Notes from PlasmaX

Perfi May 8, 2015

These notes will not be 100% comprehensive, as I'm making them mainly for my own use. However, if you spot any mistakes, feel free to catch me on the forums and I'll fix any mistakes.

1 Week 1. Description of the plasma state, with Paolo Ricci

Didn't start making notes until 1.5 so I'll be skimming the earlier topics.

1.1 Plasmas in nature and laboratory

- Plasma - the 4th state of matter. Heat stuff up to 11400K (= 1eV) and gases begin being ionized.
- The Sun is a miasma of incandescent plasma[1]
- Lightning is plasma (ionized air)
- Plasma diplays
- Nuclear fusion - can't really get there without turning stuff into plasma
- The word 'plasma' comes from greek $\pi\lambda\alpha\sigma\mu\alpha$, which means 'moldable substance' or 'jelly', though it was mentioned on the forums that it might mean 'living thing'... which is really fitting when you think about it
- A brief history:
  - 1920's-1930's: ionospheric plasma research (for radio transmission) and vacuum tubes (Langmuir)
  - 1940's: MHD plasma waves (Alfvén)
  - 1950's: research on Magnetic Fusion. Geneva UN conference on uses for atomic energy which don't kill people
- Fusion experiments: L-1, TFTR, JET, ITER tokamaks; W7-X stellarator at MPI in Germany; the NIF inertial fusion facility in US
- The Earth's magnetosphere; van Allen belts
- Jets - space plasmas
- Lots of industrial applications

1.2 Rigorous definition of plasma: Debye length

A plasma is a **globally neutral** *ionised gas* with **collective effects**

The following parameters classify plasmas:

- **Debye length**

  Distance over the potential of a charged particle decreases by a factor $1/e$ due to screening by other charged particles

[1] https://www.youtube.com/watch?v=sLkGSV9WDMA

$$\lambda_{De} = \sqrt{\frac{\epsilon_0 T_e}{e^2 n_0}}$$

(for electrons)

Solved in lecture by a statistical approach which assumed $n\frac{4}{3}\pi\lambda_{De}^3 \equiv N_D \gg 1$ (for a Debye sphere; in the lecture $n\lambda_{De}^3$ was used, which relates to a Debye cube. There's not much difference between them, a factor of 4). N_D means the number of particles inside a sphere (or cube, following the lecture) of radius equal to the Debye length. The condition means there's plenty of particles to screen our test particle. This also assumed that binary interactions between particles were weak ($\frac{e\phi}{T_e} \gg 1$)

1.3 Plasma definition: frequencies and parameters

- **Plasma frequency** Assume a plasma of same density of ions and electrons. Displace electrons by Δx. They begin to exhibit harmonic oscillations (for Δx not too large). Newton's 2nd law gives

  $$\frac{d^2\Delta x}{dt^2} + \frac{n_0 e^2}{\epsilon_0 m_e}\Delta x = 0$$

  Can define plasma frequency

  $$\omega_{pe} \equiv \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}} = \frac{v_{th,e}}{\lambda_{De}}$$

  where $v_{th,e}$ denotes the thermal speed of electrons

- **Collision frequency**

  The frequency of coulomb collisions between particles

  $$\nu_{coll} \equiv \frac{n_0 e^4}{16\pi\epsilon_0^2 m_e^2 v_{th,e}^3}$$

- Size of plasma has to be much larger than its Debye length (or there's no quasineutrality)

1.4 Particle motion in a static uniform magnetic field . Plasma magnetic properties

- Larmor radius - particles gyrate around the guiding center at this distance

  $$\rho \equiv \frac{mv_\perp}{|q|\,B}$$

- Cyclotron frequency

  $$\omega_c \equiv \frac{v_\perp}{\rho} = \frac{|q|\,B}{m}$$

  Particle rotation direction on their helical trajectory

  - $q > 0$ ('by default'): left hand rotation with respect to **B**
  - $q < 0$ (electrons): right hand rotation
  - Magnetic moment

    $$|\mu| \equiv IA = \frac{|q|\,\omega_e}{2\pi}\pi\rho^2 = \frac{mv_\perp^2}{2B} = \frac{E_{kin\perp}}{B}$$

    (direction opposite to B) is an adiabatic invariant for every particle; doesn't change under slow changes of factors. However, it will change through heat exchange, which usually operates on slower timescales than magnetic field changes.

