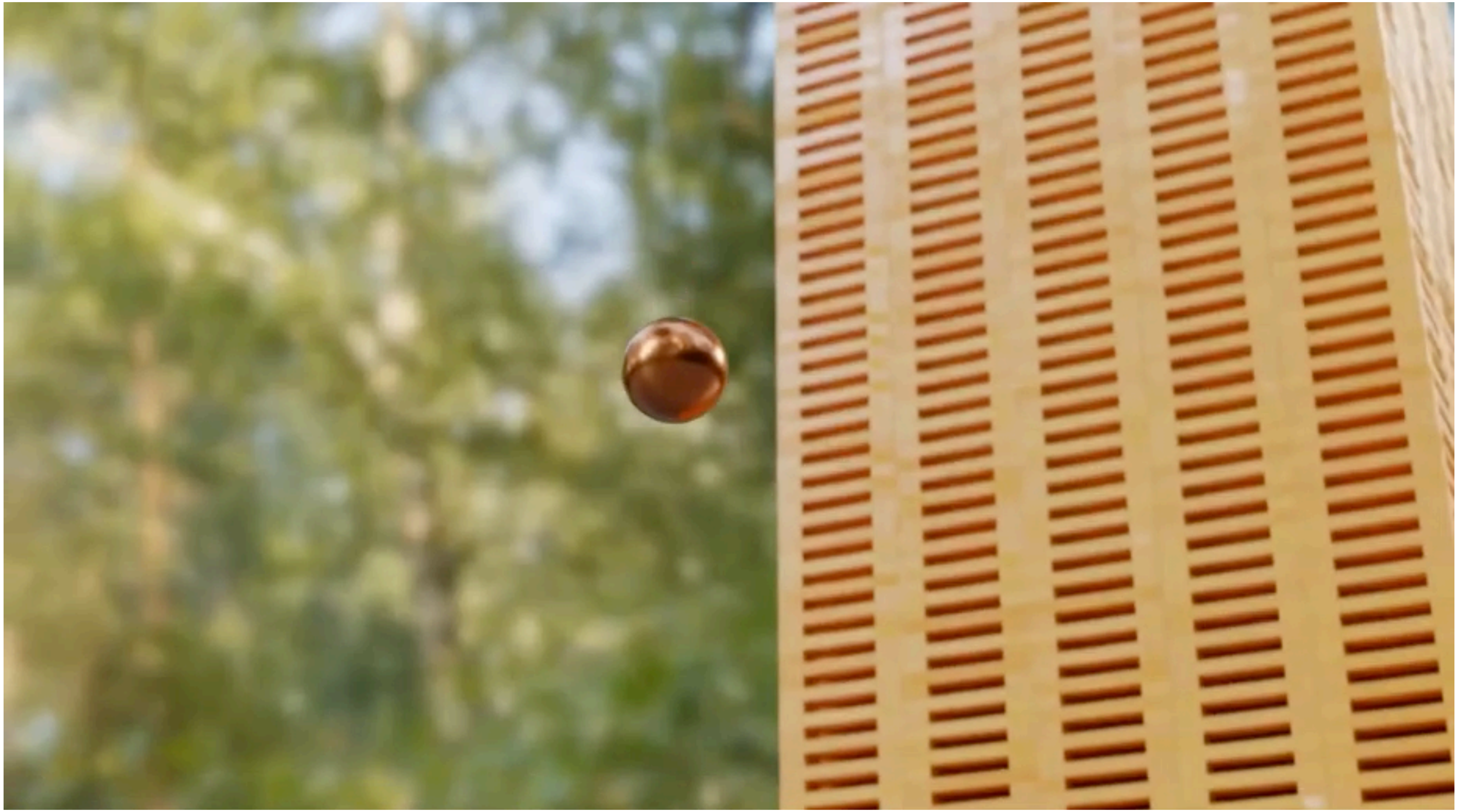
Lecture 11:

Rigid Bodies

FUNDAMENTALS OF COMPUTER GRAPHICS

Animation & Simulation

Stanford CS248B, Fall 2023



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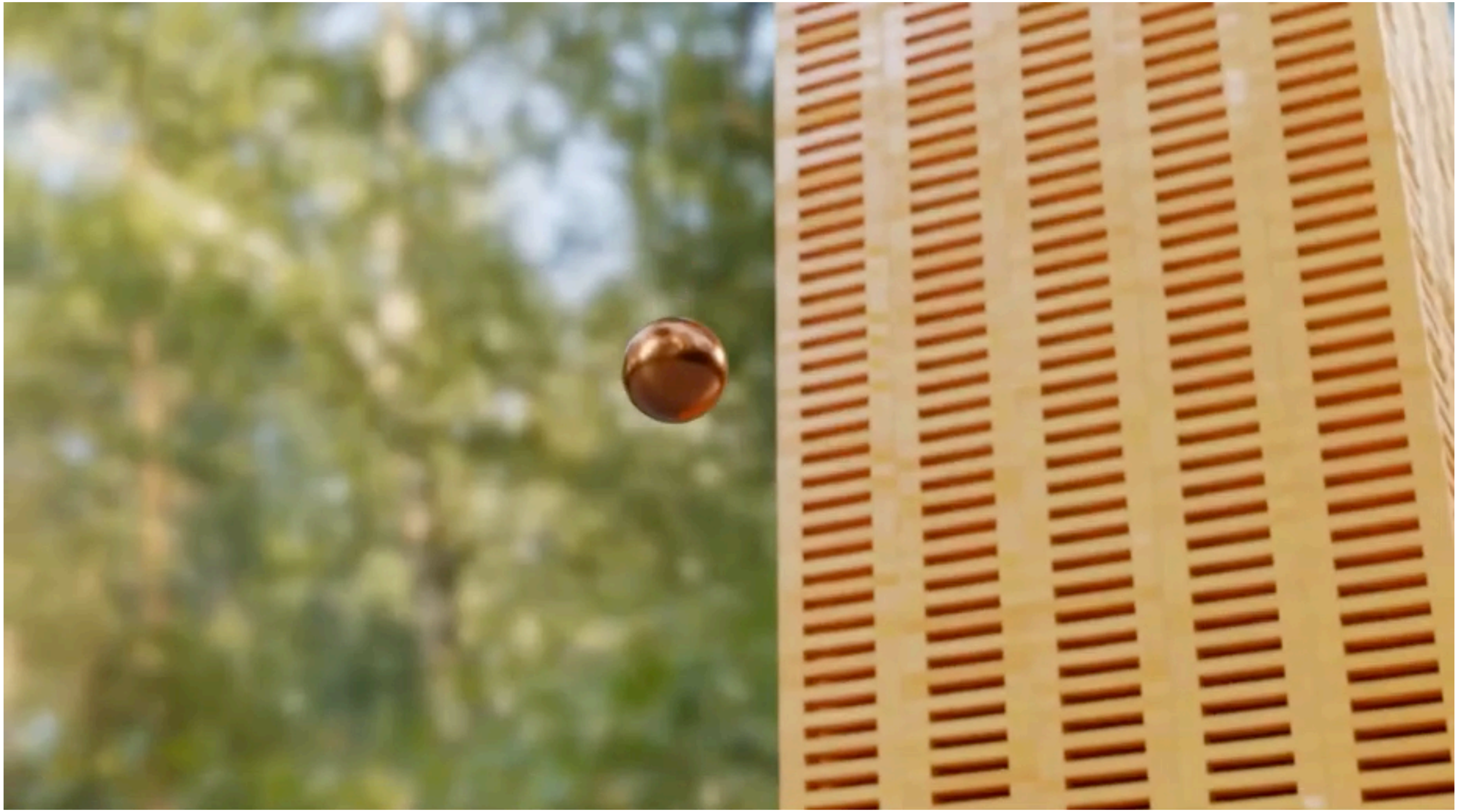

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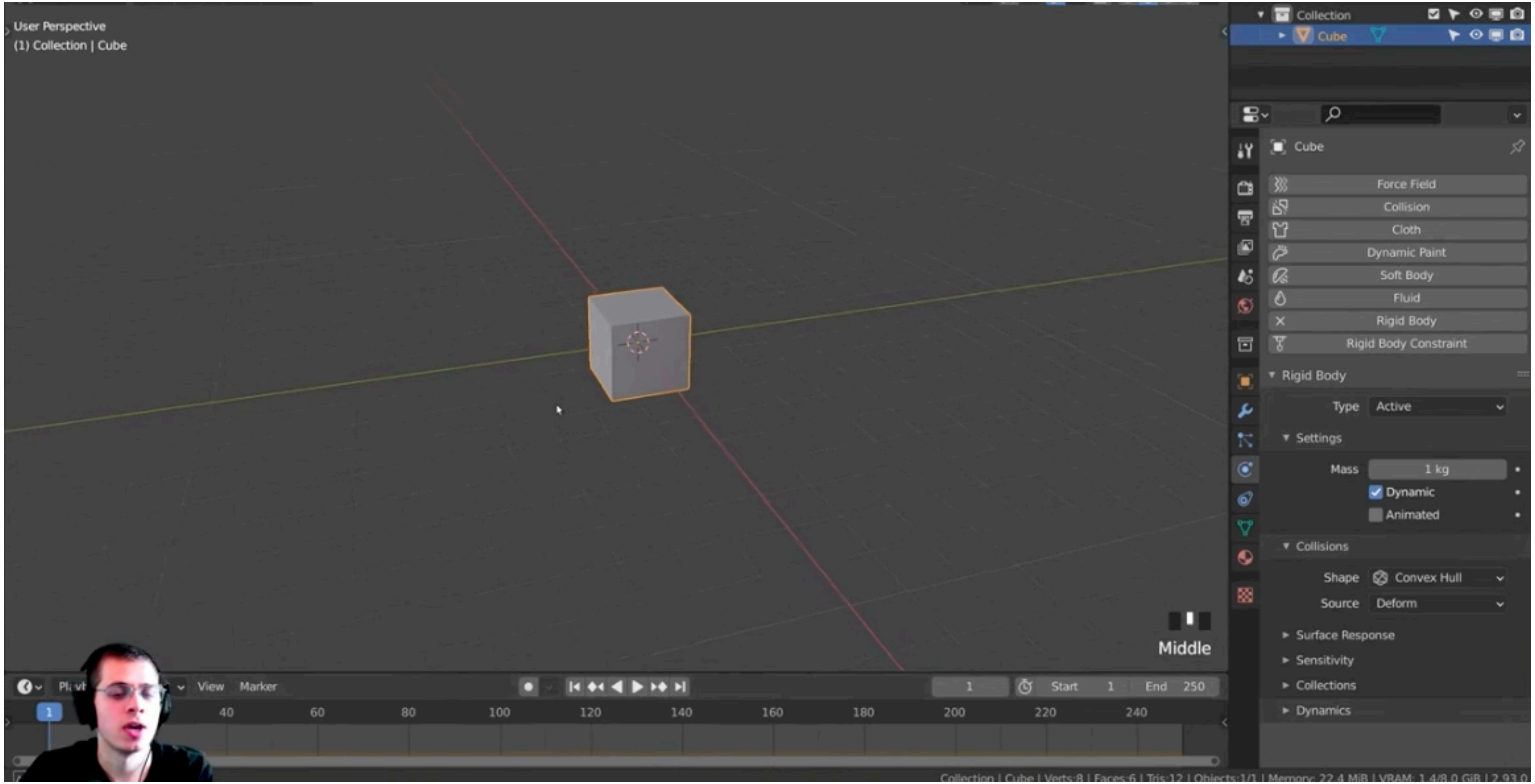

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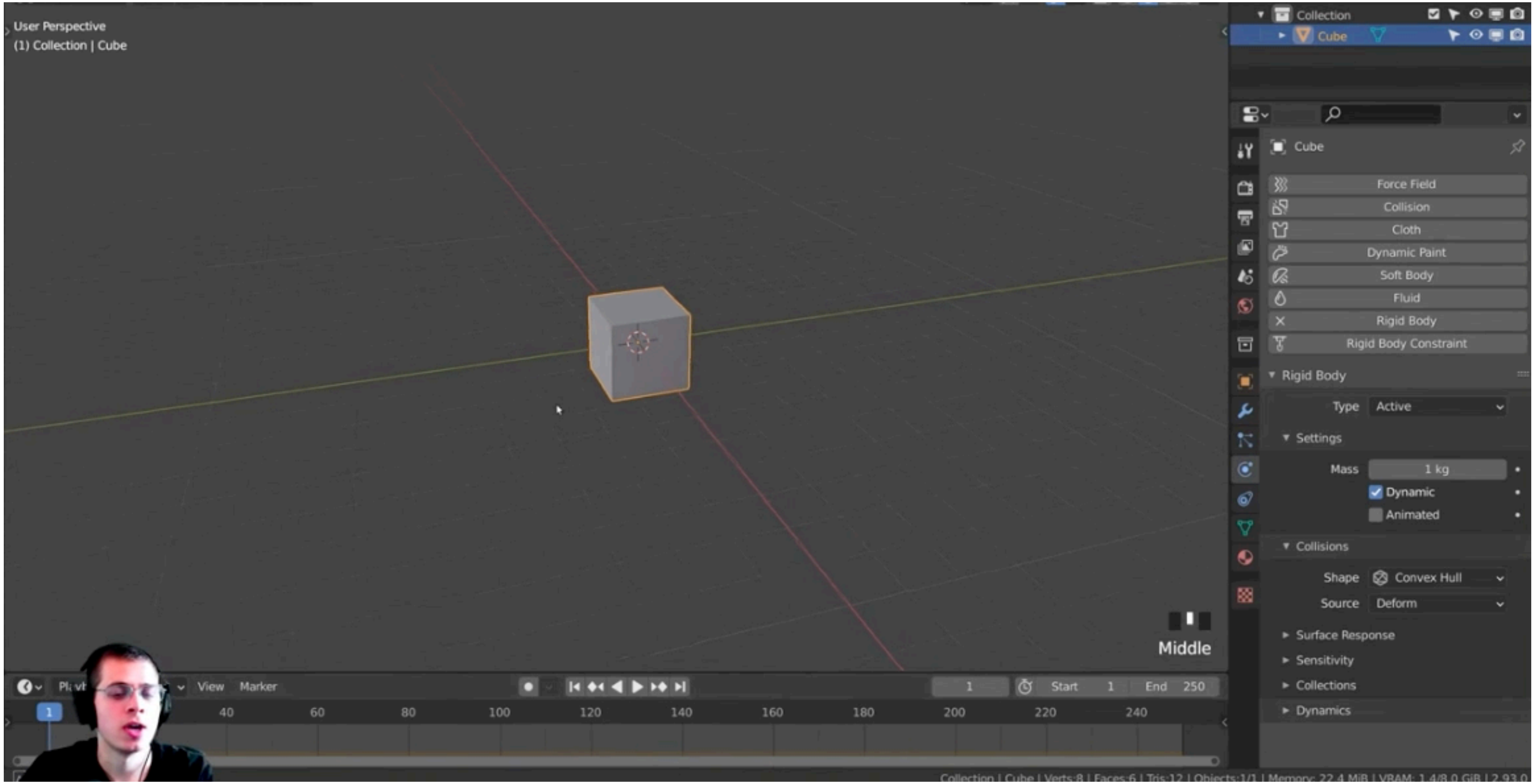

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Learning Objectives

- **Learn the representation of rigid body and its coordinate frame**
- **Understand angular position, velocity, momentum, inertia and force**
- **Understand the differential equations for rigid bodies**
- **Learn the numerical integration process for rigid bodies**

3D Translation

- **A point mass moving in 3D space only needs translational variables in the state space.**

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} \begin{matrix} \text{position} \\ \text{linear velocity} \end{matrix}$$

- **The ODE for the translation motion:**

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{bmatrix} = f(\begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}) = \begin{bmatrix} \mathbf{v} \\ \frac{\mathbf{f}}{m} \end{bmatrix}$$

- **What about an object with spatial extent? The state space should also include rotational variables.**

3D translation and orientation

Translation

- Position
- Linear velocity
- Mass
- Linear momentum
- Force

Rotation

- Orientation
- Angular velocity
- Inertia tensor
- Angular momentum
- Torque

3D translation and orientation

Translation

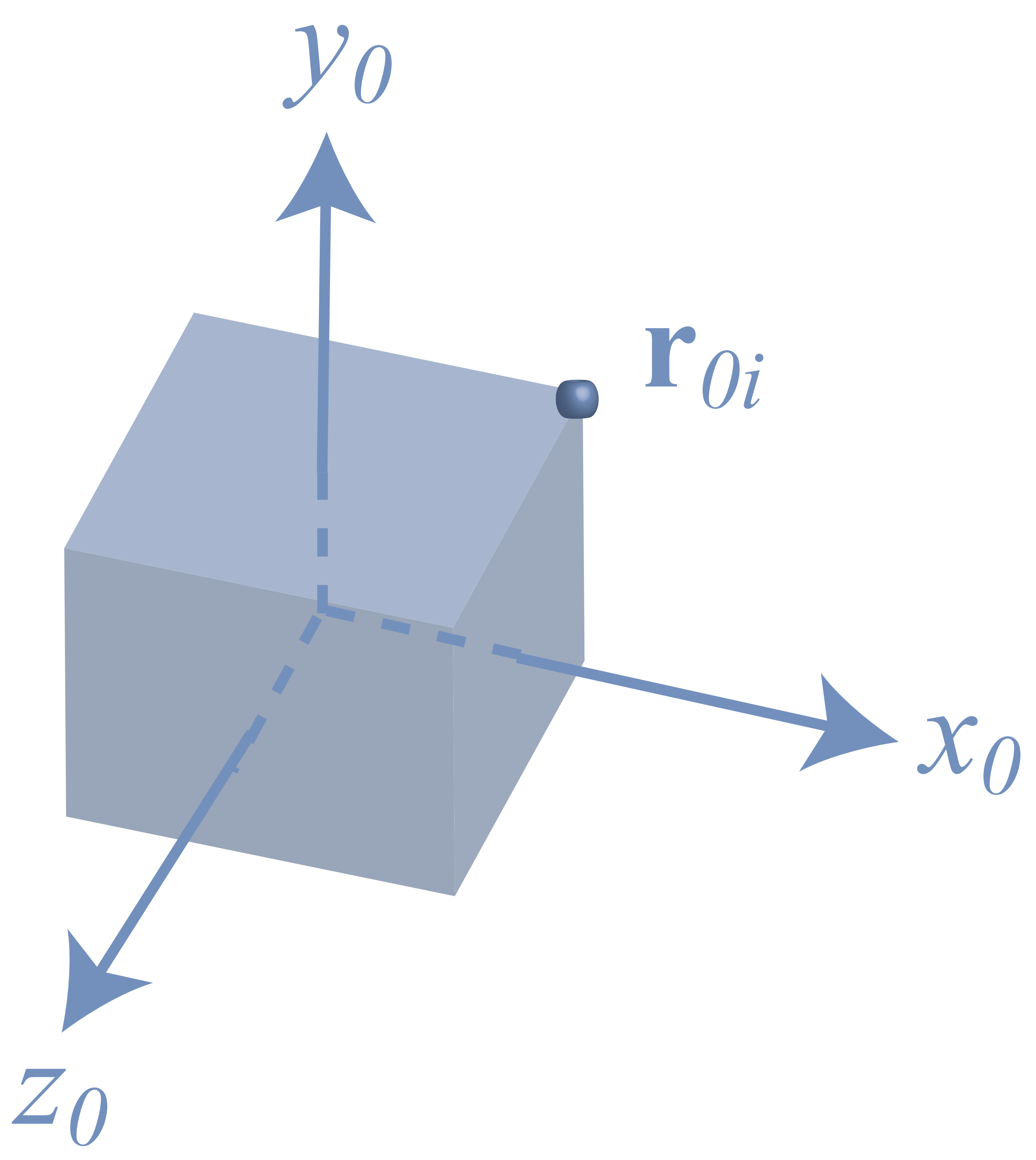
Position
Linear velocity
Mass
Linear momentum
Force

Rotation

Orientation
Angular velocity
Inertia tensor
Angular momentum
Torque

Body space

- A fixed and unchanged space where the shape of a rigid body is defined.
- The origin of the body space is attached to a point on the rigid body, e.g. the geometric center of the rigid body.



