Eady problem

Eady problem is a specific example for baroclinic instability with assumptions that:

- Coriolis effect is invariable (f-plane)
- Fluid is uniformly stratified (i.e. buoyancy frequency N^2 is constant)
- Fluid has uniform velocity shear ($U = \Lambda z = \frac{U_0}{H}z$)
- The bottom and surface are flat and rigid

1 Governing equations

For quasi-geostrophic equations, the governing equations for Eady problem are:

$$\frac{\partial q}{\partial t} + \mathbf{u}\nabla q = 0, \qquad 0 < z < H \tag{1}$$

$$\frac{\partial b}{\partial t} + \mathbf{u}\nabla b = 0, \qquad z = 0 \quad and \quad H \tag{2}$$

where q is quasi-geostrophic potential vorticity and can be written with respect to stream function Ψ:

$$q = \nabla^2\Psi + \beta y + \frac{\partial}{\partial z}\frac{f_0^2}{N^2}\frac{\partial\Psi}{\partial z} \tag{3}$$

After applying assumptions to (1),

$$q = \nabla^2\Psi + \frac{H^2}{L_d^2}\frac{\partial^2\Psi}{\partial z^2} \tag{4}$$

where $\frac{H^2}{L_d^2} = \frac{f_0^2}{N^2}$ with H and L_d^2 the height of domain and the deformation radius respectively.
b is buoyancy:

$$b = f_0\frac{\partial\Psi}{\partial z} \tag{5}$$

2 Basic state

Initially, the velocity shear is uniform,

$$U(z) = \Lambda z = \frac{U_0}{H}z \tag{6}$$

As $U = -\frac{\partial \Psi}{\partial y}$, the initial stream function is:

$$\Psi = -\Lambda z y \tag{7}$$

With these conditions, we can obtain the basic state for potential vorticity and buoyancy:

$$Q = \nabla^2 \Psi + \frac{H^2}{L_d^2} \frac{\partial}{\partial z} (\frac{\partial \Psi}{\partial z}) = 0 \tag{8}$$

$$B = f_0 \frac{\partial \Psi}{\partial z} = -f_0 \Lambda y \tag{9}$$

3 Perturbation and linearisation

With introduction of an infinitesimal perturbation Ψ' in stream function, the corresponding variations in velocities, potential vorticity and buoyancy are functions of Ψ' :

$$\begin{aligned}
\Psi' &= \Psi'(x, y, z, t) \\
u' &= -\frac{\partial \Psi'}{\partial y} \\
v' &= \frac{\partial \Psi'}{\partial x} \\
q' &= \nabla^2 \Psi' + \frac{f_0^2}{N^2} \frac{\partial^2}{\partial z^2} \Psi' \\
b' &= f_0 \frac{\partial \Psi'}{\partial z}
\end{aligned} \tag{10}$$

After perturbing and linearlising governing equations (1) and (2) of this system, equations for perturbations are:

$$\frac{\partial q'}{\partial t} + U \frac{\partial q'}{\partial x} = 0, \qquad 0 < z < H \tag{11}$$

$$\frac{\partial b'}{\partial t} + U \frac{\partial b'}{\partial x} + v'(-f_0 \Lambda) = 0, \qquad z = 0 \quad or \quad z = H \tag{12}$$

Substitute perturbations (10) then 11 and 12 become:
For interior fluid, 0< z <H,

$$(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x})(\nabla^2 \Psi' + \frac{f_0^2}{N^2} \frac{\partial^2}{\partial z^2} \Psi') = 0 \tag{13}$$

For boundaries, $z = 0$ or $z = H$,

$$(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}) \frac{\partial \Psi'}{\partial z} - \Lambda \frac{\partial \Psi'}{\partial x} = 0 \tag{14}$$

These two equations (13) and (14) are eigenfunctions for this problem.

