,r \Rightarrow q\,\mathcal{U}\,(q\,\mathcal{U}\,s)) \\
\Rightarrow & \langle \text{(3.77) Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \rangle \\
 & q\,\mathcal{U}\,(q\,\mathcal{U}\,s) \\
= & \langle \text{(36) Left absorption of } \mathcal{U} \rangle \\
 & q\,\mathcal{U}\,s \quad \blacksquare
\end{array}
$$

Most of the remaining theorems in this section are included in a single source in our survey.

(127) $\mathcal{U}$ **strengthening rule:** $\Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \Rightarrow (p\,\mathcal{U}\,q \Rightarrow r\,\mathcal{U}\,s)$

Proof:

$$
\begin{array}{ll}
 & \Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \Rightarrow (p\,\mathcal{U}\,q \Rightarrow r\,\mathcal{U}\,s) \\
= & \langle \text{(3.65) Shunting, } p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r) \rangle \\
 & \Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \wedge p\,\mathcal{U}\,q \Rightarrow r\,\mathcal{U}\,s
\end{array}
$$

And now,

$$
\begin{array}{ll}
 & \Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \wedge p \,\mathcal{U}\, q \\
= & \langle (99) \text{ Distributivity of } \Box \text{ over } \wedge \rangle \\
 & \Box(p \Rightarrow r) \wedge \Box(q \Rightarrow s) \wedge p \,\mathcal{U}\, q \\
\Rightarrow & \langle (85) \text{ Right monotonicity of } \mathcal{U} \text{ with } p, q, r := q, s, p \\
 & \quad \text{and } (4.3) \text{ Monotonicity of } \wedge \rangle \\
 & \Box(p \Rightarrow r) \wedge (p \,\mathcal{U}\, q \Rightarrow p \,\mathcal{U}\, s) \wedge p \,\mathcal{U}\, q \\
\Rightarrow & \langle (3.77) \text{ Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \text{ and } (4.3) \text{ Monotonicity of } \wedge \rangle \\
 & \Box(p \Rightarrow r) \wedge p \,\mathcal{U}\, s \\
\Rightarrow & \langle (86) \text{ Left monotonicity of } \mathcal{U} \text{ with } q, r := r, s \\
 & \quad \text{and } (4.3) \text{ Monotonicity of } \wedge \rangle \\
 & (p \,\mathcal{U}\, s \Rightarrow r \,\mathcal{U}\, s) \wedge p \,\mathcal{U}\, s \\
\Rightarrow & \langle (3.77) \text{ Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \rangle \\
 & r \,\mathcal{U}\, s \quad \blacksquare
\end{array}
$$

(128) **Induction rule $\Diamond$:** $\Box(p \vee \bigcirc q \Rightarrow q) \Rightarrow (\Diamond p \Rightarrow q)$

Proof:

$$
\begin{array}{ll}
 & \Box(p \vee \bigcirc q \Rightarrow q) \\
= & \langle (3.78)\ (p \Rightarrow r) \wedge (q \Rightarrow r) \equiv p \vee q \Rightarrow r \rangle \\
 & \Box((p \Rightarrow q) \wedge (\bigcirc q \Rightarrow q)) \\
= & \langle (99) \text{ Distributivity of } \Box \text{ over } \wedge \rangle \\
 & \Box(p \Rightarrow q) \wedge \Box(\bigcirc q \Rightarrow q) \\
\Rightarrow & \langle (58)\ \Diamond \text{ Induction and } (4.3) \text{ Monotonicity of } \wedge \rangle \\
 & \Box(p \Rightarrow q) \wedge (\Diamond q \Rightarrow q) \\
\Rightarrow & \langle (119) \text{ Monotonicity of } \Diamond \text{ and } (4.3) \text{ Monotonicity of } \wedge \rangle \\
 & (\Diamond p \Rightarrow \Diamond q) \wedge (\Diamond q \Rightarrow q) \\
\Rightarrow & \langle (3.82a) \text{ Transitivity} \rangle \\
 & \Diamond p \Rightarrow q \quad \blacksquare
\end{array}
$$

(129) **Induction rule $\Box$:** $\Box(p \Rightarrow \bigcirc p \wedge q) \Rightarrow (p \Rightarrow \Box q)$

Proof:

$$
\begin{array}{ll}
 & \mathit{true} \\
= & \langle (55)\ \mathcal{U} \text{ Induction with } r := \mathit{false} \rangle \\
 & \Box(p \Rightarrow (\bigcirc p \wedge q) \vee \mathit{false}) \Rightarrow (p \Rightarrow \Box q \vee q \,\mathcal{U}\, \mathit{false}) \\
= & \langle (11) \text{ Right zero of } \mathcal{U} \rangle \\
 & \Box(p \Rightarrow (\bigcirc p \wedge q) \vee \mathit{false}) \Rightarrow (p \Rightarrow \Box q \vee \mathit{false}) \\
= & \langle (3.30) \text{ Identity of } \vee,\ p \vee \mathit{false} \equiv p \text{ twice} \rangle \\
 & \Box(p \Rightarrow \bigcirc p \wedge q) \Rightarrow (p \Rightarrow \Box q) \quad \blacksquare
\end{array}
$$

(130) **Induction rule $\mathcal{U}$:** $\quad \Box\,(p \Rightarrow \neg q \wedge \circ\, p) \Rightarrow (p \Rightarrow \neg(r\,\mathcal{U}\,q))$

Proof:

$$
\begin{array}{ll}
 & \Box\,(p \Rightarrow \neg q \wedge \circ\, p) \\
\Rightarrow & \langle \text{(129) Induction rule } \Box \rangle \\
 & p \Rightarrow \Box \neg q \\
\Rightarrow & \langle \text{(81) with } p, q := \neg q, r, \text{ (3.12) Double negation } \neg\neg p \equiv p, \text{ and} \\
 & \quad \text{(3.82a) Transitivity} \rangle \\
 & p \Rightarrow \neg(r\,\mathcal{U}\,q) \quad \blacksquare
\end{array}
$$

(131) $\Diamond$ **Confluence:** $\quad \Box\,((p \Rightarrow \Diamond\,(q \vee r)) \wedge (q \Rightarrow \Diamond\, t) \wedge (r \Rightarrow \Diamond\, t)) \Rightarrow (p \Rightarrow \Diamond\, t)$

Proof:

$$
\begin{array}{ll}
 & \Box\,((p \Rightarrow \Diamond\,(q \vee r)) \wedge (q \Rightarrow \Diamond\, t) \wedge (r \Rightarrow \Diamond\, t)) \Rightarrow (p \Rightarrow \Diamond\, t) \\
= & \langle \text{(3.65) Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
 & \Box\,((p \Rightarrow \Diamond\,(q \vee r)) \wedge (q \Rightarrow \Diamond\, t) \wedge (r \Rightarrow \Diamond\, t)) \wedge p \Rightarrow \Diamond\, t
\end{array}
$$

And now,

$$
\begin{array}{ll}
 & \Box\,((p \Rightarrow \Diamond\,(q \vee r)) \wedge (q \Rightarrow \Diamond\, t) \wedge (r \Rightarrow \Diamond\, t)) \wedge p \\
= & \langle \text{(99) Distributivity of } \Box \text{ over } \wedge \rangle \\
 & \Box\,(p \Rightarrow \Diamond\,(q \vee r)) \wedge \Box\,(q \Rightarrow \Diamond\, t) \wedge \Box\,(r \Rightarrow \Diamond\, t) \wedge p \\
\Rightarrow & \langle \text{(76) Strengthening of } \Box \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & (p \Rightarrow \Diamond\,(q \vee r)) \wedge \Box\,(q \Rightarrow \Diamond\, t) \wedge \Box\,(r \Rightarrow \Diamond\, t) \wedge p \\
\Rightarrow & \langle \text{(3.77) Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & \Diamond\,(q \vee r) \wedge \Box\,(q \Rightarrow \Diamond\, t) \wedge \Box\,(r \Rightarrow \Diamond\, t) \\
= & \langle \text{(52) Distributivity of } \Diamond \text{ over } \vee \rangle \\
 & (\Diamond\, q \vee \Diamond\, r) \wedge \Box\,(q \Rightarrow \Diamond\, t) \wedge \Box\,(r \Rightarrow \Diamond\, t) \\
\Rightarrow & \langle \text{(119) Monotonicity of } \Diamond \text{ and (4.3) Monotonicity of } \wedge \text{, twice} \rangle \\
 & (\Diamond\, q \vee \Diamond\, r) \wedge (\Diamond\, q \Rightarrow \Diamond\Diamond\, t) \wedge (\Diamond\, r \Rightarrow \Diamond\Diamond\, t) \\
= & \langle \text{(50) Absorption of } \Diamond \text{, twice} \rangle \\
 & (\Diamond\, q \vee \Diamond\, r) \wedge (\Diamond\, q \Rightarrow \Diamond\, t) \wedge (\Diamond\, r \Rightarrow \Diamond\, t) \\
\Rightarrow & \langle \text{(3.76.3) } (p \vee q) \wedge (q \Rightarrow r) \Rightarrow p \vee r \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & (\Diamond\, r \vee \Diamond\, t) \wedge (\Diamond\, r \Rightarrow \Diamond\, t) \\
\Rightarrow & \langle \text{(3.76.3) } (p \vee q) \wedge (q \Rightarrow r) \Rightarrow p \vee r \rangle \\
 & \Diamond\, t \vee \Diamond\, t \\
= & \langle \text{(3.26) Idempotency of } \vee, \; p \vee p \equiv p \rangle \\
 & \Diamond\, t \quad \blacksquare
\end{array}
$$

(132) **Temporal generalization law:** $\quad \Box\,(\Box\, p \Rightarrow q) \Rightarrow (\Box\, p \Rightarrow \Box\, q)$

Proof:

$$
\begin{array}{ll}
 & \Box(\Box p \Rightarrow q) \\
\Rightarrow & \langle (120) \text{ Monotonicity of } \Box \text{ with } p := \Box p \rangle \\
 & \Box\Box p \Rightarrow \Box q \\
= & \langle (72) \text{ Absorption of } \Box \rangle \\
 & \Box p \Rightarrow \Box q \quad \blacksquare
\end{array}
$$

(133) **Temporal particularization law:** $\Box(p \Rightarrow \Diamond q) \Rightarrow (\Diamond p \Rightarrow \Diamond q)$

Proof:

$$
\begin{array}{ll}
 & \Box(p \Rightarrow \Diamond q) \\
\Rightarrow & \langle (119) \text{ Monotonicity of } \Diamond \text{ with } q := \Diamond q \rangle \\
 & \Diamond p \Rightarrow \Diamond\Diamond q \\
= & \langle (50) \text{ Absorption of } \Diamond \rangle \\
 & \Diamond p \Rightarrow \Diamond q \quad \blacksquare
\end{array}
$$

(134) $\Box(p \Rightarrow \bigcirc q) \Rightarrow (p \Rightarrow \Diamond q)$

Proof:

$$
\begin{array}{ll}
 & \Box(p \Rightarrow \bigcirc q) \\
\Rightarrow & \langle (76) \text{ Strengthening of } \Box \rangle \\
 & p \Rightarrow \bigcirc q \\
\Rightarrow & \langle (47) \text{ Weakening of } \Diamond \text{ and (3.82a) Transitivity} \rangle \\
 & p \Rightarrow \Diamond q \quad \blacksquare
\end{array}
$$

(135) $\Box(p \Rightarrow \bigcirc \neg p) \Rightarrow (p \Rightarrow \neg\Box p)$

Proof:

$$
\begin{array}{ll}
 & \Box(p \Rightarrow \bigcirc \neg p) \\
\Rightarrow & \langle (134) \text{ with } q := \neg p \rangle \\
 & p \Rightarrow \Diamond \neg p \\
= & \langle (60) \text{ Dual of } \Box \rangle \\
 & p \Rightarrow \neg\Box p \quad \blacksquare
\end{array}
$$

Because the implication relation is reflexive, antisymmetric, and transitive, it defines a partially ordered set on linear temporal logic expressions. Figure 2 is a collection of seven Hasse diagrams showing some implication relations. Each number in parentheses is a linear temporal logic theorem. A number that labels an edge in a Hasse diagram is an implication theorem, and a number that labels a box is an equivalence theorem. For example, edge (87) represents the theorem that $\Box\neg p$ implies $\neg\Box p$, and box (61) represents the theorem that $\neg\Diamond p$ is equivalent to $\Box\neg p$.

The collection of theorems in this paper omit some implication theorems that are trivially derived by mutual transitivity. For example, one such theorem is that $\Box\neg p$ implies

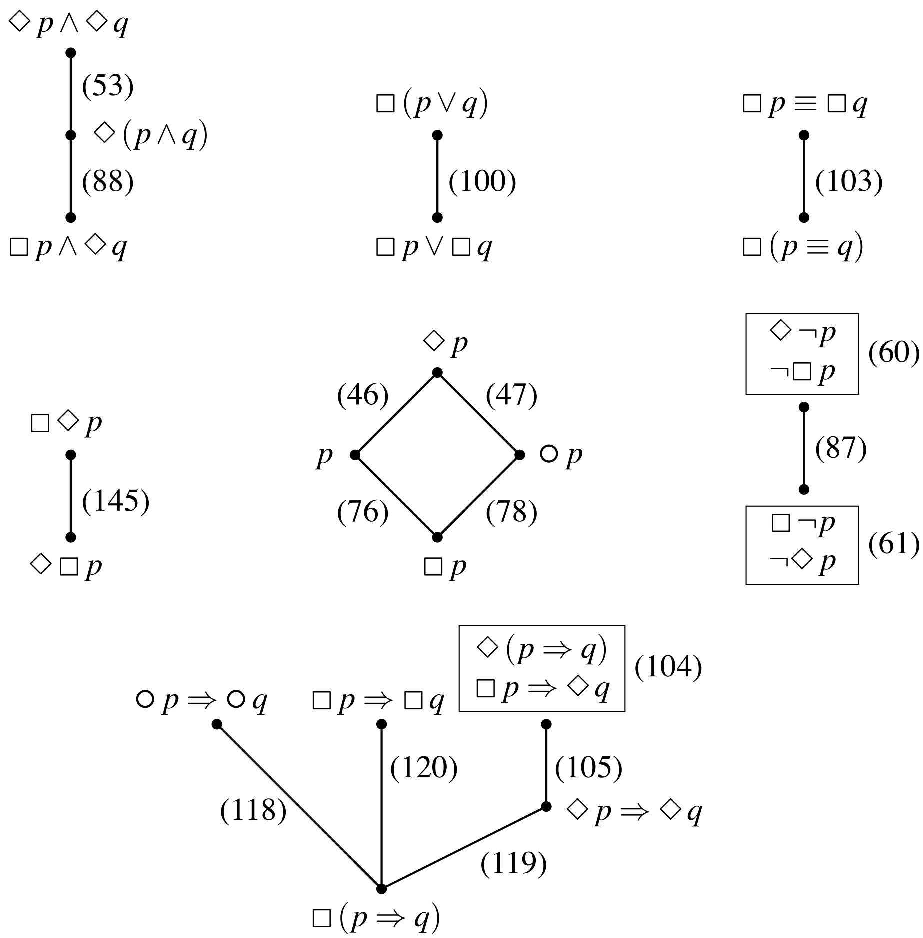

Figure 2: Seven Hasse diagrams showing some implication relations of linear temporal logic.

$\Diamond\neg p$, which follows from the theorems that $\Box\neg p$ implies $\neg\Box p$ and that $\neg\Box p$ is equivalent to $\Diamond\neg p$. The edge labeled by (87) thus represents four implication theorems, one for each combination of the two antecedents $\Box\neg p$ and $\neg\Diamond p$ and the two consequents $\Diamond\neg p$ and $\neg\Box p$. Likewise, the edge labeled by (105) represents two implication theorems.

3.7 Proof metatheorems

In the calculational system $\mathcal{E}$, Gries and Schneider (LADM) prove the metatheorem (9.16), which states that P is a theorem iff $(\forall x\,|\colon P)$ is a theorem. [11] Theorems in $\mathcal{E}$ are thus said to be "implicitly universally quantified." A similar concept applies to temporal logic theorems in $\mathcal{E}$ except that the implicit application is in the temporal dimension. The following metatheorem shows that theorems "implicitly always hold." Case 1 in the proof below is known as the temporal generalization rule [21], and Case 2 is known as the specialization rule [19].

(136) **Metatheorem:** P is a theorem iff $\Box P$ is a theorem.

Proof: The proof is by (4.7) Mutual implication. The proof of each case is by (4.4) Deduction.

Case 1. If P is a theorem then $\Box P$ is a theorem.
Suppose P is a theorem. Because all theorems are equivalent to each other, and (3.4) *true* is a theorem, P is equivalent to *true*. Then, $\Box P$ can be proved to be a theorem as follows.

$$
\begin{array}{ll}
 & \Box P \\
= & \langle P \text{ is a theorem} \rangle \\
 & \Box \, true \\
= & \langle (64) \text{ Truth of } \Box \rangle \\
 & true
\end{array}
$$

Case 2. If $\Box P$ is a theorem then P is a theorem.
Suppose $\Box P$ is a theorem. Then, $\Box P$ is equivalent to *true*. P can be proved to be a theorem by (4.7.1) Truth implication as follows.

$$
\begin{array}{ll}
 & P \\
\Leftarrow & \langle (76) \text{ Strengthening of } \Box \rangle \\
 & \Box P \\
= & \langle \Box P \text{ is a theorem} \rangle \\
 & true \quad \blacksquare
\end{array}
$$

Proofs of the following three metatheorems are similar.

(137) **Metatheorem $\bigcirc$:** If $P \Rightarrow Q$ is a theorem then $\bigcirc P \Rightarrow \bigcirc Q$ is a theorem.

Proof: The proof is by (4.4) Deduction. Suppose $P \Rightarrow Q$ is a theorem. Then, by (136) Metatheorem, $\Box (P \Rightarrow Q)$ is a theorem. Because all theorems are equivalent to each other, and (3.4) *true* is a theorem, $\Box (P \Rightarrow Q)$ is equivalent to *true*. Then, $\bigcirc P \Rightarrow \bigcirc Q$ can be proved to be a theorem by (4.7.1) Truth implication as follows.

$$
\begin{array}{ll}
 & \bigcirc P \Rightarrow \bigcirc Q \\
\Leftarrow & \langle (118) \text{ Monotonicity of } \bigcirc \rangle \\
 & \Box (P \Rightarrow Q) \\
= & \langle \Box (P \Rightarrow Q) \text{ is a theorem} \rangle \\
 & true \quad \blacksquare
\end{array}
$$

(138) **Metatheorem $\Diamond$:** If $P \Rightarrow Q$ is a theorem then $\Diamond P \Rightarrow \Diamond Q$ is a theorem.

Proof: The proof is by (4.4) Deduction. Suppose $P \Rightarrow Q$ is a theorem. Then, by (136) Metatheorem, $\Box (P \Rightarrow Q)$ is a theorem. Because all theorems are equivalent to each other, and (3.4) *true* is a theorem, $\Box (P \Rightarrow Q)$ is equivalent to *true*. Then, $\Diamond P \Rightarrow \Diamond Q$ can be proved to be a theorem by (4.7.1) Truth implication as follows.

$\quad \Diamond P \Rightarrow \Diamond Q$
$\Leftarrow \quad \langle$(119) Monotonicity of $\Diamond\rangle$
$\quad \Box(P \Rightarrow Q)$
$= \quad \langle \Box(P \Rightarrow Q)$ is a theorem$\rangle$
$\quad$ *true* ▮

(139) **Metatheorem $\Box$:** If $P \Rightarrow Q$ is a theorem then $\Box P \Rightarrow \Box Q$ is a theorem.

Proof: The proof is by (4.4) Deduction. Suppose $P \Rightarrow Q$ is a theorem. Then, by (136) Metatheorem, $\Box(P \Rightarrow Q)$ is a theorem. Because all theorems are equivalent to each other, and (3.4) *true* is a theorem, $\Box(P \Rightarrow Q)$ is equivalent to *true*. Then, $\Box P \Rightarrow \Box Q$ can be proved to be a theorem by (4.7.1) Truth implication as follows.

$\quad \Box P \Rightarrow \Box Q$
$\Leftarrow \quad \langle$(120) Monotonicity of $\Box\rangle$
$\quad \Box(P \Rightarrow Q)$
$= \quad \langle \Box(P \Rightarrow Q)$ is a theorem$\rangle$
$\quad$ *true* ▮

The proof of (142) in Section 3.8 illustrates the use of (136) Metatheorem. The proof of (166) illustrates the use of (138) Metatheorem $\Diamond$ and (139) Metatheorem $\Box$.

3.8 Always, continued

Theorems (140) and (141) do not seem to appear in the LTL literature. However, they play a key role here in the proofs of several later theorems that do exist in the literature. The proof of (140) is based on (129) Induction rule $\Box$ with $p,q := p\,\mathcal{U}\,\Box q, p\,\mathcal{U}\,q$. It establishes the truth of the antecedent with the help of two lemmas, each of which is proved with metatheorem (136). The strengthening and ordering theorems are also unique to this system.

(140) $\mathcal{U}\,\Box$ **implication:** $p\,\mathcal{U}\,\Box q \Rightarrow \Box(p\,\mathcal{U}\,q)$

Proof: Theorem (129) Induction rule $\Box$ with $p,q := p\,\mathcal{U}\,\Box q, p\,\mathcal{U}\,q$ is

$$\Box(p\,\mathcal{U}\,\Box q \Rightarrow (p\,\mathcal{U}\,q \wedge \circ(p\,\mathcal{U}\,\Box q))) \Rightarrow (p\,\mathcal{U}\,\Box q \Rightarrow \Box(p\,\mathcal{U}\,q))$$

Because the consequent is the theorem to be proved, it suffices to prove the truth of the antecedent.

