Project5 - Stability (by Kieran)

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1 General Stability Criteria

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1.1 Introduction

This paper will examine general criteria for the stability and instability of inviscid parallel flows. The first section will explore unstratified flows, deriving Rayliegh's and Fjortoft's necessary (but not sufficient) criteria for instability. The second section will analyze stratified flows and derive stability criteria related to the local Richardson number of the flow.

1.2 1. Unstratified Flows

1.2.1 1.1 Introduction

Throughout this paper parallel flows of the following form will be considered:
$ {U} = U_ (z*){i}$ for $z_{1*} \leq z_* \leq z_{2*}$
The following normalized quantities will be used throughout:

$$\mathbf{U} = \frac{\mathbf{U}_*}{max(|\mathbf{U}_*|)}$$

$$z = \frac{z_*}{(z_{2*} - z_{1*})}$$

$$p = \frac{p_*}{\rho_m}$$

Where ρ_m is the mean density of the flow.

The stability of these flows to small pertubations of the form $\mathbf{u}'(\mathbf{x}, t)$ will be examined with the aim of deriving general criteria for stability. As U arleady satisfies the boundary conditions, the pertubations are subject to the boundary conditions: $\mathbf{u}'(z = z_1) = \mathbf{u}'(z = z_2) = 0$.

1.2.2 1.2 Equations of Motion

The inviscid Navier-Stoke's equation with constant density and no external forces is:

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \cdot \mathbf{u})\mathbf{u} = -\nabla p$$

Let: $\mathbf{u} = \mathbf{U} + \mathbf{u}'$

Then:

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{u}'}{\partial t}$$

And:

$$(\nabla \cdot (\mathbf{U} + \mathbf{u}'))(\mathbf{U} + \mathbf{u}') = (\nabla \cdot (\mathbf{U} + \mathbf{u}'))(\mathbf{U} + \mathbf{u}') = \mathbf{U}\frac{\partial}{\partial x}(\mathbf{U} + \mathbf{u}') + (\nabla \cdot (\mathbf{u}')\mathbf{U} + (\nabla \cdot (\mathbf{u}')\mathbf{u}'$$

Noting that $\frac{\partial}{\partial x}\mathbf{U} = 0$ and linearising ($(\nabla \cdot (\mathbf{u}')\mathbf{u}' \approx 0$):

$$\Big(\nabla \cdot \big(\mathbf{U} + \mathbf{u}'\big)\Big)\big(\mathbf{U} + \mathbf{u}'\big) \approx \mathbf{U}\frac{\partial}{\partial x}\mathbf{u}' + \big(\nabla \cdot \mathbf{u}'\big)\mathbf{U} = \mathbf{U}\frac{\partial}{\partial x}\mathbf{u}' + w'\frac{\partial}{\partial z}\mathbf{U}$$

Thus the equation of motion simplifies to:

$$\frac{\partial \mathbf{u}'}{\partial t} + U\frac{\partial}{\partial x}\mathbf{u}' + w'\frac{\partial}{\partial z}\mathbf{U} = -\nabla p$$

1.2.3 1.3 Squires Theorem

This section will examine a proof Squire's theorem: for any 3-dimensional pertubation, there is a 2 dimensional pertubation with the same wavenumber that is more unstable.

Intuitively, this is done by taking a 2-dimensional and rotating it align with the mean flow; this allows for more energy to be trasnferred from the mean flow to the disturbance causing it to be more unstable.

Noting that coefficients of the above equation of motion only depend on z, solutions of the following form will be searched for:

$$\mathbf{u}' = \hat{\mathbf{u}}(z)e^{i(ax+by+act)}$$

$$p' = \hat{p}(z)e^{i(ax+by+act)}$$

Let:

$$\mathbf{u}' = \hat{u}(z)e^{i(ax+by-act)}\hat{\mathbf{i}} + \hat{v}(z)e^{i(ax+by+act)}\hat{\mathbf{j}} + \hat{w}(z)e^{i(ax+by-act)}\hat{\mathbf{k}}$$

In order for the initial disturbances to have finite energy a and b must be real.

