

# AA 203

## Optimal and Learning-Based Control

Pontryagin's maximum principle and indirect methods

Autonomous Systems Laboratory

Stanford University

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**Stanford**  
University

# Agenda

1. Geometry and generalizations of first-order NOCs
2. Weak Pontryagin maximum principle in discrete-time
3. Weak Pontryagin maximum principle in continuous-time
4. Pontryagin maximum principle in continuous-time
5. Indirect methods for optimal control
6. Time-optimal control problems

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## Review: First-order NOCs

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f(x) \\ & \text{subject to} && h(x) = 0 \\ & && g(x) \preceq 0 \end{aligned} \quad L(x, \lambda, \mu) := f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

### Theorem (First-order NOCs)

*Suppose  $x^* \in \mathbb{R}^n$  is a local minimum of  $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$  subject to  $h(x^*) = 0$  and  $g(x^*) \preceq 0$  with  $h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$  and  $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^r)$ . Moreover, assume*

$$\{\nabla h_i(x^*)\}_{i=1}^m \cup \{\nabla g_j(x^*)\}_{j \in \mathcal{A}_g(x^*)}$$

*are linearly independent. Then there exist unique  $\lambda^* \in \mathbb{R}^m$  and  $\mu^* \in \mathbb{R}^r$  such that*

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0, \quad \mu^* \succeq 0, \quad \mu_j^* = 0, \quad \forall j \notin \mathcal{A}_g(x^*),$$

The assumption on the constraint gradients is known as the *linear independence constraint qualification (LICQ)*.

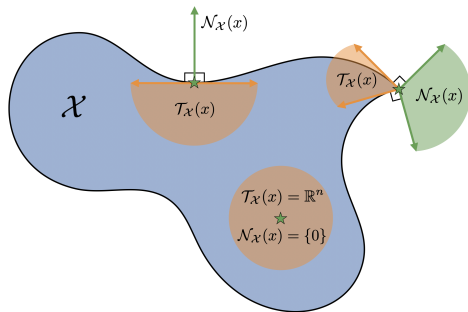
# Geometry of first-order NOCs

Tangent cone  $\mathcal{T}_{\mathcal{X}}(x)$  “vectors that stay in  $\mathcal{X}$ ”

Normal cone  $\mathcal{N}_{\mathcal{X}}(x)$  “vectors that leave  $\mathcal{X}$ ”

If  $x^*$  is a local minimum of  $f$  over  $\mathcal{X}$ , then  $-\nabla f(x^*) \in \mathcal{N}_{\mathcal{X}}(x^*)$ , i.e., there is no feasible component of  $-\nabla f(x^*)$  that would allow us to locally decrease  $f(x^*)$ .

For convenience, we write “ $-\nabla f(x^*) \perp_{x^*} \mathcal{X}$ ”.



If  $\mathcal{X} = \{x \in \mathbb{R}^n \mid h(x) = 0, g(x) \preceq 0\}$  and the LICQ holds at  $x^* \in \mathcal{X}$ , then

$$\mathcal{T}_{\mathcal{X}}(x^*) = \left\{ d \in \mathbb{R}^n \mid \frac{\partial h}{\partial x}(x^*)d = 0, \nabla g_j(x^*)^\top d \leq 0, \forall j \in \mathcal{A}_g(x^*) \right\}$$

$$\mathcal{N}_{\mathcal{X}}(x^*) = \left\{ v \in \mathbb{R}^n \mid v = \frac{\partial h}{\partial x}(x^*)^\top \lambda + \frac{\partial g}{\partial x}(x^*)^\top \mu, \mu \succeq 0, \mu_j = 0, \forall j \notin \mathcal{A}_g(x^*) \right\}$$

