Notes from PlasmaX

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These notes will not be 100% comprehensive, as I'm making them mainly for my own use. However, if you spot any mistakes, feel free to catch me on the forums and I'll fix any mistakes.

0.1 The notes' repository

The repository for these notes is maintained here. Feel free to add any changes to the .tex files (if you don't feel like compiling the pdf don't worry, I've got that covered).

0.2 The course wiki and errata pages

The course's got a wiki page which you can check out here. I try to incorporate corrections from the errata page into these notes, but obviously I may miss some of those. If you find anything wrong, add that in there and/or say so on the forums.

0.3 Qni's Julia package

Julia along with the Equations package can be used as an open source alternative to Matlab.

Official Julia website: http://julialang.org/

Equations repository: https://github.com/jhlq/Equations.jl

To install the package type, inside Julia: Pkg.clone("https://github.com/jhlq/Equations.jl")

Example use in the form of solutions to course assignments is available in notebook format in the PlasmaXNotes repository, in the examples directory of Equations and as a html at http://artai.co/Plasma.html

The Equations package is currently in a developing stage so please submit any encountered issues.

1 Week 1. Description of the plasma state, with Paolo Ricci

Didn't start making notes until 1.5 so I'll be skimming the earlier topics.

1.1 Plasmas in nature and laboratory

- Plasma - the 4th state of matter. Heat stuff up to 11400K (= 1eV) and gases begin being ionized.
- The Sun is a miasma of incandescent plasma[1]
- Lightning is plasma (ionized air)
- Plasma diplays
- Nuclear fusion - can't really get there without turning stuff into plasma
- The word 'plasma' comes from greek $\pi\lambda\alpha\sigma\mu\alpha$, which means 'moldable substance' or 'jelly', though it was mentioned on the forums that it might mean 'living thing'... which is really fitting when you think about it
- A brief history:
  - 1920's-1930's: ionospheric plasma research (for radio transmission) and vacuum tubes (Langmuir)
  - 1940's: MHD plasma waves (Alfv
  - 1950's: research on Magnetic Fusion. Geneva UN conference on uses for atomic energy which don't kill people
- Fusion experiments: L-1, TFTR, JET, ITER tokamaks; W7-X stellarator at MPI in Germany; the NIF inertial fusion facility in US
- The Earth's magnetosphere; van Allen belts
- Jets - space plasmas
- Lots of industrial applications

1.2 Rigorous definition of plasma: Debye length

A plasma is a **globally neutral** *ionised gas* with **collective effects**

The following parameters classify plasmas:

- **Debye length**

  Distance over the potential of a charged particle decreases by a factor $1/e$ due to screening by other charged particles

  $$\lambda_{De} = \sqrt{\frac{\epsilon_0 T_e}{e^2 n_0}}$$

  (for electrons)

  Solved in lecture by a statistical approach which assumed $n\frac{4}{3}\pi\lambda_{De}^3 \equiv N_D \gg 1$ (for a Debye sphere; in the lecture $n\lambda_{De}^3$ was used, which relates to a Debye cube. There's not much difference between them, a factor of 4). N_D means the number of particles inside a sphere (or cube, following the lecture) of radius equal to the Debye length. The condition means there's plenty of particles to screen our test particle. This also assumed that binary interactions between particles were weak ($\frac{e\phi}{T_e} \gg 1$)

[1] https://www.youtube.com/watch?v=sLkGSV9WDMA

1.3 Plasma definition: frequencies and parameters

- **Plasma frequency** Assume a plasma of same density of ions and electrons. Displace electrons by Δx. They begin to exhibit harmonic oscillations (for Δx not too large). Newton's 2nd law gives

$$\frac{d^2 \Delta x}{dt^2} + \frac{n_0 e^2}{\epsilon_0 m_e} \Delta x = 0$$

  Can define plasma frequency

$$\omega_{pe} \equiv \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}} = \frac{v_{th,e}}{\lambda_{De}}$$

  where $v_{th,e}$ denotes the thermal speed of electrons

- **Collision frequency**

  The frequency of coulomb collisions between particles

$$\nu_{coll} \equiv \frac{n_0 e^4}{16\pi \epsilon_0^2 m_e^2 v_{th,e}^3}$$

- Size of plasma has to be much larger than its Debye length (or there's no quasineutrality)

1.4 Particle motion in a static uniform magnetic field . Plasma magnetic properties

- Larmor radius - particles gyrate around the guiding center at this distance

$$\rho \equiv \frac{m v_\perp}{|q| B}$$

- Cyclotron frequency

$$\omega_c \equiv \frac{v_\perp}{\rho} = \frac{|q| B}{m}$$

  Particle rotation direction on their helical trajectory

  - $q > 0$ ('by default'): left hand rotation with respect to $\mathbf{B}$
  - $q < 0$ (electrons): right hand rotation
  - Magnetic moment

$$|\mu| \equiv IA = \frac{|q| \omega_e}{2\pi} \pi \rho^2 = \frac{m v_\perp^2}{2B} = \frac{E_{kin\perp}}{B}$$

    (direction opposite to B) is an adiabatic invariant for every particle; doesn't change under slow changes of factors involved in the equation for μ. However, it will change through heat exchange, which usually operates on slower timescales than magnetic field changes (see 2c).

    Plasmas are diamagnetic (they reduce externally applied magnetic fields) (because of direction of μ)

1.5 Particle motion in given electromagnetic fields: the drifts

Static and uniform E and B fields. Particles under Lorentz force which can be decomposed as:

- Parallel direction:

$$m \frac{d\mathbf{v}_\parallel}{dt} = q E_\parallel$$

  Uniform acceleration

- Perpendicular direction:

$$m\frac{d\mathbf{v}_\perp}{dt} = q(\mathbf{E}_\perp + \mathbf{V}_\perp \times \mathbf{B})$$

The many drifts in a plasma:

- $\mathbf{E} \times \mathbf{B}$ drift
  - Perpendicular component averages out over gyroperiod

$$\mathbf{v_e} = \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2}$$

  - This is a motion of the guiding center which is superposed over the gyromotion
  - Does not depend on charge, neither in magnitude nor in direction (but gyromotion direction does)
  - Guiding center moves over lines of constant electrostatic potential ϕ (the drift does not change the particle energy!)
  - A generalization of this drift for any force:

$$\mathbf{v_F} = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}$$

  - For a gravitational force (say, space plasmas), this depends on charge. Separates positive and negative charges. Polarizes the plasma, creating a $\mathbf{E}$ field and an $\mathbf{E} \times \mathbf{B}$ drift
- Curvature drift
  - $\mathbf{B}$ field curved, particle follows the B field - this happens through a centrifugal force

$$\mathbf{F_c} = \frac{mv_\parallel^2}{R_B^2}\mathbf{R_B}$$

  - This causes a drift:

$$\mathbf{v_d} = \frac{\mathbf{F_c} \times \mathbf{B}}{qB^2} = \frac{mv_\perp^2}{qB^2 R_B^2}(\mathbf{R_B} \times \mathbf{B})$$

- Gradient drift $\boldsymbol{\nabla} B \perp \vec{B}$)
  - Happens in changing (spatially) magnetic fields

$$\mathbf{v}_{\boldsymbol{\nabla}\mathbf{B}} = \frac{mv^2}{2qB^3}(\mathbf{B} \times \boldsymbol{\nabla} B)$$

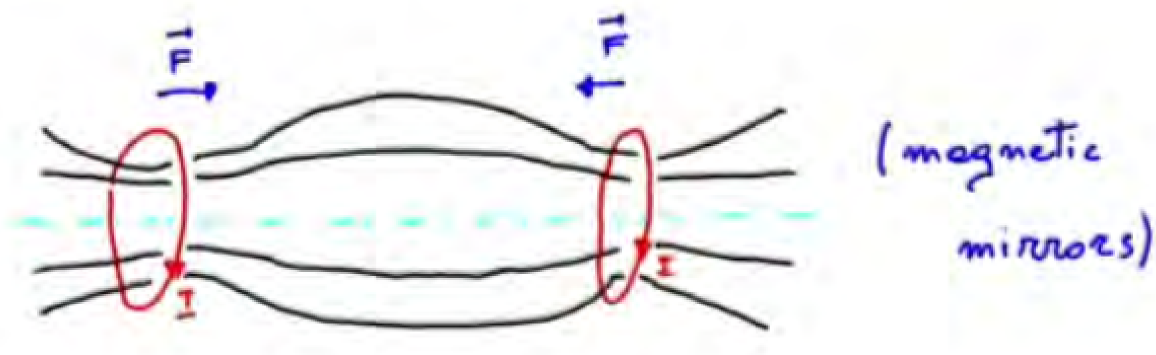
  - A derivation so complicated, it deserved a separate appendix. As particles gyrate, they move between regions of smaller and bigger B. This causes a drift in a direction perpendicular to both the B field and the gradient of its value We consider a small variation in B and expand B in a taylor series around B_0.
    Then we use that expansion to solve $m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times B$, plugging in our expansion for B.
    We also decompose the velocity: an average v_0 and a small perturbation. v_0 is the solution to the equation for constant magnetic field B_0.
    We neglect the cross product of the two small perturbations and average over a gyroperiod.
    We use our knowledge of the solution for the static magnetic field (gyration in the plane perpendicular to B) to deal with the perpendicular velocities (x and y in this decomposition under the assumption that B is along z).
    The drift velocity is the perturbation described by the formula above for an arbitrary geometry of the problem.

1.6 Plasma confinement based on single particle motion. Magnetic mirrors, stellarators, tokamaks

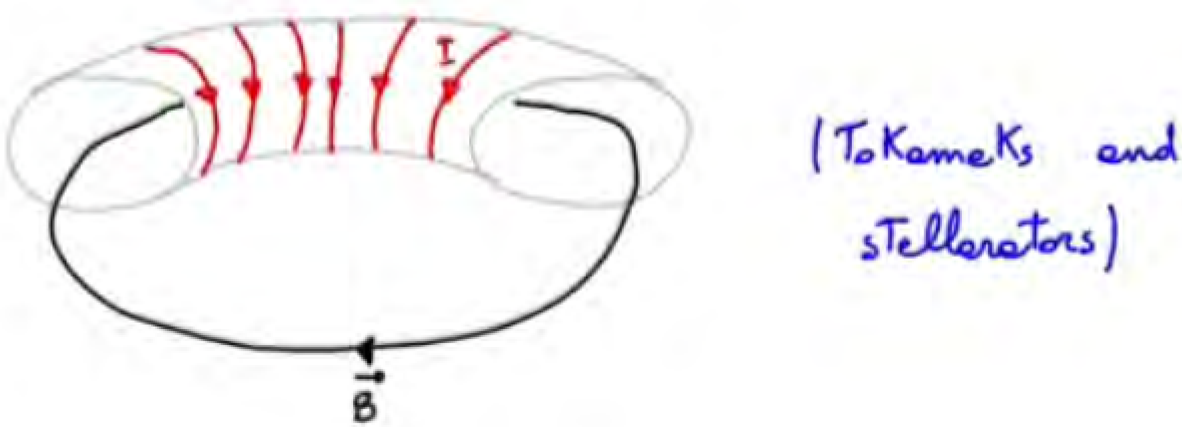
- How do you confine a plasma?

