Notes from PlasmaX

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These notes will not be 100% comprehensive, as I'm making them mainly for my own use. However, if you spot any mistakes, feel free to catch me on the forums and I'll fix any mistakes.

0.1 The notes' repository

The repository for these notes is maintained here. Feel free to add any changes to the .tex files (if you don't feel like compiling the pdf don't worry, I've got that covered).

0.2 The course wiki and errata pages

The course's got a wiki page which you can check out here. I try to incorporate corrections from the errata page into these notes, but obviously I may miss some of those. If you find anything wrong, add that in there and/or say so on the forums.

0.3 Qni's Julia package

Julia along with the Equations package can be used as an open source alternative to Matlab.

Official Julia website: http://julialang.org/

Equations repository: https://github.com/jhlq/Equations.jl

To install the package type, inside Julia: Pkg.clone("https://github.com/jhlq/Equations.jl")

Example use in the form of solutions to course assignments is available in notebook format in the PlasmaXNotes repository, in the examples directory of Equations and as a html at http://artai.co/Plasma.html

The Equations package is currently in a developing stage so please submit any encountered issues.

1 Week 1. Description of the plasma state, with Paolo Ricci

Didn't start making notes until 1.5 so I'll be skimming the earlier topics.

1.1 Plasmas in nature and laboratory

- Plasma - the 4th state of matter. Heat stuff up to 11400K (= 1eV) and gases begin being ionized.
- The Sun is a miasma of incandescent plasma[1]
- Lightning is plasma (ionized air)
- Plasma diplays
- Nuclear fusion - can't really get there without turning stuff into plasma
- The word 'plasma' comes from greek $\pi\lambda\alpha\sigma\mu\alpha$, which means 'moldable substance' or 'jelly', though it was mentioned on the forums that it might mean 'living thing'... which is really fitting when you think about it
- A brief history:
  - 1920's-1930's: ionospheric plasma research (for radio transmission) and vacuum tubes (Langmuir)
  - 1940's: MHD plasma waves (Alfven)
  - 1950's: research on Magnetic Fusion. Geneva UN conference on uses for atomic energy which don't kill people
- Fusion experiments: L-1, TFTR, JET, ITER tokamaks; W7-X stellarator at MPI in Germany; the NIF inertial fusion facility in US
- The Earth's magnetosphere; van Allen belts
- Jets - space plasmas
- Lots of industrial applications

1.2 Rigorous definition of plasma: Debye length

A plasma is a **globally neutral** *ionised gas* with **collective effects**

The following parameters classify plasmas:

- **Debye length**

  Distance over the potential of a charged particle decreases by a factor $1/e$ due to screening by other charged particles

  $$\lambda_{De} = \sqrt{\frac{\epsilon_0 T_e}{e^2 n_0}}$$

  (for electrons)

  Solved in lecture by a statistical approach which assumed $n\frac{4}{3}\pi\lambda_{De}^3 \equiv N_D \gg 1$ (for a Debye sphere; in the lecture $n\lambda_{De}^3$ was used, which relates to a Debye cube. There's not much difference between them, a factor of 4). N_D means the number of particles inside a sphere (or cube, following the lecture) of radius equal to the Debye length. The condition means there's plenty of particles to screen our test particle. This also assumed that binary interactions between particles were weak ($\frac{e\phi}{T_e} \gg 1$)

[1] https://www.youtube.com/watch?v=sLkGSV9WDMA

1.3 Plasma definition: frequencies and parameters

- **Plasma frequency** Assume a plasma of same density of ions and electrons. Displace electrons by Δx. They begin to exhibit harmonic oscillations (for Δx not too large). Newton's 2nd law gives

$$\frac{d^2\Delta x}{dt^2} + \frac{n_0 e^2}{\epsilon_0 m_e}\Delta x = 0$$

  Can define plasma frequency

$$\omega_{pe} \equiv \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}} = \frac{v_{th,e}}{\lambda_{De}}$$

  where $v_{th,e}$ denotes the thermal speed of electrons

- **Collision frequency**

  The frequency of coulomb collisions between particles

$$\nu_{coll} \equiv \frac{n_0 e^4}{16\pi\epsilon_0^2 m_e^2 v_{th,e}^3}$$

- Size of plasma has to be much larger than its Debye length (or there's no quasineutrality)

1.4 Particle motion in a static uniform magnetic field . Plasma magnetic properties

- Larmor radius - particles gyrate around the guiding center at this distance

$$\rho \equiv \frac{mv_\perp}{|q|\,B}$$

- Cyclotron frequency

$$\omega_c \equiv \frac{v_\perp}{\rho} = \frac{|q|\,B}{m}$$

  Particle rotation direction on their helical trajectory

  - $q > 0$ ('by default'): left hand rotation with respect to $\mathbf{B}$
  - $q < 0$ (electrons): right hand rotation
  - Magnetic moment

$$|\mu| \equiv IA = \frac{|q|\,\omega_e}{2\pi}\pi\rho^2 = \frac{mv_\perp^2}{2B} = \frac{E_{kin\perp}}{B}$$

    (direction opposite to B) is an adiabatic invariant for every particle; doesn't change under slow changes of factors involved in the equation for μ. However, it will change through heat exchange, which usually operates on slower timescales than magnetic field changes (see 2c).

    Plasmas are diamagnetic (they reduce externally applied magnetic fields) (because of direction of μ)

1.5 Particle motion in given electromagnetic fields: the drifts

Static and uniform E and B fields. Particles under Lorentz force which can be decomposed as:

- Parallel direction:

$$m\frac{d\mathbf{v}_\parallel}{dt} = qE_\parallel$$

  Uniform acceleration

- Perpendicular direction:
$$m\frac{d\mathbf{v}_\perp}{dt} = q(\mathbf{E}_\perp + \mathbf{V}_\perp \times \mathbf{B})$$

The many drifts in a plasma:

- $\mathbf{E} \times \mathbf{B}$ drift
  - Perpendicular component averages out over gyroperiod
  $$\mathbf{v_e} = \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2}$$
  - This is a motion of the guiding center which is superposed over the gyromotion
  - Does not depend on charge, neither in magnitude nor in direction (but gyromotion direction does)
  - Guiding center moves over lines of constant electrostatic potential ϕ (the drift does not change the particle energy!)
  - A generalization of this drift for any force:
  $$\mathbf{v_F} = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}$$
  - For a gravitational force (say, space plasmas), this depends on charge. Separates positive and negative charges. Polarizes the plasma, creating a $\mathbf{E}$ field and an $\mathbf{E} \times \mathbf{B}$ drift
- Curvature drift
  - $\mathbf{B}$ field curved, particle follows the B field - this happens through a centrifugal force
  $$\mathbf{F_c} = \frac{mv_\parallel^2}{R_B^2}\mathbf{R_B}$$
  - This causes a drift:
  $$\mathbf{v_d} = \frac{\mathbf{F_c} \times \mathbf{B}}{qB^2} = \frac{mv_\perp^2}{qB^2R_B^2}(\mathbf{R_B} \times \mathbf{B})$$
- Gradient drift $\boldsymbol{\nabla} B \perp \vec{B}$)
  - Happens in changing (spatially) magnetic fields
  $$\mathbf{v_{\boldsymbol{\nabla} B}} = \frac{mv^2}{2qB^3}(\mathbf{B} \times \boldsymbol{\nabla} B)$$
  - A derivation so complicated, it deserved a separate appendix. As particles gyrate, they move between regions of smaller and bigger B. This causes a drift in a direction perpendicular to both the B field and the gradient of its value We consider a small variation in B and expand B in a taylor series around B_0.

    Then we use that expansion to solve $m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times B$, plugging in our expansion for B.

    We also decompose the velocity: an average v_0 and a small perturbation. v_0 is the solution to the equation for constant magnetic field B_0.

    We neglect the cross product of the two small perturbations and average over a gyroperiod.

    We use our knowledge of the solution for the static magnetic field (gyration in the plane perpendicular to B) to deal with the perpendicular velocities (x and y in this decomposition under the assumption that B is along z).

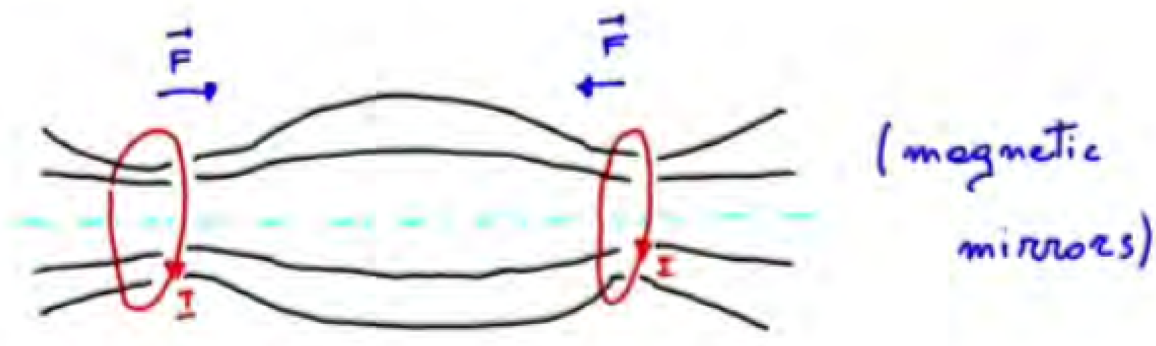
    The drift velocity is the perturbation described by the formula above for an arbitrary geometry of the problem.

1.6 Plasma confinement based on single particle motion. Magnetic mirrors, stellarators, tokamaks

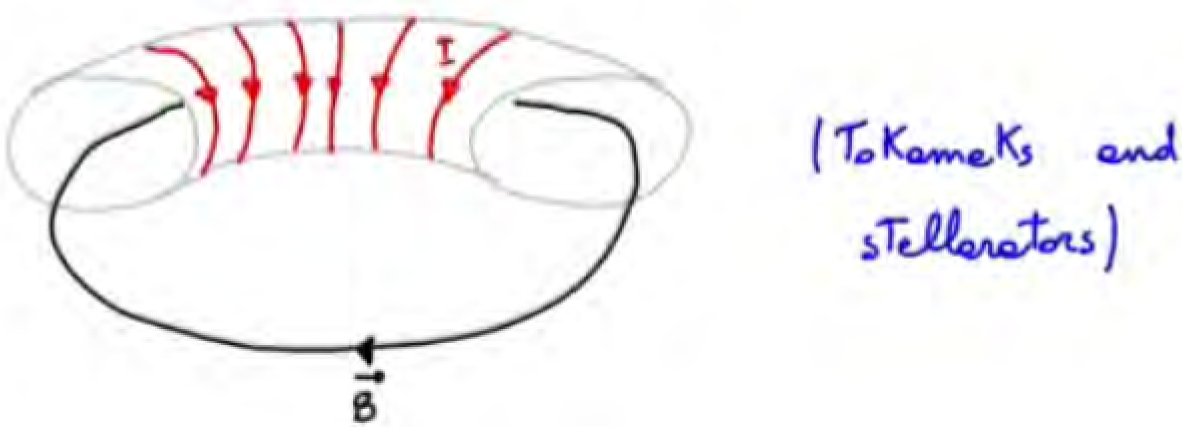
- How do you confine a plasma?

  Charged particles follow helical trajectories along B field. This confines them in the perpendicular direction. What about the parallel one?

- Can use open field lines. Take two circular coaxial electromagnets.



- Can use closed field lines. Closed geometries. Example: tokamaks (toroidal), stellarators.



- The magnetic mirror geometry is neat for particles really close to the axis. B is maximum (field density increases) near the electromagnets

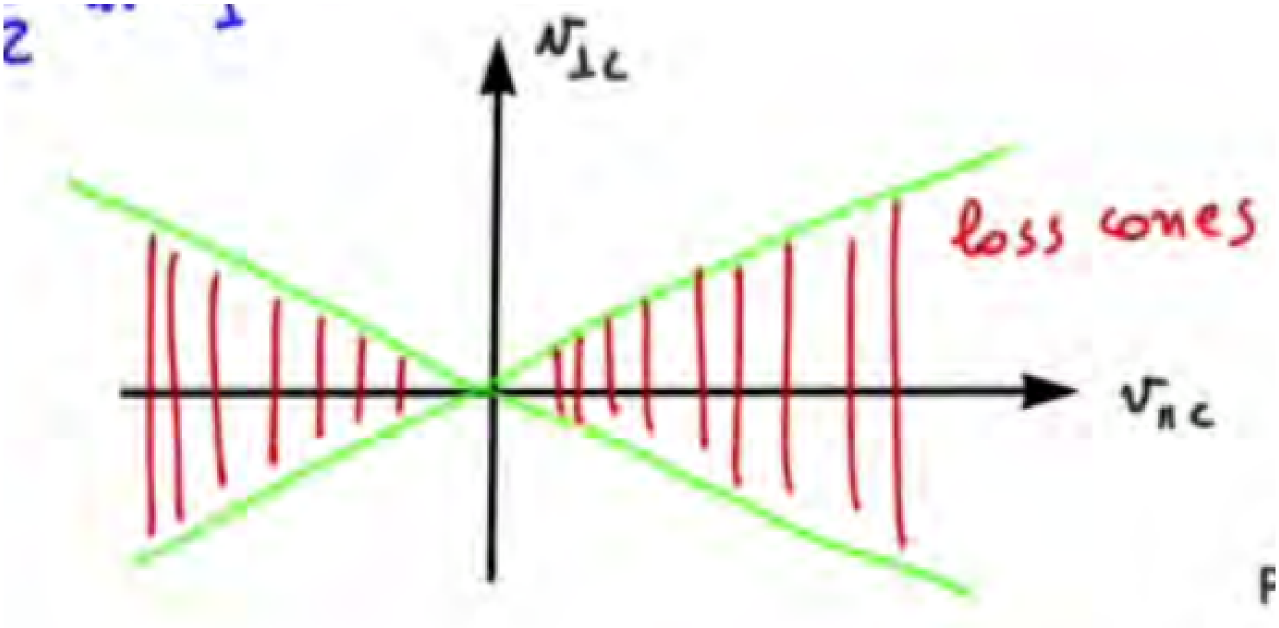
  force in the axial direction is

$$F_z = -\mu |\nabla B|$$

  $v_{\parallel}$ has to vanish at B_{max} so that the kinetic energy is just composed of the perpendicular component of velocity

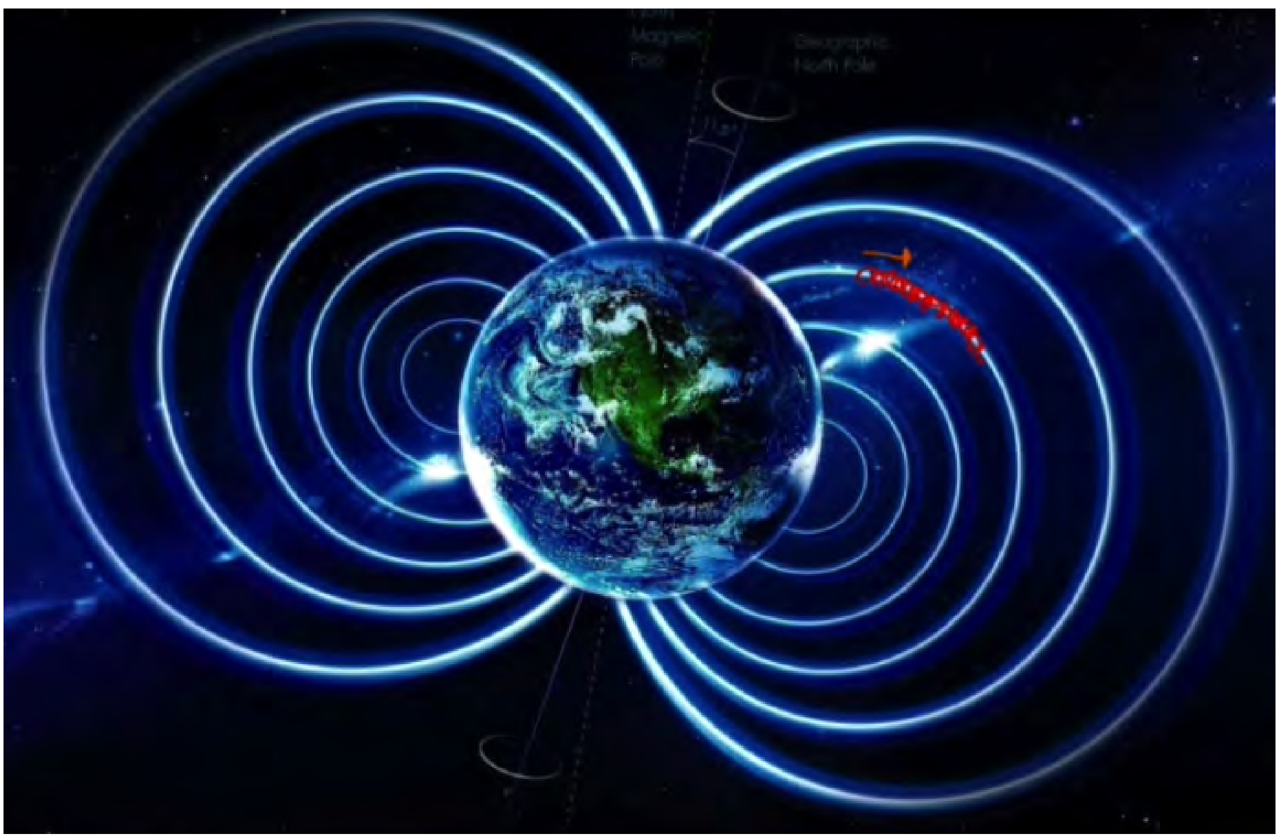
  Particle reflection condition

$$\frac{v_{\perp}^2}{v_{\perp}^2 + v_{\parallel}^2} > \frac{B_{min}}{B_{max}}$$



This means that particles in the **loss cones** in phase space (marked red; those which don't satisfy the inequality) cannot be confined in the mirror!

Neat example: the Earth's magnetic field is a magnetic mirror!



- What about closed magnetic field lines? Can those deal with loss cones?

  B is not homogeneous! Curved! Has curvature and gradient drifts!

  For a purely toroidal field, positively charged particles drift towards the bottom, while negatively charged ones drift towards the top. This polarizes the plasma and introduces the $E \times B$ drift outwards, sending the plasma crashing into the major radius wall.

  A solution: a poloidal magnetic field to short circuit the charge accumulation. Either:

– Drive a current through the plasma → Tokamaks
– Get rid of axial symmetry → Stellarators

2 Week 2: Kinetic description of plasmas, with Paolo Ricci

2.1 From single particle to kinetic description

Kinetic description of plasma. A (relatively?) complete description of plasma which covers both the particles and the fields evolving over time.

The usual diagram for a plasma description, seen often in simulations:

(a) Take Newton's equations using electric and magnetic fields **for all particles at all times** (use Lorentz force)

(b) Use positions and velocities to compute charge and current densities. Charge density given as sum over particles of their charges, localized through use of Dirac delta functions. Current density - similar, but multiplied by particle velocity vectors inside the sum.

(c) Take charge and current density, plug them into Maxwell equations, calculate E and B fields at positions

(d) Take calculated E and B fields and apply them as forces to particles. Repeat cycle until bored or simulation returns segmentation fault.

But real plasmas involve on the order of 10^{21} particles for a fusion plasma. Too much strain on our computational abilities. Impractical. We use a distribution function:

$f(\mathbf{r}, \mathbf{v}, t) d\mathbf{r} d\mathbf{v}$ = number of particles at time t, in phase space volume $d\mathbf{r}d\mathbf{v}$ located at $\mathbf{r}, \mathbf{v}$. We have a separate distribution function f_i for every species

- Total number of particles N_S given by integral of distribution function over all positions and velocities (which covers all the phase space)
- Number density of particles n_s given by integral over all velocities for a given location $\mathbf{r}$
- Average velocity given by $\frac{1}{n_s}\int \mathbf{v} f_i(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}$

Examples of distribution functions

- Maxwell-Boltzmann distribution function, for three dimensions

$$F_0(\mathbf{v}) = n_0\Big(\frac{1}{2\pi v_{thermal}}\Big)^{3/2} \exp\Big(-\frac{v^2}{2v_{thermal}^2}\Big)$$

  In 1D, only the normalization of the distribution changes from the 3D case:

$$F_0(v) = n_0\Big(\frac{1}{2\pi v_{thermal}}\Big)^{1/2} \exp\Big(-\frac{v^2}{2v_{thermal}^2}\Big)$$

- Monoenergetic beam in 1D

$$F_0(v) = n_0\delta(v - v_0)$$

- Two counterstreaming beams in 1D (two-stream instability!)

$$F_0(v) = \frac{n_0}{2}[\delta(v - v_0) + \delta(v + v_0)]$$

Conservation of particles number

If there are no sources or sinks, we have the following condition for conservation of number of particles

$$\frac{df_s}{dt} = -\nabla_{6D} \cdot (\mathbf{u} f_s)$$

where we introduce the six-dimensional nabla operator because who's gonna stop us

$$\nabla_{6D} = \Big(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{dv_x}, \frac{d}{dv_y}, \frac{d}{dv_z}\Big) = \Big(\frac{d}{d\mathbf{r}}, \frac{d}{d\mathbf{v}}\Big)$$

$$\mathbf{u} = \Big(\frac{d\mathbf{r}}{dt}, \frac{d\mathbf{v}}{dt}\Big) = \Big(\mathbf{v}, \frac{\mathbf{F}}{m_s}\Big) = \Big(\mathbf{v}, \frac{\mathbf{F}_{\mathbf{longrange}} + \mathbf{F}_{\mathbf{shortrange}}}{m_s}\Big)$$

Long range forces - collective interactions. Short range forces - binary collisions (between individual particles, like you'd have in a gas). Plugging these back into the particle conservation equation:

$$\frac{df_s}{dt} = -\frac{d}{d\mathbf{r}} \cdot (\mathbf{v} f_s) - \frac{d}{d\mathbf{v}} \cdot \Big[\frac{\mathbf{F}_{\mathbf{longrange}} + \mathbf{F}_{\mathbf{shortrange}}}{m_s} f_s\Big]$$

Boltzmann equation We can improve on the previous equation. Start out with the expanded particle conservation equation:

$$\frac{df_s}{dt} = -\frac{d}{d\mathbf{r}} \cdot (\mathbf{v} f_s) - \frac{d}{d\mathbf{v}} \cdot \Big[\frac{\mathbf{F}_{\mathbf{longrange}} + \mathbf{F}_{\mathbf{shortrange}}}{m_s} f_s\Big]$$

- In the phase space approach, velocity is treated as a completely independent variable from r (though you could consider one as a derivative of the other). Thus $\frac{d}{d\mathbf{r}} \cdot (\mathbf{v} f_s) = \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}}$
- long range force can be decomposed into electric field independent of v, and the $\mathbf{v} \times \mathbf{B}$ term - perpendicular to v. Thus, $\frac{d}{d\mathbf{v}} \cdot [\mathbf{F}_{\mathbf{longrange}} f_s] = \mathbf{F}_{\mathbf{longrange}} \cdot \frac{df_s}{d\mathbf{v}}$
- Plugging in:

$$\frac{df_s}{dt} = -\mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} - \frac{\mathbf{F}_{\mathbf{longrange}}}{m_s} \cdot \frac{df_s}{d\mathbf{v}} - \frac{d}{d\mathbf{v}} \cdot \Big(\frac{\mathbf{F}_{\mathbf{shortrange}}}{m_s} f_s\Big)$$

- Can be rewritten as:

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{\mathbf{F}_{\mathbf{longrange}}}{m_s} \cdot \frac{df_s}{d\mathbf{v}} = -\frac{d}{d\mathbf{v}} \cdot \Big(\frac{\mathbf{F}_{\mathbf{shortrange}}}{m_s} f_s\Big)$$

  Term on the right is called a 'collision operator' $(\frac{df}{dt})_c$.

