The Eady Problem in 3D

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1. Introduction

In 1949, Eric Thomas Eady first published the model which now carries his name [Eady, 1949]. [Pedlosky, 1987] stated the following:

> An elegantly simple model of baroclinic instability which shows the instability process in its purest form was introduced by Eady (1949).

It was a major contribution to the field of instability theory as well as weather forecasting, since it explained how baroclinic instability occurs. It showed that small scale perturbations can grow into large scale weather patterns. In short, baroclinic instability occurs when there is a uniform velocity shear, which creates a horizontal buoyancy gradient perpendicular to the velocity. This allows potential energy from the buoyancy gradient to be converted to kinetic energy, causing instability. The model is often used for the troposphere, since it assumes a uniform velocity shear.

In this report, first, the governing equations, the basic state, and 3-dimensional perturbations are considered. Then, edge wave solutions to the 2 dimensional perturbation equations are found by approximating them as a linear combination of the solution to the interior and top equation and the solution to the interior and bottom equation. This allows for the problem to be studied using linear algebra as a reduced 2x2 system. Finally, 3 dimensional perturbations are studied, and the consequences for the energy of the model is explored.

2. The governing equations

We have two equations that govern the behaviour of the system, assuming quasi-geostrophy and hydrostatics. These are found by manipulating the quasi-geostrophic equations.

$$(\partial_t + u \cdot \nabla)q = 0, \text{ for } 0 < z < H \tag{2.1a}$$
$$(\partial_t + u \cdot \nabla)b = 0, \text{ for } z = 0,\ H \tag{2.1b}$$
$$q = \left(\nabla_z + \frac{f_0^2}{N^2}\partial_z^2\right)\psi \tag{2.1c}$$
$$b = f_0 \partial_z \psi \tag{2.1d}$$

Here, u denotes the velocity vector $u = (u, v, w)$. The basic state is assumed to have a uniform velocity shear in u, and $\nabla_z = \partial_x^2 + \partial_y^2$.

3. The Basic State

Initially, the velocity shear wind profile is uniform. That is, the mean flow in the x-direction $U(z)$ uniformly increases with height from the bottom $U(z) = \Lambda z$. Here, Λ is a proportionality constant and has inverse time units. In general, the velocity in the x-direction, in terms of the streamfunction, is given by:

$$U(z) = -\partial_y \psi^e \tag{3.1}$$

Thus, from (3.1), it can be inferred that the the streamfunction is given by:

$$\psi^e(y,z) = -\Lambda yz \tag{3.2}$$

Hence, the mean buoyancy gradient $B(y)$, which is given by $B(y) = f_0 \partial_z \psi^e$, in terms of the streamfunction is:

$$B(y) = -f_0 \partial_z \psi^e \tag{3.3}$$

4. 3-D Perturbation and Linearisation

Here, a perturbation $\psi'(x,y,z,t)$ is assumed. Below $\psi = \psi^e + \psi'$ is substituted into the potential vorticity equation (2.1a) and the buoyancy equation (2.1b), and the equations are simplified. The buoyancy equation splits up into two cases: One for the top of the domain and one for the bottom. Putting them in order of height:

$$\left(\partial_t + (\Lambda H - \partial_y \psi')\partial_x + (\partial_x \psi')\partial_y\right)\left(-\Lambda y + \partial_z \psi'\right) = 0, \text{ for } z = H$$
$$\left(\partial_t + (\Lambda z - \partial_y \psi')\partial_x + (\partial_x \psi')\partial_y\right)\left(\partial_x^2 + \partial_y^2 + \frac{f_0^2}{N^2}\partial_z^2\right)\psi' = 0, \text{ for } 0 < z < H$$
$$\left(\partial_t - \partial_y \psi' \partial_x + \partial_x \psi' \partial_y\right)\left(-\Lambda y + \partial_z \psi'\right) = 0, \text{ for } z = H$$

After linearising these equations, the following are obtained:

$$(\partial_t + \Lambda H \partial_x)\,\partial_z \psi' - \Lambda \partial_x \psi' = 0, \text{ for } z = H$$
$$(\partial_t + \Lambda z \partial_x)\left(\partial_x^2 + \partial_y^2 + \frac{f_0^2}{N^2}\partial_z^2\right)\psi' = 0, \text{ for } 0 < z < H$$
$$\partial_t \partial_z \psi' - \Lambda \partial_x \psi' = 0, \text{ for } z = H$$