The equation of motion can be written as:

$$\Big(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\Big)\hat{u} + w'\frac{\partial U}{\partial z} = -\frac{\partial p}{\partial x}$$

$$\Big(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\Big)\hat{v} = -\frac{\partial p}{\partial y}$$

$$\Big(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\Big)\hat{w} = -\frac{\partial p}{\partial z}$$

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{w}}{\partial z} = 0$$

Substituting the disturbances into these above equations and eliminating exponentials yields:

$$ia\big(U-c\big)\hat{u}+\hat{w}U'=-ia\hat{p}$$

$$ia\big(U-c\big)\hat{v}=-ib\hat{p}$$

$$ia\big(U-c\big)\hat{w}=\frac{d\hat{p}}{dz}$$

$$ia\hat{u}+ib\hat{v}+\frac{\partial\hat{w}}{\partial z}=0$$

Summing a times the first equation and b times the second equations yields:

$$ia\big(U-c\big)\big(a\hat{u}+b\hat{v}\big)+a\hat{w}U'=-i\big(a^2+b^2\big)\hat{p}$$

Dividing through by a:

$$i\big(U-c\big)\big(a\hat{u}+b\hat{v}\big)+\hat{w}U'=-i\frac{\big(a^2+b^2\big)}{a}\hat{p}$$

Let $k=\sqrt{\big(a^2+b^2\big)}$
Let:

$$\tilde{u}=\frac{\big(a\hat{u}+b\hat{v}\big)}{k}$$

$$\tilde{v}=0$$

$$\tilde{w}=\hat{w}$$

Let $\tilde{p}=\dfrac{k\hat{p}}{a}$
Then the above equations can be re-written as:

$$ik\big(U-c\big)\tilde{u}+\tilde{w}U'=-ik\hat{p}$$

$$ik\big(U-c\big)\tilde{w}=-ik\frac{d\hat{p}}{dz}$$

$$k\tilde{u}+\frac{\partial\tilde{w}}{\partial z}=0$$

Thus, if a 3-dimensional disturbance of the form satisfies the equations of motion:

$$\mathbf{u}'=\hat{u}(z)e^{i(ax+by-act)}\hat{\mathbf{i}}+\hat{v}(z)e^{i(ax+by+act)}\hat{\mathbf{j}}+\hat{w}(z)e^{i(ax+by-act)}\hat{\mathbf{k}}$$

Then following 2-dimensional disturbance with the same wave number also satisfies the equations of motion:

$$\tilde{\mathbf{u}}=\tilde{u}(z)e^{i(kx-kct)}\hat{\mathbf{i}}+\hat{w}(z)e^{i(k-kct)}\hat{\mathbf{k}}$$

As $|k|\geq|a|$, $|kc|\geq|ac|$ thus this 2-dimensional distrubance is at least as unstable as the 3-dimensional disturbance.

The significance of this result is that when studying general criterea for modal instability only 2-dimensional disturbances need be considered. A flow is considered unstable if there exists any disturbance that will grow exponentially with time, thus if a 2-dimensional disturbance is found to grow exponentially with time then flow is unstable. Conversly, a flow is considered stable only if all disturbances do not grow exponentially with time; if a flow is found to be stable in response to all 2-dimensional disturbances then using squires theorem it must also be stable in response to all 3-dimensional disturbances and thus the flow is considered stable.

1.2.4 1.4 Rayleigh's Equation

This section will derive Rayleigh's equation for the stream function of 2-dimensional disturbances. It is convenient when studying 2-dimensional flows to use the stream function: $\psi(\mathbf{x}, t)$.

$$u' = \frac{\partial \psi}{\partial z}$$

$$w' = -\frac{\partial \psi}{\partial x}$$

Substituting these into the equations of motion:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x}\frac{\partial U}{\partial z} = -\frac{\partial p}{\partial x}$$

$$-\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial \psi}{\partial x} = -\frac{\partial p}{\partial z}$$

Once again, the coefficients only depend on z, thus solutions of the form $\psi = \phi e^{ik(x-ct)}$ will be searched for. Substituting this into the first equation yields:

$$ik(U-c)\phi' - ik\phi U' = -ik\tilde{p}$$

$$\tilde{p} = \phi U' - (U-c)\phi'$$

Differentiating $\tilde{p}$ with the purpose of substituting into the z-equation of motion:

$$\tilde{p}' = \phi U'' + \phi' U' - (U-c)\phi'' - U'\phi'$$

$$\tilde{p}' = \phi U'' - (U-c)\phi''$$

Substituing into the z equation of motion:

$$ik(U-c)(-ik\phi) = -\phi U'' + (U-c)\phi''$$

$$(U-c)\left(\phi'' - k^2\phi\right) - \phi U'' = 0$$

This last equation is known as Rayleigh's equation. Notice that if k satisfies the above equation then so does $-k$. Thus it can be taken without loss of generality that $k \geq 0$. Taking the complex conjugate of the entire equation yields:

$$(U-c^*)\left(\phi''^* - k^2\phi^*\right) - \phi^* U'' = 0$$