    Plasmas are diamagnetic (they reduce externally applied magnetic fields) (because of direction of μ)

1.5 Particle motion in given electromagnetic fields: the drifts

Static and uniform E and B fields. Particles under Lorentz force which can be decomposed as:

- Parallel direction:

$$m\frac{d\mathbf{v}_\parallel}{dt} = qE_\parallel$$

  Uniform acceleration

- Perpendicular direction:

$$m\frac{d\mathbf{v}_\perp}{dt} = q(\mathbf{E}_\perp + \mathbf{V}_\perp \times \mathbf{B})$$

The many drifts in a plasma:

- $\mathbf{E} \times \mathbf{B}$ drift
  - Perpendicular component averages out over gyroperiod

$$\mathbf{v_e} = \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2}$$

  - This is a motion of the guiding center which is superposed over the gyromotion
  - Does not depend on charge, neither in magnitude nor in direction (but gyromotion direction does)
  - Guiding center moves over lines of constant electrostatic potential ϕ (the drift does not change the particle energy!)
  - A generalization of this drift for any force:

$$\mathbf{v_F} = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}$$

  - For a gravitational force (say, space plasmas), this depends on charge. Separates positive and negative charges. Polarizes the plasma, creating a $\mathbf{E}$ field and an $\mathbf{E} \times \mathbf{B}$ drift

- Curvature drift
  - $\mathbf{B}$ field curved, particle follows the B field - this happens through a centrifugal force

$$\mathbf{F_c} = \frac{mv_\parallel^2}{R_B^2}\mathbf{R_B}$$

  - This causes a drift:

$$\mathbf{v_d} = \frac{\mathbf{F_c} \times \mathbf{B}}{qB^2} = \frac{mv_\perp^2}{qB^2R_B^2}(\mathbf{R_B} \times \mathbf{B})$$

- Gradient drift $\boldsymbol{\nabla} B \perp \vec{B})$
  - Happens in changing (spatially) magnetic fields

$$\mathbf{v}_{\boldsymbol{\nabla}\mathbf{B}} = \frac{mv^2}{qcB^2}(\mathbf{B} \times \boldsymbol{\nabla} B)$$

– A derivation so complicated, it deserved a separate appendix. As particles gyrate, they move between regions of smaller and bigger B. This causes a drift in a direction perpendicular to both the B field and the gradient of its value We consider a small variation in B and expand B in a taylor series around B_0.

Then we use that expansion to solve $m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times B$, plugging in our expansion for B.

We also decompose the velocity: an average v_0 and a small perturbation. v_0 is the solution to the equation for constant magnetic field B_0.

We neglect the cross product of the two small perturbations and average over a gyroperiod.

We use our knowledge of the solution for the static magnetic field (gyration in the plane perpendicular to B) to deal with the perpendicular velocities (x and y in this decomposition under the assumption that B is along z).

