Project1 - stratified KH (by Nick)

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1 The stratified Kelvin-Helmholtz instability

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Earlier in the course, the unstratified Kelvin Helmholtz instability (see the the cases on the left side of the figure below) was analysed analytically and numerically. That is, density is constant throughout the system. Here the stratified Kelvin-Helmholtz instability in two dimensions is considered where the stratification consists of a constant density top layer and a constant density bottom layer. Firstly, the discrete density profile case will be solved analytically, and secondly, the continuous density profile case will be solved numerically. The basic picture is visualised by the cases on the right hand side of the figure below.

The basic state for this system is:

$$\boldsymbol{u} = U_0 \,\mathrm{sign}(z)\, \widehat{\boldsymbol{x}} \quad , \quad \rho = \rho_m + \rho_0 \mathrm{sign}(z). \tag{1a}$$

with a pressure of:

$$p_0(z) = P_0 - \rho_0 g z. \tag{1b}$$

Physical mechanism Before doing any mathematical analysis, it is interesting to consider what might be expected from physical analysis of stratification. Suppose the bottom layer is infinitely dense. Suppose the interface between the layers is perturbed, and a wave begins to propagate. It must have some energy to displace the particles from their initial position. However, due to the bottom layer being infinitely dense, an infinite amount of energy is required to displace it. Thus, no wave can propagate. Hence, physically it would be expected that stratification makes the basic state more stable. This will later be discussed mathematically once the growth factor has been found.

1.2 Study of the Instability

The Set-Up Just to be clear, the assumptions regarding the velocity, pressure, and density perturbations are the following:

$$\boldsymbol{u}(x,z,t) = \left[U_0 \,\mathrm{sign}(z) + u'(x,z,t)\right] \widehat{\boldsymbol{x}} + w'(x,z,t)\, \widehat{\boldsymbol{z}}, \quad p = p_0 + p', \quad \rho = \rho_m + \rho_0 + \rho', \tag{2}$$

However, the most important equation compared to the other instability projects is the density or thermodynamic equation. Since here the stratified Kelvin-Helmholtz instability is considered, there is a difference in density, as well as potentially a density perturbation. Assuming inviscid flow, the density is governed by the following equation:

$$\partial_t \rho' + \boldsymbol{u} \cdot \nabla \rho' + w \frac{d\rho_0}{dz} = 0$$

Note that since the density in the discrete profile is constant above and below z = 0, the $\frac{d\rho_0}{dz}$ term is 0 above and below z = 0. Thus for all z, except z = 0, we have:

$$\partial_t \rho' + \boldsymbol{u} \cdot \nabla \rho' = 0$$

Boundary Conditions The boundary conditions used for this problem can be found below (see Navid's notes for the derivation). The first is based on analysis of the vortex sheet in between the two layers, while the second is based on the fact that total pressure must be continuous across the boundary.

$$\widehat{\boldsymbol{\eta}} = \frac{ik\widehat{\psi}_1(\eta_+)}{\sigma + ikU_0} = \frac{ik\widehat{\psi}_2(\eta_-)}{\sigma - ikU_0} \tag{3}$$

$$\rho_1 \left[(\sigma + ikU_0)\partial_z \widehat{\psi}_1(0) + \frac{k^2 g \widehat{\psi}_1(0)}{\sigma + ikU_0} \right] = \rho_2 \left[(\sigma - ikU_0)\partial_z \widehat{\psi}_2(0) + \frac{k^2 g \widehat{\psi}_2(0)}{\sigma - ikU_0} \right] \tag{4}$$

The streamfunction and the density equation Note that since the problem is considered in two dimensions, the velocities can be written in terms of a stream function as follows:

$$u' = -\partial_z \psi'$$
$$w' = \partial_x \psi'$$

Assuming a solution to the stream function of the form $\psi = \widehat{\psi}(z)e^{ikx+\sigma t}$, we then have:

$$w' = ik\psi'$$

From the density equation, and linearisation:

$$\partial_t \rho' + (U_0 + u')\partial_x \rho' = \partial_t \rho' + U_0 \partial_x \rho'$$

Substituting the stream function expressions, we have that:

$$\sigma\rho' + ikU_0\rho' = (\sigma + ikU_0)\rho' = 0$$

Since $\sigma = -ikU_0$ is a trivial solution, we assume $\rho' = 0$.