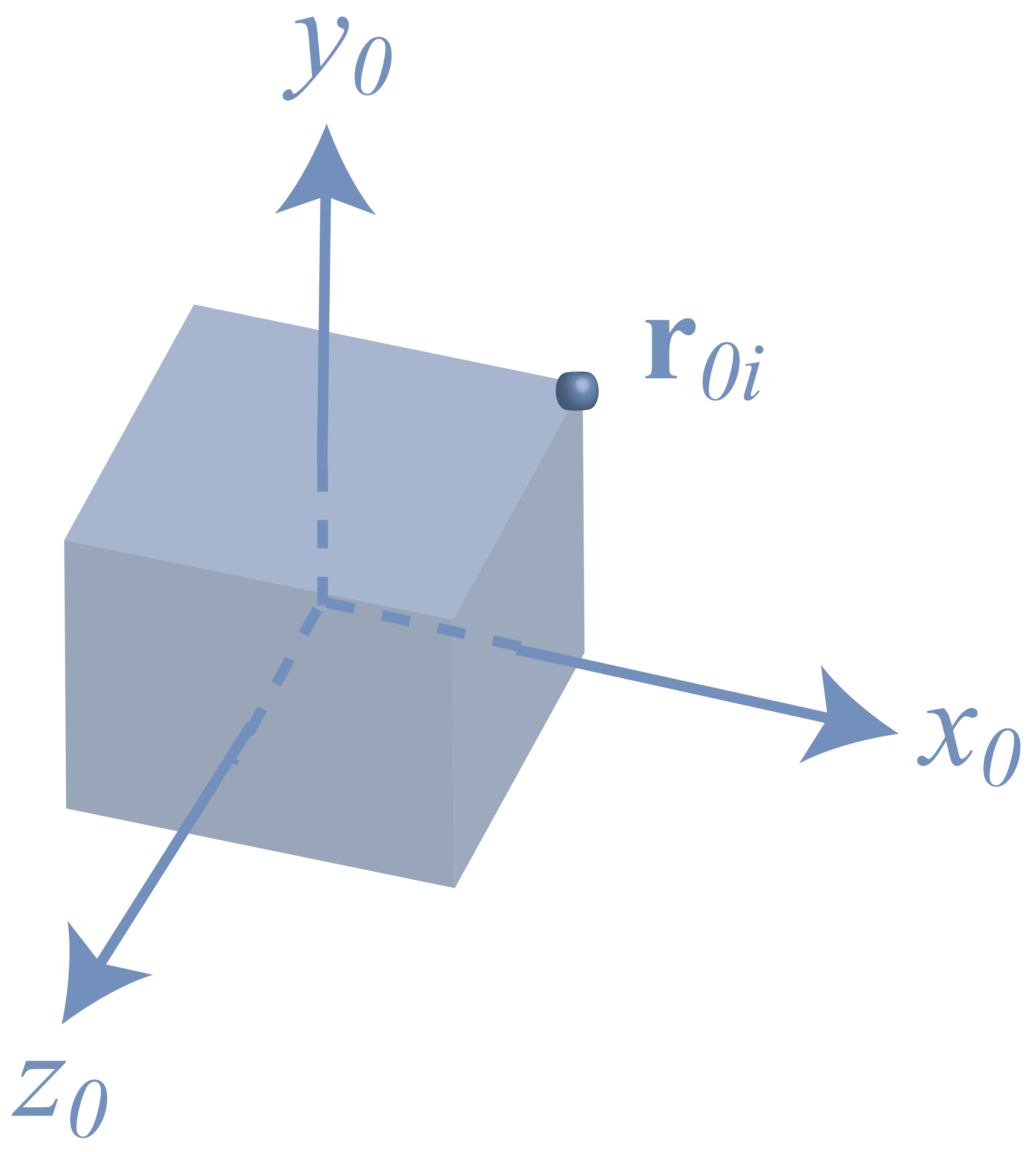
Spatial variables

- Spatial variables of a rigid body include:
  - Translation of the body space

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

  - Rotation of the body space

$$\mathbf{R}(t) = \begin{bmatrix} r_{xx} & r_{yx} & r_{zx} \\ r_{xy} & r_{yy} & r_{zy} \\ r_{xz} & r_{yz} & r_{zz} \end{bmatrix}$$



World space

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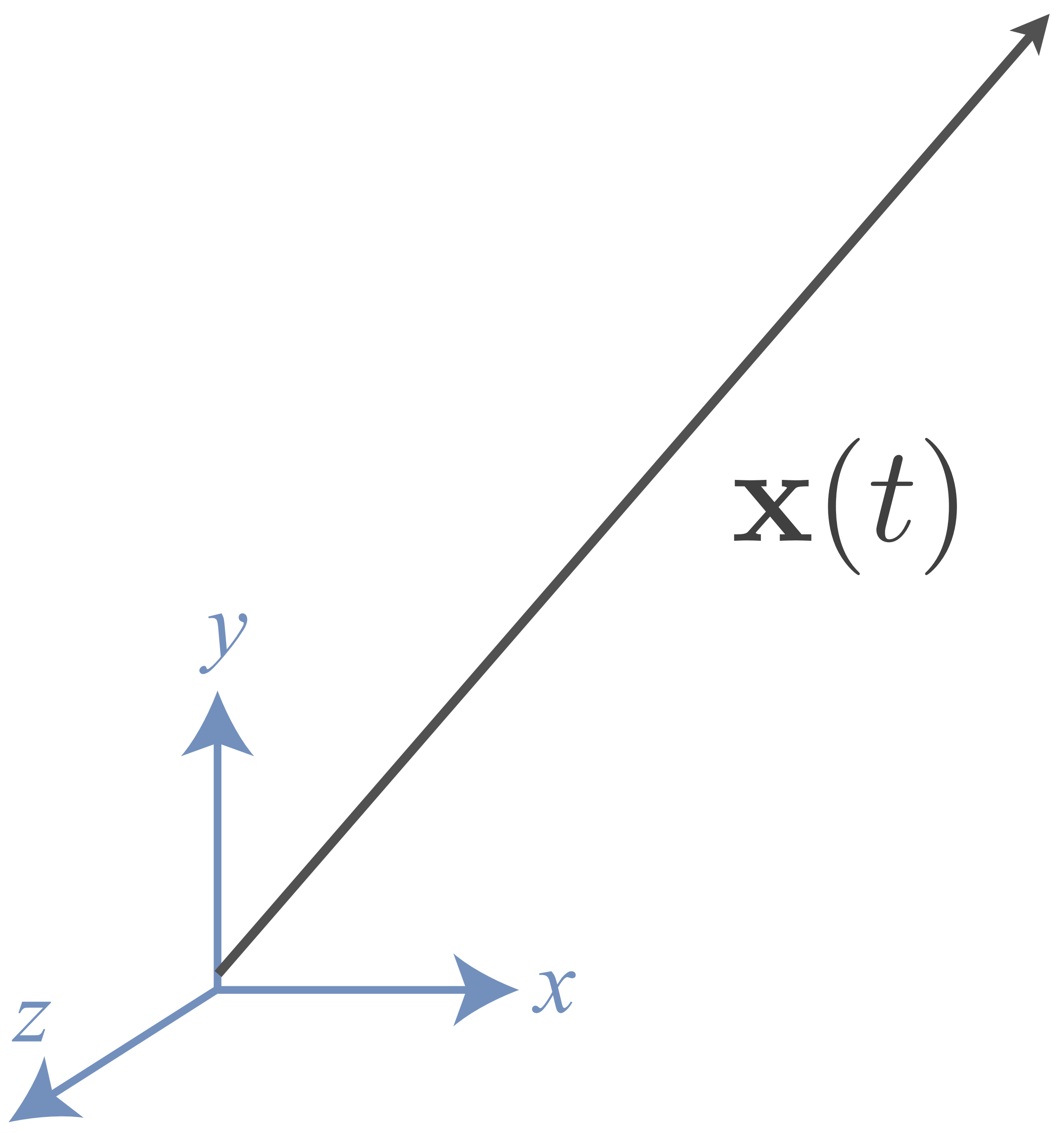

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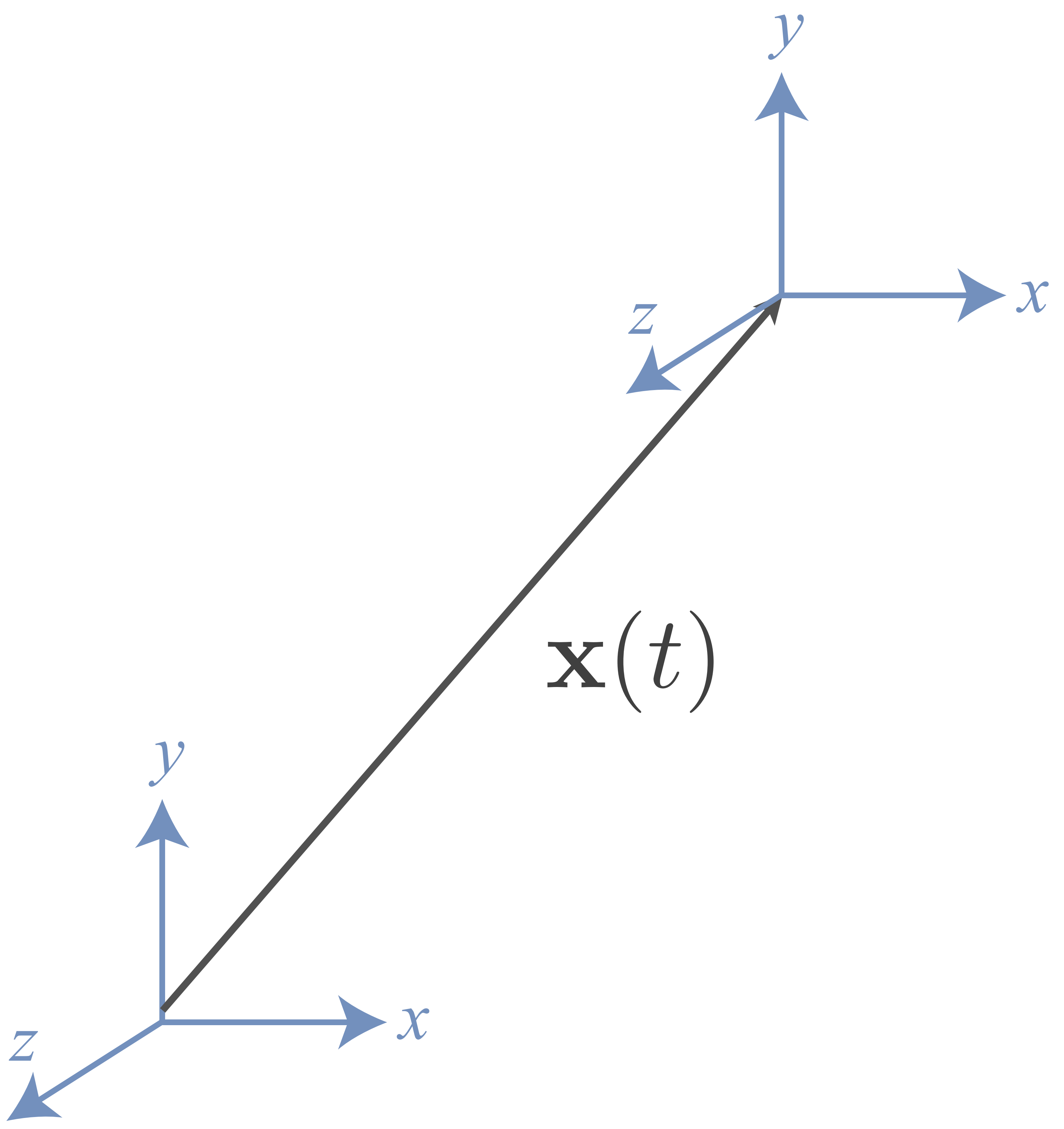

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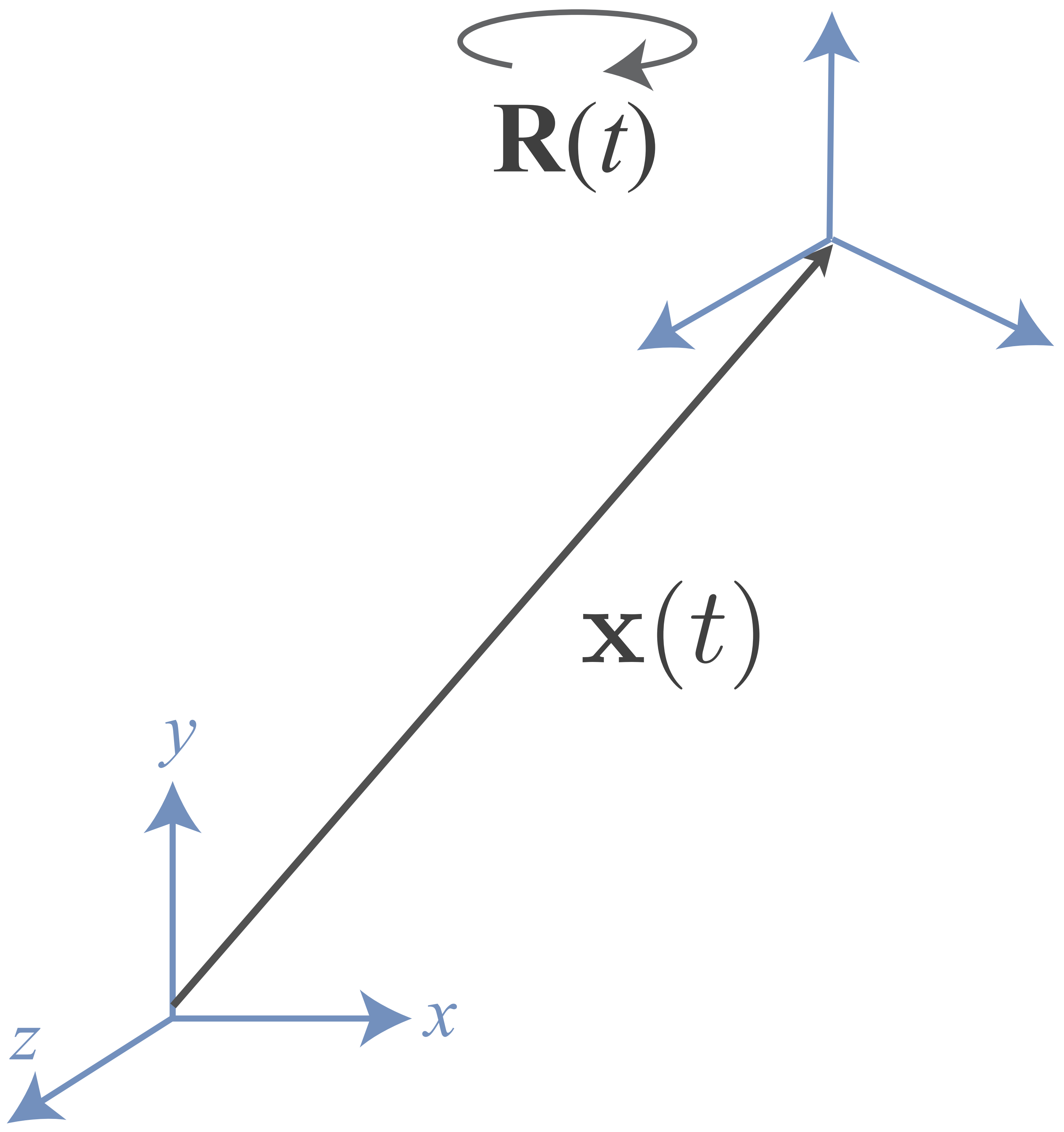