  Charged particles follow helical trajectories along B field. This confines them in the perpendicular direction. What about the parallel one?

- Can use open field lines. Take two circular coaxial electromagnets.



- Can use closed field lines. Closed geometries. Example: tokamaks (toroidal), stellarators.



- The magnetic mirror geometry is neat for particles really close to the axis. B is maximum (field density increases) near the electromagnets

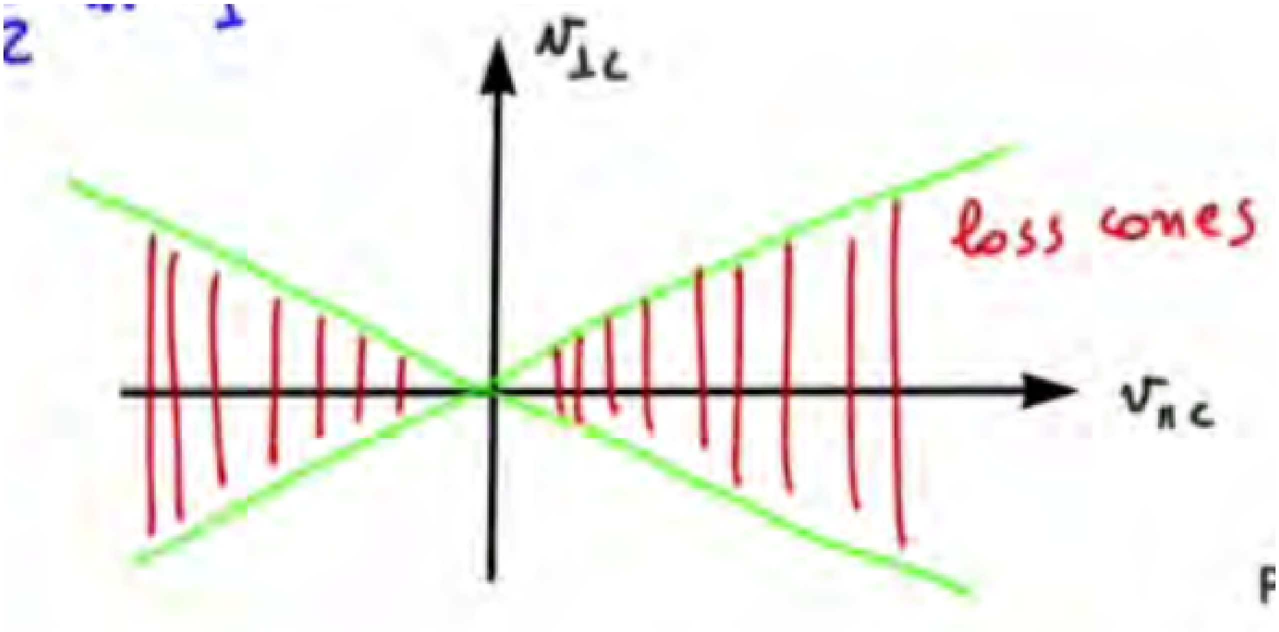
  force in the axial direction is

  $$F_z = -\mu \left| \nabla B \right|$$

  $v_{\parallel}$ has to vanish at B_{max} so that the kinetic energy is just composed of the perpendicular component of velocity

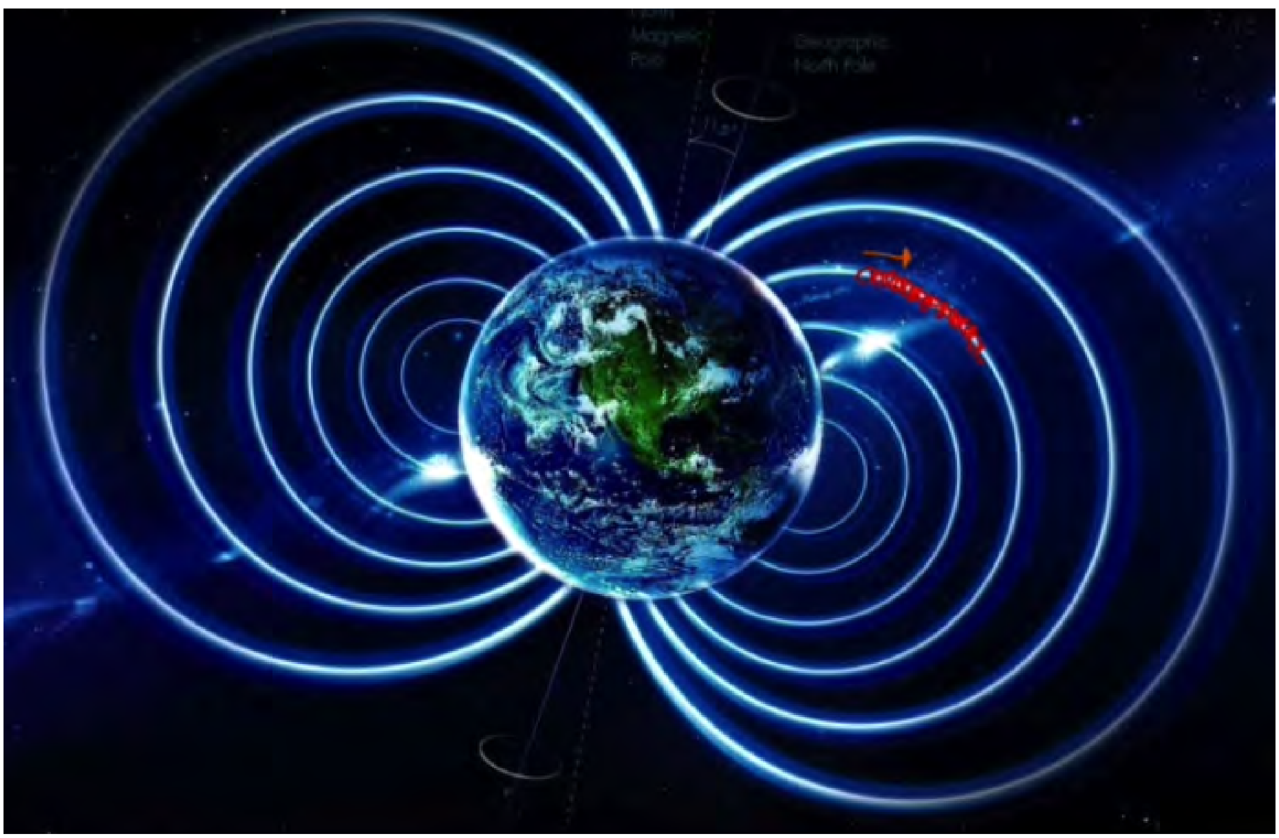
  Particle reflection condition

  $$\frac{v_{\perp}^2}{v_{\perp}^2 + v_{\parallel}^2} > \frac{B_{min}}{B_{max}}$$



This means that particles in the **loss cones** in phase space (marked red; those which don't satisfy the inequality) cannot be confined in the mirror!

Neat example: the Earth's magnetic field is a magnetic mirror!



- What about closed magnetic field lines? Can those deal with loss cones?

  B is not homogeneous! Curved! Has curvature and gradient drifts!

  For a purely toroidal field, positively charged particles drift towards the bottom, while negatively charged ones drift towards the top. This polarizes the plasma and introduces the $E \times B$ drift outwards, sending the plasma crashing into the major radius wall.

  A solution: a poloidal magnetic field to short circuit the charge accumulation. Either:

- Drive a current through the plasma $\rightarrow$ Tokamaks
- Get rid of axial symmetry $\rightarrow$ Stellarators

2 Week 2: Kinetic description of plasmas, with Paolo Ricci

2.1 From single particle to kinetic description

Kinetic description of plasma. A (relatively?) complete description of plasma which covers both the particles and the fields evolving over time.

The usual diagram for a plasma description, seen often in simulations:

(a) Take Newton's equations using electric and magnetic fields **for all particles at all times** (use Lorentz force)

(b) Use positions and velocities to compute charge and current densities. Charge density given as sum over particles of their charges, localized through use of Dirac delta functions. Current density - similar, but multiplied by particle velocity vectors inside the sum.

(c) Take charge and current density, plug them into Maxwell equations, calculate E and B fields at positions

(d) Take calculated E and B fields and apply them as forces to particles. Repeat cycle until bored or simulation returns segmentation fault.