- And we get the **Boltzmann equation**:

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E}_{\mathbf{longrange}} + \mathbf{v} \times \mathbf{B}_{\mathbf{longrange}}) \cdot \frac{df_s}{d\mathbf{v}} = \Big(\frac{df_s}{dt}\Big)_c$$

2.2 2b) Coulomb collisions in plasmas. Bonus module.

We use Boltzmann equation and look into the short range interactions

An electron with charge $-e$ approaches a positive ion (assumed immobile) with charge Ze. Electron trajectory changes. $\mathbf{v_e}$ - initial electron velocity b - impact parameter, shortest distance between extrapolated line of initial electron trajectory and ion position

$$\frac{\text{Coulomb interaction energy}}{\text{Kinetic energy}} \sim \frac{\frac{Ze^2}{4\pi\epsilon_0 b}}{m_e v_e^2} \sim 1$$

(similar to one so collision interaction is important)

$$b \sim \frac{Ze^2}{4\pi\epsilon_0 m_e v_e^2} = b_{\pi/2}$$

Coulomb cross section: $\sigma_{\pi/2} = \pi b_{\pi/2}^2 = \frac{\pi Z^2 e^4}{(4\pi\epsilon_0)^2 m_e^2 v_e^4}$

Collision frequency: $\nu_{\pi/2} = n_i v_e \sigma_{\pi_2} = fracn_i \pi Z^2 e^4 (4\pi\epsilon_0)^2 m_e^2 v_e^3$

Is this a correct estimate? Do collective small angle deflections matter in a plasma? How can we take the interaction with many particles into account properly? Average over all phase space somehow?

Take the electron-ion collision again. Denote θ - angle between initial and final electron velocity.

Particles interact through Coulomb force. Angular momentum and energy - conserved (if electron is much lighter than ion, $\frac{m_e}{m_i} \ll 1$).

$$\mathrm{tg}(\theta/2) = \frac{b_{\pi/2}}{b} = \frac{Ze^2}{4\pi\epsilon_0 m_e v_e^2 b}$$

$b_{\pi/2}$ - impact parameter at which collision deflects electron by 90°.

Cumulative effect for many collisions? Imagine electron moving towards ion cloud.

Due to symmetry we take $\langle \Delta \mathbf{v}_{\perp \mathbf{e}} \rangle = 0$ but $\langle \Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} \rangle \neq 0$. So magnitude could change, but there will be no preferred direction. $\perp$ stands for parallel to initial velocity.

$$\frac{d \langle \Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} \rangle}{dt} = \int db \qquad n_i v_e 2\pi b$$

(we integrate over all possible impact parameters

$$\Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} = v_e^2 \sin\theta^2 = \Delta v_e^2 \,\mathrm{tg}\,\theta/2^2 [1 + \mathrm{tg}\,\theta/2^2]^{-2}$$

Plugging into the integral:

$$\frac{d \langle \Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} \rangle}{dt} = 8\pi n_i v_e^3 \int_0^{\lambda_D} db \qquad \frac{(b_{\pi/2}/b)^2 b}{(1 + (b_{\pi/2}/b)^2)^2} \pi b$$

We neglect quantum effects (thus integrating from 0) and integrate up to Debye length as coulomb interactions are screened beyond it. Finally, we get:

$$\frac{d \langle \Delta \mathbf{v}_{\perp \mathbf{e}}^{\mathbf{2}} \rangle}{dt} = 8\pi n_i v_e^3 b_{\pi/2}^2 \ln \frac{\lambda_D}{b_{\pi/2}} (\text{if } \lambda_D \gg b_{\pi/2}$$

Following section may have some 4's swapped for Δ's.

- Note that electrons do not lose much energy as $m_e \ll m_i$. Basically reflected balls from a wall. Thus

$$v_e(\Delta v_{\parallel e}) + 0.5 \Delta v_{\perp e}^2 = 0$$

  And

$$\frac{d \left\langle \Delta v_{\parallel e} \right\rangle}{dt} = -4\pi n_i v_e^2 b_{\pi/2}^2 \ln \frac{\lambda_D}{b_{\pi/2}}$$

- We define the coulomb logarithm:

$$\ln \Lambda \equiv \ln \frac{\lambda_D}{b_{\pi/2}} \sim \text{In most plasmas equals 15 to 25}$$

-

$$\frac{d \left\langle \Delta v_{\parallel e} \right\rangle}{dt} = -\nu_{ei} v_e$$

  Collision frequency of electrons against ions:

$$\nu_{ei} = 4\pi n_i b_{\pi/2}^2 v_e \ln \Lambda = n_i \sigma_{ei} v_e$$

  Whereas

$$\sigma_{ei} = 4\pi b_{\pi/2}^2 \ln \Lambda$$

  Can compare

$$\frac{\sigma_{\pi/2}}{\sigma_{ei}} = \frac{\pi b_{\pi/2}^2}{4\pi b_{\pi/2}^2 \ln \Lambda} \ll 1$$

  Much smaller than 1! So small angle deflections dominate over large scale deflections!

2.3 2c) Collisional processes in plasmas

2.3.1 Slowing down of an electron beam

$$\frac{d\left\langle \Delta v_{\parallel e}\right\rangle}{dt} = -\nu_{ei} v_e = -\frac{n_i Z^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_e^2 v_e^3}$$

We could use this to calculate how an electron beam slows in a plasma. Assume a Maxwellian distribution of electron velocities with mean velocity $u_e \ll v_{thermal,e}$ in 1D:

$$f_e(v) = n_0\Big(\frac{m_e}{2\pi v_{thermal,e}}\Big)^{1/2} \exp\Big(\frac{-m_e(v_{\parallel e} - u_e)^2}{2v_{thermal,e}^2}\Big)$$

$$\frac{du_e}{dt} = -\left\langle \nu_{ei} v_{\parallel e}\right\rangle = -\frac{1}{n_0}\int \nu_{ei} v_{\parallel e} f_e(v_{\parallel e}) dv_{\parallel e} \simeq -\langle \nu_{ei}\rangle\, u_e (\text{if } u_e \ll v_{thermal,e})$$

The average collision frequency between electrons and ions ν_{ei} is

$$\nu_{ei} = \frac{\sqrt{2}}{12\pi^{3/2}} \frac{n_i Z^2 e^4 \ln\Lambda}{\epsilon_0^2 m_e^{1/2} T_e^{3/2}}$$

There are also collisions between electrons coming from the beam and electrons in the plasma:

$$\nu_{ee} = \frac{\sqrt{2}}{12\pi^{3/2}} \frac{n_e e^4 \ln\Lambda}{\epsilon_0^2 m_e^{1/2} T_e^{3/2}} \sim \frac{\langle \nu_{ei}\rangle n_e}{Z^2 n_i}$$

2.3.2 Plasma resistivity

Take a cloud of ions and electrons. Apply electric field **E**. Ions will move in direction of E, whereas electrons will move in opposite direction. E then drives a current in a plasma - charges are moving!

We neglect the slow and heavy electrons and focus on electron movement. From Newton's second law:

$$m_e n_e \frac{d\mathbf{u_e}}{dt} = -en_e\mathbf{E} + \mathbf{R_{ei}}$$

R_{ei} is the collision term we have just calculated. This slows down the current.

$$\mathbf{R_{ei}} = -m_e n_e \langle \nu_{ei}\rangle (\mathbf{u_e} - \mathbf{u_i}) \text{ (assuming } u_e \ll v_{th,e})$$

- After a transient, we'll reach steady state operation and $\frac{d}{dt} = 0$
- The current can be depicted as $\mathbf{j} = -n_e e(\mathbf{u_e} - \mathbf{u_i})$

Thus:

$$e^2 n_e \mathbf{E} = m_e \langle \nu_{ei}\rangle \mathbf{j}$$

$$\mathbf{E} = \frac{m_e \langle \nu_{ei}\rangle}{e^2 n_e}\mathbf{j} \equiv \eta\mathbf{j}$$

By comparison with Ohm's law we can define the plasma resistivity:

$$\eta \equiv \frac{m_e \langle \nu_{ei}\rangle}{e^2 n_e} = \frac{\sqrt{2m_e} Z e^2 \ln\Lambda}{12\pi^{3/2}\epsilon_0^2 T_e^{3/2}}$$

The bigger the temperature, the lower the resistivity. Unlike in metals. It's also independent of density! The contributions of increasing the number of carriers and increasing the number of collisions cancel each other out exactly.

2.3.3 Overview of plasma collision frequencies

- Electron - ion collision frequency $nu_{ei} =$
- Electron - electron collision frequency nu_{ei}
- Ion - ion collision frequency nu_{ii}.

Ions gain energy when you fire an electron beam into a plasma (could be heated this way?).

$$m_e \Delta \mathbf{v_e} = m_i \Delta \mathbf{v_i}$$
$$0.5 m_i \left| \Delta \mathbf{v_i} \right|^2 = \frac{m_e^2}{2m_i} \left| \Delta \mathbf{v_e} \right|^2 \sim \frac{m_e^2}{2m_i} \left| \Delta \mathbf{v}_{\perp \mathbf{e}}{}^2 \right|$$

(as we can ignore the change in parallel electron velocity)

Rate of exchange of energy (between species! This equalizes the temperatures between electrons and ions!):

$$\langle \nu_E \rangle = \frac{n_i Z^2 e^4 \sqrt{m_e} \ln \Lambda}{3\pi \sqrt{2\pi} \epsilon_0^2 m_i T_e^{3/2}} \sim Z \frac{m_e}{m_i} \langle \nu_{ei} \rangle$$

The electrons have a similar, very fast rate of collisions with each other and with ions. The rate of collisions between ions happens 40 times slower, and then the rate of energy exchange is 40 times slower than that. At a similar rate to that of energy exchange is the rate of ions colliding with electrons.

2.4 2d) Vlasov equation

2.4.1 Derivation from Boltzmann equation

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s} (\vec{E} + \vec{v} \times \mathbf{B}) \cdot \frac{df_s}{d\mathbf{v}} = \left(\frac{df_s}{dt} \right)_c$$

If we can assume that the number of particles in a Debye cube is REALLY HIGH: $n\lambda_D^3 \ggg$, so that $(\frac{df}{dt})_c = 0$, then the **Vlasov equation** holds:

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s} (\vec{E} + \vec{v} \times \mathbf{B}) \cdot \frac{df_s}{d\mathbf{v}} = 0$$

E and B here represent the long range interactions. The charge density is computed as indicated before, integrating out all the velocities. The currents are likewise obtained by summing over the species and calculating the average velocities at each positions.

2.4.2 Conservation laws for the Vlasov equation

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s} (\vec{E} + \vec{v} \times \mathbf{B}) \cdot \frac{df_s}{d\mathbf{v}} = 0$$

This satisfies the following conservation properties:

- Number of particles - we can integrate the Vlasov equation over all positions and velocities. Integrating $\frac{df_s}{dt}$ gives us $\frac{dN_s}{dt}$. The second term gives us, by means of Gauss (divergence) theorem and pushing the boundaries out to infinity, where f_s should decay to zero, zero. In the third term we have a velocity divergence. Since no particles have infinite velocities[2], we can once more use the divergence theorem *(in velocity space!!!)* to eliminate the third term and we reach $\frac{dN_s}{dt} = 0$. Particles are conserved.

[2] Einstein says hi.

- Momentum, which we calculate as the sum of particle and field momenta. No actual derivation is given except for the formula:

$$\mathbf{P_{tot}} = \sum_s m_s \int d\mathbf{r} \int d\mathbf{v}\, \mathbf{v} f_s + \epsilon_0 \int d\mathbf{r}(\mathbf{E} \times \mathbf{B}) = \text{const}$$

- Total energy can be decomposed into energy of particles and energy of field.

$$E = \sum_s \int d\mathbf{r} \int d\mathbf{v} \frac{1}{2} m_s v^2 f_s + \frac{1}{2} \int d\mathbf{r}(\epsilon_0 E^2 + B^2/\mu_0) = \text{const}$$

- Entropy, as given by information theory

$$S = \sum_s \int d\mathbf{r} \int d\mathbf{v} f_s \ln f_s = \text{const}$$

  This is because collisions are neglected by the Vlasov equation. Therefore it is time-reversible!

2.4.3 Interpretation of Vlasov equation

- f_s has incompressible motion (in phase space) - it can be considered as an incompressible fluid (moving in phase space - it's going to obey Liouville's theorem)
- As seen by particle along orbit

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m_s} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}}$$

  And the funny thing is, that's just the Vlasov equation itself. Thus along particle orbits, $f_s = \text{const}$.

2.4.4 (Formal) solutions of Vlasov equation

If c_j is a constant of motion, then any distribution function being a function of any number of constants of motion c_j is a solution.

This is unlike the Boltzmann equation, where only the Maxwellian distribution was a stationary solution.

It can be really difficult to find constants of motion, as well. It seems to be implied that practical solutions rely on numerical methods - that way you need not specify constants of motion for a formal solution.

2.5 The two stream instability!

We make this our testing ground for the Vlasov equation. The situation is two beams of electrons moving in opposite directions in 1D. Spoiler alert - WE'RE ACTUALLY GOING TO MESS AROUND WITH THE MATLAB CODE FOR THIS KIND OF SIMULATION IN THE HOMEWORK, THIS IS AWESOME. Ahem.

2.5.1 Simplifications

We take the Vlasov equation and Maxwell's equations

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s}(\vec{E} + \vec{v} \times \mathbf{B}) \cdot \frac{df_s}{d\mathbf{v}} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0 \frac{d\mathbf{E}}{dt})$$

We simplify the situation. Set $\mathbf{B} = 0$ for an electrostatic situation.

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s} \vec{E} \cdot \frac{df_s}{d\mathbf{v}} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = 0$$

which implies

$$\mathbf{E} = -\nabla \phi$$

$$\Delta \phi = -\frac{\rho}{\epsilon_0}$$

We also assume that ions are stationary in the background with density n_0 (electrons move at a much faster time scale). This means we only have to solve Vlasov for the electron motion and can replace f_s by f for brevity:

$$\frac{df}{dt} + \mathbf{v} \cdot \frac{df}{d\mathbf{r}} - \frac{e\vec{E}}{m_s} \cdot \frac{df}{d\mathbf{v}} = 0$$

$$\Delta \phi = \frac{e}{\epsilon_0} \int f d\mathbf{v} - \frac{e}{\epsilon_0} n_0$$

2.5.2 Linearisation

We're going to be analysing small perturbations from equilibrium. This means basically expanding the quality of interest in a short Taylor series:

$$g = g_0 \text{ (equilibrium)} + g_1 \text{ (perturbation, } g_1 \ll g_0)$$

In our case, f_0 being the initial distribution functions which is isotropic over all position space (thus no dependence on $\mathbf{r}$) and f_1 being our small perturbation:

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t)$$

$$\phi = \phi_1(\mathbf{r}, t) \text{ as we can set } \phi_0 = 0 \text{ since it's constant anyway}$$

$$\mathbf{E} = \mathbf{E_1}(\mathbf{r}, t)$$

We plug these into the Vlasov equation:

$$\frac{\partial f_0 + f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial (f_0 + f_1)}{\partial \mathbf{r}} - \frac{e\mathbf{E_1}}{m_e} \cdot \frac{\partial (f_0 + f_1)}{\partial \mathbf{v}} = 0$$

Simplifying and neglecting $\mathbf{E_1} \cdot \frac{\partial f_1}{\partial \mathbf{v}}$ as small times small:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e\mathbf{E_1}}{m_e} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

Also plugging in $\mathbf{E_1} = -\nabla \phi$:

$$\Delta \phi_1 = \frac{e}{\epsilon_0} \int f_1 d\mathbf{v}$$

Thus:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m_e} \frac{\partial \phi_1}{\partial \mathbf{r}} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

$$\Delta \phi_1 = \frac{e}{\epsilon_0} \int f_1 d\mathbf{v}$$

We now apply Fourier analysis to f_1:

$$f_1(\mathbf{r}, \mathbf{v}, t) = \int d\mathbf{k} \int d\omega \tilde{f}_1(\mathbf{k}, \mathbf{v}, \omega) \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

This is neat because:

- The time derivative is simple:

$$\frac{\partial f_1}{\partial t}(\mathbf{r}, \mathbf{v}, t) = \int d\mathbf{k} \int d\omega (-i\omega) \tilde{f}_1(\mathbf{k}, \mathbf{v}, \omega) \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

- The spatial derivative is also pretty simple:

$$\frac{\partial f_1}{\partial \mathbf{r}}(\mathbf{r}, \mathbf{v}, t) = \int d\mathbf{k} \int d\omega (i\mathbf{k}) \tilde{f}_1(\mathbf{k}, \mathbf{v}, \omega) \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

So we go back to the Vlasov equation and plug in the expressions above:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m_e} \frac{\partial \phi_1}{\partial \mathbf{r}} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0$$

$$\int d\mathbf{k} \int d\omega [(-i\omega + i\mathbf{k} \cdot \mathbf{v}) \tilde{f}_1 + \frac{ie\tilde{\phi}_1}{m_e} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}] \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t) = 0$$

This can be true only if all the coefficients vanish:

$$(-i\omega + i\mathbf{k} \cdot \mathbf{v}) \tilde{f}_1 + \frac{ie\tilde{\phi}_1}{m_e} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0$$

$$\tilde{f}_1 = \frac{e}{m_e} \frac{\tilde{\phi}_1}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

Plugging this result into the Fourier transform of the Poission equation above:

$$\Delta \phi_1 = \frac{e}{\epsilon_0} \int f_1 d\mathbf{v}$$

$$-k^2 \tilde{\phi}_1 = \frac{e}{\epsilon_0} \int \tilde{f}_1 d\mathbf{v} = \frac{e^2 \tilde{\phi}_1}{\epsilon_0 m_e} \int \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v}$$

Which then implies

$$\tilde{\phi}_1 k^2 [1 + \frac{e^2}{\epsilon_0 m_e k^2} \int \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v}] = 0$$

We denote the bulky part as the dispersion function

$$D(\omega, \mathbf{k}) \equiv 1 + \frac{e^2}{\epsilon_0 m_e k^2} \int \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v}$$

Which we can find the roots of, and the solutions (values of ω and $\mathbf{k}$) gives us the normal modes of our plasma.

There's a singularity at $\omega = \mathbf{k} \cdot \mathbf{v}$. This is when particles match velocities with wave velocities in the plasmas... this may or may not be connected to Landau damping. This topic will not be further developed in the course.

2.5.3 Getting to the two-stream instability

The two-stream instability is thermodynamically weird. The velocity distribution is non-maxwellian. It's just two sharp peaks (low entropy). Could there be intrinsic modes in the system that restore thermodynamic equilibrium (high entropy)?

We take a 1D system. The derivation above is general (plenty of vectors brought to you by yours truly).

$$f = f_0(v_x) = f(u)$$

$$D(\omega, k) = 1 + \frac{e^2}{\epsilon_0 m_e} \frac{1}{k} \int \frac{df_0}{du} \frac{du}{\omega - ku}$$

I'm pretty sure there's a mistake in the lecture and a $1/k$ factor was lost there (it should be $1/k^2$).