## Example: A problem with linearly dependent constraints

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad f(x) := x_1 + x_2$$

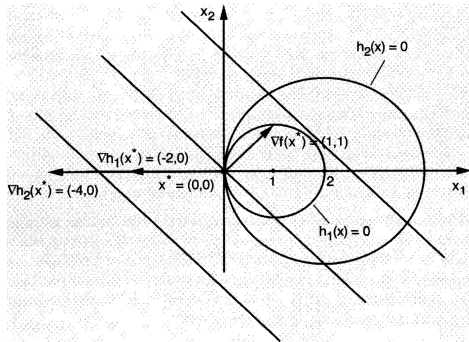
$$\text{subject to} \quad h_1(x) := (x_1 - 1)^2 + x_2^2 - 1 = 0$$

$$h_2(x) := (x_1 - 2)^2 + x_2^2 - 4 = 0$$

At the only feasible point  $x^* = 0$ , we have

$$\nabla f(x^*) = (1, 1)$$

$$\nabla h_1(x^*) = (-2, 0), \quad \nabla h_2(x^*) = (-4, 0)$$



The constraint gradients are linearly dependent (i.e., the LICQ does not hold), so we cannot write  $\nabla f(x^*) + \lambda_1^* \nabla h_1(x^*) + \lambda_2^* \nabla h_2(x^*) = 0$ .

In essence, the constraints “pinch together” so that just one  $x^*$  is feasible, regardless of the objective value.

## Theorem (Fritz John first-order NOCs)

Let  $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ ,  $h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ , and  $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^r)$ . Suppose  $x^* \in \mathbb{R}^n$  is a local minimum of the problem

$$\begin{aligned} & \underset{x \in \mathcal{S}}{\text{minimize}} && f(x) \\ & \text{subject to} && h(x) = 0 \cdot \\ & && g(x) \preceq 0 \end{aligned}$$

Then there exist  $(\eta, \lambda^*, \mu^*) \in \{0, 1\} \times \mathbb{R}^m \times \mathbb{R}^r$  such that

$$\begin{aligned} & (\eta, \lambda^*, \mu^*) \neq 0 && \text{non-triviality} \\ & -\nabla_x L_\eta(x^*, \lambda^*, \mu^*) \perp_{x^*} \mathcal{S} && \text{stationarity} \\ & \mu_j^* \geq 0, \mu_j^* g_j(x^*) = 0, \forall j \in \{1, 2, \dots, r\} && \text{complementarity} \end{aligned}$$

where  $L_\eta(x, \lambda, \mu)$  is the partial Lagrangian

$$L_\eta(x, \lambda, \mu) := \eta f(x) + \lambda^\top h(x) + \mu^\top g(x).$$

## Theorem (Fritz John first-order NOCs)

If  $x^*$  is a local minimum, there exist  $(\eta, \lambda^*, \mu^*) \in \{0, 1\} \times \mathbb{R}^m \times \mathbb{R}^r$  such that

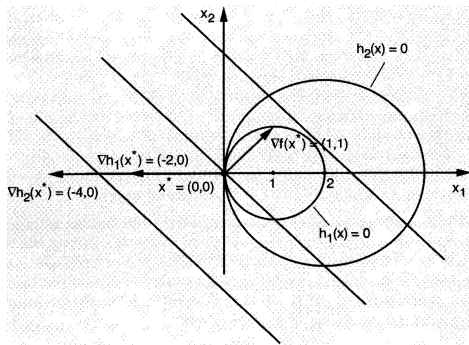
$$(\eta, \lambda^*, \mu^*) \neq 0$$

$$-\nabla_x L_\eta(x^*, \lambda^*, \mu^*) \perp_{x^*} \mathcal{S}$$

$$\mu_j^* \geq 0, \mu_j^* g_j(x^*) = 0, \forall j \in \{1, 2, \dots, r\}$$

where  $L_\eta(x, \lambda, \mu)$  is the partial Lagrangian

$$L_\eta(x, \lambda, \mu) := \eta f(x) + \lambda^\top h(x) + \mu^\top g(x).$$



The “abnormal case”  $\eta = 0$  yields necessary conditions independent of the objective  $f$ .