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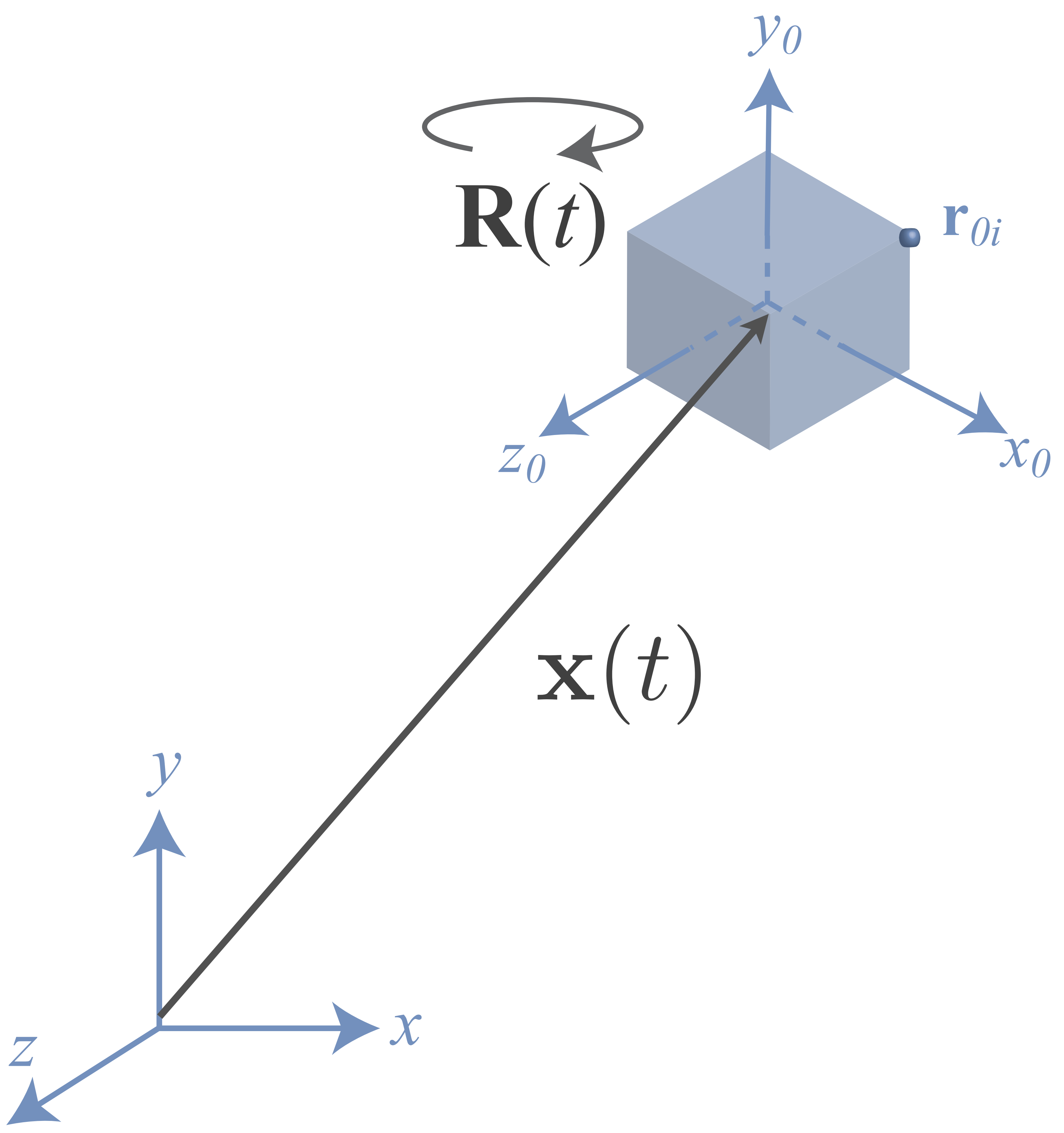


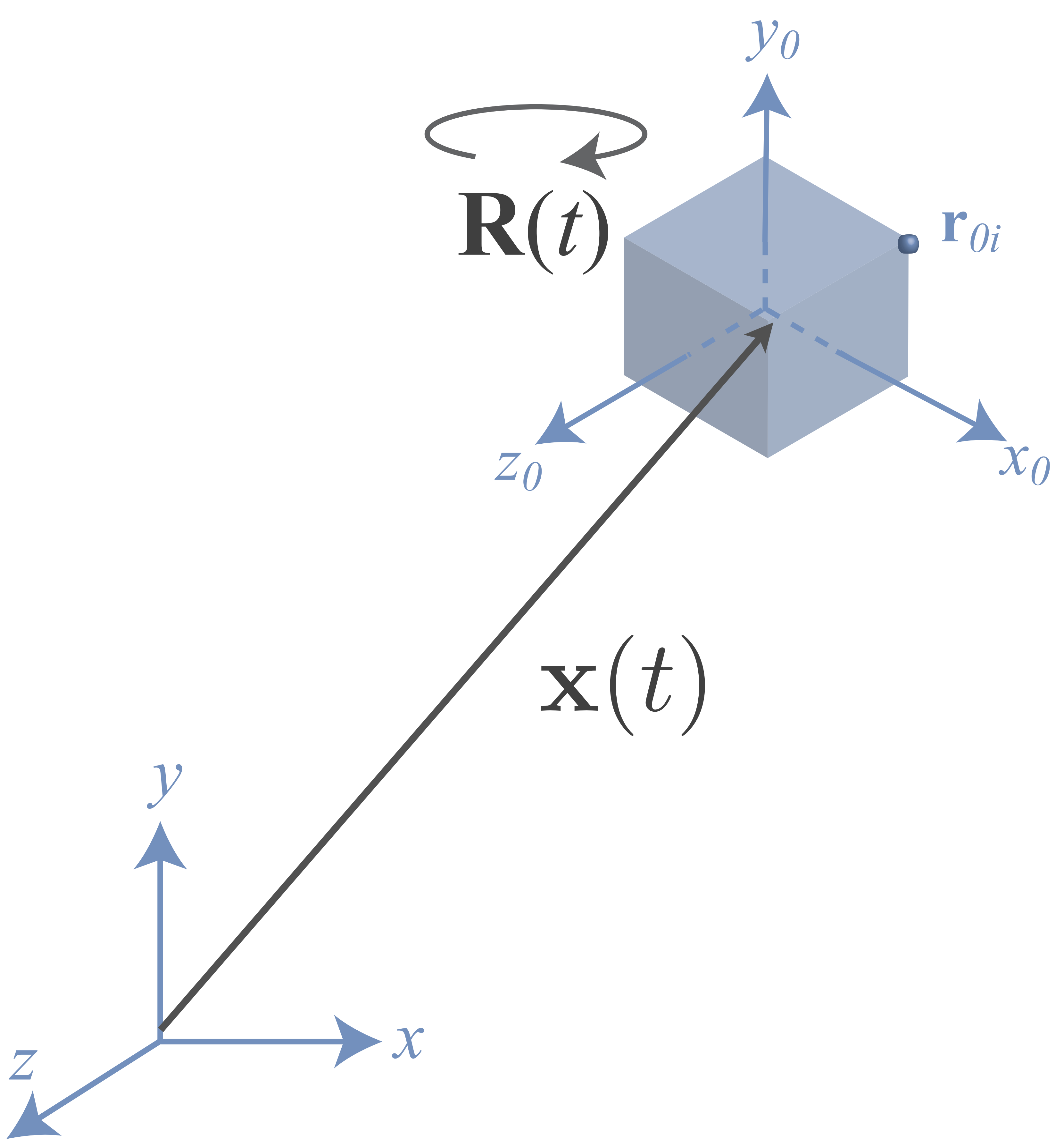
World space

- Use $\mathbf{x}(t)$ and $\mathbf{R}(t)$ to transform the body space into world space.
- What are the world coordinate of an arbitrary point $\mathbf{r}_{0i}$ on the body?

$$\mathbf{r}_i(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_{0i}$$

the same point in the world space

a point in the body space



Position and orientation

- **Assume the rigid body has uniform density, what is the physical meaning of $\mathbf{x}(t)$?**
  - **The center of mass over time**
- **What is the physical meaning of $\mathbf{R}(t)$?**
  - **Consider the x-axis in body space, (1, 0, 0), what is the direction of this vector in world space at time t?**

$$\mathbf{R}(t)\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}r_{xx}\\r_{xy}\\r_{xz}\end{bmatrix}$$

The first column of $\mathbf{R}(t)$

  - **$\mathbf{R}(t)$ represents directions of x, y, and z axes of the body space in world space at time t.**

3D translation and orientation

Translation

Position
Linear velocity
Mass
Linear momentum
Force

Rotation

Orientation
Angular velocity
Inertia tensor
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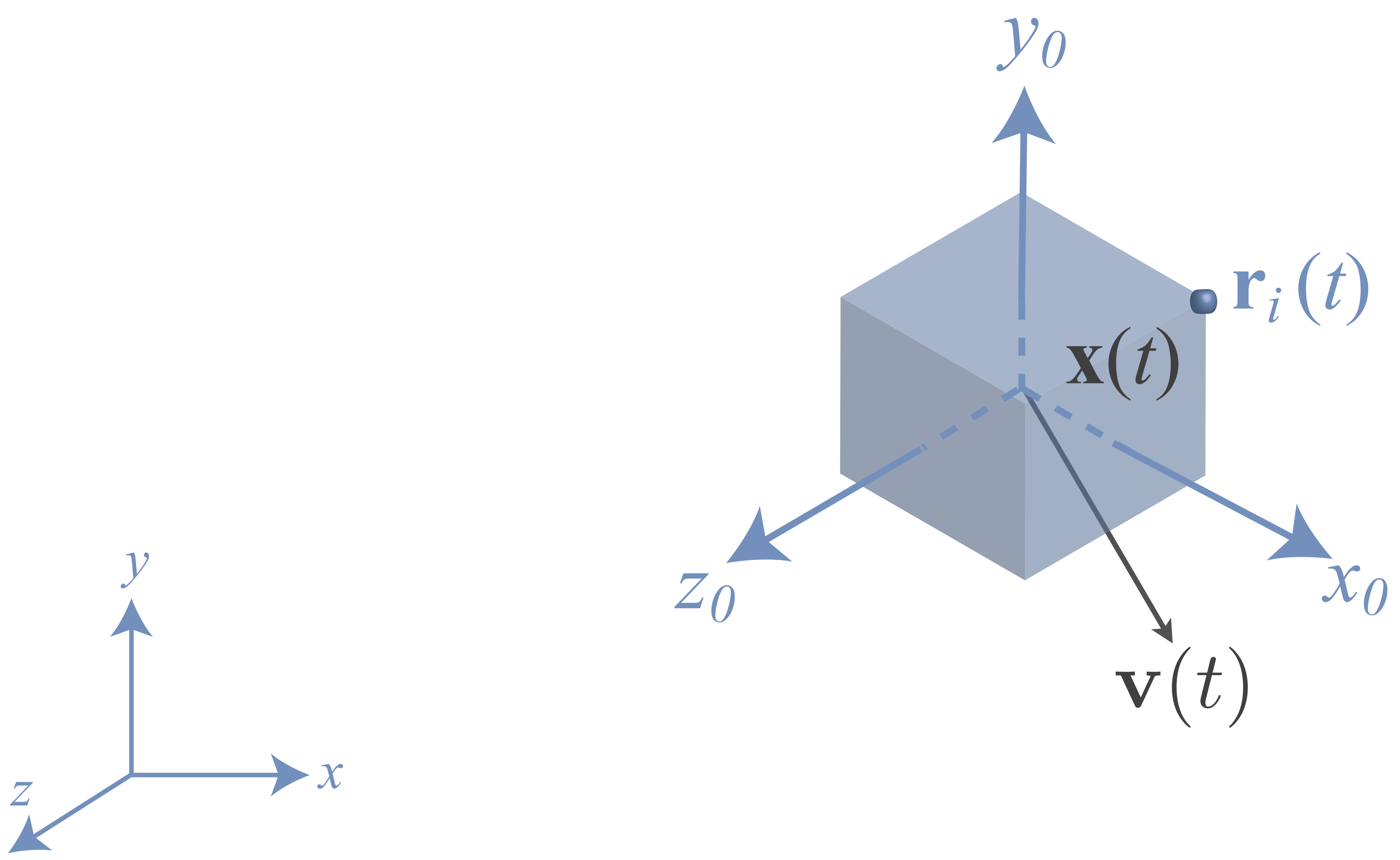
Torque

Linear velocity

Since $\mathbf{x}(t)$ is the position of the center of mass in world space, $\dot{\mathbf{x}}(t)$ is the velocity of the center of mass in world space

$$\mathbf{v}(t) = \dot{\mathbf{x}}(t)$$



Angular velocity

- If we freeze the position of the COM in space, then any movement is due to the body spinning about some axis that passes through the COM (Otherwise, the COM would itself be moving).
- So we define the spin as angular velocity, a vector $\omega(t)$
    - Direction of $\omega(t)$ is the axis the object spins about in world space.
    - Magnitude of $\omega(t)$ is the speed of the object spins.
- Using this notion, any movement of COM is due to the linear velocity and angular velocity only accounts for motion relative to COM.

Quiz

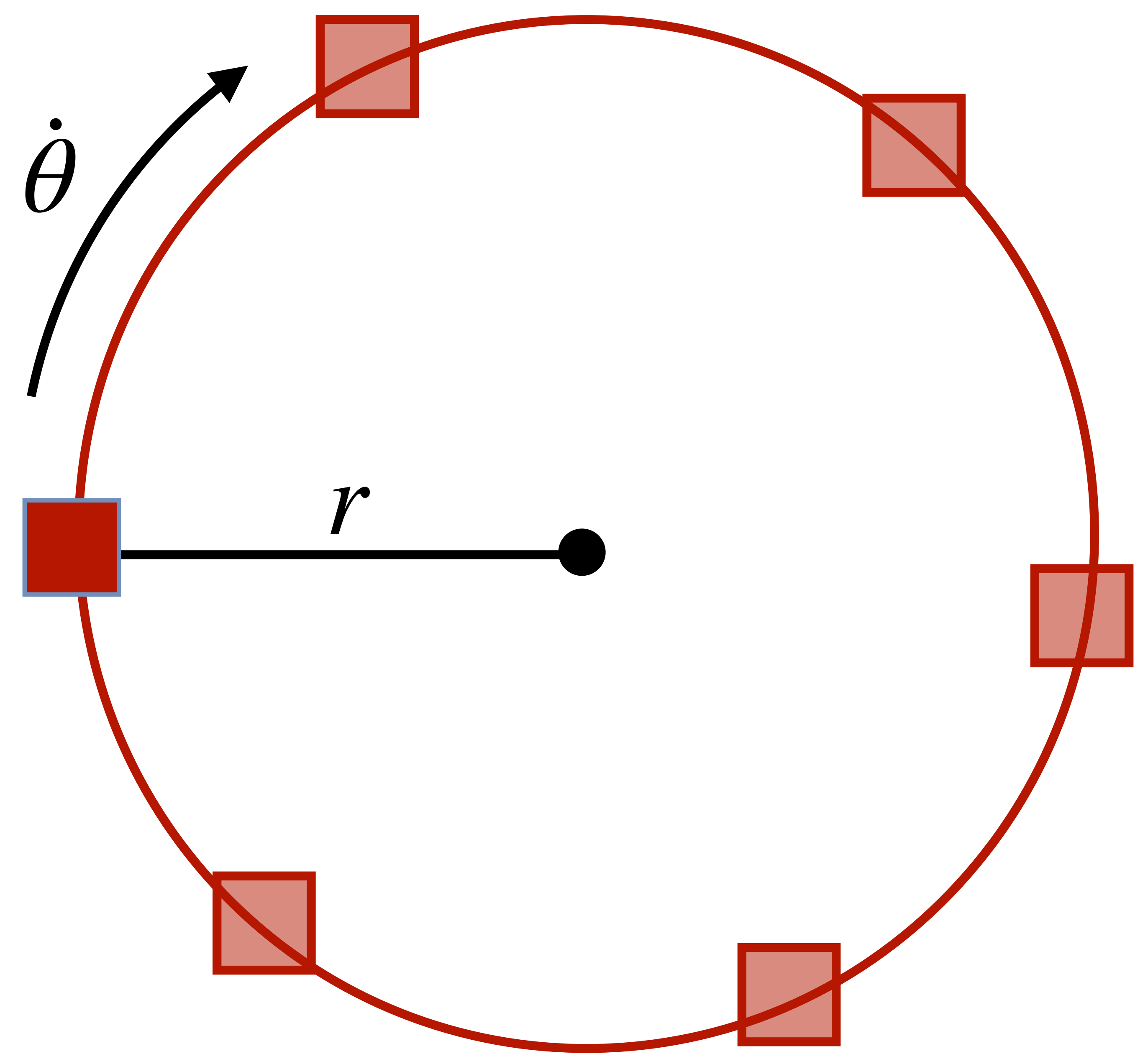
- A 2D rigid body is circling around a point with a distance r and spinning speed $\dot{\theta}$.

  - What's the linear velocity?

$$\|\mathbf{v}\| = r\dot{\theta}$$

  - What's the angular velocity?

zero



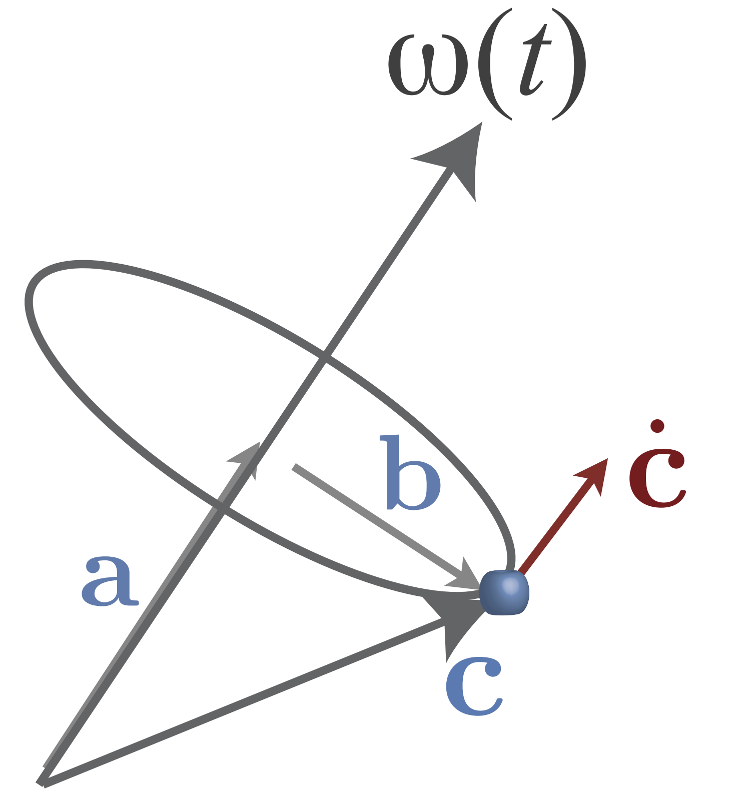
Orientation and angular velocity

- **Linear position and velocity are related by**

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{x}(t) = \dot{\mathbf{x}}$$

- **How are angular position (orientation) and velocity related?**
  - $\omega(t) = \dot{\mathbf{R}}(t)$ **is clearly incorrect!**
- **What is the correct relation between $\mathbf{R}(t)$ and $\omega(t)$?**

Consider a vector $\mathbf{c}(t)$ at time t specified in world space. How do we express $\dot{\mathbf{c}}(t)$ in terms of $\boldsymbol{\omega}(t)$?