$\quad \Box(p\,\mathcal{U}\,\Box q \Rightarrow (p\,\mathcal{U}\,q \wedge \circ(p\,\mathcal{U}\,\Box q)))$
$= \quad \langle$(3.63.1) Distributivity of $\Rightarrow$ over $\wedge$, $p \Rightarrow q \wedge r \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)\rangle$
$\quad \Box((p\,\mathcal{U}\,\Box q \Rightarrow p\,\mathcal{U}\,q) \wedge (p\,\mathcal{U}\,\Box q \Rightarrow \circ(p\,\mathcal{U}\,\Box q)))$
$= \quad \langle$(99) Distributivity of $\Box$ over $\wedge\rangle$
$\quad \Box(p\,\mathcal{U}\,\Box q \Rightarrow p\,\mathcal{U}\,q) \wedge \Box(p\,\mathcal{U}\,\Box q \Rightarrow \circ(p\,\mathcal{U}\,\Box q))$

$= \quad \langle \text{Lemma A: } \Box(p \,\mathcal{U}\, \Box q \Rightarrow p \,\mathcal{U}\, q) \rangle$
$\quad \mathit{true} \wedge \Box(p \,\mathcal{U}\, \Box q \Rightarrow \bigcirc(p \,\mathcal{U}\, \Box q))$
$= \quad \langle \text{Lemma B: } \Box(p \,\mathcal{U}\, \Box q \Rightarrow \bigcirc(p \,\mathcal{U}\, \Box q)) \rangle$
$\quad \mathit{true} \wedge \mathit{true}$
$= \quad \langle \text{(3.38) Idempotency of } \wedge,\ p \wedge p \equiv p \rangle$
$\quad \mathit{true}$ ∎

Lemma A: $\Box(p \,\mathcal{U}\, \Box q \Rightarrow p \,\mathcal{U}\, q)$
Proof: The proof is by (4.7.1) Truth implication.

$\quad \mathit{true}$
$= \quad \langle \text{(76) Strengthening of } \Box \text{ with } p := q \rangle$
$\quad \Box q \Rightarrow q$
$= \quad \langle \text{(136) Metatheorem with the above theorem} \rangle$
$\quad \Box(\Box q \Rightarrow q)$
$\Rightarrow \quad \langle \text{(85) Right monotonicity of } \mathcal{U} \text{ with } p, q, r := \Box q, q, p \rangle$
$\quad p \,\mathcal{U}\, \Box q \Rightarrow p \,\mathcal{U}\, q$
$= \quad \langle \text{(136) Metatheorem with the above theorem} \rangle$
$\quad \Box(p \,\mathcal{U}\, \Box q \Rightarrow p \,\mathcal{U}\, q)$ ∎

Lemma B: $\Box(p \,\mathcal{U}\, \Box q \Rightarrow \bigcirc(p \,\mathcal{U}\, \Box q))$
Proof: By (136) Metatheorem, Lemma B is valid iff

$$p \,\mathcal{U}\, \Box q \Rightarrow \bigcirc(p \,\mathcal{U}\, \Box q)$$

is valid.

$\quad p \,\mathcal{U}\, \Box q$
$= \quad \langle \text{(10) Expansion of } \mathcal{U} \rangle$
$\quad \Box q \vee (p \wedge \bigcirc(p \,\mathcal{U}\, \Box q))$
$= \quad \langle \text{(3.45) Distributivity of } \vee \text{ over } \wedge,\ p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \rangle$
$\quad (\Box q \vee p) \wedge (\Box q \vee \bigcirc(p \,\mathcal{U}\, \Box q))$
$\Rightarrow \quad \langle \text{(3.76b) Strengthening, } p \wedge q \Rightarrow p \rangle$
$\quad \Box q \vee \bigcirc(p \,\mathcal{U}\, \Box q)$
$= \quad \langle \text{(66) Expansion of } \Box \rangle$
$\quad (q \wedge \bigcirc\Box q) \vee \bigcirc(p \,\mathcal{U}\, \Box q)$
$= \quad \langle \text{(3.45) Distributivity of } \vee \text{ over } \wedge,\ p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \rangle$
$\quad (q \vee \bigcirc(p \,\mathcal{U}\, \Box q)) \wedge (\bigcirc\Box q \vee \bigcirc(p \,\mathcal{U}\, \Box q))$
$\Rightarrow \quad \langle \text{(3.76b) Strengthening, } p \wedge q \Rightarrow p \rangle$
$\quad \bigcirc\Box q \vee \bigcirc(p \,\mathcal{U}\, \Box q)$
$= \quad \langle \text{(9) Distributivity of } \bigcirc \text{ over } \mathcal{U} \rangle$

$\quad \bigcirc \Box q \vee \bigcirc p \,\mathcal{U} \bigcirc \Box q$
$= \quad \langle$(32) Absorption with $p, q := \bigcirc p, \bigcirc \Box q\rangle$
$\quad \bigcirc p \,\mathcal{U} \bigcirc \Box q$
$= \quad \langle$(9) Distributivity of $\bigcirc$ over $\mathcal{U}\rangle$
$\quad \bigcirc (p \,\mathcal{U}\, \Box q)$ ■

(141) **Absorption of $\mathcal{U}$ into $\Box$:** $\quad p \,\mathcal{U}\, \Box p \equiv \Box p$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$\quad p \,\mathcal{U}\, \Box p$
$\Rightarrow \quad \langle$(140) $\mathcal{U}\,\Box$ implication, with $q := p\rangle$
$\quad \Box (p \,\mathcal{U}\, p)$
$= \quad \langle$(22) Idempotency of $\mathcal{U}\rangle$
$\quad \Box p$

The proof in the second direction follows.

$\quad \Box p$
$\Rightarrow \quad \langle$(29) $\mathcal{U}$ Insertion, with $q := \Box p\rangle$
$\quad p \,\mathcal{U}\, \Box p$ ■

(142) **Right $\wedge\,\mathcal{U}$ strengthening:** $\quad p \,\mathcal{U}\, (q \wedge r) \Rightarrow p \,\mathcal{U}\, (q \,\mathcal{U}\, r)$

Proof: The proof is by (4.7.1) Truth implication.

$\quad$ *true*
$= \quad \langle$(136) Metatheorem and (30)$\rangle$
$\quad \Box (q \wedge r \Rightarrow q \,\mathcal{U}\, r)$
$\Rightarrow \quad \langle$(85) Right monotonicity of $\mathcal{U}$ with $p, q, r := q \wedge r, q \,\mathcal{U}\, r, p\rangle$
$\quad p \,\mathcal{U}\, (q \wedge r) \Rightarrow p \,\mathcal{U}\, (q \,\mathcal{U}\, r)$ ■

(143) **Left $\wedge\,\mathcal{U}$ strengthening:** $\quad (p \wedge q) \,\mathcal{U}\, r \Rightarrow (p \,\mathcal{U}\, q) \,\mathcal{U}\, r$

Proof: The proof is by (4.7.1) Truth implication.

$\quad$ *true*
$= \quad \langle$(136) Metatheorem and (30)$\rangle$
$\quad \Box (p \wedge q \Rightarrow p \,\mathcal{U}\, q)$
$\Rightarrow \quad \langle$(86) Left monotonicity of $\mathcal{U}$ with $p, q := p \wedge q, p \,\mathcal{U}\, q\rangle$
$\quad (p \wedge q) \,\mathcal{U}\, r \Rightarrow (p \,\mathcal{U}\, q) \,\mathcal{U}\, r$ ■

(144) **Left $\wedge\,\mathcal{U}$ ordering:** $\quad (p \wedge q) \,\mathcal{U}\, r \Rightarrow p \,\mathcal{U}\, (q \,\mathcal{U}\, r)$

Proof: Theorem (127) $\mathcal{U}$ strengthening rule with $p, q, r, s := p \wedge q, r, p, q \,\mathcal{U}\, r$ is

$$\Box((p \wedge q \Rightarrow p) \wedge (r \Rightarrow q \,\mathcal{U}\, r)) \Rightarrow (p \wedge q) \,\mathcal{U}\, r \Rightarrow p \,\mathcal{U}\, (q \,\mathcal{U}\, r)$$

Because the consequent is the theorem to be proved, it suffices to prove the truth of the antecedent.

$$\begin{array}{ll}
 & \Box((p \wedge q \Rightarrow p) \wedge (r \Rightarrow q \,\mathcal{U}\, r)) \\
= & \langle \text{(3.76b) Strengthening, } p \wedge q \Rightarrow p \text{ and (29) } \mathcal{U} \text{ insertion} \rangle \\
 & \Box(\mathit{true} \wedge \mathit{true}) \\
= & \langle \text{(3.38) Idempotency of } \wedge,\ p \wedge p \equiv p \rangle \\
 & \Box\, \mathit{true} \\
= & \langle \text{(64) Truth of } \Box \rangle \\
 & \mathit{true} \quad \blacksquare
\end{array}$$

The $\Diamond\Box$ implication theorem states that $\Diamond\Box p$ ensures that p will always eventually hold, but not the converse. Suppose, for example, that p continually oscillates between true and false over time. Then, the consequent of (145) is true, but the antecedent is false. Its proof uses (140). Theorem (146) is a second version of (95) $\Box\Diamond$ excluded middle, and is perhaps less intuitive. Its proof uses (145).

(145) $\Diamond\Box$ **implication:** $\Diamond\Box p \Rightarrow \Box\Diamond p$

Proof:

$$\begin{array}{ll}
 & \Diamond\Box p \\
= & \langle \text{(38) Definition of } \Diamond \rangle \\
 & \mathit{true} \,\mathcal{U}\, \Box p \\
\Rightarrow & \langle \text{(140) } \mathcal{U}\Box \text{ implication} \rangle \\
 & \Box(\mathit{true} \,\mathcal{U}\, p) \\
= & \langle \text{(38) Definition of } \Diamond \rangle \\
 & \Box\Diamond p \quad \blacksquare
\end{array}$$

(146) $\Box\Diamond$ **excluded middle:** $\Box\Diamond p \vee \Box\Diamond \neg p$

Proof:

$$\begin{array}{ll}
 & \Box\Diamond p \vee \Box\Diamond \neg p \\
= & \langle \text{(62) Dual of } \Diamond\Box \rangle \\
 & \Box\Diamond p \vee \neg\Diamond\Box p \\
= & \langle \text{(3.59) Implication, } p \Rightarrow q \ \equiv\ \neg p \vee q \rangle \\
 & \Diamond\Box p \Rightarrow \Box\Diamond p \quad \text{—(145) } \Diamond\Box \text{ implication} \quad \blacksquare
\end{array}$$

(147) $\Diamond\Box$ **contradiction:** $\Diamond\Box p \wedge \Diamond\Box \neg p \ \equiv\ \mathit{false}$

Proof:

$$
\begin{array}{ll}
 & \Diamond\Box p \wedge \Diamond\Box\neg p \;\equiv\; \mathit{false} \\
= & \langle (3.15)\ \neg p \equiv p \equiv \mathit{false} \rangle \\
 & \neg(\Diamond\Box p \wedge \Diamond\Box\neg p) \\
= & \langle (3.47a)\ \text{De Morgan } \neg(p \wedge q) \equiv \neg p \vee \neg q \rangle \\
 & \neg\Diamond\Box p \vee \neg\Diamond\Box\neg p \\
= & \langle (62)\ \text{Dual of } \Diamond\Box \text{ and } (63)\ \text{Dual of } \Box\Diamond \rangle \\
 & \Box\Diamond\neg p \vee \Box\Diamond\neg\neg p \\
= & \langle (3.12)\ \text{Double negation, } \neg\neg p \equiv p \rangle \\
 & \Box\Diamond\neg p \vee \Box\Diamond p \quad \text{—(146) } \Box\Diamond \text{ excluded middle} \quad \blacksquare
\end{array}
$$

(148) $\mathcal{U}$ **frame law of** $\bigcirc$**:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc(p\,\mathcal{U}\,q))$

Proof:

$$
\begin{array}{ll}
 & \Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc(p\,\mathcal{U}\,q)) \\
= & \langle (2)\ \text{Distributivity of } \bigcirc \text{ over } \Rightarrow \rangle \\
 & \Box p \Rightarrow \bigcirc(q \Rightarrow p\,\mathcal{U}\,q) \\
= & \langle (29)\ \mathcal{U} \text{ insertion} \rangle \\
 & \Box p \Rightarrow \bigcirc\,\mathit{true} \\
= & \langle (7)\ \text{Truth of } \bigcirc \rangle \\
 & \Box p \Rightarrow \mathit{true} \quad \text{—(3.72) Right zero of } \Rightarrow,\ p \Rightarrow \mathit{true} \;\equiv\; \mathit{true} \quad \blacksquare
\end{array}
$$

(149) $\mathcal{U}$ **frame law of** $\Diamond$**:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p\,\mathcal{U}\,q))$

Proof:

$$
\begin{array}{ll}
 & \Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p\,\mathcal{U}\,q)) \\
= & \langle (3.65)\ \text{Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
 & \Box p \wedge \Diamond q \Rightarrow \Diamond(p\,\mathcal{U}\,q)
\end{array}
$$

And now,

$$
\begin{array}{ll}
 & \Box p \wedge \Diamond q \\
\Rightarrow & \langle (84)\ \mathcal{U} \text{ implication} \rangle \\
 & p\,\mathcal{U}\,q \\
\Rightarrow & \langle (46)\ \text{Weakening of } \Diamond,\ p \Rightarrow \Diamond p \text{ with } p := p\,\mathcal{U}\,q \rangle \\
 & \Diamond(p\,\mathcal{U}\,q) \quad \blacksquare
\end{array}
$$

(150) $\mathcal{U}$ **frame law of** $\Box$**:** $\Box p \Rightarrow (\Box q \Rightarrow \Box(p\,\mathcal{U}\,q))$

Proof:

$$
\begin{array}{ll}
 & \Box p \Rightarrow (\Box q \Rightarrow \Box(p\,\mathcal{U}\,q)) \\
= & \langle (3.65)\ \text{Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
 & \Box p \wedge \Box q \Rightarrow \Box(p\,\mathcal{U}\,q))
\end{array}
$$

And now,

$\Box\, p \wedge \Box\, q$
$\Rightarrow\ \langle$(3.76b) Strengthening, $p \wedge q \Rightarrow p\rangle$
$\Box\, q$
$\Rightarrow\ \langle$(29) $\mathcal{U}$ Insertion with $q := \Box\, q\,\rangle$
$p\,\mathcal{U}\,\Box\, q$
$\Rightarrow\ \langle$(140) $\mathcal{U}\,\Box$ implication$\rangle$
$\Box\,(p\,\mathcal{U}\,q)$ ∎

The absorption theorems, (151) and (152), together with absorption theorems (50) and (72), allow any arbitrary string of $\Diamond$ and $\Box$ operators of any arbitrary length to be collapsed into one of four expressions: $\Diamond\, p$, $\Box\, p$, $\Box\Diamond\, p$, or $\Diamond\Box\, p$. These theorems are common to all systems. The remaining absorption theorems (153) to (156) are simple extensions mentioned in a single source in our survey.

(151) **Absorption of $\Diamond$ into $\Box\Diamond$:** $\Diamond\Box\Diamond\, p \equiv \Box\Diamond\, p$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$\Diamond\Box\Diamond\, p$
$\Rightarrow\ \langle$(145) $\Diamond\Box$ implication$\rangle$
$\Box\Diamond\Diamond\, p$
$=\ \langle$(50) Absorption of $\Diamond\,\rangle$
$\Box\Diamond\, p$

The proof in the second direction follows.

$\Box\Diamond\, p$
$\Rightarrow\ \langle$(46) Weakening of $\Diamond\,\rangle$
$\Diamond\Box\Diamond\, p$ ∎

(152) **Absorption of $\Box$ into $\Diamond\Box$:** $\Box\Diamond\Box\, p \equiv \Diamond\Box\, p$

Proof:

$\Box\Diamond\Box\, p$
$=\ \langle$(3.12) Double negation, $\neg\neg p \equiv p\rangle$
$\Box\Diamond\Box\neg\neg p$
$=\ \langle$(63) Dual of $\Box\Diamond$ and (61) Dual of $\Diamond\,\rangle$
$\neg\Diamond\Box\Diamond\neg p$
$=\ \langle$(151) Absorption of $\Diamond$ into $\Box\Diamond\,\rangle$

$\neg\Box\Diamond\neg p$
$= \quad \langle(62)\text{ Dual of }\Diamond\Box\rangle$
$\neg\neg\Diamond\Box p$
$= \quad \langle(3.12)\text{ Double negation, }\neg\neg p \equiv p\rangle$
$\Diamond\Box p \quad \blacksquare$

(153) **Absorption of $\Box\Diamond$:** $\Box\Diamond\Box\Diamond p \equiv \Box\Diamond p$

Proof:

$\Box\Diamond\Box\Diamond p$
$= \quad \langle(152)\text{ Absorption of }\Box\text{ into }\Diamond\Box\text{ with }p := \Diamond p\rangle$
$\Diamond\Box\Diamond p$
$= \quad \langle(151)\text{ Absorption of }\Diamond\text{ into }\Box\Diamond\rangle$
$\Box\Diamond p \quad \blacksquare$

(154) **Absorption of $\Diamond\Box$:** $\Diamond\Box\Diamond\Box p \equiv \Diamond\Box p$

Proof:

$\Diamond\Box\Diamond\Box p$
$= \quad \langle(151)\text{ Absorption of }\Diamond\text{ into }\Box\Diamond\text{ with }p := \Box p\rangle$
$\Box\Diamond\Box p$
$= \quad \langle(152)\text{ Absorption of }\Box\text{ into }\Diamond\Box\rangle$
$\Diamond\Box p \quad \blacksquare$

(155) **Absorption of $\bigcirc$ into $\Box\Diamond$:** $\bigcirc\Box\Diamond p \equiv \Box\Diamond p$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$\bigcirc\Box\Diamond p$
$\Rightarrow \quad \langle(47)\text{ Weakening of }\Diamond\text{ with }p := \Box\Diamond p\rangle$
$\Diamond\Box\Diamond p$
$= \quad \langle(151)\text{ Absorption of }\Diamond\text{ into }\Box\Diamond\rangle$
$\Box\Diamond p$

The proof in the second direction follows.

$\Box\Diamond p$
$\Rightarrow \quad \langle(79)\text{ Strengthening of }\Box\rangle$
$\bigcirc\Box\Diamond p \quad \blacksquare$

(156) **Absorption of $\bigcirc$ into $\Diamond\Box$:** $\bigcirc\Diamond\Box p \equiv \Diamond\Box p$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$$
\begin{array}{ll}
 & \bigcirc\Diamond\Box p \\
\Rightarrow & \langle (47)\text{ Weakening of } \Diamond \text{ with } p := \Diamond\Box p \rangle \\
 & \Diamond\Diamond\Box p \\
= & \langle (50)\text{ Absorption of } \Diamond \rangle \\
 & \Diamond\Box p
\end{array}
$$

The proof in the second direction follows.

$$
\begin{array}{ll}
 & \Diamond\Box p \\
= & \langle (152)\text{ Absorption of } \Box \text{ into } \Diamond\Box \rangle \\
 & \Box\Diamond\Box p \\
\Rightarrow & \langle (78)\text{ Strengthening of } \Box\text{, with } p := \Diamond\Box p \rangle \\
 & \bigcirc\Diamond\Box p \quad \blacksquare
\end{array}
$$

The proof of monotonicity theorem (157) uses (139) Metatheorem $\Box$ and (119) Monotonicity of $\Diamond$. The proof of monotonicity theorem (158) does the same with (120) Monotonicity of $\Box$.

(157) **Monotonicity of $\Box\Diamond$:** $\Box(p \Rightarrow q) \Rightarrow (\Box\Diamond p \Rightarrow \Box\Diamond q)$

Proof: The proof is by (4.7.1) Truth implication.

$$
\begin{array}{ll}
 & \mathit{true} \\
= & \langle (139)\text{ Metatheorem } \Box \text{ and (119) Monotonicity of } \Diamond \rangle \\
 & \Box\Box(p \Rightarrow q) \Rightarrow \Box(\Diamond p \Rightarrow \Diamond q) \\
= & \langle (72)\text{ Absorption of } \Box \rangle \\
 & \Box(p \Rightarrow q) \Rightarrow \Box(\Diamond p \Rightarrow \Diamond q) \\
= & \langle (3.39)\text{ Identity of } \wedge,\ p \wedge \mathit{true} \equiv p \text{ and (120) Monotonicity of } \Box \\
 & \quad \text{with } p, q := \Diamond p, \Diamond q \rangle \\
 & (\Box(p \Rightarrow q) \Rightarrow \Box(\Diamond p \Rightarrow \Diamond q)) \wedge (\Box(\Diamond p \Rightarrow \Diamond q) \Rightarrow (\Box\Diamond p \Rightarrow \Box\Diamond q)) \\
\Rightarrow & \langle (3.82a)\text{ Transitivity} \rangle \\
 & \Box(p \Rightarrow q) \Rightarrow (\Box\Diamond p \Rightarrow \Box\Diamond q) \quad \blacksquare
\end{array}
$$

(158) **Monotonicity of $\Diamond\Box$:** $\Box(p \Rightarrow q) \Rightarrow (\Diamond\Box p \Rightarrow \Diamond\Box q)$

Proof: The proof is by (4.7.1) Truth implication.

$$
\begin{array}{ll}
& \textit{true} \\
= & \langle (139)\ \text{Metatheorem}\ \Box\ \text{and}\ (120)\ \text{Monotonicity of}\ \Box \rangle \\
& \Box\,\Box\,(p \Rightarrow q) \Rightarrow \Box\,(\Box\,p \Rightarrow \Box\,q) \\
= & \langle (72)\ \text{Absorption of}\ \Box \rangle \\
& \Box\,(p \Rightarrow q) \Rightarrow \Box\,(\Box\,p \Rightarrow \Box\,q) \\
= & \langle (3.39)\ \text{Identity of}\ \wedge,\ p \wedge \textit{true} \equiv p\ \text{and}\ (119)\ \text{Monotonicity of}\ \Diamond \\
& \quad \text{with}\ p,q := \Box\,p, \Box\,q \rangle \\
& (\Box\,(p \Rightarrow q) \Rightarrow \Box\,(\Box\,p \Rightarrow \Box\,q))\ \wedge\ (\Box\,(\Box\,p \Rightarrow \Box\,q) \Rightarrow (\Diamond\,\Box\,p \Rightarrow \Diamond\,\Box\,q)) \\
\Rightarrow & \langle (3.82a)\ \text{Transitivity} \rangle \\
& \Box\,(p \Rightarrow q) \Rightarrow (\Diamond\,\Box\,p \Rightarrow \Diamond\,\Box\,q) \quad \blacksquare
\end{array}
$$

The next group of four distributivity theorems show how $\Box\Diamond$ and $\Diamond\Box$ distribute over conjunction and disjunction. Theorem (159) shows that $\Box\Diamond$ distributes over conjunction only in one direction. Similarly, Theorem (160) shows that $\Diamond\Box$ distributes over disjunction only in one direction. However, Theorems (161) and (162) show that $\Box\Diamond$ distributes over disjunction and $\Diamond\Box$ distributes over conjunction in both directions.