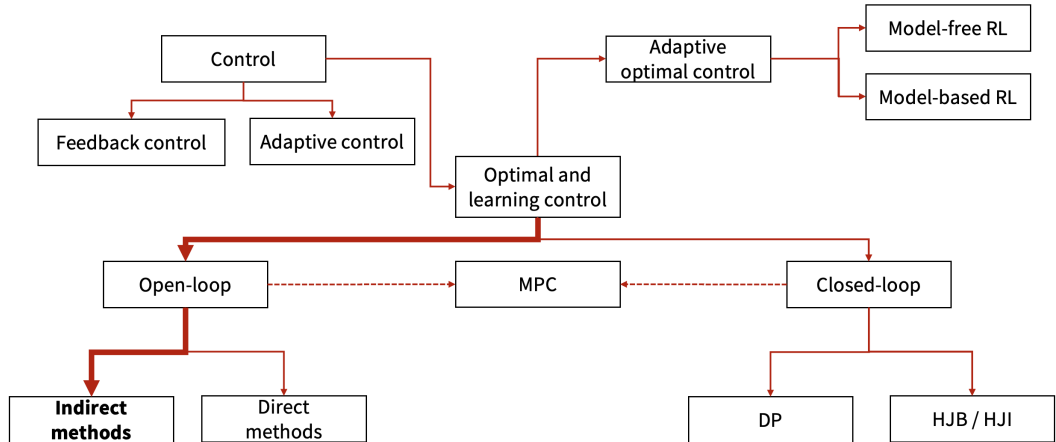
## Corollary

If  $\mathcal{S} = \mathbb{R}^n$  and the LICQ holds, then  $\eta = 1$  and  $\nabla_x L_1(x^*, \lambda^*, \mu^*) = 0$ .



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# Optimal control problem (discrete-time)

Consider the discrete-time optimal control problem (OCP)

$$\begin{aligned} & \underset{x,u}{\text{minimize}} \quad \ell_T(x_T) + \sum_{t=0}^{T-1} \ell(t, x_t, u_t) && \text{cost (terminal + stage)} \\ & \text{subject to} \quad x_{t+1} = f(t, x_t, u_t), \quad \forall t \in \{0, 1, \dots, T-1\} && \text{dynamical feasibility} \\ & \quad \quad \quad x_0 = \bar{x}_0 && \text{initial condition} \\ & \quad \quad \quad x_T \in \mathcal{X}_T && \text{terminal condition} \\ & \quad \quad \quad u_t \in \mathcal{U}, \quad \forall t \in \{0, 1, \dots, T-1\} && \text{input constraints} \end{aligned}$$

An optimal control  $u^* = \{u_t^*\}_{t=0}^{T-1}$  for a specific initial state  $\bar{x}_0$  is an *open-loop* input.

An optimal control of the form  $u_t^* = \pi^*(t, x_t)$  is a *closed-loop* input.

## Lagrangian, Hamiltonian, and the adjoint equation (discrete-time)

The partial Lagrangian is

$$\begin{aligned} L_\eta(x, u, p) &= \eta \ell_T(x_T) + \underbrace{p_0^\top (x_0 - \bar{x}_0)}_{\text{initial condition}} + \sum_{t=0}^T \left( \eta \ell(t, x_t, u_t) + \underbrace{p_{t+1}^\top (x_{t+1} - f(t, x_t, u_t))}_{\text{dynamical feasibility}} \right), \\ &= \eta \ell_T(x_T) + p_0^\top (x_0 - \bar{x}_0) + \sum_{t=0}^T \left( p_{t+1}^\top x_{t+1} - H_\eta(t, x_t, u_t, p_{t+1}) \right) \end{aligned}$$

with normality  $\eta \in \{0, 1\}$ , Lagrange multipliers  $\{p_t\}_{t=0}^N \subset \mathbb{R}^n$ , and *Hamiltonian*

$$H_\eta(t, x, u, p) := p^\top f(t, x, u) - \eta \ell(t, x, u).$$

Setting  $\nabla_{x_t} L(x^*, u^*) = 0$  for  $t \in \{0, 1, \dots, T-1\}$  yields

$$p_t^* = \nabla_x H_\eta(t, x_t^*, u_t^*, p_{t+1}^*), \quad \forall t \in \{0, 1, \dots, T-1\},$$

which is a *backwards recursion* for the *adjoint* or *co-state*  $p_t^*$ .