For two counter streaming beams:

$$f_0(u) = \frac{n}{2}[\delta(u - v_0) + \delta(u + v_0)]$$

The distribution function 'luckily' avoids the singularity. We calculate the dispersion function and arrive at

$$D(\omega, k) = 1 - \frac{ne^2}{2\epsilon_0} m_e [\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}]$$

(note that the plasma frequency pops up: $\omega_{pe}^2 = \frac{ne^2}{\epsilon_0 m_e}$)

This is a 4th order polynomial in the nominator. The function has two vertical asymptotes at $\omega = \pm kv_0$ and a horizontal one at 1.

Depending on the parameters, if $D(\omega = 0, k) \geqslant 0$, there's 4 real roots. The modes will be oscillatory instead of exponentially growing.

Otherwise, if $D(\omega = 0, k) < 0$, there's 2 real roots corresponding to oscillations and 2 complex roots corresponding to exponential explosions.

$$D(\omega = 0, k) = 1 - \frac{\omega_{pe}^2}{k^2 v_0^2} < 0 \rightarrow k^2 v_0^2 < \omega_{pe}^2$$

Unstable modes are those that have sufficiently long wavelengths. That's all we can tell analytically. IT'S PARTICLE IN CELL TIME!

2.6 2f) Kinetic plasma simulations. The Particle in Cell method

2.6.1 The various time scales in a plasma

- $10^{-10}s$ - electron cyclotron motion
- $10^{-7}s$ - ion cyclotron motion
- $10^{-5}s$ - microturbulence
- $10^{-3}s$ - fast global instabilities
- $10^{-1}s$ - slow global instabilities
- $1s$ - energy confinement time
- 10^3s - gas equilibrisation

It's extremely hard to simulate all of these at once! Accurate modelling of cyclotron motion (say, 10^{-14} timestep) would mean observing **fast** global instabilities after 10^11 timesteps. That's a lot of timesteps.

2.6.2 Simulation approach - Particle in Cell (PIC) method

One could solve the entire Vlasov equation by separating the whole $6D$ phase space into a grid and solving the partial differential equation. Possible. But usually, one uses the particle in cell method.

f is constant along particle trajectories. This translates directly into Newton's equations for a single particle:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

In the PIC (Particle in Cell) method, we approximate the distribution function by a discrete sum

$$f(\mathbf{r}, \mathbf{v}, t) \simeq \sum_\alpha f_\alpha(\mathbf{r} - \mathbf{r}_{\alpha\mathbf{0}}, \mathbf{v} - \mathbf{v}_{\alpha\mathbf{0}})$$

The f_α's are called **superparticles**; they have compact support - they're zero everywhere but on a small domain in phase space around $r_{\alpha 0}$ and $v_{\alpha 0}$.

We introduce the integral $I_\alpha = \int\int d\mathbf{r}d\mathbf{v} f_\alpha$ which is done over the small domain of the particle.

At all times, the superparticles satisfy

$$f(\mathbf{r}, \mathbf{v}, t) \simeq \sum_\alpha f_\alpha(\mathbf{r} - \mathbf{r}_\alpha(t), \mathbf{v} - \mathbf{v}_\alpha(t))$$

If we have that

$$\frac{d\mathbf{r}_\alpha}{dt} = \mathbf{v}_\alpha$$

$$\frac{d\mathbf{v}_\alpha}{dt} = \frac{q_\alpha}{m}(\mathbf{E}_\alpha + \mathbf{v}_\alpha \times \mathbf{B}_\alpha)$$

where

$$\mathbf{E}_\alpha = \frac{1}{I_d}\int\int \mathbf{E} f_\alpha d\mathbf{r}\, d\mathbf{v}$$

$$\mathbf{B}_\alpha = \frac{1}{I_d}\int\int \mathbf{B} f_\alpha d\mathbf{r}\, d\mathbf{v}$$

and we have some initial conditions

$$\mathbf{r}_\alpha(t = 0) = \mathbf{r}_{\alpha_\mathbf{0}}$$

$$\mathbf{v}_\alpha(t = 0) = \mathbf{v}_{\alpha_\mathbf{0}}$$

Then the distribution function defined as the sum of all superparticles (their distribution functions) also solves the Vlasov equation.[3]

2.6.3 Practical PIC

In one dimension, we have to:

(a) Solve Poisson's equation to get the potential ϕ and field $\mathbf{E}$

This is done by discretizing space and time. We get discrete approximations to charge densities, potentials and fields.

Derivatives are discretized using central differences.

(b) Evaluate electric fields acting on superparticles

$$E_\alpha = E_j \text{ if } |x_j - x_\alpha| \leqslant \Delta x/2$$

[3] This is a complicated way of saying that we can use particles to approximate the motion of the plasma - see Birdsall and Langdon.

(c) Apply fields to superparticles. Solve for the motion of superparticles by a numerical algorithm (Euler's or Runge-Kutta's)

(d) Assign charge to discrete locations on grid. We sum the charges of superparticles in the j-th cell (in $x_{j-1/2} < x_\alpha < x_{j+1/2}$) and divide by the cell size.

On a closing note: PICs are awesome, they're used in all kinds of fields. Electromagnetic fields, gravitational fields... go learn them. :)

3 Week 3. Fluid description of plasmas, with Paolo Ricci

3.1 From Vlasov to two-fluid

- We'll turn the computationally expensive kinetic model into a simple fluid model (can be solved with partial differential function)
- We'll derive fluid quantities (pressure, etc) from distribution functions
- We'll take a moment for moments of the kinetic equation
- We'll apply the continuity equation (conservation of mass etc)
- ... and we'll get the two fluid model!

3.1.1 Fluid quantities from the distribution function

In the fluid model we only really look at quantities that depend on position. This means we'll have to integrate out all velocity dependence.

Number density - the integral of distribution function over all velocities. Simple. We just care about the density *at a location.*

$$n_s(r,t) = \int dv f_s(r,v,t)$$

For any function $g(r,v)$, it's average value is the integral of the function weighted by distribution function, normalized by dividing the integral by the number density of the species.

$$\langle g(r) \rangle = \frac{1}{n_s} \int g(r,v) f_s(r,v,t) d_v f$$

Examples:

- Average fluid velocity, $g(v) = v$
$$u_s(r,t) = \frac{1}{n_s} \int v f_s(r,v,t) dv$$
- Average kinetic energy density, $g(v) = \frac{1}{2} m_s v^2$
$$w_s(r,t) = \frac{1}{n_s} \int \frac{1}{2} m_s v^2 f_s(r,v,t) dv$$
- Pressure **tensor** (as pressure may depend on direction), $g(v) = m_s(\mathbf{v}-\mathbf{u_s})(\mathbf{v}-\mathbf{u_s})$ - a tensor (dyadic) quantity)
$$\hat{P} = \int m_s(\mathbf{v}-\mathbf{u_s})(\mathbf{v}-\mathbf{u_s}) f_s(r,v,t)$$

3.1.2 Examples of fluid averaged quantities

(a) Beam density. Distribution function with uniform spread in $1D$ positions and a delta function dependence on velocity (only one velocity in the whole beam

$$f = n_0 \delta(\mathbf{v} - \mathbf{v_0})$$

The density ends up being just n_0 because of the Dirac delta velocity dependence.

The fluid velocity is just the velocity of the particles, as the delta function times velocity gives us just the particle velocity upon integrating out.

The kinetic energy density is just $mv_0^2/2$, obviously.

The pressure tensor ends up as zero! Both the vector quantities end up being zero as $\mathbf{v} = \mathbf{v_0}$. Reasonable. If everything's moving at the same velocity there's not going to be anything pushing anything else.

(b) Maxwellian velocity distribution with uniform position distribution

$$f = n_0\Big(\frac{1}{2\pi v_{th}^2}\Big)^{3/2}\exp\frac{-(\mathbf{v}-\mathbf{v_0})^2}{2v_{th}^2}$$

Note that the lecture had a version with masses. This was corrected in the errata - I suppose because we're working in position-velocity phase space instead of position-momentum phase space.

Number density is just n_0, as the velocity distribution is normalized to one anyway.

Average fluid velocity - gaussian average value v_0.

Average kinetic energy density: kinetic energy from average velocity plus $\frac{3}{2}k_bT$ (thermal velocity)

Pressure tensor is a diagonal matrix.

$$\hat{P} = \begin{pmatrix} n_0T & 0 & 0 \\ 0 & n_0T & 0 \\ 0 & 0 & n_0T \end{pmatrix}$$

Note that in the errata, n_0K_BT (from the lecture) was corrected to n_0T, as in this course temperatures are defined thermodynamically, through energy, including the Boltzmann constant.

3.1.3 The moments of the kinetic equation

We take the Boltzmann equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{v}} = \Big(\frac{\partial f_s}{\partial t}\Big)_c$$

Shift the collision term to the left side and denote the whole thing, which is now equal to zero for all positions, times and velocities (I'll denote this as $BOLT$):

$$BOLT = \frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{v}} - \Big(\frac{\partial f_s}{\partial t}\Big)_c = 0$$

And take averages - 'moments' - of that by integrating $BOLT$ with a weight g. This will give us the following equations:

- Continuity for $g = 1$
- Momentum for $g = mv$
- Energy for $g = mv^2/2$

The continuity equation will be done here. The rest are, in true physics tradition, easy to derive and as such are left to the reader.

3.1.4 The continuity equation

$$\int BOLT dv = \int\Big(\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{v}} - \Big(\frac{\partial f_s}{\partial t}\Big)_c\Big)dv = 0$$

- The first term ends up as just the time derivative of average number density

$$\int\frac{\partial f_s}{\partial t}dv = \frac{\partial}{\partial t}\int f_s dv = \frac{\partial n_s}{\partial t}$$

- The second term is the time derivative of average fluid velocity times average number density

$$\int \mathbf{v}\cdot\frac{\partial \mathbf{f_s}}{\partial \mathbf{r}}dv = \frac{\partial}{\partial r}\int v f_s dv = \frac{\partial(n_s\mathbf{u_s})}{\partial \mathbf{r}}$$

- The third term: the forces can be brought under the velocity derivative. We then use the divergence theorem in velocity space and since particles aren't going to have infinite velocities, the surface integral over the region in velocity space is zero.
- The fourth collision term is zero on the grounds that 'collisions do not create nor destroy particles'. Not sure how that works...

On the whole, we get the neat continuity equation:

$$\boxed{\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (n_s \mathbf{u_s}) = 0} \tag{1}$$

3.1.5 The two fluid model!

We treat electrons as one fluid, and electrons as another, separate fluid. The two fluids are in the same region in space and interacting with each other (not much of a plasma otherwise!).

We take the continuity equation derived above

$$\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (n_s \mathbf{u_s}) = 0$$

We also take the momentum equation, which was left for the student to derive

$$m_s n_s \frac{d\mathbf{u_s}}{dt} = q_s n_s \Big(\mathbf{E} + \mathbf{u_s} \times \mathbf{B}\Big) - \frac{\partial}{\partial \mathbf{r}} \cdot \hat{P}_s + \mathbf{R_s}$$

Where $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u_s} \cdot \frac{\partial}{\partial \mathbf{r}}$ is the total time derivative (streaming included), and R is a collision term

$$\mathbf{R_s} = \int m_s(\mathbf{v} - \mathbf{u_s})\Big(\frac{\partial f}{\partial t}\Big)_c dv$$

We also take the energy equation[4]

$$\frac{3}{2} n_s \frac{dT_s}{dt} + \hat{P}_s \cdot \frac{\partial}{\partial \mathbf{r}} \mathbf{u_s} + \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{q_s} = Q_s$$

Where T_s is the temperature factor related to spread of velocity

$$\frac{1}{3n_s} \int m_s(\mathbf{v} - \mathbf{u_s})^2 f_s dv$$

q_s is the heat flux vector, the kinetic energy transported by the velocity of the fluid

$$\frac{m_s}{2} \int (\mathbf{v} - \mathbf{u_s})^2 (\mathbf{v} - \mathbf{u_s}) f_s dv$$

And Q_s is the heat generated by viscous forces (basically friction, collisions?)

$$Q_s = \int \frac{m_s}{2} (\mathbf{v} - \mathbf{u_s})^2 \Big(\frac{\partial f}{\partial t}\Big)_c dv$$

The system of equation is **not closed**, we need an expression for the heat flux, which requires knowing the distribution function. This is a *closure problem.* Closures can be difficult to find and people are actively sciencing that as hard as they can. The problem is that you'd need to use the distribution function to get quantities we'd like to use instead of the distribution function.

Coupling with Maxwell equations is achieved by noting that the charge density is just the species number density times species charge, summed over all species. Current density, likewise, is the sum of species number density times species charge times average species velocity. With these two (and a set of initial conditions), Maxwell equations can be solved for the forces on our fluids.

[4] I'm not sure about the equations on this page. May have taken a tensor for a vector or vice versa. If anyone can confirm they're correct or spot any mistakes, please say so on the forums.

3.1.6 Two fluid simulation of plasma turbulence

Does it make sense to use the two-fluid model for simulations to describe plasmas instead of, y'know, particles?[5]

It does make sense - it's relatively easy to find a good closure to close the set of equations. It's helpful when plasmas are *fairly collisional*. In that regime, the distribution function is usually close to Maxwellian and that can help find a closure.

Two fluid simulations are especially helpful in the periphery of magnetic fusion devices, where plasmas are relatively cold. This actually helps with turbulence, and turbulence is HARD to deal with:

> 'I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.'
>
> - Horace Lamb -

3.2 The two-fluid dispersion relation

3.2.1 Linearization and fourier mode analysis - a review

What we want to do right now is study the evolution of small amplitude perturbations applied to equilibrium states.

Take the continuum equation

$$\frac{\partial n}{\partial t} + n\boldsymbol{\nabla}\cdot\mathbf{u} = 0$$

We take the case of static equilibrium:
$n = n_0 + n_1$, $u = u_0 + u_1 = u_1$
(as we're in a static equilibrium, the equilibrium average velocity is 0)
The linearized continuum equation

$$\frac{\partial n_1}{\partial t} + n_0\boldsymbol{\nabla}\cdot\mathbf{u_1} = 0$$

($n_1\boldsymbol{\nabla}\cdot\mathbf{u_1}$ was neglected as the product of two small quantities)

To begin fourier mode analysis, we write

$$n_1(r,t) = \int\int d\mathbf{k}d\omega\tilde{n}_1(\mathbf{k},\omega)\exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

Thus we get the substitutions $\nabla \to i\mathbf{k}$, $\frac{\partial}{\partial t} \to -i\omega$
So our linearized continuum equation becomes

$$-i\omega\tilde{n}_1 + n_0 i\mathbf{k}\cdot\tilde{\mathbf{u_1}} = 0$$

3.2.2 Linearization of Maxwell equations

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

We take the curl of the first equation, substitute the second and get

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{j}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$$

[5] If the author of the notes seems prejudiced in favor of using particles, it may be related to the fact that particle simulations are **awesome**.

We use our Fourier transform substitutions from earlier and the vector property

$$\mathbf{k}\times\mathbf{k}\times\tilde{\mathbf{E}} = k^2(\frac{\mathbf{kk}}{k^2}-1)\tilde{\mathbf{E}}$$

and write

$$-\frac{c^2k^2}{\omega^2}(\frac{\mathbf{kk}}{k^2}-1)\tilde{\mathbf{E}} = \frac{i}{\epsilon_0\omega}\tilde{\mathbf{j}}+\tilde{\mathbf{E}}$$

Where it turns out that $\frac{c^2k^2}{\omega^2} = N^2$, N is the index of refraction!

3.2.3 Linearization of two-fluid equations

We take the limit of low ($T\to 0$) temperature. What we want to get is $\tilde{\mathbf{j}} = \hat{\sigma}\tilde{\mathbf{E}}$

Using the momentum equation, removing the pressure term due to low temperature limit

$$m_s(\frac{\partial \mathbf{u_s}}{\partial t}+(\mathbf{u_s}\cdot\nabla)\mathbf{u_s}) = q_s(\mathbf{E}+\mathbf{u_s}\times\mathbf{B})$$

We assume that there's equilibrium electric field E_0 (static case, remember?), no movement ($u_{s0}=0$), $n_{s0}=n_0$, and we've got a uniform magnetic field in the z direction $\mathbf{B}=B_0\hat{\mathbf{z}}$

Linearizing, fourier transforming the momentum equation, introducing the cyclotron frequency $\Omega_s = \frac{q_sB_0}{m_s}$, and skipping writing the tildes (basically everything's in fourier space now)

$$\begin{pmatrix} -i\omega & -\Omega_s & 0 \\ \Omega_s & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix}\mathbf{u_{s1}} = \frac{q_s}{m_s}\mathbf{E_1}$$

From this we can get the fluid velocity by inverting the matrix and sticking all the constants in a tensor we call the *mobility tensor* μ_s:

$$\mathbf{u_{s1}} = \frac{q_s}{m_s}\begin{pmatrix} -i\omega & -\Omega_s & 0 \\ \Omega_s & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix}^{-1}\mathbf{E_1} \equiv \hat{\mu_s}\mathbf{E_1}$$

$$\hat{\mu_s} = \begin{pmatrix} \frac{-i\omega}{\Omega_s^2-\omega^2} & \frac{\Omega_s}{\Omega_s^2-\omega^2} & 0 \\ \frac{-\Omega_s}{\Omega_s^2-\omega^2} & \frac{-i\omega}{\Omega_s^2-\omega^2} & 0 \\ 0 & 0 & \frac{-i}{\omega} \end{pmatrix}$$

And we have the current! It's the sum over all species of velocities times densities times charges!

$$\mathbf{j} = \sum_s q_s n_{s0}\mathbf{u_{s1}} = \Big(\sum_s q_s n_{s0}\hat{\mu_s}\Big)\mathbf{E_1} \equiv \hat{\sigma}\mathbf{E_1}$$

Where we introduced $\hat{\sigma}$ as the conductivity tensor.

3.2.4 The two-fluid dispersion relation

$$-N^2(\frac{\mathbf{kk}}{k^2}-1)\mathbf{E_1} = \frac{i}{\epsilon_0\omega}\mathbf{j}+\mathbf{E_1}$$

We define another tensor to help us write all this, noting that $\mathbf{j_1} = \hat{\sigma}\mathbf{E_1}$:

$$\hat{\varepsilon} = \frac{i\hat{\sigma}}{\epsilon_0\omega}+1$$

$$\Big(N^2\Big(\frac{\mathbf{kk}}{k^2}-1\Big)+\hat{\varepsilon}\Big)\mathbf{E_1}$$

And we get the dispersion relation

$$\boxed{\det\Big(N^2\Big(\frac{\mathbf{kk}}{k^2}-1\Big)+\hat{\varepsilon}\Big)=D(\omega,\mathbf{k})=0} \tag{2}$$

Writing $\hat{\varepsilon}$ out explicitly:

$$\hat{\varepsilon}=\begin{pmatrix}\varepsilon_1 & -i\varepsilon_2 & 0\\ i\varepsilon_2 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_3\end{pmatrix}$$

$$\varepsilon_1=1+\sum_s\frac{\omega_{ps}^2}{\Omega_s^2-\omega^2}$$

$$\varepsilon_2=-\sum_s\frac{\Omega_s}{\omega}\frac{\omega_{ps}^2}{\Omega_s^2-\omega^2}$$

$$\varepsilon_3=1-\sum_s\frac{\omega_{ps}^2}{\omega^2}$$

3.3 Two fluid waves, parallel and perpendicular propagation

3.3.1 Two fluid cold plasma dispersion relation

We're once more taking the equilibrium at zero temperature: no fluid velocity, uniform density, no electric field, magnetic field uniform in the $\hat{\mathbf{z}}$ direction.

We use the dispersion relation above. We note by inspecting the matrix[6] that the z direction - the one parallel to the magnetic field - is very different from the perpendicular directions.

In the limit of zero magnetic field, the cyclotron frequency also vanishes. The diagonal epsilons simplify and ε_2 vanishes, turning $\hat{\varepsilon}$ into a diagonal matrix - the plasma becomes isotropic.

If we assume that the wave is propagating in a direction perpendicular to the B field, in the YZ plane, with the direction given by the angle from the z axis

$$\mathbf{k_0}=k\sin\vartheta\hat{\mathbf{y}}+k\cos\vartheta\hat{\mathbf{z}}$$

Then our dispersion relation takes the form:

$$\Big(N^2\Big(\frac{\mathbf{kk}}{k^2}-1\Big)+\hat{\varepsilon}\Big)=\begin{pmatrix}\varepsilon_1-N^2 & -i\varepsilon_2 & 0\\ i\varepsilon_2 & \varepsilon_1-N^2\cos\vartheta^2 & N^2\sin\vartheta\cos\vartheta\\ 0 & N^2\sin\vartheta\cos\vartheta & -N^2\sin\vartheta^2+\varepsilon_3\end{pmatrix}$$

3.3.2 Waves propagating parallel to B

Our K vector takes the neat form $k=k\hat{\mathbf{z}}$

This simplifies the $\hat{\varepsilon}$ matrix:

$$\det\begin{pmatrix}\varepsilon_1-N^2 & -i\varepsilon_2 & 0\\ i\varepsilon_2 & \varepsilon_1-N^2 & 0\\ 0 & 0 & \varepsilon_3\end{pmatrix}=0$$

In the case of $\varepsilon_3=0$, taking the matrix's determinant gives the solution $\omega\simeq\omega_{pe}$. These are exactly the plasma waves we've been considering earlier, in the first lecture! The electric field E is also parallel to both K and B. These are the waves related to restoring quasineutrality in the plasma.

The other solution is when

[6] Unfortunately, no one can be told what the matrix is. You have to see it for yourself in the previous section.