(159) **Distributivity of $\Box\Diamond$ over $\wedge$:** $\Box\Diamond\,(p \wedge q) \Rightarrow \Box\Diamond\,p \wedge \Box\Diamond\,q$

Proof:

$$
\begin{array}{ll}
& \Box\Diamond\,(p \wedge q) \\
\Rightarrow & \langle (139)\ \text{Metatheorem}\ \Box\ \text{and}\ (53)\ \text{Distributivity of}\ \Diamond\ \text{over}\ \wedge, \\
& \quad \Box\Diamond\,(p \wedge q) \Rightarrow \Box\,(\Diamond\,p \wedge \Diamond\,q) \rangle \\
& \Box\,(\Diamond\,p \wedge \Diamond\,q) \\
= & \langle (99)\ \text{Distributivity of}\ \Box\ \text{over}\ \wedge \rangle \\
& \Box\Diamond\,p \wedge \Box\Diamond\,q \quad \blacksquare
\end{array}
$$

(160) **Distributivity of $\Diamond\Box$ over $\vee$:** $\Diamond\Box\,p \vee \Diamond\Box\,q \Rightarrow \Diamond\Box\,(p \vee q)$

Proof:

$$
\begin{array}{ll}
& \Diamond\Box\,p \vee \Diamond\Box\,q \\
= & \langle (52)\ \text{Distributivity of}\ \Diamond\ \text{over}\ \vee \rangle \\
& \Diamond\,(\Box\,p \vee \Box\,q) \\
\Rightarrow & \langle (138)\ \text{Metatheorem}\ \Diamond\ \text{and}\ (100)\ \text{Distributivity of}\ \Box\ \text{over}\ \vee, \\
& \quad \Diamond\,(\Box\,p \vee \Box\,q) \Rightarrow \Diamond\Box\,(p \vee q) \rangle \\
& \Diamond\Box\,(p \vee q) \quad \blacksquare
\end{array}
$$

(161) **Distributivity of $\Box\Diamond$ over $\vee$:** $\Box\Diamond\,(p \vee q) \equiv \Box\Diamond\,p \vee \Box\Diamond\,q$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$\quad \Box \Diamond p \vee \Box \Diamond q$
$\Rightarrow \quad \langle$(100) Distributivity of $\Box$ over $\vee$ with $p,q := \Diamond p, \Diamond q\rangle$
$\quad \Box(\Diamond p \vee \Diamond q)$
$= \quad \langle$(52) Distributivity of $\Diamond$ over $\vee\rangle$
$\quad \Box \Diamond (p \vee q)$

The proof in the second direction is by (4.7.1) Truth implication.

$\quad$ *true*
$= \quad \langle$(88) with $p,q := \Diamond(p \vee q), \Box \neg p\rangle$
$\quad \Box \Diamond (p \vee q) \wedge \Diamond \Box \neg p \Rightarrow \Diamond(\Diamond(p \vee q) \wedge \Box \neg p)$
$\Rightarrow \quad \langle$Lemma: $\Diamond(\Diamond(p \vee q) \wedge \Box \neg p) \Rightarrow \Diamond q$ and (3.82a) Transitivity$\rangle$
$\quad \Box \Diamond (p \vee q) \wedge \Diamond \Box \neg p \Rightarrow \Diamond q$
$= \quad \langle$(139) Metatheorem $\Box$ with the above theorem$\rangle$
$\quad \Box(\Box \Diamond (p \vee q) \wedge \Diamond \Box \neg p) \Rightarrow \Box \Diamond q$
$= \quad \langle$(99) Distributivity of $\Box$ over $\wedge\rangle$
$\quad \Box \Box \Diamond (p \vee q) \wedge \Box \Diamond \Box \neg p \Rightarrow \Box \Diamond q$
$= \quad \langle$(72) Absorption of $\Box$ and (152) Absorption of $\Box$ into $\Diamond \Box\rangle$
$\quad \Box \Diamond (p \vee q) \wedge \Diamond \Box \neg p \Rightarrow \Box \Diamond q$
$= \quad \langle$(3.65) Shunting, $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)\rangle$
$\quad \Box \Diamond (p \vee q) \Rightarrow (\Diamond \Box \neg p \Rightarrow \Box \Diamond q)$
$= \quad \langle$(63) Dual of $\Box \Diamond\rangle$
$\quad \Box \Diamond (p \vee q) \Rightarrow (\neg \Box \Diamond p \Rightarrow \Box \Diamond q)$
$= \quad \langle$(3.59) Implication $p \Rightarrow q \equiv \neg p \vee q\rangle$
$\quad \Box \Diamond (p \vee q) \Rightarrow \Box \Diamond p \vee \Box \Diamond q \quad \blacksquare$

Lemma: $\Diamond(\Diamond(p \vee q) \wedge \Box \neg p) \Rightarrow \Diamond q$
Proof: The proof is by (4.7.1) Truth implication.

$\quad$ *true*
$= \quad \langle$(88) Distributivity of $\Diamond$ over $\wedge$ with $p,q := \neg p, p \vee q\rangle$
$\quad \Box \neg p \wedge \Diamond(p \vee q) \Rightarrow \Diamond(\neg p \wedge (p \vee q))$
$= \quad \langle$(3.44a) Absorption, $p \wedge (\neg p \vee q) \equiv p \wedge q\rangle$
$\quad \Box \neg p \wedge \Diamond(p \vee q) \Rightarrow \Diamond(\neg p \wedge q)$
$= \quad \langle$(138) Metatheorem $\Diamond$ with the above theorem$\rangle$
$\quad \Diamond(\Box \neg p \wedge \Diamond(p \vee q)) \Rightarrow \Diamond \Diamond(\neg p \wedge q)$
$= \quad \langle$(50) Absorption of $\Diamond$ and (3.36) Symmetry of $\wedge\rangle$
$\quad \Diamond(\Diamond(p \vee q) \wedge \Box \neg p) \Rightarrow \Diamond(\neg p \wedge q)$
$\Rightarrow \quad \langle$(53) Distributivity of $\Diamond$ over $\wedge$ and (3.82a) Transitivity$\rangle$
$\quad \Diamond(\Diamond(p \vee q) \wedge \Box \neg p) \Rightarrow \Diamond \neg p \wedge \Diamond q$

$\Rightarrow \quad \langle$(3.76b) Strengthening, $p \wedge q \Rightarrow p$ and (3.82a) Transitivity$\rangle$
$\quad \Diamond(\Diamond(p \vee q) \wedge \Box \neg p) \Rightarrow \Diamond q \quad \blacksquare$

(162) **Distributivity of $\Diamond\Box$ over $\wedge$:** $\quad \Diamond\Box(p \wedge q) \equiv \Diamond\Box p \wedge \Diamond\Box q$

Proof:

$\quad \Diamond\Box(p \wedge q)$
$= \quad \langle$(3.12) Double negation, $\neg\neg p \equiv p$, twice$\rangle$
$\quad \Diamond\Box(\neg\neg p \wedge \neg\neg q)$
$= \quad \langle$(3.47b) De Morgan, $\neg(p \vee q) \equiv \neg p \wedge \neg q\rangle$
$\quad \Diamond\Box\neg(\neg p \vee \neg q)$
$= \quad \langle$(63) Dual of $\Box\Diamond\rangle$
$\quad \neg\Box\Diamond(\neg p \vee \neg q)$
$= \quad \langle$(161) Distributivity of $\Box\Diamond$ over $\vee\rangle$
$\quad \neg(\Box\Diamond\neg p \vee \Box\Diamond\neg q)$
$= \quad \langle$(62) Dual of $\Diamond\Box\rangle$
$\quad \neg(\neg\Diamond\Box p \vee \neg\Diamond\Box q)$
$= \quad \langle$(3.47a) De Morgan $\neg(p \wedge q) \equiv \neg p \vee \neg q\rangle$
$\quad \neg\neg(\Diamond\Box p \wedge \Diamond\Box q)$
$= \quad \langle$(3.12) Double negation, $\neg\neg p \equiv p\rangle$
$\quad \Diamond\Box p \wedge \Diamond\Box q \quad \blacksquare$

Theorem (163) is Problem 4.2 in Manna and Pnueli. [19] Theorems (164) and (165) are Exercise 14.6 and 14.7 respectively in Ben-Ari. [2]

(163) **Eventual latching:** $\quad \Diamond\Box(p \Rightarrow \Box q) \equiv \Diamond\Box\neg p \vee \Diamond\Box q$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$\quad \Diamond\Box(p \Rightarrow \Box q)$
$\Rightarrow \quad \langle$Lemma: $\Diamond\Box(p \Rightarrow \Box q) \Rightarrow \Diamond(\Box\Diamond p \Rightarrow \Diamond\Box q)\rangle$
$\quad \Diamond(\Box\Diamond p \Rightarrow \Diamond\Box q)$
$= \quad \langle$(104) Distributivity of $\Diamond$ over $\Rightarrow\rangle$
$\quad \Box\Box\Diamond p \Rightarrow \Diamond\Diamond\Box q$
$= \quad \langle$(72) Absorption of $\Box$ and (50) Absorption of $\Diamond\rangle$
$\quad \Box\Diamond p \Rightarrow \Diamond\Box q$
$= \quad \langle$(3.59) Implication $p \Rightarrow q \equiv \neg p \vee q\rangle$
$\quad \neg\Box\Diamond p \vee \Diamond\Box q$
$= \quad \langle$(63) Dual of $\Box\Diamond\rangle$
$\quad \Diamond\Box\neg p \vee \Diamond\Box q$

The proof in the second direction follows.

$$
\begin{array}{ll}
 & \Diamond\Box(p \Rightarrow \Box q) \\
= & \langle (3.59)\ \text{Implication},\ p \Rightarrow q \ \equiv\ \neg p \vee q \rangle \\
 & \Diamond\Box(\neg p \vee \Box q) \\
\Leftarrow & \langle (160)\ \text{Distributivity of } \Diamond\Box \text{ over } \vee \rangle \\
 & \Diamond\Box\neg p \vee \Diamond\Box\Box q \\
= & \langle (72)\ \text{Absorption of } \Box \rangle \\
 & \Diamond\Box\neg p \vee \Diamond\Box q \quad \blacksquare
\end{array}
$$

Lemma: $\Diamond\Box(p \Rightarrow \Box q) \Rightarrow \Diamond(\Box\Diamond p \Rightarrow \Diamond\Box q)$
Proof: The proof is by (4.7.1) Truth implication.

$$
\begin{array}{ll}
 & \mathit{true} \\
= & \langle (119)\ \text{Monotonicity of } \Diamond \text{ with } q := \Box q \rangle \\
 & \Box(p \Rightarrow \Box q) \Rightarrow (\Diamond p \Rightarrow \Diamond\Box q) \\
= & \langle (136)\ \text{Metatheorem with the above theorem} \rangle \\
 & \Box(\Box(p \Rightarrow \Box q) \Rightarrow (\Diamond p \Rightarrow \Diamond\Box q)) \\
\Rightarrow & \langle (157)\ \text{Monotonicity of } \Box\Diamond \rangle \\
 & \Box\Diamond\Box(p \Rightarrow \Box q) \Rightarrow \Box\Diamond(\Diamond p \Rightarrow \Diamond\Box q) \\
= & \langle (152)\ \text{Absorption of } \Box \text{ into } \Diamond\Box \rangle \\
 & \Diamond\Box(p \Rightarrow \Box q) \Rightarrow \Box\Diamond(\Diamond p \Rightarrow \Diamond\Box q) \\
\Rightarrow & \langle (76)\ \text{Strengthening of } \Box \text{ and (3.82a) Transitivity} \rangle \\
 & \Diamond\Box(p \Rightarrow \Box q) \Rightarrow \Diamond(\Diamond p \Rightarrow \Diamond\Box q) \\
= & \langle (104)\ \text{Distributivity of } \Diamond \text{ over } \Rightarrow \rangle \\
 & \Diamond\Box(p \Rightarrow \Box q) \Rightarrow (\Box\Diamond p \Rightarrow \Diamond\Diamond\Box q) \\
= & \langle (50)\ \text{Absorption of } \Diamond \rangle \\
 & \Diamond\Box(p \Rightarrow \Box q) \Rightarrow (\Box\Diamond p \Rightarrow \Diamond\Box q) \\
\Rightarrow & \langle (46)\ \text{Weakening of } \Diamond \text{ and (3.82a) Transitivity} \rangle \\
 & \Diamond\Box(p \Rightarrow \Box q) \Rightarrow \Diamond(\Box\Diamond p \Rightarrow \Diamond\Box q) \quad \blacksquare
\end{array}
$$

(164) $\Box(\Box\Diamond p \Rightarrow \Diamond q) \equiv \Diamond\Box\neg p \vee \Box\Diamond q$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$$
\begin{array}{ll}
 & \Box(\Box\Diamond p \Rightarrow \Diamond q) \\
\Rightarrow & \langle (120)\ \text{Monotonicity of } \Box \text{ with } p, q := \Box\Diamond p, \Diamond q \rangle \\
 & \Box\Box\Diamond p \Rightarrow \Box\Diamond q \\
= & \langle (72)\ \text{Absorption of } \Box \rangle \\
 & \Box\Diamond p \Rightarrow \Box\Diamond q
\end{array}
$$

$$
\begin{array}{ll}
= & \langle (3.59) \text{ Implication, } p \Rightarrow q \equiv \neg p \vee q \rangle \\
 & \neg \Box \Diamond p \vee \Box \Diamond q \\
= & \langle (63) \text{ Dual of } \Box \Diamond \rangle \\
 & \Diamond \Box \neg p \vee \Box \Diamond q
\end{array}
$$

The proof in the second direction follows.

$$
\begin{array}{ll}
 & \Box (\Box \Diamond p \Rightarrow \Diamond q) \\
= & \langle (3.59) \text{ Implication, } p \Rightarrow q \equiv \neg p \vee q \rangle \\
 & \Box (\neg \Box \Diamond p \vee \Diamond q) \\
\Leftarrow & \langle (100) \text{ Distributivity of } \Box \text{ over } \vee \rangle \\
 & \Box \neg \Box \Diamond p \vee \Box \Diamond q \\
= & \langle (63) \text{ Dual of } \Box \Diamond \rangle \\
 & \Box \Diamond \Box \neg p \vee \Box \Diamond q \\
= & \langle (152) \text{ Absorption of } \Box \text{ into } \Diamond \Box \rangle \\
 & \Diamond \Box \neg p \vee \Box \Diamond q \quad \blacksquare
\end{array}
$$

(165) $\Box ((p \vee \Box q) \wedge (\Box p \vee q)) \equiv \Box p \vee \Box q$

Proof: The proof is by (4.7) Mutual implication and is based on the following lemmas, where R is defined as the expression

$$R : (p \vee \Box q) \wedge (\Box p \vee q)$$

With this definition, the theorem to be proved is

$$\Box R \equiv \Box p \vee \Box q$$

Lemma A: $R \equiv \Box p \vee \Box q \vee (p \wedge q)$

Proof:

$$
\begin{array}{ll}
 & R \\
= & \langle \text{Definition of } R \rangle \\
 & (p \vee \Box q) \wedge (\Box p \vee q) \\
= & \langle (3.46) \text{ Distributivity of } \wedge \text{ over } \vee, \ p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \rangle \\
 & (p \wedge \Box p) \vee (p \wedge q) \vee (\Box p \wedge \Box q) \vee (q \wedge \Box q) \\
= & \langle (68) \text{ Absorption of } \wedge \text{ into } \Box \text{, twice} \rangle \\
 & \Box p \vee (p \wedge q) \vee (\Box p \wedge \Box q) \vee \Box q \\
= & \langle (3.43b) \text{ Absorption } p \vee (p \wedge q) \equiv p \rangle \\
 & \Box p \vee \Box q \vee (p \wedge q) \quad \blacksquare
\end{array}
$$

Lemma B: $\Box R \wedge \neg \Box p \wedge \neg \Box q \Rightarrow \circ (\Box R \wedge \neg \Box p \wedge \neg \Box q)$

Proof:

$\square R \wedge \neg\square p \wedge \neg\square q$
$= \quad \langle$(66) Expansion of $\square\rangle$
$R \wedge \circ\square R \wedge \neg\square p \wedge \neg\square q$
$= \quad \langle$(3.36) Symmetry of $\wedge$, $p \wedge q \equiv q \wedge p\rangle$
$\circ\square R \wedge R \wedge \neg\square p \wedge \neg\square q$
$= \quad \langle$Lemma A: $R \equiv \square p \vee \square q \vee (p \wedge q)\rangle$
$\circ\square R \wedge (\square p \vee \square q \vee (p \wedge q)) \wedge \neg\square p \wedge \neg\square q$
$= \quad \langle$(3.44a) Absorption, $p \wedge (\neg p \vee q) \equiv p \wedge q$, twice$\rangle$
$\circ\square R \wedge (\neg\square p \wedge \neg\square q \wedge (p \wedge q))$
$= \quad \langle$(66) Expansion of $\square$ and (3.47a) De Morgan $\neg(p \wedge q) \equiv \neg p \vee \neg q$, twice$\rangle$
$\circ\square R \wedge (p \wedge q) \wedge (\neg p \vee \neg\circ\square p) \wedge (\neg q \vee \neg\circ\square q)$
$= \quad \langle$(1) Self dual, twice$\rangle$
$\circ\square R \wedge (p \wedge q) \wedge (\neg p \vee \circ\neg\square p) \wedge (\neg q \vee \circ\neg\square q)$
$= \quad \langle$(3.44a) Absorption, $p \wedge (\neg p \vee q) \equiv p \wedge q$, twice$\rangle$
$\circ\square R \wedge p \wedge \circ\neg\square p \wedge q \wedge \circ\neg\square q$
$\Rightarrow \quad \langle$(3.76b) Strengthening, $p \wedge q \Rightarrow p\rangle$
$\circ\square R \wedge \circ\neg\square p \wedge \circ\neg\square q$
$= \quad \langle$(5) Distributivity of $\circ$ over $\wedge\rangle$
$\circ(\square R \wedge \neg\square p \wedge \neg\square q)$ ∎

Lemma C: $\square R \wedge \neg\square p \wedge \neg\square q \Rightarrow \square(\square R \wedge \neg\square p \wedge \neg\square q)$
Proof: The proof is by (4.7.1) Truth implication.

true
$= \quad \langle$Lemma B: $\square R \wedge \neg\square p \wedge \neg\square q \Rightarrow \circ(\square R \wedge \neg\square p \wedge \neg\square q)\rangle$
$\square R \wedge \neg\square p \wedge \neg\square q \Rightarrow \circ(\square R \wedge \neg\square p \wedge \neg\square q)$
$= \quad \langle$(136) Metatheorem with the above theorem$\rangle$
$\square(\square R \wedge \neg\square p \wedge \neg\square q \Rightarrow \circ(\square R \wedge \neg\square p \wedge \neg\square q))$
$\Rightarrow \quad \langle$(57) $\square$ Induction$\rangle$
$\square R \wedge \neg\square p \wedge \neg\square q \Rightarrow \square(\square R \wedge \neg\square p \wedge \neg\square q)$ ∎

Lemma D: $\square(\square R \wedge \neg\square p \wedge \neg\square q) \Rightarrow \square p \wedge \square q$
Proof: The proof is by (4.7.1) Truth implication.

true
$= \quad \langle$(3.71) Reflexivity of $\Rightarrow$, $p \Rightarrow p\rangle$
$\square R \wedge \neg\square p \wedge \neg\square q \Rightarrow \square R \wedge \neg\square p \wedge \neg\square q$
$\Rightarrow \quad \langle$(76) Strengthening of $\square$ and (4.3) Monotonicity of $\wedge$
and (3.82a) Transitivity$\rangle$
$\square R \wedge \neg\square p \wedge \neg\square q \Rightarrow R \wedge \neg\square p \wedge \neg\square q$

$=\quad\langle$Lemma A: $R \equiv \Box p \vee \Box q \vee (p \wedge q)\rangle$
$\Box R \wedge \neg\Box p \wedge \neg\Box q \Rightarrow (\Box p \vee \Box q \vee (p \wedge q)) \wedge \neg\Box p \wedge \neg\Box q$
$=\quad\langle$(3.44a) Absorption, $p \wedge (\neg p \vee q) \equiv p \wedge q$, twice$\rangle$
$\Box R \wedge \neg\Box p \wedge \neg\Box q \Rightarrow \neg\Box p \wedge \neg\Box q \wedge (p \wedge q)$
$\Rightarrow\quad\langle$(3.76b) Strengthening, $p \wedge q \Rightarrow p$ and (3.82a) Transitivity$\rangle$
$\Box R \wedge \neg\Box p \wedge \neg\Box q \Rightarrow p \wedge q$
$=\quad\langle$(136) Metatheorem with the above theorem$\rangle$
$\Box(\Box R \wedge \neg\Box p \wedge \neg\Box q \Rightarrow p \wedge q)$
$\Rightarrow\quad\langle$(120) Monotonicity of $\Box\rangle$
$\Box(\Box R \wedge \neg\Box p \wedge \neg\Box q) \Rightarrow \Box(p \wedge q)$
$=\quad\langle$(99) Distributivity of $\Box$ over $\wedge\rangle$
$\Box(\Box R \wedge \neg\Box p \wedge \neg\Box q) \Rightarrow \Box p \wedge \Box q \quad \blacksquare$

Lemma E: $\Box R \wedge \neg\Box p \wedge \neg\Box q \Rightarrow \Box p \wedge \Box q$
Proof:

$\Box R \wedge \neg\Box p \wedge \neg\Box q$
$\Rightarrow\quad\langle$Lemma C: $\Box R \wedge \neg\Box p \wedge \neg\Box q \Rightarrow \Box(\Box R \wedge \neg\Box p \wedge \neg\Box q)\rangle$
$\Box(\Box R \wedge \neg\Box p \wedge \neg\Box q)$
$\Rightarrow\quad\langle$Lemma D: $\Box(\Box R \wedge \neg\Box p \wedge \neg\Box q) \Rightarrow \Box p \wedge \Box q\rangle$
$\Box p \wedge \Box q \quad \blacksquare$