## Transversality and the maximum condition (discrete-time)

The partial Lagrangian is

$$L_\eta(x, u, p) = \eta \ell_T(x_T) + p_0^\top (x_0 - \bar{x}_0) + \sum_{t=0}^{T-1} \left( p_{t+1}^\top x_{t+1} - H_\eta(t, x_t, u_t, p_{t+1}) \right)$$

where we left out  $x_T \in \mathcal{X}_T$  and  $u_t \in \mathcal{U}$ . Setting  $-\nabla_{x_T} L_\eta(x^*, u^*) \perp_{x_T^*} \mathcal{X}_T$  yields the *transversality condition*

$$-p_T^* - \eta \nabla \ell_T(x_T^*) \perp_{x_T^*} \mathcal{X}_T,$$

and setting  $-\nabla_{u_t} L(x^*, u^*) \perp_{u_t^*} \mathcal{U}$  yields the *weak maximum condition*

$$\nabla_u H_\eta(t, x_t^*, u_t^*, p_{t+1}^*) \perp_{u_t^*} \mathcal{U}, \quad \forall t \in \{0, 1, \dots, T-1\}.$$

We refer to this condition as “weak” since it is a necessary, but not sufficient condition for a solution of the problem

$$\underset{u \in \mathcal{U}}{\text{maximize}} \quad H_\eta(t, x_t^*, u, p_{t+1}^*).$$

## Pontryagin maximum principle (discrete-time)

Collect all of these necessary conditions together to get the Pontryagin maximum principle (PMP).

### Theorem (Pontryagin maximum principle (discrete-time))

Let  $(x^*, u^*)$  be a local minimum of the discrete-time OCP with terminal set  $\mathcal{X}_T$  and control set  $\mathcal{U}$ . Then  $\eta \in \{0, 1\}$  and  $\{p_t^*\}_{t=0}^T \subset \mathbb{R}^n$  exist such that

$$(\eta, p_0^*, p_1^*, \dots, p_T^*) \neq 0 \quad \text{non-triviality}$$

$$p_t^* = \nabla_x H_\eta(t, x_t^*, u_t^*, p_{t+1}^*), \quad \forall t \in \{0, 1, \dots, T-1\} \quad \text{adjoint equation}$$

$$-p_T^* - \eta \nabla \ell_T(x_T^*) \perp_{x_T^*} \mathcal{X}_T \quad \text{transversality}$$

$$\nabla_u H_\eta(t, x_t^*, u_t^*, p_{t+1}^*) \perp_{u_t^*} \mathcal{U}, \quad \forall t \in \{0, 1, \dots, T-1\} \quad \text{maximum condition (weak)}$$

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## Optimal control problem (continuous-time)

Consider the continuous-time optimal control problem (OCP)

$$\begin{array}{ll}\underset{x,u}{\text{minimize}} & \ell_T(x(T)) + \int_0^T \ell(t, x(t), u(t)) dt & \text{cost (terminal + stage)} \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t)), \quad \forall t \in [0, T] & \text{dynamical feasibility} \\ & x(0) = x_0 & \text{initial condition} \\ & x(T) \in \mathcal{X}_T & \text{terminal condition} \\ & u(t) \in \mathcal{U}, \quad \forall t \in [0, T] & \text{input constraints}\end{array}$$

An optimal control  $u^*(t)$  for a specific initial state  $x_0$  is an *open-loop* input.

An optimal control of the form  $u^*(t) = \pi^*(t, x(t))$  is a *closed-loop* input.



Consider piecewise continuous trajectories such that  $x(t) = x(t_k)$  and  $u(t) = u(t_k)$  for  $t \in [t_k, t_{k+1})$ , with  $k \in \{0, 1, \dots, N-1\}$ ,  $t_0 = 0$  and  $t_N = T$ .

Define  $\Delta t_k := t_{k+1} - t_k$  such that  $\Delta t_k > 0$  for all  $k \in \{0, 1, \dots, N-1\}$ .

