The Orr solution:

Physical interpretation and relevance to physical problems

Shangyu Hu – U6301345

1. The Orr solution

Problem set up

For linear Couette flow without sidewalls (infinite z-domain), the mean zonal current is:

$$\vec{U}(z) = \Lambda z\hat{x}$$

Now take two-dimensional perturbations:

$$\vec{u} = \vec{U} + \vec{u}' = [\Lambda z + u'(x,z,t)]\hat{x} + w'(x,z,t)\hat{z}$$

Linearized equations

Combined with the incompressibility, the governing equations are:

$$\begin{cases} \left(\partial_t + \Lambda z\partial_x\right)u' + w'\dfrac{\partial U}{\partial z} = -\dfrac{\partial P}{\partial x} \\ \left(\partial_t + \Lambda z\partial_x\right)w' = -\dfrac{\partial P}{\partial z} \\ \dfrac{\partial u'}{\partial x} + \dfrac{\partial w'}{\partial z} = 0 \end{cases}$$

The stream function & vorticity

Note that since we are in the two-dimensional field, the perturbation velocities can be represented by the stream function:

$$u' = -\partial_z\psi'$$

$$w' = \partial_x\psi'$$

So we can get:

$$(\partial_t + \Lambda z\partial_x)(\partial_z u' - \partial_x w') = (\partial_t + \Lambda z\partial_x)\zeta' = 0,$$

Where the vorticity ζ' is related to the stream function via

$$\zeta' = \partial_z u' - \partial_x w' = -\nabla^2\psi'$$

Given the initial vorticity perturbation is a simple harmonic function

$$\zeta'(x,z,t=0)=\zeta_0\ e^{i(k_0x+m_0z)}$$

Combined with $(\partial_t+\Lambda z\partial_x)\zeta'=0$,

We can get:

$$\zeta'(x,z,t)=\hat{\zeta}(z,t)e^{ik_0x}=\zeta_0\ e^{i[k_0x+(m_0-k\Lambda t)z]}$$

Similarly,

$$\psi'(x,z,t)=\psi_0\ e^{i[k_0x+(m_0-k\Lambda t)z]}$$

Combined with $\zeta'=\partial_z u'-\partial_x w'=-\nabla^2\psi'$, we can get:

$$\psi_0=\frac{\zeta_0}{{k_0}^2+(m_0-k\Lambda t)^2}$$

$$\psi'=\frac{\zeta_0}{{k_0}^2+(m_0-k\Lambda t)^2}e^{i[k_0x+(m_0-k\Lambda t)z]}$$

The contour plots of stream function

Mathematically, the stream function is defined in such way that the constant lines of stream function are streamlines in a 2D incompressible fluid flow, and its derivatives give velocity components of a particular perturbation flow situation. And from the solution of the stream function ($\psi'=\frac{\zeta_0}{{k_0}^2+(m_0-k\Lambda t)^2}e^{i[k_0x+(m_0-k\Lambda t)z]}$) we can see that if we can fix the $k_0, m_0, \Lambda, \zeta_0$ and allow t to increas, we can see how the stream function evolves with time. The following Fig 1 just shows some of the conyour plots of snapshots of ψ', which can correspond to the time that pointed in Fig 2.

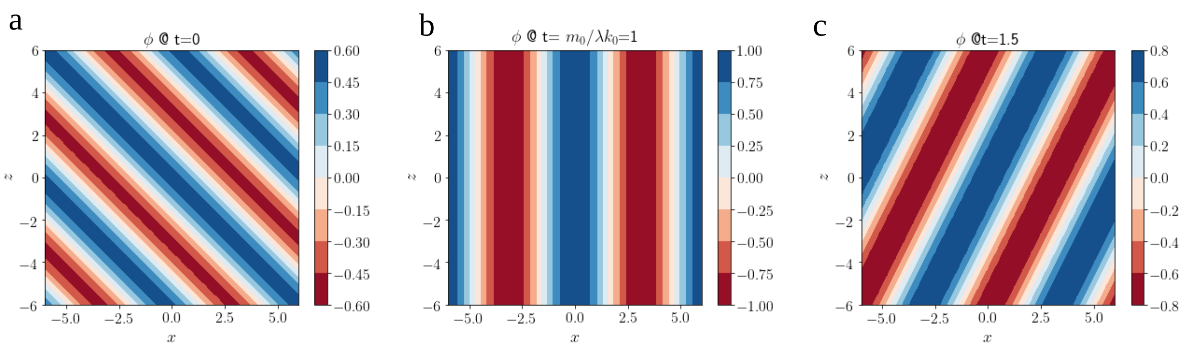


Fig 1. The stream function for the simple Orr solution, at three different times: (a) t=0, (b) $t=\frac{m_0}{k_0\Lambda}=1$, (c) t=1.5, all with $k_0=m_0=1, \Lambda=1, \zeta_0=1$.

