Instability in Two-layer Phillips' Model

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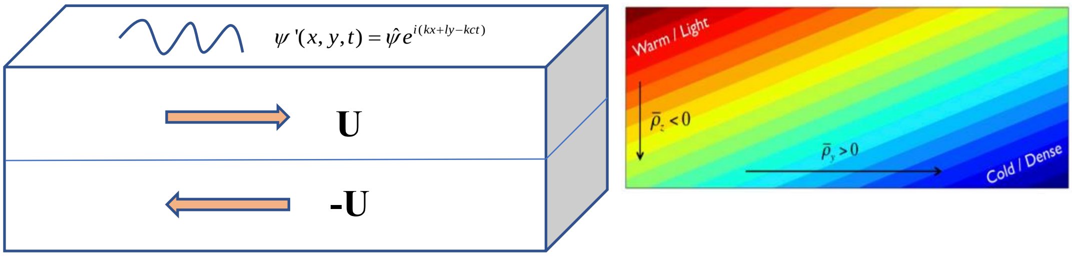
1. Introduction to Phillips problem

To study the baroclinic instability in the earth system, the Phillips' model might be the simplest example to test whether a two-layer, quasi-geostrophic system has the potential to develop instability (two-layer represents the atmosphere-ocean system, and quasi-geostrophic allow us to study small motion and perturbation under geostrophic field).



The Phillips model is an idealized two-layer model with equal thickness and different shear flow in each layer and no boundary (here we consider this problem in the stratified fluid field on β plane ($f = f_0 + \beta y$)). Firstly, we can pose the problem by starting with the potential vorticity equations:

$$q = \nabla^2\psi + \beta y + \frac{\partial}{\partial z}\left(\frac{f_0^{\,2}}{N^2}\frac{\partial \psi}{\partial z}\right) \tag{1}$$

Because the potential vorticity equals the curl of velocity plus the Coriolis effects and the buoyancy effects on the vorticity, and either the velocity field or buoyancy field can be expressed by the stream function, so the potential vorticity can be expressed by the stream function ψ. Below are the quasi-geostrophic potential vorticity equations for each layer:

$$\begin{cases} q_1 = \nabla^2\psi_1 + \beta y + \dfrac{1}{2}k_d^{\,2}\left(\psi_2 - \psi_1\right) \\ q_2 = \nabla^2\psi_2 + \beta y + \dfrac{1}{2}k_d^{\,2}\left(\psi_1 - \psi_2\right) \end{cases} \tag{2}$$

Here $\frac{1}{2}k_d{}^2 = \left(\frac{2f_0}{NH}\right)^2$, H is the total depth of the domain, N is the Brunt-Väisälä frequency and k_d can also expressed by the deformation radius L_d ($k_d = \sqrt{8}\frac{f_0}{NH} = \frac{\sqrt{8}}{L_d}$).

Given the velocity field of each layer ($U_1 = U\hat{x}, U_2 = -U\hat{x}$), so the basic state of stream function can be solved by this velocity field ($u = \frac{\partial \psi}{\partial y}$):

$$\psi_1 = -Uy, \quad \psi_2 = Uy \tag{3}$$

The basic state potential vorticity is then given by:

$$Q_1 = \beta y + k_d{}^2 Uy, \quad Q_2 = \beta y - k_d{}^2 Uy \tag{4}$$

And because of the potential vorticity conservation,

$$\frac{dq}{dt} = 0 \tag{5}$$

The linearized potential vorticity equation is

$$\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + v\frac{\partial Q}{\partial y} = 0 \tag{6}$$

So for each layer:

$$\begin{aligned}(\partial_t + U\partial_x)[\nabla^2\psi_1' + \frac{1}{2}k_d{}^2(\psi_2' - \psi_1')] + (\beta + k_d{}^2U)\partial_x\psi_1' = 0\\ (\partial_t - U\partial_x)[\nabla^2\psi_2' + \frac{1}{2}k_d{}^2(\psi_1' - \psi_2')] + (\beta - k_d{}^2U)\partial_x\psi_2' = 0\end{aligned} \tag{7}$$

For simplicity, here we set the problem in a square, doubly periodic domain and seek for solutions of the following form:

$$\psi' = \hat{\psi}e^{i(kx+ly-kct)} \tag{8}$$

Here k is the x- wavenumber and l- is the y- wavenumber.