$N^2 = \varepsilon_1 + \varepsilon_2 \equiv \varepsilon_R$ (right handed waves) or $N^2 = \varepsilon_1 - \varepsilon_2 \equiv \varepsilon_R$ (left handed waves)

For right handed waves ($N^2 = \varepsilon_R$), the dispersion relation gives

$$E_x = -iE_y$$

What's physically interesting is the real part of the expression. Denoting $E_x \equiv E$:

$$\mathbf{E} = E\cos\omega t\hat{\mathbf{x}} + E\sin\omega t\hat{\mathbf{y}}$$

So the wave propagates along the B field and its electric field has a right handed rotation about the B field vector. Left handed waves are, thus, self explanatory.

Plugging $N^2 = \varepsilon_R$ into the dispersion relation:

$$N^2\varepsilon_R \simeq \frac{(\omega-\omega_R)(\omega-\omega_L)}{(\omega-|\Omega_E|)}$$

The expressions for left and right handed wave frequencies come out as

$$\boxed{\omega_R = \frac{1}{2}\Big(|\Omega_e| + \sqrt{\Omega_e^2+\omega_{pe}^2}\Big) > |\Omega_e|} \tag{3}$$

$$\boxed{\omega_L = \frac{1}{2}\Big(-|\Omega_e| + \sqrt{\Omega_e^2+\omega_{pe}^2}\Big) > 0} \tag{4}$$

Note that they've got completely different values.

If, instead, you plug in $N^2 = \varepsilon_L$, the equation you get is

$$N^2 = \frac{k^2c^2}{\omega^2} = \frac{(\omega+\omega_R)(\omega-\omega_L)}{(\omega+|\Omega_e|)(\omega-\Omega_i)}$$

3.3.3 Waves propagating perpendicular to equilibrium B field

The determinant can be written as

$$(-N^2+\varepsilon_3)(\varepsilon_1(\varepsilon_1-N^2)-\varepsilon_2^2)$$

This gives us the so called *ordinary mode (OM)* $N^2 = \varepsilon_3$. E is parallel to B and the equation for frequencies, in the limit of low ion plasma frequency (which usually is pretty low), is:

$$\frac{k^2c^2}{\omega^2} \simeq 1 - \frac{\omega_{pe}^2}{\omega}$$

The other mode is the *extraordinary mode (XM)*. The E field is in the XY plane. $N^2 = \frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_1-\varepsilon_2)}{\varepsilon_1} = \frac{\varepsilon_L\varepsilon_R}{\varepsilon_1}$. This can be written as

$$N^2 \simeq \frac{(\omega^2-\omega_R^2)(\omega^2-\omega_L^2)}{\omega^2-\omega_{LH}^2)(\omega^2-\omega_{UH}^2}$$

$$\boxed{\omega_{UH}^2 = \Omega_e^2 + \omega_{pe}^2} \tag{5}$$

This is called the upper hybrid frequency - hybrid as it's a combination of the electron cyclotron and plasma frequencies.[7]

$$\boxed{\omega_{LH}^2 = \frac{\Omega_i\,|\Omega_e|\left(1+\frac{m_e\Omega_e^2}{m_i\omega_{pe}^2}\right)}{1+\frac{\Omega_e^2}{\omega_{pe}^2}}} \tag{6}$$

This is called the lower hybrid frequency.

[7] See Chen, chapter 4, 'Waves in Plasmas'

3.3.4 The main takeaway

Even in the dramatically simplified cold plasma two-fluid model, plasmas are complex beasts with plenty of intrinsic modes. Plasma waves, right and left handed waves, extraordinary modes, ordinary modes... On to the next module, where the physics of those will be developed!

3.4 Properties of two-fluid waves (Resonanses, cutoffs, etc)

3.4.1 Properties of right and left handed waves

For right handed waves, the dispersion relation takes the form

$$(\frac{kc}{\omega})^2 = \frac{(\omega - \omega_R)(\omega + \omega_L)}{(\omega - |\Omega_e|)(\omega + \Omega_i)}$$

Whereas for left handed waves

$$\frac{kc}{\omega})^2 = \frac{(\omega + \omega_R)(\omega - \omega_L)}{(\omega + |\Omega_e|)(\omega - \Omega_i)}$$

where

$$\omega_R = \frac{1}{2}(|\Omega_e| + \sqrt{\Omega_e^2 + \omega_{pe}^2}) > |\Omega_e|$$

$$\omega_L = \frac{1}{2}(-|\Omega_e| + \sqrt{\Omega_e^2 + \omega_{pe}^2}) > 0$$

To propagate, the LHS expression $(\frac{kc}{\omega})^2$ must be positive. This implies the right hand side has to be positive. So in the interval

$$|Omega_e| < \omega < \omega_R$$

R-waves do not propagate, and in the interval

$$\Omega_i < \omega < \omega_L$$

L-waves do not propagate. Thus, in general

$$\Omega_i < \omega_L < |\Omega_e| < \omega_R$$

In the limit of $\omega \to +\infty$, the dispersion relation for both cases simplifies to

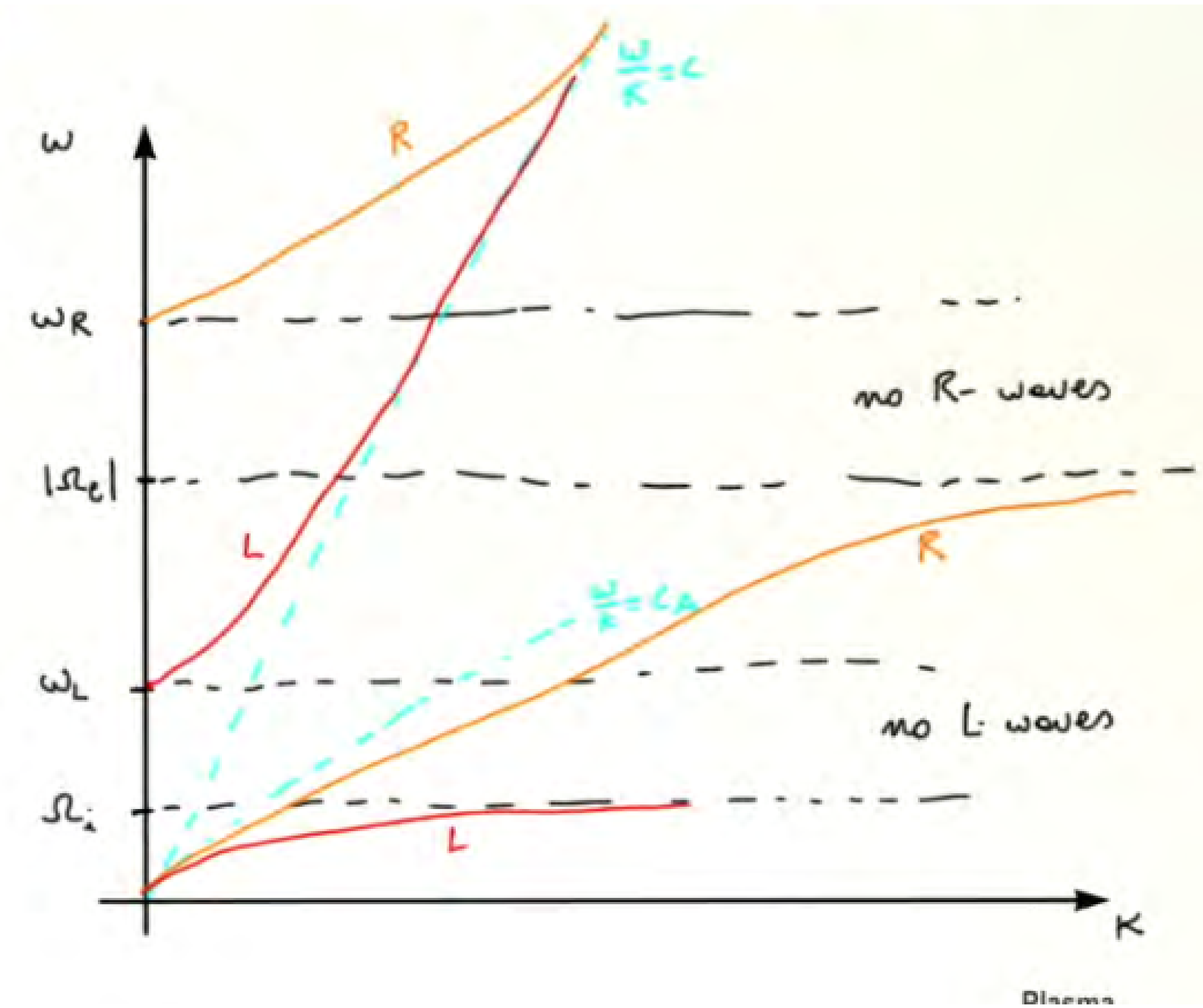
$$\frac{\omega}{k} = v_{propagation} = c$$

This describes lightspeed waves, electromagnetic waves. The plasma can't respond to waves with such a high frequency! Likewise, in the limit of $\omega \to 0$, $k \to 0$, we get

$$\left(\frac{kc}{\omega}\right)^2 = \frac{\omega_R\omega_L}{|\Omega_E|\,\Omega_i} = \frac{\omega_{pe}^2}{|\Omega_E|\,\Omega_i} = \frac{m_i n_0}{\epsilon_0 B_0^2} = \frac{c^2\mu_0 m_i n_0}{B_0^2} = \frac{c^2}{c_A^2}$$

Where $c_A = \frac{B_0}{\sqrt{\mu_0 m_i n_0}}$ is the so called Alfven speed, related to Alfven magnetohydrodynamic waves. So what we've got is $\omega = \pm c_a k$.

All this means that the waves have asymptotic k/ω relations:



3.4.2 Whistler waves

We take the case of R-waves with $\omega/k \ll c \ll \omega_{pe} < |\Omega_e| < \omega_R$ - we're looking at the bottom part of the graph above.

$$(\frac{kc}{\omega})^2 = \frac{(\omega - \omega_R)(\omega + \omega_L)}{(\omega - |\Omega_e|)(\omega + \Omega_i)} = \frac{\omega^2 + \omega(\omega_L - \omega_R) - \omega_L \omega_R}{(\omega - |\Omega_e|)(\omega + \Omega_i)}$$

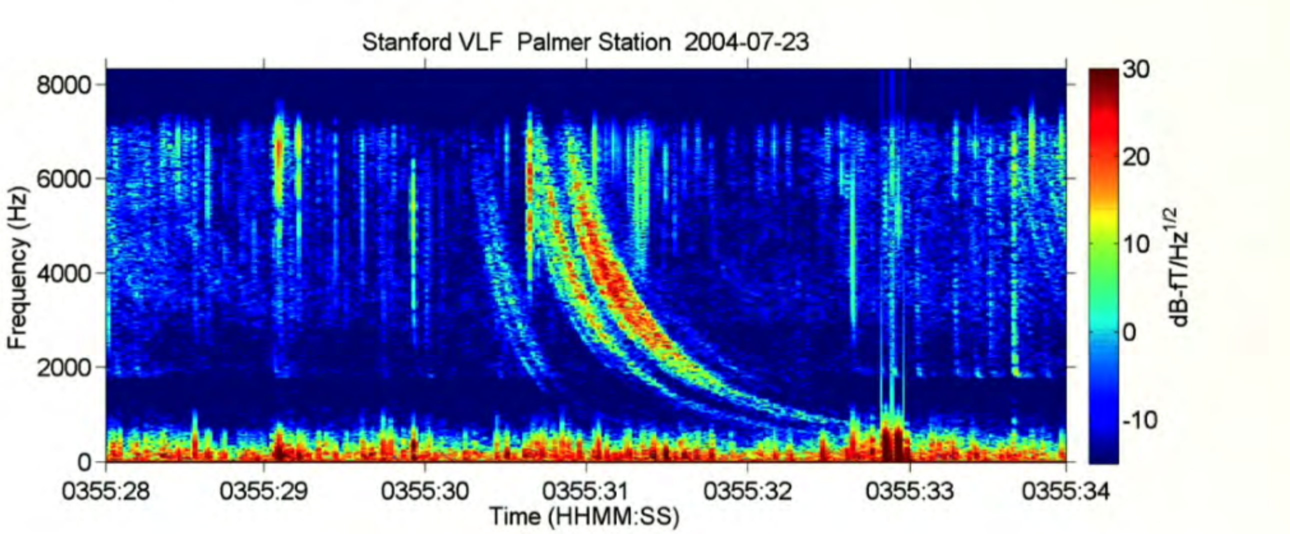
This can be simplified by neglecting some terms due to the regime we're in:

$$\frac{kc}{\omega})^2 = \frac{\omega_{pe}^2}{\omega\,|\Omega_e|}$$

$$\frac{\omega}{k} = \frac{c}{\omega_{pe}}\sqrt{|\Omega_e|\,\omega}$$

Functionally this shows that $v_{propagation} \propto \sqrt{\omega}$ and $\frac{\partial \omega}{\partial k} \propto \sqrt{\omega}$ - the propagation (group) velocity increases with frequency. Higher frequency waves propagate faster.

An example - if lightning strikes somewhere on Earth, waves will be produced in the ionosphere and higher frequency ones will propagate faster than slower ones. Here's a neat graph showing that phenomenon:



That actually does sound like whistling! The waves are in the acoustic range!

3.4.3 Cut-off and resonance frequencies

For right handed waves:

$$(\frac{kc}{\omega})^2 = \frac{(\omega - \omega_R)(\omega + \omega_L)}{(\omega - |\Omega_e|)(\omega + \Omega_i)}$$

ω_R and $|\Omega_i|$ are frequencies which the waves approach asymptotically. ω_R is the cutoff frequency: as k goes to 0, ω/k goes to infinity. k turns from real to imaginary - exponential decay of waves - they don't propagate! Waves are reflected.

Likewise, $|\Omega_i|$ is the resonance frequency: as k goes to infinity, ω/k goes to zero. Wavelengths decrease, small dissipative processes become important - the wave is absorbed into the plasma.

Of course, for L-waves the cutoff is at and the resonance is at .

3.4.4 Properties of waves propagating perpendicular to equilibrium B

For the ordinary mode OM

$$\omega^2 = \omega_{pe}^2 + k^2 c^2$$

The cutoff frequency is at ω_{pe}. There's no resonance frequency. As ω goes to infinity, the propagation velocity approaches c. The limit ω and k both going to zero cannot be approached by the ordinary mode.

For the extraordinary mode XM

$$N^2 = \frac{k^2 c^2}{\omega^2} = \frac{(\omega^2 - \omega_R^2)(\omega^2 - \omega_L^2)}{\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2}$$

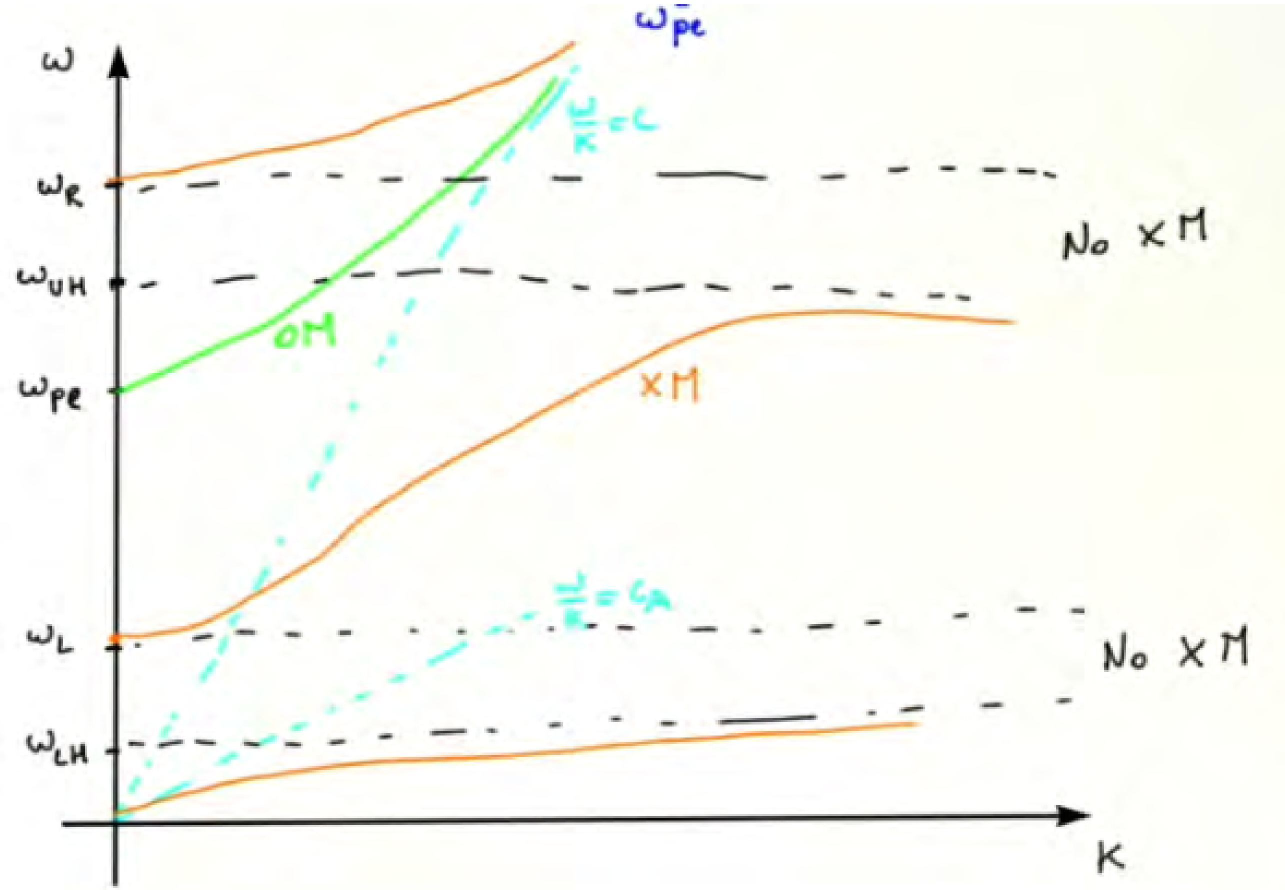
Where the lower and upper hybrid frequencies are

$$\omega_{UH}^2 = \Omega_e^2 + \omega_{pe}^2$$

$$\omega_{LH}^2 = \frac{\Omega_i \, |\Omega_e| \left(1 + \frac{m_e \Omega_e^2}{m_i \omega_{pe}^2}\right)}{1 + \frac{\Omega_e^2}{\omega_{pe}^2}}$$

There are two cutoff frequencies: ω_R and ω_L. Likewise, there's two resonances: ω_{LH} and ω_{UH}. As before, as ω goes to infinity, the propagation velocity approaches c. As k and ω approach zero, $\omega = c_A k$ - the Alfven velocity appears!

As before, there's two asymptotes for propagation velocity, this time they're the speed of light and the speed of Alfven:



XM waves have two regions where they cannot propagate - between lower hybrid/left and upper hybrid/right hand frequencies.

4 Week 4. Fluid description of plasmas part II, with Paolo Ricci

4.1 Single fluid and MHD

4.1.1 The two fluid model review

Each model has:

- a **continuity equation**

$$\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (n_s \mathbf{u_s}$$

- A **momentum equation**

$$m_s n_s \frac{d\mathbf{u_s}}{dt} = q_s n_s (\mathbf{E} + \mathbf{u_s} \times \mathbf{B}) - \mathbf{\nabla} \cdot \mathbf{\hat{P}_s} + \mathbf{R_s}$$

  (where $\frac{d}{dt}$ is the *convective derivative* and $\mathbf{R_s} = \int m_s(\mathbf{v} - \mathbf{u_s})(\frac{\partial f}{\partial t})_c d\mathbf{v}$

- An equation for the pressure - a closure equation, which we won't write here
- And there's the Maxwell equations to which the fluid model is coupled by

$$\rho = \sum_s q_s n_s$$

$$\mathbf{j} = \sum_s q_s n_s \mathbf{u_s}$$

4.1.2 One-fluid variables

- Mass density

$$\rho_M(\mathbf{r}, t) = \sum_s n_s m_s = n_e m_e + n_i m_i$$

- Center-of-mass velocity

$$\mathbf{V}(\mathbf{r}, t) = \frac{\sum_s m_s n_s \mathbf{u_s}}{\sum_s n_s m_s}$$

- Total electric charge

$$\rho = \sum_s n_s q_s = e(n_i - n_e)$$

- Total electric current

$$\mathbf{J} = \sum_s q_s n_s \mathbf{u_s}$$

- Pressure in center of mass frame

$$P_s^{\hat{cm}} = n_s m_s \int (\mathbf{v} - \mathbf{V})((\mathbf{v} - \mathbf{V}) f_s d\mathbf{v}$$

$$P^{\hat{c}}m = \sum_s P_s^{\hat{cm}}$$

4.1.3 One-fluid equations

We multiply the continuity equations for both species by their respective masses, add both up and we get

$$\boxed{\frac{\partial \rho_M}{\partial t} + \frac{\partial}{\partial \mathbf{r}(\rho_M \mathbf{V}) = 0}} \tag{7}$$

Likewise, by multiplying by the species' charge we can get an equation for charge density

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{J} = 0} \tag{8}$$

Taking the momentum equations and directly summing them to acquire the center of mass velocity equation

$$\boxed{\rho_M \frac{d\mathbf{V}}{dt} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} - \frac{\partial}{\partial \mathbf{r}} \cdot P^{\hat{cM}}} \tag{9}$$

Dividing each of the momentum equations by species charge times number density, and also multiplying the ion equation by the ratio m_e/m_i **and assuming that**

- $n_e = n_i \equiv n_0$
- $m_e \ll m_i$
- current much smaller than current carried by each species
  $\mathbf{j} \ll en\mathbf{u_e}, en\mathbf{u_i}$

We get an **electron current equation** (also called Ohm's law!)

$$\boxed{\frac{m_e}{n_e^2}\Big[\frac{\partial \mathbf{J}}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{VJ} + \mathbf{JV})\Big] = \mathbf{E} + \mathbf{V} \times \mathbf{B} - \frac{1}{en}\frac{\partial}{\partial \mathbf{r}} P_e^{\hat{cm}} - \eta \mathbf{J}} \tag{10}$$

4.1.4 Summary of the one fluid model

- Mass continuity equation

$$\boxed{\frac{\partial rho_M}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\rho_M \mathbf{V}) = 0} \tag{11}$$

- Charge continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{J} = 0} \tag{12}$$

- Momentum equation

$$\boxed{\rho_M \frac{d\mathbf{V}}{dt} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} - \frac{\partial}{\partial \mathbf{r}} \cdot P^{\hat{cM}}} \tag{13}$$

- Ohm's law

$$\boxed{\frac{m_e}{n_e^2}\Big[\frac{\partial \mathbf{J}}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{VJ} + \mathbf{JV})\Big] = \mathbf{E} + \mathbf{V} \times \mathbf{B} - \frac{1}{en}\frac{\partial}{\partial \mathbf{r}} P_e^{\hat{cm}} - \eta \mathbf{J}} \tag{14}$$

- Closure, under the assumption of sufficient collisions (γ is 1 for isothermal setting, $\frac{c_P}{c_V}$ for adiabatic, $+\infty$ for incompressible plasmas):

$$\boxed{\frac{\partial}{\partial \mathbf{r}} \hat{P_{cm}} = \frac{\partial}{\partial \mathbf{r}} P \ , \ \frac{d}{dt}(P \rho_M^{-\gamma})} \tag{15}$$

- Maxwell equations

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0} \tag{16}$$

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{B} = 0} \tag{17}$$

$$\boxed{\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{18}$$

$$\boxed{\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 (\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t})} \tag{19}$$

This is a completely closed system of equations!