Lemma F: $\Box p \vee \Box q \Rightarrow \bigcirc(\Box p \vee \Box q)$
Proof:

$\Box p \vee \Box q$
$=\quad\langle$(66) Expansion of $\Box$, twice$\rangle$
$(p \wedge \bigcirc\Box p) \vee (q \wedge \bigcirc\Box q)$
$\Rightarrow\quad\langle$(3.76b) Strengthening, $p \wedge q \Rightarrow p$ and (4.2) Monotonicity of $\vee$, twice$\rangle$
$\bigcirc\Box p \vee \bigcirc\Box q$
$=\quad\langle$(4) Distributivity of $\bigcirc$ over $\vee\rangle$
$\bigcirc(\Box p \vee \Box q) \quad \blacksquare$

Lemma G: $\Box(\Box p \vee \Box q) \Rightarrow \Box R$
Proof: The proof is by (4.7.1) Truth implication.

true
$=\quad\langle$(3.76a) Weakening, $p \Rightarrow p \vee q\rangle$
$\Box p \vee \Box q \Rightarrow \Box p \vee \Box q \vee (p \wedge q)$
$=\quad\langle$Lemma A: $R \equiv \Box p \vee \Box q \vee (p \wedge q)\rangle$
$\Box p \vee \Box q \Rightarrow R$

$$
\begin{array}{ll}
= & \langle (136) \text{ Metatheorem with the above theorem}\rangle \\
 & \Box(\Box p \vee \Box q \Rightarrow R) \\
\Rightarrow & \langle (120) \text{ Monotonicity of } \Box \rangle \\
 & \Box(\Box p \vee \Box q) \Rightarrow \Box R \quad \blacksquare
\end{array}
$$

The proof of (165) in the first direction is by (4.9) Proof by contradiction. To prove

$$\Box R \Rightarrow \Box p \vee \Box q$$

by contradiction, prove that

$$\neg(\Box R \Rightarrow \Box p \vee \Box q) \Rightarrow \mathit{false}$$

Proof:

$$
\begin{array}{ll}
 & \neg(\Box R \Rightarrow \Box p \vee \Box q) \\
= & \langle (3.59) \text{ Implication, } p \Rightarrow q \;\equiv\; \neg p \vee q\rangle \\
 & \neg(\neg\Box R \vee \Box p \vee \Box q) \\
= & \langle (3.47\text{b}) \text{ De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q\rangle \\
 & \Box R \wedge \neg\Box p \wedge \neg\Box q \\
= & \langle (3.38) \text{ Idempotency of } \wedge,\ p \wedge p \equiv p\rangle \\
 & (\Box R \wedge \neg\Box p \wedge \neg\Box q) \wedge (\Box R \wedge \neg\Box p \wedge \neg\Box q) \\
\Rightarrow & \langle \text{Lemma E: } \Box R \wedge \neg\Box p \wedge \neg\Box q \Rightarrow \Box p \wedge \Box q \\
 & \quad \text{and (4.3) Monotonicity of } \wedge\rangle \\
 & \Box R \wedge \neg\Box p \wedge \neg\Box q \wedge \Box p \wedge \Box q \\
= & \langle (3.42) \text{ Contradiction, } p \wedge \neg p \;\equiv\; \mathit{false}, \text{twice}\rangle \\
 & \Box R \wedge \mathit{false} \wedge \mathit{false} \\
= & \langle (3.40) \text{ Zero of } \wedge,\ p \wedge \mathit{false} \equiv \mathit{false}\rangle \\
 & \mathit{false}
\end{array}
$$

The proof of (165) in the second direction

$$\Box p \vee \Box q \Rightarrow \Box R$$

is by (4.7.1) Truth implication.
Proof:

$$
\begin{array}{ll}
 & \mathit{true} \\
= & \langle \text{Lemma F: } \Box p \vee \Box q \Rightarrow \bigcirc(\Box p \vee \Box q)\rangle \\
 & \Box p \vee \Box q \Rightarrow \bigcirc(\Box p \vee \Box q) \\
= & \langle (136) \text{ Metatheorem with the above theorem}\rangle \\
 & \Box(\Box p \vee \Box q \Rightarrow \bigcirc(\Box p \vee \Box q)) \\
\Rightarrow & \langle (57)\ \Box \text{ Induction}\rangle
\end{array}
$$

$\square p \vee \square q \Rightarrow \square(\square p \vee \square q)$
$\Rightarrow \quad \langle$Lemma G: $\square(\square p \vee \square q) \Rightarrow \square R$ and (3.82a) Transitivity$\rangle$
$\square p \vee \square q \Rightarrow \square R$ ∎

The metatheorems and absorption laws imply the following intuitive theorem. If p will eventually be always true, and it is always the case that q will be eventually true, then it is always the case that $p \wedge q$ will eventually be true.

(166) $\diamond \square p \wedge \square \diamond q \Rightarrow \square \diamond (p \wedge q)$

Proof:

true
$= \quad \langle$(88) Distributivity of $\diamond$ over $\wedge$ and (139) Metatheorem $\square\rangle$
$\square(\square p \wedge \diamond q) \Rightarrow \square \diamond (p \wedge q)$
$= \quad \langle$(99) Distributivity of $\square$ over $\wedge\rangle$
$\square \square p \wedge \square \diamond q \Rightarrow \square \diamond (p \wedge q)$
$= \quad \langle$(72) Absorption of $\square\rangle$
$\square p \wedge \square \diamond q \Rightarrow \square \diamond (p \wedge q)$
$= \quad \langle$(3.65) Shunting, $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)\rangle$
$\square p \Rightarrow (\square \diamond q \Rightarrow \square \diamond (p \wedge q))$
$= \quad \langle$(138) Metatheorem $\diamond$ with the above theorem$\rangle$
$\diamond \square p \Rightarrow \diamond(\square \diamond q \Rightarrow \square \diamond (p \wedge q))$
$= \quad \langle$(104) Distributivity of $\diamond$ over $\Rightarrow\rangle$
$\diamond \square p \Rightarrow (\square \square \diamond q \Rightarrow \diamond \square \diamond (p \wedge q))$
$= \quad \langle$(72) Absorption of $\square$ and (151) Absorption of $\diamond$ into $\square\rangle$
$\diamond \square p \Rightarrow (\square \diamond q \Rightarrow \square \diamond (p \wedge q))$
$= \quad \langle$(3.65) Shunting, $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)\rangle$
$\diamond \square p \wedge \square \diamond q \Rightarrow \square \diamond (p \wedge q)$ ∎

(167) $\square((\square p \Rightarrow \diamond q) \wedge (q \Rightarrow \circ r)) \Rightarrow (\square p \Rightarrow \circ \square \diamond r)$

Proof:

$\square((\square p \Rightarrow \diamond q) \wedge (q \Rightarrow \circ r)) \Rightarrow (\square p \Rightarrow \circ \square \diamond r)$
$= \quad \langle$(3.65) Shunting, $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)\rangle$
$\square((\square p \Rightarrow \diamond q) \wedge (q \Rightarrow \circ r)) \wedge \square p \Rightarrow \circ \square \diamond r$

And now,

$\square((\square p \Rightarrow \diamond q) \wedge (q \Rightarrow \circ r)) \wedge \square p$
$= \quad \langle$(99) Distributivity of $\square$ over $\wedge\rangle$

$\Box(\Box p \Rightarrow \Diamond q) \wedge \Box(q \Rightarrow \bigcirc r) \wedge \Box p$
$\Rightarrow$ $\langle$(46) Weakening of $\Diamond$, with $p := \Box p$ and (4.3) Monotonicity of $\wedge$ $\rangle$
$\Box(\Box p \Rightarrow \Diamond q) \wedge \Box(q \Rightarrow \bigcirc r) \wedge \Diamond \Box p$
$\Rightarrow$ $\langle$(157) Monotonicity of $\Box\Diamond$ and (4.3) Monotonicity of $\wedge$ $\rangle$
$(\Box\Diamond\Box p \Rightarrow \Box\Diamond\Diamond q) \wedge \Box(q \Rightarrow \bigcirc r) \wedge \Diamond \Box p$
$=$ $\langle$(152) Absorption of $\Box$ into $\Diamond\Box$ and (50) Absorption of $\Diamond$ $\rangle$
$(\Diamond\Box p \Rightarrow \Box\Diamond q) \wedge \Box(q \Rightarrow \bigcirc r) \wedge \Diamond \Box p$
$\Rightarrow$ $\langle$(3.77) Modus ponens, $p \wedge (p \Rightarrow q) \Rightarrow q$ and (4.3) Monotonicity of $\wedge$$\rangle$
$\Box\Diamond q \wedge \Box(q \Rightarrow \bigcirc r)$
$\Rightarrow$ $\langle$(157) Monotonicity of $\Box\Diamond$ and (4.3) Monotonicity of $\wedge$ $\rangle$
$\Box\Diamond q \wedge (\Box\Diamond q \Rightarrow \Box\Diamond\bigcirc r)$
$\Rightarrow$ $\langle$(3.77) Modus ponens, $p \wedge (p \Rightarrow q) \Rightarrow q$$\rangle$
$\Box\Diamond\bigcirc r$
$=$ $\langle$(51) Exchange of $\bigcirc$ and $\Diamond$ and (73) Exchange of $\bigcirc$ and $\Box$$\rangle$
$\bigcirc\Box\Diamond r$ ∎

(168) **Progress proof rule:** $\Diamond\Box p \wedge \Box(\Box p \Rightarrow \Diamond q) \Rightarrow \Diamond q$

Proof:

$\Diamond\Box p \wedge \Box(\Box p \Rightarrow \Diamond q)$
$\Rightarrow$ $\langle$(119) Monotonicity of $\Diamond$ and (4.3) Monotonicity of $\wedge$$\rangle$
$\Diamond\Box p \wedge (\Diamond\Box p \Rightarrow \Diamond\Diamond q)$
$=$ $\langle$(50) Absorption of $\Diamond$$\rangle$
$\Diamond\Box p \wedge (\Diamond\Box p \Rightarrow \Diamond q)$
$\Rightarrow$ $\langle$(3.77) Modus ponens, $p \wedge (p \Rightarrow q) \Rightarrow q$$\rangle$
$\Diamond q$ ∎

3.9 Wait

$p \,\mathcal{U}\, q$ requires p to be true until q, which is guaranteed to eventually be true. $p \,\mathcal{W}\, q$ has no such guarantee. That is, if q is eventually true then p is true until that time. But, if q is not eventually true then p must be true always. Accordingly, Kröger and Merz [18] refer to $\mathcal{W}$ as the *weak* version of $\mathcal{U}$. Equations (169) and (170) are the only defining axioms for the *wait* operator.

(169) **Definition of $\mathcal{W}$:** $p \,\mathcal{W}\, q \equiv \Box p \vee p \,\mathcal{U}\, q$

(170) **Axiom, Distributivity of $\neg$ over $\mathcal{W}$:** $\neg(p \,\mathcal{W}\, q) \equiv \neg q \,\mathcal{U}\, (\neg p \wedge \neg q)$

The defining equation gives $\mathcal{W}$ in terms of $\mathcal{U}$. The following theorem gives $\mathcal{U}$ in terms of $\mathcal{W}$.

(171) $\mathcal{U}$ **in terms of** $\mathcal{W}$**:** $p \,\mathcal{U}\, q \equiv p \,\mathcal{W}\, q \wedge \Diamond q$

Proof:

$$
\begin{array}{ll}
 & p\,\mathcal{U}\,q \equiv p\,\mathcal{W}\,q \wedge \Diamond q \\
= & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
 & p\,\mathcal{U}\,q \equiv (\Box p \vee p\,\mathcal{U}\,q) \wedge \Diamond q \\
= & \langle (3.46) \text{ Distributivity of } \wedge \text{ over } \vee,\ p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \rangle \\
 & p\,\mathcal{U}\,q \equiv (\Box p \wedge \Diamond q) \vee (p\,\mathcal{U}\,q \wedge \Diamond q) \\
= & \langle (39) \text{ Absorption of } \Diamond \text{ into } \mathcal{U} \rangle \\
 & p\,\mathcal{U}\,q \equiv (\Box p \wedge \Diamond q) \vee p\,\mathcal{U}\,q \\
= & \langle (3.57) \text{ Definition of implication, } p \Rightarrow q \ \equiv\ p \vee q \ \equiv\ q \rangle \\
 & \Box p \wedge \Diamond q \Rightarrow p\,\mathcal{U}\,q \quad \text{—(84) } \mathcal{U} \text{ implication} \quad \blacksquare
\end{array}
$$

(172) $p\,\mathcal{W}\,q \equiv \Box(p \wedge \neg q) \vee p\,\mathcal{U}\,q$

Proof:

$$
\begin{array}{ll}
 & \Box(p \wedge \neg q) \vee p\,\mathcal{U}\,q \\
= & \langle (99) \text{ Distributivity of } \Box \text{ over } \wedge \rangle \\
 & (\Box p \wedge \Box \neg q) \vee p\,\mathcal{U}\,q \\
= & \langle (3.45) \text{ Distributivity of } \vee \text{ over } \wedge,\ p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \rangle \\
 & (\Box p \vee p\,\mathcal{U}\,q) \wedge (\Box \neg q \vee p\,\mathcal{U}\,q) \\
= & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
 & p\,\mathcal{W}\,q \wedge (\Box \neg q \vee p\,\mathcal{U}\,q) \\
= & \langle (171)\ \mathcal{U} \text{ in terms of } \mathcal{W} \rangle \\
 & p\,\mathcal{W}\,q \wedge (\Box \neg q \vee (p\,\mathcal{W}\,q \wedge \Diamond q)) \\
= & \langle (3.45) \text{ Distributivity of } \vee \text{ over } \wedge,\ p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \rangle \\
 & p\,\mathcal{W}\,q \wedge ((\Box \neg q \vee p\,\mathcal{W}\,q) \wedge (\Box \neg q \vee \Diamond q)) \\
= & \langle (89)\ \Diamond \text{ excluded middle and (3.39) Identity of } \wedge,\ p \wedge \mathit{true} \equiv p \rangle \\
 & p\,\mathcal{W}\,q \wedge (\Box \neg q \vee p\,\mathcal{W}\,q) \\
= & \langle (3.43a) \text{ Absorption, } p \wedge (p \vee q) \equiv p \rangle \\
 & p\,\mathcal{W}\,q \quad \blacksquare
\end{array}
$$

The proof of (173) Distributivity of $\neg$ over $\mathcal{U}$ uses (170) Axiom, Distributivity of $\neg$ over $\mathcal{W}$. Apparently, the reverse is not possible. That is, if (173) is taken as the axiom instead of (170), we were not able to prove (170) as a theorem.

(173) **Distributivity of $\neg$ over $\mathcal{U}$:** $\neg(p\,\mathcal{U}\,q) \equiv \neg q\,\mathcal{W}\,(\neg p \wedge \neg q)$

Proof: (Michael Ortiz)

$$
\begin{array}{ll}
 & \neg q\,\mathcal{W}\,(\neg p \wedge \neg q) \\
= & \langle (169) \text{ Definition of } \mathcal{W} \rangle
\end{array}
$$

$\square \neg q \vee \neg q \ \mathcal{U}\ (\neg p \wedge \neg q)$
$= \quad \langle(170)$ Distributivity of $\neg$ over $\mathcal{W}\rangle$
$\square \neg q \vee \neg(p \ \mathcal{W}\ q)$
$= \quad \langle(61)$ Dual of $\diamond\rangle$
$\neg(p \ \mathcal{W}\ q) \vee \neg \diamond q$
$= \quad \langle(3.47a)$ De Morgan $\neg(p \wedge q) \equiv \neg p \vee \neg q\rangle$
$\neg(p \ \mathcal{W}\ q \wedge \diamond q)$
$= \quad \langle(171)\ \mathcal{U}$ in terms of $\mathcal{W}\rangle$
$\neg(p \ \mathcal{U}\ q) \quad \blacksquare$

Theorem (174) $\mathcal{U}$ implication shows that $p \ \mathcal{U}\ q$ is stronger than $p \ \mathcal{W}\ q$. Theorem (175) Distributivity of $\wedge$ over $\mathcal{W}$ corresponds to, and is derived from, (83) Distributivity of $\wedge$ over $\mathcal{U}$. Theorem (176) $\mathcal{W} \diamond$ equivalence comes from Manna and Pnueli [19] where it is used as the normal form for simple obligation formulas.

(174) $\mathcal{U}$ **implication:** $\quad p \ \mathcal{U}\ q \Rightarrow p \ \mathcal{W}\ q$

Proof:

$p \ \mathcal{U}\ q$
$= \quad \langle(171)\ \mathcal{U}$ in terms of $\mathcal{W}\rangle$
$p \ \mathcal{W}\ q \wedge \diamond q$
$\Rightarrow \quad \langle(3.76b)$ Strengthening, $p \wedge q \Rightarrow p\rangle$
$p \ \mathcal{W}\ q \quad \blacksquare$

(175) **Distributivity of $\wedge$ over $\mathcal{W}$:** $\quad \square p \wedge q \ \mathcal{W}\ r \Rightarrow (p \wedge q) \ \mathcal{W}\ (p \wedge r)$

Proof:

$\square p \wedge q \ \mathcal{W}\ r$
$= \quad \langle(169)$ Definition of $\mathcal{W}\rangle$
$\square p \wedge (\square q \vee q \ \mathcal{U}\ r)$
$= \quad \langle(3.46)$ Distributivity of $\wedge$ over $\vee$, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)\rangle$
$(\square p \wedge \square q) \vee (\square p \wedge q \ \mathcal{U}\ r)$
$\Rightarrow \quad \langle(83)$ Distributivity of $\wedge$ over $\mathcal{U}$ and (4.2) Monotonicity of $\vee\rangle$
$(\square p \wedge \square q) \vee (p \wedge q) \ \mathcal{U}\ (p \wedge r)$
$= \quad \langle(99)$ Distributivity of $\square$ over $\wedge\rangle$
$\square (p \wedge q) \vee (p \wedge q) \ \mathcal{U}\ (p \wedge r)$
$= \quad \langle(169)$ Definition of $\mathcal{W}\rangle$
$(p \wedge q) \ \mathcal{W}\ (p \wedge r) \quad \blacksquare$

(176) $\mathcal{W} \diamond$ **equivalence:** $\quad p \ \mathcal{W} \diamond q \equiv \square p \vee \diamond q$

Proof:

$p \, \mathcal{W} \, \Diamond q$
$= \quad \langle (169) \text{ Definition of } \mathcal{W} \rangle$
$\Box p \lor p \, \mathcal{U} \, \Diamond q$
$= \quad \langle (41) \text{ Absorption of } \mathcal{U} \text{ into } \Diamond \rangle$
$\Box p \lor \Diamond q \quad \blacksquare$

Theorem (177) $\mathcal{W} \, \Box$ implication corresponds to (140) $\mathcal{U} \, \Box$ implication. Theorem (178) Absorption of $\mathcal{W}$ into $\Box$ corresponds to (141) Absorption of $\mathcal{U}$ into $\Box$. Theorem (179) Perpetuity for the *wait* operator corresponds to (42) Eventuality for the *until* operator. The *always* operator, which is universal, is in the antecedent of the implication in Perpetuity, while the *eventually* operator, which is existential, is in the consequent of the implication in Eventuality.

(177) $\mathcal{W} \, \Box$ **implication:** $\quad p \, \mathcal{W} \, \Box q \Rightarrow \Box (p \, \mathcal{W} \, q)$

Proof:

$\Box (p \, \mathcal{W} \, q)$
$= \quad \langle (169) \text{ Definition of } \mathcal{W} \rangle$
$\Box (\Box p \lor p \, \mathcal{U} \, q)$
$\Leftarrow \quad \langle (100) \text{ Distributivity of } \Box \text{ over } \lor \rangle$
$\Box \Box p \lor \Box (p \, \mathcal{U} \, q)$
$= \quad \langle (72) \text{ Absorption of } \Box \rangle$
$\Box p \lor \Box (p \, \mathcal{U} \, q)$
$\Leftarrow \quad \langle (140) \; \mathcal{U} \, \Box \text{ implication and (4.2) Monotonicity of } \lor \rangle$
$\Box p \lor p \, \mathcal{U} \, \Box q$
$= \quad \langle (169) \text{ Definition of } \mathcal{W} \text{ with } q := \Box q \rangle$
$p \, \mathcal{W} \, \Box q \quad \blacksquare$

(178) **Absorption of** $\mathcal{W}$ **into** $\Box$**:** $\quad p \, \mathcal{W} \, \Box p \equiv \Box p$

Proof:

$p \, \mathcal{W} \, \Box p$
$= \quad \langle (169) \text{ Definition of } \mathcal{W} \rangle$
$\Box p \lor p \, \mathcal{U} \, \Box p$
$= \quad \langle (141) \text{ Absorption of } \mathcal{U} \text{ into } \Box \rangle$
$\Box p \lor \Box p$
$= \quad \langle (3.26) \text{ Idempotency of } \lor, \, p \lor p \equiv p \rangle$
$\Box p \quad \blacksquare$

(179) **Perpetuity:** $\quad \Box p \Rightarrow p \, \mathcal{W} \, q$

Proof:

$\quad \Box p$
$\Rightarrow \quad \langle$(3.76a) Weakening, $p \Rightarrow p \vee q\rangle$
$\quad \Box p \vee p\,\mathcal{U}\,q$
$= \quad \langle$(169) Definition of $\mathcal{W}\rangle$
$\quad p\,\mathcal{W}\,q \quad \blacksquare$

(180) **Distributivity of $\circ$ over $\mathcal{W}$:** $\quad \circ(p\,\mathcal{W}\,q) \equiv \circ p\,\mathcal{W} \circ q$

Proof:

$\quad \circ(p\,\mathcal{W}\,q)$
$= \quad \langle$(169) Definition of $\mathcal{W}\rangle$
$\quad \circ(\Box p \vee p\,\mathcal{U}\,q)$
$= \quad \langle$(4) Distributivity of $\circ$ over $\vee\rangle$
$\quad \circ\Box p \vee \circ(p\,\mathcal{U}\,q)$
$= \quad \langle$(73) Exchange of $\circ$ and $\Box$ and (9) Distributivity of $\circ$ over $\mathcal{U}\rangle$
$\quad \Box\circ p \vee \circ p\,\mathcal{U} \circ q$
$= \quad \langle$(169) Definition of $\mathcal{W}$ with $p,q := \circ p, \circ q\rangle$
$\quad \circ p\,\mathcal{W} \circ q \quad \blacksquare$

Expansion of the *wait* operator (181) corresponds to expansion of the *until* operator (10). Excluded middle for the *wait* operator (182) corresponds to excluded middle for the *until* operator (23). Left zero of the *wait* operator (183) corresponds to right zero of the *until* operator (11).