These can be non-dimensionalised by taking the following:

$$x = \frac{NH}{f_0}\tilde{x} \tag{4.1a}$$
$$z = H\tilde{z} \tag{4.1b}$$
$$t = \frac{N}{\Lambda f_0} \tag{4.1c}$$
$$y = H\tilde{y} \tag{4.1d}$$

By substituting the expressions on the right hand side and simplifying, the non-dimensionalised governing equations are found. Note that these equations are eigenfunctions. That is, they can be written in the form: $D\tilde{\psi}' = \Lambda \tilde{\psi}'$.

$$\left((\partial_{\tilde{t}} + \partial_{\tilde{x}})\,\partial_{\tilde{z}} - \partial_{\tilde{x}}\right)\tilde{\psi}' = 0, \text{ for } z = H \tag{4.2a}$$
$$(\partial_{\tilde{t}} + \tilde{z}\partial_{\tilde{x}})\left(\partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2 + \partial_{\tilde{z}}^2\right)\tilde{\psi}' = 0, \text{ for } 0 < z < H \tag{4.2b}$$
$$(\partial_{\tilde{t}}\partial_{\tilde{z}} - \partial_{\tilde{x}})\,\tilde{\psi}' = 0, \text{ for } z = 0 \tag{4.2c}$$

From here onward, the non-dimensional parameters and variables are no longer denoted by a tilde. That is, they are all assumed to be non-dimensional, unless otherwise stated.

5. Physical Mechanism and Eady Modes

Before considering the full 3D-solution, it is helpful to understand the physical mechanism behind the propagation of the edge waves waves. These exist at the top and bottom boundary. To find the edge wave solution at the bottom boundary, the equation for the interior (4.2b) together with the equation for the bottom boundary (4.2c) is solved in two dimensions. It is assumed that the perturbation is periodic in x. Due to this homogeneity, the perturbation can be assumed to be of the form:

$$\psi'(x,z,t) = \hat{\psi}(z,t)e^{ikx} \tag{5.1}$$

Thus, the equations of motion can be found to be given by:

$$((\partial_{\tilde{t}} + ik)\,\partial_{\tilde{z}} - ik)\,\tilde{\psi}' = 0, \text{ for } z = H \tag{5.2a}$$

$$(\partial_{\tilde{t}} + \tilde{z}ik)\left(\partial_{\tilde{z}}^2 - k^2\right)\tilde{\psi}' = 0, \text{ for } 0 < z < H \tag{5.2b}$$

$$(\partial_{\tilde{t}}\partial_{\tilde{z}} - ik)\,\tilde{\psi}' = 0, \text{ for } z = 0 \tag{5.2c}$$

Note that the $\partial_y^2\psi' = 0$ since here it is assumed ψ' is not dependent on y. From (6.2b), it can be seen that $\psi'_-(x,z,t) = \hat{\psi_-}'(t)e^{-kz}e^{ikx}$. The reason for choosing $-z$ in the exponent is that the solution has to go to 0 as $z \to \infty$ at the top, and be of the highest amplitude at the bottom. By considering (6.2c), it can be seen that $\partial_t\psi' = i\psi'$. Therefore, the bottom edge wave $\psi_{\prime}$ is given by:

$$\psi'_-(x,z,t) = e^{-kz}e^{ik(x-\frac{1}{k}t)} \tag{5.3}$$

Similarly, for the top, it can be found that

$$\psi'_+(x,z,t) = e^{ik(1-z)}e^{ik\left(x-(1-\frac{1}{k})t\right)} \tag{5.4}$$

since this solution has to go to 0 as $z \to \infty$, and be at its highest amplitude at the top boundary $z = 1$. Note that the solution near the top and near the bottom have different phase speeds, which comes from the extra term of $ik\partial_z\psi'$. That is, at the top, the mean flow contributes to the overall phase speed, in addition to the perturbation, but at the bottom, only the perturbation contributes to the phase speed.