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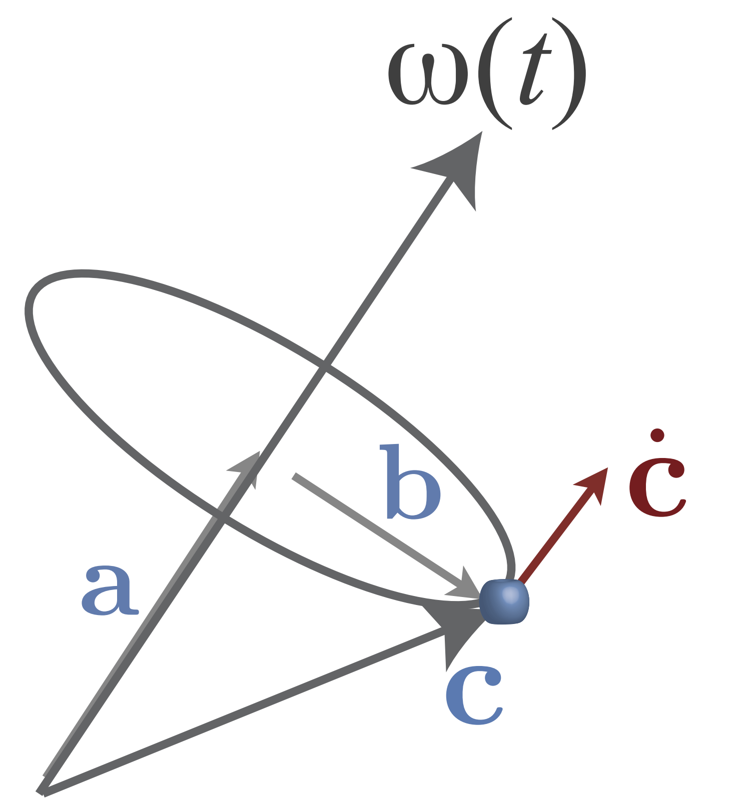
Consider a vector $\mathbf{c}(t)$ at time t specified in world space. How do we express $\dot{\mathbf{c}}(t)$ in terms of $\boldsymbol{\omega}(t)$?

$$\|\dot{\mathbf{c}}\| = \|\mathbf{b}\|\|\boldsymbol{\omega}(t)\| = \|\boldsymbol{\omega}(t) \times \mathbf{b}\|$$



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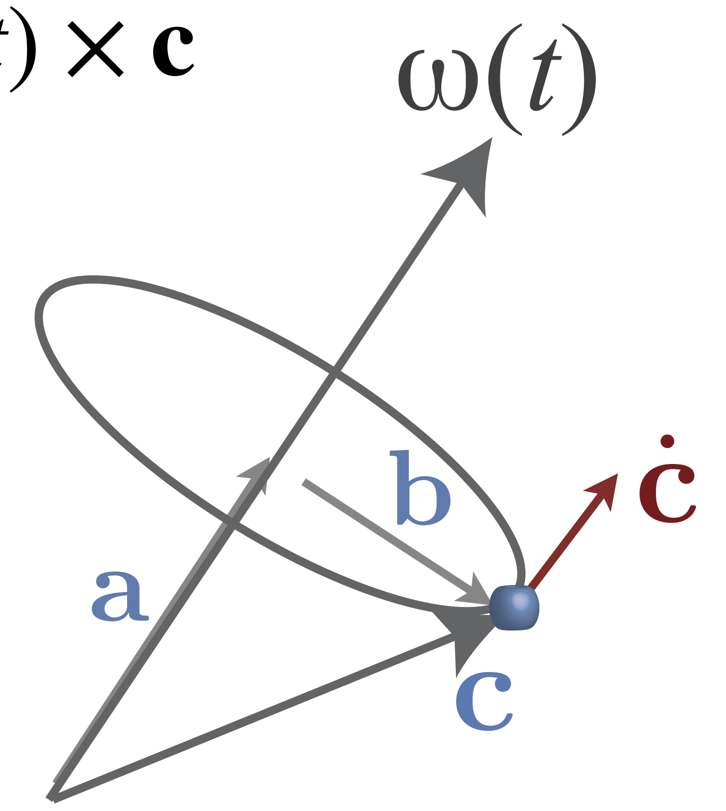
$$\dot{\mathbf{c}}(t) = \boldsymbol{\omega}(t) \times \mathbf{b} = \boldsymbol{\omega}(t) \times \mathbf{b} + \boldsymbol{\omega}(t) \times \mathbf{a}$$

$$= \boldsymbol{\omega}(t) \times (\mathbf{b} + \mathbf{a}) = \boldsymbol{\omega}(t) \times \mathbf{c}$$



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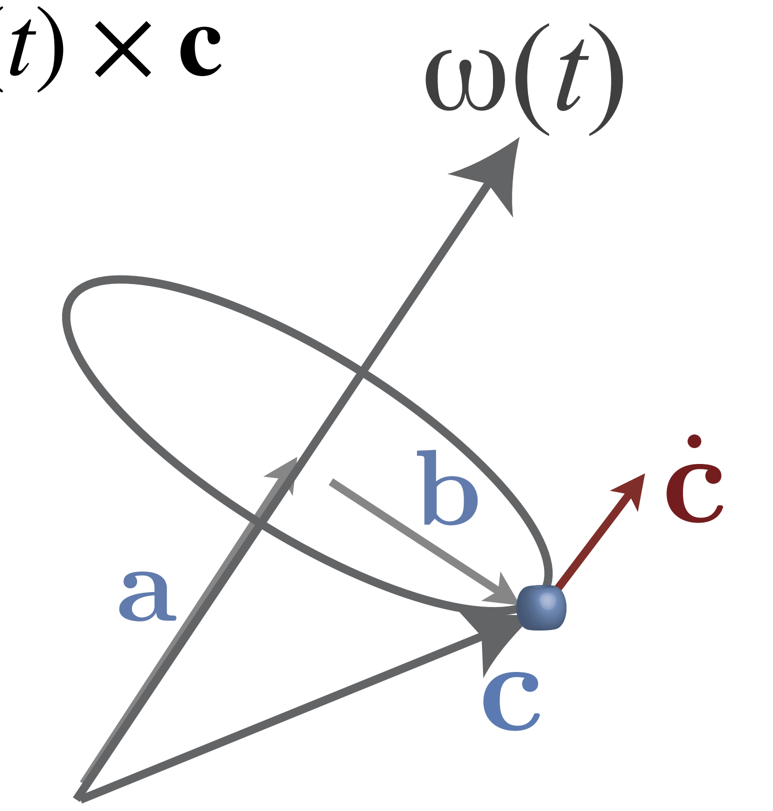


Orientation and angular velocity

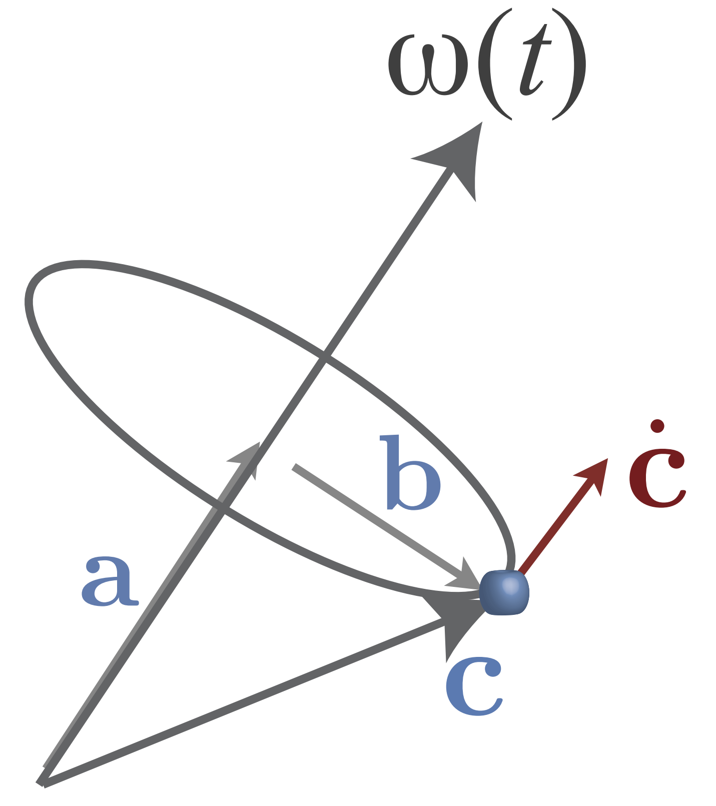
- Given the physical meaning of $\mathbf{R}(t)$, what does each column of $\dot{\mathbf{R}}(t)$ mean?
  - At time t, the direction of x-axis of the rigid body in world space is the first column of $\mathbf{R}(t)$:

$$\begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix}$$

  - Then, at time t, what is the derivative of the first column of $\mathbf{R}(t)$?

$$\begin{bmatrix} \dot{r}_{xx} \\ \dot{r}_{xy} \\ \dot{r}_{xz} \end{bmatrix} = \boldsymbol{\omega}(t) \times \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix}$$

$$\dot{\mathbf{c}}(t) = \boldsymbol{\omega}(t) \times \mathbf{c}$$



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$$\begin{bmatrix} \dot{r}_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix} = \boldsymbol{\omega}(t) \times \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix} \quad \Longrightarrow \quad \dot{\mathbf{R}}(t) = \left[\boldsymbol{\omega}(t) \times \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix} \quad \boldsymbol{\omega}(t) \times \begin{bmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{bmatrix} \quad \boldsymbol{\omega}(t) \times \begin{bmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{bmatrix} \right]$$

Orientation and angular velocity

Consider $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^3$. The cross product of them is

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{bmatrix}$$

$$\dot{\mathbf{R}}(t) = \begin{bmatrix} \boldsymbol{\omega}(t) \times \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix} & \boldsymbol{\omega}(t) \times \begin{bmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{bmatrix} & \boldsymbol{\omega}(t) \times \begin{bmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{bmatrix} \end{bmatrix}$$

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Given $\mathbf{a}$, let's define $[\mathbf{a}]$ to be a skew symmetric matrix:

$$[\mathbf{a}] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

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Then, the cross product of two vectors can be expressed as a matrix-vector multiplication.

$$[\mathbf{a}]\mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \mathbf{a} \times \mathbf{b}$$

Orientation and angular velocity

Consider $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^3$. The cross product of them is

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$$\boxed{\dot{\mathbf{R}}(t) = [\omega(t)]\mathbf{R}(t)}$$

A point on rigid body

- **Imagine a rigid body is composed of a large number of small particles, indexed from 1 to N**
- **Each particle has a constant location $\mathbf{r}_{0i}$ in body space**
- **The location of i-th particle in world space at time t is $\mathbf{r}_i(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_{0i}$**
- **The velocity of i-th particle in world space at time t :**

$$\dot{\mathbf{r}}_i(t) = \frac{d}{dt}\mathbf{r}_i(t) = \mathbf{v}(t) + [\boldsymbol{\omega}(t)]\mathbf{R}(t)\mathbf{r}_{0i} = \mathbf{v}(t) + [\boldsymbol{\omega}(t)]\left(\mathbf{R}(t)\mathbf{r}_{0i} + \mathbf{x}(t) - \mathbf{x}(t)\right)$$

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$$= \mathbf{v}(t) + [\boldsymbol{\omega}(t)]\Big(\mathbf{r}_i(t) - \mathbf{x}(t)\Big) = \mathbf{v}(t) + \boldsymbol{\omega}(t) \times \Big(\mathbf{r}_i(t) - \mathbf{x}(t)\Big)$$

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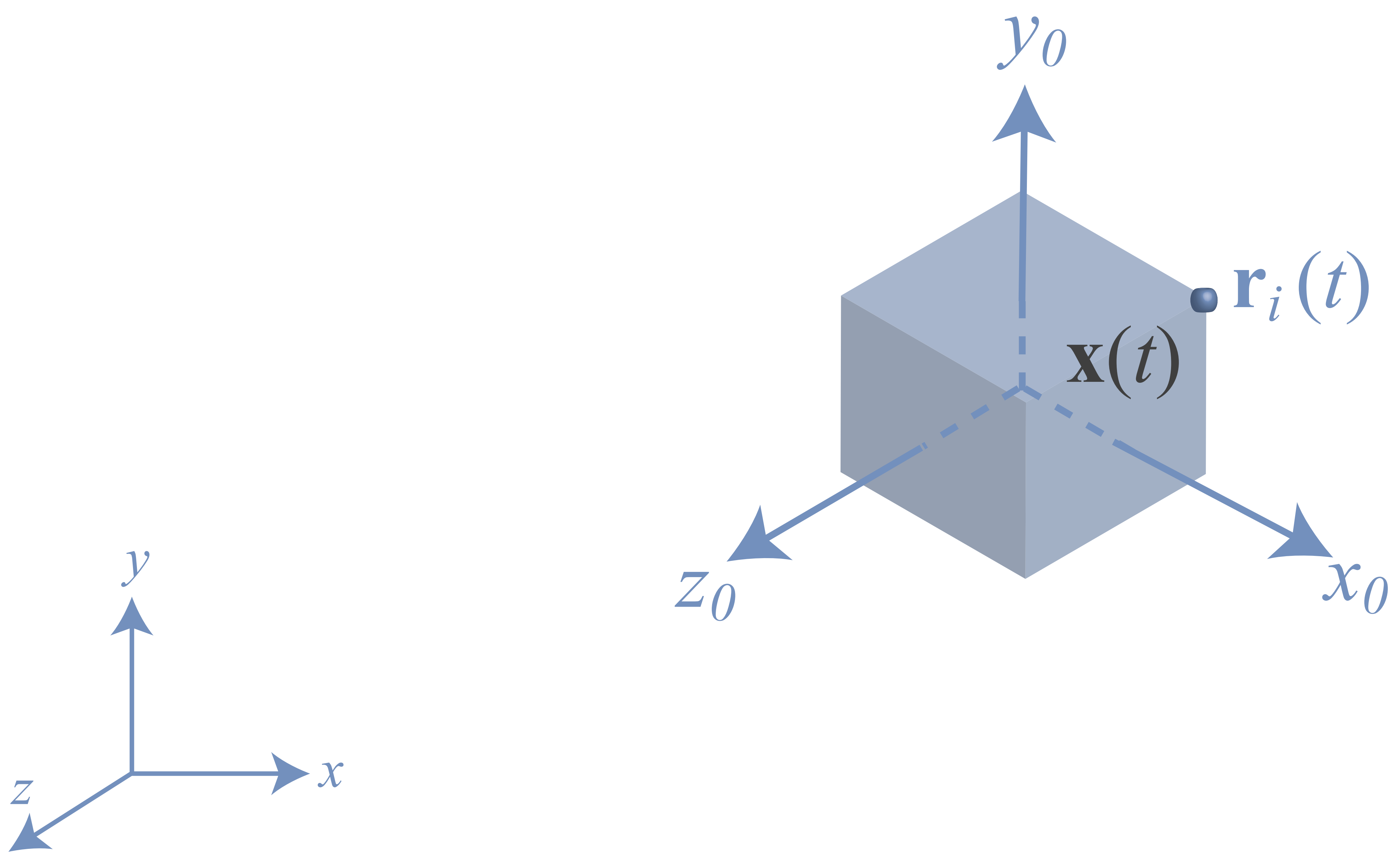
$$
\begin{aligned}
\dot{\mathbf{r}}_i(t) &= \frac{d}{dt}\mathbf{r}_i(t) = \mathbf{v}(t) + [\boldsymbol{\omega}(t)]\mathbf{R}(t)\mathbf{r}_{0i} = \mathbf{v}(t) + [\boldsymbol{\omega}(t)]\big(\mathbf{R}(t)\mathbf{r}_{0i} + \mathbf{x}(t) - \mathbf{x}(t)\big) \\
&= \mathbf{v}(t) + [\boldsymbol{\omega}(t)]\big(\mathbf{r}_i(t) - \mathbf{x}(t)\big) = \overbrace{\mathbf{v}(t)}^{\text{linear component}} + \overbrace{\boldsymbol{\omega}(t) \times \big(\mathbf{r}_i(t) - \mathbf{x}(t)\big)}^{\text{angular component}}
\end{aligned}
$$

A point on rigid body



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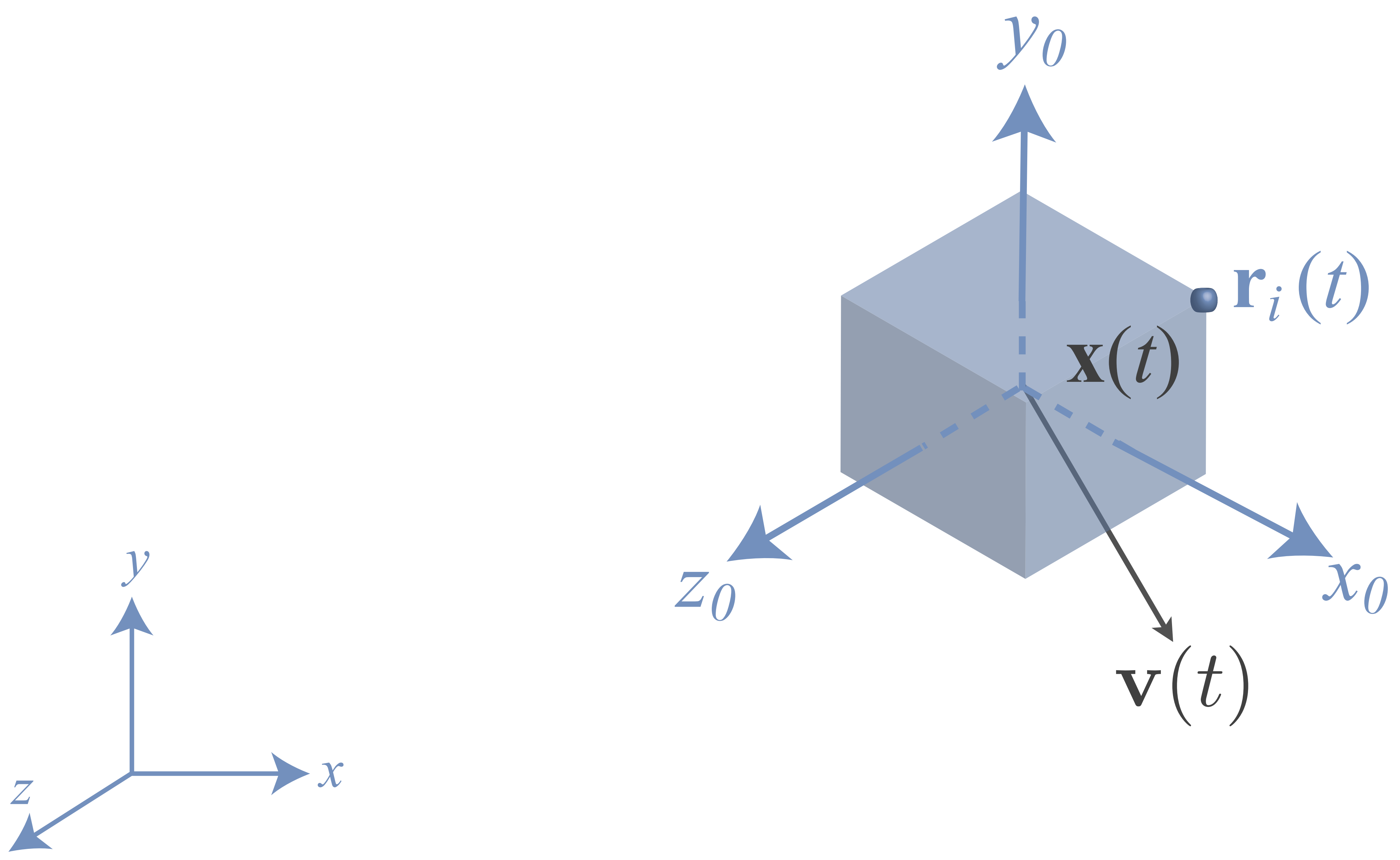