4.1.5 Four simplifying assumptions for the one-fluid model

- Negligible electron inertia $m_e \to 0$. This simplifies Ohm's law to

$$0 = \mathbf{E} + \mathbf{V} \times \mathbf{B} - \frac{1}{en} \frac{\partial}{\partial \mathbf{r}} \hat{P_e^{cm}} - \eta \mathbf{J}$$

- Assume quasineutrality ($L \gg \lambda_D$, $\omega \ll \omega_{pe}, \omega \ll \omega_{ci}$ (ion cyclotron frequency)), $\rho \ll ne$

  This lets us neglect Gauss' law, the $\frac{\partial \rho}{\partial t}$ term in the continuity equation and the $\rho \mathbf{E}$ term in the momentum equation

- Neglect EM waves: $\left| \epsilon_0 \frac{\partial \mathbf{E} t}{\partial t} \right| \ll |\mathbf{j}|$

  This lets us neglect the displacement current in $\boldsymbol{\nabla} \times \mathbf{B}$. This implies $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$ and lets us remove the other term in the charge continuity equation

- Small larmor radius (to neglect finite larmor radius effects): $L \gg \rho_i$, or in other words $|\boldsymbol{\nabla} \times \mathbf{B}| \gg$, which lets us remove the pressure term in Ohm's law.

You have successfully crafted a +3 Magnetohydrodynamics Model of Quality!

4.1.6 The MHD model

- Continuity equation (for mass density)

$$\boxed{\frac{\partial \rho_M}{\partial t} + \frac{\partial}{\partial \mathbf{r}} (\rho_M \mathbf{V})} \tag{20}$$

- Momentum equation

$$\boxed{\rho_M \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \frac{\partial P}{\partial \mathbf{r}}} \tag{21}$$

- Ohm's law

$$\boxed{\mathbf{E} + \boldsymbol{\nabla} \times \mathbf{B} = \eta \mathbf{J}} \tag{22}$$

- Closure (pressure, TD) equation

$$\boxed{\frac{d}{dt}(P\rho_M^{-\gamma}) = 0} \tag{23}$$

- Maxwell equations

$$\boxed{\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{24}$$

$$\boxed{\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}} \tag{25}$$

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{B} = 0} \tag{26}$$

  The last one is an initial condition for B.

This is a system of 14 equations in 14 unknowns ρ_M, $\mathbf{V}$, $\mathbf{J}$, $\mathbf{B}$, P, $\mathbf{E}$.

For sufficiently hot temperatures, the resistivity η can be neglected for the *ideal MHD model*. If not, it's a *resistive MHD model*.

4.1.7 MHD simulations

Good for global dynamics of a plasma, especially for fusion plasma confinement devices. Example - W7-X MHD simulation done by EPFL group - equilibrium magnetic field.

4.2 MHD equilibrium and applications in one dimension

4.2.1 Static ideal MHD equilibria

The MHD equations are

$$\frac{\partial \rho_M}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\rho_M \mathbf{V})$$
$$\rho_M \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \frac{\partial P}{\partial \mathbf{r}}$$
$$\mathbf{E} + \boldsymbol{\nabla} \times \mathbf{B} = \eta \mathbf{J} \overset{\text{for ideal MHD } \eta=0}{=} 0$$

$$\frac{d}{dt}(P\rho_M^{-\gamma}) = 0$$
$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

In a static equilibrium, $\frac{\partial}{\partial t} = 0$. We'll also focus on $\mathbf{V} = 0$. Transforming the equations, the continuity equation, the $\boldsymbol{\nabla} \times \mathbf{E}$ Maxwell equation and the closure equation drop out. The momentum equation becomes the force balance equation

$$0 = \mathbf{J} \times \mathbf{B} - \frac{\partial P}{\partial \mathbf{r}}$$

From Ohm's law

$$\mathbf{E} = 0$$
$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

These four equations give the conditions for static equilibrium in ideal MHD.

4.2.2 Force balance equation analysis

If we dot $\mathbf{B} \cdot \nabla \mathbf{P} = \mathbf{B} \cdot (\mathbf{j} \times \mathbf{B}) \stackrel{\perp}{=} 0$. Likewise, $\mathbf{j} \cdot \nabla \mathbf{P} = \mathbf{j} \cdot (\mathbf{j} \times \mathbf{B}) \stackrel{\perp}{=} 0$. Thus, both the magnetic field lines and current run along isobaric (constant pressure) surfaces.

Substituting $\nabla \times \mathbf{B}$ into the force balance equation, we get $\frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \sim \frac{B^2}{mu_0 l} = \nabla P \sim \frac{P}{l}$, where l is a characteristic length.

We define

$$\boxed{\beta = \frac{P}{(\frac{B^2}{2\mu_0})} \ll 1} \tag{27}$$

The condition $\beta \ll 1$ is fulfilled for a typical lab plasma. For high β, the plasma's kinetic energy is much higher than the energy in its magnetic field. For the Earth's magnetosphere, $\beta \sim 1$. In some astrophysical systems, β can be larger than 1.

$$\boxed{(\nabla \times \mathbf{B}) \times \mathbf{B} = 0} \tag{28}$$

This is called the **force free equilibrium**, for very low β. For higher β, we get the **force balance equilibrium**

$$\boxed{\frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla P} \tag{29}$$

4.2.3 Force-free equilibria

One example of force free equilibrium is the surface of the sun. Since the curl of B is parallel to B itself,

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}$$

$$\nabla \cdot (\alpha \mathbf{B}) = \mathbf{B} \cdot \nabla \alpha = 0$$

This means that B runs along surfaces of constant α.

For cylindrically symmetric situations (ϑ symmetry), $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ is satisfied by

$$\boxed{vB = B_\vartheta \hat{\vartheta} + B_z \hat{\mathbf{z}} = \frac{B_0 K r}{1 + k^2 r^2} \hat{\vartheta} + \frac{B_0}{1 + k^2 r^2} \hat{\mathbf{z}}} \tag{30}$$

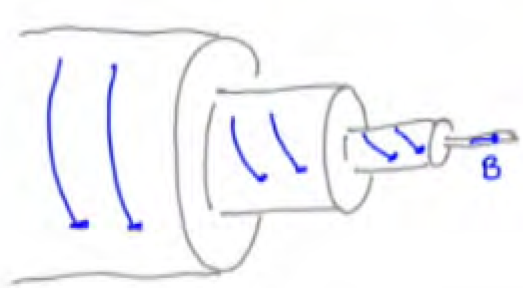
This gives

$$\alpha = \frac{\mu_0 j_z}{B_z}$$

$$j_z = \frac{2kB_0/\mu_0}{(1 + k^2 r^2)^2}$$

K is a $1/m$ constant that sets the scale of the equilibrium. r is the radial distance.

We imagine nested cylinders of decreasing radius. The B field runs along their surfaces. At small r, B is mostly along $\hat{\mathbf{z}}$. At higher radii, B is along $\hat{\vartheta}$. This is a so called **flux rope**. These are the flumes on the surface of the sun!



4.2.4 Force-balanced equilibria

We plug in

$$(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \times \mathbf{B} - \frac{1}{2}\nabla B^2$$

Which gives us

$$\boxed{\boldsymbol{\nabla}(P + \frac{B^2}{2\mu_0}) = \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}} \tag{31}$$

The terms above are the so called plasma pressure, magnetic pressure and field line tension.

4.2.5 Force-free and force-balanced equilibria

For a cylindrical symmetry as before, and pressure and magnetic field components depending only on radial distance

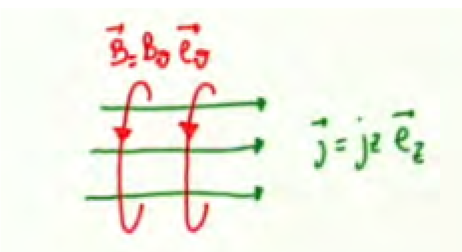
$$\frac{d}{dr}\Big(P + \frac{B_\vartheta^2 + B_z^2}{2\mu_0}\Big) = -\frac{B_\vartheta^2}{\mu_0 r}$$

This allows us to get the current, as $\mathbf{j} = \frac{1}{\mu_0}\boldsymbol{\nabla} B$:

$$\mathbf{j} = j_\vartheta \hat{\vartheta} + j_z \hat{\mathbf{z}} = -\frac{1}{\mu_0}\frac{dB_z}{dz} + \frac{1}{\mu_0 r}\frac{d}{dr}(rB_\vartheta)$$

Two of these functions - B_ϑ, B_z, P (all depending on r) can be specified arbitrarily, the third will get determined by the relations above and boundary conditions.

4.2.6 Z pinch, theta pinch



The Z, or Bennett pinch has $B_z = 0$. The plasma is confined by the $\hat{\vartheta}$ magnetic field. The current runs in the z direction.

One example is a uniform current running through a plasma:

$$j_z = j_{z0} \text{ for } r \leqslant a \text{ and beyond that } 0$$

$$I = \int_0^a j_z 2\pi r dr = \pi a^2 j_{z0}$$

This implies

$$B_\vartheta = \frac{\mu_0 I r}{2\pi a^2}$$

and outside the plasma

$$B_\vartheta = \frac{\mu_0 I}{2\pi r}$$

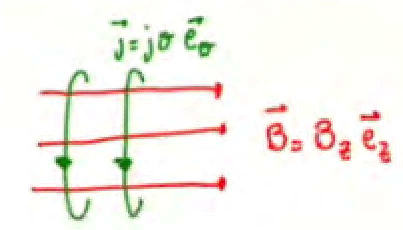
We can write the force balance equation

$$\frac{d}{dr}\Big(P+\frac{B_\vartheta^2}{2\mu_0}\Big)=-\frac{B_\vartheta^2}{\mu_0 r}$$

And from the condition that outside the plasma, its pressure is zero, we get

$$P(r)=\mu_0(\frac{I}{2\pi a})^2(1-\frac{r^2}{a^2})$$

There's also the theta ϑ pinch:



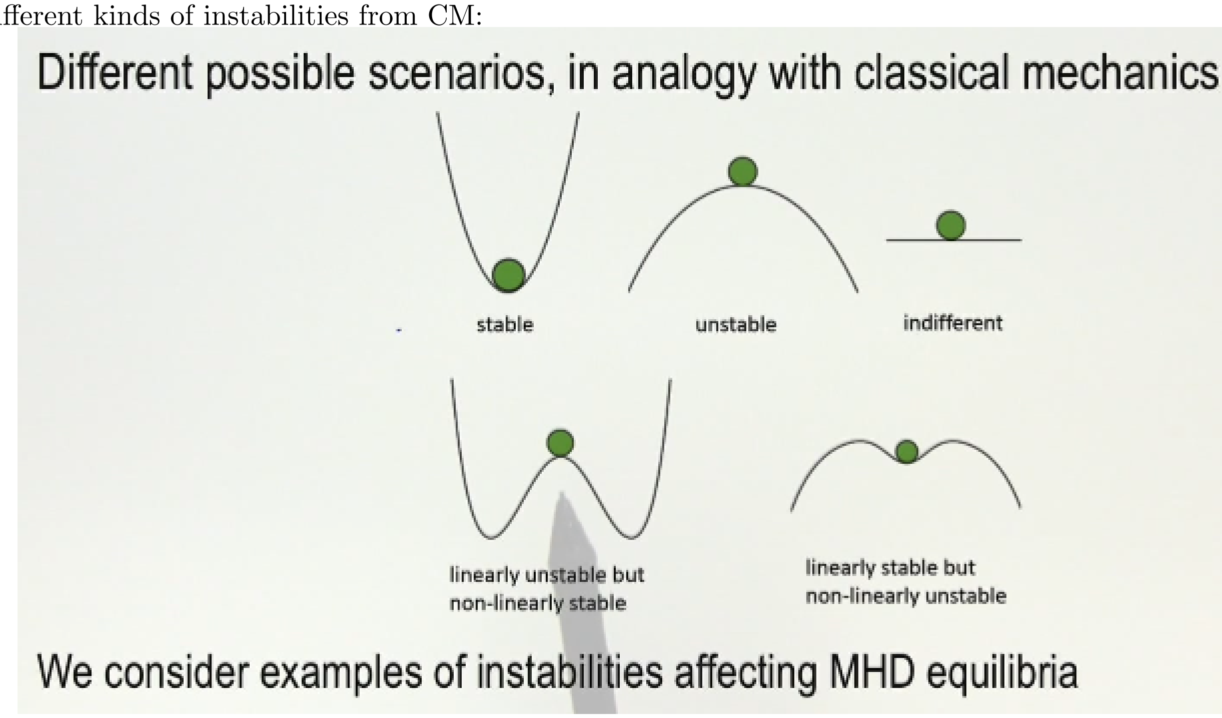
For which the force balance equation is very simple:

$$\frac{d(P+\frac{B_z^2}{2\mu_0})}{dz}=0$$

4.3 MHD stability, with Duccio Testa

4.3.1 What if a MHD equilibrium plasma is perturbed?

Different kinds of instabilities from CM:



What we're interested in is unstable equilibria.

We first examine the wonderfully named **sausage instability**. In a Z-pinch (axial current, azimuthal B field), the balance is between the magnetic field lines' tension, the gradient of the pressure, and the magnetic field itself (B_ϑ^2 term).

At a compression point, the radius of the cross section is smaller; B_ϑ increases, the pressure also increases and this squeezes the plasma.

Likewise, at a bulge, B decreases, the pressure decreases and the plasma tends to expand.

Kink instabilities form when Z-pinches are bent; at the point of negative curvature, field lines come closer together, the B field increases, the pressure increases and the plasma is pushed towards having even more curvature.

Rayleigh-Taylor instabilities. Assume a plasma above a region of vacuum (or air), with a small ripple at the boundary. The $\mathbf{g} \times \mathbf{B}$ drift will separate electrons and ions. There's a charge separation and a resulting E field. A $\mathbf{E} \times \mathbf{B}$ drift at the ripples will tend to magnify them, leading to instability.

This is just like in fluids!

Consider the curvature drift $F_c \sim \mathbf{R_c}$. If the B field surrounds a plasma (concave towards plasma), the drift will magnify the instability. Otherwise, it will bring the plasma towards equilibrium ('good curvature').

4.3.2 Wall effect on MHD instabilities

Axial current, azimuthal B field. Plasma surrounded by ideal (no resistivity) wall which - we assume - it cannot penetrate.

We consider the conservation of magnetic flux $\Phi = \int_S B \cdot dS$.

$$\frac{d\Phi}{dt} = \int_S \frac{\partial B}{\partial t} \cdot dS + \int_S B \cdot dS$$

Plug in Faraday's law and Stokes' theorem, we get

$$\int_S \frac{\partial B}{\partial t} \cdot dS = -\oint_C E \cdot dL$$

$$\int_S B \cdot \frac{d}{dt} dS = \oint_C \mathbf{B} \cdot \mathbf{V} \times \mathbf{dl}$$

Summing up the two terms and using a vector identity, we actually get Ohm's law. But since we're in ideal MHD, the integrand is zero! The magnetic flux is indeed constant.

The magnetic field is compressed at the plasma-wall boundary. B increases, P increases and pushes the plasma back towards the center. Surround a plasma with a wall and it gets stabilized! However, in practice the wall's resistivity is finite and it is only stabilised for some time.

4.3.3 General methods for MHD stability analysis

For uniform plasmas - use fourier mode analysis and if $Im(\omega) > 0$, we get exponential perturbation growth and instability.

For non-uniform plasmas - fourier analysis in time for Newton's law in fluid displacement gives a harmonic oscillator equation where the sign of ω^2 determines stability. Can also do energy analysis:

This is the change in potential energy. Sign determines stability.

4.3.4 MHD instability control

Passive methods, such as walls. Active control - when we notice an instability begins to grow, we use active feedback (lithium? magnetic resonance for ELMs?).

4.4 MHD waves

The handouts have gotten amazing so I will be doing less transcription of equations and more commentary now.

Plasma waves in the MHD model include: shear Alfven waves, fast compressional Alfven waves and slow magnetosonic waves. It's the same old story - use ideal MHD equations, add small perturbations, linearize about the equilibrium, fourier transform, look at frequencies and see whether they suggest exponential growth.

The sound speed squared depends on the equilibrium value of pressure.

We're working on a set of 4 equations.

To pick a geometry, we choose a wave propagating in the XZ plane and a B field along the Z axis. We can have either transverse waves - velocity oscillates in the Y direction - or longitudinal waves - along k in the XZ plane.

4.4.1 Transverse - shear Alfven - waves

Transverse waves are not compressible ($\rho_{M1} = 0$)!

Nontrivial solutions give the shear alfven wave dispersion relation using

$$\omega^2 = k_z^2 c_A^2 = k^2 c_A^2 \cos\theta^2$$

These are important in DT fusion plasmas. Velocity of α particles at $E = 3.5MeV$ is greater than c_A, they're said to be superAlfvenic and they become resonant. Whatever that means.

4.4.2 Longitudinal - fast compressional Alfven - waves

Redoing all the earlier calculation and neglecting terms of high order in the $\frac{c_S^2}{c_A^2}$ ratio, we get two ($\pm$) solutions. Denote ω_+ as the solution to the + equation. Neglecting high order terms once more, we get

$$\omega_+^2 = k^2 c_A^2 (1 + \frac{c_S^2}{c_A^2} \sin\theta^2)$$

This is called the fast compressional Alfven wave - fast in phase velocity, relative to sound speed. The sound speed in fact only enters as a small correction.

Taking the $-$ solution we get the slow magneto-sonic wave

$$\omega_-^2 = k^2 c_S^2 \cos\theta^2$$

5 Week 5. SPAAAAAAAAAAAAACE, with Ivo Furno

5.1 The Sun

5.1.1 Solar properties

- Mean distance $d_0 = 1.5e11m = 1AU$
- Mass $M_0 = 2e30kg$, typical for *main sequence stars*
- Radius $R_0 = 7e8m$. Sharply defined (photosphere)
- Luminosity $L_0 = 3.84e26W$. Can use $E = mc^2$ to get $\frac{dm}{dt} = 10^9 kg/s$
- About 80% hydrogen

Logarithmic star luminosity grows linearly with logarithmic mass

$$L/L_0 = (M/M_0)^\alpha$$
$$3 < \alpha < 4$$

A star 10 times heavier will burn mass 1000 times faster. Lower lifetime.

5.1.2 Hydrostatic equilibrium

Assume: Sun is perfectly spherical. Chop it up in spherical shells, radial distance r, thickness dr. Local matter density ρ. M_r - total mass inside shell.

Use force balance to get relation between acceleration, gravity and radial change in pressure (pressure forces).

Set acceleration to zero for hydrostatic equilibrium and solar death rays to kill for maximum awesome,

$$\frac{dP}{dr} = -GM_r\rho/r^2$$

Integrate shells ($4\pi r^2$) from 0 to R_0.

$$3\langle P\rangle V = \int_0^{R_0} \rho G M_0 4\pi r dr$$

Right side is gravitational energy. Assume volume is filled with a plasma, N protons, N electrons, temperature T.

$$\langle P\rangle = 2Nk_bT/V$$

Integrating, we get an average temperature $2.3e6K$. Maximum can reach $1.6e7K$

5.1.3 Proton proton chain reaction (pp I chain)

Total energy release - $26.22MeV$.

5.1.4 Solar luminosity

Energy density proportional to T^4. Total radiative energy you get by multiplying by volume. Luminosity can be gotten as total energy over average time to reach solar surface.

Simplest assumption for a photon going straight from solar center gets you a photon getting there in $500s$.

The plasma is not transparent! Photons get absorbed, scattered, reemitted...