But real plasmas involve on the order of 10^{21} particles for a fusion plasma. Too much strain on our computational abilities. Impractical. We use a distribution function:

$f(\mathbf{r}, \mathbf{v}, t)d\mathbf{r}d\mathbf{v}$ = number of particles at time t, in phase space volume $d\mathbf{r}d\mathbf{v}$ located at $\mathbf{r}, \mathbf{v}$. We have a separate distribution function f_i for every species

- Total number of particles N_S given by integral of distribution function over all positions and velocities (which covers all the phase space)
- Number density of particles n_s given by integral over all velocities for a given location $\mathbf{r}$
- Average velocity given by $\frac{1}{n_s}\int \mathbf{v} f_i(\mathbf{r}, \mathbf{v}, t)d\mathbf{v}$

Examples of distribution functions

- Maxwell-Boltzmann distribution function, for three dimensions

$$F_0(\mathbf{v}) = n_0(\frac{1}{2\pi v_{thermal}})^{3/2} \exp -\frac{v^2}{2v_{thermal}^2}$$

  In 1D, only the normalization of the distribution changes from the 3D case:

$$F_0(v) = n_0(\frac{1}{2\pi v_{thermal}})^{1/2} \exp -\frac{v^2}{2v_{thermal}^2}$$

- Monoenergetic beam in 1D

$$F_0(v) = n_0\delta(v - v_0)$$

- Two counterstreaming beams in 1D (two-stream instability!)

$$F_0(v) = \frac{n_0}{2}[\delta(v - v_0) + \delta(v + v_0)]$$

Conservation of particles number

If there are no sources or sinks, we have the following condition for conservation of number of particles

$$\frac{df_s}{dt} = -\nabla_{6D} \cdot (\mathbf{u}f_s)$$

where we introduce the six-dimensional nabla operator because who's gonna stop us

$$\nabla_{6D} = (\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{dv_x}, \frac{d}{dv_y}, \frac{d}{dv_z}) = (\frac{d}{d\mathbf{r}}, \frac{d}{d\mathbf{v}})$$

$$\mathbf{u} = (\frac{d\mathbf{r}}{dt}, \frac{d\mathbf{v}}{dt}) = (\mathbf{v}, \frac{\mathbf{F}}{m_s}) = (\mathbf{v}, \frac{\mathbf{F_{longrange}} + \mathbf{F_{shortrange}}}{m_s})$$

Long range forces - collective interactions. Short range forces - binary collisions (between individual particles, like you'd have in a gas). Plugging these back into the particle conservation equation:

$$\frac{df_s}{dt} = -\frac{d}{d\mathbf{r}} \cdot (\mathbf{v} f_s) - \frac{d}{d\mathbf{v}} \cdot [\frac{\mathbf{F_{longrange}} + \mathbf{F_{shortrange}}}{m_s} f_s]$$

Boltzmann equation We can improve on the previous equation. Start out with the expanded particle conservation equation:

$$\frac{df_s}{dt} = -\frac{d}{d\mathbf{r}} \cdot (\mathbf{v} f_s) - \frac{d}{d\mathbf{v}} \cdot [\frac{\mathbf{F_{longrange}} + \mathbf{F_{shortrange}}}{m_s} f_s]$$

- In the phase space approach, velocity is treated as a completely independent variable from r (though you could consider one as a derivative of the other). Thus $\frac{d}{d\mathbf{r}} \cdot (\mathbf{v} f_s) = \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}}$
- long range force can be decomposed into electric field independent of v, and the $\mathbf{v} \times \mathbf{B}$ term - perpendicular to v. Thus, $\frac{d}{d\mathbf{v}} \cdot [\mathbf{F_{longrange}} f_s] = \mathbf{F_{longrange}} \cdot \frac{df_s}{d\mathbf{v}}$
- Plugging in:

$$\frac{df_s}{dt} = -\mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} - \frac{\mathbf{F_{longrange}}}{m_s} \cdot \frac{df_s}{d\mathbf{v}} - \frac{d}{d\mathbf{v}} \cdot (\frac{\mathbf{F_{shortrange}}}{m_s} f_s)$$

- Can be rewritten as:

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{\mathbf{F_{longrange}}}{m_s} \cdot \frac{df_s}{d\mathbf{v}} = -\frac{d}{d\mathbf{v}} \cdot (\frac{\mathbf{F_{shortrange}}}{m_s} f_s)$$

  Term on the right is called a 'collision operator' $(\frac{df}{dt})_c$.

- And we get the **Boltzmann equation**:

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E_{longrange}} + \mathbf{v} \times \mathbf{B_{longrange}}) \cdot \frac{df_s}{d\mathbf{v}} = (\frac{df_s}{dt})_c$$

2.2 2b) Coulomb collisions in plasmas. Bonus module.

We use Boltzmann equation and look into the short range interactions

An electron with charge $-e$ approaches a positive ion (assumed immobile) with charge Ze. Electron trajectory changes. $\mathbf{v_e}$ - initial electron velocity b - impact parameter, shortest distance between extrapolated line of initial electron trajectory and ion position

$$\frac{\text{Coulomb interaction energy}}{\text{Kinetic energy}} \sim \frac{\frac{Ze^2}{4\pi\epsilon_0 b}}{m_e v_e^2} \sim 1$$

(similar to one so collision interaction is important)

$$b \sim \frac{Ze^2}{4\pi\epsilon_0 m_e v_e^2} = b_{\pi/2}$$

Coulomb cross section: $\sigma_{\pi/2} = \pi b_{\pi/2}^2 = \frac{\pi Z^2 e^4}{(4\pi\epsilon_0)^2 m_e^2 v_e^4}$

Collision frequency: $\nu_{\pi/2} = n_i v_e \sigma_{\pi_2} = fracn_i \pi Z^2 e^4 (4\pi\epsilon_0)^2 m_e^2 v_e^3$

Is this a correct estimate? Do collective small angle deflections matter in a plasma? How can we take the interaction with many particles into account properly? Average over all phase space somehow?

Take the electron-ion collision again. Denote θ - angle between initial and final electron velocity.

Particles interact through Coulomb force. Angular momentum and energy - conserved (if electron is much lighter than ion, $\frac{m_e}{m_i} \ll 1$).

$$tan(\theta/2) = \frac{b_{\pi/2}}{b} = \frac{Ze^2}{4\pi\epsilon_0 m_e v_e^2 b}$$

$b_{\pi/2}$ - impact parameter at which collision deflects electron by 90°.

Cumulative effect for many collisions? Imagine electron moving towards ion cloud.

Due to symmetry we take $\langle \Delta \mathbf{v}_{\perp \mathbf{e}} \rangle = 0$ but $\left\langle \Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} \right\rangle \neq 0$. So magnitude could change, but there will be no preferred direction. $\perp$ stands for parallel to initial velocity.

$$\frac{d\left\langle \Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} \right\rangle}{dt} = \int db n_i v_e 2\pi b$$

(we integrate over all possible impact parameters

$$\Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} = v_e^2 \sin\theta^2 = \Delta v_e^2 \tan\theta/2^2 [1 + \tan\theta/2^2]^{-2}$$

Plugging into the integral:

$$\frac{d\left\langle \Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} \right\rangle}{dt} = 8\pi n_i v_e^3 \int_0^{\lambda_D} \frac{(b_{\pi/2}/b)^2 b}{(1 + (b_{\pi/2}/b)^2)^2} db\pi b$$

We neglect quantum effects (thus integrating from 0) and integrate up to Debye length as coulomb interactions are screened beyond it. Finally, we get:

$$\frac{d\left\langle \Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} \right\rangle}{dt} = 8\pi n_i v_e^3 b_{\pi/2}^2 \ln\frac{\lambda_D}{b_{\pi/2}} (\text{if } \lambda_D \gg b_{\pi/2}$$

Following section may have some 4's swapped for Δ's.

- Note that electrons do not lose much energy as $m_e \ll m_i$. Basically reflected balls from a wall. Thus

  $$v_e(\Delta v_{\parallel e}) + 0.5\Delta v_{\perp e}^2 = 0$$

  And

  $$\frac{d\left\langle \Delta v_{\parallel e} \right\rangle}{dt} = -4\pi n_i v_e^2 b_{\pi/2}^2 \ln\frac{\lambda_D}{b_{\pi/2}}$$

- We define the coulomb logarithm:

  $$\ln\Lambda \equiv \ln\frac{\lambda_D}{b_{\pi/2}} \sim \text{In most plasmas equals 15 to 25}$$

- $$\frac{d\left\langle \Delta v_{\parallel e}\right\rangle}{dt} = -\nu_{ei} v_e$$

  Collision frequency of electrons against ions:

  $$\nu_{ei} = 4\pi n_i b_{\pi/2}^2 v_e \ln\Lambda = n_i \sigma_{ei} v_e$$

  Whereas

  $$\sigma_{ei} = 4\pi b_{\pi/2}^2 \ln\Lambda$$

  Can compare

  $$\frac{\sigma_{\pi/2}}{\sigma_{ei}} = \frac{\pi b_{\pi/2}^2}{4\pi b_{\pi/2}^2 \ln\Lambda} \ll 1$$

  Much smaller than 1! So small angle deflections dominate over large scale deflections!

2.3 2c) Collisional processes in plasmas

2.3.1 Slowing down of an electron beam

$$\frac{d\left\langle \Delta v_{\parallel e}\right\rangle}{dt} = -\nu_{ei} v_e = -\frac{n_i Z^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_e^2 v_e^3}$$

We could use this to calculate how an electron beam slows in a plasma. Assume a Maxwellian distribution of electron velocities with mean velocity $u_e \ll v_{thermal,e}$ in 1D:

$$f_e(v) = n_0\left(\frac{m_e}{2\pi v_{thermal,e}}\right)^{1/2} \exp\frac{-m_e(v_{\parallel e} - u_e)^2}{2v_{thermal,e}^2}$$

$$\frac{du_e}{dt} = -\left\langle \nu_{ei} v_{\parallel e}\right\rangle = \frac{-1}{n_0}\int \nu_{ei} v_{\parallel e} f_e(v_{\parallel e}) dv_{\parallel e} \simeq -\left\langle \nu_{ei}\right\rangle u_e (\text{if } u_e \ll v_{thermal,e})$$

The average collision frequency between electrons and ions ν_{ei} is

$$\nu_{ei} = \frac{\sqrt{2}}{12\pi^(3/2)} \frac{n_i Z^2 e^4 \ln\Lambda}{\epsilon_0^2 m_e^(1/2) T_e^(3/2)}$$

There are also collisions between electrons coming from the beam and electrons in the plasma:

$$\nu_{ee} = \frac{\sqrt{2}}{12\pi^(3/2)} \frac{n_e e^4 \ln\Lambda}{\epsilon_0^2 m_e^(1/2) T_e^(3/2)} \sim \frac{\left\langle \nu_{ei}\right\rangle n_e}{Z^2 n_i}$$

2.3.2 Plasma resistivity

Take a cloud of ions and electrons. Apply electric field $\mathbf{E}$. Ions will move in direction of E, whereas electrons will move in opposite direction. E then drives a current in a plasma - charges are moving!

We neglect the slow and heavy electrons and focus on electron movement. From Newton's second law:

$$m_e n_e \frac{d\mathbf{u_e}}{dt} = -en_e\mathbf{E} + \mathbf{R_{ei}}$$

R_{ei} is the collision term we have just calculated. This slows down the current.

$$\mathbf{R_{ei}} = -m_e n_e \left\langle \nu_{ei}\right\rangle (\mathbf{u_e} - \mathbf{u_i}) \text{ (assuming } u_e \ll v_{th,e})$$

- After a transient, we'll reach steady state operation and $\frac{d}{dt} = 0$
- The current can be depicted as $\mathbf{j} = -n_e e(\mathbf{u_e} - \mathbf{u_i})$

Thus:

$$e^2 n_e \mathbf{E} = m_e \langle \nu_{ei} \rangle \mathbf{j}$$

$$\mathbf{E} = \frac{m_e \langle \nu_{ei} \rangle}{e^2 n_e} \mathbf{j} \equiv \eta \mathbf{j}$$

By comparison with Ohm's law we can define the plasma resistivity:

$$\eta \equiv \frac{m_e \langle \nu_{ei} \rangle}{e^2 n_e} = \frac{\sqrt{2 m_e} Z e^2 \ln \Lambda}{12 \pi^{3/2} \epsilon_0^2 T_e^{3/2}}$$

The bigger the temperature, the lower the resistivity. Unlike in metals. It's also independent of density! The contributions of increasing the number of carriers and increasing the number of collisions cancel each other out exactly.