Consider the discretized OCP

$$\begin{aligned} & \underset{x,u}{\text{minimize}} \quad \ell_T(x(t_N)) + \sum_{k=0}^{N-1} \Delta t_k \ell(t_k, x(t_k), u(t_k)) \\ & \text{subject to} \quad x(t_{k+1}) = x(t_k) + \Delta t_k f(t_k, x(t_k), u(t_k)), \quad \forall k \in \{0, 1, \dots, N-1\} \\ & \quad \quad \quad x(t_0) = x_0 \\ & \quad \quad \quad x(t_N) \in \mathcal{X}_T \\ & \quad \quad \quad u(t_k) \in \mathcal{U}, \quad \forall k \in \{0, 1, \dots, N-1\} \end{aligned}$$

Use the discrete-time PMP on a local minimum  $(x^*, u^*)$  of the discretized OCP to get

$$(\eta, p(t_0), p(t_1), \dots, p(t_N)) \neq 0$$

$$- \frac{(p^*(t_{k+1}) - p^*(t_k))}{\Delta t_k} = \nabla_x H_\eta(t_k, x^*(t_k), u^*(t_k), p^*(t_{k+1})), \quad \forall k \in \{0, 1, \dots, N-1\}$$

$$- p^*(t_N) - \eta \nabla \ell_T(x^*(t_N)) \perp_{x^*(t_N)} \mathcal{X}_T$$

$$\nabla_u H_\eta(t_k, x^*(t_k), u^*(t_k), p^*(t_{k+1})) \perp_{u_t^*} \mathcal{U}, \quad \forall k \in \{0, 1, \dots, N-1\}$$

where we use the *continuous-time Hamiltonian*

$$H_\eta(t, x, u, p) := p^\top f(t, x, u) - \eta \ell(t, x, u).$$

## Pontryagin maximum principle (continuous-time, weak)

The above conditions suggest the following continuous-time PMP as  $\Delta t_k \rightarrow 0$ .

### Theorem (Pontryagin maximum principle (continuous-time, weak))

Let  $(x^*, u^*)$  be a local minimum of the continuous-time optimal control problem with terminal set  $\mathcal{X}_T$  and control set  $\mathcal{U}$ . Then  $\eta \in \{0, 1\}$  and  $p^* : [0, T] \rightarrow \mathbb{R}^n$  exist such that

$$(\eta, p(t)) \not\equiv 0 \quad \text{non-triviality}$$

$$-\dot{p}^*(t) = \nabla_x H_\eta(t, x^*(t), u^*(t), p^*(t)), \quad \forall t \in [0, T] \quad \text{adjoint equation}$$

$$-p^*(T) - \eta \nabla \ell_T(x^*(T)) \perp_{x^*(T)} \mathcal{X}_T \quad \text{transversality}$$

$$H_\eta(t, x^*(t), u^*(t), p^*(t)) \perp_{u^*(t)} \mathcal{U}, \quad \forall t \in [0, T] \quad \text{maximum condition}$$

“( $\eta, p(t)$ )  $\not\equiv 0$ ” means there exists at least one  $t \in [0, T]$  such that  $(\eta, p(t)) \neq 0$ .

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Recall that  $(x^*, u^*)$  is a *local minimum* of  $J(x^*, u^*)$  if there exists  $\varepsilon > 0$  such that  $J(x^*, u^*) \leq J(x, u)$  for all  $(x, u)$  in the  $\varepsilon$ -sized norm ball around  $(x^*, u^*)$ .

In using the discrete-time PMP as a heuristic to obtain the continuous-time PMP, we are implicitly using the  $\mathcal{C}^0$ -norm for both  $x^*$  and  $u^*$ , i.e.,

$$\|x - x^*\|_{\mathcal{C}^0} := \max_{t \in [0, T]} \|x(t) - x^*(t)\|, \quad \|u - u^*\|_{\mathcal{C}^0} := \max_{t \in [0, T]} \|u(t) - u^*(t)\|.$$

We can strengthen the continuous-time PMP if we use the  $\mathcal{C}^0$ -norm for  $x^*$  and the  $\mathcal{L}^1$ -norm for  $u^*$ , i.e.,

$$\|x - x^*\|_{\mathcal{C}^0} := \max_{t \in [0, T]} \|x(t) - x^*(t)\|, \quad \|u - u^*\|_{\mathcal{L}^1} := \int_0^T \|u(t) - u^*(t)\| dt.$$

## Strengthening the maximum condition via needle perturbations

In general, the  $\mathcal{L}^1$ -norm ball for  $u^*$  allows for *large pointwise variations* at each time  $t$ . Suppose the control set  $\mathcal{U}$  is bounded, i.e.,  $\|u - v\| \leq c$  for all  $u, v \in \mathcal{U}$  and some  $c > 0$ .