As we can see in Fig 1, "blue belts" mark stream function ridges, and those blue contour regions denote positive stream function, red contour area denote negative steam function. The contour lines always remain straight but rotated clockwise in the x-z plane for positive Λ where the mean flow is $U = \Lambda z$. Moreover, if you pay attention to the magnitude of the colorbar, you would notice that the stream function has time dependent amplitudes (as time increase, the amplitude of the stream function starts to increase; when $t = \frac{m_0}{k_0 \Lambda} = 1$, the system reaches its maximum perturbation energy, and the amplitude would also reach its maximum; then the amplitude would start to decay). This is mainly because the perturbation energy $E = \int dx \int dz \ \frac{1}{2}(u'^2 + w'^2) = \int dx \int dz \ \frac{1}{2}(\partial_z \psi'^2 + \partial_x \psi'^2)$, so the trend of the amplitude of the stream function is consistent with the trend of perturbation energy change.

The instability in this problem

When averaged over a wavelength in x,

$$E(t) = \frac{1}{2}(u'^2 + w'^2) = \frac{1}{4}\frac{\zeta_0{}^2}{k^2 + (m_0 - k_0 \Lambda t)^2}$$

$$\Rightarrow \frac{E(t)}{E(0)} = \frac{k_0{}^2 + m_0{}^2}{k_0{}^2 + (m_0 - k_0 \Lambda t)^2}$$

From the above relationship, if we fix some values ($k_0 = m_0 = 1, \Lambda = 1$), we can compute how the perturbation energy varies with time.

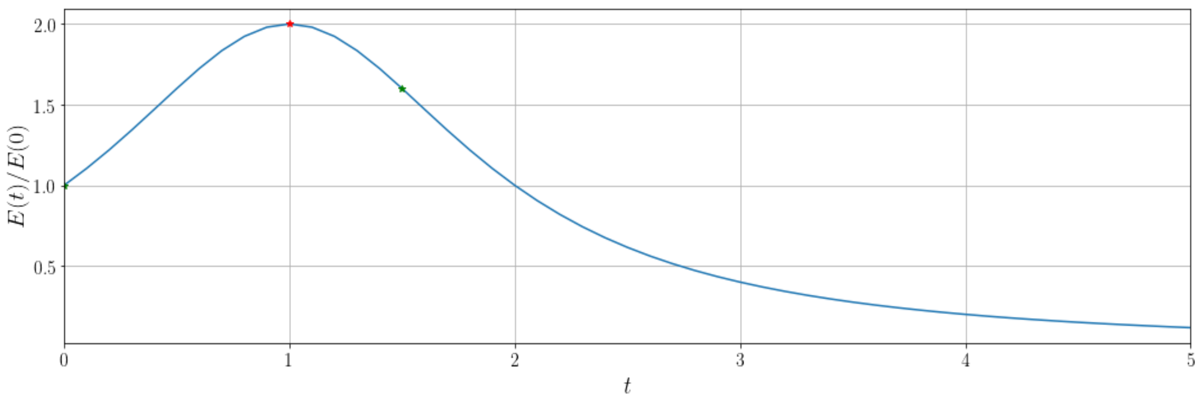


Fig 2. How the perturbation energy evolves with time. Two green dots show two cases when t=0 and t=1.5, which are corresponding to the stream function in Fig 1(a) and 1(c); whereas the red dot denotes the case when the system reaches its maximum perturbation energy when $t = \frac{m_0}{k_0 \Lambda} = 1$,which corresponds to then stream function in Fig 1(b).

As we can see in Fig 2, the perturbation energy initially grows, when $t = \frac{m_0}{k_0 \Lambda}$, the perturbation energy reaches its maximum $E_{\max} = 1 + \frac{k_0^{\ 2}}{m_0^{\ 2}}$. And as time continues to increase, the perturbation energy starts to decay.

And what if we change the x-wavenumber k? Will k effect the perturbation energy?