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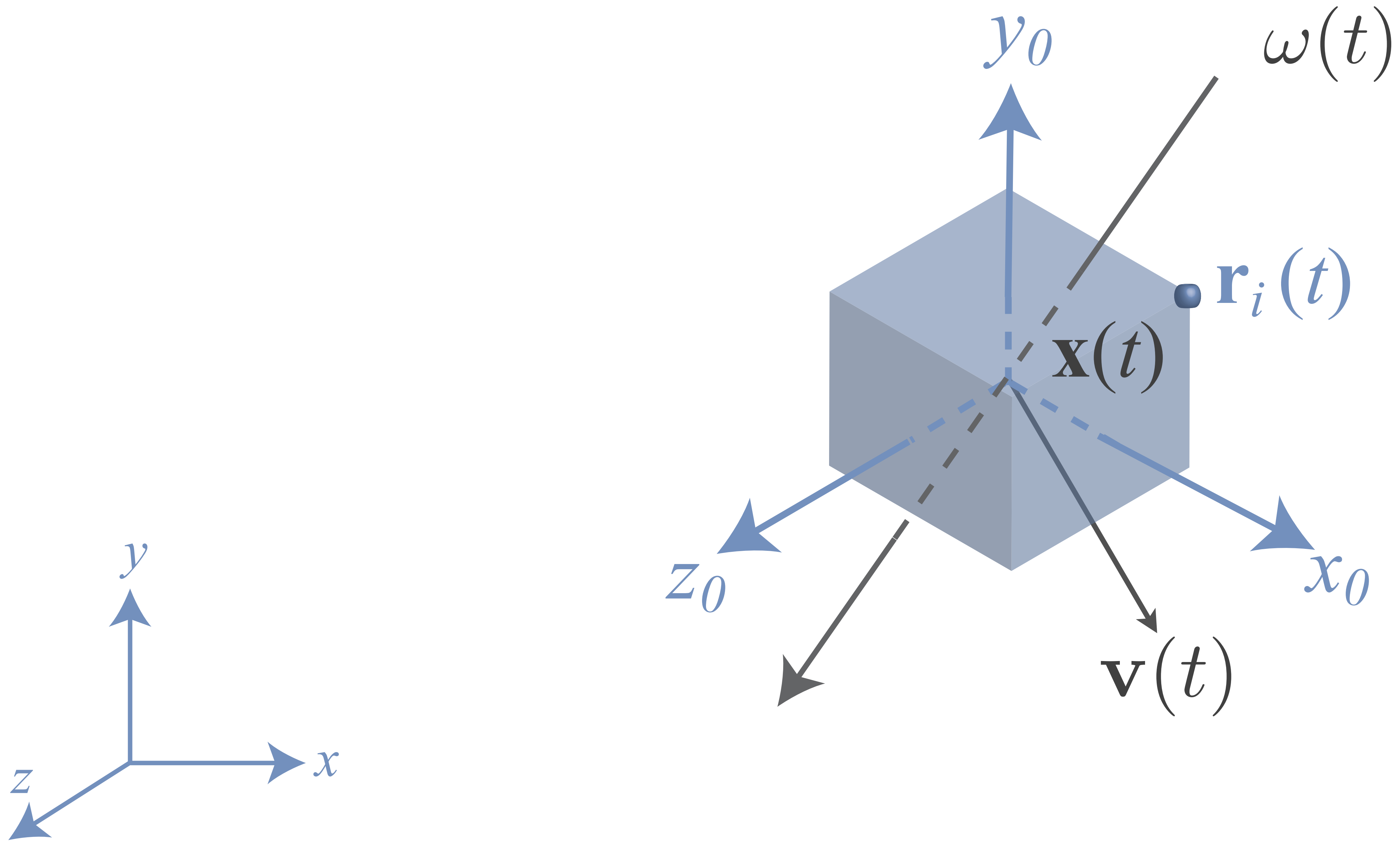

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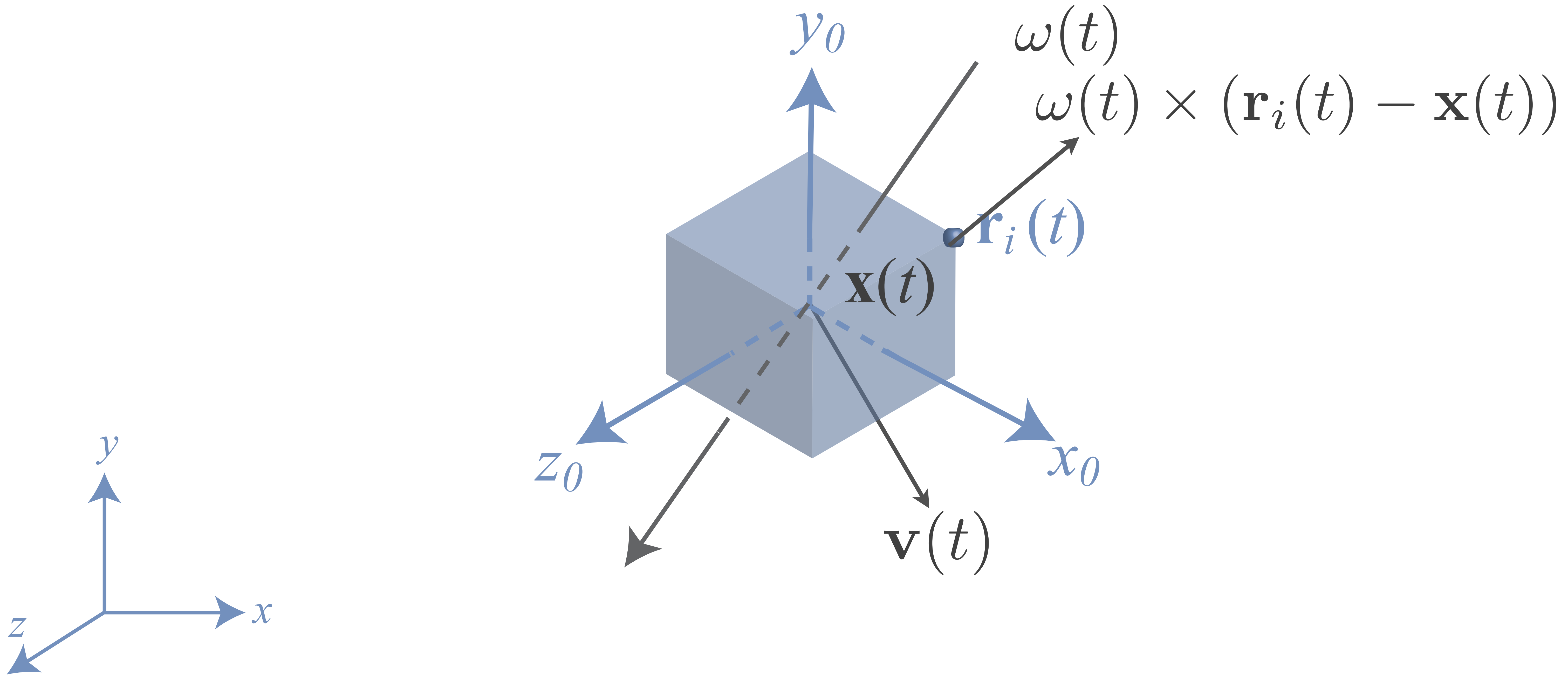

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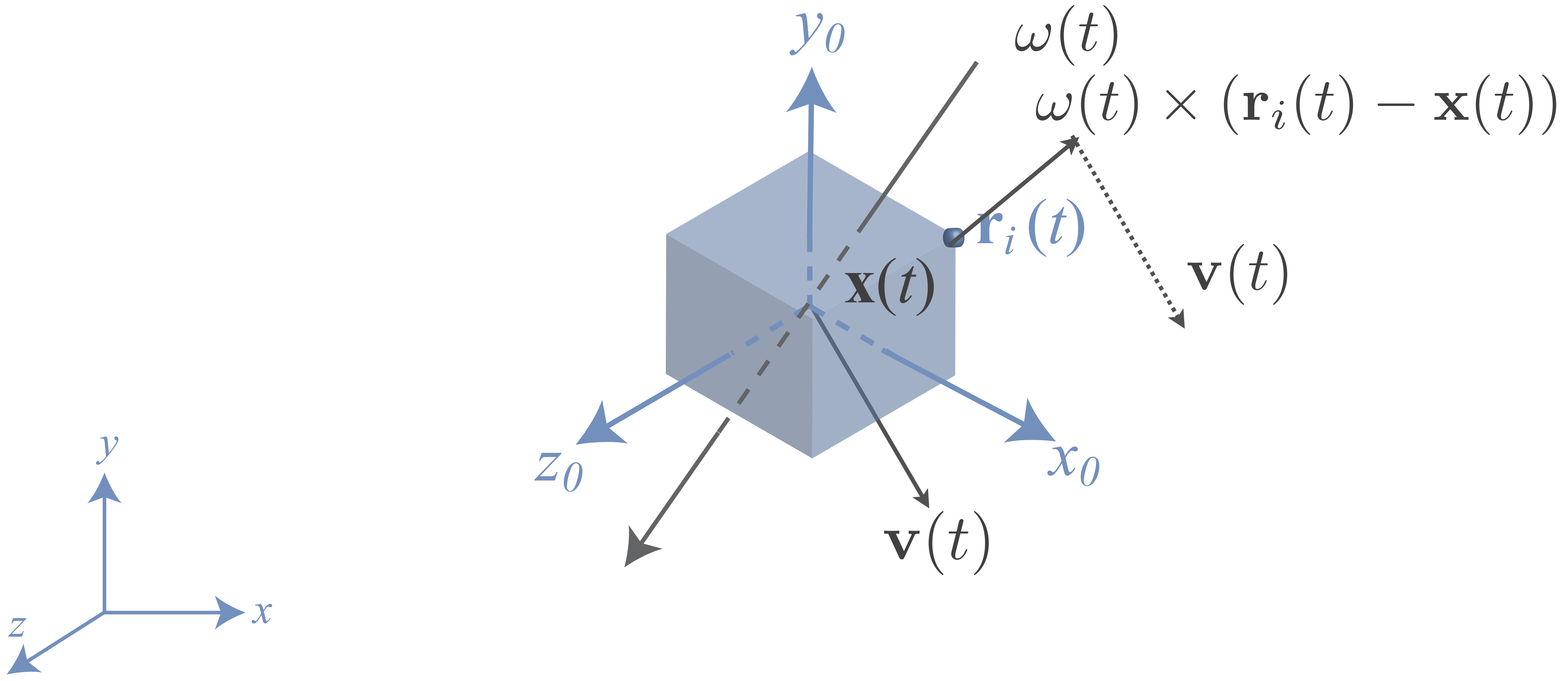

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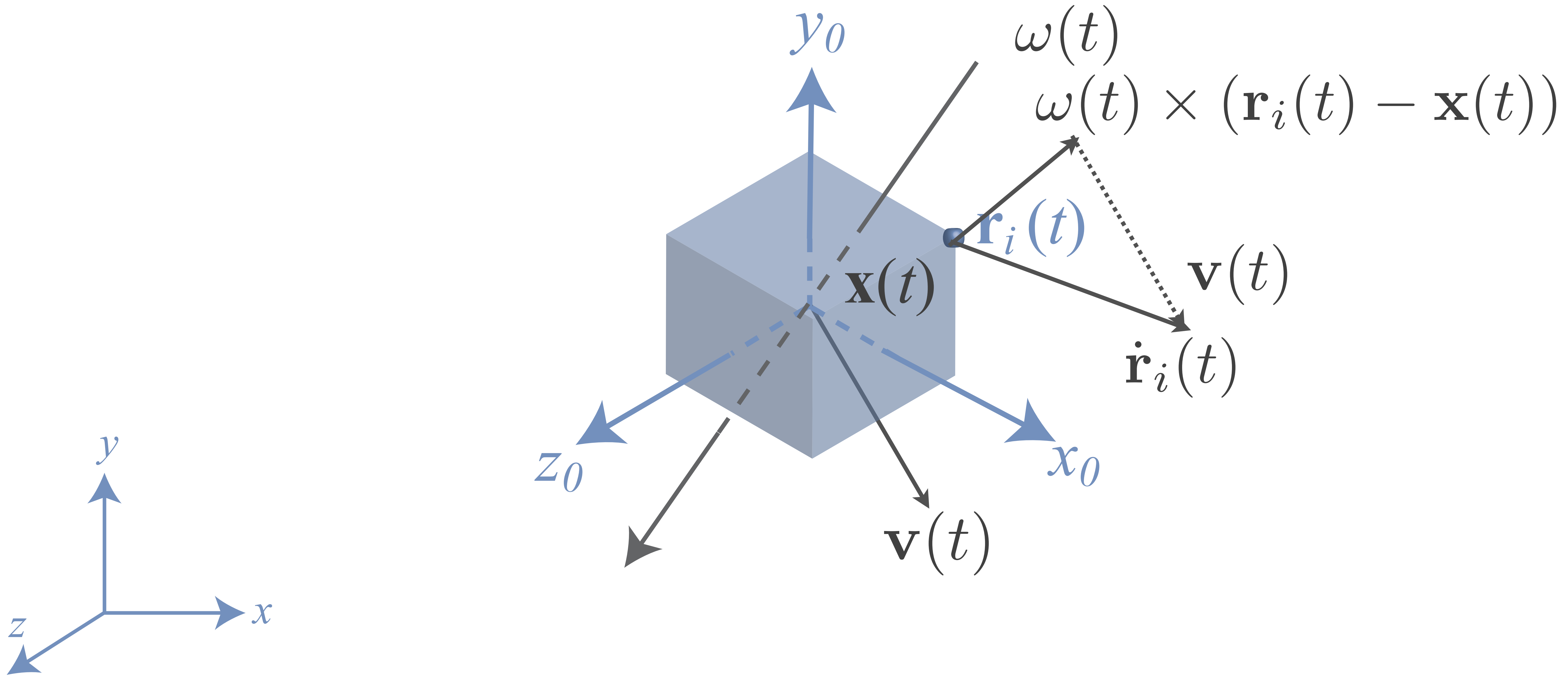

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Quiz

- True or False
  - If a cube has non-zero angular velocity, a corner point always moves faster than the COM
  - If a cube has zero angular velocity, a corner point always moves at the same speed as the COM
  - If a cube has non-zero angular velocity and zero linear velocity, the COM may or may not be moving

3D translation and orientation

Translation

- Position
- Linear velocity
- Mass
- Linear momentum
- Force

Rotation

- Orientation
- Angular velocity
- Inertia tensor
- Angular momentum
- Torque

Mass

- **The mass of the i-th particle is m_i, what is the total mass of rigid body?**

$$M = \sum_{i=1}^{N} m_i$$

- **What is the center of mass of rigid body in world space?**

$$\sum_{i}^{N} \mathbf{r}_{0i} = \mathbf{0}$$

$$\frac{\sum_{i=1}^{N} m_i \mathbf{r}_i(t)}{M} = \frac{m_i}{M} \sum_{i}^{N} (\mathbf{x}(t) + \mathbf{R}(t)\mathbf{r}_{0i}) = \frac{m_i}{M}(N\mathbf{x}(t) + \mathbf{R}(t) \sum_{i}^{N} \mathbf{r}_{0i}) = \mathbf{x}(t)$$

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- **What about center of mass in body space?**

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$$(0,0,0)$$

Inertia tensor

- **Inertia tensor describes how the mass of a rigid body is distributed relative to a reference point, often defined as the center of mass for convenience.**

$$\mathbf{I}(t) = \sum_{i=1}^{N} \begin{bmatrix} m_i(r_{iy}^{'2} + r_{iz}^{'2}) & -m_i r_{ix}^{'} r_{iy}^{'} & m_i r_{ix}^{'} r_{iz}^{'} \\ -m_i r_{iy}^{'} r_{ix}^{'} & m_i(r_{ix}^{'2} + r_{iz}^{'2}) & -m_i r_{iy}^{'} r_{iz}^{'} \\ -m_i r_{iz}^{'} r_{ix}^{'} & -m_i r_{iz}^{'} r_{iy}^{'} & m_i(r_{ix}^{'2} + r_{iy}^{'2}) \end{bmatrix}, \text{ where } \mathbf{r}_i^{'} = \mathbf{r}_i(t) - \mathbf{x}(t)$$

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principal moments of inertia

products of inertia

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principal moments of inertia

products of inertia

- For an actual implementation, we replace the finite sum with the integrals over a body's volume in world space.
- $\mathbf{I}(t)$ depends on the orientation of a body, but not the translation.
- Inertia tensors vary in world space over time, but are constant in the body space.