Given some  $u^* : [0, T] \rightarrow \mathcal{U}$ , any  $\tau \in [0, T)$  and  $\varepsilon > 0$  such that  $[\tau, \tau + \varepsilon) \subset [0, T]$ , and any  $v \in \mathcal{U}$ , define

$$u(t) = \begin{cases} v, & t \in [\tau, \tau + \varepsilon) \\ u^*(t), & t \in [0, \tau) \cup [\tau + \varepsilon, T] \end{cases}$$

This is a *spatial needle perturbation* of  $u^*(t)$ . Then it can be shown that

$$\begin{aligned} \|u - u^*\|_{\mathcal{L}^1} &:= \int_0^T \|u(t) - u^*(t)\| dt = \int_{\tau}^{\tau+\varepsilon} \|v - u^*(t)\| dt \leq \int_{\tau}^{\tau+\varepsilon} c dt = \varepsilon c. \\ x(T) &\approx x^*(T) + \varepsilon d, \quad d \in \mathcal{T}_{\mathcal{X}_T}(x^*(T)) \end{aligned}$$

for small enough  $\varepsilon$ . Overall, a large temporal perturbation in  $u^*(t)$  can correspond to small feasible perturbations to both  $x^*$  and  $u^*$ .

## Pontryagin maximum principle (continuous-time)

The possibility of large temporal control perturbations still corresponding to “feasible neighbours” of  $(x^*, u^*)$  suggests the following strengthened PMP.

### Theorem (Pontryagin maximum principle (continuous-time))

*Let  $(x^*, u^*)$  be a local minimum (using the  $\mathcal{C}^0$ -norm and  $\mathcal{L}^1$ -norm, respectively) of the continuous-time OCP with terminal set  $\mathcal{X}_T$  and bounded control set  $\mathcal{U}$ . Then  $\eta \in \{0, 1\}$  and  $p^* : [0, T] \rightarrow \mathbb{R}^n$  exist such that*

$$(\eta, p^*(t)) \neq 0 \quad \text{non-triviality}$$

$$-\dot{p}^*(t) = \nabla_x H_\eta(t, x^*(t), u^*(t), p^*(t)), \quad \forall t \in [0, T] \quad \text{adjoint equation}$$

$$-p^*(T) - \eta \nabla \ell_T(x^*(T)) \perp_{x^*(T)} \mathcal{X}_T \quad \text{transversality}$$

$$H_\eta(t, x^*(t), u^*(t), p^*(t)) = \sup_{u \in \mathcal{U}} H_\eta(t, x^*(t), u, p^*(t)), \quad \forall t \in [0, T] \quad \text{maximum condition}$$

A rigorous proof relies on variational calculus ([Liberzon, 2012](#); [Clarke, 2013](#)).

## Example: Minimum fuel for a control-affine system

Consider the continuous-time OCP

$$\underset{x,u}{\text{minimize}} \quad \int_0^T \sum_{j=1}^m \alpha_j |u_j(t)| dt$$

$$\text{subject to } \dot{x}(t) = a(t, x(t)) + \sum_{j=1}^m u_j(t) b_j(t, x(t)), \quad \forall t \in [0, T]$$

$$x(0) = x_0$$

$$x(T) = 0$$

$$\underline{u} \preceq u(t) \preceq \bar{u}, \quad \forall t \in [0, T]$$

The Hamiltonian is

$$H_\eta(t, x, u, p) = p^\top \left( a(t, x) + \sum_{j=1}^m u_j b_j(t, x) \right) - \eta \sum_{i=1}^m \alpha_i |u_i|$$