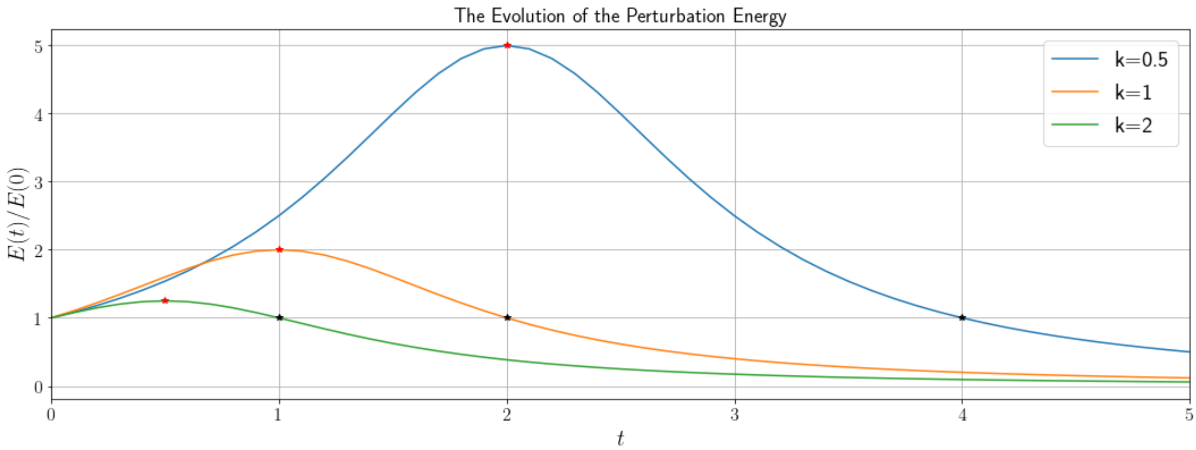
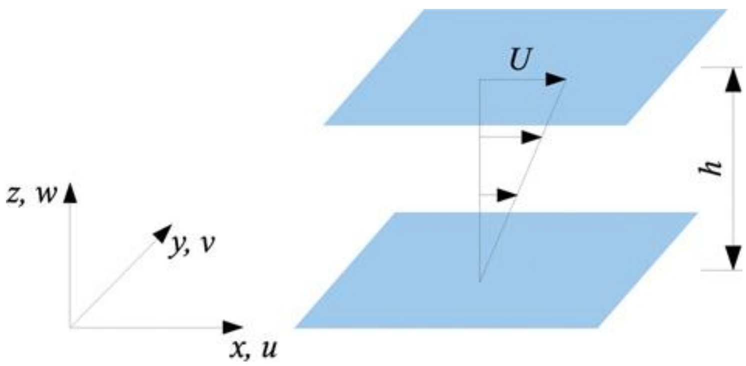


Fig 3. The evolution of the perturbation energy with different k. All with $m_0 = 1, \Lambda = 1$.

The answer is yes. As we can see in the time ($t|_{E_{\max}} = \frac{m_0}{k_0 \Lambda}$) and energy ($\left.\frac{E(t)}{E(0)}\right|_{\max} = 1 + \frac{m_0^{\ 2}}{k_0^{\ 2}}$) equations that the system reaches it maximum perturbation energy, if we increase k, the $t|_{E_{\max}}$, $\left.\frac{E(t)}{E(0)}\right|_{\max}$ would all decrease. Which means that if k increase, the system has less time to take to reach its maximum energy and also the magnitude of the maximum energy would shrink.

2. Couette flow with boundaries at z=-0.5h, 0.5h



In this case, we add boundaries to see how the system would change with time.

Governing equations & Operator

From the Orr solution we know that:

$$\begin{cases} \psi' = \hat{\psi}(z,t)e^{ikx} = \psi_0 e^{i[kx+(m-k\Lambda t)z]} \\ \\ (\partial_t + \Lambda z \partial_x)(\partial_z u' - \partial_x w') = 0 \end{cases}$$

So,

$$\partial_t \hat{\psi} = A\hat{\psi} = -ik\nabla^{-2}\Lambda z \nabla^2 \hat{\psi}$$

Here we can obtain the operator A:

$$A = -ik(\partial_z{}^2 - k^2)^{-1}\Lambda z(\partial_z{}^2 - k^2).$$

Here the $\partial_z{}^2$ operator we use the zero boundry condition (for display convinence, I just show 5 rows and 5 columns):

```
[[-2.  1.  0.  0.  0.]
 [ 1. -2.  1.  0.  0.]
 [ 0.  1. -2.  1.  0.]
 [ 0.  0.  1. -2.  1.]
 [ 0.  0.  0.  1. -2.]]
```

Moreover, since the perturbation energy

$$E = \frac{1}{2}(u'^2 + w'^2) = \psi^{?} M \psi$$

which gives M:

$$M = -(\partial_z{}^2 - k^2).$$

Thus we can get A_M and do the further eigen analysis.

The instability in this problem

If we perturb the system with the operator A and do the eigen analysis, the maximum real part of the eigen value would be the growth rate σ ($\psi' = \psi(z)e^{ikx+\sigma t}$). Then if we fix the Λ and allow k to vary, we can see how σ change with k.