Thus, the perturbation buoyancy, which is essentially the same as temperature, is given by $b' = f_0\partial_z\psi' = -ik\psi'$ which is periodic in x. Hence, there will be an infinite sequence of cold and warm regions. First, consider the top boundary (see Fig. 1). Around a cold region, the flow is anticyclonic or counter-clockwise in the Southern Hemisphere, since it is higher in pressure relative to its surroundings. Thus, $v < 0$ to the left of the cold region, and $v > 0$ to the right. Supposing this is indeed in the Southern Hemisphere, warm air from the North flows to the left of the cold region, while cold air from the South flows to the right of the cold region. Hence, the cold region moves to the left over time. A warm region is a low-pressure area, and thus has cyclonic, clockwise flow in the Southern Hemisphere. Hence, $v > 0$ to the left of the region, while $v < 0$ to the right. Thus, cold is coming in to the left of the region, while warm air is coming in to the right. Therefore, the region moves to the left with time also. The cold regions and the warm regions move to the left. However, near the top boundary, there also exists a mean flow to the right. Hence, the net flow depends on the sum of the wave and the mean flow velocities. Near the bottom boundary, there is no mean flow. Note that at the bottom, the streamfunction has the

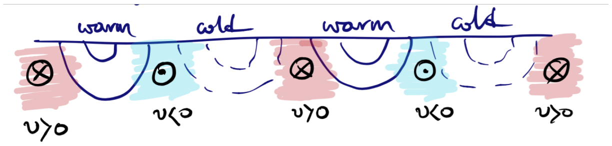

FIG. 1. Top edge wave solution

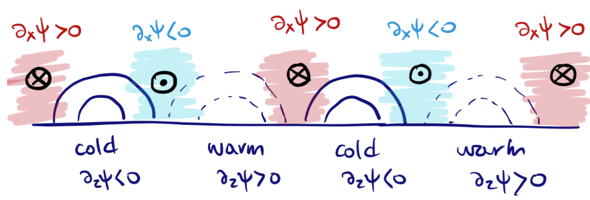

FIG. 2. Bottom edge wave solution

opposite relation to height, and therefore, the pressure is opposite, which means that the perturbation flow rotates in the opposite direction (see Fig. 2).

An Eady mode exists when the velocity at the top matches the velocity at the bottom. This allows the perturbations to either mutually reinforce each other (stable Eady mode) or cancel each other out (unstable Eady mode). Consider the unstable Eady mode first (see Fig 3). A cold region at the bottom gives rise to cold air coming in to the right. This process occurs across the entire height, and thus makes the cold region at the top even colder. Hence, the cold region at the top causes even more cold air to come in to its left, which then makes the cold region at the bottom even colder. Hence, the perturbation at the top amplifies the perturbation at the bottom. And the perturbation at the bottom amplifies the perturbation at the top. Therefore, over time, the instability grows and the system becomes unstable. Now consider the the stable Eady mode (see Fig 4). A cold region at the bottom gives rise to cold air to the right. The warm region at the top becomes colder. Simultaneously, the warm region gives rise to warm air to its left, which causes the cold region at the top to become warmer. Hence, over time, the system stabilizes. There is always likely to be an unstable Eady mode for some wavenumber due to waves consisting of an infinite number of different wavenumbers (though potentially of very small amplitudes). The solution to the system of equations can be approximated, as stated above, by taking

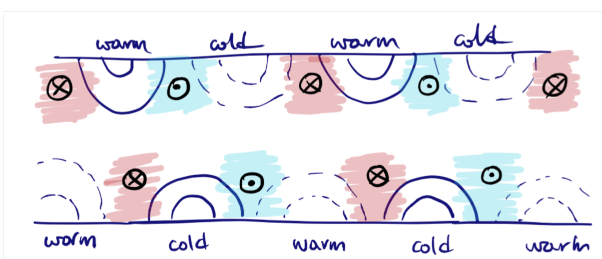

FIG. 3. Unstable Eady mode

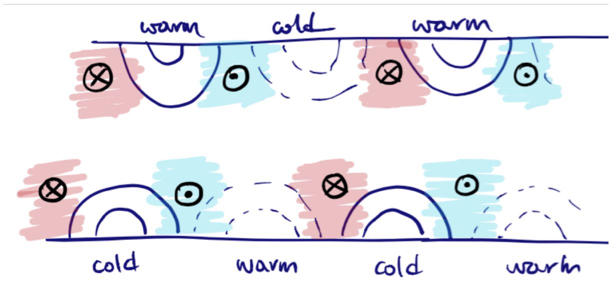

FIG. 4. Stable Eady mode

a linear combination of the top solution and the bottom solution. That is:

$$\psi' = \alpha(t)\psi'_{-} + \beta(t)\psi'_{+} \tag{5.5}$$

Substituting this into the bottom and top equation respectively gives:

$$-\alpha_t + \beta_t e^{-k} = i\alpha + i\beta e^{-k}, \text{ for } z = 0 \tag{5.6a}$$

$$-\alpha_t e^{-k} + \beta_t = \alpha(1+i)e^{-k} + \beta(1-i), \text{ for } z = 1 \tag{5.6b}$$