2. Solutions in Phillips problem

From (7) and (8) we can convert the potential vorticity equations into the eigenvalue problem:

$$\begin{aligned} &ik(-c+U)[(-k^2-l^2)\hat{\psi}_1+\frac{1}{2}k_d{}^2(\hat{\psi}_2-\hat{\psi}_1)]+ik(\beta+k_d{}^2U)\hat{\psi}_1=0 \\ &ik(-c-U)[(-k^2-l^2)\hat{\psi}_2+\frac{1}{2}k_d{}^2(\hat{\psi}_1-\hat{\psi}_2)]+ik(\beta-k_d{}^2U)\hat{\psi}_2=0 \end{aligned} \tag{9}$$

Use the horizontal wave number $K^2=k^2+l^2$ to represent the x- and y- wavenumber re-arrange the equations, which turns out to be:

$$\begin{pmatrix} (U-c)(\frac{k_d^2}{2}+K^2)-(\beta+k_d^2U) & -\frac{k_d^2}{2}(U-c) \\ -\frac{k_d^2}{2}(U+c) & (U+c)(\frac{k_d^2}{2}+K^2)+(\beta-k_d^2U) \end{pmatrix}\begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix}=0. \tag{10}$$

which can be simplified as:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix}=0. \tag{11}$$

For non-trivial solution, [A][D] - [B][C] = 0, so we can solve c:

$$c=-\frac{\beta}{K^2+k_d{}^2}\left(1+\frac{k_d{}^2}{2K^2}\pm\frac{k_d{}^2}{2K^2}\left[1+4U^2\frac{K^4\left(K^4-k_d{}^4\right)}{\beta^2k_d{}^4}\right]^{\frac{1}{2}}\right) \tag{12}$$

The eigenvector $\begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{pmatrix}$ has multiple solutions, one of them is:

$$\begin{cases} \psi_1'(x,y,t)=Be^{i(kx+ly-kct)} \\ \psi_2'(x,y,t)=-Ae^{i(kx+ly-kct)} \end{cases} \tag{13}$$

Because there is no boundary conditional in Phillips problem, so that is not enough conditions to constrain the perturbation in one particular form.

3. Discussion

3.1 Conditions for instability

From Kelvin-Helmholtz instability we know that when $\mathrm{Re}(\sigma) > 0$ we have the instability because then a factor $e^{\mathrm{Re}(\sigma)t}$ appears to grow exponentially. Similarly, in Phillips problem, we have a factor e^{-ikct}. So in order to get instability, c must have imaginary part, then $\sigma = -ikct = -\mathrm{Im}(c)kt$ is a real number and it is positive.

From (12) we know that c is determined by β, K, k_d and U, so if we want to get the instability, it might constrain the relationship of the above four variables.

3.1.1 Constrain for U – minimum shear for instability

If c has imaginary part, the value inside the sqrt needs to be negative. So:

$$1 + 4U^2 \frac{K^4\left(K^4 - k_d{}^4\right)}{\beta^2 k_d{}^4} < 0$$
$$\Rightarrow K^4\left(k_d{}^4 - K^4\right) > \frac{1}{4U^2}\beta k_d{}^4$$

$$\xrightarrow{LHS \quad \max:\ K^4 = k_d{}^4 - K^4 \Rightarrow K^4 = \frac{1}{2}k_d{}^4} U > \frac{\beta}{k_d{}^2}$$

So the minimum shear flow difference:

$$dU = U_1 - U_2 = 2U > \frac{2\beta}{k_d{}^2},$$

which means the instability only occurs when the shear difference between the top and bottom layer is greater than $\frac{2\beta}{k_d{}^2}$.