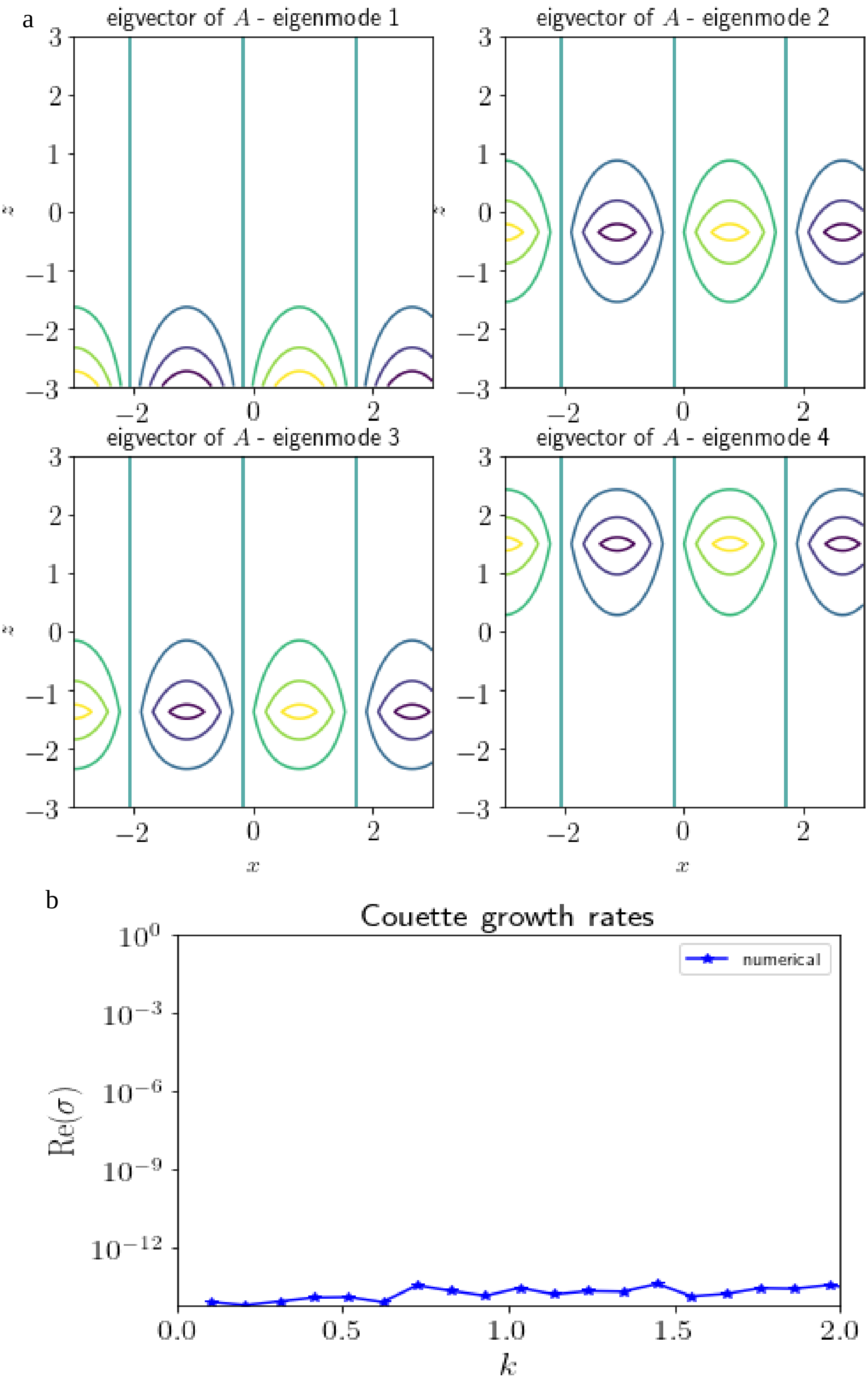


Fig 4. (a)Some of the eigenvectors of A. (b) The growth rate σ (maximum real part of the eigenvalue when doing the eigen analysis of A) with changing k.

By doing the eigen analysis of operator A, we can get multiple eigenvectors (eigenmodes in Fig 4b), but if we perturb the system with all the eigenmodes, there are no model instability in this couette problem as the growth rate is always near 0.

Perturb the system with multiple operators ($A_M, A_M{}^{\dagger}, A_M + A_M{}^{\dagger}, e^{A_M{}^{\dagger}t}e^{A_M t}$)

We know from the previous that A has multiple eigenvectors, here we use $A_M = M^{\frac{1}{2}} A M^{-\frac{1}{2}}$ to represent the A_{mean}.

By doing the eigen analysis in each case, we can see how the system evolves with time.