- Free free absorption (inverse bremsstrahlung)

- Bound free absorption (photoionization)
- bound bound absorption (photoexcitation)
- electron scattering (Thompson, Compton scattering)

Mean free path for Thomson scattering in solar interior is $2cm$, that's awfully low. This means photons don't go in a straight line but diffuse in a random walk

$$\left\langle R^2 \right\rangle = D\tau_{ph}$$

Under this assumption, a photon takes $10e8y$ to escape. And this fits the experimental data..... Whoa.

We get the result that luminosity scales as M^3. Just what we began with for main sequence stars!

5.2 The solar cycle. Solar magnetic fields

Hydrodynamic dynamo process generates the magnetic fields

5.2.1 Sunspots

Sunspots - dark patches through which magnetic fields emerge from the interior. The number of sunspots is periodic.

International Sunspot Number, an average value from observatories over the globe.

This is strikingly periodic!

Sunspots larger than the Earth. AB/944

Dark region - umbra, lighter around it - penumbra. Umbrae are characterized by % of hemisphere surface, and latidude positions.

Solar rotation time: 1 month approximatley

5.2.2 The butterfly diagram

Sunspots range from -30 to $+30$ degrees in latitude. At the beginning of each cycle, sunspots occur at maximum values and converge towards the equator (which, however, stays free of sunspots)

5.2.3 Magnetic field measurements

Started in 1908 by Hale. Zeeman splitting implied magnetic fields about $0.2T$.

Binary groups - fields leaving and entering Sun, closed field loops. Groups on northern hemisphere have symmetric groups on the southern hemisphere, with switched order of entering/leaving sun by mag fields.

5.2.4 Hale's law

Leading sunspot (in rotation) tends to be closer to the equator

Between periods, polarities of sunspots reverse!

5.2.5 Global magnetic field structure

At solar minimum, polar fields about 10 Gauss - much lower than in sunspots. With no sunspots, we'd get a dipole field.

At solar maximum, multipolar field, lots of twisted lines leaving and entering solar surface

At next solar minimum, reversed polarity.

5.3 Dynamo - from plasma flow to magnetic fields

Mechanical energy (fluid movement) converted into stretching and twisting B lines

Faraday rotor - homopolar disk dynamo. Metal disk rotating at constant angular velocity. Wire transfers charge to central rod.

Above a critical value of angular velocity, there's flux amplification and dynamo action.

Dynamos result from asymmetries of internal motion - fluid movement inside the Sun

Theorems:

(a) there are no 2D dynamo fields. It always depends on all 3 coordinates.

(b) Can't make a dynamo with a toroidal flow.

(c) Can't have a field vanishing at infinity (e.g. physical) for axisymmetric

(d) High degree of symmetry stops dynamo action: stationary axisymmetric, stationary centrally symmetric, planar velocity field

5.3.1 Surface rotation

Non-uniform rotation of solar surface - equator rotates with period of 25 days. Closer to the pole, period increases. The Sun is not a solid body, but a miasma of incandescent plasma, of course.

5.3.2 Helioseismology

Acoustic waves in the sun allows to determine rotation rate relation to depth

Up to $0.7R_0$ the sun rotates as a solid body

There's a $0.04R_0$ area called the **tachocline** where the sun experiences extreme shear forces

5.3.3 Meridional circulation

Flow from equator to poles

I skipped making notes in this section. If anyone wants to fill this part out, feel free to!

5.4 Reconnection: from magnetic fields to flows

Likewise, I skipped making notes in this chapter.

5.5 How does the solar wind blow?

We'll use an MHD description to describe the solar wind.

Cometary tails pointing away from Sun regardless of velocity imply ionized gas pushed away from the comet by the solar wind.

Example - Hale-Bopp comet. White tail - light pressure, radially away from Sun. Blue tail - not radially away from Sun, points in another direction - it's solar wind. Doesn't necessarily have to blow radially, right?

Solar wind speed measured for the first time in 1962 by Mariner 2 probe. Measured speed - $300 - 700km/s$.

1990 ESA/NASA Ulysses mission detected strong correlation of solar wind to number of sunspots. It also detected a variation in power (with solar activity) of a factor of 4.

5.5.1 Fluid model of the solar wind

Note that this neglects kinetic effects and is not completely correct!

We'll work, obviously, in spherical coordinates. Assume solar wind velocity $v(r)$ is purely radial and dependent only on radial distance. Our parameters then are pressure P, temperature T, density ρ.

Use mass conservation and equation of motion, including force of solar gravity. Assume negligible $\mathbf{j} \times \mathbf{B}$ term. Assume also that since we're looking for stationary solutions, time derivatives are negligible.

Assume also a hydrostatic solution - zero wind velocity. Then we can use the ideal gas law for protons and electrons together (neglecting electron mass compared to ion mass for averaging). Assume isothermal gas (constant temperature). It turns out that the pressure falls inversely with radius. The result is rather nonphysical because we get a finite pressure at inifinity... so the hydrostatic solution is a failure.

Roll back the zero wind velocity, but carry on with the isothermal plasma assumption.

Define sound speed as $v_s^2 = \frac{K_b T}{\mu}$, μ being the average particle mass.

Define also a critical radius $r_g = \frac{GM_{sum}}{2v_s^2}$

5.5.2 Solar wind

$$\frac{1}{v}\frac{dv}{dr}(\frac{v^2}{v_s^2} - 1) = \frac{2}{r}(1 - \frac{r_c}{r})$$

At small r, $1 - r_c/r < 0$ and for assumed small starting velocity $(v/v_s)^2 - 1 < 0$. Thus $\frac{dv}{dr} > 0$ - the plasma accelerates and the solar wind starts to blow.

The phase space is divided into four regions by the two critical values v_s, r_c.

(a) If a trajectory reaches $v = v_s$ at $r < r_c$ it accelerates while turning back. This is nonphysical.

(b) If $v < v_s$ at $r = r_c$, the wind accelerates until the critical radius (minimum derivative and then begins slowing down. This is called a subsonic breeze.

(c) If v_s at $r = r_c$, the wind keeps accelerating afterwards. The acceleration is always positive, though may become small (asympotically zero). This is a supersonic wind.

5.5.3 Typical values for solar wind

$$r_c = \frac{M_s umG}{2v_s^2}$$

$$v_s = \sqrt{\frac{k_b T}{\mu}}$$

Average mass $\mu = 0.6m_p$ due to helium presence

Temperature $1.5e6K$.

Boltzmann constant $k_b = 1.38e - 23J/K$.

This gives us $v_s = 1.4e5m/s$, $r_c = 3.4e9m = 4.5R_{sun}$. This is where the wind becomes supersonic.

We can use these velocities with the equation of motion (assuming isothermality), integrating over trajectories to get energy conservation (a version of Bernoulli's theorem). Our calculated values give us initial conditions.

5.5.4 Solar mass loss due to solar wind

It turns out to be $1.58e9kg/s$ - negligible compared to solar mass.

Plasma density in solar corona - $10e14m_p$.

5.5.5 Solar wind at Earth position

$1AU \sim 214R_0 \sim 48r_c$, which gives a velocity of $4v_s = 5.6e5m/s$. About 500 kilometers a second. It needs a good couple days to reach Earth and a good couple **months** to reach the outer planets.

The solar wind strikes the magnetic field and, through the frozen flux theorem, pushes it into the outer Solar System.

The interplanetary magnetic field is weak in magnitude, but it influences our magnetosphere. As the solar wind reaches our magnetosphere, it has a density of about $3 - 8$ ions per cubic centimeter. The magnetosphere carves a cavity free from those ions and deflects them. This forms a $17km$-thich bow shock layer at a distance of about $90000km$ away from the Earth.

Some solar wind plasma penetrates the bow shock and forms the **magnetosheath**.The **magnetopause** is the abrupt boundary where the planetary magnetic field pressure and solar wind pressure equalize.

The solar wind creates a neutral sheet behind the Earth, where B lines in opposite directions lie close to each other. This is just the condition for magnetic reconnection! This happens often during CMEs as the tail gets essentially cut off by the increased solar wind pressure. This pumps energetic particles into our atmosphere, producing auroras.

6 Week 6. Plasma applications in industry and medicine, with Alan Howling

6.1 Survey of plasmas in industry and medicine

6.1.1 What's the difference

Industrial plasmas tend to have a low degree of ionization ($\frac{1}{1000000}$). Lots of neutrals, and lots of collisional damping, thus no waves. They're made with no magnetic fields (no need to confine the plasma).

6.1.2 Why plasma chemistry is difficult

There's chemical reactions between different cations, anions and radicals. This creates a complex soup of different plasma species in bulk plasmas. There's little simplicity here.

6.1.3 Different types of reactions

Homogeneous reactions in the gas phase:

- Ionization - electron hits compound, leaves cation and 2 electrons
- Dissociation - electron breaks down a compound and leaves
- Attachment - electron hits compound, breaks it down and attaches itself to one of the resulting molecules leaving an anion

Heterogeneous reaction between the gas and the surface:

- Ion neutralization - cations and electrons recombine on the surface with high efficiency
- Association - hydrogen atoms recombine into H_2 on surfaces, as in volume processes it's difficult for them to conserve momentum and energy
- Secondary emission - positive ions bombard surfaces and release secondary electrons
- Sputtering - positive ions bombard surfaces and release atoms of the surface compound
- Deposition - gaseous A reacts with surface B, leaves compound AB on the surface
- Etching - gaseous A reacts with surface B, leaves compound AB in a gas form (can be pumped away)

Transport of species to surfaces - diffusion (Fick's law) (neutral flux) and, for ions, Bohmian diffusion.

6.1.4 Elastic collisions - only kinetic energy exchanged (no losses)

A ball of speed V, mass m strikes a stationary mass M. Due to conservation of momentum and energy, maximum fraction of energy transferred to M is $\delta_{max} = \frac{4mM}{(m+M)^2}$. For an electron striking an atom, $\delta_{max} = \frac{4m}{M} \ll 1$. For an ion striking an atom, the masses are similar and $\delta_{max} \sim 1$.

Thus, electrons don't heat the gas efficiently, while ions do.

6.1.5 Power balance

Electrons in an electric field E gain energy with average power $P = Eeu$, u being their drift velocity and proportional to mu_eE, mu_e being the electron mobility, so $P = \frac{e^2E^2}{m_e\nu_m}$, ν_m being the collision frequency between neutrals and electrons (thermalization).

Average energy loss per collision $= 1.5\delta e(T_e - T_gas)$. Multiply by neutral-electron collision frequency for average power. Balance out the two terms and we have

$$T_e - T_{gas} = \frac{2eE^2}{3\delta m_e \nu_m^2}$$

This scales as $1/p^2$. So we have a nonequilibrium plasma, much higher electron temperature than ion temperature, at low pressures ($1mbar$). This equithermalizes at atmospheric pressures.

6.1.6 Inelastic collisions - with excited atoms

Same situation and before. Conservation of energy now involves term for internal energy of atom M, which is excited. It turns out that for internal energy, $delta_{max} \sim 1$ for electrons while still $\ll 1$ for kinetic energy! So you can use high energy electrons to modify chemical bonds, but leave the gas temperature unchanged.

6.1.7 Key to industrial plasma processing

In $2eV$ (low temperature), $T_e \gg T_{gas}$ (nonequilibrium) plasmas, we can do **high temperature plasma chemistry on low temperature substrates**. Substrates like glass, plastics or **people**.

6.1.8 Plasmas in medicine

Take a low temperature **dielectric barrier discharge** plasma jet in air - just flow a rare gas into a glass tube (to separate gas from electrodes, which have a voltage applied to them). Gas gets ionized and accelerated, resulting in a centimeter length plasma jet.

This can be used for sterilization. This is an active research area - the plasma can create **reactive oxygen and nitrogen radicals**, there's ultraviolet photons, electric fields and electrical charges, and they may work together in ways we don't understand. It's difficult to pick out just one of those factors and see what kind of changes it causes.

This is kind of awesome.

6.2 Breakdown in low pressure gases: part 1

Our example will be a communication satellite powered by photovoltaic solar panels. These rotate around the communications device by a slip ring assembly. We want to design that one to avoid electrical breakdown. How does the latter work?

6.2.1 Background ionization

We take a small vacuum vessel (a few cm long) and put in rarified argon at a few $mbar$. Apply DC voltage to the walls. How does the plasma start?

There's always background radiation, cosmic rays, radioactivity. That means electrons are randomly appearing everywhere all the time. The plasma begins to exist spontaneously, given the applied voltage!

Photoemission intensity leads to a saturation current $i_0 = \dot{N}_0 e$ - the maximum value of a $1 - \exp(-t)$ increase. Small current, $10pA$, at voltages of about $10V$.

We now increase the voltage...

6.2.2 First Townsend coefficient

Electron impact ionizes atoms. Current exceeds saturation value, as each electron creates an avalanche (standard exponential growth).

Townsend's first ionization coefficient - $\frac{\text{number of ionization collisions}}{\text{number of electrons, unit length along E}}$. Thus $d\dot{N} = \dot{N}\alpha dx$, leading to $i = i_0 \exp(\alpha d)$.

Every ionization creates positive ions! This means that the current is independent of the x position (along vacuum vessel).

So the $I - V$ curve is a $1 - \exp(-t)$ decaying increase until Townsend discharge begins.

6.2.3 Second Townsend coefficient

A start electron hits an ion, this returns to the cathode. The electron instead avalanches towards the anode. If an electron hits a returning ion, it's called a secondary emission event. This is described by Townsend's second ionization coefficient γ - number of electrons emitted per incident ion. About 0.01.

How does this impact the current?

1st ion current given by $i_0(\exp(\alpha d) - 1)$. 1st secondary emission - that times γ. These cause an avalanche... it's a geometric series.

In sum, the current is $i = i_0 \frac{\exp(\alpha d)}{1-\gamma(\exp(\alpha d)-1)}$. This would increase to infinity as the denominator goes to zero... that's a spark!

6.2.4 Breakdown criterion in gases

$$1 - \gamma(\exp(\alpha d) - 1) = 0$$

This is when the gas is said to **break down**. It's a self sustaining discharge. In other words, $\gamma \exp(\alpha d) = \gamma + 1$ - every electron creates another to replace it before it gets absorbed.

This opens a current limited only by external resistance. That can lead to very high currents, and even melting of metals.

6.3 Low Pressure Gases 2: Electrical Breakdown

We now investigate a lab plasma in a vacuum chamber device controlled by a laptop with a windows system.

We'll derive Paschen's low, see how breakdown is measured, investigate vacuum breakdown and see a numerical model used to descrive this.

$$\gamma \exp(\alpha d) = 1 + \gamma$$

6.3.1 Townsend's first coefficient, alpha

α stood for number of ionization collisions per electron per distance. Use definition of mean free path λ as number of collisions per distance $= 1/\lambda$.

Ionization probability of collision is a Boltzmann factor. ε_i - ionization energy. exp()

We can multiply that and then equate to alpha, getting (as pressure inversely proportional to λ)

$$\alpha = Ap \exp(-Bp/E)$$

6.3.2 Paschen's law

For parallel plates $E = V/d$

We can thus get a breakdown voltage - Paschen's law

$$V_B = \frac{Bpd}{\ln Apd - \ln \ln 1 + 1/\gamma}$$

Can be simplified by introducing another constant, $C = A/\ln 1 + 1/\gamma$

Plotting the curve, we get that at low pd there can be no breakdown (infinite V_B). There's a minimum at $\ln Cpd = 1$.

α depends on gas type, γ - gas type and electrode material.

At low pressures - few collisions, so you need high ionization probability, high voltage.

At high pressures - short mean free path, high collision energy ($eE\lambda$), need high voltages again.

6.3.3 Vacuum discharge

Measuring this experimentally we notice that Paschen's law golds ok but not at the 'infinite V_B' region. This is due to the fact that plasma can form **inside the metal electrodes** in vacuum at really high voltage. Can be proven by spectroscopy. **The electrodes themselves get ionized**. This is related to **thermionic field emissions**.

Another issue is that the minimum region of gas discharge when measured experimentally is wider and flatter than the Paschen curve.

6.3.4 Breakdown numerical simulation by a two fluid model for arbitrary geometry

$$\frac{dn_j}{dt} + \mathbf{\nabla} \cdot \mathbf{\Gamma_j} = S_j$$

S_j is a source term. Γ_j is a flux therm, can be decomposed into diffusive ($-D_j \mathbf{\nabla} n_j$) and convective ($n_j u_j = \pm n_j \mu_j E$) terms.

$S_j = n_e \alpha u_e$, ionization rate due to collisions.

Boundary conditions - zero flux of ions leaving anode. For ions - flux of electrons leaving cathode is related to flux of ions by $-\gamma_{second}$ And we also have the applied voltage.

Results - for a $1mm$ single gap , the Paschen curve agrees with results. At $100mm$, we still have agreement.

Having a model with multiple electrode gaps gives a wide and flat curve. All in all, it's a geometry issue.

6.4 Sheaths and plasma etching - part 1

6.4.1 How does a plasma form?

Transition from Townsend discharge (weak space charge, Laplace equation) to self sustaining plasmas (Poisson equation)

Note that the transition curve is only for parallel plates!

6.4.2 Formation of a sheath

Ion thermal flux, given by kinetic theory, through any boundary in a plasma - completely negated by ion flux from the other side.

Electron flux (also from kinetic theory) - much larger than ion flux due to larger mass. Temperature also a factor, but not as big.

$$\Gamma_e \gg \Gamma_i$$

But this analysis has been isotropic, no boundary conditions taken into account. But you need wall interactions for deposition, etching and surface modding.

Suppose this imaginary surface was replaced by metal wall?

Electrons will flow to the wall and form a negative charge on the wall. A positive layer will form near the wall (and stay there for a time, due to lower ion mobility). The plasma potential rises in the direction into the plasma, to slow down electron escape. There's a **sheath potential drop**.

If there's no net current, there's a flux equilibrium.

Now, use a Gaussian pillbox perpendicular to the wall. As electric field is zero (if pillbox of enough length), the charge inside is zero!

So:

- transition region
- bulk positive charge equal in magnitude to negative surface charge

- dynamic equilibrium of ion and electron fluxes at the wall
- plasma potential always positive with respect to most positive surface
- thickness - several λ_D
- the sheath is a dark layer because there's few electrons in it (in bulk)
- strong electric field $\rightarrow$ positive ion flux directed towards the wall

6.4.3 Plasma etch applications

A plasma formed between two electrodes. At the sheath, photoresist (much smaller than sheath thickness) deposited on a silicon substrate. Ions accelerated perpendicular to substrate.

This allows high aspect ratio etching - requires high precision. Cannot be achieved with a wet process. Plasmas are necessary because of the vertical ion flux.

Transistor densities steadily increasing, photoresist feature size steadily increasing.

This is how Moore's law comes about!

6.5 Sheaths and plasma etching - part 2 - a mathematical description

6.5.1 Sheath potential

V_{plasma} at large distance from wall. $V = 0$ we define to be at sheath edge. $V_{plasma} - 0 = \Delta V_{presheath}$. u_s - speed of initially stationary ions at sheath.

Use conservation of energy: $0.5Mu^2 + eV = 0.5Mu_s^2$

Conservation of ion flux, stating that no ionization occurs in sheath: $n_i u = n_s u_s$

Can eliminate unknown ion speed u, $n_i = n_s(1 - \frac{2eV}{Mu_s^2})^{-0.5}$

6.5.2 Ion densities

We can use a piecewise model.

In the bulk plasma $n_0 = n_e = n_i$.

In the presheath $n_e = n_i$ approaches n_s towards the sheath edge.

Up to now, this is the quasi neutral plasma.

Inside the sheath, $n_i > n_e$. They both vary with x (actual dependence is through dependence on V).

Electron sheath density is a Boltzmann: $n_e = n_s \exp(V/T_e)$

6.5.3 Bohm criterion

Using 1D Poisson inside the sheath:

$$\frac{d^2V}{dx^2} = -(n_i - n_e)e/\epsilon_0$$

Insert our expressions for density from above:

At plasma sheath interface, we expand the Taylor series to get

$$\frac{d^2V}{dx^2} = \frac{en_s}{\epsilon_0}\Big(\frac{1}{T_e} - \frac{e}{Mu_s^2}\Big)$$

Assume physically that the behavior is non-oscillatory. Thus the factor $\left(\frac{1}{T_e} - \frac{e}{Mu_s^2}\right) > 0$.

From this, we can get a bound:

$$u_s \geqslant u_B = \sqrt{\frac{eT_e}{M}}$$

This is Bohm's criterion for ion velocity leaving the plasma. It depends on the *electron temperature*.

So the ions are accelerated to a kinetic energy $0.5Mu_B^2$ at the sheath, the potential drop is $T_e/2$

Bohm's criterion is kind of universal.

6.5.4 Consequences of Bohm's criterion

Ion flux into sheath $= 0.61n_0u_B$

Ion saturation current - ion current to a probe area A, seen very often

We can calculate the ion and electron fluxes (electron flux from kinetic theory). We can then calculate the sheet voltage drop and combine it with the presheath voltage drop, to get the total ion energy. Turns out to be $\sim 5.2T_e$ in this case.