## Example: Minimum fuel for a control-affine system

The Hamiltonian is

$$H_\eta(t, x, u, p) = p^\top a(t, x) + \sum_{j=1}^m \left( u_j p^\top b_j(t, x) - \eta \alpha_j |u_j| \right)$$

The adjoint equation is

$$\dot{p}^* = -\nabla_x H_\eta(t, x, u, p) = -\frac{\partial a}{\partial x}(t, x)p - \sum_{j=1}^m u_j \frac{\partial b_j}{\partial x}(t, x)p$$

The maximum condition is

$$u_j^* = \arg \max_{u_j \in [\underline{u}_j, \bar{u}_j]} \left( u_j p^\top b_j(t, x) + \eta \alpha_j |u_j| \right) = \begin{cases} \underline{u}_j, & p^\top b_j(t, x) > \eta \alpha_j \\ 0, & p^\top b_j(t, x) \in [-\eta \alpha_j, \eta \alpha_j] \\ \bar{u}_j, & p^\top b_j(t, x) < -\eta \alpha_j \end{cases}$$

which for  $\eta = 1$  is an example of “bang-off-bang” control.

## Example: Minimum fuel for a control-affine system

Assume  $\eta = 1$ , i.e., the “normal” case. Altogether, we have the *boundary value problem (BVP)*

$$\begin{pmatrix} \dot{x}^* \\ \dot{p}^* \end{pmatrix} = \begin{pmatrix} a(t, x^*) + \sum_{j=1}^m u_j^* b_j(t, x^*) \\ -\frac{\partial a}{\partial x}(t, x^*)p - \sum_{j=1}^m u_j^* \frac{\partial b_j}{\partial x}(t, x^*)p^* \end{pmatrix}, \quad u_j^* = \begin{cases} \underline{u}_j, & p^\top b_j(t, x) > \alpha_j \\ 0, & p^\top b_j(t, x) \in [-\alpha_j, \alpha_j] \\ \bar{u}_j, & p^\top b_j(t, x) < -\alpha_j \end{cases}$$

with boundary conditions  $x(0) = x_0$  and  $x(T) = 0$ .

Transversality did not factor into this problem, since the normal cone of the singleton  $\mathcal{X}_T = \{0\}$  is just  $\mathbb{R}^n$  (i.e., any direction “leaves” the terminal set).

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An *indirect method* generally focuses on solving the BVP

$$\begin{pmatrix} \dot{x}^* \\ \dot{p}^* \end{pmatrix} = \begin{pmatrix} f(t, x^*, u^*) \\ -\nabla_x H_\eta(t, x^*, u^*(t, x^*, p^*), p^*) \end{pmatrix}, \quad x^*(0) = x_0, \quad h(x^*(T), p^*(T)) = 0.$$

where  $h(x^*(T), p^*(T)) \in \mathbb{R}^n$ . The *open-loop* optimal control candidate  $u^*(t, x^*(t), p^*(t))$  is then extracted.

The boundary condition  $h(x^*(T), p^*(T)) = 0$  is determined by the terminal set constraint  $x^*(T) \in \mathcal{X}_T$  and the transversality condition  $-p^*(T) - \eta \nabla \ell_T(x^*(T)) \perp_{x^*(T)} \mathcal{X}_T$ .

We are implicitly assuming an optimal control exists. Even then, there may be multiple local optima.

## Shooting methods

To solve the BVP

$$\begin{pmatrix} \dot{x}^* \\ \dot{p}^* \end{pmatrix} = \begin{pmatrix} f(t, x^*, u^*) \\ -\nabla_x H_\eta(t, x^*, u^*(t, x^*, p^*), p^*) \end{pmatrix}, \quad x^*(0) = x_0, \quad h(x^*(T), p^*(T)) = 0,$$

we consider the associated initial value problem (IVP)

$$\begin{pmatrix} \dot{x}^* \\ \dot{p}^* \end{pmatrix} = \begin{pmatrix} f(t, x^*, u^*) \\ -\nabla_x H_\eta(t, x^*, u^*(t, x^*, p^*), p^*) \end{pmatrix}, \quad x^*(0) = x_0, \quad p^*(0) = p_0.$$

We can integrate the IVP forward in time to get  $x^*(T; p_0)$  and  $p^*(T; p_0)$ , which are parameterized by  $p_0$ .