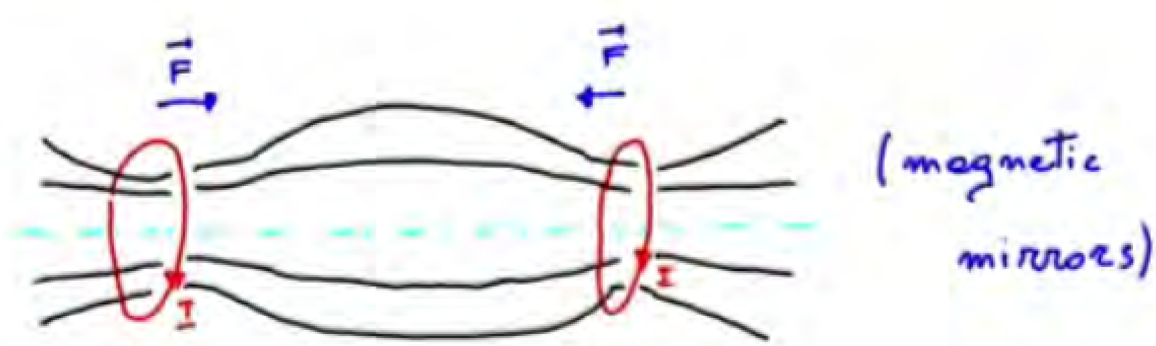
The drift velocity is the perturbation described by the formula above for an arbitrary geometry of the problem.

1.6 Plasma confinement based on single particle motion. Magnetic mirrors, stellarators, tokamaks

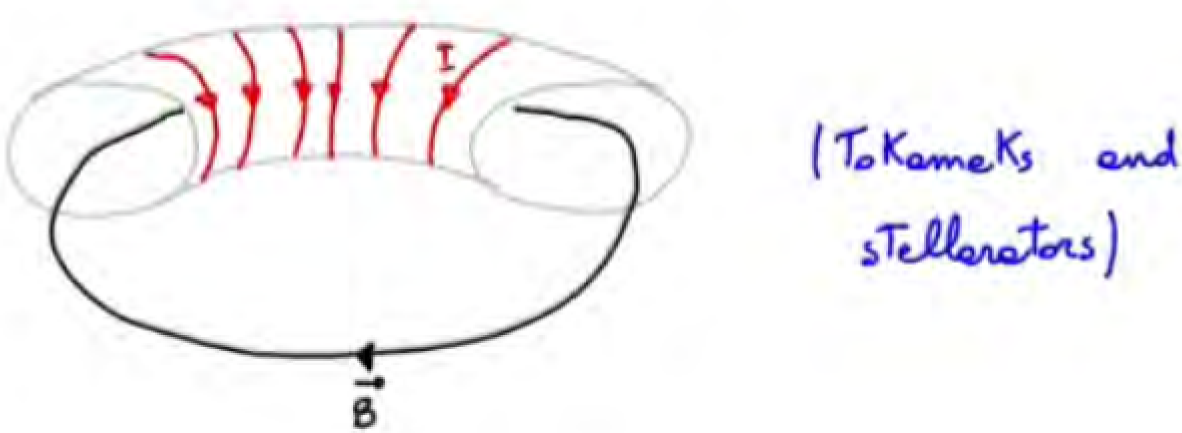
- How do you confine a plasma?

  Charged particles follow helical trajectories along B field. This confines them in the perpendicular direction. What about the parallel one?

- Can use open field lines. Take two circular coaxial electromagnets.



- Can use closed field lines. Closed geometries. Example: tokamaks (toroidal), stellarators.



- The magnetic mirror geometry is neat for particles really close to the axis. B is maximum (field density increases) near the electromagnets