Inertia tensor

We can precompute the integral part in the body space to save time

$$\mathbf{I}(t) = \sum_{i=1}^{N} \begin{bmatrix} m_i(r_{iy}^{\prime 2} + r_{iz}^{\prime 2}) & -m_i r_{ix}^{\prime} r_{iy}^{\prime} & m_i r_{ix}^{\prime} r_{iz}^{\prime} \\ -m_i r_{iy}^{\prime} r_{ix}^{\prime} & m_i(r_{ix}^{\prime 2} + r_{iz}^{\prime 2}) & -m_i r_{iy}^{\prime} r_{iz}^{\prime} \\ -m_i r_{iz}^{\prime} r_{ix}^{\prime} & -m_i r_{iz}^{\prime} r_{iy}^{\prime} & m_i(r_{ix}^{\prime 2} + r_{iy}^{\prime 2}) \end{bmatrix}$$, where $\mathbf{r}_i^{\prime} = \mathbf{r}_i(t) - \mathbf{x}(t) = \mathbf{R}(t)\mathbf{r}_{0i}$

$$\begin{aligned} \mathbf{I}(t) &= \sum_i m_i\big((\mathbf{r}_i^{\prime T}\mathbf{r}_i^{\prime})\mathbf{1} - \mathbf{r}_i^{\prime}\mathbf{r}_i^{\prime T}\big) \\ &= \sum_i m_i\Big(\big(\mathbf{R}(t)\mathbf{r}_{0i}\big)^T\big((\mathbf{R}(t)\mathbf{r}_{0i}\big)\mathbf{1} - \big(\mathbf{R}(t)\mathbf{r}_{0i}\big)\big(\mathbf{R}(t)\mathbf{r}_{0i}\big)^T\Big) \\ &= \sum_i m_i\big(\mathbf{R}(t)(\mathbf{r}_{0i}^T\mathbf{r}_{0i})\mathbf{R}(t)^T\mathbf{1} - \mathbf{R}(t)\mathbf{r}_{0i}\mathbf{r}_{0i}^T\mathbf{R}(t)^T\big) \\ &= \mathbf{R}(t)\Big(\sum_i m_i\big((\mathbf{r}_{0i}^T\mathbf{r}_{0i})\mathbf{1} - \mathbf{r}_{0i}\mathbf{r}_{0i}^T\big)\Big)\mathbf{R}(t)^T \end{aligned}$$

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$$\begin{aligned} \mathbf{I}(t) &= \sum_i m_i\big((\mathbf{r}_i'^T \mathbf{r}_i')\mathbf{1} - \mathbf{r}_i' \mathbf{r}_i'^T\big) \\ &= \sum_i m_i\Big(\big(\mathbf{R}(t)\mathbf{r}_{0i}\big)^T\big(\mathbf{R}(t)\mathbf{r}_{0i}\big)\mathbf{1} - \big(\mathbf{R}(t)\mathbf{r}_{0i}\big)\big(\mathbf{R}(t)\mathbf{r}_{0i}\big)^T\Big) \\ &= \sum_i m_i\big(\mathbf{R}(t)(\mathbf{r}_{0i}^T \mathbf{r}_{0i})\mathbf{R}(t)^T\mathbf{1} - \mathbf{R}(t)\mathbf{r}_{0i}\mathbf{r}_{0i}^T\mathbf{R}(t)^T\big) \\ &= \mathbf{R}(t)\Big(\sum_i m_i\big((\mathbf{r}_{0i}^T \mathbf{r}_{0i})\mathbf{1} - \mathbf{r}_{0i}\mathbf{r}_{0i}^T\big)\Big)\mathbf{R}(t)^T \end{aligned}$$

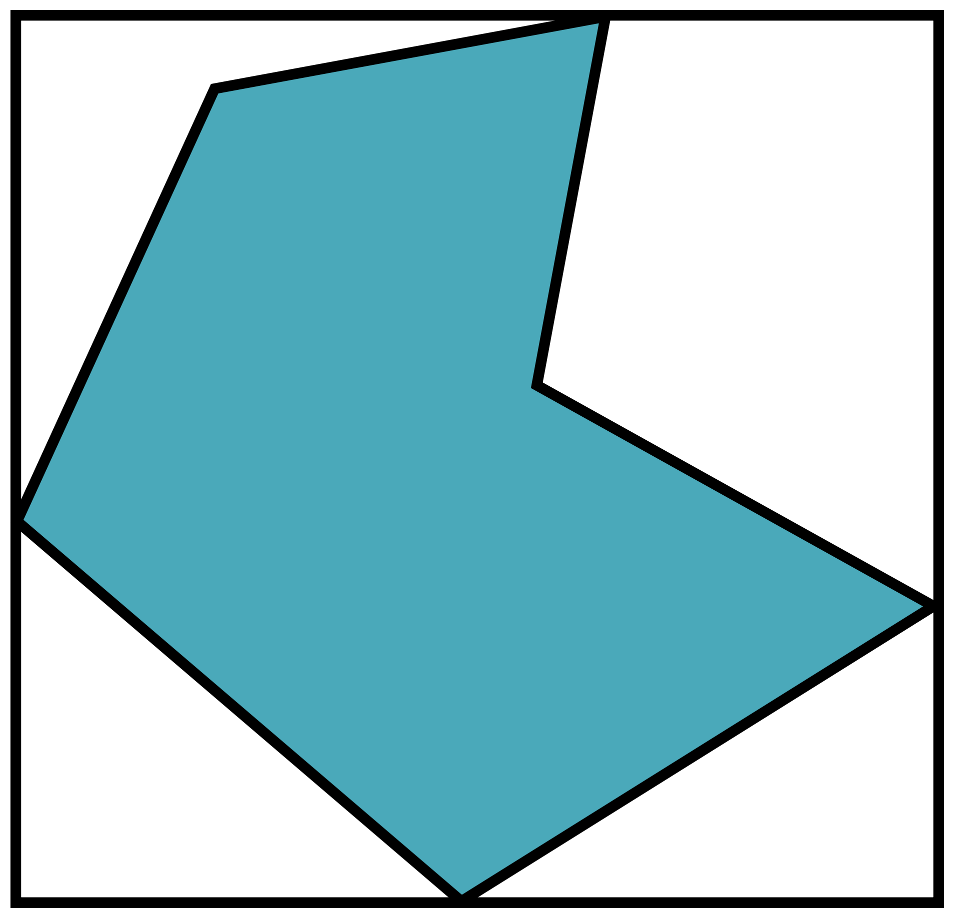
$$\mathbf{I}(t) = = \mathbf{R}(t)\mathbf{I}_b\mathbf{R}(t)^T$$

$$\mathbf{I}_b = \sum_i m_i\big((\mathbf{r}_{0i}^T \mathbf{r}_{0i})\mathbf{1} - \mathbf{r}_{0i}\mathbf{r}_{0i}^T\big)$$

Approximate inertia tensor

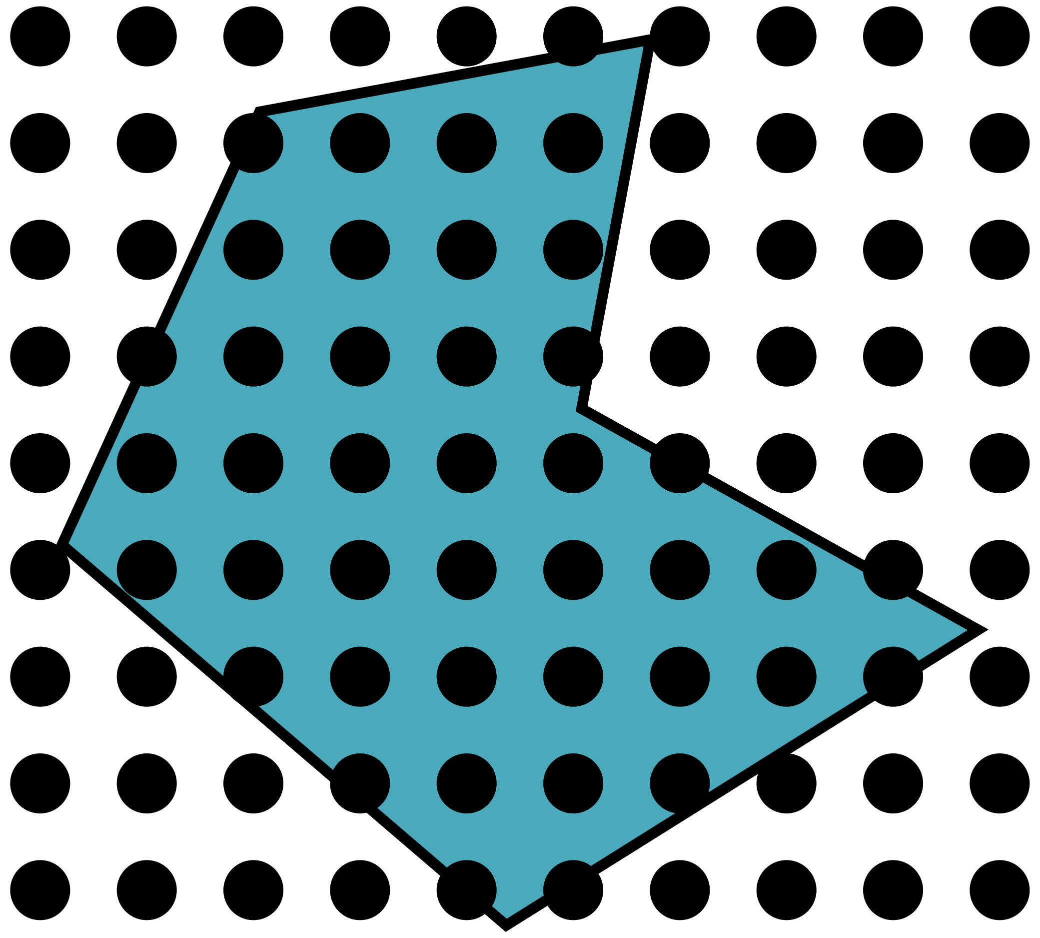
- Closed-form solutions exist for primitive shapes.
- For arbitrary geometry, we can approximate it by

Bounding box



Simple but inaccurate

Point sampling



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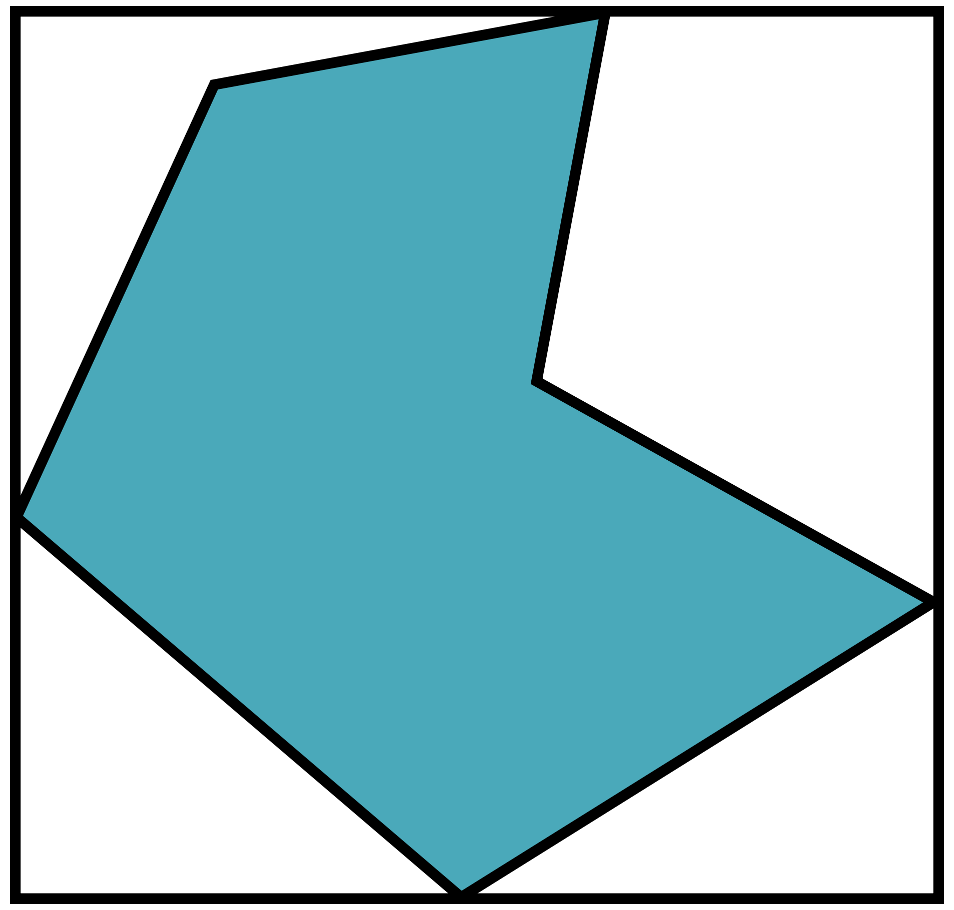

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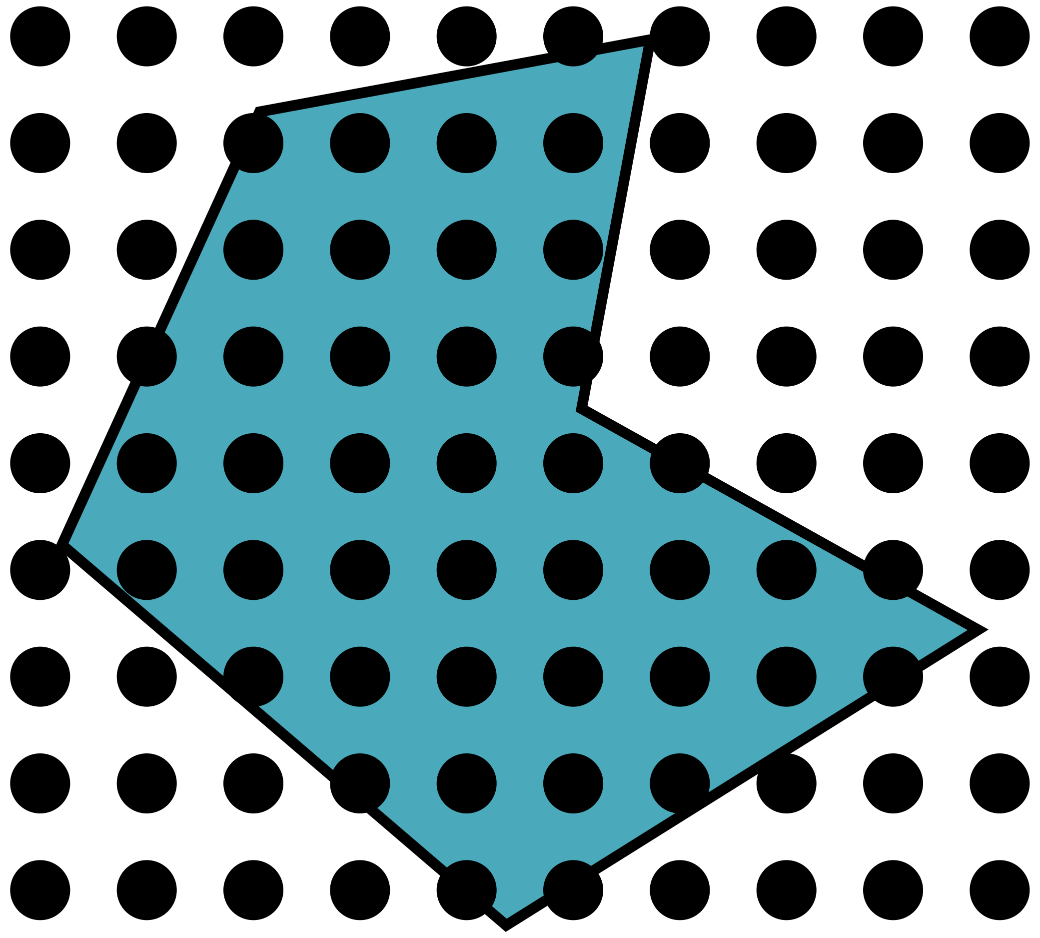
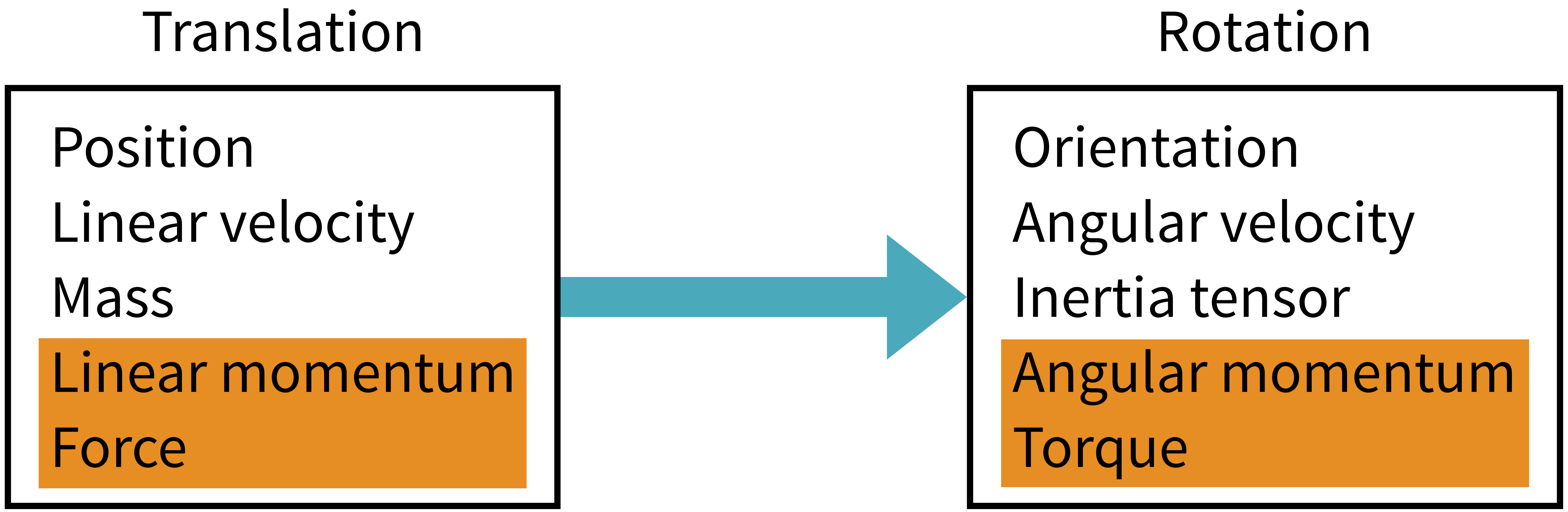