2.3.3 Overview of plasma collision frequencies

- Electron - ion collision frequency $nu_{ei} =$
- Electron - electron collision frequency nu_{ei}
- Ion - ion collision frequency nu_{ii}.

Ions gain energy when you fire an electron beam into a plasma (could be heated this way?).

$$m_e \Delta \mathbf{v_e} = m_i \Delta \mathbf{v_i}$$

$$0.5 m_i \left| \Delta \mathbf{v_i} \right|^2 = \frac{m_e^2}{2 m_i} \left| \Delta \mathbf{v_e} \right|^2 \sim \frac{m_e^2}{2 m_i} \left| \Delta \mathbf{v_{\perp e}}^2 \right|$$

(as we can ignore the change in parallel electron velocity)

Rate of exchange of energy (between species! This equalizes the temperatures between electrons and ions!):

$$\langle \nu_E \rangle = \frac{n_i Z^2 e^4 \sqrt{m_e} \ln \Lambda}{3 \pi \sqrt{2 \pi} \epsilon_0^2 m_i T_e^{3/2}} \sim Z \frac{m_e}{m_i} \langle \nu_{ei} \rangle$$

The electrons have a similar, very fast rate of collisions with each other and with ions. The rate of collisions between ions happens 40 times slower, and then the rate of energy exchange is 40 times slower than that. At a similar rate to that of energy exchange is the rate of ions colliding with electrons.

2.4 2d) Vlasov equation

2.4.1 Derivation from Boltzmann equation

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s} (\vec{E} + \vec{v} \times \mathbf{B}) \cdot \frac{df_s}{d\mathbf{v}} = (\frac{df_s}{dt})_c$$

If we can assume that the number of particles in a Debye cube is REALLY HIGH: $n \lambda_D^3 \gg\gg$, so that $(\frac{df}{dt})_c = 0$, then the **Vlasov equation** holds:

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s} (\vec{E} + \vec{v} \times \mathbf{B}) \cdot \frac{df_s}{d\mathbf{v}} = 0$$

E and B here represent the long range interactions. The charge density is computed as indicated before, integrating out all the velocities. The currents are likewise obtained by summing over the species and calculating the average velocities at each positions.

2.4.2 Conservation laws for the Vlasov equation

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s}(\vec{E} + \vec{v} \times \mathbf{B}) \cdot \frac{df_s}{d\mathbf{v}} = 0$$

This satisfies the following conservation properties:

- Number of particles - we can integrate the Vlasov equation over all positions and velocities. Integrating $\frac{df_s}{dt}$ gives us $\frac{dN_s}{dt}$. The second term gives us, by means of Gauss (divergence) theorem and pushing the boundaries out to infinity, where f_s should decay to zero, zero. In the third term we have a velocity divergence. Since no particles have infinite velocities[2], we can once more use the divergence theorem *(in velocity space!!!)* to eliminate the third term and we reach $\frac{dN_s}{dt} = 0$. Particles are conserved.

- Momentum, which we calculate as the sum of particle and field momenta. No actual derivation is given except for the formula:

$$\mathbf{P_{tot}} = \sum_s m_s \int d\mathbf{r} \int d\mathbf{v}\, \mathbf{v} f_s + \epsilon_0 \int d\mathbf{r}(\mathbf{E} \times \mathbf{B}) = \text{const}$$

- Total energy can be decomposed into energy of particles and energy of field.

$$E = \sum_s \int d\mathbf{r} \int d\mathbf{v} \frac{1}{2} m_s v^2 f_s + \frac{1}{2} \int d\mathbf{r}(\epsilon_0 E^2 + B^2/\mu_0) = \text{const}$$

- Entropy, as given by information theory

$$S = \sum_s \int d\mathbf{r} \int d\mathbf{v} f_s \ln f_s = \text{const}$$

  This is because collisions are neglected by the Vlasov equation. Therefore it is time-reversible!

2.4.3 Interpretation of Vlasov equation

- f_s has incompressible motion (in phase space) - it can be considered as an incompressible fluid (moving in phase space - it's going to obey Liouville's theorem)

- As seen by particle along orbit

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m_s} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}}$$

  And the funny thing is, that's just the Vlasov equation itself. Thus along particle orbits, $f_s = 0$.

2.4.4 (Formal) solutions of Vlasov equation

If c_j is a constant of motion, then any distribution function being a function of any number of constants of motion c_j is a solution.

This is unlike the Boltzmann equation, where only the Maxwellian distribution was a stationary solution.

It can be really difficult to find constants of motion, as well. It seems to be implied that practical solutions rely on numerical methods - that way you need not specify constants of motion for a formal solution.

2.5 The two stream instability!

We make this our testing ground for the Vlasov equation. The situation is two beams of electrons moving in opposite directions in 1D. Spoiler alert - WE'RE ACTUALLY GOING TO MESS AROUND WITH THE MATLAB CODE FOR THIS KIND OF SIMULATION IN THE HOMEWORK, THIS IS AWESOME. Ahem.

[2] Einstein says hi.

2.5.1 Simplifications

We take the Vlasov equation and Maxwell's equations

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s}(\vec{E} + \vec{v} \times \mathbf{B}) \cdot \frac{df_s}{d\mathbf{v}} = 0$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0 \frac{d\mathbf{E}}{dt})$$

We simplify the situation. Set $\mathbf{B} = 0$ for an electrostatic situation.

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s}\vec{E} \cdot \frac{df_s}{d\mathbf{v}} = 0$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$\boldsymbol{\nabla} \times \mathbf{E} = 0$$

which implies

$$\mathbf{E} = -\boldsymbol{\nabla}\phi$$

$$\Delta\phi = -\frac{\rho}{\epsilon_0}$$

We also assume that ions are stationary in the background with density n_0 (electrons move at a much faster time scale). This means we only have to solve Vlasov for the electron motion and can replace f_s by f for brevity:

$$\frac{df}{dt} + \mathbf{v} \cdot \frac{df}{d\mathbf{r}} - \frac{e\vec{E}}{m_s} \cdot \frac{df}{d\mathbf{v}} = 0$$

$$\Delta\phi = \frac{e}{\epsilon_0} \int f d\mathbf{v} - \frac{e}{\epsilon_0} n_0$$

2.5.2 Linearisation

We're going to be analysing small perturbations from equilibrium. This means basically expanding the quality of interest in a short Taylor series:

$$g = g_0 \text{ (equilibrium)} + g_1 \text{ (perturbation, } g_1 \ll g_0\text{)}$$

In our case, f_0 being the initial distribution functions which is isotropic over all position space (thus no dependence on $\mathbf{r}$) and f_1 being our small perturbation:

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t)$$

$$\phi = \phi_1(\mathbf{r}, t) \text{ as we can set } \phi_0 = 0 \text{ since it's constant anyway}$$

$$\mathbf{E} = \mathbf{E_1}(\mathbf{r}, t)$$

We plug these into the Vlasov equation:

$$\frac{\partial f_0 + f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial(f_0 + f_1)}{\partial \mathbf{r}} - \frac{e\mathbf{E_1}}{m_e} \cdot \frac{\partial(f_0 + f_1)}{\partial \mathbf{v}} = 0$$

Simplifying and neglecting $\mathbf{E_1} \cdot \frac{\partial f_1}{\partial \mathbf{v}}$ as small times small:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e\mathbf{E_1}}{m_e} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

Also plugging in $\mathbf{E_1} = -\boldsymbol{\nabla}\phi$:

$$\Delta\phi_1 = \frac{e}{\epsilon_0} \int f_1 d\mathbf{v}$$

Thus:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m_e} \frac{\partial \phi_1}{\partial \mathbf{r}} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

$$\Delta\phi_1 = \frac{e}{\epsilon_0} \int f_1 d\mathbf{v}$$

We now apply Fourier analysis to f_1:

$$f_1(\mathbf{r}, \mathbf{v}, t) = \int d\mathbf{k} \int d\omega \tilde{f}_1(\mathbf{k}, \mathbf{v}, \omega) \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

This is neat because:

- The time derivative is simple:

$$\frac{\partial f_1}{\partial t}(\mathbf{r}, \mathbf{v}, t) = \int d\mathbf{k} \int d\omega (-i\omega) \tilde{f}_1(\mathbf{k}, \mathbf{v}, \omega) \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

- The spatial derivative is also pretty simple:

$$\frac{\partial f_1}{\partial \mathbf{r}}(\mathbf{r}, \mathbf{v}, t) = \int d\mathbf{k} \int d\omega (i\mathbf{k}) \tilde{f}_1(\mathbf{k}, \mathbf{v}, \omega) \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

So we go back to the Vlasov equation and plug in the expressions above:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m_e} \frac{\partial \phi_1}{\partial \mathbf{r}} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0$$

$$\int d\mathbf{k} \int d\omega [(-i\omega + i\mathbf{k} \cdot \mathbf{v})\tilde{f}_1 + \frac{ie\tilde{\phi_1}}{m_e} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}] \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t) = 0$$

This can be true only if all the coefficients vanish:

$$(-i\omega + i\mathbf{k} \cdot \mathbf{v})\tilde{f}_1 + \frac{ie\tilde{\phi_1}}{m_e} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0$$

$$\tilde{f}_1 = \frac{e}{m_e} \frac{\tilde{\phi_1}}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

Plugging this result into the Fourier transform of the Poission equation above:

$$\Delta\phi_1 = \frac{e}{\epsilon_0} \int f_1 d\mathbf{v}$$

$$-k^2 \tilde{\phi_1} = \frac{e}{\epsilon_0} \int \tilde{f}_1 d\mathbf{v} = \frac{e^2 \tilde{\phi_1}}{\epsilon_0 m_e} \int \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v}$$

Which then implies

$$\tilde{\phi_1} k^2 [1 + \frac{e^2}{\epsilon_0 m_e k^2} \int \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v}] = 0$$

We denote the bulky part as the dispersion function

$$D(\omega, \mathbf{k}) \equiv 1 + \frac{e^2}{\epsilon_0 m_e k^2} \int \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v}$$

Which we can find the roots of, and the solutions (values of ω and $\mathbf{k}$) gives us the normal modes of our plasma.

