RP Optomechanics theory questions

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(Dated: October 25, 2024)

I. THEORY QUESTIONS

A. II.A1

$\delta = \frac{4\pi}{\lambda} n' L \cos\theta'$ with $n' = 1$ and $\theta' = 0$ gives $\delta = \frac{4\pi L}{\lambda}$. To have constructive interference we must have $\delta = 2\pi m$ where $m \in \mathbb{N}$. This gives $L = \frac{m\lambda}{2}$. For destructive interference we must have $\delta = \pi(2m+1)$ where $m \in \mathbb{N}$. This gives $L = \frac{(2m+1)\lambda}{4}$

B. II.B1

Substituting $Z(x,t) = u(x)e^{i\omega t}$ In the Differential equation gives $EI_z \frac{\partial^4 \left(u(x)e^{i\omega t}\right)}{\partial x^4} + \rho A \frac{\partial^2 \left(u(x)e^{i\omega t}\right)}{\partial t^2} = 0$. Calculating the derivatives gives $EI_z u(x)'''' e^{i\omega t} - \rho A \omega^2 u(x) e^{i\omega t} = 0$. Rearranging and dividing by $e^{i\omega t}$ gives $u'''' - \frac{\rho A \omega^2}{EI_z} u = 0$. So from this it follows that $k = \sqrt[4]{\frac{\rho A \omega^2}{EI_z}}$. This gives $\omega = \sqrt{\frac{k^4 EI_z}{\rho A}}$.

C. II.B2

Since there are 4 unknown parameters there must be (at least) 4 boundary conditions (BCs). At the fixed end ($x = 0$) we have $u(0) = 0$ (1) and $u'(0) = 0$ (2) and at the free end ($x = L$) we have $u''(L) = 0$ (3) and $u'''(L) = 0$ (4), as given by the question. In order to solve this we need the derivatives of the general solution for $u(x)$. Solving this gives:

$u(x) = a_1 \cosh(kx) + a_2 \sinh(kx) + a_3 \cos(kx) + a_4 \sin(kx)$

$u'(x) = a_1 k \sinh(kx) + a_2 k \cosh(kx) - a_3 k \sin(kx) + a_4 k \cos(kx)$

$u''(x) = a_1 k^2 \cosh(kx) + a_2 k^2 \sinh(kx) - a_3 k^2 \cos(kx) - a_4 k^2 \sin(kx)$

$u'''(x) = a_1 k^3 \sinh(kx) + a_2 k^3 \cosh(kx) + a_3 k^3 \sin(kx) - a_4 k^3 \cos(kx)$

BC 1 gives that $a_1 + a_3 = 0$ and BC 2 gives that $a_2 + a_4 = 0$. BC 3 together with $a_1 = -a_3$ and $a_2 = -a_4$ gives that $a_3 (\cosh(kL) + \cos(kL)) + a_4 (\sinh(kL) + \sin(kL)) = 0$. BC 4 together with $a_1 = -a_3$ and $a_2 = -a_4$ gives that $a_3 (\sinh(kL) - \sin(kL)) + a_4 (\cosh(kL) + \cos(kL)) = 0$ Since we do not want to the find the trivial solutions the determinant of the following matrix must be zero:

$$\begin{bmatrix} \cosh(kL) + \cos(kL) & \sinh(kL) + \sin(kL) \\ \sinh(kL) - \sin(kL) & \cosh(kL) + \cos(kL) \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \end{bmatrix}$$

Calculating the determinant we find $\cosh^2(kL) + \cos^2(kL) + 2\cosh(kL)\cos(kL) - \sinh^2(kL) + \sin^2(kL)$. Using the identity $\sin^2(x) + \cos^2(x) = 1$ and $\cosh^2(x) - \sinh^2(x) = 1$ we find that $1 + \cosh(kL)\cos(kL) = 0$ so for $\alpha_n = k_n L = kL$ we see that $1 + \cosh(\alpha_n)\cos(\alpha_n) = 0$ is valid for arbitrary numbers of L.

D. II.B3

Numerically we can solve the equation $1 + \cosh(\alpha_n)\cos(\alpha_n) = 0$ using the following code:

```
import numpy as np
from scipy.optimize import brentq

def f(alpha):
    return np.cos(alpha) * np.cosh(alpha) + 1

def roots(f):
    roots = []
    alpha_range = np.linspace(0,30,1000)

    # since there are multiple roots we can't
    # use brentq immidiatelly but must split it
    # in for example multiple intervals, and
    # check  if these contain a 0
    for i in range(len(alpha_range)-1):
        alpha0 = alpha_range[i]
        alpha1 = alpha_range[i+1]


        # just to chech if it went through the
        # zero-line
         if f(alpha0)*f(alpha1) < 0:
             root = brentq(f,alpha0,alpha1)
             roots.append(root)

    return roots

alpha_roots = roots(f)
print(alpha_roots)
```

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this gives use the following results for α_0 within the range $[0, 30]$:

```
[1.875, 4.694, 7.855, 10.996, 14.137,
17.279, 20.420, 23.562, 26.706, 29.845]
```

E. II.B4

The moment of inertia I_z is $I_z = \int_A Rz^2\,dA = \int_0^w R\,dy \int_{-h/2}^{h/2} z^2\,dz = [y]_0^{\omega}[z^2/3]_{-h/2}^{h/2} = \frac{wh^3}{12}$

F. II.B5

Using $\omega = 2\pi f$ and $A = wh$, $k = k_n/L$, $I_z = \frac{wh^3}{12}$ and the formula found in IB, we find that $f = \frac{\omega}{2\pi} = \sqrt{\frac{k^4 E I_z}{4\pi^2 \rho A}} = \sqrt{\frac{k_n^4 E w h^3}{4\pi^2 \cdot 12 L^4 \rho w h}} = \frac{k_n^2}{2\pi}\sqrt{\frac{Eh^3}{12\rho L^4}}$

G. II.B6

Using the following code the first 5 resonance frequencies of the cantilever are calculated. Here the α_0 from ID are used and its assumed that $E = 300 \cdot 10^9\ Pa$ and $rho = 3170\ kg/m^3$.

```
def frequency(kn,E,rho,L,h):
    f = (kn**2 / (2 * np.pi)) * np.sqrt((E *
    h**2) / (12 * rho * L**4))
    return f

alpha_roots= alpha_roots[0:5]
w = 20e-6           #micro m
L = 100e-6          #micro m
h = 0.8e-6          #micro m
E = 300e9           #Pa
rho = 3170          #kg/m3

frequencies = frequency(np.array(alpha_roots)
,E,rho,L,h)
print(frequencies)
```

From this code it follows that the first 5 resonance frequencies are:

```
[125719,  787868, 2206054, 4322984, 7146203]
```