(181) **Expansion of $\mathcal{W}$:** $\quad p\,\mathcal{W}\,q \equiv q \vee (p \wedge \circ(p\,\mathcal{W}\,q))$

Proof:

$\quad q \vee (p \wedge \circ(p\,\mathcal{W}\,q))$
$= \quad \langle$(169) Definition of $\mathcal{W}\rangle$
$\quad q \vee (p \wedge \circ(\Box p \vee p\,\mathcal{U}\,q))$
$= \quad \langle$(4) Distributivity of $\circ$ over $\vee\rangle$
$\quad q \vee (p \wedge (\circ\Box p \vee \circ(p\,\mathcal{U}\,q)))$
$= \quad \langle$(3.46) Distributivity of $\wedge$ over $\vee$, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)\rangle$
$\quad q \vee (p \wedge \circ\Box p) \vee (p \wedge \circ(p\,\mathcal{U}\,q))$
$= \quad \langle$(66) Expansion of $\Box\rangle$
$\quad q \vee \Box p \vee (p \wedge \circ(p\,\mathcal{U}\,q))$
$= \quad \langle$(10) Expansion of $\mathcal{U}\rangle$
$\quad \Box p \vee p\,\mathcal{U}\,q$
$= \quad \langle$(169) Definition of $\mathcal{W}\rangle$
$\quad p\,\mathcal{W}\,q \quad \blacksquare$

(182) $\mathcal{W}$ **excluded middle:** $p\,\mathcal{W}\,q \vee p\,\mathcal{W}\,\neg q$

Proof:

$$\begin{array}{ll} & p\,\mathcal{W}\,q \vee p\,\mathcal{W}\,\neg q \\ = & \langle (181) \text{ Expansion, twice} \rangle \\ & q \vee (p \wedge \circ\,(p\,\mathcal{W}\,q)) \vee \neg q \vee (p \wedge \circ\,(p\,\mathcal{W}\,\neg q)) \\ = & \langle (3.28) \text{ Excluded middle, } p \vee \neg p \rangle \\ & \mathit{true} \vee (p \wedge \circ\,(p\,\mathcal{W}\,q)) \vee (p \wedge \circ\,(p\,\mathcal{W}\,\neg q)) \\ = & \langle (3.29) \text{ Zero of } \vee,\ p \vee \mathit{true} \equiv \mathit{true} \rangle \\ & \mathit{true} \quad \blacksquare \end{array}$$

(183) **Left zero of** $\mathcal{W}$ **:** $\mathit{true}\,\mathcal{W}\,q \equiv \mathit{true}$

Proof:

$$\begin{array}{ll} & \mathit{true}\,\mathcal{W}\,q \\ = & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\ & \Box\,\mathit{true} \vee \mathit{true}\,\mathcal{U}\,q \\ = & \langle (64) \text{ Truth of } \Box \rangle \\ & \mathit{true} \vee \mathit{true}\,\mathcal{U}\,q \\ = & \langle (3.29) \text{ Zero of } \vee,\ p \vee \mathit{true} \equiv \mathit{true} \rangle \\ & \mathit{true} \quad \blacksquare \end{array}$$

The next four distributive theorems for the *wait* operator correspond to, and are proved from, the distributive axioms for the *until* operator (12), (13), (14), and (15). The proof of (187) Right distributivity of $\mathcal{W}$ over $\wedge$ is an example of one benefit of $\mathcal{E}$, with its emphasis on equality, over $\mathcal{H}$, with its emphasis on implication and modus ponens. Before we formulated the calculational 8-step proof of (187), we had an earlier proof based on mutual implication that required 20 steps.

(184) **Left distributivity of** $\mathcal{W}$ **over** $\vee$**:** $p\,\mathcal{W}\,(q \vee r) \equiv p\,\mathcal{W}\,q \vee p\,\mathcal{W}\,r$

Proof:

$$\begin{array}{ll} & p\,\mathcal{W}\,q \vee p\,\mathcal{W}\,r \\ = & \langle (169) \text{ Definition of } \mathcal{W} \text{, twice} \rangle \\ & \Box\,p \vee p\,\mathcal{U}\,q \vee \Box\,p \vee p\,\mathcal{U}\,r \\ = & \langle (3.26) \text{ Idempotency of } \vee,\ p \vee p \equiv p \rangle \\ & \Box\,p \vee p\,\mathcal{U}\,q \vee p\,\mathcal{U}\,r \\ = & \langle (12) \text{ Left Distributivity of } \mathcal{U} \text{ over } \vee \rangle \\ & \Box\,p \vee p\,\mathcal{U}\,(q \vee r) \\ = & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\ & p\,\mathcal{W}\,(q \vee r) \quad \blacksquare \end{array}$$

(185) **Right distributivity of $\mathcal{W}$ over $\vee$:** $p\,\mathcal{W}\,r \vee q\,\mathcal{W}\,r \Rightarrow (p \vee q)\,\mathcal{W}\,r$

Proof:

$$
\begin{array}{ll}
 & p\,\mathcal{W}\,r \vee q\,\mathcal{W}\,r \\
= & \langle \text{(169) Definition of } \mathcal{W} \rangle \\
 & \Box p \vee \Box q \vee p\,\mathcal{U}\,r \vee q\,\mathcal{U}\,r \\
\Rightarrow & \langle \text{(100) Distributivity of } \Box \text{ over } \vee \text{ and (4.2) Monotonicity of } \vee, \\
 & \quad (p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r) \rangle \\
 & \Box (p \vee q) \vee p\,\mathcal{U}\,r \vee q\,\mathcal{U}\,r \\
\Rightarrow & \langle \text{(13) Right distributivity of } \mathcal{U} \text{ over } \vee \text{ and (4.2) Monotonicity of } \vee \rangle \\
 & \Box (p \vee q) \vee (p \vee q)\,\mathcal{U}\,r \\
= & \langle \text{(169) Definition of } \mathcal{W} \rangle \\
 & (p \vee q)\,\mathcal{W}\,r \quad \blacksquare
\end{array}
$$

(186) **Left distributivity of $\mathcal{W}$ over $\wedge$:** $p\,\mathcal{W}\,(q \wedge r) \Rightarrow p\,\mathcal{W}\,q \wedge p\,\mathcal{W}\,r$

Proof:

$$
\begin{array}{ll}
 & p\,\mathcal{W}\,(q \wedge r) \\
= & \langle \text{(169) Definition of } \mathcal{W} \rangle \\
 & \Box p \vee p\,\mathcal{U}\,(q \wedge r) \\
\Rightarrow & \langle \text{(14) Left Distributivity of } \mathcal{U} \text{ over } \wedge \text{ and (4.2) Monotonicity of } \vee \rangle \\
 & \Box p \vee (p\,\mathcal{U}\,q \wedge p\,\mathcal{U}\,r) \\
= & \langle \text{(3.45) Distributivity of } \vee \text{ over } \wedge,\ p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \rangle \\
 & (\Box p \vee p\,\mathcal{U}\,q) \wedge (\Box p \vee p\,\mathcal{U}\,r) \\
= & \langle \text{(169) Definition of } \mathcal{W}\text{, twice} \rangle \\
 & p\,\mathcal{W}\,q \wedge p\,\mathcal{W}\,r \quad \blacksquare
\end{array}
$$

(187) **Right distributivity of $\mathcal{W}$ over $\wedge$:** $(p \wedge q)\,\mathcal{W}\,r \equiv p\,\mathcal{W}\,r \wedge q\,\mathcal{W}\,r$

Proof:

$$
\begin{array}{ll}
 & p\,\mathcal{W}\,r \wedge q\,\mathcal{W}\,r \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p \rangle \\
 & \neg\neg(p\,\mathcal{W}\,r \wedge q\,\mathcal{W}\,r) \\
= & \langle \text{(3.47a) De Morgan, } \neg(p \wedge q) \equiv \neg p \vee \neg q \rangle \\
 & \neg(\neg(p\,\mathcal{W}\,r) \vee \neg(q\,\mathcal{W}\,r)) \\
= & \langle \text{(170) Distributivity of } \neg \text{ over } \mathcal{W}\text{, twice} \rangle \\
 & \neg(\neg r\,\mathcal{U}\,(\neg p \wedge \neg r) \vee \neg r\,\mathcal{U}\,(\neg q \wedge \neg r)) \\
= & \langle \text{(12) Left Distributivity of } \mathcal{U} \text{ over } \vee \rangle \\
 & \neg(\neg r\,\mathcal{U}\,((\neg p \wedge \neg r) \vee (\neg q \wedge \neg r)))
\end{array}
$$

$=$ $\langle$(3.46) Distributivity of $\wedge$ over $\vee$, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)\rangle$
$\neg(\neg r \,\mathcal{U}\, (\neg r \wedge (\neg p \vee \neg q)))$
$=$ $\langle$(3.47a) De Morgan, $\neg(p \wedge q) \equiv \neg p \vee \neg q\rangle$
$\neg(\neg r \,\mathcal{U}\, (\neg r \wedge \neg(p \wedge q)))$
$=$ $\langle$(170) Distributivity of $\neg$ over $\mathcal{W}$ with $p, q := p \wedge q, r\rangle$
$\neg\neg((p \wedge q) \,\mathcal{W}\, r)$
$=$ $\langle$(3.12) Double negation, $\neg\neg p \equiv p\rangle$
$(p \wedge q) \,\mathcal{W}\, r$ ∎

Theorem (188) for $\mathcal{W}$ is identical to (19) for $\mathcal{U}$. Both $\mathcal{W}$ and $\mathcal{U}$ obey the disjunction and conjunction rules—(189), (190), (195), and (196)—which give rise, in turn, to expanded distributivity theorems of $\neg$ over $\mathcal{W}$ and $\mathcal{U}$, (197) to (202). The conjunction and disjunction rules do not appear in other LTL systems.

(188) **Right distributivity of $\mathcal{W}$ over $\Rightarrow$:** $(p \Rightarrow q) \,\mathcal{W}\, r \Rightarrow (p \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, r)$

Proof:

$(p \Rightarrow q) \,\mathcal{W}\, r \Rightarrow (p \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, r)$
$=$ $\langle$(3.65) Shunting, $p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r)\rangle$
$(p \Rightarrow q) \,\mathcal{W}\, r \wedge p \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, r$

And now,

$(p \Rightarrow q) \,\mathcal{W}\, r \wedge p \,\mathcal{W}\, r$
$=$ $\langle$(187) Right distributivity of $\mathcal{W}$ over $\wedge\rangle$
$(p \wedge (p \Rightarrow q)) \,\mathcal{W}\, r$
$=$ $\langle$(3.66) $p \wedge (p \Rightarrow q) \ \equiv\ p \wedge q\rangle$
$(p \wedge q) \,\mathcal{W}\, r$
$=$ $\langle$(187) Right distributivity of $\mathcal{W}$ over $\wedge\rangle$
$p \,\mathcal{W}\, r \wedge q \,\mathcal{W}\, r$
$\Rightarrow$ $\langle$(3.76b) Strengthening, $p \wedge q \Rightarrow p\rangle$
$q \,\mathcal{W}\, r$ ∎

(189) **Disjunction rule of $\mathcal{W}$:** $p \,\mathcal{W}\, q \equiv (p \vee q) \,\mathcal{W}\, q$

Proof:

$p \,\mathcal{W}\, q$
$=$ $\langle$(3.12) Double negation, $\neg\neg p \equiv p\rangle$
$\neg\neg(p \,\mathcal{W}\, q)$
$=$ $\langle$(170) Distributivity of $\neg$ over $\mathcal{W}\rangle$
$\neg(\neg q \,\mathcal{U}\, (\neg p \wedge \neg q))$

$= \quad \langle$(173) Distributivity of $\neg$ over $\mathcal{U}\rangle$
$\neg(\neg p \wedge \neg q)\ \mathcal{W}\ (q \wedge \neg(\neg p \wedge \neg q))$
$= \quad \langle$(3.47a) De Morgan, $\neg(p \wedge q) \equiv \neg p \vee \neg q$, twice$\rangle$
$(p \vee q)\ \mathcal{W}\ (q \wedge (p \vee q))$
$= \quad \langle$(3.43a) Absorption, $p \wedge (p \vee q) \equiv p\rangle$
$(p \vee q)\ \mathcal{W}\ q$ ∎

(190) **Disjunction rule of $\mathcal{U}$:** $p\ \mathcal{U}\ q \equiv (p \vee q)\ \mathcal{U}\ q$

Proof:

$p\ \mathcal{U}\ q$
$= \quad \langle$(3.12) Double negation, $\neg\neg p \equiv p\rangle$
$\neg\neg(p\ \mathcal{U}\ q)$
$= \quad \langle$(173) Distributivity of $\neg$ over $\mathcal{U}\rangle$
$\neg(\neg q\ \mathcal{W}\ (\neg p \wedge \neg q))$
$= \quad \langle$(170) Distributivity of $\neg$ over $\mathcal{W}\rangle$
$\neg(\neg p \wedge \neg q)\ \mathcal{U}\ (q \wedge \neg(\neg p \wedge \neg q))$
$= \quad \langle$(3.47a) De Morgan, $\neg(p \wedge q) \equiv \neg p \vee \neg q$, twice$\rangle$
$(p \vee q)\ \mathcal{U}\ (q \wedge (p \vee q))$
$= \quad \langle$(3.43a) Absorption, $p \wedge (p \vee q) \equiv p\rangle$
$(p \vee q)\ \mathcal{U}\ q$ ∎

(191) **Rule of $\mathcal{W}$:** $\neg q\ \mathcal{W}\ q$

Proof:

$\neg q\ \mathcal{W}\ q$
$= \quad \langle$(189) Disjunction rule of $\mathcal{W}\rangle$
$(\neg q \vee q)\ \mathcal{W}\ q$
$= \quad \langle$(3.28) Excluded middle $p \vee \neg p\rangle$
true $\mathcal{W}\ q$
$= \quad \langle$183) Left zero of $\mathcal{W}\rangle$
true ∎

(192) **Rule of $\mathcal{U}$:** $\neg q\ \mathcal{U}\ q \equiv \Diamond q$

Proof:

$\neg q\ \mathcal{U}\ q$
$= \quad \langle$(190) Disjunction rule of $\mathcal{U}\rangle$
$(\neg q \vee q)\ \mathcal{U}\ q$
$= \quad \langle$(3.28) Excluded middle $p \vee \neg p\rangle$

$true \,\mathcal{U}\, q$
$= \quad \langle(38)\text{ Definition of } \Diamond\rangle$
$\Diamond q \quad \blacksquare$

(193) $(p \Rightarrow q)\, \mathcal{W}\, p$

Proof: The proof is by (4.7.1) Truth implication.

$(p \Rightarrow q)\, \mathcal{W}\, p$
$= \quad \langle(3.59)\text{ Implication } p \Rightarrow q \equiv \neg p \vee q\rangle$
$(\neg p \vee q)\, \mathcal{W}\, p$
$\Leftarrow \quad \langle(185)\text{ Right distributivity of } \mathcal{W} \text{ over } \vee\rangle$
$\neg p\, \mathcal{W}\, p \vee q\, \mathcal{W}\, p$
$= \quad \langle(191)\text{ Rule of } \mathcal{W}\rangle$
$true \vee q\, \mathcal{W}\, p$
$= \quad \langle(3.29)\text{ Zero of } \vee\rangle$
$true \quad \blacksquare$

(194) $\Diamond p \Rightarrow (p \Rightarrow q)\, \mathcal{U}\, p$

Proof:

$(p \Rightarrow q)\, \mathcal{U}\, p$
$= \quad \langle(3.59)\text{ Implication } p \Rightarrow q \equiv \neg p \vee q\rangle$
$(\neg p \vee q)\, \mathcal{U}\, p$
$\Leftarrow \quad \langle(13)\text{ Right distributivity of } \mathcal{U} \text{ over } \vee\rangle$
$\neg p\, \mathcal{U}\, p \vee q\, \mathcal{U}\, p$
$\Leftarrow \quad \langle(3.76a)\text{ Weakening, } p \Rightarrow p \vee q\rangle$
$\neg p\, \mathcal{U}\, p$
$= \quad \langle(192)\text{ Rule of } \mathcal{U}\rangle$
$\Diamond p \quad \blacksquare$

(195) **Conjunction rule of $\mathcal{W}$:** $p\, \mathcal{W}\, q \equiv (p \wedge \neg q)\, \mathcal{W}\, q$

Proof:

$(p \wedge \neg q)\, \mathcal{W}\, q$
$= \quad \langle(187)\text{ Right distributivity of } \mathcal{W} \text{ over } \wedge\rangle$
$p\, \mathcal{W}\, q \wedge \neg q\, \mathcal{W}\, q$
$= \quad \langle(191)\text{ Rule of } \mathcal{W}\rangle$
$p\, \mathcal{W}\, q \wedge true$
$= \quad \langle(3.39)\text{ Identity of } \wedge,\ p \wedge true \equiv p\rangle$
$p\, \mathcal{W}\, q \quad \blacksquare$

(196) **Conjunction rule of $\mathcal{U}$:** $p\,\mathcal{U}\,q \equiv (p \wedge \neg q)\,\mathcal{U}\,q$

Proof:

$$
\begin{array}{ll}
 & (p \wedge \neg q)\,\mathcal{U}\,q \\
= & \langle (171)\ \mathcal{U}\ \text{in terms of}\ \mathcal{W} \rangle \\
 & (p \wedge \neg q)\,\mathcal{W}\,q \wedge \Diamond q \\
= & \langle (195)\ \text{Conjunction rule of}\ \mathcal{W} \rangle \\
 & p\,\mathcal{W}\,q \wedge \Diamond q \\
= & \langle (171)\ \mathcal{U}\ \text{in terms of}\ \mathcal{W} \rangle \\
 & p\,\mathcal{U}\,q \quad \blacksquare
\end{array}
$$

(197) **Distributivity of $\neg$ over $\mathcal{W}$:** $\neg(p\,\mathcal{W}\,q) \equiv (p \wedge \neg q)\,\mathcal{U}\,(\neg p \wedge \neg q)$

Proof:

$$
\begin{array}{ll}
 & \neg(p\,\mathcal{W}\,q) \\
= & \langle (170)\ \text{Distributivity of}\ \neg\ \text{over}\ \mathcal{W} \rangle \\
 & \neg q\,\mathcal{U}\,(\neg p \wedge \neg q) \\
= & \langle (196)\ \text{Conjunction rule of}\ \mathcal{U}\ \text{with}\ p,q := \neg q, \neg p \wedge \neg q \rangle \\
 & (\neg q \wedge \neg(\neg p \wedge \neg q))\,\mathcal{U}\,(\neg p \wedge \neg q) \\
= & \langle (3.47a)\ \text{De Morgan},\ \neg(p \wedge q)\ \equiv\ \neg p \vee \neg q \rangle \\
 & (\neg q \wedge (p \vee q))\,\mathcal{U}\,(\neg p \wedge \neg q) \\
= & \langle (3.44a)\ \text{Absorption},\ p \wedge (\neg p \vee q) \equiv p \wedge q\ \text{with}\ p,q := \neg q, p \rangle \\
 & (p \wedge \neg q)\,\mathcal{U}\,(\neg p \wedge \neg q) \quad \blacksquare
\end{array}
$$

(198) **Distributivity of $\neg$ over $\mathcal{U}$:** $\neg(p\,\mathcal{U}\,q) \equiv (p \wedge \neg q)\,\mathcal{W}\,(\neg p \wedge \neg q)$

Proof:

$$
\begin{array}{ll}
 & \neg(p\,\mathcal{U}\,q) \\
= & \langle (173)\ \text{Distributivity of}\ \neg\ \text{over}\ \mathcal{U} \rangle \\
 & \neg q\,\mathcal{W}\,(\neg p \wedge \neg q) \\
= & \langle (195)\ \text{Conjunction rule of}\ \mathcal{W}\ \text{with}\ p,q := \neg q, \neg p \wedge \neg q \rangle \\
 & (\neg q \wedge \neg(\neg p \wedge \neg q))\,\mathcal{W}\,(\neg p \wedge \neg q) \\
= & \langle (3.47a)\ \text{De Morgan},\ \neg(p \wedge q)\ \equiv\ \neg p \vee \neg q \rangle \\
 & (\neg q \wedge (p \vee q))\,\mathcal{W}\,(\neg p \wedge \neg q) \\
= & \langle (3.44a)\ \text{Absorption},\ p \wedge (\neg p \vee q) \equiv p \wedge q\ \text{with}\ p,q := \neg q, p \rangle \\
 & (p \wedge \neg q)\,\mathcal{W}\,(\neg p \wedge \neg q) \quad \blacksquare
\end{array}
$$

(199) **Dual of $\mathcal{U}$:** $\neg(\neg p\,\mathcal{U}\,\neg q) \equiv q\,\mathcal{W}\,(p \wedge q)$

Proof:

$\neg(\neg p \,\mathcal{U}\, \neg q)$
$=\quad \langle$(173) Distributivity of $\neg$ over $\mathcal{U}$, with $p,q := \neg p, \neg q\rangle$
$q \,\mathcal{W}\, (p \wedge q)$ ∎

(200) **Dual of $\mathcal{U}$:** $\neg(\neg p \,\mathcal{U}\, \neg q) \equiv (\neg p \wedge q) \,\mathcal{W}\, (p \wedge q)$

Proof:

$\neg(\neg p \,\mathcal{U}\, \neg q)$
$=\quad \langle$(198) Distributivity of $\neg$ over $\mathcal{U}$, with $p,q := \neg p, \neg q\rangle$
$(\neg p \wedge q) \,\mathcal{W}\, (p \wedge q)$ ∎

(201) **Dual of $\mathcal{W}$:** $\neg(\neg p \,\mathcal{W}\, \neg q) \equiv q \,\mathcal{U}\, (p \wedge q)$

Proof:

$\neg(\neg p \,\mathcal{W}\, \neg q)$
$=\quad \langle$(170) Distributivity of $\neg$ over $\mathcal{W}$, with $p,q := \neg p, \neg q\rangle$
$q \,\mathcal{U}\, (p \wedge q)$ ∎

(202) **Dual of $\mathcal{W}$:** $\neg(\neg p \,\mathcal{W}\, \neg q) \equiv (\neg p \wedge q) \,\mathcal{U}\, (p \wedge q)$

Proof:

$\neg(\neg p \,\mathcal{W}\, \neg q)$
$=\quad \langle$(197) Distributivity of $\neg$ over $\mathcal{W}$, with $p,q := \neg p, \neg q\rangle$
$(\neg p \wedge q) \,\mathcal{U}\, (p \wedge q)$ ∎

Theorem (203) shows that the *wait* operator, like the *until* operator, is idempotent. Theorem (204) shows that *true* is the right zero of *wait*, as it is for *until*. Theorem (205) shows that *false* is the left identity of *wait*, as it is for *until*. Theorem (206) for the *wait* operator corresponds to theorem (28) for the *until* operator and shows that $p \,\mathcal{W}\, q$ is stronger than $p \vee q$. Theorem (207) shows that $\Box(p \vee q)$ is stronger than $p \,\mathcal{W}\, q$. Theorem (209), insertion for the *wait* operator, corresponds to (29), insertion for the *until* operator.