There's a singularity at $\omega = \mathbf{k} \cdot \mathbf{v}$. This is when particles match velocities with wave velocities in the plasmas... this may or may not be connected to Landau damping. This topic will not be further developed in the course.

2.5.3 Getting to the two-stream instability

The two-stream instability is thermodynamically weird. The velocity distribution is non-maxwellian. It's just two sharp peaks (low entropy). Could there be intrinsic modes in the system that restore thermodynamic equilibrium (high entropy)?

We take a 1D system. The derivation above is general (plenty of vectors brought to you by yours truly).

$$f = f_0(v_x) = f(u)$$

$$D(\omega, k) = 1 + \frac{e^2}{\epsilon_0 m_e} \frac{1}{k} \int \frac{df_0}{du} \frac{du}{\omega - ku}$$

I'm pretty sure there's a mistake in the lecture and a $1/k$ factor was lost there (it should be $1/k^2$).

For two counter streaming beams:

$$f_0(u) = \frac{n}{2}[\delta(u - v_0) + \delta(u + v_0)]$$

The distribution function 'luckily' avoids the singularity. We calculate the dispersion function and arrive at

$$D(\omega, k) = 1 - \frac{ne^2}{2\epsilon_0} m_e [\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}]$$

(note that the plasma frequency pops up: $\omega_{pe}^2 = \frac{ne^2}{\epsilon_0 m_e}$)

This is a 4th order polynomial in the nominator. The function has two vertical asymptotes at $\omega = \pm kv_0$ and a horizontal one at 1.

Depending on the parameters, if $D(\omega = 0, k) \geq 0$, there's 4 real roots. The modes will be oscillatory instead of exponentially growing.

Otherwise, if $D(\omega = 0, k) < 0$, there's 2 real roots corresponding to oscillations and 2 complex roots corresponding to exponential explosions.

$$D(\omega = 0, k) = 1 - \frac{\omega_{pe}^2}{k^2 v_0^2} < 0 \rightarrow k^2 v_0^2 < \omega_{pe}^2$$

Unstable modes are those that have sufficiently long wavelengths. That's all we can tell analytically. IT'S PARTICLE IN CELL TIME!

2.6 2f) Kinetic plasma simulations. The Particle in Cell method

2.6.1 The various time scales in a plasma

- $10^{-10}s$ - electron cyclotron motion
- $10^{-7}s$ - ion cyclotron motion

- $10^{-5}s$ - microturbulence
- $10^{-3}s$ - fast global instabilities
- $10^{-1}s$ - slow global instabilities
- $1s$ - energy confinement time
- $10^{3}s$ - gas equilibrisation

It's extremely hard to simulate all of these at once! Accurate modelling of cyclotron motion (say, 10^{-14} timestep) would mean observing **fast** global instabilities after $10^{1}1$ timesteps. That's a lot of timesteps.

2.6.2 Simulation approach - Particle in Cell (PIC) method

One could solve the entire Vlasov equation by separating the whole $6D$ phase space into a grid and solving the partial differential equation. Possible. But usually, one uses the particle in cell method.

f is constant along particle trajectories. This translates directly into Newton's equations for a single particle:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

In the PIC (Particle in Cell) method, we approximate the distribution function by a discrete sum

$$f(\mathbf{r}, \mathbf{v}, t) \simeq \sum_{\alpha} f_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha\mathbf{0}}, \mathbf{v} - \mathbf{v}_{\alpha\mathbf{0}})$$

The f_{α}'s are called **superparticles**; they have compact support - they're zero everywhere but on a small domain in phase space around $r_{\alpha 0}$ and $v_{\alpha 0}$.

We introduce the integral $I_{\alpha} = \int\int d\mathbf{r} d\mathbf{v} f_{\alpha}$ which is done over the small domain of the particle.

At all times, the superparticles satisfy

$$f(\mathbf{r}, \mathbf{v}, t) \simeq \sum_{\alpha} f_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha}(t), \mathbf{v} - \mathbf{v}_{\alpha}(t))$$

If we have that

$$\frac{d\mathbf{r}_{\alpha}}{dt} = \mathbf{v}_{\alpha}$$

$$\frac{d\mathbf{v}_{\alpha}}{dt} = \frac{q_{\alpha}}{m}(\mathbf{E}_{\alpha} + \mathbf{v}_{\alpha} \times \mathbf{B}_{\alpha})$$

where

$$\mathbf{E}_{\alpha} = \frac{1}{I_d}\int\int \mathbf{E} f_{\alpha} d\mathbf{r}\, d\mathbf{v}$$

$$\mathbf{B}_{\alpha} = \frac{1}{I_d}\int\int \mathbf{B} f_{\alpha} d\mathbf{r}\, d\mathbf{v}$$

and we have some initial conditions

$$\mathbf{r}_{\alpha}(t=0) = \mathbf{r}_{\alpha_{\mathbf{0}}}$$

$$\mathbf{v}_{\alpha}(t=0) = \mathbf{v}_{\alpha_{\mathbf{0}}}$$

Then the distribution function defined as the sum of all superparticles (their distribution functions) also solves the Vlasov equation.[3]

[3] This is a complicated way of saying that we can use particles to approximate the motion of the plasma - see Birdsall and Langdon.

2.6.3 Practical PIC

In one dimension, we have to:

(a) Solve Poisson's equation to get the potential ϕ and field $\mathbf{E}$

This is done by discretizing space and time. We get discrete approximations to charge densities, potentials and fields.

Derivatives are discretized using central differences.

(b) Evaluate electric fields acting on superparticles

$$E_\alpha = E_j \text{ if } |x_j - x_\alpha| \leq \Delta x/2$$

(c) Apply fields to superparticles. Solve for the motion of superparticles by a numerical algorithm (Euler's or Runge-Kutta's)

(d) Assign charge to discrete locations on grid. We sum the charges of superparticles in the j-th cell (in $x_{j-1/2} < x_\alpha < x_{j+1/2}$) and divide by the cell size.

On a closing note: PICs are awesome, they're used in all kinds of fields. Electromagnetic fields, gravitational fields... go learn them. :)

3 Week 3. Fluid description of plasmas, with Paolo Ricci

3.1 From Vlasov to two-fluid

- We'll turn the computationally expensive kinetic model into a simple fluid model (can be solved with partial differential function)
- We'll derive fluid quantities (pressure, etc) from distribution functions
- We'll take a moment for moments of the kinetic equation
- We'll apply the continuity equation (conservation of mass etc)
- ... and we'll get the two fluid model!

3.1.1 Fluid quantities from the distribution function

In the fluid model we only really look at quantities that depend on position. This means we'll have to integrate out all velocity dependence.

Number density - the integral of distribution function over all velocities. Simple. We just care about the density *at a location*.

$$n_s(r,t) = \int dv f_s(r,v,t)$$

For any function $g(r,v)$, it's average value is the integral of the function weighted by distribution function, normalized by dividing the integral by the number density of the species.

$$\langle g(r) \rangle = \frac{1}{n_s} \int g(r,v) f_s(r,v,t) d_v f$$

Examples:

- Average fluid velocity, $g(v) = v$
$$u_s(r,t) = \frac{1}{n_s} \int v f_s(r,v,t) dv$$
- Average kinetic energy density, $g(v) = \frac{1}{2} m_s v^2$
$$w_s(r,t) = \frac{1}{n_s} \int \frac{1}{2} m_s v^2 f_s(r,v,t) dv$$
- Pressure **tensor** (as pressure may depend on direction), $g(v) = m_s(\mathbf{v} - \mathbf{u_s})(\mathbf{v} - \mathbf{u_s})$ - a tensor (dyadic) quantity)
$$\hat{P} = \int m_s(\mathbf{v} - \mathbf{u_s})(\mathbf{v} - \mathbf{u_s}) f_s(r,v,t)$$

3.1.2 Examples of fluid averaged quantities

(a) Beam density. Distribution function with uniform spread in $1D$ positions and a delta function dependence on velocity (only one velocity in the whole beam

$$f = n_0 \delta(\mathbf{v} - \mathbf{v_0})$$

The density ends up being just n_0 because of the Dirac delta velocity dependence.

The fluid velocity is just the velocity of the particles, as the delta function times velocity gives us just the particle velocity upon integrating out.

The kinetic energy density is just $mv_0^2/2$, obviously.

The pressure tensor ends up as zero! Both the vector quantities end up being zero as $\mathbf{v} = \mathbf{v_0}$. Reasonable. If everything's moving at the same velocity there's not going to be anything pushing anything else.

(b) Maxwellian velocity distribution with uniform position distribution

$$f = n_0\Big(\frac{1}{2\pi v_{th}^2}\Big)^{3/2} \exp\frac{-(\mathbf{v}-\mathbf{v_0})^2}{2v_{th}^2}$$

Note that the lecture had a version with masses. This was corrected in the errata - I suppose because we're working in position-velocity phase space instead of position-momentum phase space.

Number density is just n_0, as the velocity distribution is normalized to one anyway.

Average fluid velocity - gaussian average value v_0.

Average kinetic energy density: kinetic energy from average velocity plus $\frac{3}{2}k_bT$ (thermal velocity)

Pressure tensor is a diagonal matrix.

$$\hat{P} = \begin{pmatrix} n_0T & 0 & 0 \\ 0 & n_0T & 0 \\ 0 & 0 & n_0T \end{pmatrix}$$

Note that in the errata, n_0K_BT (from the lecture) was corrected to n_0T, as in this course temperatures are defined thermodynamically, through energy, including the Boltzmann constant.

3.1.3 The moments of the kinetic equation

We take the Boltzmann equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{v}} = \Big(\frac{\partial f_s}{\partial t}\Big)_c$$

Shift the collision term to the left side and denote the whole thing, which is now equal to zero for all positions, times and velocities (I'll denote this as $BOLT$):

$$BOLT = \frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{v}} - \Big(\frac{\partial f_s}{\partial t}\Big)_c = 0$$

And take averages - 'moments' - of that by integrating $BOLT$ with a weight g. This will give us the following equations:

- Continuity for $g = 1$
- Momentum for $g = mv$
- Energy for $g = mv^2/2$

The continuity equation will be done here. The rest are, in true physics tradition, easy to derive and as such are left to the reader.