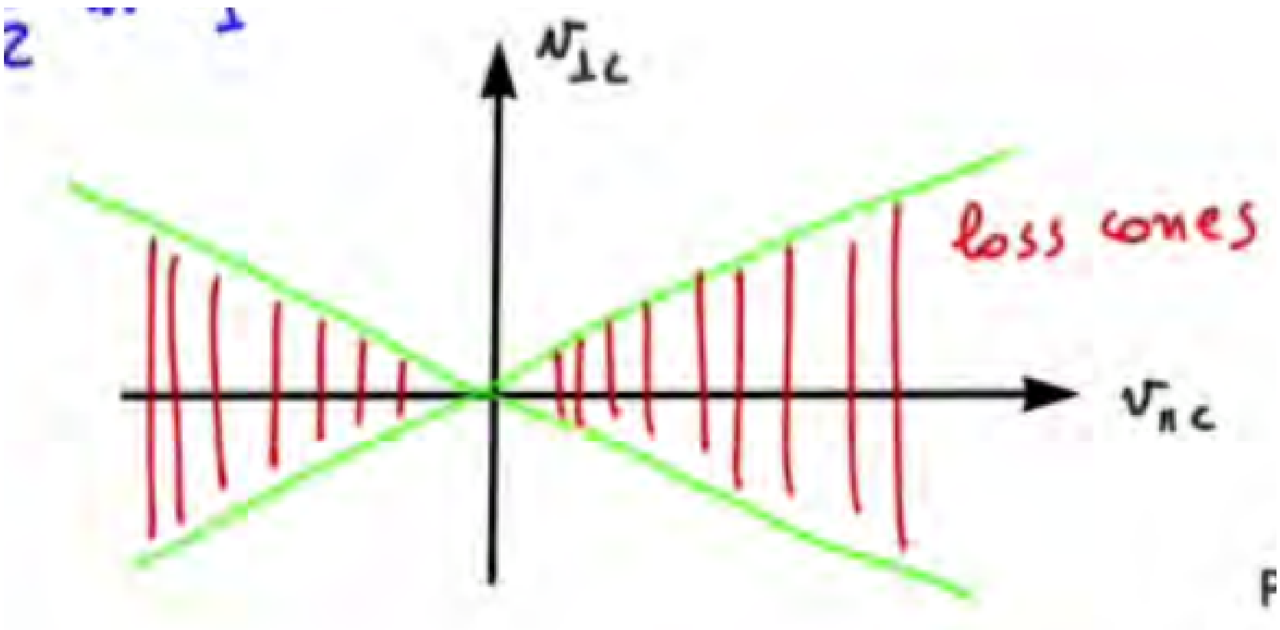
  force in the axial direction is

$$F_z = -\mu \, |\boldsymbol{\nabla} B|$$

  $v_\parallel$ has to vanish at B_{max} so that the kinetic energy is just composed of the perpendicular component of velocity