(203) **Idempotency of $\mathcal{W}$:** $p \,\mathcal{W}\, p \equiv p$

Proof:

$p \,\mathcal{W}\, p$
$=\quad \langle$(169) Definition of $\mathcal{W}\rangle$
$\Box p \vee p \,\mathcal{U}\, p$
$=\quad \langle$(22) Idempotency of $\mathcal{U}\rangle$
$\Box p \vee p$
$=\quad \langle$(69) Absorption of $\Box$ into $\vee\rangle$
p ∎

(204) **Right zero of $\mathcal{W}$:** $p\ \mathcal{W}\ true \equiv true$

Proof:

$$
\begin{aligned}
& p\ \mathcal{W}\ true \\
= \quad & \langle (169)\text{ Definition of } \mathcal{W} \rangle \\
& \Box\, p \vee p\ \mathcal{U}\ true \\
= \quad & \langle (20)\text{ Right zero of } \mathcal{U} \rangle \\
& \Box\, p \vee true \\
= \quad & \langle (3.29)\text{ Zero of } \vee,\ p \vee true \equiv true \rangle \\
& true \quad \blacksquare
\end{aligned}
$$

(205) **Left identity of $\mathcal{W}$:** $false\ \mathcal{W}\ q \equiv q$

Proof:

$$
\begin{aligned}
& false\ \mathcal{W}\ q \\
= \quad & \langle (169)\text{ Definition of } \mathcal{W} \rangle \\
& \Box\, false \vee false\ \mathcal{U}\ q \\
= \quad & \langle (21)\text{ Left identity of } \mathcal{U} \rangle \\
& \Box\, false \vee q \\
= \quad & \langle (65)\text{ Falsehood of } \Box \rangle \\
& false \vee q \\
= \quad & \langle (3.30)\text{ Identity of } \vee,\ p \vee false \equiv p \rangle \\
& q \quad \blacksquare
\end{aligned}
$$

(206) $p\ \mathcal{W}\ q \Rightarrow p \vee q$

Proof:

$$
\begin{aligned}
& p\ \mathcal{W}\ q \\
= \quad & \langle (181)\text{ Expansion of } \mathcal{W} \rangle \\
& q \vee (p \wedge \circ\,(p\ \mathcal{W}\ q)) \\
\Rightarrow \quad & \langle (3.76\text{d})\ p \vee (q \wedge r) \Rightarrow p \vee q \rangle \\
& p \vee q \quad \blacksquare
\end{aligned}
$$

(207) $\Box\,(p \vee q) \Rightarrow p\ \mathcal{W}\ q$

Proof:

$$
\begin{aligned}
& \Box\,(p \vee q) \\
\Rightarrow \quad & \langle (179)\text{ Perpetuity} \rangle \\
& (p \vee q)\ \mathcal{W}\ q \\
= \quad & \langle (189)\text{ Disjunction rule of } \mathcal{W} \rangle \\
& p\ \mathcal{W}\ q \quad \blacksquare
\end{aligned}
$$

(208) $\square(\neg q \Rightarrow p) \Rightarrow p\,\mathcal{W}\,q$

Proof:

$$\begin{aligned}
& \square(\neg q \Rightarrow p) \\
= \quad & \langle (3.59)\text{ Implication, } p \Rightarrow q \equiv \neg p \vee q \text{ and } (3.24)\text{ Symmetry of } \vee,\ p \vee q \equiv q \vee p \rangle \\
& \square(p \vee q) \\
\Rightarrow \quad & \langle (207)\ \square(p \vee q) \Rightarrow p\,\mathcal{W}\,q \rangle \\
& p\,\mathcal{W}\,q \quad \blacksquare
\end{aligned}$$

(209) $\mathcal{W}$ **insertion:** $q \Rightarrow p\,\mathcal{W}\,q$

Proof:

$$\begin{aligned}
& p\,\mathcal{W}\,q \\
= \quad & \langle (181)\text{ Expansion of } \mathcal{W} \rangle \\
& q \vee (p \wedge \circ(p\,\mathcal{W}\,q)) \\
\Leftarrow \quad & \langle (3.76a)\text{ Weakening, } p \Rightarrow p \vee q \rangle \\
& q \quad \blacksquare
\end{aligned}$$

The next three theorems (210), (211), and (212) complete the set of frame laws which included the theorems from (106) to (117) for binary propositional operators, and the $\mathcal{U}$ frame laws from theorems (148), (149), and (150).

(210) $\mathcal{W}$ **frame law of** $\circ$**:** $\square p \Rightarrow (\circ q \Rightarrow \circ(p\,\mathcal{W}\,q))$

Proof:

$$\begin{aligned}
& \square p \Rightarrow (\circ q \Rightarrow \circ(p\,\mathcal{W}\,q)) \\
= \quad & \langle (2)\text{ Distributivity of } \circ \text{ over } \Rightarrow \rangle \\
& \square p \Rightarrow \circ(q \Rightarrow p\,\mathcal{W}\,q) \\
= \quad & \langle (209)\ \mathcal{W}\text{ insertion} \rangle \\
& \square p \Rightarrow \circ\, true \\
= \quad & \langle (7)\text{ Truth of } \circ \rangle \\
& \square p \Rightarrow true \quad \text{—(3.72) Right zero of } \Rightarrow,\ p \Rightarrow true \equiv true \quad \blacksquare
\end{aligned}$$

(211) $\mathcal{W}$ **frame law of** $\diamond$**:** $\square p \Rightarrow (\diamond q \Rightarrow \diamond(p\,\mathcal{W}\,q))$

Proof:

$$\begin{aligned}
& \square p \Rightarrow (\diamond q \Rightarrow \diamond(p\,\mathcal{W}\,q)) \\
= \quad & \langle (3.65)\text{ Shunting, } p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r) \rangle \\
& \square p \wedge \diamond q \Rightarrow \diamond(p\,\mathcal{W}\,q)
\end{aligned}$$

And now,

$\Box p \wedge \Diamond q$
$\Rightarrow$ $\langle$(84) $\mathcal{U}$ implication $\rangle$
$p \,\mathcal{U}\, q$
$\Rightarrow$ $\langle$(174) $p \,\mathcal{U}\, q \Rightarrow p \,\mathcal{W}\, q$ $\rangle$
$p \,\mathcal{W}\, q$
$\Rightarrow$ $\langle$(46) Weakening of $\Diamond$, $p \Rightarrow \Diamond p$ with $p := p \,\mathcal{W}\, q$ $\rangle$
$\Diamond (p \,\mathcal{W}\, q)$ ∎

(212) $\mathcal{W}$ **frame law of** $\Box$**:** $\Box p \Rightarrow (\Box q \Rightarrow \Box (p \,\mathcal{W}\, q))$

Proof:

$\Box p \Rightarrow (\Box q \Rightarrow \Box (p \,\mathcal{W}\, q))$
$=$ $\langle$(3.65) Shunting, $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$$\rangle$
$\Box p \wedge \Box q \Rightarrow \Box (p \,\mathcal{W}\, q))$

And now,

$\Box p \wedge \Box q$
$\Rightarrow$ $\langle$(3.76b) Strengthening, $p \wedge q \Rightarrow p$ with $p, q := \Box (p \,\mathcal{W}\, q), \Box p$$\rangle$
$\Box q$
$\Rightarrow$ $\langle$(209) $\mathcal{W}$ Insertion with $q := \Box q$ $\rangle$
$p \,\mathcal{W}\, \Box q$
$\Rightarrow$ $\langle$(177) $\mathcal{W} \Box$ implication$\rangle$
$\Box (p \,\mathcal{W}\, q)$ ∎

Here are four induction theorems for $\mathcal{W}$, the last two of which are unique to this system.

(213) $\mathcal{W}$ **induction:** $\Box (p \Rightarrow (\bigcirc p \wedge q) \vee r) \Rightarrow (p \Rightarrow q \,\mathcal{W}\, r)$

Proof:

$\Box (p \Rightarrow (\bigcirc p \wedge q) \vee r) \Rightarrow (p \Rightarrow q \,\mathcal{W}\, r)$
$=$ $\langle$(169) Definition of $\mathcal{W}$ $\rangle$
$\Box (p \Rightarrow (\bigcirc p \wedge q) \vee r) \Rightarrow (p \Rightarrow \Box q \vee q \,\mathcal{U}\, r)$ —(55) $\mathcal{U}$ Induction ∎

(214) $\mathcal{W}$ **induction:** $\Box (p \Rightarrow \bigcirc (p \vee q)) \Rightarrow (p \Rightarrow p \,\mathcal{W}\, q)$

Proof:

$\Box (p \Rightarrow \bigcirc (p \vee q))$
$\Rightarrow$ $\langle$(56) $\mathcal{U}$ Induction$\rangle$
$p \Rightarrow \Box p \vee p \,\mathcal{U}\, q$
$=$ $\langle$(169) Definition of $\mathcal{W}$ $\rangle$
$p \Rightarrow p \,\mathcal{W}\, q$ ∎

(215) $\mathcal{W}$ **induction:** $\quad \Box(p \Rightarrow \bigcirc p) \Rightarrow (p \Rightarrow p\,\mathcal{W}\,q)$

Proof:

$$\begin{array}{ll} & \Box(p \Rightarrow \bigcirc p) \\ \Rightarrow & \langle (57)\ \Box \text{ Induction} \rangle \\ & p \Rightarrow \Box p \\ \Rightarrow & \langle (179) \text{ Perpetuity and (3.82a) Transitivity} \rangle \\ & p \Rightarrow p\,\mathcal{W}\,q \quad \blacksquare \end{array}$$

(216) $\mathcal{W}$ **induction:** $\quad \Box(p \Rightarrow q \wedge \bigcirc p) \Rightarrow (p \Rightarrow p\,\mathcal{W}\,q)$

Proof:

$$\begin{array}{ll} & \Box(p \Rightarrow q \wedge \bigcirc p) \\ = & \langle (3.63.1) \text{ Distributivity of} \Rightarrow \text{over} \wedge,\ \ p \Rightarrow q \wedge r \ \equiv\ (p \Rightarrow q) \wedge (p \Rightarrow r) \rangle \\ & \Box((p \Rightarrow q) \wedge (p \Rightarrow \bigcirc p)) \\ = & \langle (99) \text{ Distributivity of } \Box \text{ over } \wedge \rangle \\ & \Box(p \Rightarrow q) \wedge \Box(p \Rightarrow \bigcirc p) \\ \Rightarrow & \langle (3.76b) \text{ Strengthening, } p \wedge q \Rightarrow p \rangle \\ & \Box(p \Rightarrow \bigcirc p) \\ \Rightarrow & \langle (57)\ \Box \text{ Induction} \rangle \\ & p \Rightarrow \Box p \\ \Rightarrow & \langle (179) \text{ Perpetuity and (3.82a) Transitivity} \rangle \\ & p \Rightarrow p\,\mathcal{W}\,q \quad \blacksquare \end{array}$$

The next five absorption theorems correspond to the five absorption theorems for *until*, (31), (32), (33), (35), and (34) respectively. None appear in other LTL systems.

(217) **Absorption:** $\quad p \vee p\,\mathcal{W}\,q \equiv p \vee q$

Proof:

$$\begin{array}{ll} & p \vee p\,\mathcal{W}\,q \\ = & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\ & p \vee p\,\mathcal{U}\,q \vee \Box p \\ = & \langle (69) \text{ Absorption of } \vee \text{ into } \Box \rangle \\ & p \vee p\,\mathcal{U}\,q \\ = & \langle (31) \text{ Absorption} \rangle \\ & p \vee q \quad \blacksquare \end{array}$$

(218) **Absorption:** $\quad p\,\mathcal{W}\,q \vee q \equiv p\,\mathcal{W}\,q$

Proof: (Ravi Mohan)

(11) Right zero of $\mathcal{U}$: $p\,\mathcal{U}\,\mathit{false} \equiv \mathit{false}$	(183) Left zero of $\mathcal{W}$: $\mathit{true}\,\mathcal{W}\,q \equiv \mathit{true}$
(20) Right zero of $\mathcal{U}$: $p\,\mathcal{U}\,\mathit{true} \equiv \mathit{true}$	(204) Right zero of $\mathcal{W}$: $p\,\mathcal{W}\,\mathit{true} \equiv \mathit{true}$
(21) Left identity of $\mathcal{U}$: $\mathit{false}\,\mathcal{U}\,q \equiv q$	(205) Left identity of $\mathcal{W}$: $\mathit{false}\,\mathcal{W}\,q \equiv q$
(38) $\mathit{true}\,\mathcal{U}\,q \equiv \Diamond q$	(224) $\Box$ to $\mathcal{W}$ law: $p\,\mathcal{W}\,\mathit{false} \equiv \Box p$

Figure 3: The eight possibilities of *true* and *false* on the left hand side and right hand side of $\mathcal{U}$ and $\mathcal{W}$.

$$\begin{array}{ll} & p\,\mathcal{W}\,q \vee q \equiv p\,\mathcal{W}\,q \\ = & \langle (3.57) \text{ Definition of implication}, p \Rightarrow q \;\equiv\; p \vee q \;\equiv\; q \rangle \\ & q \Rightarrow p\,\mathcal{W}\,q \quad \text{—(209) } \mathcal{W} \text{ Insertion} \quad \blacksquare \end{array}$$

(219) **Absorption:** $p\,\mathcal{W}\,q \wedge q \equiv q$

Proof: (Ravi Mohan)

$$\begin{array}{ll} & p\,\mathcal{W}\,q \wedge q \equiv q \\ = & \langle (3.60) \text{ Implication}, p \Rightarrow q \;\equiv\; p \wedge q \;\equiv\; p \rangle \\ & q \Rightarrow p\,\mathcal{W}\,q \quad \text{—(209) } \mathcal{W} \text{ Insertion} \quad \blacksquare \end{array}$$

(220) **Absorption:** $p\,\mathcal{W}\,q \wedge (p \vee q) \equiv p\,\mathcal{W}\,q$

Proof: (Ravi Mohan)

$$\begin{array}{ll} & p\,\mathcal{W}\,q \wedge (p \vee q) \equiv p\,\mathcal{W}\,q \\ = & \langle (3.60) \text{ Implication}, p \Rightarrow q \;\equiv\; p \wedge q \;\equiv\; p \rangle \\ & p\,\mathcal{W}\,q \Rightarrow p \vee q \quad \text{—(206)} \quad \blacksquare \end{array}$$

(221) **Absorption:** $p\,\mathcal{W}\,q \vee (p \wedge q) \equiv p\,\mathcal{W}\,q$

Proof:

$$\begin{array}{ll} & p\,\mathcal{W}\,q \vee (p \wedge q) \\ = & \langle (181) \text{ Expansion of } \mathcal{W} \rangle \\ & q \vee (p \wedge \circ\,(p\,\mathcal{W}\,q)) \vee (p \wedge q) \\ = & \langle (3.43b) \text{ Absorption } p \vee (p \wedge q) \equiv p \text{ with } p, q := q, p \rangle \\ & q \vee (p \wedge \circ\,(p\,\mathcal{W}\,q)) \\ = & \langle (181) \text{ Expansion of } \mathcal{W} \rangle \\ & p\,\mathcal{W}\,q \quad \blacksquare \end{array}$$

The left and right absorption theorems for $\mathcal{W}$, (222) and (223), correspond to, and are proved from, the left and right absorption theorems for $\mathcal{U}$, (36) and (37). Theorem

(224) corresponds to the definition of the $\Diamond$ operator (38). Figure 3 summarizes the eight possibilities of *true* and *false* on the left hand side and right hand side of $\mathcal{U}$ and $\mathcal{W}$.

(222) **Left absorption of $\mathcal{W}$:** $p\,\mathcal{W}\,(p\,\mathcal{W}\,q) \equiv p\,\mathcal{W}\,q$

Proof:

$$\begin{array}{ll}
 & p\,\mathcal{W}\,(p\,\mathcal{W}\,q) \\
= & \langle\text{(169) Definition of } \mathcal{W}\text{, twice}\rangle \\
 & p\,\mathcal{U}\,(p\,\mathcal{U}\,q \vee \Box p) \vee \Box p \\
= & \langle\text{(12) Left distributivity of } \mathcal{U} \text{ over } \vee\rangle \\
 & p\,\mathcal{U}\,(p\,\mathcal{U}\,q) \vee p\,\mathcal{U}\,\Box p \vee \Box p \\
= & \langle\text{(141) Absorption of } \mathcal{U} \text{ into } \Box \text{ and (3.26) Idempotency of } \vee,\ p \vee p \equiv p\rangle \\
 & p\,\mathcal{U}\,(p\,\mathcal{U}\,q) \vee \Box p \\
= & \langle\text{(36) Left absorption of } \mathcal{U}\rangle \\
 & p\,\mathcal{U}\,q \vee \Box p \\
= & \langle\text{(169) Definition of } \mathcal{W}\rangle \\
 & p\,\mathcal{W}\,q \quad \blacksquare
\end{array}$$

(223) **Right absorption of $\mathcal{W}$:** $(p\,\mathcal{W}\,q)\,\mathcal{W}\,q \equiv p\,\mathcal{W}\,q$

Proof: The proof is by (4.7) Mutual implication.
The proof in the first direction follows.

$$\begin{array}{ll}
 & (p\,\mathcal{W}\,q)\,\mathcal{W}\,q \\
= & \langle\text{(169) Definition of } \mathcal{W}\rangle \\
 & (p\,\mathcal{U}\,q \vee \Box p)\,\mathcal{W}\,q \\
\Leftarrow & \langle\text{(185) Right distributivity of } \mathcal{W} \text{ over } \vee\rangle \\
 & (p\,\mathcal{U}\,q)\,\mathcal{W}\,q \vee \Box p\,\mathcal{W}\,q \\
\Leftarrow & \langle\text{(174) } \mathcal{U} \text{ implication and (4.2) Monotonicity of } \vee\rangle \\
 & (p\,\mathcal{U}\,q)\,\mathcal{U}\,q \vee \Box p\,\mathcal{W}\,q \\
= & \langle\text{(37) Right absorption of } \mathcal{U}\rangle \\
 & p\,\mathcal{U}\,q \vee \Box p\,\mathcal{W}\,q \\
\Leftarrow & \langle\text{(179) Perpetuity and (4.2) Monotonicity of } \vee\rangle \\
 & p\,\mathcal{U}\,q \vee \Box\Box p \\
= & \langle\text{(72) Absorption of } \Box\rangle \\
 & p\,\mathcal{U}\,q \vee \Box p \\
= & \langle\text{(169) Definition of } \mathcal{W}\rangle \\
 & p\,\mathcal{W}\,q
\end{array}$$

The proof in the second direction follows.

$$
\begin{array}{ll}
 & (p \,\mathcal{W}\, q) \,\mathcal{W}\, q \\
\Rightarrow & \langle (206)\ p \,\mathcal{W}\, q \Rightarrow p \vee q \rangle \\
 & p \,\mathcal{W}\, q \vee q \\
= & \langle (218) \text{ Absorption} \rangle \\
 & p \,\mathcal{W}\, q \quad \blacksquare
\end{array}
$$

(224) $\square$ **to** $\mathcal{W}$ **law:** $\square p \equiv p \,\mathcal{W}\, \mathit{false}$

Proof:

$$
\begin{array}{ll}
 & p \,\mathcal{W}\, \mathit{false} \\
= & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
 & \square p \vee p \,\mathcal{U}\, \mathit{false} \\
= & \langle (11) \text{ Right zero of } \mathcal{U} \rangle \\
 & \square p \vee \mathit{false} \\
= & \langle (3.30) \text{ Identity of } \vee,\ p \vee \mathit{false} \equiv p \rangle \\
 & \square p \quad \blacksquare
\end{array}
$$

(225) $\lozenge$ **to** $\mathcal{W}$ **law:** $\lozenge p \equiv \neg(\neg p \,\mathcal{W}\, \mathit{false})$

Proof:

$$
\begin{array}{ll}
 & \neg(\neg p \,\mathcal{W}\, \mathit{false}) \\
= & \langle (224)\ \square \text{ to } \mathcal{W} \text{ law with } p := \neg p \rangle \\
 & \neg \square \neg p \\
= & \langle (59)\ \lozenge p \equiv \neg \square \neg p \rangle \\
 & \lozenge p \quad \blacksquare
\end{array}
$$

Theorem (226) for $\mathcal{W}$ corresponds to (84) $\mathcal{U}$ implication. It is also known as an entailment law for the *wait* operator. [21] The following four absorption theorems combine an *until* operation with a *wait* operation. Theorem (231) corresponds to (141) Absorption of $\mathcal{U}$ into $\square$. Theorem (232) corresponds to (40) Absorption of $\mathcal{U}$ into $\lozenge$. Theorem (233) corresponds to (39) Absorption of $\lozenge$ into $\mathcal{U}$.