Simple, more accurate, but
requires expensive volume test

3D translation and orientation



Force and torque

- $\mathbf{f}_i(t)$ **denotes the total force from external forces acting on the i-th particle at time t.**

  - **Total force on rigid body:** $$\mathbf{f}(t) = \sum_i \mathbf{f}_i(t)$$

  - **Total torque on rigid body:** $$\boldsymbol{\tau}(t) = \sum_i \big(\mathbf{r}_i(t) - \mathbf{x}(t)\big) \times \mathbf{f}_i(t)$$

- **Torque depends on the points of application but force does not.**
- **Force that passes through COM does not induce torque.**

Momentum

- $\mathbf{p}(t)$: **Total linear moment of the rigid body is the same as if the body was simply a particle with mass M and velocity $\mathbf{v}(t)$.**

$$\mathbf{p}(t) = \sum_i m_i \dot{\mathbf{r}}_i(t)$$

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$$\mathbf{L}(t) = \mathbf{I}(t)\boldsymbol{\omega}(t)$$

Derivative of momentum

- **Change in linear momentum is equivalent to the total forces acting on the rigid body.**

$$\dot{\mathbf{p}}(t) = M\dot{\mathbf{v}}(t) = \mathbf{f}(t)$$

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Here is the proof:

$$\begin{aligned}\boldsymbol{\tau}(t) &= \sum_i \mathbf{r}'_i \times \mathbf{F}_i \\ &= \sum_i \mathbf{r}'_i \times m_i \ddot{\mathbf{r}}_i = \sum_i \mathbf{r}'_i \times m_i(\dot{\mathbf{v}} - \dot{\mathbf{r}}'_i \times \boldsymbol{\omega} - \mathbf{r}'_i \times \dot{\boldsymbol{\omega}}) \\ &= -\left(\sum_i m_i[\mathbf{r}'_i][\dot{\mathbf{r}}'_i]\right)\boldsymbol{\omega} - \left(\sum_i m_i[\mathbf{r}'_i][\mathbf{r}'_i]\right)\dot{\boldsymbol{\omega}}\end{aligned}$$

Recall $\mathbf{I}(t) = -\sum_i m_i[\mathbf{r}'_i][\mathbf{r}'_i]$

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Recall $\mathbf{I}(t) = -\sum_i m_i[\mathbf{r}_i'][\mathbf{r}_i']$, so $\dot{\mathbf{I}}(t) = \sum_i -m_i[\mathbf{r}_i'][\dot{\mathbf{r}}_i'] - m_i[\dot{\mathbf{r}}_i'][\mathbf{r}_i']$

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Because $m_i[\dot{\mathbf{r}}'_i][\mathbf{r}'_i]\boldsymbol{\omega} = m_i[\boldsymbol{\omega} \times \mathbf{r}'_i](-\boldsymbol{\omega} \times \mathbf{r}'_i) = -m_i(\boldsymbol{\omega} \times \mathbf{r}'_i) \times (\boldsymbol{\omega} \times \mathbf{r}'_i) = \mathbf{0}$

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- **Change in linear momentum is equivalent to the total forces acting on the rigid body.**

$$\dot{\mathbf{p}}(t) = M\dot{\mathbf{v}}(t) = \mathbf{f}(t)$$

- **The relation between angular momentum and the total torque is analogous to the linear case.**

$$\dot{\mathbf{L}}(t) = \mathbf{I}(t)\dot{\boldsymbol{\omega}} + \dot{\mathbf{I}}(t)\boldsymbol{\omega} = \boldsymbol{\tau}(t)$$

Here is the proof:

$$\begin{aligned}
\boldsymbol{\tau}(t) &= \sum_i \mathbf{r}'_i \times \mathbf{F}_i \\
&= \sum_i \mathbf{r}'_i \times m_i \ddot{\mathbf{r}}_i = \sum_i \mathbf{r}'_i \times m_i(\dot{\mathbf{v}} - \dot{\mathbf{r}}'_i \times \boldsymbol{\omega} - \mathbf{r}'_i \times \dot{\boldsymbol{\omega}}) \\
&= -\Big(\sum_i m_i[\mathbf{r}'_i][\dot{\mathbf{r}}'_i]\Big)\boldsymbol{\omega} - \Big(\sum_i m_i[\mathbf{r}'_i][\mathbf{r}'_i]\Big)\dot{\boldsymbol{\omega}} \\
&= \dot{\mathbf{I}}(t)\boldsymbol{\omega} + \mathbf{I}(t)\dot{\boldsymbol{\omega}} = \frac{d}{dt}\mathbf{I}(t)\boldsymbol{\omega} = \dot{\mathbf{L}}(t)
\end{aligned}$$

Recall $\mathbf{I}(t) = -\sum_i m_i[\mathbf{r}'_i][\mathbf{r}'_i]$, so $\dot{\mathbf{I}}(t) = \sum_i - m_i[\mathbf{r}'_i][\dot{\mathbf{r}}'_i] - m_i[\dot{\mathbf{r}}'_i][\mathbf{r}'_i]$

Drop $\dot{\mathbf{I}}(t)\boldsymbol{\omega} = \sum_i - m_i[\mathbf{r}'_i][\dot{\mathbf{r}}'_i]\boldsymbol{\omega} \cancel{- m_i[\dot{\mathbf{r}}'_i][\mathbf{r}'_i]\boldsymbol{\omega}}$

Because $m_i[\dot{\mathbf{r}}'_i][\mathbf{r}'_i]\boldsymbol{\omega} = m_i[\boldsymbol{\omega} \times \mathbf{r}'_i](-\boldsymbol{\omega} \times \mathbf{r}'_i) = -m_i(\boldsymbol{\omega} \times \mathbf{r}'_i) \times (\boldsymbol{\omega} \times \mathbf{r}'_i) = \mathbf{0}$

Put it all together

$$\mathbf{Y}(t) = \begin{array}{cl} \left[\begin{array}{c} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{p}(t) \\ \mathbf{L}(t) \end{array}\right] & \begin{array}{l} \text{position} \\ \text{orientation} \\ \text{linear momentum} \\ \text{angular momentum} \end{array} \end{array}$$

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Given the current state $\mathbf{Y}_n$, how to evaluate $\dot{\mathbf{Y}}_n$, assuming the mass, M, and inertia in the body space, $\mathbf{I}_b$ are known?

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$$\dot{\mathbf{Y}}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \\ \\ \\ \end{bmatrix}$$

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Evaluate all the forces, $\mathbf{f}_1, \cdots, \mathbf{f}_n$ currently applied on the rigid body.

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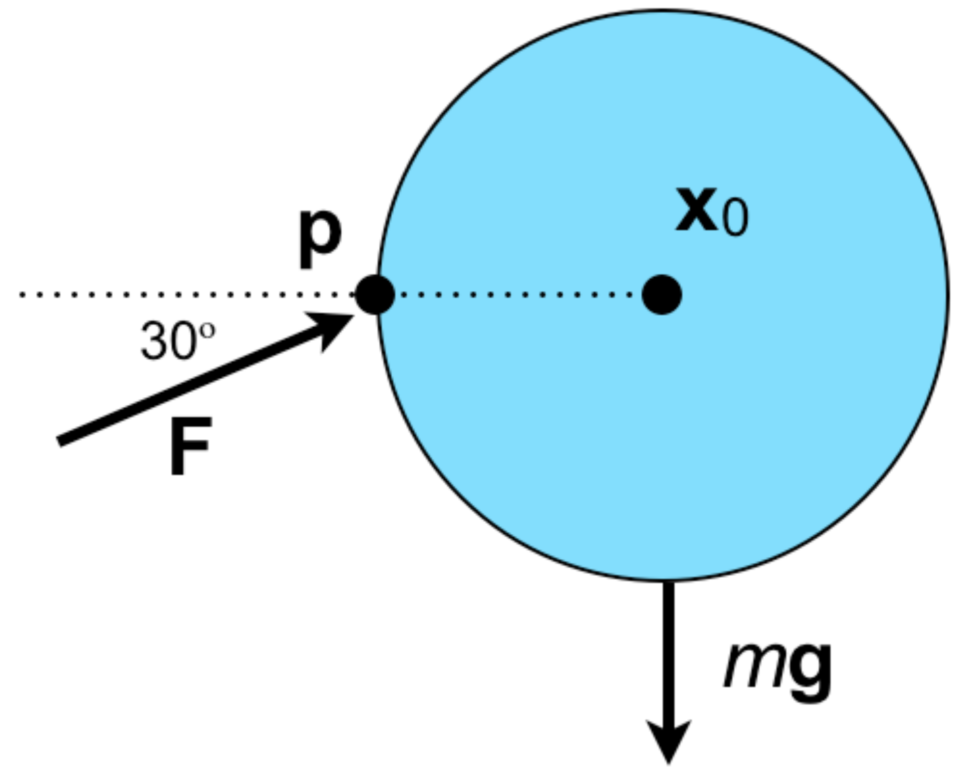
$$\boldsymbol{\tau}(t) = \sum_{i=1}^{n} \big(\mathbf{r}_i(t) - \mathbf{x}(t)\big) \times \mathbf{f}_i(t)\big)$$

Point of application must be known

Quiz

- **Consider a 3D sphere with radius 1m, mass 1kg, and inertia I_{body}. The initial linear and angular velocity are both zero. The initial position and the initial orientation are x_0 and R_0. The forces applied on the sphere include gravity (g) and an initial push F applied at point p. Note that F is only applied for one time step at t_0. If we use Explicit Euler method with time step h to integrate , what are the position and the orientation of the sphere at t_2? Use the actual numbers defined as below to compute your solution (except for g and h).**



$\mathbf{x}_0 = (0, 0, 0)$

$$\mathbf{R}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{p} = (-1, 0, 0)$

$\mathbf{F} = (4\cos(30^\circ), 4\sin(30^\circ), 0)$

$m = 1$

$$\mathbf{I}_{body} = \begin{pmatrix} 2/5 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 2/5 \end{pmatrix}$$

Issues with rotation matrix

- The rotational matrix might no longer be orthonormal due to accumulated numerical errors.
- Rectifying a rotational matrix is not trivial.
  - Could use Gram-Schmidt process to make $\mathbf{R}$ orthonormal.

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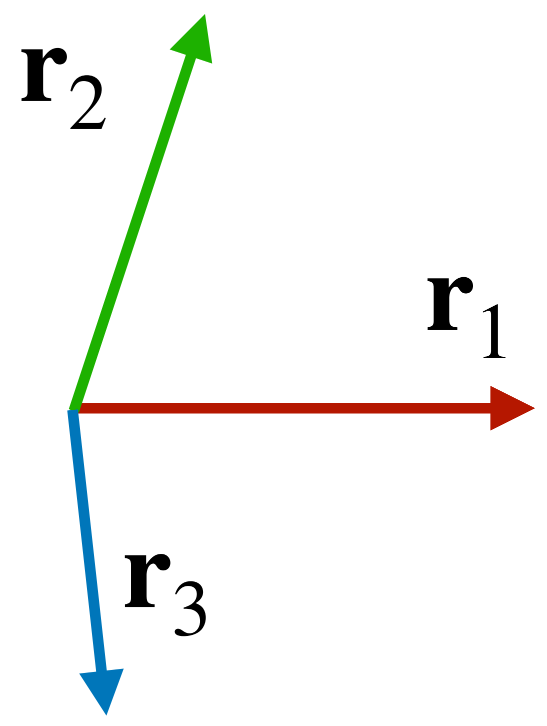

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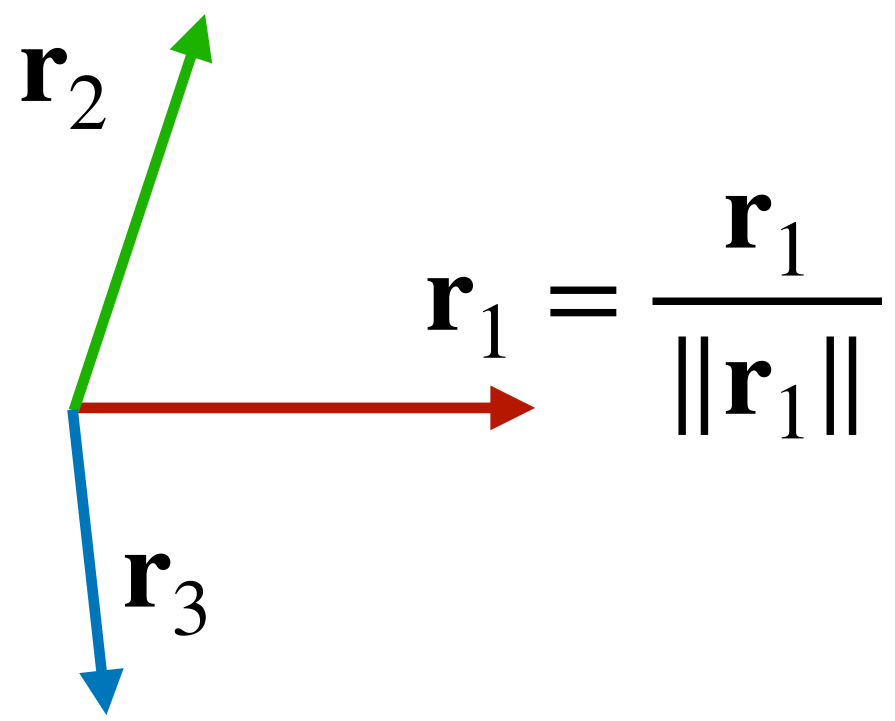

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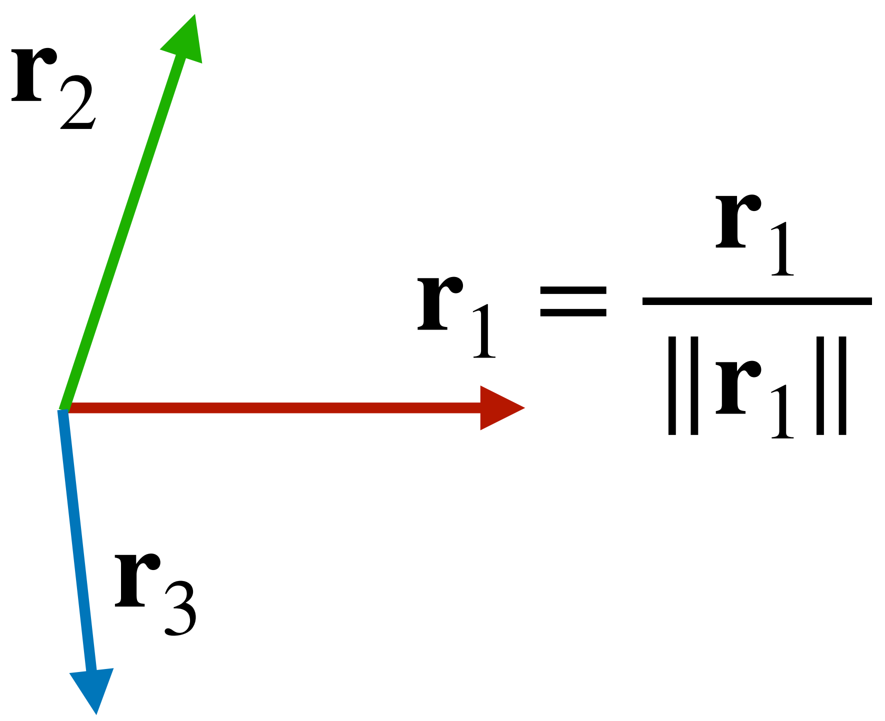

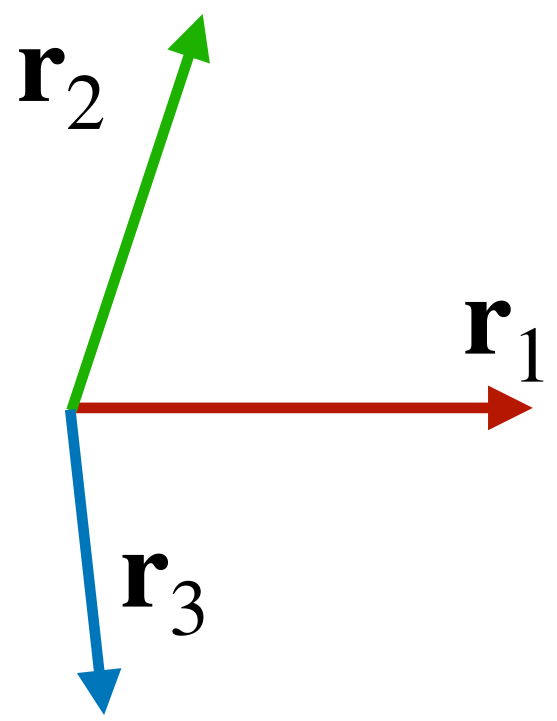

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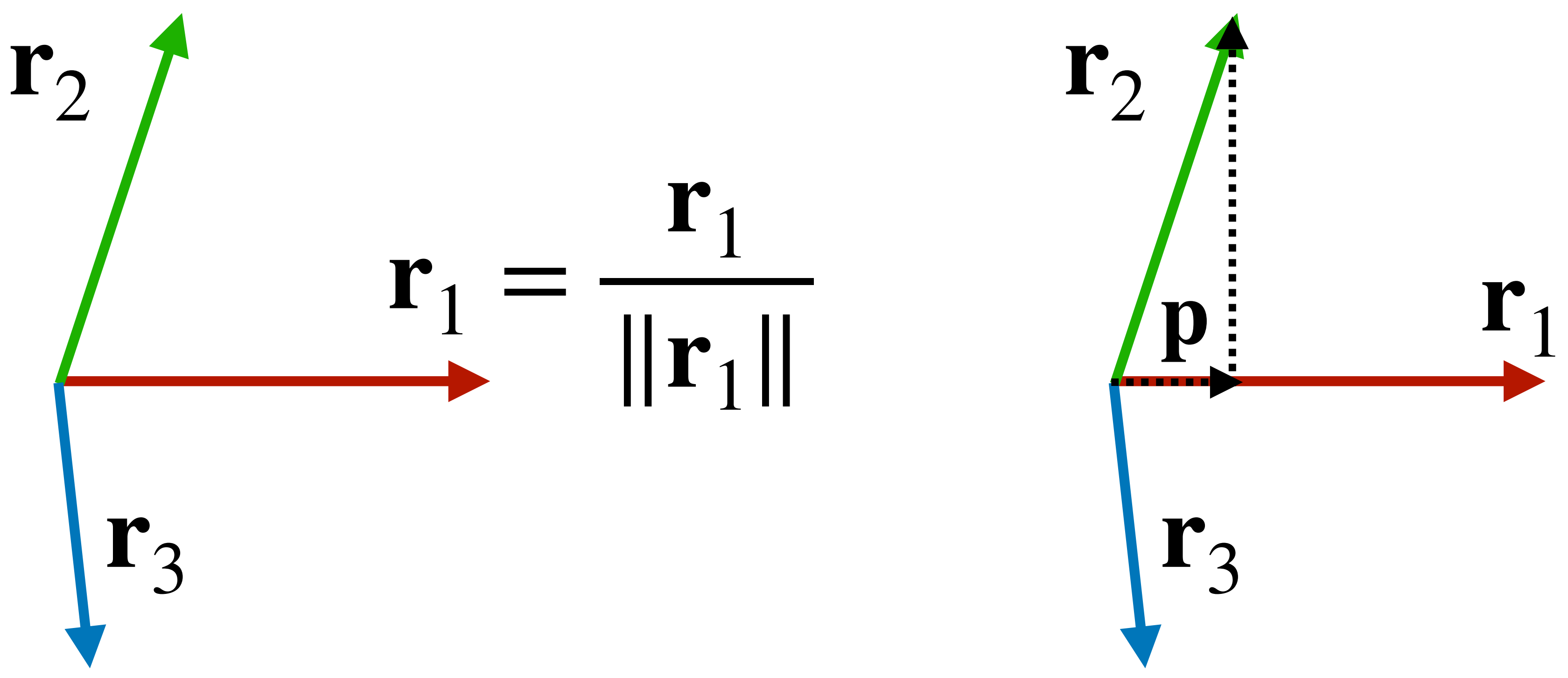