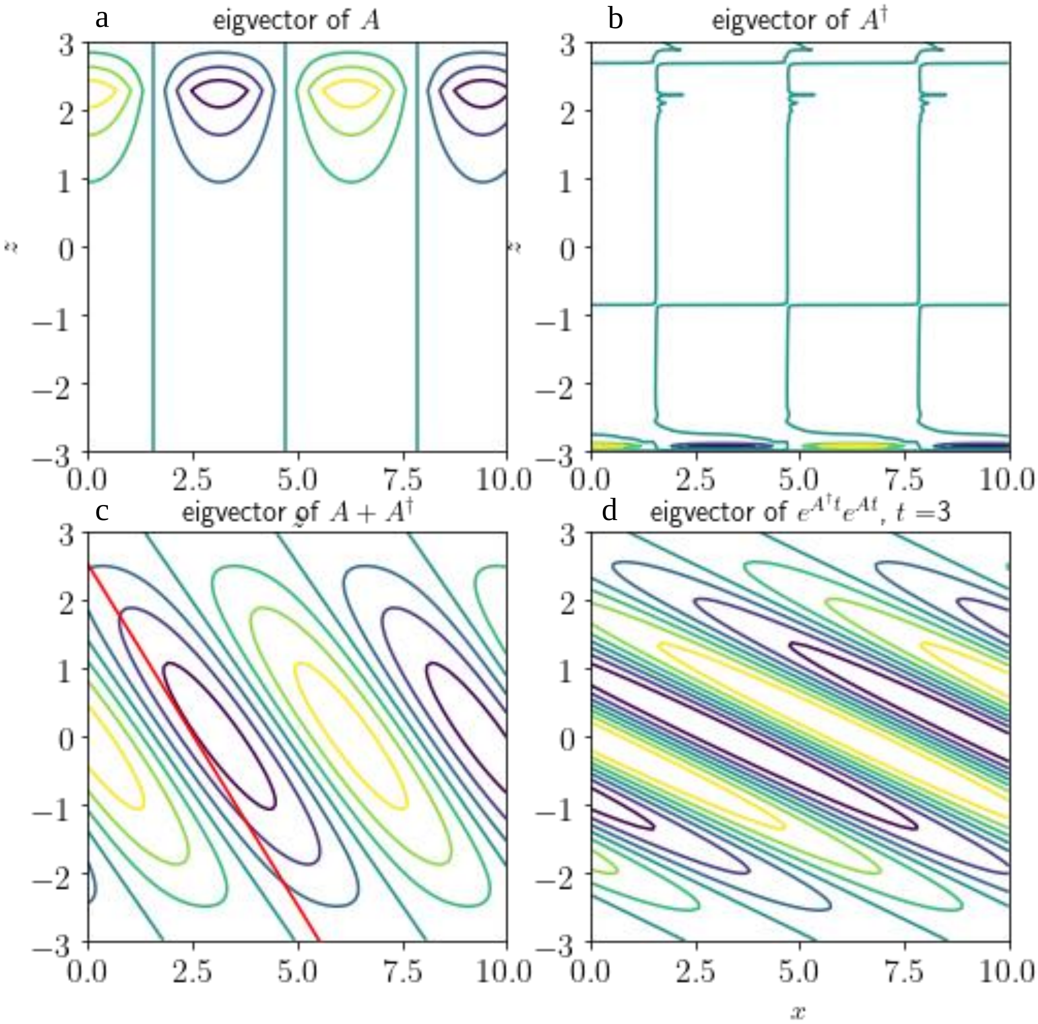


Fig 5. Eigenvector of (a) A_M . (b) $A_M{}^{\dagger}$ (it seems wrong which may caused by some numerical problem). (c) $A_M + A_M{}^{\dagger}$ (the red solid line represents a particular angle of 45°). (d) $e^{A_M{}^{\dagger}t}e^{A_M t}$.

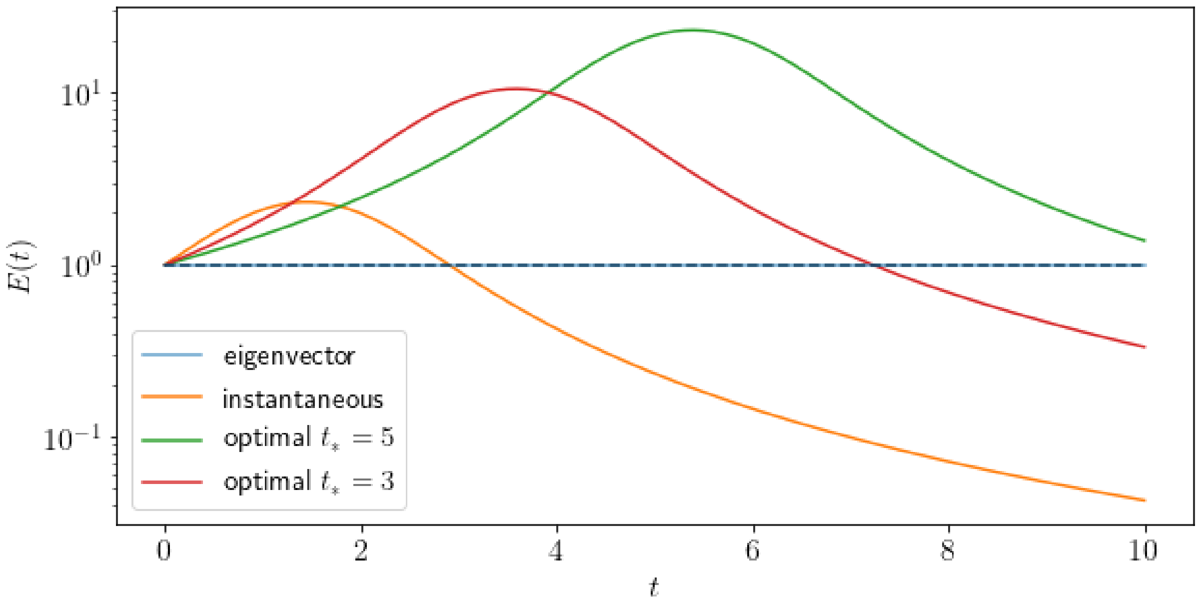


Fig 6. The evolution of the perturbation energy in each case. Blue line corresponds to Fig 5a; black dash line corresponds to Fig 5b; orange line corresponds to Fig 5c; red line corresponds to Fig 5d; green line corresponds to the Discussion part Fig 7 (Here I just plot together to see the difference).