We can use a root-finding method (e.g., bisection search, Newton-Raphson method) to find  $p_0$  such that  $h(x^*(T; p_0), p^*(T; p_0)) = 0$ . This is called *single shooting* and gives us a solution of the BVP.

# Agenda

1. Geometry and generalizations of first-order NOCs
2. Weak Pontryagin maximum principle in discrete-time
3. Weak Pontryagin maximum principle in continuous-time
4. Pontryagin maximum principle in continuous-time
5. Indirect methods for optimal control
6. Time-optimal control problems

Consider the continuous-time OCP

$$\begin{array}{ll}\underset{x,u,T \geq 0}{\text{minimize}} & \ell_T(T, x(T)) + \int_0^T \ell(t, x(t), u(t)) dt & \text{cost (terminal + stage)} \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t)), \quad \forall t \in [0, T] & \text{dynamical feasibility} \\ & x(0) = x_0 & \text{initial condition} \\ & x(T) \in \mathcal{X}_T & \text{terminal condition} \\ & u(t) \in \mathcal{U}, \quad \forall t \in [0, T] & \text{input constraints}\end{array}$$

The final time  $T$  is now a *free variable* (subject to  $T \geq 0$ ).

Use the change of variables  $t(s) = Ts$  with  $s \in [0, 1]$  to get

$$\begin{aligned} & \underset{(x,t),(u,T)}{\text{minimize}} && \ell_T(t(1), x(1)) + T \int_0^1 \ell(t(s), x(s), u(s)) ds && \text{cost (terminal + stage)} \\ & \text{subject to} && \dot{x}(s) = Tf(t(s), x(s), u(s)), \quad \dot{t}(s) = T, \quad \forall s \in [0, 1] && \text{dynamical feasibility} \\ & && x(0) = x_0, \quad t(0) = 0 && \text{initial condition} \\ & && x(1) \in \mathcal{X}_T && \text{terminal condition} \\ & && u(s) \in \mathcal{U}, \quad T \in [0, \infty), \quad \forall s \in [0, 1] && \text{input constraints} \end{aligned}$$

We treat  $t$  and  $T$  as a new state and input, respectively. We can then apply the PMP.



## Theorem (Pontryagin maximum principle (continuous-time, free final time))

Let  $(x^*, u^*, T^*)$  be a local minimum (using the  $\mathcal{C}^0$ -norm,  $\mathcal{L}^1$ -norm, and absolute value, respectively) of the continuous-time OCP with terminal set  $\mathcal{X}_T$ , bounded control set  $\mathcal{U}$ , and free final time  $T \geq 0$ . Then  $\eta \in \{0, 1\}$  and  $p^* : [0, T^*] \rightarrow \mathbb{R}^n$  exist such that

$$(\eta, p^*(t)) \neq 0 \quad \text{non-triviality}$$

$$-\dot{p}^*(t) = \nabla_x H_\eta(t, x^*(t), u^*(t), p^*(t)), \quad \forall t \in [0, T^*] \quad \text{adjoint equation}$$

$$-p^*(T^*) - \eta \nabla \ell_T(x^*(T^*)) \perp_{x^*(T)} \mathcal{X}_T \quad \text{transversality}$$

$$H_\eta(t, x^*(t), u^*(t), p^*(t)) = \sup_{u \in \mathcal{U}} H_\eta(t, x^*(t), u, p^*(t)), \quad \forall t \in [0, T^*] \quad \text{maximum condition}$$

$$H_\eta(T^*, x^*(T^*), u^*(T^*), p^*(T^*)) = \eta \frac{\partial \ell_T}{\partial t}(T^*, x^*(T^*)) \quad \begin{array}{l} \text{maximum condition} \\ \text{(boundary)} \end{array}$$

Direct methods for optimal control  
(i.e., solving discretized optimal control problems directly)

- F. Clarke. *Functional Analysis, Calculus of Variations and Optimal Control*. Springer, 2013.
- D. Liberzon. *Calculus of Variations and Optimal Control Theory: A Concise Introduction*. Princeton University Press, 2012.