(226) $\mathcal{W}$ **implication:** $p \,\mathcal{W}\, q \Rightarrow \square p \vee \lozenge q$

Proof:

$$
\begin{array}{ll}
 & p \,\mathcal{W}\, q \\
= & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
 & \square p \vee p \,\mathcal{U}\, q \\
\Rightarrow & \langle (42) \text{ Eventuality and } (4.2) \text{ Monotonicity of } \vee \rangle \\
 & \square p \vee \lozenge q \quad \blacksquare
\end{array}
$$

(227) **Absorption:** $p \mathcal{W} (p \mathcal{U} q) \equiv p \mathcal{W} q$

Proof:

$$
\begin{aligned}
& p \mathcal{W} (p \mathcal{U} q) \\
= \quad & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
& \Box p \vee p \mathcal{U} (p \mathcal{U} q) \\
= \quad & \langle (36) \text{ Left absorption of } \mathcal{U} \rangle \\
& \Box p \vee p \mathcal{U} q \\
= \quad & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
& p \mathcal{W} q \quad \blacksquare
\end{aligned}
$$

(228) **Absorption:** $(p \mathcal{U} q) \mathcal{W} q \equiv p \mathcal{U} q$

Proof:

$$
\begin{aligned}
& (p \mathcal{U} q) \mathcal{W} q \\
= \quad & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
& \Box (p \mathcal{U} q) \vee (p \mathcal{U} q) \mathcal{U} q \\
= \quad & \langle (37) \text{ Right absorption of } \mathcal{U} \rangle \\
& \Box (p \mathcal{U} q) \vee p \mathcal{U} q \\
= \quad & \langle (69) \text{ Absorption of } \Box \text{ into } \vee \text{ with } p := p \mathcal{U} q \rangle \\
& p \mathcal{U} q \quad \blacksquare
\end{aligned}
$$

(229) **Absorption:** $p \mathcal{U} (p \mathcal{W} q) \equiv p \mathcal{W} q$

Proof:

$$
\begin{aligned}
& p \mathcal{U} (p \mathcal{W} q) \\
= \quad & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
& p \mathcal{U} (\Box p \vee p \mathcal{U} q) \\
= \quad & \langle (12) \text{ Left Distributivity of } \mathcal{U} \text{ over } \vee \rangle \\
& p \mathcal{U} \Box p \vee p \mathcal{U} (p \mathcal{U} q) \\
= \quad & \langle (36) \text{ Left absorption of } \mathcal{U} \rangle \\
& p \mathcal{U} \Box p \vee p \mathcal{U} q \\
= \quad & \langle (141) \text{ Absorption} \rangle \\
& \Box p \vee p \mathcal{U} q \\
= \quad & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
& p \mathcal{W} q \quad \blacksquare
\end{aligned}
$$

(230) **Absorption:** $(p \mathcal{W} q) \mathcal{U} q \equiv p \mathcal{U} q$

Proof:

$$
\begin{array}{ll}
& (p \,\mathcal{W}\, q) \,\mathcal{U}\, q \\
= & \langle (171)\ \mathcal{U} \text{ in terms of } \mathcal{W} \rangle \\
& (p \,\mathcal{W}\, q) \,\mathcal{W}\, q \wedge \Diamond q \\
= & \langle (223) \text{ Right absorption of } \mathcal{W} \rangle \\
& p \,\mathcal{W}\, q \wedge \Diamond q \\
= & \langle (171)\ \mathcal{U} \text{ in terms of } \mathcal{W} \rangle \\
& p \,\mathcal{U}\, q \quad \blacksquare
\end{array}
$$

(231) **Absorption of $\mathcal{W}$ into $\Diamond$:** $\Diamond q \,\mathcal{W}\, q \equiv \Diamond q$

Proof:

$$
\begin{array}{ll}
& \Diamond q \,\mathcal{W}\, q \\
= & \langle (38) \text{ Definition of } \Diamond \rangle \\
& (\mathit{true} \,\mathcal{U}\, q) \,\mathcal{W}\, q \\
= & \langle (228) \text{ Absorption} \rangle \\
& \mathit{true} \,\mathcal{U}\, q \\
= & \langle (38) \text{ Definition of } \Diamond \rangle \\
& \Diamond q \quad \blacksquare
\end{array}
$$

(232) **Absorption of $\mathcal{W}$ into $\Box$:** $\Box p \wedge p \,\mathcal{W}\, q \equiv \Box p$

Proof:

$$
\begin{array}{ll}
& \Box p \wedge p \,\mathcal{W}\, q \\
= & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
& \Box p \wedge (\Box p \vee p \,\mathcal{U}\, q) \\
= & \langle (3.43a) \text{ Absorption } p \wedge (p \vee q) \equiv p \rangle \\
& \Box p \quad \blacksquare
\end{array}
$$

(233) **Absorption of $\Box$ into $\mathcal{W}$:** $\Box p \vee p \,\mathcal{W}\, q \equiv p \,\mathcal{W}\, q$

Proof:

$$
\begin{array}{ll}
& \Box p \vee p \,\mathcal{W}\, q \\
= & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
& \Box p \vee \Box p \vee p \,\mathcal{U}\, q \\
= & \langle (3.26) \text{ Idempotency of } \vee,\ p \vee p \equiv p \rangle \\
& \Box p \vee p \,\mathcal{U}\, q \\
= & \langle (169) \text{ Definition of } \mathcal{W} \rangle \\
& p \,\mathcal{W}\, q \quad \blacksquare
\end{array}
$$

The next two pair of theorems correspond. The *wait* versions are in the temporal logic literature. The *until* versions are unique to this system. The monotonicity theorems and the strengthening and catenation rules are common. The implication rules (242) to (245) are unique to this system.

(234) $p \mathcal{W} q \land \square \neg q \Rightarrow \square p$

Proof:

$$
\begin{array}{ll}
 & p \mathcal{W} q \land \square \neg q \\
\Rightarrow & \langle (226)\ \mathcal{W}\ \text{implication and (4.3) Monotonicity of } \land \rangle \\
 & (\square p \lor \Diamond q) \land \square \neg q \\
= & \langle (61)\ \text{Dual of } \Diamond \rangle \\
 & (\square p \lor \Diamond q) \land \neg \Diamond q \\
= & \langle (3.44a)\ \text{Absorption, } p \land (\neg p \lor q) \equiv p \land q \rangle \\
 & \square p \land \neg \Diamond q \\
\Rightarrow & \langle (3.76b)\ \text{Strengthening, } p \land q \Rightarrow p \rangle \\
 & \square p \quad \blacksquare
\end{array}
$$

(235) $\square p \Rightarrow p \mathcal{U} q \lor \square \neg q$

Proof:

$$
\begin{array}{ll}
 & p \mathcal{U} q \lor \square \neg q \\
\Leftarrow & \langle (84)\ \mathcal{U}\ \text{implication and (4.2) Monotonicity of } \lor \rangle \\
 & (\square p \land \Diamond q) \lor \square \neg q \\
= & \langle (61)\ \text{Dual of } \Diamond \rangle \\
 & (\square p \land \Diamond q) \lor \neg \Diamond q \\
= & \langle (3.44b)\ \text{Absorption, } p \lor (\neg p \land q) \equiv p \lor q \rangle \\
 & \square p \lor \neg \Diamond q \\
\Leftarrow & \langle (3.76a)\ \text{Weakening, } p \Rightarrow p \lor q \rangle \\
 & \square p \quad \blacksquare
\end{array}
$$

(236) $\neg \square p \land p \mathcal{W} q \Rightarrow \Diamond q$

Proof:

$$
\begin{array}{ll}
 & \neg \square p \land p \mathcal{W} q \Rightarrow \Diamond q \\
= & \langle (3.65)\ \text{Shunting, } p \land q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r) \rangle \\
 & p \mathcal{W} q \Rightarrow (\neg \square p \Rightarrow \Diamond q) \\
= & \langle (3.59)\ \text{Implication } p \Rightarrow q \ \equiv\ \neg p \lor q \rangle \\
 & p \mathcal{W} q \Rightarrow \square p \lor \Diamond q \quad \text{—(226) } \mathcal{W} \text{ implication} \quad \blacksquare
\end{array}
$$

(237) $\Diamond q \Rightarrow \neg\Box p \vee p \,\mathcal{U}\, q$

Proof:

$$
\begin{array}{ll}
& \Diamond q \Rightarrow \neg\Box p \vee p \,\mathcal{U}\, q \\
= & \langle (3.59)\ \text{Implication}\ p \Rightarrow q \ \equiv\ \neg p \vee q \rangle \\
& \Diamond q \Rightarrow (\Box p \Rightarrow p \,\mathcal{U}\, q) \\
= & \langle (3.65)\ \text{Shunting},\ p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r) \rangle \\
& \Diamond q \wedge \Box p \Rightarrow p \,\mathcal{U}\, q \quad \text{—(84)}\ \mathcal{U}\ \text{implication} \quad \blacksquare
\end{array}
$$

(238) **Left monotonicity of $\mathcal{W}$:** $\Box(p \Rightarrow q) \Rightarrow (p \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, r)$

Proof:

$$
\begin{array}{ll}
& \Box(p \Rightarrow q) \Rightarrow (p \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, r) \\
= & \langle (3.65)\ \text{Shunting},\ p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r) \rangle \\
& \Box(p \Rightarrow q) \wedge p \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, r
\end{array}
$$

And now,

$$
\begin{array}{ll}
& \Box(p \Rightarrow q) \wedge p \,\mathcal{W}\, r \\
\Rightarrow & \langle (175)\ \text{Distributivity of} \wedge \text{over}\ \mathcal{W} \rangle \\
& ((p \Rightarrow q) \wedge p) \,\mathcal{W}\, ((p \Rightarrow q) \wedge r) \\
= & \langle (3.66)\ p \wedge (p \Rightarrow q) \ \equiv\ p \wedge q \rangle \\
& ((p \wedge q) \,\mathcal{W}\, ((p \Rightarrow q) \wedge r) \\
\Rightarrow & \langle (186)\ \text{Left Distributivity of}\ \mathcal{W}\ \text{over} \wedge \rangle \\
& ((p \wedge q) \,\mathcal{W}\, (p \Rightarrow q) \wedge (p \wedge q) \,\mathcal{W}\, r \\
= & \langle (187)\ \text{Right Distributivity of}\ \mathcal{W}\ \text{over} \wedge \rangle \\
& ((p \wedge q) \,\mathcal{W}\, (p \Rightarrow q) \wedge p \,\mathcal{W}\, r \wedge q \,\mathcal{W}\, r \\
\Rightarrow & \langle (3.76b)\ \text{Strengthening},\ p \wedge q \Rightarrow p \rangle \\
& q \,\mathcal{W}\, r \quad \blacksquare
\end{array}
$$

(239) **Right monotonicity of $\mathcal{W}$:** $\Box(p \Rightarrow q) \Rightarrow (r \,\mathcal{W}\, p \Rightarrow r \,\mathcal{W}\, q)$

Proof:

$$
\begin{array}{ll}
& r \,\mathcal{W}\, p \Rightarrow r \,\mathcal{W}\, q \\
= & \langle (169)\ \text{Definition of}\ \mathcal{W}\text{, twice} \rangle \\
& \Box r \vee r \,\mathcal{U}\, p \Rightarrow \Box r \vee r \,\mathcal{U}\, q \\
\Leftarrow & \langle (4.2)\ \text{Monotonicity of} \vee,\ (p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r) \rangle \\
& r \,\mathcal{U}\, p \Rightarrow r \,\mathcal{U}\, q \\
\Leftarrow & \langle (85)\ \text{Right monotonicity of}\ \mathcal{U} \rangle \\
& \Box(p \Rightarrow q) \quad \blacksquare
\end{array}
$$

(240) $\mathcal{W}$ **strengthening rule:** $\Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \Rightarrow (p \,\mathcal{W}\, q \Rightarrow r \,\mathcal{W}\, s)$

Proof:

$$
\begin{array}{ll}
 & \Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \Rightarrow (p \,\mathcal{W}\, q \Rightarrow r \,\mathcal{W}\, s) \\
= & \langle \text{(3.65) Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
 & \Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \wedge p \,\mathcal{W}\, q \Rightarrow r \,\mathcal{W}\, s
\end{array}
$$

And now,

$$
\begin{array}{ll}
 & \Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \wedge p \,\mathcal{W}\, q \\
= & \langle \text{(99) Distributivity of } \Box \text{ over } \wedge \rangle \\
 & \Box(p \Rightarrow r) \wedge \Box(q \Rightarrow s) \wedge p \,\mathcal{W}\, q \\
\Rightarrow & \langle \text{(239) Right monotonicity of } \mathcal{W} \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & \Box(p \Rightarrow r) \wedge (p \,\mathcal{W}\, q \Rightarrow p \,\mathcal{W}\, s) \wedge p \,\mathcal{W}\, q \\
\Rightarrow & \langle \text{(3.77) Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & \Box(p \Rightarrow r) \wedge p \,\mathcal{W}\, s \\
\Rightarrow & \langle \text{(238) Left monotonicity of } \mathcal{W} \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & (p \,\mathcal{W}\, s \Rightarrow r \,\mathcal{W}\, s) \wedge p \,\mathcal{W}\, s \\
\Rightarrow & \langle \text{(3.77) Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \rangle \\
 & r \,\mathcal{W}\, s \quad \blacksquare
\end{array}
$$

(241) $\mathcal{W}$ **catenation rule:** $\Box((p \Rightarrow q \,\mathcal{W}\, r) \wedge (r \Rightarrow q \,\mathcal{W}\, s)) \Rightarrow (p \Rightarrow q \,\mathcal{W}\, s)$

Proof:

$$
\begin{array}{ll}
 & \Box((p \Rightarrow q \,\mathcal{W}\, r) \wedge (r \Rightarrow q \,\mathcal{W}\, s)) \Rightarrow (p \Rightarrow q \,\mathcal{W}\, s) \\
= & \langle \text{(3.65) Shunting, } p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r) \rangle \\
 & \Box((p \Rightarrow q \,\mathcal{W}\, r) \wedge (r \Rightarrow q \,\mathcal{W}\, s)) \wedge p \Rightarrow q \,\mathcal{W}\, s
\end{array}
$$

And now,

$$
\begin{array}{ll}
 & \Box((p \Rightarrow q \,\mathcal{W}\, r) \wedge (r \Rightarrow q \,\mathcal{W}\, s)) \wedge p \\
= & \langle \text{(99) Distributivity of } \Box \text{ over } \wedge \rangle \\
 & \Box(p \Rightarrow q \,\mathcal{W}\, r) \wedge \Box(r \Rightarrow q \,\mathcal{W}\, s) \wedge p \\
\Rightarrow & \langle \text{(76) Strengthening of } \Box \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & (p \Rightarrow q \,\mathcal{W}\, r) \wedge \Box(r \Rightarrow q \,\mathcal{W}\, s) \wedge p \\
\Rightarrow & \langle \text{(3.77) Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \text{ and (4.3) Monotonicity of } \wedge \rangle \\
 & q \,\mathcal{W}\, r \wedge \Box(r \Rightarrow q \,\mathcal{W}\, s) \\
\Rightarrow & \langle \text{(239) Right monotonicity of } \mathcal{W} \text{ with } p,q,r := r, q \,\mathcal{W}\, s, q \rangle \\
 & q \,\mathcal{W}\, r \wedge (q \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, (q \,\mathcal{W}\, s)) \\
\Rightarrow & \langle \text{(3.77) Modus ponens, } p \wedge (p \Rightarrow q) \Rightarrow q \rangle \\
 & q \,\mathcal{W}\, (q \,\mathcal{W}\, s)
\end{array}
$$

$=$ $\langle$(222) Left absorption of $\mathcal{W}$ $\rangle$
$q \mathcal{W} s$ ∎

(242) **Left** $\mathcal{U}$ $\mathcal{W}$ **implication:** $(p \mathcal{U} q) \mathcal{W} r \Rightarrow (p \mathcal{W} q) \mathcal{W} r$

Proof: The proof is by (4.7.1) Truth implication.

true
$=$ $\langle$(136) Metatheorem and (174) $\mathcal{U}$ implication$\rangle$
$\Box (p \mathcal{U} q \Rightarrow p \mathcal{W} q)$
$\Rightarrow$ $\langle$(238) Left monotonicity of $\mathcal{W}$ with $p, q := p \mathcal{U} q, p \mathcal{W} q\rangle$
$(p \mathcal{U} q) \mathcal{W} r \Rightarrow (p \mathcal{W} q) \mathcal{W} r$ ∎

(243) **Right** $\mathcal{W}$ $\mathcal{U}$ **implication:** $p \mathcal{W} (q \mathcal{U} r) \Rightarrow p \mathcal{W} (q \mathcal{W} r)$

Proof: The proof is by (4.7.1) Truth implication.

true
$=$ $\langle$(136) Metatheorem and (174) $\mathcal{U}$ implication$\rangle$
$\Box (q \mathcal{U} r \Rightarrow q \mathcal{W} r)$
$\Rightarrow$ $\langle$(239) Right monotonicity of $\mathcal{W}$ with $p, q, r := q \mathcal{U} r, q \mathcal{W} r, p\rangle$
$p \mathcal{W} (q \mathcal{U} r) \Rightarrow p \mathcal{W} (q \mathcal{W} r)$ ∎

(244) **Right** $\mathcal{U}$ $\mathcal{U}$ **implication:** $p \mathcal{U} (q \mathcal{U} r) \Rightarrow p \mathcal{U} (q \mathcal{W} r)$

Proof: The proof is by (4.7.1) Truth implication.

true
$=$ $\langle$(136) Metatheorem and (174) $\mathcal{U}$ implication$\rangle$
$\Box (q \mathcal{U} r \Rightarrow q \mathcal{W} r)$
$\Rightarrow$ $\langle$(85) Right monotonicity of $\mathcal{U}$ with $p, q, r := q \mathcal{U} r, q \mathcal{W} r, p\rangle$
$p \mathcal{U} (q \mathcal{U} r) \Rightarrow p \mathcal{U} (q \mathcal{W} r)$ ∎

(245) **Left** $\mathcal{U}$ $\mathcal{U}$ **implication:** $(p \mathcal{U} q) \mathcal{U} r \Rightarrow (p \mathcal{W} q) \mathcal{U} r$

Proof: The proof is by (4.7.1) Truth implication.

true
$=$ $\langle$(136) Metatheorem and (174) $\mathcal{U}$ implication$\rangle$
$\Box (p \mathcal{U} q \Rightarrow p \mathcal{W} q)$
$\Rightarrow$ $\langle$(86) Left monotonicity of $\mathcal{U}$ with $p, q := p \mathcal{U} q, p \mathcal{W} q\rangle$
$(p \mathcal{U} q) \mathcal{U} r \Rightarrow (p \mathcal{W} q) \mathcal{U} r$ ∎

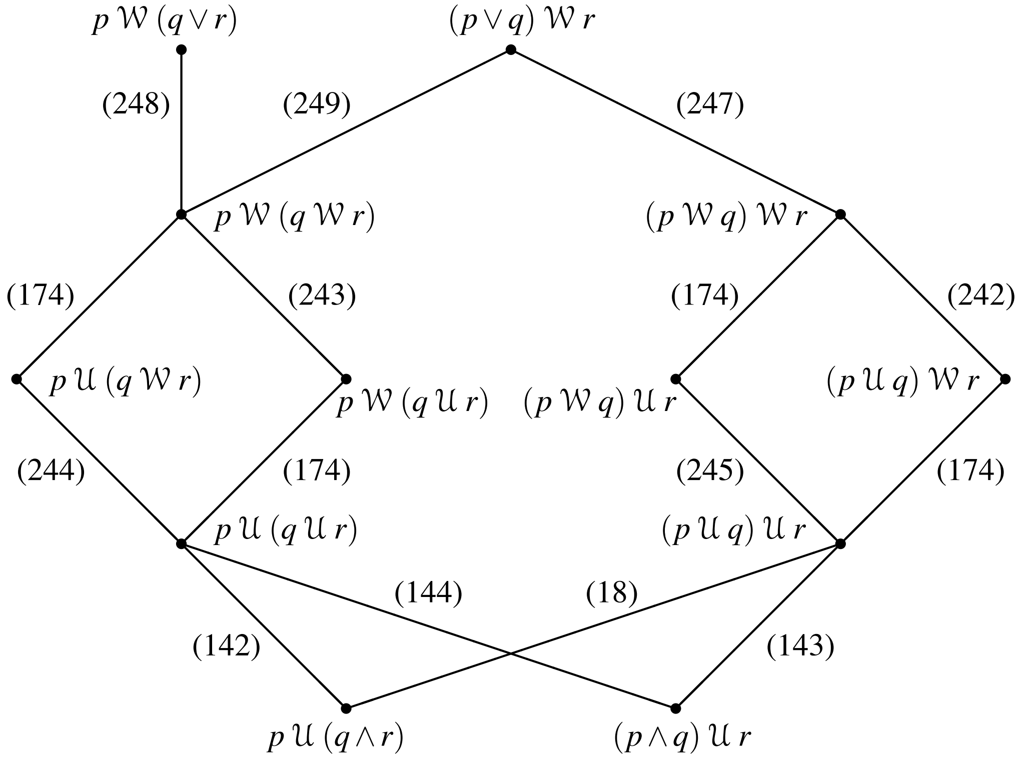

Figure 4: A Hasse diagram showing some implication relations of linear temporal logic.

Of the three strengthening rules, only (247) appears in the LTL literature. All of the following ordering rules are in the literature except for (251) $\mathcal{U}$ ordering. Figure 4 is a Hasse diagram showing some implication relations of these theorems.