According to Fig 5 & 6, we can see that if we give the system perturbation like $A_M, A_M{}^{\dagger}$, the system would not have energy grow (the system has no instability). But if we perturb the system by using operator $A_M + A_M{}^{\dagger}, e^{A_M{}^{\dagger}t}e^{A_M t}$, the system would have energy grow. Moreover, the bigger optimal time we set to make the energy grows the most at this optimal time, the more flatter ψ' at the beginning (the angle that initially the stream function twists would become smaller); The more time it takes for the system to reaches its maximum energy; And the larger maximum energy value the system could get, which means that the system has more energy growth and optimal time increase. Eventually the energy would decay and the system would become stable, but at the same time, such big energy growth may also affect the system and mess things up before the system eventually becomes stable.

3. Discussion

Links between the Orr solution and couette problem (with sidewalls)

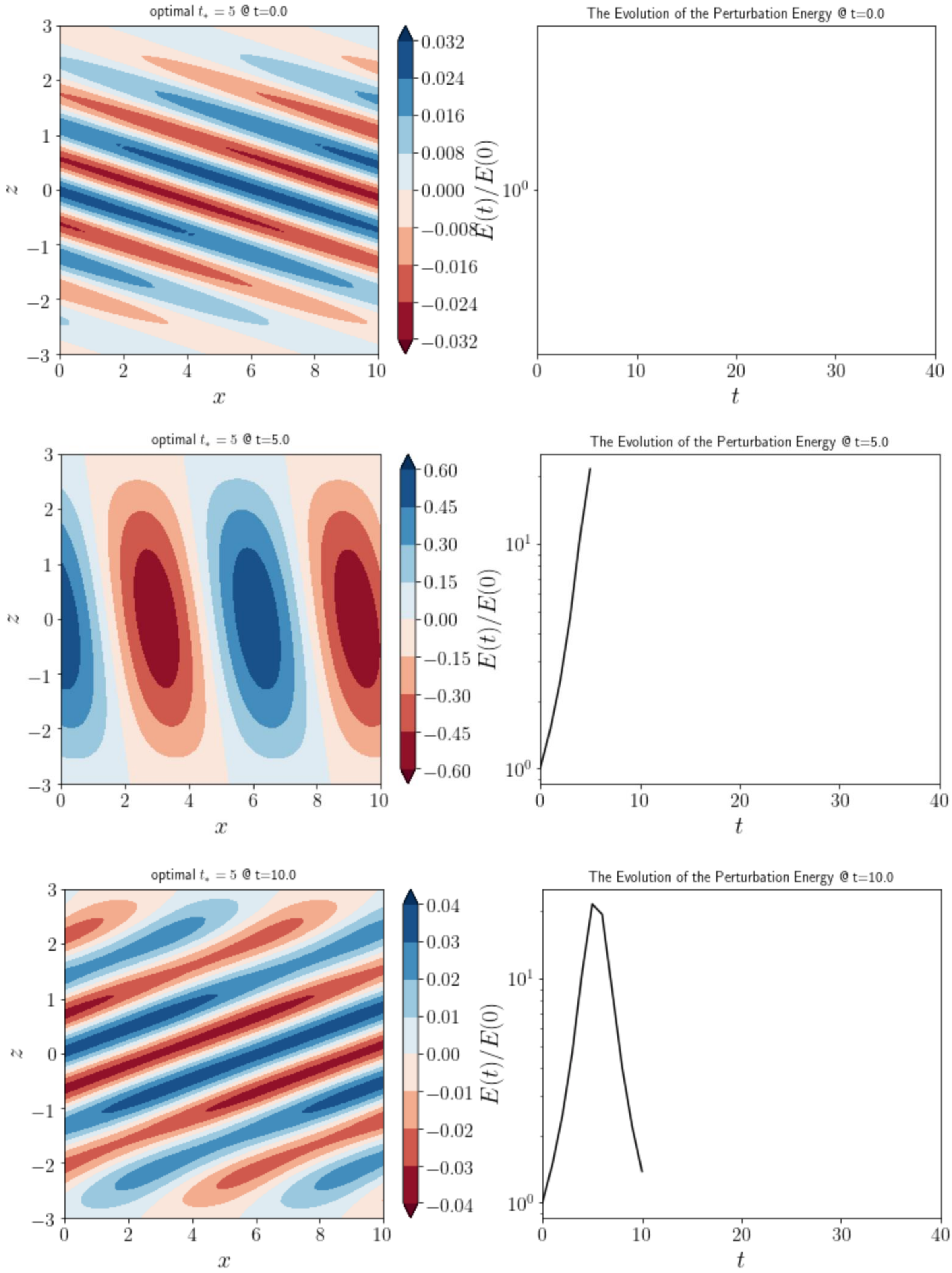


Fig 7. How the Couette system evolves with time by perturb the system using $e^{A_M^\dagger t}e^{A_M t}, t = 5$.