3.1.4 The continuity equation

$$\int BOLT dv = \int \Big(\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{v}} - \Big(\frac{\partial f_s}{\partial t}\Big)_c\Big)dv = 0$$

- The first term ends up as just the time derivative of average number density

$$\int \frac{\partial f_s}{\partial t}dv = \frac{\partial}{\partial t}\int f_s dv = \frac{\partial n_s}{\partial t}$$

- The second term is the time derivative of average fluid velocity times average number density

$$\int \mathbf{v} \cdot \frac{\partial \mathbf{f_s}}{\partial \mathbf{r}} dv = \frac{\partial}{\partial r} \int v f_s dv = \frac{\partial (n_s \mathbf{u_s})}{\partial \mathbf{r}}$$

- The third term: the forces can be brought under the velocity derivative. We then use the divergence theorem in velocity space and since particles aren't going to have infinite velocities, the surface integral over the region in velocity space is zero.

- The fourth collision term is zero on the grounds that 'collisions do not create nor destroy particles'. Not sure how that works...

On the whole, we get the neat continuity equation:

$$\boxed{\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (n_s \mathbf{u_s}) = 0} \tag{1}$$

3.1.5 The two fluid model!

We treat electrons as one fluid, and electrons as another, separate fluid. The two fluids are in the same region in space and interacting with each other (not much of a plasma otherwise!).

We take the continuity equation derived above

$$\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (n_s \mathbf{u_s}) = 0$$

We also take the momentum equation, which was left for the student to derive

$$m_s n_s \frac{d\mathbf{u_s}}{dt} = q_s n_s \Big(\mathbf{E} + \mathbf{u_s} \times \mathbf{B}\Big) - \frac{\partial}{\partial \mathbf{r}} \cdot \hat{P}_s + \mathbf{R_s}$$

Where $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u_s} \cdot \frac{\partial}{\partial \mathbf{r}}$ is the total time derivative (streaming included), and R is a collision term

$$\mathbf{R_s} = \int m_s (\mathbf{v} - \mathbf{u_s}) \Big(\frac{\partial f}{\partial t}\Big)_c dv$$

We also take the energy equation[4]

$$\frac{3}{2} n_s \frac{dT_s}{dt} + \hat{P}_s \cdot \frac{\partial}{\partial \mathbf{r}} \mathbf{u_s} + \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{q_s} = Q_s$$

Where T_s is the temperature factor related to spread of velocity

$$\frac{1}{3n_s} \int m_s (\mathbf{v} - \mathbf{u_s})^2 f_s dv$$

q_s is the heat flux vector, the kinetic energy transported by the velocity of the fluid

$$\frac{m_s}{2} \int (\mathbf{v} - \mathbf{u_s})^2 (\mathbf{v} - \mathbf{u_s}) f_s dv$$

And Q_s is the heat generated by viscous forces (basically friction, collisions?)

$$Q_s = \int \frac{m_s}{2} (\mathbf{v} - \mathbf{u_s})^2 \Big(\frac{\partial f}{\partial t}\Big)_c dv$$

The system of equation is **not closed**, we need an expression for the heat flux, which requires knowing the distribution function. This is a *closure problem.* Closures can be difficult to find and people are actively

[4] I'm not sure about the equations on this page. May have taken a tensor for a vector or vice versa. If anyone can confirm they're correct or spot any mistakes, please say so on the forums.

sciencing that as hard as they can. The problem is that you'd need to use the distribution function to get quantities we'd like to use instead of the distribution function.

Coupling with Maxwell equations is achieved by noting that the charge density is just the species number density times species charge, summed over all species. Current density, likewise, is the sum of species number density times species charge times average species velocity. With these two (and a set of initial conditions), Maxwell equations can be solved for the forces on our fluids.

3.1.6 Two fluid simulation of plasma turbulence

Does it make sense to use the two-fluid model for simulations to describe plasmas instead of, y'know, particles?[5]

It does make sense - it's relatively easy to find a good closure to close the set of equations. It's helpful when plasmas are *fairly collisional*. In that regime, the distribution function is usually close to Maxwellian and that can help find a closure.

Two fluid simulations are especially helpful in the periphery of magnetic fusion devices, where plasmas are relatively cold. This actually helps with turbulence, and turbulence is HARD to deal with:

> 'I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.'
> - Horace Lamb -

3.2 The two-fluid dispersion relation

3.2.1 Linearization and fourier mode analysis - a review

What we want to do right now is study the evolution of small amplitude perturbations applied to equilibrium states.

Take the continuum equation

$$\frac{\partial n}{\partial t} + n\mathbf{\nabla} \cdot \mathbf{u} = 0$$

We take the case of static equilibrium:
$n = n_0 + n_1$, $u = u_0 + u_1 = u_1$
(as we're in a static equilibrium, the equilibrium average velocity is 0)
The linearized continuum equation

$$\frac{\partial n_1}{\partial t} + n_0\mathbf{\nabla} \cdot \mathbf{u_1} = 0$$

($n_1\mathbf{\nabla} \cdot \mathbf{u_1}$ was neglected as the product of two small quantities)

To begin fourier mode analysis, we write

$$n_1(r,t) = \int\int d\mathbf{k} d\omega \tilde{n_1}(\mathbf{k},\omega) \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

Thus we get the substitutions $\nabla \rightarrow i\mathbf{k}$, $\frac{\partial}{\partial t} \rightarrow -i\omega$
So our linearized continuum equation becomes

$$-i\omega\tilde{n}_1 + n_0 i\mathbf{k}\cdot\tilde{\mathbf{u_1}} = 0$$

[5] If the author of the notes seems prejudiced in favor of using particles, it may be related to the fact that particle simulations are **awesome**.

3.2.2 Linearization of Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

We take the curl of the first equation, substitute the second and get

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{j}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

We use our Fourier transform substitutions from earlier and the vector property

$$\mathbf{k} \times \mathbf{k} \times \tilde{\mathbf{E}} = k^2(\frac{\mathbf{kk}}{k^2} - 1)\tilde{\mathbf{E}}$$

and write

$$-\frac{c^2k^2}{\omega^2}(\frac{\mathbf{kk}}{k^2} - 1)\tilde{\mathbf{E}} = \frac{i}{\epsilon_0 \omega}\tilde{\mathbf{j}} + \tilde{\mathbf{E}}$$

Where it turns out that $\frac{c^2k^2}{\omega^2} = N^2$, N is the index of refraction!

3.2.3 Linearization of two-fluid equations

We take the limit of low ($T \to 0$) temperature. What we want to get is $\tilde{\mathbf{j}} = \hat{\sigma}\tilde{\mathbf{E}}$

Using the momentum equation, removing the pressure term due to low temperature limit

$$m_s(\frac{\partial \mathbf{u_s}}{\partial t} + (\mathbf{u_s} \cdot \nabla)\mathbf{u_s}) = q_s(\mathbf{E} + \mathbf{u_s} \times \mathbf{B})$$

We assume that there's equilibrium electric field E_0 (static case, remember?), no movement ($u_{s0} = 0$), $n_{s0} = n_0$, and we've got a uniform magnetic field in the z direction $\mathbf{B} = B_0\hat{\mathbf{z}}$

Linearizing, fourier transforming the momentum equation, introducing the cyclotron frequency $\Omega_s = \frac{q_s B_0}{m_s}$, and skipping writing the tildes (basically everything's in fourier space now)

$$\begin{pmatrix} -i\omega & -\Omega_s & 0 \\ \Omega_s & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix} \mathbf{u_{s1}} = \frac{q_s}{m_s}\mathbf{E_1}$$

From this we can get the fluid velocity by inverting the matrix and sticking all the constants in a tensor we call the *mobility tensor* μ_s:

$$\mathbf{u_{s1}} = \frac{q_s}{m_s}\begin{pmatrix} -i\omega & -\Omega_s & 0 \\ \Omega_s & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix}^{-1} \mathbf{E_1} \equiv \hat{\mu_s}\mathbf{E_1}$$

$$\hat{\mu_s} = \begin{pmatrix} \frac{-i\omega}{\Omega_s^2 - \omega^2} & \frac{\Omega_s}{\Omega_s^2 - \omega^2} & 0 \\ \frac{-\Omega_s}{\Omega_s^2 - \omega^2} & \frac{-i\omega}{\Omega_s^2 - \omega^2} & 0 \\ 0 & 0 & \frac{-i}{\omega} \end{pmatrix}$$

And we have the current! It's the sum over all species of velocities times densities times charges!

$$\mathbf{j} = \sum_s q_s n_{s0}\mathbf{u_{s1}} = \Big(\sum_s q_s n_{s0}\hat{\mu_s}\Big)\mathbf{E_1} \equiv \hat{\sigma}\mathbf{E_1}$$

Where we introduced $\hat{\sigma}$ as the conductivity tensor.

3.2.4 The two-fluid dispersion relation

$$-N^2(\frac{\mathbf{kk}}{k^2} - 1)\mathbf{E_1} = \frac{i}{\epsilon_0\omega}\mathbf{j} + \mathbf{E_1}$$

We define another tensor to help us write all this, noting that $\mathbf{j_1} = \hat{\sigma}\mathbf{E_1}$:

$$\hat{\varepsilon} = \frac{i\hat{\sigma}}{\epsilon_0\omega} + 1$$

$$\left(N^2\left(\frac{\mathbf{kk}}{k^2} - 1\right) + \hat{\varepsilon}\right)\mathbf{E_1}$$

And we get the dispersion relation

$$\boxed{\det\left(N^2\left(\frac{\mathbf{kk}}{k^2} - 1\right) + \hat{\varepsilon}\right) = D(\omega, \mathbf{k}) = 0} \tag{2}$$

Writing $\hat{\varepsilon}$ out explicitly:

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

$$\varepsilon_1 = 1 + \sum_s \frac{\omega_{ps}^2}{\Omega_s^2 - \omega^2}$$

$$\varepsilon_2 = -\sum_s \frac{\Omega_s}{\omega}\frac{\omega_{ps}^2}{\Omega_s^2 - \omega^2}$$

$$\varepsilon_3 = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

3.3 Two fluid waves, parallel and perpendicular propagation

3.3.1 Two fluid cold plasma dispersion relation

We're once more taking the equilibrium at zero temperature: no fluid velocity, uniform density, no electric field, magnetic field uniform in the $\hat{\mathbf{z}}$ direction.