(246) **Left** $\mathcal{U} \vee$ **strengthening:** $(p \,\mathcal{U}\, q) \,\mathcal{U}\, r \Rightarrow (p \vee q) \,\mathcal{U}\, r$

Proof: The proof is by (4.7.1) Truth implication.

$$\begin{aligned} & \quad \textit{true} \\ = & \quad \langle (136) \text{ Metatheorem and } (28) \rangle \\ & \quad \Box (p \,\mathcal{U}\, q \Rightarrow p \vee q) \\ \Rightarrow & \quad \langle (86) \text{ Left monotonicity of } \mathcal{U} \text{ with } p,q := p \,\mathcal{U}\, q, p \vee q \rangle \\ & \quad (p \,\mathcal{U}\, q) \,\mathcal{U}\, r \Rightarrow (p \vee q) \,\mathcal{U}\, r \quad \blacksquare \end{aligned}$$

(247) **Left** $\mathcal{W} \vee$ **strengthening:** $(p \,\mathcal{W}\, q) \,\mathcal{W}\, r \Rightarrow (p \vee q) \,\mathcal{W}\, r$

Proof: The proof is by (4.7.1) Truth implication.

$$\begin{aligned} & \quad \textit{true} \\ = & \quad \langle (136) \text{ Metatheorem and } (206) \rangle \\ & \quad \Box (p \,\mathcal{W}\, q \Rightarrow p \vee q) \\ \Rightarrow & \quad \langle (238) \text{ Left monotonicity of } \mathcal{W} \text{ with } p,q := p \,\mathcal{W}\, q, p \vee q \rangle \\ & \quad (p \,\mathcal{W}\, q) \,\mathcal{W}\, r \Rightarrow (p \vee q) \,\mathcal{W}\, r \quad \blacksquare \end{aligned}$$

(248) **Right** $\mathcal{W} \vee$ **strengthening:** $p\,\mathcal{W}\,(q\,\mathcal{W}\,r) \Rightarrow p\,\mathcal{W}\,(q \vee r)$

Proof: The proof is by (4.7.1) Truth implication.

$$\begin{array}{ll}
& \textit{true} \\
= & \langle \text{(136) Metatheorem and (206)} \rangle \\
& \Box\,(q\,\mathcal{W}\,r \Rightarrow q \vee r) \\
\Rightarrow & \langle \text{(239) Right monotonicity of } \mathcal{W} \text{ with } p,q,r := q\,\mathcal{W}\,r, q \vee r, p \rangle \\
& p\,\mathcal{W}\,(q\,\mathcal{W}\,r) \Rightarrow p\,\mathcal{W}\,(q \vee r) \quad \blacksquare
\end{array}$$

(249) **Right** $\mathcal{W} \vee$ **ordering:** $p\,\mathcal{W}\,(q\,\mathcal{W}\,r) \Rightarrow (p \vee q)\,\mathcal{W}\,r$

Proof:

$$\begin{array}{ll}
& p\,\mathcal{W}\,(q\,\mathcal{W}\,r) \\
= & \langle \text{(169) Definition of } \mathcal{W}\text{, twice} \rangle \\
& \Box\, p \vee p\,\mathcal{U}\,(\Box\, q \vee q\,\mathcal{U}\,r) \\
= & \langle \text{(12) Left distributivity of } \mathcal{U} \text{ over } \vee \rangle \\
& \Box\, p \vee p\,\mathcal{U}\,\Box\, q \vee p\,\mathcal{U}\,(q\,\mathcal{U}\,r) \\
\Rightarrow & \langle \text{(140) } \mathcal{U}\,\Box \text{ implication and (4.2) Monotonicity of } \vee \rangle \\
& \Box\, p \vee \Box\,(p\,\mathcal{U}\,q) \vee p\,\mathcal{U}\,(q\,\mathcal{U}\,r) \\
\Rightarrow & \langle \text{(17) Right } \mathcal{U} \vee \text{ ordering and (4.2) Monotonicity of } \vee \rangle \\
& \Box\, p \vee \Box\,(p\,\mathcal{U}\,q) \vee (p \vee q)\,\mathcal{U}\,r \\
\Rightarrow & \langle \text{(100) Distributivity of } \Box \text{ over } \vee \text{ and (4.2) Monotonicity of } \vee \rangle \\
& \Box\,(p \vee p\,\mathcal{U}\,q) \vee (p \vee q)\,\mathcal{U}\,r \\
= & \langle \text{(31) Absorption} \rangle \\
& \Box\,(p \vee q) \vee (p \vee q)\,\mathcal{U}\,r \\
= & \langle \text{(169) Definition of } \mathcal{W} \rangle \\
& (p \vee q)\,\mathcal{W}\,r \quad \blacksquare
\end{array}$$

(250) **Right** $\wedge\,\mathcal{W}$ **ordering:** $p\,\mathcal{W}\,(q \wedge r) \Rightarrow (p\,\mathcal{W}\,q)\,\mathcal{W}\,r$

Proof:

$$\begin{array}{ll}
& (p\,\mathcal{W}\,q)\,\mathcal{W}\,r \\
= & \langle \text{(169) Definition of } \mathcal{W} \rangle \\
& (\Box\, p \vee p\,\mathcal{U}\,q)\,\mathcal{W}\,r \\
\Leftarrow & \langle \text{(185) Right distributivity of } \mathcal{W} \text{ over } \vee \rangle \\
& (\Box\, p\,\mathcal{W}\,r) \vee (p\,\mathcal{U}\,q)\,\mathcal{W}\,r \\
= & \langle \text{(169) Definition of } \mathcal{W}\text{, twice} \rangle \\
& \Box\,\Box\, p \vee \Box\, p\,\mathcal{U}\,r \vee \Box\,(p\,\mathcal{U}\,q) \vee (p\,\mathcal{U}\,q)\,\mathcal{U}\,r \\
= & \langle \text{(72) Absorption of } \Box \rangle
\end{array}$$

$\quad \Box\, p \vee \Box\, p\, \mathcal{U}\, r \vee \Box\, (p\, \mathcal{U}\, q) \vee (p\, \mathcal{U}\, q)\, \mathcal{U}\, r$
$\Leftarrow \quad \langle$(3.76a) Weakening, $p \Rightarrow p \vee q\rangle$
$\quad \Box\, p \vee (p\, \mathcal{U}\, q)\, \mathcal{U}\, r$
$\Leftarrow \quad \langle$(18) Right $\wedge\, \mathcal{U}$ ordering and (4.2) Monotonicity of $\vee\rangle$
$\quad \Box\, p \vee p\, \mathcal{U}\, (q \wedge r)$
$= \quad \langle$(169) Definition of $\mathcal{W}\rangle$
$\quad p\, \mathcal{W}\, (q \wedge r)$ ∎

(251) $\mathcal{U}$ **ordering:** $\neg p\, \mathcal{U}\, q \vee \neg q\, \mathcal{U}\, p \equiv \Diamond\, (p \vee q)$

Proof:

$\quad \neg p\, \mathcal{U}\, q \vee \neg q\, \mathcal{U}\, p$
$= \quad \langle$(196) Conjunction rule of $\mathcal{U}$, twice$\rangle$
$\quad (\neg p \wedge \neg q)\, \mathcal{U}\, q \vee (\neg q \wedge \neg p)\, \mathcal{U}\, p$
$= \quad \langle$(12) Left distributivity of $\mathcal{U}$ over $\vee\rangle$
$\quad (\neg p \wedge \neg q)\, \mathcal{U}\, (p \vee q)$
$= \quad \langle$(3.47b) De Morgan, $\neg(p \vee q) \;\equiv\; \neg p \wedge \neg q\rangle$
$\quad \neg(p \vee q)\, \mathcal{U}\, (p \vee q)$
$= \quad \langle$(192) Rule of $\mathcal{U}$ with $q := p \vee q\rangle$
$\quad \Diamond\, (p \vee q)$ ∎

(252) $\mathcal{W}$ **ordering:** $\neg p\, \mathcal{W}\, q \vee \neg q\, \mathcal{W}\, p$

Proof:

$\quad \neg p\, \mathcal{W}\, q \vee \neg q\, \mathcal{W}\, p$
$= \quad \langle$(195) Conjunction rule of $\mathcal{W}$, twice$\rangle$
$\quad (\neg p \wedge \neg q)\, \mathcal{W}\, q \vee (\neg q \wedge \neg p)\, \mathcal{W}\, p$
$= \quad \langle$(184) Left distributivity of $\mathcal{W}$ over $\vee\rangle$
$\quad (\neg p \wedge \neg q)\, \mathcal{W}\, (p \vee q)$
$= \quad \langle$(3.47b) De Morgan, $\neg(p \vee q) \;\equiv\; \neg p \wedge \neg q\rangle$
$\quad \neg(p \vee q)\, \mathcal{W}\, (p \vee q)$
$= \quad \langle$(191) Rule of $\mathcal{W}$ with $q := p \vee q\rangle$
$\quad$ *true* ∎

(253) $\mathcal{W}$ **implication ordering:** $p\, \mathcal{W}\, q \wedge \neg q\, \mathcal{W}\, r \Rightarrow p\, \mathcal{W}\, r$

Proof: The proof is based on the following lemmas.

Lemma A: $p\, \mathcal{U}\, q \wedge \Box\, \neg q \equiv \mathit{false}$
Proof:

$$
\begin{array}{ll}
& p\,\mathcal{U}\,q \wedge \Box\neg q \\
= & \langle (171)\ \mathcal{U}\ \text{in terms of}\ \mathcal{W} \rangle \\
& p\,\mathcal{W}\,q \wedge \Diamond q \wedge \Box\neg q \\
= & \langle (92)\ \Diamond\ \text{contradiction} \rangle \\
& p\,\mathcal{W}\,q \wedge \mathit{false} \\
= & \langle (3.40)\ \text{Zero of}\ \wedge,\ p \wedge \mathit{false} \equiv \mathit{false} \rangle \\
& \mathit{false} \quad \blacksquare
\end{array}
$$

Lemma B: $\Box p \wedge \neg\Box q \Rightarrow p\,\mathcal{W}\,r$

Proof:

$$
\begin{array}{ll}
& \Box p \wedge \neg\Box q \\
\Rightarrow & \langle (3.76b)\ \text{Strengthening},\ p \wedge q \Rightarrow p \rangle \\
& \Box p \\
\Rightarrow & \langle (179)\ \text{Perpetuity} \rangle \\
& p\,\mathcal{W}\,r \quad \blacksquare
\end{array}
$$

Lemma C: $\Box p \wedge \neg q\,\mathcal{U}\,r \Rightarrow p\,\mathcal{W}\,r$

Proof:

$$
\begin{array}{ll}
& \Box p \wedge \neg q\,\mathcal{U}\,r \\
\Rightarrow & \langle (3.76b)\ \text{Strengthening},\ p \wedge q \Rightarrow p \rangle \\
& \Box p \\
\Rightarrow & \langle (179)\ \text{Perpetuity} \rangle \\
& p\,\mathcal{W}\,r \quad \blacksquare
\end{array}
$$

And now,

$$
\begin{array}{ll}
& p\,\mathcal{W}\,q \wedge \neg q\,\mathcal{W}\,r \\
= & \langle (169)\ \text{Definition of}\ \mathcal{W}\text{, twice} \rangle \\
& (\Box p \vee p\,\mathcal{U}\,q)\ \wedge\ (\Box\neg q \vee \neg q\,\mathcal{U}\,r) \\
= & \langle (3.46)\ \text{Distributivity of}\ \wedge\ \text{over}\ \vee,\ p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \rangle \\
& (\Box p \wedge \Box\neg q)\ \vee\ (\Box p \wedge \neg q\,\mathcal{U}\,r)\ \vee\ (p\,\mathcal{U}\,q \wedge \Box\neg q)\ \vee\ (p\,\mathcal{U}\,q \wedge \neg q\,\mathcal{U}\,r) \\
= & \langle \text{Lemma A} \rangle \\
& (\Box p \wedge \Box\neg q)\ \vee\ (\Box p \wedge \neg q\,\mathcal{U}\,r)\ \vee\ \mathit{false}\ \vee\ (p\,\mathcal{U}\,q \wedge \neg q\,\mathcal{U}\,r) \\
= & \langle (3.30)\ \text{Identity of}\ \vee,\ p \vee \mathit{false} \equiv p \rangle \\
& (\Box p \wedge \Box\neg q)\ \vee\ (\Box p \wedge \neg q\,\mathcal{U}\,r)\ \vee\ (p\,\mathcal{U}\,q \wedge \neg q\,\mathcal{U}\,r) \\
\Rightarrow & \langle \text{Lemma B, Lemma C, and (4.2) Monotonicity of}\ \vee \rangle \\
& p\,\mathcal{W}\,r\ \vee\ p\,\mathcal{W}\,r\ \vee\ (p\,\mathcal{U}\,q \wedge \neg q\,\mathcal{U}\,r) \\
\Rightarrow & \langle (16)\ \mathcal{U}\ \text{implication ordering and (4.2) Monotonicity of}\ \vee \rangle \\
& p\,\mathcal{W}\,r\ \vee\ p\,\mathcal{W}\,r\ \vee\ p\,\mathcal{U}\,r
\end{array}
$$

$\Rightarrow \quad \langle (174)\ \mathcal{U}$ implication and (4.2) Monotonicity of $\vee \rangle$
$\quad p\,\mathcal{W}\,r \vee p\,\mathcal{W}\,r \vee p\,\mathcal{W}\,r$
$= \quad \langle$(3.26) Idempotency of $\vee$, $p \vee p \equiv p \rangle$
$\quad p\,\mathcal{W}\,r \quad \blacksquare$

(254) **Lemmon formula:** $\Box(\Box p \Rightarrow q) \vee \Box(\Box q \Rightarrow p)$

Proof: The proof is by (4.7.1) Truth implication.

$\quad \Box(\Box p \Rightarrow q) \vee \Box(\Box q \Rightarrow p)$
$= \quad \langle$(3.59) Implication $p \Rightarrow q \equiv \neg p \vee q$, twice$\rangle$
$\quad \Box(\neg\Box p \vee q) \vee \Box(\neg\Box q \vee p)$
$\Leftarrow \quad \langle$(206) twice, (120) Monotonicity of $\Box$, and (4.2) Monotonicity of $\vee \rangle$
$\quad \Box(\neg\Box p\,\mathcal{W}\,q) \vee \Box(\neg\Box q\,\mathcal{W}\,p)$
$\Leftarrow \quad \langle (177)\ \mathcal{W}\,\Box$ implication and (4.2) Monotonicity of $\vee \rangle$
$\quad \neg\Box p\,\mathcal{W}\,\Box q \vee \neg\Box q\,\mathcal{W}\,\Box p$
$= \quad \langle (252)\ \mathcal{W}$ ordering with $p, q := \Box p, \Box q \rangle$
$\quad$ *true* $\blacksquare$

The ten axioms that define the behavior of the *until* operator are (9), (10), (11), (12), (13), (14), (15), (16), (17), and (18). The corresponding theorems for the *wait* operator are (180), (181), (224), (184), (185), (186), (187), (253), (249), and (250). Of these ten theorems, nine are identical, with the substitution of $\mathcal{W}$ for $\mathcal{U}$, to the corresponding axioms that define the *until* operator. In addition, (170), the axiom that describes the distributivity of $\neg$ over $\mathcal{W}$, is identical to (173) with the interchange of $\mathcal{W}$ and $\mathcal{U}$. The one theorem that distinguishes $\mathcal{W}$ from the defining axioms of $\mathcal{U}$ is

(11) **Axiom, Right zero of $\mathcal{U}$:** $p\,\mathcal{U}\,\mathit{false} \equiv \mathit{false}$

for the *until* operator versus

(224) **$\Box$ to $\mathcal{W}$ law:** $\Box p \equiv p\,\mathcal{W}\,\mathit{false}$

for the *wait* operator.

4 Modal logic

Linear temporal logic is just one of many different modal logic systems. This section shows how linear temporal logic fits in general systems of modal logic and how calculation logic applies to modal logic.

4.1 Modal logic systems

In the terminology of modal logic, each state of an anchored sequence is called a *world*. In LTL, each world represents the state of a computation at each discrete point of time, and so one world is related to another world as one point in time is related to another point in time, *i.e.*, as occurring before or after it. In general, worlds need not have such an interpretation. Rather, a specific modal logic system is defined by a nonempty set of worlds W and a relation ρ over W. A frame is the ordered pair $\langle W, \rho \rangle$, and different modal systems are specified by different frames.

The most general modal system, known as K, extends propositional calculus by adding the unary operator $\Box$ together with the axiom $\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$, which is our theorem (120) Monotonicity of $\Box$. The Hilbert style inference rules of Uniform Substitution and Modus Ponens are extended by adding the rule of Necessitation: If P is a theorem then so is $\Box P$, which is Case 1 of our (136) Metatheorem. The operator $\Diamond p$ is defined to be an abbreviation for $\neg \Box \neg p$ as in (59). With these extensions, there is no restriction on relation ρ in the frame $\langle W, \rho \rangle$ that defines K.

A stronger modal system, known as T, extends K by adding the axiom $\Box p \Rightarrow p$, sometimes called the Axiom of Necessity, which is our theorem (76). Every theorem in K is also a theorem in T, but not every theorem in T is a theorem in K. It has been shown that theorems in T are valid on every frame $\langle W, \rho \rangle$ in which ρ is reflexive, *i.e.*, in which for all $w \in W$, $w \rho w$. [16]

A stronger yet modal system, known as S4, extends T by adding the single extra axiom $\Box p \Rightarrow \Box \Box p$, which is part of our theorem (72) Absorption of $\Box$, $\Box \Box p \equiv \Box p$ by virtue of mutual implication. Theorems in S4 are valid on every frame $\langle W, \rho \rangle$ in which ρ is both reflexive and transitive, *i.e.*, in which for all $w, x, y \in W$, $w \rho x \wedge x \rho y \Rightarrow w \rho y$.

Linear temporal systems model time as discrete points on the number line. Each world is on a time line and is either contemporaneous with, or earlier than, itself and all worlds that come after it. This "earlier than" relation is defined formally as a connected relation, in which for all $w, x, y \in W$, $w \rho x \wedge w \rho y \Rightarrow x \rho y \vee y \rho x$. In other words, if w is contemporaneous with or earlier than x, and w is contemporaneous with or earlier than y, then either x is contemporaneous with or earlier than y, or y is contemporaneous with or earlier than x. The temporal modal system, known as S4.3 [6], extends S4 by the axiom $\Box(\Box p \Rightarrow q) \vee \Box(\Box q \Rightarrow p)$, which is our theorem (254) Lemmon formula. Theorems in S4.3 are valid on every frame $\langle W, \rho \rangle$ in which ρ is reflexive, transitive, and connected. [16]

The Lemmon formula imposes linearity of the time line. The Dummett formula

$$\Box(\Box(p \Rightarrow \Box p) \Rightarrow \Box p) \Rightarrow (\Diamond \Box p \Rightarrow \Box p)$$

imposes discreteness of the time line. [7, 20] The modal system S4.3 with the addition of the Dummett formula is known as S4.3*Dum*. Our LTL system contains S4.3, because it includes as theorems the axioms of S4.3. The LTL systems we survey are close to S4.3*Dum* but do not contain the Dummett formula as an axiom. Instead, discreteness of the time line is implicit in the definition of the *next* operator $\circ$. The binary operators *until* $\mathcal{U}$ and *wait* $\mathcal{W}$ are absent in classical modal logic systems.

4.2 Calculational modal logic systems

Gries and Schneider [15, 12] describe the benefits of the calculational system for propositional logic and its application to discrete math. Gries and Schneider [14] and Tourlakis [22] demonstrate the soundness and completeness of such systems. The calculational method is the subject of a special issue in [1].

Gries and Schneider [13] extend calculational logic to the Carnapian modal system known as C. Theorems in C are valid on every frame $\langle W, \rho \rangle$ in which ρ is the universal accessibility relation, *i.e.*, each state is related to all states. Because C is not a linear temporal logic, the operator $\Box$ has a different interpretation from that in LTL. Denoting the Carnapian operator as $\Box_c$ the interpretation of $\Box_c P$, where P is a propositional logic expression, is "P is true in all states," or, equivalently, "P is valid." Gries and Schneider point out that this logic "... can be used for proving theorems that could otherwise be handled only at the meta-level, and most likely informally." As an example, (2.3b) Metatheorem Duality, $P \equiv Q$ is valid iff $P_D \equiv Q_D$ is valid, can be written as $\Box_c(P \equiv Q) \equiv \Box_c(P_D \equiv Q_D)$. In such a system, metatheorems become formulas in the logic and are thus directly available for use in calculational reasoning.

Schneider's graduate text *On Concurrent Programming* [21] extends the calculational logic system to linear temporal logic. It claims, and we concur, that "... proofs in Temporal Logic are often easier to read and to construct when an equational format is available." As justification for the extension, it gives a temporal logic analogue of (I1) Leibniz, TL Substitution of Equals. For proof steps that assert $p \Rightarrow q$, it gives two additional temporal logic rules – TL Monotonicity Rule and TL Antimonotonicity Rule. These rules are the basis for the justification of temporal logic calculational proofs.

5 Verification tools

[*To be written*.]

5.1 ACL2

[*To be written*.]

5.2 Coq

[*To be written*.]

6 Conclusion

[*Needs revising*]

Dijkstra and Scholten [5], and Feijen [9] originally developed $\mathcal{E}$ as a logic system to prove program correctness based on a calculational style. Gries and Schneider [11] extend that system to a theory of sets, a theory of sequences, relations and functions, a theory of integers, recurrence relations, modern algebra, and a theory of graphs. Similarly, this system extends $\mathcal{E}$ to a theory of linear temporal logic. It takes unary operator *next* $\circ$ and binary operator *until* $\mathcal{U}$ as primitives and defines *eventually* $\diamond$, *always* $\square$, and *wait* $\mathcal{W}$ in terms of them.

The calculational deductive system $\mathcal{E}$ has several advantages over other logic systems. The primary advantage is that the calculational system has only four inference rules. Consequently, proofs of theorems are easier to understand and more intuitive to those schooled in that system. One goal of this basic introduction is to make linear temporal logic accessible at the undergraduate level.

Many users of LTL are concerned with the scalability of reasoning and automated theorem proving. In contrast, the goal of this project is to make manual proofs accessible to human users. It would be an interesting area for future research to determine if and/or how the calculational system might be applied to LTL synthesis.

In our judgement, the progress we were able to make in exploring the structure of linear temporal systems is directly attributable to our training in the calculational deductive system $\mathcal{E}$. We believe the advantages of $\mathcal{E}$ over other logic systems is so substantial that it should be the tool of choice for computer science theory. We hope that this extension of $\mathcal{E}$ to linear temporal logic will not only be of use in the temporal logic community, but will serve as an example to promote $\mathcal{E}$ in the broader computer science community.

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[*Needs revising*]

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