Note that ψ'_{-} and ψ'_{+} are evaluated at $z = 0$ and $z = 1$ respectively. After rearranging, the following are obtained:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} -1 & e^{-k} \\ -e^{-k} & 1 \end{pmatrix}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} i & ie^{-k} \\ (1+i)e^{-k} & -1-i \end{pmatrix}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

Thus, by finding the inverse of the matrix on the left hand-side and multiplying both sides by this matrix, the eigenfunction can be obtained. This matrix can be found to be given by:

$$B^{-1} = \frac{1}{-1+e^{-2k}}\begin{pmatrix} 1 & i-e^{-k} \\ e^{-k} & -1 \end{pmatrix} \tag{5.7}$$

Hence, the eigenfunction is given by:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \frac{1}{-1+e^{2k}}\begin{pmatrix} i(1-e^{2k}+k) & -ike^{k} \\ ike^{k} & i(-1+e^{2k}-ke^{2k}) \end{pmatrix}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \tag{5.8}$$

6. 3D Perturbations

To obtain the solution for 3D perturbations, now a perturbation that is assumed to be periodic in x and y is postulated. Due to homogeneity in x and y, the perturbation can be put in the form of:

$$\psi'(x,y,z,t) = \hat{\psi}(z,t)e^{ikx}e^{ily} \tag{6.1}$$

Then the equations of motions are:

$$((\partial_{\tilde{t}} + ik)\,\partial_{\tilde{z}} - ik)\,\tilde{\psi}' = 0, \text{ for } z = H \tag{6.2a}$$

$$(\partial_{\tilde{t}} + \tilde{z}ik)\left(\partial_{\tilde{z}}^2 - k^2 - l^2\right)\tilde{\psi}' = 0, \text{ for } 0 < z < H \tag{6.2b}$$

$$(\partial_{\tilde{t}}\partial_{\tilde{z}} - ik)\,\tilde{\psi}' = 0, \text{ for } z = 0 \tag{6.2c}$$

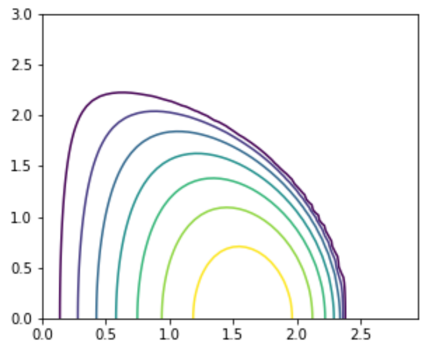

FIG. 5. Eigenmode growth rate

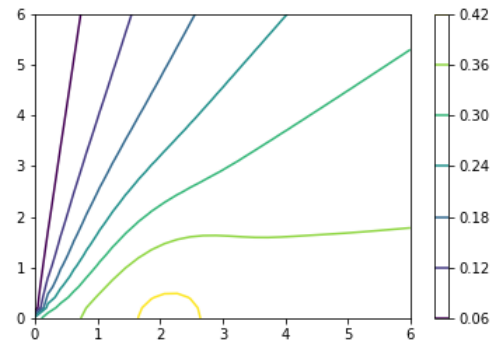

FIG. 6. Optimal growth rate for $t = 5$

The energy integrated over a wavelength in the x-direction and in the y-direction is then given by:

$$E' = \frac{NH^3\Lambda^2}{2f_0L}\int(k^2+l^2)|\hat{\psi}|^2 + |\partial_z\hat{\psi}|^2 dz \tag{6.3}$$

One of the questions to ask when working in three dimensions as opposed to two is how this affects the energy. From considering various parameters, it was found that by increasing the value of the wavenumber in the y-direction, l, the energy growth of the system is always slower. For example, see Fig. 5 for a contour plot of the highest eigenvalues for the energy growth of the Eady eigenmodes. These generally give the lowest energy growths. In addition, consider the contour plot in Fig. 6 of the optimal growth rate for $t = 5$, which shows the same result.

7. Conclusion

The governing equations of a 3-D perturbation about the basic state of the Eady model were solved for the top and bottom, and an approximation to the solution to the whole system was made by separately solving the potential vorticity equation and the buoyancy equation at the top, and the potential vorticity equation and the buoyancy equation at the bottom. The linear combination of these solutions are a good approximation for the 2D system. In addition, a conceptual explanation of the physical mechanism behind the Eady mode was given. Finally, it was found that by making the system 3-D, the energy growth decreases as the wavenumber in the y-direction is increased.

References

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