4 Solution

Suppose the solution is periodic in both x and y directions with the form:

$$\Psi' = \phi(z)e^{ily}e^{ik(x-ct)} \tag{15}$$

Therefore (11) and (12) become:

$$(\Lambda z - c)(\frac{H^2}{L_d^2}\frac{d^2}{dz^2} - (k^2+l^2))\phi = 0 \tag{16}$$

$$\begin{aligned} c\frac{d}{dz}\phi(0) + \Lambda\phi(0) &= 0 \\ (c-U_0)\frac{d}{dz}\phi(H) + \Lambda\phi(H) &= 0 \end{aligned} \tag{17}$$

According to (16), if $\Lambda z \neq c$, then

$$\frac{H^2}{L_d^2}\frac{d^2}{dz^2}\phi - (k^2+l^2)\phi = 0 \tag{18}$$

so the solution is

$$\phi(z) = Acosh\frac{\mu}{H}z + Bsinh\frac{\mu}{H}z \tag{19}$$

where $\mu^2 = L_d^2(k^2+l^2)$and μ is known as 'horizontal wavenumber'. However, coefficients A and B still remain unknown. To explore A and B, substitute this solution into boundary conditions (17) and this leads to:

$$\begin{aligned} U_0A + \mu cB &= 0 \\ [(c-U_0)\mu sinh\mu + U_0cosh\mu]A + [(c-U_0)\mu cosh\mu + U_0sinh\mu]B &= 0 \end{aligned} \tag{20}$$

Although we are still unable to find solutions for A and B because (20) are homogenous functions, if A and B have non-trivial solutions, the determinant of coefficient matrix should be zero, resulting to a constrain condition on wave speed c:

$$c^2 + U_0^2[\mu^{-1}coth\mu - \mu^{-2}] - U_0c = 0 \tag{21}$$

the solution for (21) is

$$c = \frac{U_0}{2} \pm \frac{U_0}{\mu}\sqrt{\frac{\mu^2}{4} - \mu coth\mu + 1} \tag{22}$$

It can be rearranged because

$$\begin{aligned} \frac{\mu^2}{4} - \mu coth\mu + 1 &= \frac{\mu^2}{4} - \frac{\mu}{2}(tanh\frac{\mu}{2} + coth\frac{\mu}{2}) + coth\frac{\mu}{2}tanh\frac{\mu}{2} \\ &= (\frac{\mu}{2} - coth\frac{\mu}{2})(\frac{\mu}{2} - tanh\frac{\mu}{2}) \end{aligned} \tag{23}$$

(21) becomes:

$$c = \frac{U_0}{2} \pm \frac{U_0}{\mu}[(\frac{\mu}{2} - coth\frac{\mu}{2})(\frac{\mu}{2} - tanh\frac{\mu}{2})]^{1/2} \tag{24}$$

c can be complex if $(\frac{\mu}{2} - coth\frac{\mu}{2})(\frac{\mu}{2} - tanh\frac{\mu}{2})$ is negative. Rewrite Ψ' as:

$$\Psi' = \phi(z)e^{ily}e^{kc_i t}e^{ik(x-c_r t)} \tag{25}$$

Imaginary part c_i is associated with grow rate while real part c_r is related to phase speed. The instability exists on condition that c has imaginary part, so $(\frac{\mu}{2} - coth\frac{\mu}{2})(\frac{\mu}{2} - tanh\frac{\mu}{2})$ should be less than zero. Since $\frac{\mu}{2} - tanh\frac{\mu}{2} > 0$ is always true, we need $\frac{\mu}{2} < coth\frac{\mu}{2}$ for instability. Therefore the condition for μ that must be satisfied is $\mu < 2.4$.

5 Results

To simulate this problem numerically, we follow the same typical atmosphere parameters as Vallis [2017].

$$H = 10km \qquad U_0 = 0.1m/s \qquad N = 10^{-2} \qquad L_d = 1000km \tag{26}$$

Variation of growth rate (kc_i) with nondimensional zonal wave number (kL_d) is shown in Fig. 1 (meridional wave number l is zero). Solid line is analytical solution (21) while stars are solutions of eigenfunctions using numerical method.

Growth rate increases with zonal wave number at the beginning and decreases to zero after peaking at above 0.3 when zonal wave number is around 1.6. Therefore when $l = 0$, the greatest instability is gained at $k = 1.6/L_d$.