Insert in EOM and linearise In lectures it was shown that the Boussinesq approximation can be used under the perturbation conditions in a constant density fluid. Hence, it can be applied to \$ z > 0\$ and \$ z < 0\$ respectively. Thus, for z > 0:

$$\partial_t u_1' + U_0 \partial_x u_1' = -\frac{-\partial_x p'}{\rho_1}$$

$$\partial_t w_1' + U_0 \partial_x w_1' = -\frac{\partial_z p'}{\rho_1}$$

$$\partial_x u_1' + \partial_z w_1' = 0$$

And for z < 0:

$$\partial_t u_2' + U_0 \partial_x u_2' = -\frac{-\partial_x p'}{\rho_2}$$

$$\partial_t w_2' + U_0 \partial_x w_2' = -\frac{\partial_z p'}{\rho_2}$$

$$\partial_x u_2' + \partial_z w_2' = 0$$

Take ∂_z of the first equation gives:

$$(\sigma + ikU0)\partial_z^2 \widehat{\psi}_1 - \partial ik\rho_1 \partial_z p_1(z) = 0$$

And ∂_x of the second equation gives:

$$-(\sigma + ikU0)k^2 \widehat{\psi}_1 + \partial ik\rho_1 \partial_z p_1(z) = 0$$

By adding the two equations, the pressure cancels, and thus:

$$(\partial_z^2 - k^2)\widehat{\psi}_1(z) = 0$$

From this equation, and the fact the boundary condition that $\psi_1' \to 0$ as $z \to \infty$, it can be inferred that the form the solution must take is:

$$\widehat{\psi}_1(z) = Ae^{-kz}$$

By symmetry for \$ z < 0\$, and the boundary condition that $\psi_2' \to 0$ as $z \to -\infty$, it must be that:

$$\widehat{\psi}_2(z) = Be^{kz}$$

Substituting the newly found expressions into (3), and cancelling the ik:

$$\widehat{\boldsymbol{\eta}} = \frac{Ae^{-kz}}{\sigma + ikU_0} = \frac{Be^{kz}}{\sigma - ikU_0}$$

By substituting the expressions found above for $\widehat{\psi}_1(z)$ and $\widehat{\psi}_2(z)$ into the (4):

$$\rho_1 \left[(\sigma + ikU_0)(-kAe^{-kz}) + \frac{k^2 gAe^{-kz}}{\sigma + ikU_0} \right] = \rho_2 \left[(\sigma - ikU_0)kBe^{kz} + \frac{k^2 gBe^{kz}}{\sigma - ikU_0} \right]$$

Evaluating at $z = 0$, which is where the boundary condition holds, and dividing all by k:

$$A\rho_1 \left[-(\sigma + ikU_0) + \frac{gk}{\sigma + ikU_0} \right] = B\rho_2 \left[(\sigma - ikU_0) + \frac{gk}{\sigma - ikU_0} \right]$$

Using the equation found above relating A and B, we have that:

$$\frac{(\sigma + ikU_0)}{(\sigma - ikU_0)}\rho_1 \left[-(\sigma + ikU_0) + \frac{kg}{\sigma + ikU_0} \right] = \rho_2 \left[(\sigma - ikU_0) + \frac{kg}{\sigma - ikU_0} \right]$$

Which gives:

$$(\sigma + ikU_0)^2 + \rho_2(\sigma - ikU_0)^2 = gk(\rho_1 - \rho_2)$$