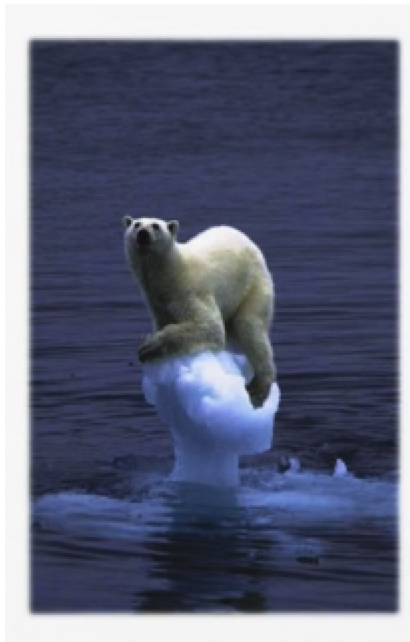
7 Week 7. Thermonuclear fusion: an overview, with Ambrogio Fasoli

7.1 The basics

7.1.1 Why fusion?

HDI correlated with electricity use per capita (though Canada is using almost three times as much as the Netherlands so there's room for improvement there). So to develop humankind we need better power sources.

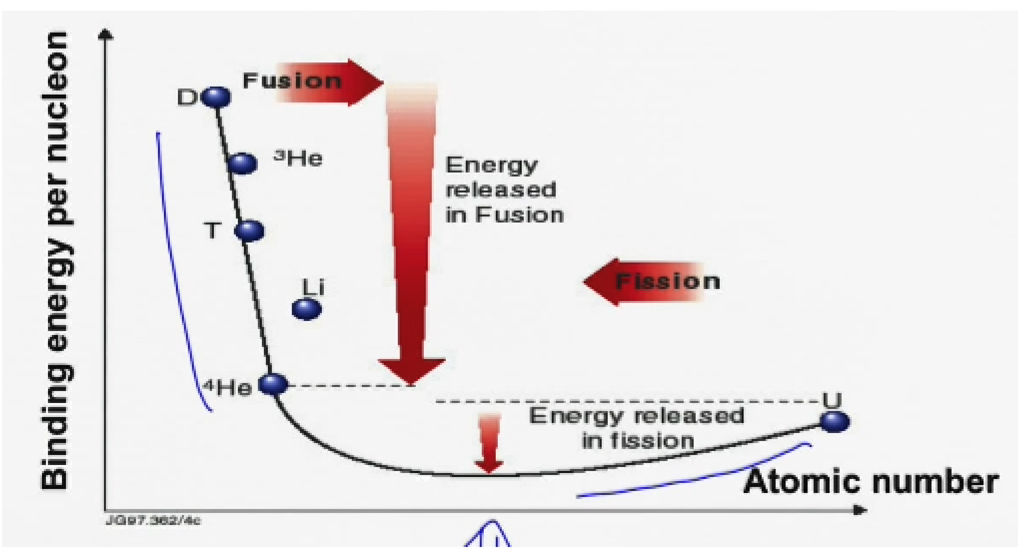
We have to fight global warming, stop using fossil fuels and save this polar bear.



We also have to make sure the power sources are safe for the population. Energy sources that endanger people are **bad**.

And fusion fulfils all of those criteria!

7.1.2 How does it work



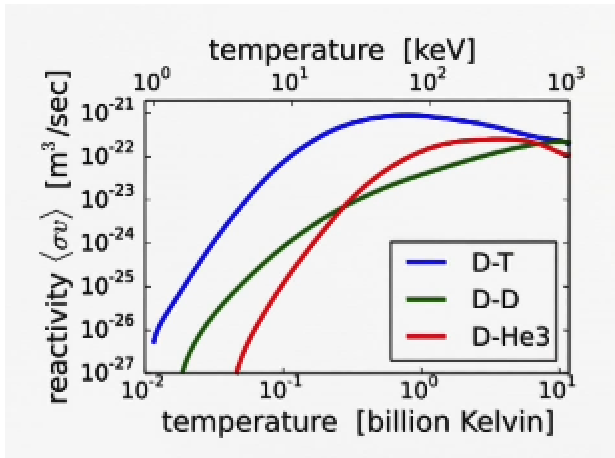
Two possible approaches to nuclear energy generation - split really heavy nuclei, like uranium, or merge small nuclei. Note that as the slope is sharp on the left, so you get a lot of energy per each fusion.

7.1.3 Fusion reactions for a terrestrial reactor

Three possibilities: DT, DHe3, DD fuel cycles.

At low kinetic energies coulomb repulsion deflects particles. This prevents fusion from occuring (essentially interception by atomic forces). High kinetic energies are equivalent to saying we need high temperature.

You don't actually need $380keV$ to get into the potential well on the left - you can do with $10keV$ due to tunneling (a quantum effect).

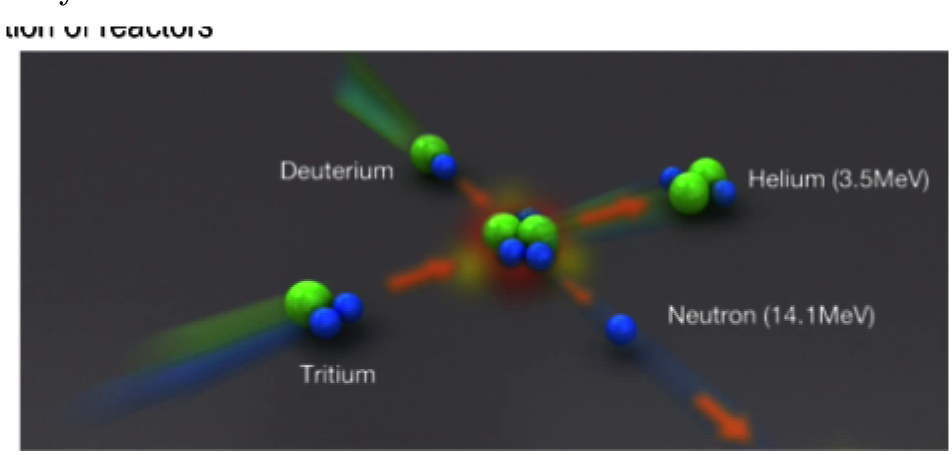


Cross section times average velocity over distribution functions (AKA **reactivity**) - this is a parameter that describes how easy it is for particles to fuse. The DT reaction has the largest one at that for the smallest temperatures, so we're focusing on that for now.

At these temperatures ($10keV$ or so), pretty much all matter is in the plasma state! So we're gonna need to go through this to understand that.

Thermonuclear fusion - as Coulomb collisions have a humongous cross section, any monoenergetic beam would be scattered in a plasma before the particles have time to fuse. Thus, an important condition is that our plasma is in a **thermalized** Maxwellian distribution (this would be mean that the plasma is kind of in a stationary state).

7.1.4 The DT cycle



This is the current approach to terrestrial fusion. Deuterium is ubiquitous in the oceans. Tritium has a half-life of 12.5 years and has to be **bred** from lithium. That can also be found in the ocean.

We look into the $Li-6$ reaction due to its large cross-section. Smack a neutron into the $Li-6$ atom and one T and one $He-4$ pops out. Lithium can be supplied by using a **blanket** mounted on the inside wall of the vacuum vessel.

7.1.5 Schematic of a fusion reactor

We inject deuterium into the reactor and lithium into the blanket. The tritium stays in a closed system. Helium can be bottled and made into balloons or sold at high prices. The energy is removed from the system by a conventional fluid based system.

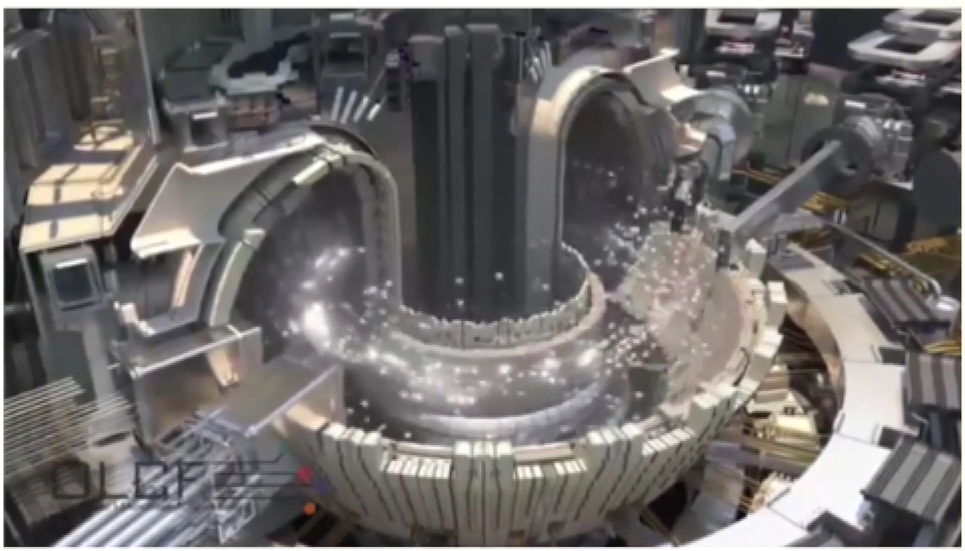
7.1.6 Why fusion is neat

Fusion energy density: $3.5e8 MJ/kg$. Compare to $30 MJ/kg$ for coal, $50 MJ/kg$ for oil, $8.5e7 MJ/kg$ for uranium fission. $9e10 MJ/kg$ for $E = mc^2$ total conversion.

The fuels are virtually inexhaustible. Lithium specifically can be mined (one example - Nevada), both lithium and deuterium can be obtained from sea water. This is so uniformly spread over the Earth that you could almost call it open source.

The environmental impact? No greenhouse gases whatsoever, helium is neat. The radioactive elements, depending on the materials used, take a short time to become safe.

It is impossible to weaponize a fusion reactor. You can't lose control of the reaction (no criticality) and there's very little fuel in the reactor at any given time (about $1g$) .



Let's go do that.

7.2 Power balance in reactors

Each DT reaction gives $3.5 MeV$ to the α and $14.1 MeC$ to the neutron, for a total of $17.6 MeV$.

Fusion power density for a particular relative velocity (or energy). n_D, n_T are densities of deuterium and tritium. σ is the reaction cross-section.

$$R_{DT}(v)\Delta E_f = (n_D n_T \sigma_{DT}(v) v)\Delta E_f \tag{32}$$

We can integrate that over velocity distributions for both D and T.

Assuming neutrality ($n_D = n_T = n_e/2 = n/2$), neglecting impurities and density of alpha particles:

$$\text{Power density} = n_D n_T \langle \sigma v \rangle_{DT} \Delta E_f = \frac{1}{4} n^2 \langle \sigma v \rangle_{DT} \Delta E_f \tag{33}$$

To produce $1GW$ of fusion power at $T = 20keV$, $n = 5e20m^{-3}$, use tabularized data for the average $\langle \sigma v \rangle$. Divide desired fusion power by fusion power density. The result: $14m^3$ - apparently a reasonable volume.

7.3 Generalities

For steady state operation - conservation of energy. The losses include radiation, convection and conduction ('direct losses'). Our power input is whatever we do to heat the plasma and the fusion power produced inside.

The radiative lossses include bremsstrahlung - electrons are accelerated in an electric field, producing x-ray radiation (which can pass through the plasma and the containment vessel - thus escaping). Power density of that $\sim n^2 T_e^{1/2}$.

The direct losses are described by the energy confinement time τ_E. Power density of that $= 3nT/\tau_E$.

7.4 Calculation of power balance - breakeven

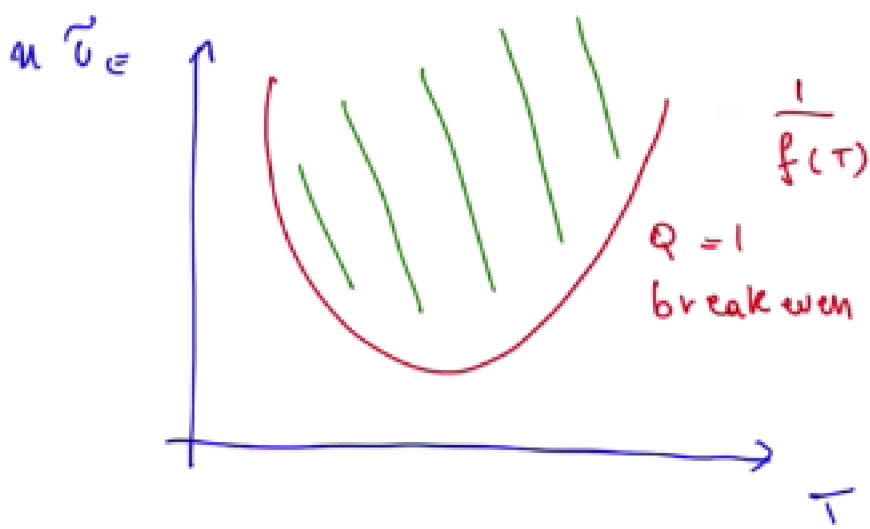
Def. physics fusion gain Q = Fusion power/Input power $= P_f/P_in$. A reactor needs $Q > 1$. At $Q = 1$ we have a breakeven situation. $P_in + P_\alpha = P_{losses} = P_{directlosses} + P_{bremsstrahlung}$, so we get $Q = \frac{P_f}{P_{directlosses}+P_{bremsstrahlung}-P_\alpha}$.

Inserting the previously derived expressions:

$$n\tau_E \geqslant \frac{12T}{\frac{6}{5}\left\langle \sigma v\right\rangle_{DT}\Delta E_F + 4AZ_{eff}T^{1/2}} \tag{34}$$

At large temperatures (over $1keV$), we can neglect the second term to get

$$\frac{1}{n\tau_E} = f(T) = \frac{\left\langle \sigma v\right\rangle_{DT}\Delta E_F}{10T} \tag{35}$$



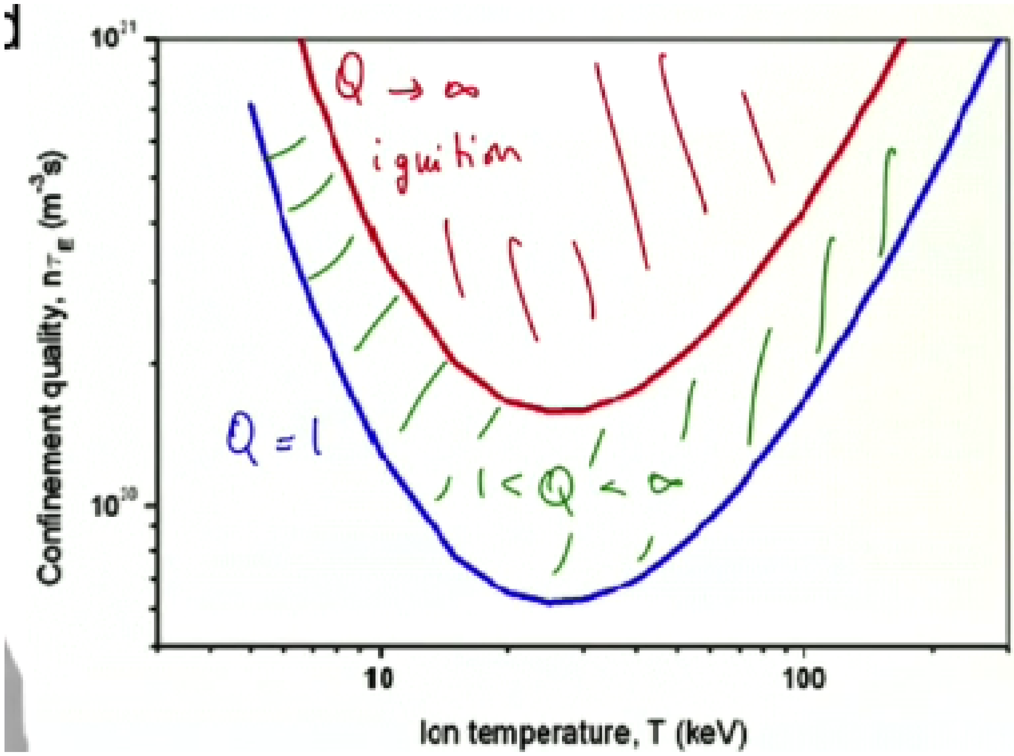
Reactors can work in the green area above the breakeven curve.

7.4.1 Ignition

This is the limit where all the heating the plasma needs comes from the fusion reactors. This would be the Sun on Earth scenario. $P_F/P_{in} = +\infty$ as $P_{in} \to 0$

$P_in = 0$ means that $P_\alpha \geqslant P_{losses} \sim P_{dl}$ (we neglect bremsstrahlung).

$n\tau_E$at ignition $\geqslant \frac{12T}{\left\langle \sigma v\right\rangle_{DT}\Delta E_\alpha} = 6/f(t)$. It's just a higher parabola!



The burning plasma regime is when alpha particle heating is greater than the input heating, $Q \geqslant 5$.

7.4.2 Engineering fusion gain

Def. Q_E - ratio of electric power output to input, η_e - efficiency of plasma heating electrical power supply, η_t - efficiency of thermal fusion power conversion into energy. $\eta = \eta_e \eta_t$, $Q_E = \frac{\eta(P_f + P_{in}) - P_{in}}{P_{in}} = \eta Q - (1 - \eta)$. $Q \gg Q_E$, sadly.

7.5 Plasma confinement approaches

We need both $n\tau_E = 10^{20} m^{-3} s$ and $T \geqslant 10 keV$. Magnetic confinement has τ_E about $1s$. Inertial confinement goes with $10^{-9}s$ and much higher densities.

7.6 Inertial confinement

We consider the ion confinement time $\tau_i = R/c_s$, R being the pellet radius and c_s the speed of sound, at which the ions tend to move.

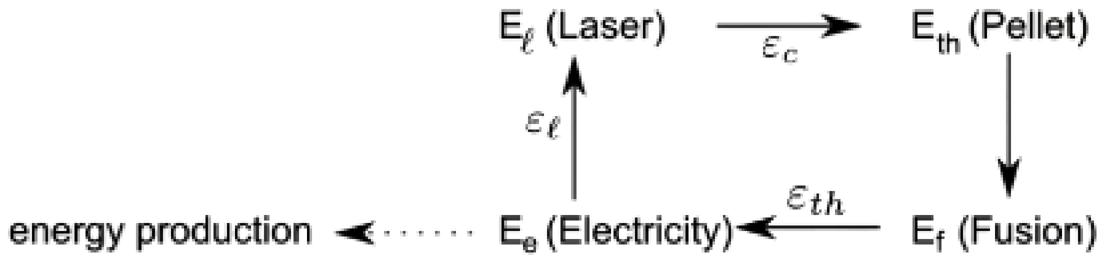
We also define the heating time $\tau_h = \frac{\text{energy in pellet}}{\text{power deposited on surface}} = \frac{3nT4\pi R^3/3}{F4\pi R^2} = \frac{nTR}{F}$, F being the energy flux

We must have heating faster than the confinement time. $\tau_h < \tau_i \rightarrow F > mT^{3/2}/\sqrt{m_i}$. The size of the pellet cancels out. Note that we neglected all compressional effects (shock waves...).

For a standard example, F is about $5e19 W/m^2$. This means lasers. Really big lasers.

7.6.1 ICF energy balance

Laser inputs energy E_L - I took the liberty of changing notation. Thermal energy is scaled by efficiency ϵ_c: $E_{th} = E_L \epsilon_c$. This produces fusion energy E_f and that process also has a finite efficiency in conversion to electricity, ϵ_{th}. We can also recirculate some of the fusion energy to drive the reactor.



It turns out there's a simple condition for the laser energy sufficient for fusion. For typical values (solid deuterium), and relatively low efficiencies (5%, 10%) it turns out to give $10^{15}J$ - 50 times the Hiroshima bomb. We can currently achieve $2MJ$. This requires a density of $3000n_0$, n_0 being solid deuterium density. This means we have to have a pressure of $3000n_0T$, which is $2e17$ Pascal. That's a lot. Can't do that just by firing lasers.

7.6.2 The rocket effect

Generate an implosion by launching a lot of energy into the external surface. Conservation of momentum will cause an inward motion of the external layers and this will compress the plasma.

Pressure $P_{rocket} = \frac{\frac{d(mv)}{dt}}{\text{surface}} \sim \frac{V\frac{dm}{dt}}{S}$ at constant v

The energy flux F, at constant v, turns out to be $\frac{v^2 dm}{2Sdt}$ and we can derive the $\frac{dm}{dt} = \frac{2SF}{v^2}$.

Cancelling the terms, we get $\frac{P_{rocket}}{P_{laser}} = \frac{2c}{v}$. This is much bigger than 1 and thus a considerable boost

7.6.3 Additional ICF effects

The shockwave, if it goes symmetrically into the center, helps increase pressure.

Alpha production can also help heat the core, but the pellet must be larger than alpha mean free path.

However, instabilities, as always, occur and the laser light can't be too intense. Also, too much density means the laser beam actually gets reflected (plasma frequency cutoff occurs)

7.6.4 Summary of physics

- Radiation phase

  Surface heated by laser, gets plasmified

- Blowoff phase

  This is where the surface is blown off and the rest is pushed inside.

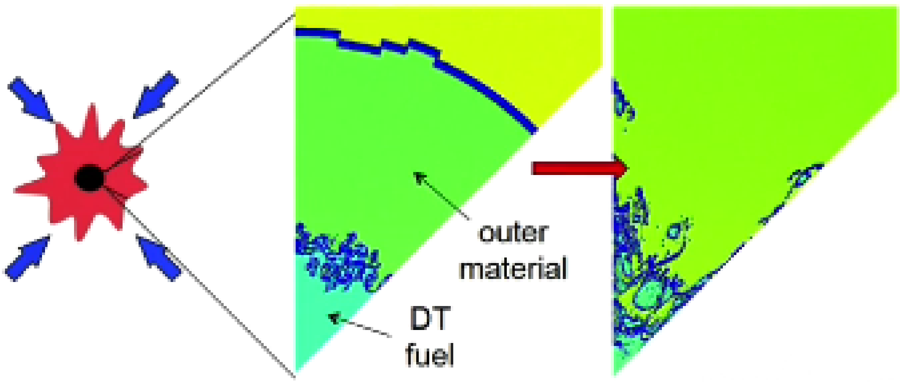
- Implosion phase

  Thermal energy is transported to the center, the core is heated

- Thernomuclear burning

The challenges here are as follows: The core cannot be heated before the shock wave hits it (separate it from the rest somehow?). Imposes constraints on laser pulse timing and pellet design.