3.1.2 Constrain for K – high and low wavenumber cut-off

If c has imaginary part:

$$\frac{4U^2K^4\left(K^4 - k_d{}^4\right)}{\beta^2 k_d{}^4} < -1$$
$$\Rightarrow 4K^4\left(k_d{}^4 - K^4\right) > \frac{\beta^2}{U^2}k_d{}^4$$
$$\Rightarrow 4U^2K^4k_d{}^4 - \beta^2 k_d{}^4 > 4U^2K^8$$

$$\xrightarrow{ocean:\ \beta\sim10^{-10}\ll U\sim10^{-1}} k_d^{\ 4} > K^4$$
$$\Rightarrow K < k_d$$

This is the high-wavenumber cut-off.

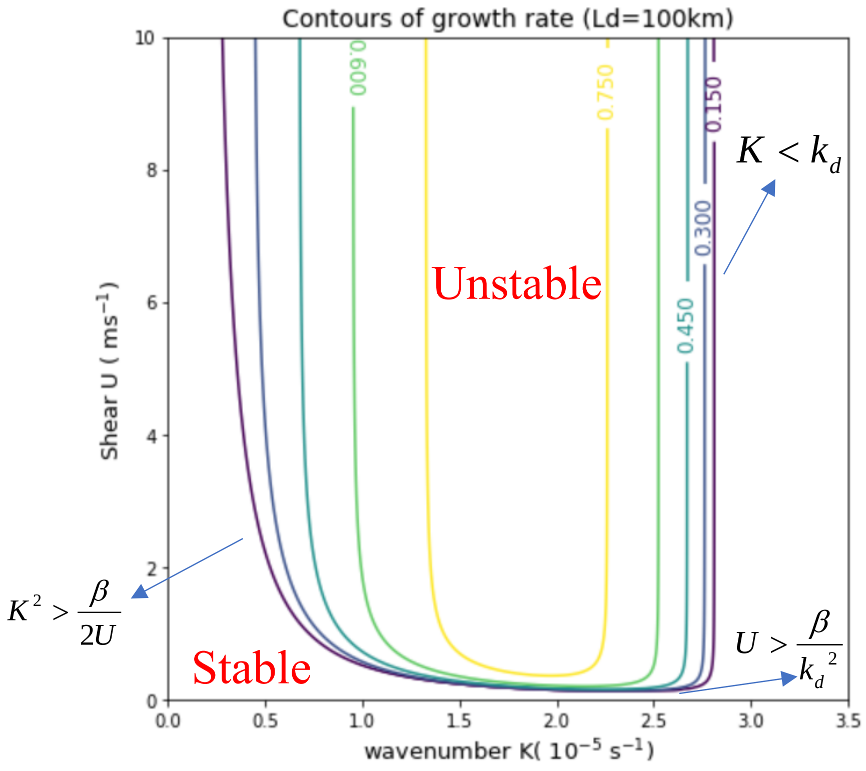
Moreover, if the horizontal wavenumber is small ($K \ll k_d$):

$$\frac{4U^2K^4\left(K^4-k_d^{\ 4}\right)}{\beta^2 k_d^{\ 4}} < -1$$
$$\xrightarrow{K\ll k_d} 4U^2K^4k_d^{\ 4} > \beta^2 k_d^{\ 4}$$
$$\Rightarrow 4U^2K^4 > \beta^2$$
$$\Rightarrow K^2 > \frac{\beta}{2U}$$

This is the low-wavenumber but off.

The above two conditions suggest that if the horizontal wavenumber K satisfies $\sqrt{\frac{\beta}{2U}} < K < k_d$, instability occurs.