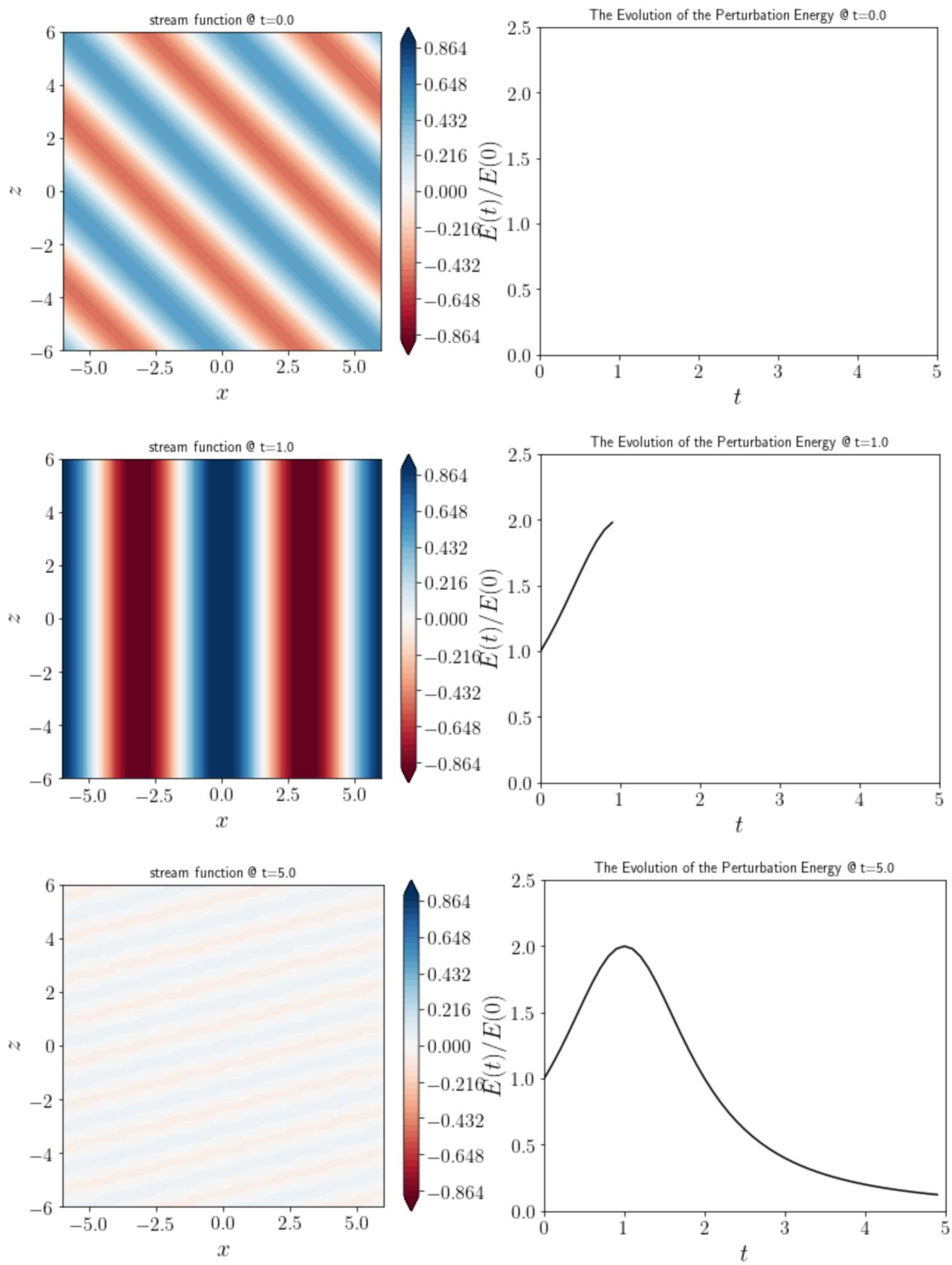


Fig 8. In the Orr solution, how does the system evolve with time. Here the three contour plots have the same colorbar magnitude, so the different color shades indicate different amplitudes of stream function.

According to Fig 7 & 8, in the two problems (Orr solution and Couette flow with boundaries), as time increase, the contour lines of the stream function rotate clockwise in the x-z plane for positive Λ where the mean flow is $U = \Lambda z$. Besides, the stream function in the two problems all have time dependent amplitudes (as time increase, the amplitude of the stream function starts to increase) and the reason is the same as I mentioned in Fig 1 analysis. Moreover, the energy in the two problems have similar trend (initially they all start to grow, then when time equals some particular time point, the energy reaches its maximum and start to decay).