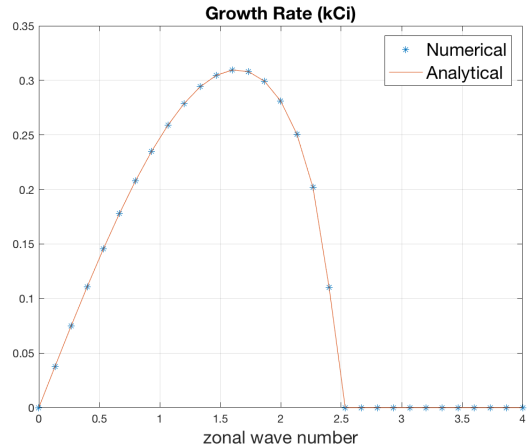

Figure 1: Growth rate computed from numerical and analytical method. Wave number is multiplied by L_d and growth rate is multiplied by L_d/U_0 in this plot for nondimensionalisation.

The effect of wave number in both zonal and meridional dimension on growth rate is shown in Fig. 1. For any given k, the maximum growth rate occurs when l is small, indicating the meridional scale is large.

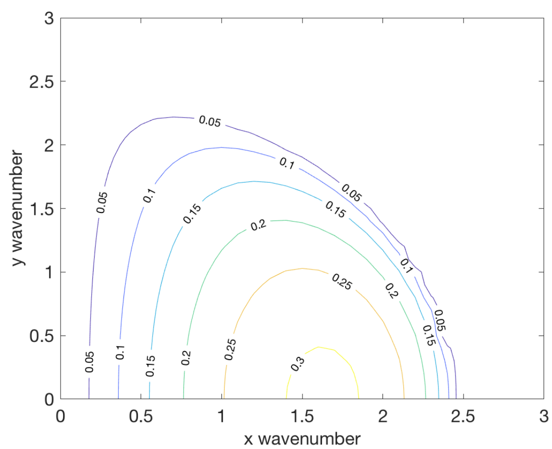

Figure 2: Growth rate contour.Wave number is multiplied by L_d and growth rate is multiplied by L_d/U_0 in this plot for nondimensionalisation.

Using numerical method, imaginary and real part of c against zonal wave number is shown in Fig. 3. For instability, c_r is 0.5 which is consistent with the analytical solution (21). This indicates that the perturbation propagates eastwards at the speed of $\frac{1}{2}U_0$. c_i declines with zonal wavenumber and vanishes around $k = 2.5/L_d$.

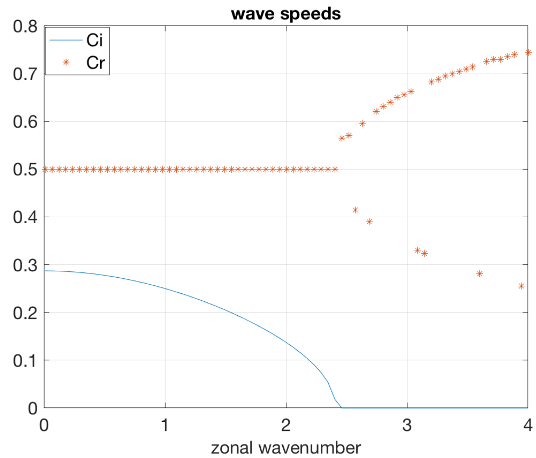

Figure 3: Real and imaginary part of wave speed as a function of zonal wave number.Wave number is multiplied by L_d and growth rate is multiplied by L_d/U_0 in this plot for nondimensionalisation.

Stream function Ψ', meridional velocity v' and potential vorticity q' are calculated for modally unstable(Fig. 4), netural(Fig. 5) and stable (Fig. 6). These figures are plotted with $l = 0$ and $k = 1.6/L_d$ but different eigenvalues (positive for unstable, nearly zero for netural and negative for stable) and corresponding eigenvectors. For unstable mode, stream functions tilts westwards while stable mode tilts eastwards. Netural mode is straight in each plot.