Also, there's the question of hydrodynamic stability:



The pellet must be very symmetric and the beams have to be really carefully aligned. Shown above is a mixing of the DT fuel and core material due to the Rayleigh Taylor instability - we don't really want that.

7.6.5 Direct and indirect drive

The direct drive is the simpler one - we just smack the pellet with lasers, as symmetrically as possible. There's multiple layers - a low density ablator on the outside, a ball of solid or liquid fuel and a gas layer.

The indirect drive is intended to lower the symmetry requirements. We have a small chamber - the so called hohlraum - around the pellet. It receives laser beams through two holes and acts as a black body source of x-rays. This is inherently more symmetric than injecting the pellet with laser energy.

The Lawrence Livermore Laboratory's National Ignition Facility uses 192 lasers to achieve pulse energy of $2MJ$, with about $500TW$ of power. These fire UV light ($352nm$).

7.6.6 Engineering challenges

Cost, efficiency, reliability. Difficult to find materials for first vacuum chamber wall. Capsules are really complex (3d printing, perhaps?).

A reactor would need to fire a few times a second to function as a power generator.

7.7 Magnetic confinement

Confine particles by making them rotate around magnetic field lines. Macroscopic confinement times allow lower densities.

There's many schemes for this:

- Linear devices - such as magnetic mirrors
- Toroidal configurations
- Stellarators

7.7.1 Simple toroidal device example

A simple toroidal field has curvature and an intensity gradient. This causes drifts - positive charges go to the top, negative charges to the bottom. This causes first an E field, then an ambient magnetic field and this means loss of confinement.

What we do is impose a poloidal field. Through their motions particles stay in regions of different drifts directions.

There's the so called hoop force which tends to move field lines to the outer region of the tokamak. To avoid this, we add a vertical field which counters that.

To sum up:

- Toroidal field coils
- Plasma current generates poloidal field
- Outer poloidal field coils that generate the vertical field, can also help positioning

How to drive the current in the plasma? That's a neat idea - it's basically a transformer. The plasma is the secondary circuit and the primary circuit is a coil inside the core - center - of the tokamak. Drive AC through that and you drive AC through the plasma by Faraday's law.

7.7.2 Progress

Density times confinement time with respect to temperature - we're steadily moving towards the high energy multiplication and sustained burn region. $Q = 5$ is getting really close now. ITER should have Q of about 10 - a reactor needs about 30 or 40. TFTR and JET actually got close to breakeven by significantly increasing the confinement time. ITER aims to push that to several hundreds of second.

ITER is being built in Cadarache in the south of France. A really global project.[8]

And now, back to our regularly scheduled programme on magnetic confinement. 5

7.8 Simple design of a magnetic fusion reactor

Constraints on the design result from nuclear physics and engineering.

We put lithium in the blanket, which also collects the heat (usual water energy transfer system).

Tritium is produced in the blanket - has to be kept in a closed system, as it's used in the DT plasma for fusion purposes.

Around the blanket we need magnets for B field generation.

We want $1 < Q < \infty$, between breakeven and ignition. $n\tau_E$ is thus between $1e20$ and $6e20 m^{-3}s$. $T \geqslant 10keV$.

We want to minimize the cost of the reactor and the requirements on plasma performance. This means minimizing τ_E and β - ratio of plasma pressure and magnetic pressure.

We'll be optimizing:

- the size of the reactor
- reactor geometry
- magnitude of B field
- n
- T
- τ_E

Our constraints are those resulting from both physics and engineering.

7.8.1 Simplified geometry for magnetic fusion reactor

Assume toroidal general shape, circular cross section. Plasma on the inside, radius a. Blanket of thickness b. Magnet thickness c.

From engineering:

We want to produce $P_E = 1GW$ for now.

Wall loading L_W - this tells us how much power we can be depositing on the wall. Usually about $5MW/m^2$.

[8] Really makes you wonder why the victory condition in Civilization games is the space race to Alpha Centauri, huh?

The magnets have to be superconducting in a certain region in BJT space - in practice this says that we can only have $13T$ at the coil.

From nuclear physics:

Value of fusion rate determined by fusion cross-section.

Processes in the blanket also have their cross-sections:

Neutron multiplication (has to occur before breeding tritium, has to occur at the same rate).

Neutron slowing down - at high neutron energies tritium production cross-section decreases.

Note that we only consider lithium 6 - at low temperatures the cross section for Li-7 is negligibly small. We don't really have to worry about that, it seems (at least at this basic level).

We must also shield the coils from the neutrons - don't want those to become radioactive, of course!

7.8.2 Neutron physics - blanket and shielt thickness

For each process, the thickness required is the inverse of number density of targets times cross section.

Neutron multiplication can be done on beryllium. Thickness comes out to be $13cm$.

Neutrum slowing down can be done on lithium itself - that's $20cm$.

Tritium breding at room temperature $0.025eV$ with $7.5\% Li^6$ turns out to be $0.2cm$. Note that the 7.5% purity of lithium comes into the number density. So we don't have to add more than necessary for slowing down, it seems.

For coil shielding, say we want to reduce the flux 100 times. Assume only the slowdown layer (which is the largest) - we need $1m$.

Total thickness about $1.2m$ - it's not direct addition as layers overlap in function.

7.8.3 Minimisation of cost of electricity

Minimising reactor cost over power is equivalent in this kind of work to minimizing volume of complex systems over power. This means - everything but the plasma, that's not something you build.

Volume of torus $2\pi R_0 \times \pi((a+b+c)^2 - a^2)$. Electrical power produces is 1/4 efficiency of power input times sum of energies in alpha, neutron and tritium (in blanket!) production, times density squared, times DT cross-section, times plasma volume $(2\pi R_0 \pi a^2)$

We use the wall loading constraint to get R_0 - we take the $5MW/m^2$ value times plasma surface area.

Neutron power is fusion power over efficiency in transformation into electric energy, times fraction of neutron energy over total produced energy.

From the equality between the latter two pops out (upon assuming $40\% = \eta_t$) an expression for R_0.

We can now take the complex system volume over power fraction - the electrical power actually drops out - and we're left with an expression:

$$\frac{V}{P} = 0.8\frac{(a+b+c)^2 - a^2}{aL_W^{max}}$$

Thus the better materials we have (more wall loading), the lower the volume and cost. b is already set - due to blanket physics - so we're optimizing in 2-space, over a and c.

7.8.4 Magnet thickness

That should also be minimum. There's the $\mathbf{J} \times \mathbf{B}$ force on the magnet (superconducting coil) and it has to be able to withstand the stress. A calculation with tensile stress gives $c = \frac{2\alpha}{1-\alpha}(a+b)$, alpha being $\frac{B_{coil}^2}{4\mu_0\sigma_{max}}$, σ_{max} being the maximum stress the magnet can endure.

It's difficult to reach high β in the plasma. So we need the maximum field at the coil - $13T$ (at higher values they stop superconducting, at least with today's technology).

Maximum stress is about $200MPa$. $\alpha \sim 0.1$. So $c = 0.22(a+b)$ - it simplifies out of our optimization problem! It's a one variable problem, in this simplified version.

$$\frac{V}{P} = 0.8\frac{(a+b+c)^2 - a^2}{aL_W^{max}}$$

Taking the derivative we get $a = \sqrt{3}b$. $c = 0.22(\sqrt{3}+1)b = 0.6b$. $b = 1.2m$, $a = 2.1m$, $c = 0.7m$.

7.8.5 Resulting reactor geometry and plasma parameters

Taking $L_W^{max} = 4.5MW/m^2$ we get $R_0 = 4.2m$. The plasma surface becomes $350m^2$, volume $365m^2$. The power density in that - power of alphas and neutrons over volume - $6MW/m^3$. B_0 being the magnetic field in the plasma, at the major radius. $B \propto 1/r$ and at the coil it's $13T$, so we'll get $2.7T$ at the major radius, assuming coil located at $R_0 - a - b$.

Assuming $\tau_E = 2s$, $n = 1e20m^-3$, $T = 15keV$, β turns out to be $= \frac{nT}{B_0^2/2\mu_0} = 8\%$.

7.9 Plasma-wall interactions

7.9.1 The first wall

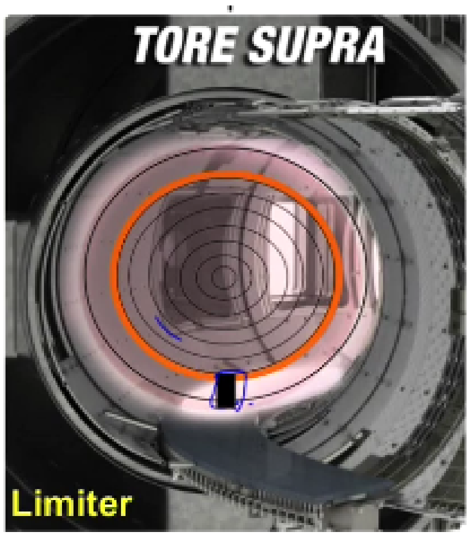
We now turn our investigation to the requirements faced[9] by the reactor wall. This has to take $10MW/m^2$ of heat! $1MW/m^2$ is felt by reentering spacecraft. $80MW/m^2$? A live rocket engine. What we also want to make sure is that the plasma is not contaminated - impurities are bad in that they increase ignition temperature, and radiate energy away (line and brehmstrahlung radiation, which goes as $Z_eff\sqrt{T_e}$). Note that this means we have to use low Z materials, it seems.

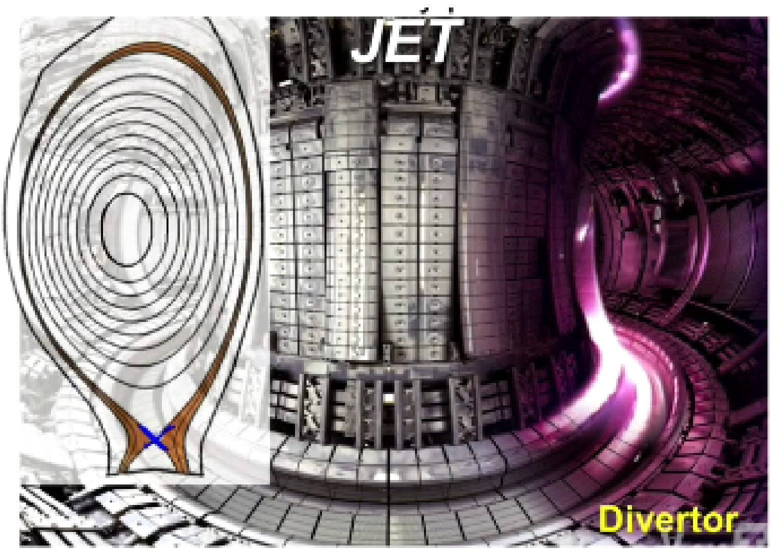
We also have to stop retention of T in the wall. Tritium slowly accumulates during the discharge. We can tolerate up to 700g of T in the wall. Carbon materials accumulate it much quicker than if you made it all out of tungsten, which seems like a neat solution.

The walls must also minimize dust production, that leads to instabilities, allow containment, refueling and removal of helium ashes - low energetic alpha particles. We have to remove those as fast as possible.

7.9.2 Limiters and divertors

Direct contact should be avoided, but is ultimately necessary (helium, energy removal, etc). People use limiters and divertors for that. Limiters stick out into the plasma and intercept particles. Divertors use an X shaped magnetic field configuration which has a null point for the poloidal field.





[9] Sorry.

7.9.3 The scrape off layer

This is the layer of the plasma which is in direct contact with the wall. The parallel flow in that strikes the limiter. Using a diffusion argument (Fick's law), we get a $1cm$ thickness for ITER - a lot of power striking a small surface!

7.9.4 Divertor advantages

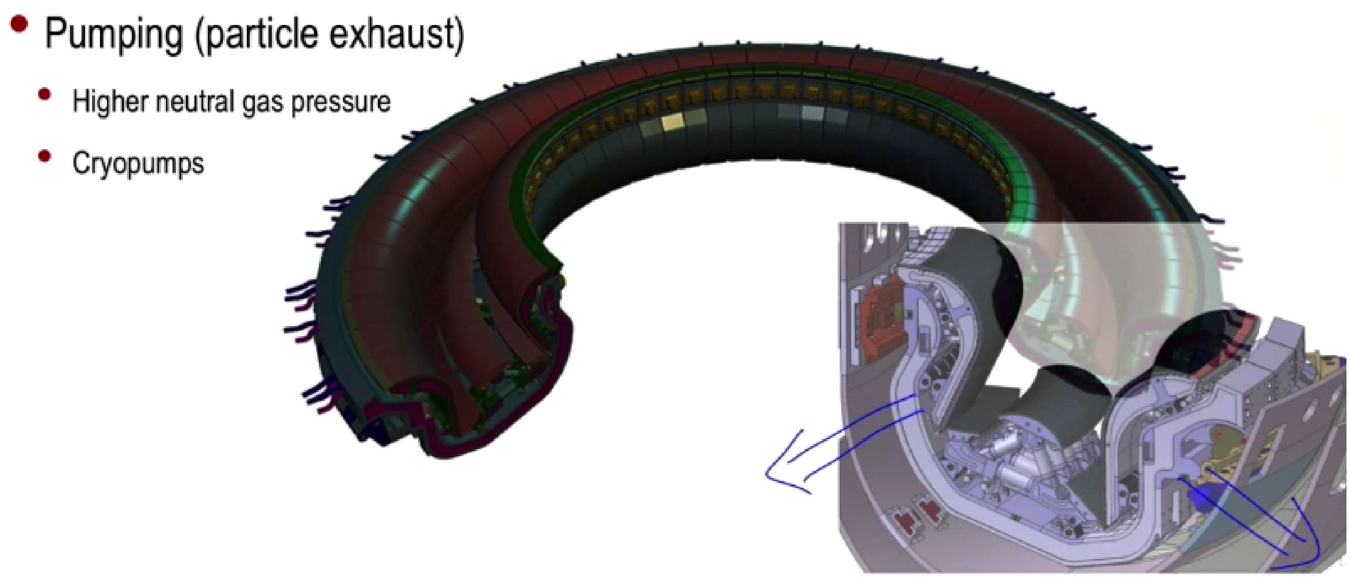
Particles move along field lines. The 'connection length' for the ITER divertor will be about $150m$, this slows down the flux of particles and reduces parallel power flux. There is also a temperature gradient - the plasma is colder towards the divertor while being thermonuclear near the core.

The slowdown of particles means that there's less sputtering and charge exchange collisions. This also means that there's less impurities in the main plasma chamber - it seems to be mostly flowing outside to the divertor chamber.

High confinement regimes are allowed by the divertor concept - they have low turbulence and are easier to confine. Transport barrier - a calm layer of plasma that stops transport.

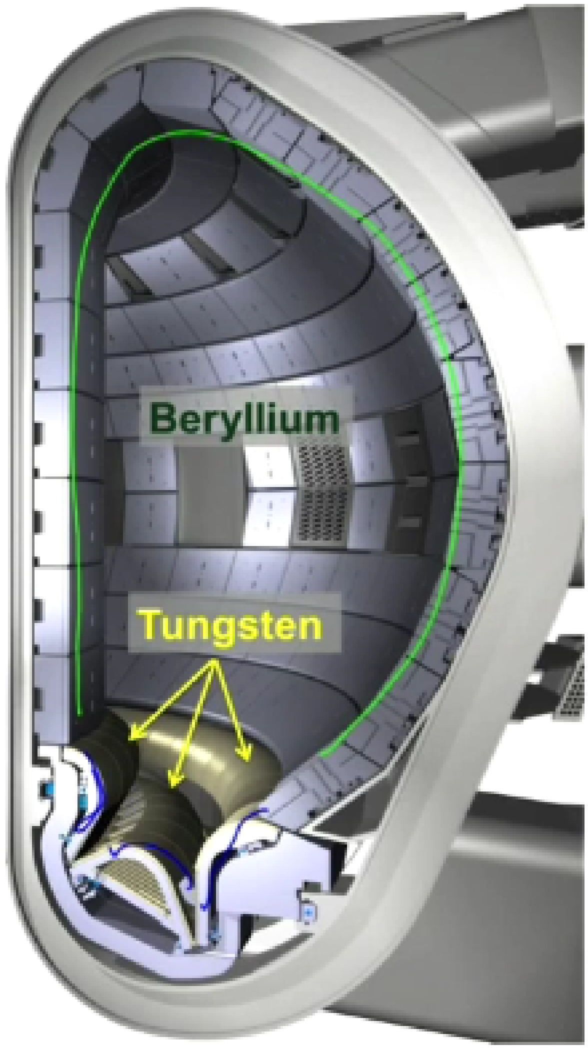
L-mode, AKA low confinement mode - turbulent edge. Small amplitude pressure gaussian, particles sorta tunnel through.. In the H-mode, the pressure gaussian is much taller and this effectively stops particle transport.

Due to the high pressure region, pumps can exhaust particles easier from there. Cryopumps can also be used to exhaust material.



The divertor allows the plasma to cool down to about 5 eV. This lowers ionisation cross-section. Allows easier change exchange collisions. This transfers the energy from ions to neutrals, forming a so-called neutral cushion in the divertor chamber. Energy is mostly dissipated here in radiation and not through wall interactions, which is neat! Otherwise this would hit the wall.

In ITER, the divertor will be made of tungsten. The walls will be made of beryllium. Both have low tritium retention. Tungsten has high Z but that's okay because the sputtering threshold is high, so not much of it will be injected into the plasma. These materials are also resistant to neutron irradiation.



Transients include ELMs - Edge Localised Modes. Steep gradients of pressure near the edge - large thermodynamic potential - can drive violent instabilities - $15MJ$ for ITER in a single burst over $0.2ms$ deposited over $6m^2$ - $10GW/m^2$ of power. Heats wall to $6000C$ - melts the metal.

Another issue are disruptions. A disruption is a sudden loss of plasma control. Deposits energy on the wall, on very specific areas. In ITER, if we have disruptions the divertor will fail after about 300 of them.

We can't fix this right now via materials! We have to control the plasma itself. DEMO will need a lot of work. There's plenty of alternative concepts, some of them: liquid metal walls - a thin layer of melted lithium circulating around the plasma. Super-X divertor designs - expand the divertor to get more volume for the plasma chamber, from which we can radiate energy. Snowflake divertor - tested at EPFL - give the divertor four legs instead of two. Longer connection length. Can change plasma stability, must be investigated. Not sure if this will work better than what current divertors.

7.10 Structural materials

Large fluxes of $14.1MeV$ neutrons in reactors (results of fusion). We must thus minimize activation of materials. Fusion is at least an order of magnitude lower in thermal power than fission. Ferritic steel is nice, but silicon carbide and vanadium alloys are really amazing - they get cold relatively quickly.

Use Carnot efficiency, $\eta_{Carnot} = 1 - T_{cold}/T_{hot}$. Higher plasma temperatures and lower coolant temperatures lead to increased η.

Fission reactors go steady state quickly, but temperatures are relatively low in our terms. Fusion reactors would have much higher temperatures, but violent transfers. Main difference: fission has average neutron energy $2MeV$. In fusion $14.1MeV$. Causes requirement for more complex materials devices.

7.10.1 Physical picture of energetic neutron collisions

A E_n neutron hits mass M atom, resting in a lattice. Maximum transferred energy $E_{max} = E_n \frac{4m_n M}{(m_n+M)^2} \sim \frac{E_n 4m_n}{M}$. For iron, $M = 56amu$. $E_max = 1MeV$. Much higher than the so called Wigner energy - threshold value for energy, at which atom is displaced in the lattice (usually about $25eV$). The iron atom gets bumped out of the lattice! What's worse, the process cascades. The pair between the hole and the displaced atom is called a Frenkel pair. We define displacements per atom as measure of damage.

Not as simple as that though - the neutron also causes nuclear transmutation. It generates hydrogen and helium atoms - this causes the grain boundaries to become brittle. Measured in atomic pair per million (appm) of He or H.

This causes changes in the material microstructure. Degrades macroscopic properties. Defects migrate across the lattice. For a $3GW$ fusion reactor, we expect $20 - 30dpa/y$ in steels. ITER is expected to have $3dpa$ at end of life cycle. Appm/dpa is much higher than in fission! This causes:

- chemical composition changes
- changes in physical properties such as electrical and thermal conductivity for functional materials
- changes in mechanical properties for structural materials
- changes in mechanical dimensions - stuff starts deforming, swelling, growing in length... wacky.

Thus we are constrained to considering low activation elements. Currently we're looking into:

- Reduced activation ferritic or martensitic steels - least advanced option
- Vanadium alloys
- Tungsten alloys
- SiC or its composites

We have to test the materials! We know little experimentally about the behavior of those materials under such strong neutron fluxes[10]. We know much more about the plasma parameters at fusion regimes than about material behavior in those regimes. Have to extrapolate from small samples.

Current ideas for testing those are low fusion gain tokamaks and accelerator based irradiation test facilities, such as the IFMIF project - uses a deuterium ion beam at $10MW$ to hit a liquid lithuim target, which produces lots of neutrons to strike a small solid material target. Can extrapolate results for larger samples.

[10]High energy neutron gun recently developed at IFPILM may help, go Poland, can into fusion project!