We use the dispersion relation above. We note by inspecting the matrix[6] that the z direction - the one parallel to the magnetic field - is very different from the perpendicular directions.

In the limit of zero magnetic field, the cyclotron frequency also vanishes. The diagonal epsilons simplify and ε_2 vanishes, turning $\hat{\varepsilon}$ into a diagonal matrix - the plasma becomes isotropic.

If we assume that the wave is propagating in a direction perpendicular to the B field, in the YZ plane, with the direction given by the angle from the z axis

$$\mathbf{k_0} = k\sin\vartheta\hat{\mathbf{y}} + k\cos\vartheta\hat{\mathbf{z}}$$

Then our dispersion relation takes the form:

$$\left(N^2\left(\frac{\mathbf{kk}}{k^2} - 1\right) + \hat{\varepsilon}\right) = \begin{pmatrix} \varepsilon_1 - N^2 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 - N^2\cos\vartheta^2 & N^2\sin\vartheta\cos\vartheta \\ 0 & N^2\sin\vartheta\cos\vartheta & -N^2\sin\vartheta^2 + \varepsilon_3 \end{pmatrix}$$

[6]Unfortunately, no one can be told what the matrix is. You have to see it for yourself in the previous section.

3.3.2 Waves propagating parallel to $\mathbf{B_0}$

Our K vector takes the neat form $k = k\hat{\mathbf{z}}$

This simplifies the $\hat{\varepsilon}$ matrix:

$$\det\begin{pmatrix} \varepsilon_1 - N^2 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 - N^2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} = 0$$

In the case of $\varepsilon_3 = 0$, taking the matrix's determinant gives the solution $\omega \simeq \omega_{pe}$. These are exactly the plasma waves we've been considering earlier, in the first lecture! The electric field E is also parallel to both K and B. These are the waves related to restoring quasineutrality in the plasma.

The other solution is when

$N^2 = \varepsilon_1 + \varepsilon_2 \equiv \varepsilon_R$ (right handed waves) or $N^2 = \varepsilon_1 - \varepsilon_2 \equiv \varepsilon_R$ (left handed waves)

For right handed waves ($N^2 = \varepsilon_R$), the dispersion relation gives

$$E_x = -iE_y$$

What's physically interesting is the real part of the expression. Denoting $E_x \equiv E$:

$$\mathbf{E} = E\cos\omega t\hat{\mathbf{x}} + E\sin\omega t\hat{\mathbf{y}}$$

So the wave propagates along the B field and its electric field has a right handed rotation about the B field vector. Left handed waves are, thus, self explanatory.

Plugging $N^2 = \varepsilon_R$ into the dispersion relation:

$$N^2\varepsilon_R \simeq \frac{(\omega - \omega_R)(\omega - \omega_L)}{(\omega - |\Omega_E|)}$$

The expressions for left and right handed wave frequencies come out as

$$\boxed{\omega_R = \frac{1}{2}\Big(|\Omega_e| + \sqrt{\Omega_e^2 + \omega_{pe}^2}\Big) > |\Omega_e|} \tag{3}$$

$$\boxed{\omega_L = \frac{1}{2}\Big(-|\Omega_e| + \sqrt{\Omega_e^2 + \omega_{pe}^2}\Big) > 0} \tag{4}$$

Note that they've got completely different values.

If, instead, you plug in $N^2 = \varepsilon_L$, the equation you get is

$$N^2 = \frac{k^2c^2}{\omega^2} = \frac{(\omega + \omega_R)(\omega - \omega_L)}{(\omega + |\Omega_e|)(\omega - \Omega_i)}$$

3.3.3 Waves propagating perpendicular to B_0

The determinant can be written as

$$(-N^2 + \varepsilon_3)(\varepsilon_1(\varepsilon_1 - N^2) - \varepsilon_2^2)$$

This gives us the so called *ordinary mode (OM)* $N^2 = \varepsilon_3$. E is parallel to B and the equation for frequencies, in the limit of low ion plasma frequency (which usually is pretty low), is:

$$\frac{k^2c^2}{\omega^2} \simeq 1 - \frac{\omega_{pe}^2}{\omega}$$

The other mode is the *extraordinary mode (XM)*. The E field is in the XY plane. $N^2 = \frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_1-\varepsilon_2)}{\varepsilon_1} = \frac{\varepsilon_L\varepsilon_R}{\varepsilon_1}$. This can be written as

$$N^2 \simeq \frac{(\omega^2-\omega_R^2)(\omega^2-\omega_L^2)}{\omega^2-\omega_{LH}^2)(\omega^2-\omega_{UH}^2}$$

$$\boxed{\omega_{UH}^2 = \Omega_e^2 + \omega_{pe}^2} \tag{5}$$

This is called the upper hybrid frequency - hybrid as it's a combination of the electron cyclotron and plasma frequencies.[7]

$$\boxed{\omega_{LH}^2 = \frac{\Omega_i\left|\Omega_e\right|\left(1+\frac{m_e\Omega_e^2}{m_i\omega_{pe}^2}\right)}{1+\frac{\Omega_e^2}{\omega_{pe}^2}}} \tag{6}$$

This is called the lower hybrid frequency.

3.3.4 The main takeaway

Even in the dramatically simplified cold plasma two-fluid model, plasmas are complex beasts with plenty of intrinsic modes. Plasma waves, right and left handed waves, extraordinary modes, ordinary modes... On to the next module, where the physics of those will be developed!

3.4 Properties of two-fluid waves (Resonanses, cutoffs, etc)

3.4.1 Properties of right and left handed waves

For right handed waves, the dispersion relation takes the form

$$(\frac{kc}{\omega})^2 = \frac{(\omega-\omega_R)(\omega+\omega_L)}{(\omega-|\Omega_e|)(\omega+\Omega_i)}$$

Whereas for left handed waves

$$\frac{kc}{\omega})^2 = \frac{(\omega+\omega_R)(\omega-\omega_L)}{(\omega+|\Omega_e|)(\omega-\Omega_i)}$$

where

$$\omega_R = \frac{1}{2}(|\Omega_e| + \sqrt{\Omega_e^2+\omega_{pe}^2}) > |\Omega_e|$$

$$\omega_L = \frac{1}{2}(-|\Omega_e| + \sqrt{\Omega_e^2+\omega_{pe}^2}) > 0$$

To propagate, the LHS expression $(\frac{kc}{\omega})^2$ must be positive. This implies the right hand side has to be positive. So in the interval

$$|Omega_e| < \omega < \omega_R$$

R-waves do not propagate, and in the interval

$$\Omega_i < \omega < \omega_L$$

L-waves do not propagate. Thus, in general

$$\Omega_i < \omega_L < |\Omega_e| < \omega_R$$

In the limit of $\omega \to +\infty$, the dispersion relation for both cases simplifies to

[7] See Chen, chapter 4, 'Waves in Plasmas'

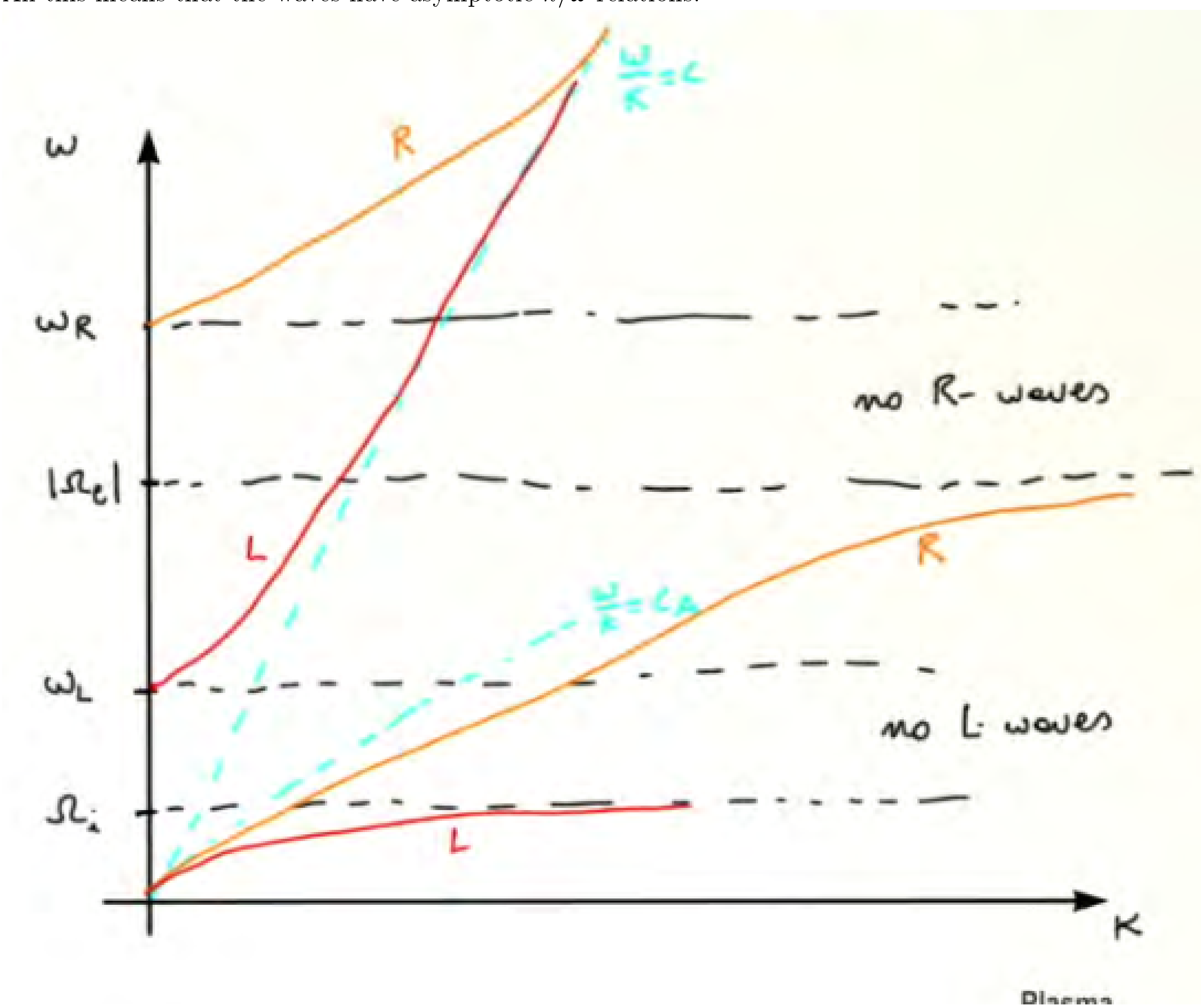
$$\frac{\omega}{k} = v_{propagation} = c$$

This describes lightspeed waves, electromagnetic waves. The plasma can't respond to waves with such a high frequency! Likewise, in the limit of $\omega \to 0$, $k \to 0$, we get

$$\frac{kc}{\omega})^2 = \frac{\omega_R \omega_L}{|\Omega_E|\,\Omega_i} = \frac{\omega_{pe}^2}{|\Omega_E|\,\Omega_i} = \frac{m_i n_0}{\epsilon_0 B_0^2} = \frac{c^2 \mu_0 m_i n_0}{B_0^2} = \frac{c^2}{c_A^2}$$

Where $c_A = \frac{B_0}{\sqrt{\mu_0 m_i n_0}}$ is the so called Alfvèn speed, related to Alfvèn magnetohydrodynamic waves. So what we've got is $\omega = \pm c_a k$.