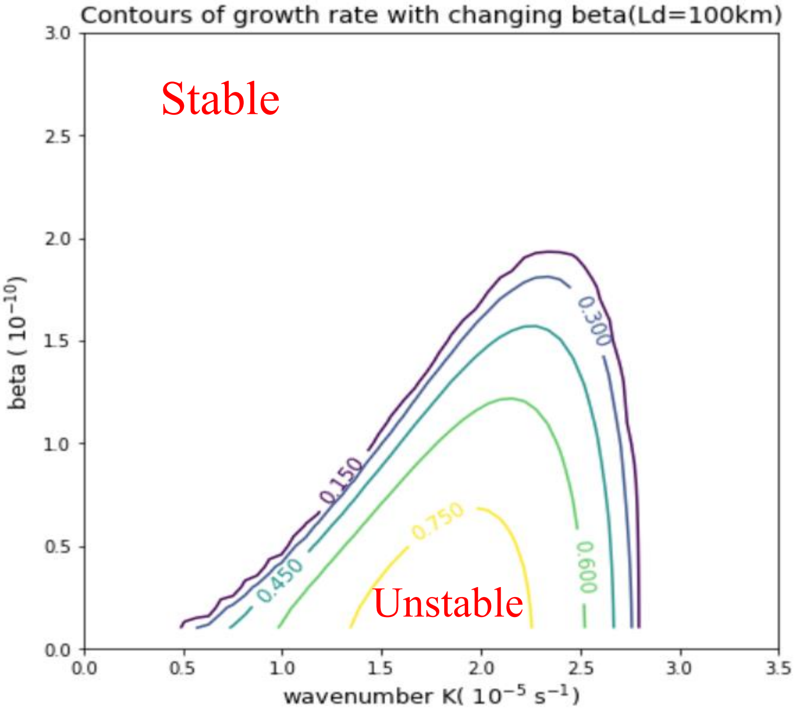
Combine the three constrains mentioned above, we can draw the growth rate as following (fix the $L_d \sim 10^5,\ \beta \sim 10^{-10}$):



From this figure, we can see that as the shear increases, the wavenumber at which the growth rate is maximum decreases slightly (comment made by Vallis, 2017. But I only can see this phenomenon when $K < 2\times10^{-5}s^{-1}$. And what about the other part?).

3.2 Instability influenced by varying β

Fix the $L_d = 10^5 m$, $U = 0.25ms^{-1}$:



As we can see from the figure, increasing β can stabilize the system, which means that if the earth rotates faster, the earth system is more likely to be stable.

3.3 Special cases

In general case ($U \neq 0, \beta \neq 0$),

$$c = -\frac{\beta}{K^2 + k_d^{\ 2}}\left(1 + \frac{k_d^{\ 2}}{2K^2} \pm \frac{k_d^{\ 2}}{2K^2}\left[1 + 4U^2\frac{K^4\left(K^4 - k_d^{\ 4}\right)}{\beta^2 k_d^{\ 4}}\right]^{\frac{1}{2}}\right)$$

As c is determined by β, K, k_d and U, here we discuss what is going to happen if we set one of these variables to be 0.

3.3.1 No shear: $U = 0, \beta \neq 0$

$$c_1 = -\frac{\beta}{K^2}, c_2 = -\frac{\beta}{K^2 + k_d{}^2}$$

As the system doesn't satisfy the condition of minimum shear difference, so c has no imaginary part, which means that there is no instability. And as c<0, the given perturbation would propagate westwards.

3.3.2 No buoyancy change: $U \neq 0, \beta \neq 0, k_d = 0$

$$c = -\frac{\beta}{K^2}$$

Also no instability, and perturbation would propagate westwards.

3.3.3 No β: $U \neq 0, \beta = 0$

$$c = \pm U \left(\frac{K^2 - k_d{}^2}{K^2 + k_d{}^2} \right)^{\frac{1}{2}}$$

So the minimum shear $U > \frac{\beta}{k_d{}^2} = 0$, which mean there is an instability in all values of U. Moreover, $\sqrt{\frac{\beta}{2U}} = 0 < K < k_d$, so the instability only occurs when $K < k_d$ (no low-wavenumber cut-off, no minimum shear flow difference).

For growth rate,

$$\sigma = Uk \left(\frac{K^2 - k_d{}^2}{K^2 + k_d{}^2} \right)^{\frac{1}{2}}$$

So for a given k, the maximum growth rate occurs when l=0. So the maximum growth rate occurs when $k = 2^{-\frac{1}{4}} k_d \sim 0.634 k_d = 1.79 / L_d$.

Reference

Vallis 2017, ch. 9, pp. 349-369 (or in the 2006 edition, ch. 6.4-6.7)