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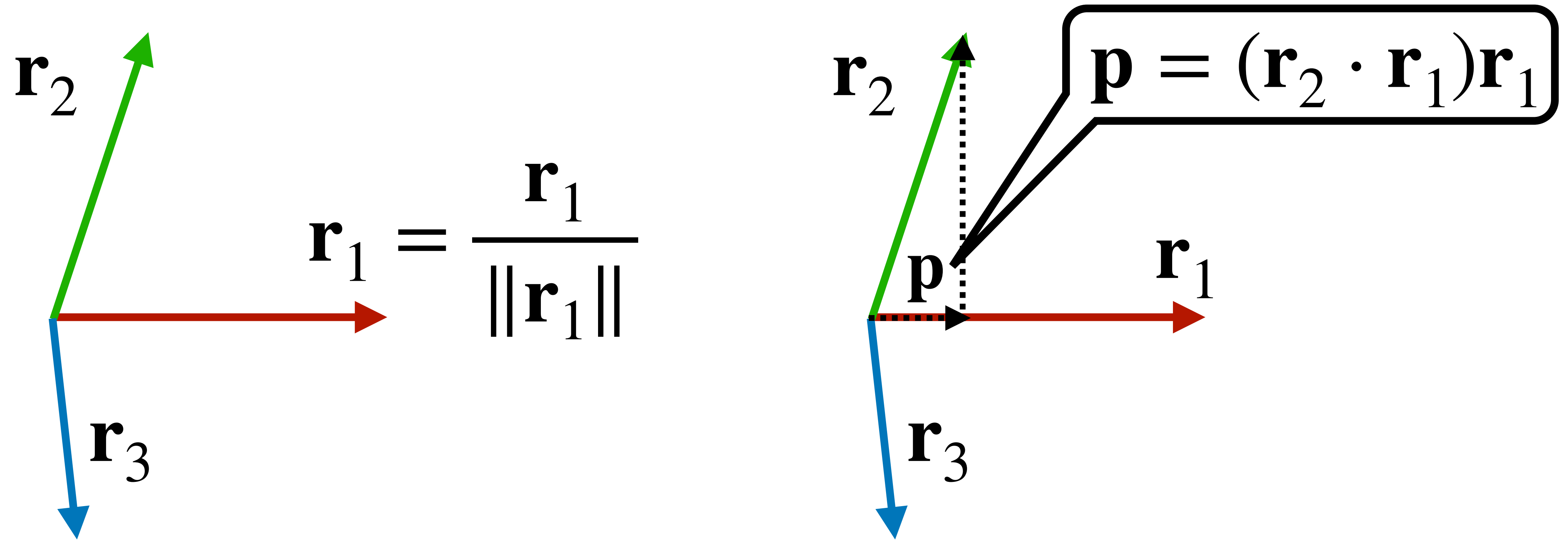

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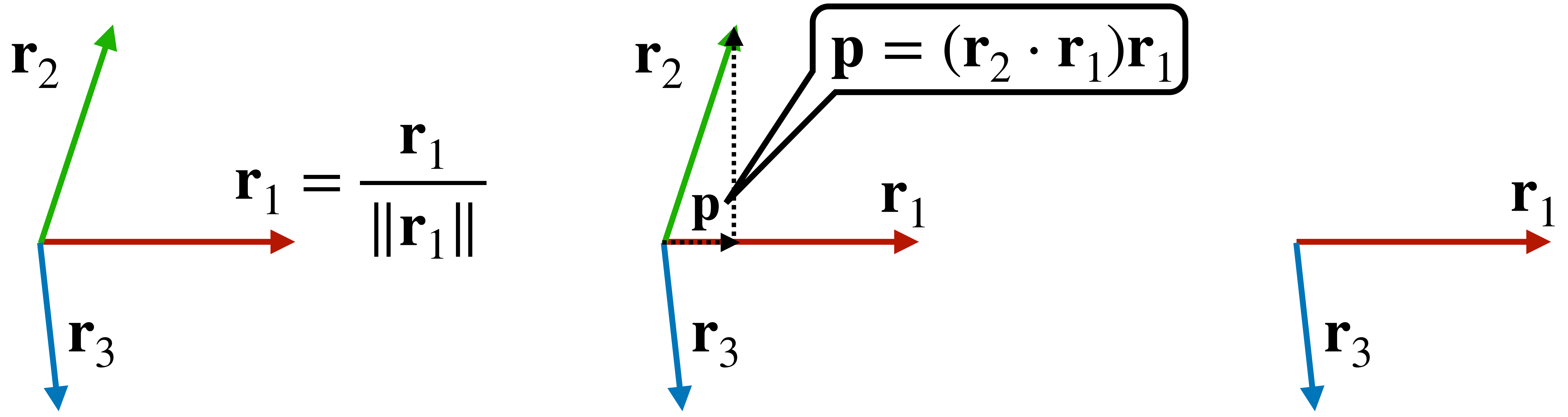

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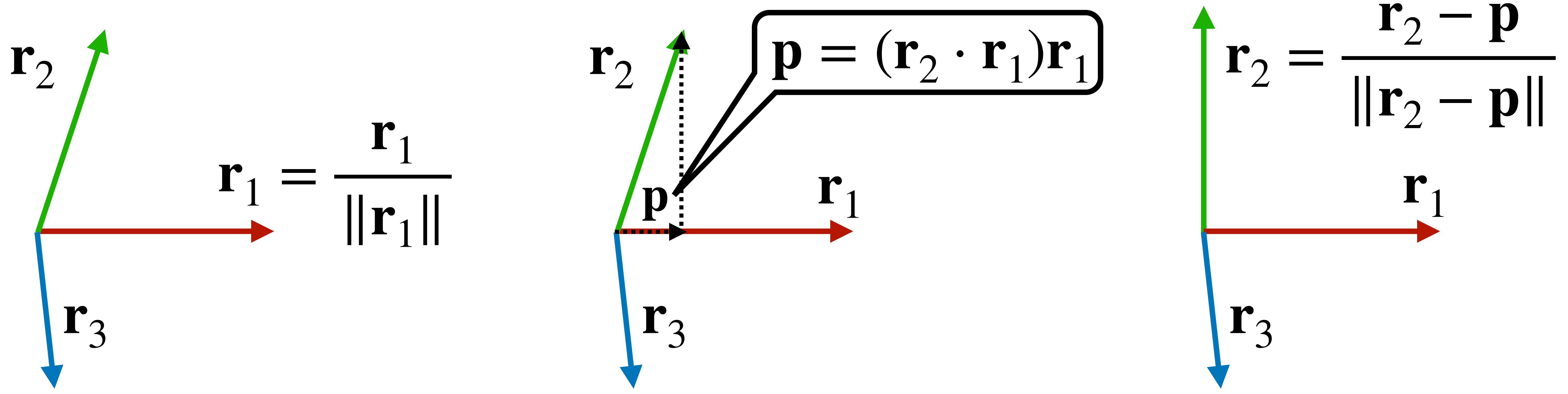

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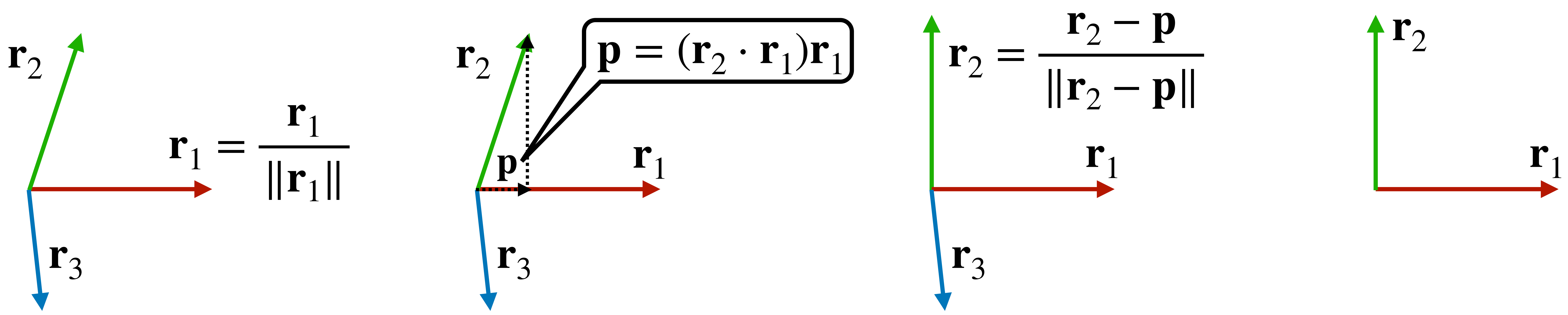

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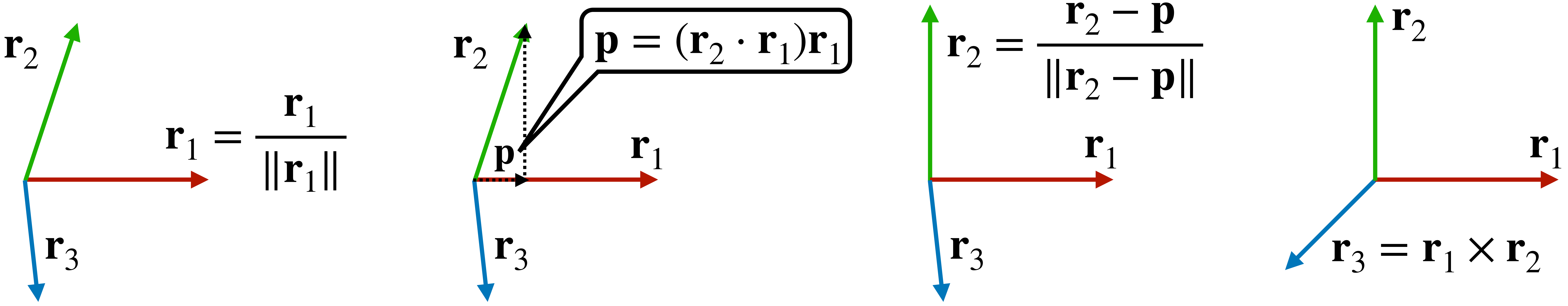

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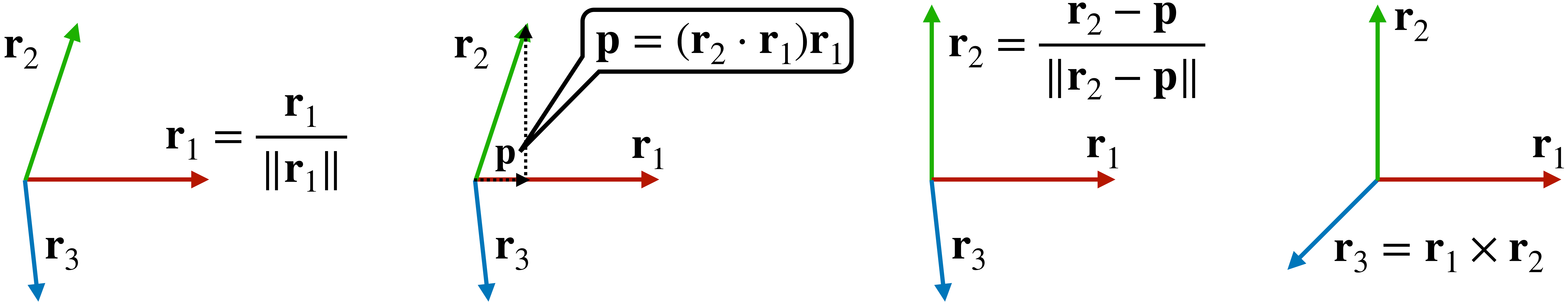

- We will use Quaternion representation for 3D orientation instead.

Momentum vs velocity

- Why do we use momentum in the state space instead of velocity?
  - Because the relation of angular momentum and torque is simpler.

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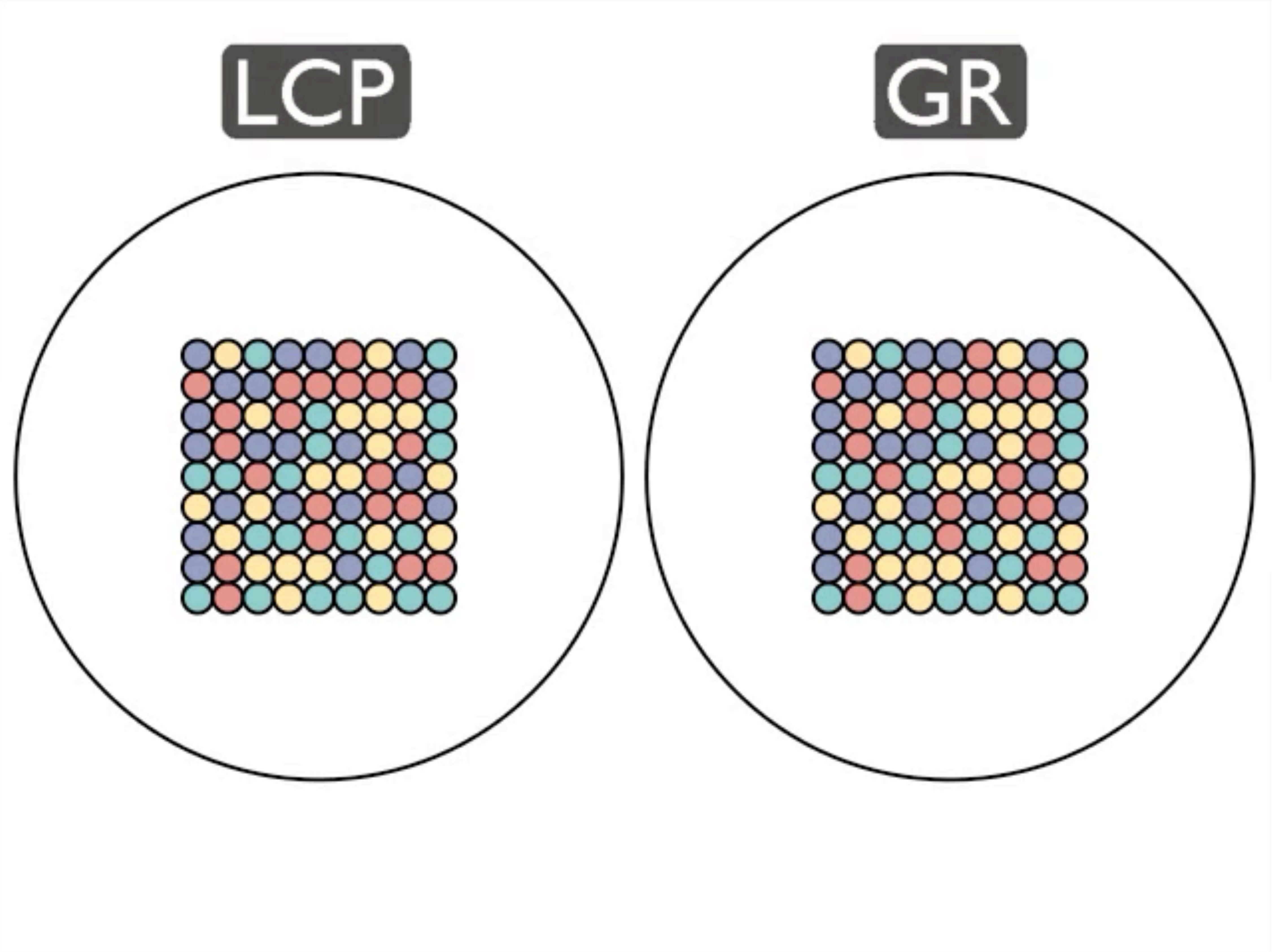
Momentum vs velocity

- Why do we use momentum in the state space instead of velocity?
  - Because the relation of angular momentum and torque is simpler.
  - Because the angular momentum is constant when there is no torques acting on the object.
- Use linear momentum $\mathbf{p}(t)$ to be consistent with angular velocity.

Constrained rigid body simulation

- Handling contacts and collisions is a very important topic that will be partially covered in later lectures.
- Idealized contact models can produce visually plausible results for graphics applications, but they are often a major source of error when predicting the motion of real-world objects.



Additional reading

- Skew symmetric matrix: https://en.wikipedia.org/wiki/Skew-symmetric_matrix
- Rigid body lecture notes from David Baraff:
  - https://www.cs.cmu.edu/~baraff/sigcourse/notesd1.pdf
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- Brian Mirtich's thesis
  - https://people.eecs.berkeley.edu/~jfc/mirtich/thesis/mirtichThesis.pdf

Logistics

- Homework 3: Rigid Bodies, will be out on 11/9
- Project 3: Inverse Kinematics, will be out on 11/16
- Homework 4: Animation Control, will be out on 11/30