One main difference that we can see in the plots is that, in the Orr solution, the contour lines always remain straight whereas in the other problem, the contours perform more like oval (or some other irregular shape). This is mainly because in the Couette problem we add the boundaries at the z=0.5h and z=-0.5h (whereas in the Orr solution it is in infinite z-domain). As you get closer to boundaries, the flow starts to be effected by the boundaries so the contour lines of the stream function cannot remain straight.

Why is the angle of the twisting is nearly parallel to 45° line

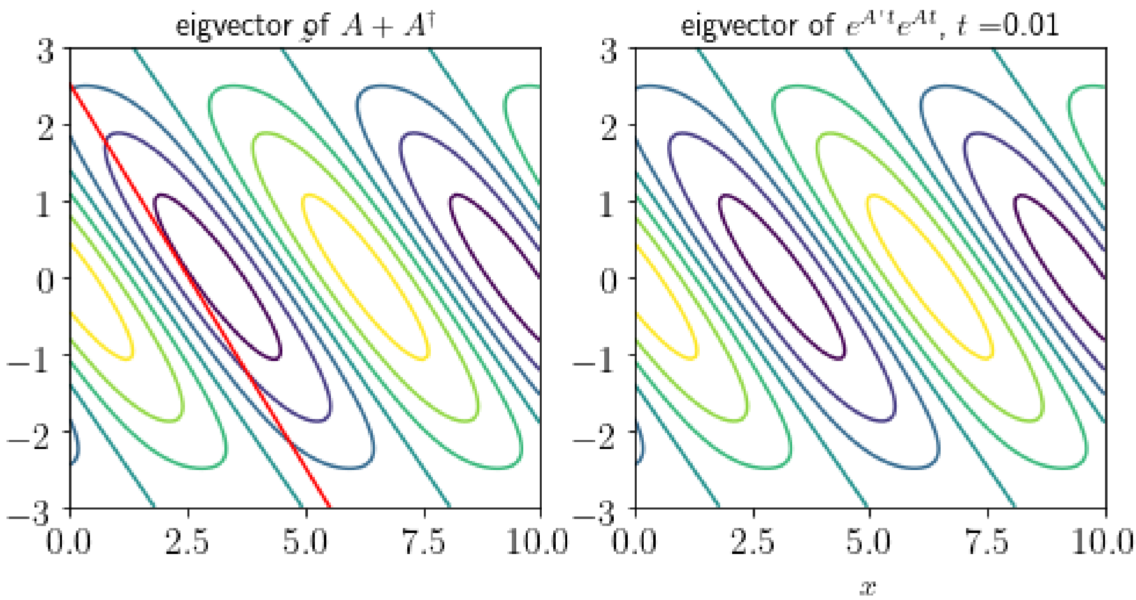


Fig 9. When $t \to 0, e^{A_M{}^{\dagger}t}e^{A_M t} \to A_M + A_M{}^{\dagger}$, which represents the maximum energy growth at the beginning (t=0).

If we want the energy grows the most at the instantaneous time:

$$g(t)=\frac{E(t)}{E(0)}=\frac{k^2+m^2}{k^2+(m-k\Lambda t)^2}$$

$$\frac{\partial g(t)}{\partial t}=\frac{\partial}{\partial t}\left(\frac{k^2+m^2}{k^2+(m-k\Lambda t)^2}\right)=\left.\frac{2k(m-k\Lambda t)\left(k^2+m^2\right)}{\left(k^2+(m-k\Lambda t)^2\right)^2}\right|_{t=0}=\frac{2km}{k^2+m^2}$$

In order to get maximum value, k and m should be equal. If k=m, the straight contour line of stream function $\psi'(x,z,t)=\psi_0\ e^{i[k_0x+(m_0-k\Lambda t)z]}$ should parallel to 45° (if we set x and z domain are of same length, we can see the exact 45°).

Why stream function inclines to left correspond to energy growth, no inclines correspond to energy maximum, and inclines to right correspond to energy decay?

First we need to see how the energy is calculated:

$$E(t)=\int dx\int dz\ \ \frac{1}{2}(u'^2+w'^2)$$

$$\begin{aligned}\frac{dE}{dt}&=\int dxdz\ \ (u'\partial_t u'+w'\partial_t w')\\&=-\int dxdz\ [u'U\partial_x u'+u'w'\partial_z U+u'\partial_x P'+w'U\partial_x w'+w'\partial_z P']\\&=-\int dxdz\ u'w'\Lambda\\&=\Lambda\int dxdz\ \ km\sin^2(kx+mz)\\&=km\Lambda\ \ \frac{1}{2}\frac{2\pi}{k}\end{aligned}$$

Here we fix $\Lambda=1$, which is positive; over one wave length $\frac{2\pi}{k}$ is also positive; so whether is $k\cdot m$ positive, zero, or negative correspond to energy grows, stops to grow (reaches its maximum) and decays ($z=\frac{const}{m}-\frac{k}{m}x$).

$k\cdot m>0$ $\qquad$ $m=0$ $\qquad$ $k\cdot m<0$