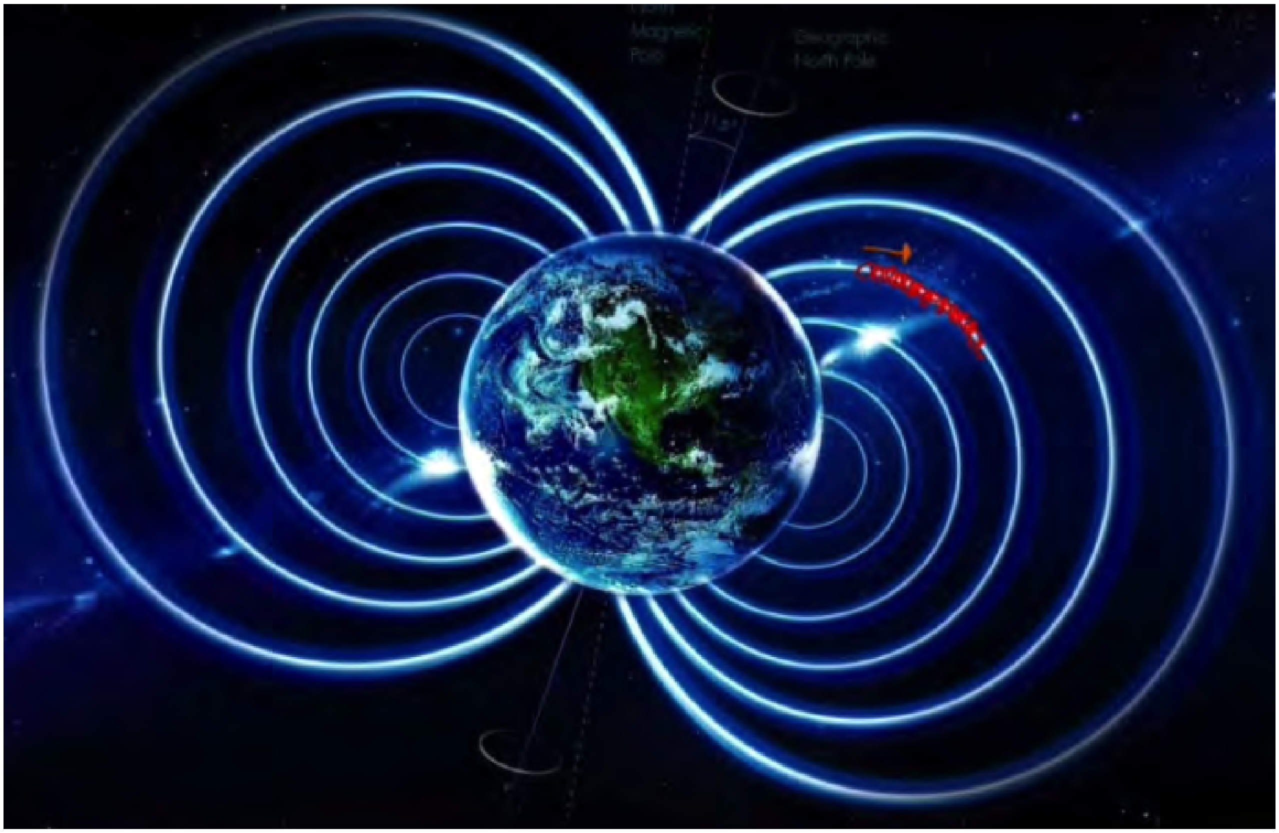
  Particle reflection condition

$$\frac{v_\perp^2}{v_\perp^2 + v_\parallel^2} > \frac{B_{min}}{B_{max}}$$



  This means that particles in the **loss cones** in phase space (marked red; those which don't satisfy the inequality) cannot be confined in the mirror!

  Neat example: the Earth's magnetic field is a magnetic mirror!



- What about closed magnetic field lines? Can those deal with loss cones?

  B is not homogeneous! Curved! Has curvature and gradient drifts!

  For a purely toroidal field, positively charged particles drift towards the bottom, while negatively charged ones drift towards the top. This polarizes the plasma and introduces the $E \times B$ drift outwards, sending the plasma crashing into the major radius wall.

  A solution: a poloidal magnetic field to short circuit the charge accumulation. Either:

  – Drive a current through the plasma $\rightarrow$ Tokamaks
  – Get rid of axial symmetry $\rightarrow$ Stellarators

2 Week 2: Kinetic description of plasmas, with Paolo Ricci

2.1 2a) From single particle to kinetic description

Kinetic description of plasma. A (relatively?) complete description of plasma which covers both the particles and the fields evolving over time.

The usual diagram for a plasma description, seen often in simulations:

(a) Take Newton's equations using electric and magnetic fields **for all particles at all times** (use Lorentz force)

(b) Use positions and velocities to compute charge and current densities. Charge density given as sum over particles of their charges, localized through use of Dirac delta functions. Current density - similar, but multiplied by particle velocity vectors inside the sum.

(c) Take charge and current density, plug them into Maxwell equations, calculate E and B fields at positions

(d) Take calculated E and B fields and apply them as forces to particles. Repeat cycle until bored or simulation returns segmentation fault.

But real plasmas involve on the order of 10^21 particles for a fusion plasma. Too much strain on our computational abilities. Impractical. We use a distribution function:

$f(\mathbf{r}, \mathbf{v}, t) d\mathbf{r} d\mathbf{v}$ = number of particles at time t, in phase space volume $d\mathbf{r}d\mathbf{v}$ located at $\mathbf{r}, \mathbf{v}$. We have a separate distribution function f_i for every species

- Total number of particles N_S given by integral of distribution function over all positions and velocities (which covers all the phase space)
- Number density of particles n_s given by integral over all velocities for a given location $\mathbf{r}$
- Average velocity given by $\frac{1}{n_s}\int \mathbf{v} f_i(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}$