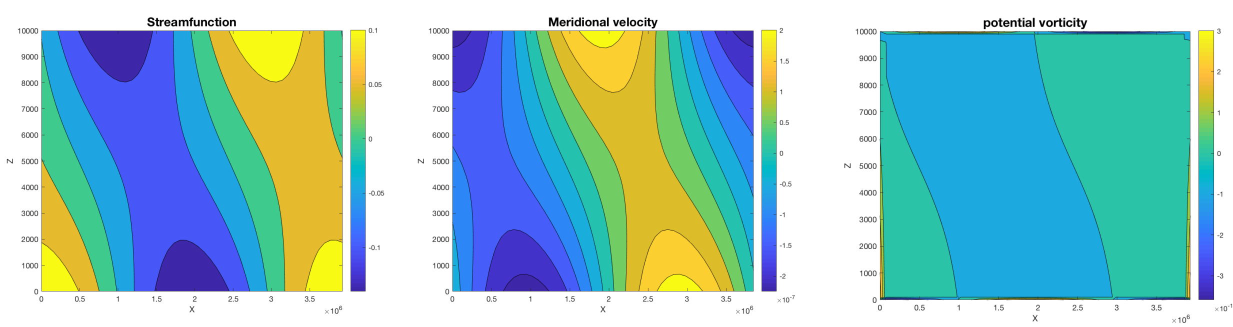

Figure 4: Stream function(left), meridional velocity v'(middle) and potential vorticity (right) for the most unstable mode.

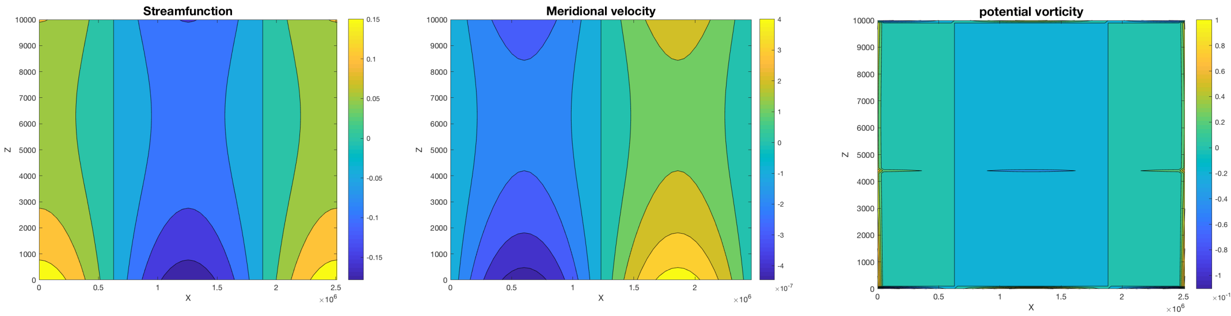

Figure 5: Stream function(left), meridional velocity v'(middle) and potential vorticity (right) for netural mode.

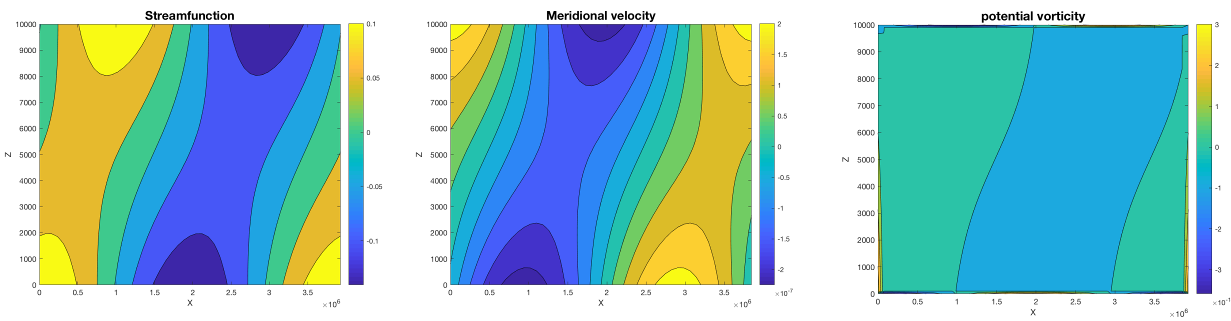

Figure 6: Stream function(left), meridional velocity v'(middle) and potential vorticity (right) for stable mode.

References

Geoffrey K. Vallis. *Barotropic and Baroclinic Instability*, page 335?378. Cambridge University Press, 2 edition, 2017. doi: 10.1017/9781107588417.010.