All this means that the waves have asymptotic k/ω relations:



3.4.2 Whistler waves

We take the case of R-waves with $\omega/k \ll c \ll \omega_{pe} < |\Omega_e| < \omega_R$ - we're looking at the bottom part of the graph above.

$$(\frac{kc}{\omega})^2 = \frac{(\omega - \omega_R)(\omega + \omega_L)}{(\omega - |\Omega_e|)(\omega + \Omega_i)} = \frac{\omega^2 + \omega(\omega_L - \omega_R) - \omega_L \omega_R}{(\omega - |\Omega_e|)(\omega + \Omega_i)}$$

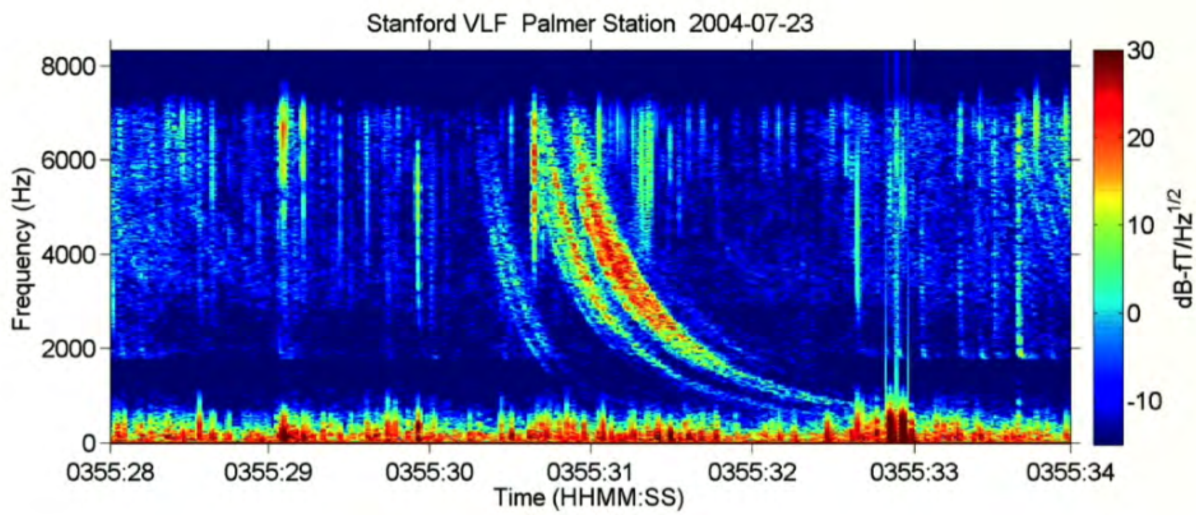
This can be simplified by neglecting some terms due to the regime we're in:

$$\frac{kc}{\omega})^2 = \frac{\omega_{pe}^2}{\omega\,|\Omega_e|}$$

$$\frac{\omega}{k} = \frac{c}{\omega_{pe}}\sqrt{|\Omega_e|\,\omega}$$

Functionally this shows that $v_{propagation} \propto \sqrt{\omega}$ and $\frac{\partial\omega}{\partial k} \propto \sqrt{\omega}$ - the propagation (group) velocity increases with frequency. Higher frequency waves propagate faster.

An example - if lightning strikes somewhere on Earth, waves will be produced in the ionosphere and higher frequency ones will propagate faster than slower ones. Here's a neat graph showing that phenomenon:



That actually does sound like whistling! The waves are in the acoustic range!

3.4.3 Cut-off and resonance frequencies

For right handed waves:

$$(\frac{kc}{\omega})^2 = \frac{(\omega-\omega_R)(\omega+\omega_L)}{(\omega-|\Omega_e|)(\omega+\Omega_i)}$$

ω_R and $|\Omega_i|$ are frequencies which the waves approach asymptotically. ω_R is the cutoff frequency: as k goes to 0, ω/k goes to infinity. k turns from real to imaginary - exponential decay of waves - they don't propagate! Waves are reflected.

Likewise, $|\Omega_i|$ is the resonance frequency: as k goes to infinity, ω/k goes to zero. Wavelengths decrease, small dissipative processes become important - the wave is absorbed into the plasma.

Of course, for L-waves the cutoff is at and the resonance is at .

3.4.4 Properties of waves propagating perpendicular to B_0

For the ordinary mode OM

$$\omega^2 = \omega_{pe}^2 + k^2c^2$$

The cutoff frequency is at ω_{pe}. There's no resonance frequency. As ω goes to infinity, the propagation velocity approaches c. The limit ω and k both going to zero cannot be approached by the ordinary mode.

For the extraordinary mode XM

$$N^2 = \frac{k^2c^2}{\omega^2} = \frac{(\omega^2 - \omega_R^2)(\omega^2 - \omega_L^2)}{\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2}$$

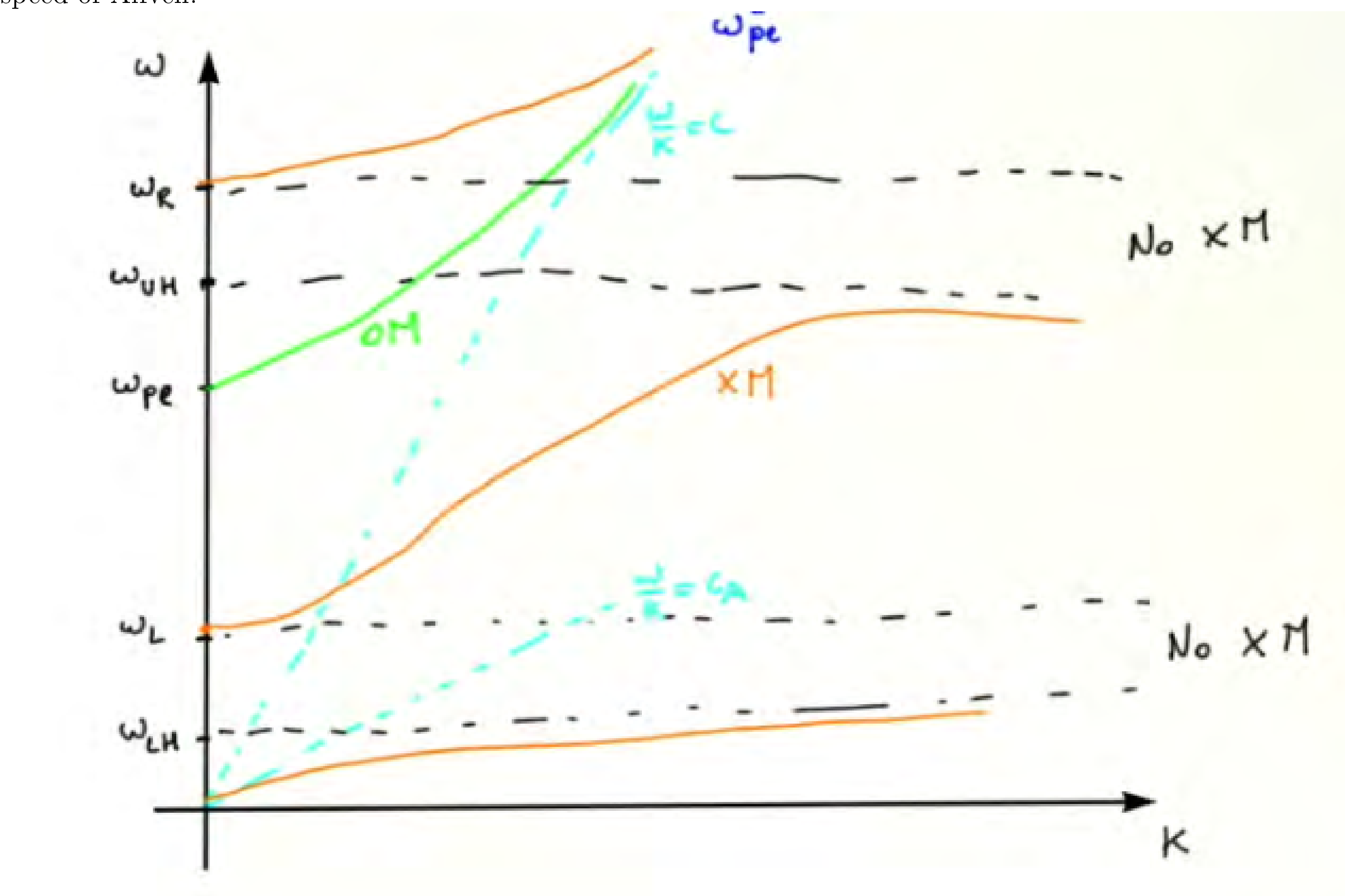
Where the lower and upper hybrid frequencies are

$$\omega_{UH}^2 = \Omega_e^2 + \omega_{pe}^2$$

$$\omega_{LH}^2 = \frac{\Omega_i \left|\Omega_e\right| \left(1 + \frac{m_e \Omega_e^2}{m_i \omega_{pe}^2}\right)}{1 + \frac{\Omega_e^2}{\omega_{pe}^2}}$$

There are two cutoff frequencies: ω_R and ω_L. Likewise, there's two resonances: ω_{LH} and ω_{UH}. As before, as ω goes to infinity, the propagation velocity approaches c. As k and ω approach zero, $\omega = c_A k$ - the Alfvèn velocity appears!

As before, there's two asymptotes for propagation velocity, this time they're the speed of light and the speed of Alfvèn:



XM waves have two regions where they cannot propagate - between lower hybrid/left and upper hybrid/right hand frequencies.