Examples of distribution functions

- Maxwell-Boltzmann distribution function, for three dimensions

$$F_0(\mathbf{v}) = n_0(\frac{1}{2\pi v_{thermal}})^(3/2) exp(-\frac{v^2}{2v_{thermal}^2})$$

In 1D, only the normalization of the distribution changes from the 3D case:

$$F_0(v) = n_0(\frac{1}{2\pi v_{thermal}})^(1/2) exp(-\frac{v^2}{2v_{thermal}^2})$$

- Monoenergetic beam in 1D

$$F_0(v) = n_0 \delta(v - v_0)$$

- Two counterstreaming beams in 1D (two-stream instability!)

$$F_0(v) = \frac{n_0}{2}[\delta(v - v_0) + \delta(v + v_0)]$$

Conservation of particles number

If there are no sources or sinks, we have the following condition for conservation of number of particles

$$\frac{df_s}{dt} = -\nabla_{ED} \cdot (\mathbf{u} f_s)$$

where

$$\nabla_{ED} = (\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{dv_x}, \frac{d}{dv_y}, \frac{d}{dv_z}) = (\frac{d}{d\mathbf{r}}, \frac{d}{d\mathbf{v}})$$

$$\mathbf{u} = (\frac{d\mathbf{r}}{dt}, \frac{d\mathbf{v}}{dt}) = (\mathbf{v}, \frac{\mathbf{F}}{m_s}) = (\mathbf{v}, \frac{\mathbf{F_{longrange}} + \mathbf{F_{shortrange}}}{m_s})$$

Long range forces - collective interactions. Short range forces - binary collisions (between individual particles, like you'd have in a gas). Plugging these back into the particle conservation equation:

$$\frac{df_s}{dt} = -\frac{d}{d\mathbf{r}} \cdot (\mathbf{v} f_s) - \frac{d}{d\mathbf{v}} \cdot [\frac{\mathbf{F_{longrange}} + \mathbf{F_{shortrange}}}{m_s} f_s]$$

Boltzmann equation We can improve on the previous equation. Start out with the expanded particle conservation equation:

$$\frac{df_s}{dt} = -\frac{d}{d\mathbf{r}} \cdot (\mathbf{v} f_s) - \frac{d}{d\mathbf{v}} \cdot [\frac{\mathbf{F_{longrange}} + \mathbf{F_{shortrange}}}{m_s} f_s]$$

- In the phase space approach, velocity is treated as a completely independent variable than v (though you could consider one as a derivative of the other). Thus $\frac{d}{d\mathbf{r}} \cdot (\mathbf{v} f_s) = \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}}$
- long range force can be decomposed into electric field independent of v, and the $\mathbf{v} \times \mathbf{B}$ term - perpendicular to v. Thus, $\frac{d}{d\mathbf{v}} \cdot [\mathbf{F_{longrange}} f_s] = \mathbf{F_{longrange}} \cdot \frac{df_s}{d\mathbf{v}}$
- Plugging in:

$$\frac{df_s}{dt} = -\mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} - \frac{\mathbf{F_{longrange}}}{m_s} \cdot \frac{df_s}{d\mathbf{v}} - \frac{d}{d\mathbf{v}} \cdot (\frac{\mathbf{F_{shortrange}}}{m_s} f_s)$$

- Can be rewritten as:

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{\mathbf{F_{longrange}}}{m_s} \cdot \frac{df_s}{d\mathbf{v}} = -\frac{d}{d\mathbf{v}} \cdot (\frac{\mathbf{F_{shortrange}}}{m_s} f_s)$$

  Term on the right is called a ‘collision operator’ $(\frac{df}{dt})_c$.

- And we get the **Boltzmann equation**:

$$\frac{df_s}{dt} + \mathbf{v} \cdot \frac{df_s}{d\mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E_{longrange}} + \mathbf{v} \times \mathbf{B_{longrange}}) \cdot \frac{df_s}{d\mathbf{v}} = (\frac{df_s}{dt})_c$$