AA203
Optimal and Learning-based Control

Course overview; control, stability, performance metrics

Stanford University

Autonomous Systems Lab

Course mechanics

Teaching team:

- Instructors: Ed Schmerling (OH: W 11am-12pm; Project OH: W 4:30-5:30pm)
  James Harrison (OH: M 10-11am; Project OH: Th 2-3pm)
- CAs: Matt Tsao and Spencer M. Richards (OH: Tu 4-6pm, Th 8:30-10:30am)

Logistics:

- Class info, lectures, and homework assignments on class web page: http://asl.stanford.edu/aa203/
- Forum: http://piazza.com/stanford/spring2021/aa203
- For urgent questions: aa203-spr2021-staff@lists.stanford.edu

Course requirements

- Homework: there will be a total of four problem sets
- Homework submissions: https://www.gradescope.com/courses/257531
- Final project (details on the course website)
- Grading:
  - homework 60% (15% per HW)
  - final project 40%

Course material

- Course notes: a set of course notes will be provided covering all the content presented in the lectures

- Recitations: Friday lecture sessions (F 10:30-11:50AM, weeks 2—5) led by the CAs covering relevant tools (computational and mathematical)

- Textbooks that may be valuable for context or further reference are listed in the syllabus

Prerequisites

- Strong familiarity with calculus (e.g., CME100)
- Strong familiarity with linear algebra (e.g., EE263 or CME200)
- Familiarity with optimization (e.g., EE364a, CME307, CS269o, AA222)
- To get the most out of this class, at least one of:
  - A course in machine learning (e.g., CS229, CS230, CS231n)
    or
  - A course in control (e.g., ENGR105, ENGR205, AA212)

Homework 0 (ungraded) is out now to gauge preparedness.

Today's Outline

1. Context and course goals

2. State-space models

3. Problem formulation for optimal control

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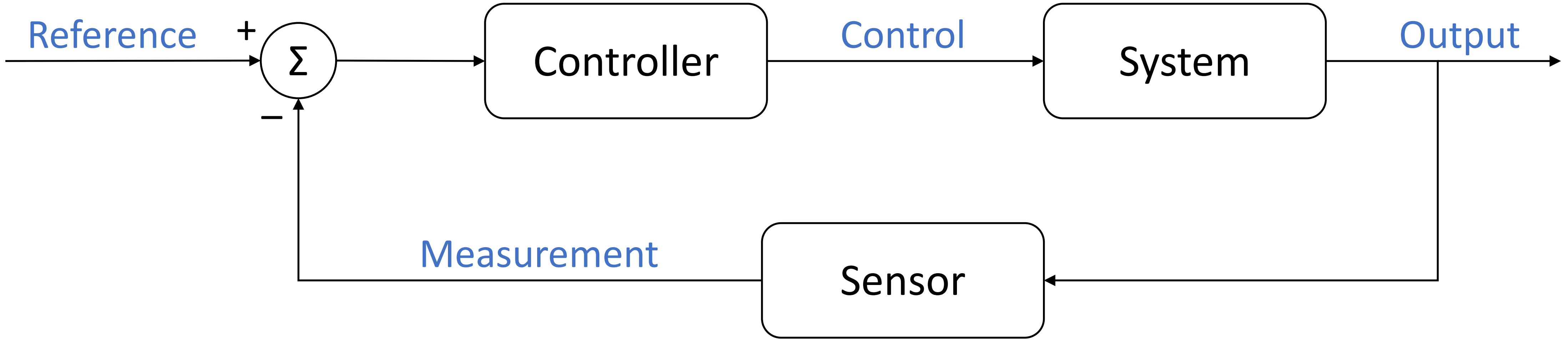
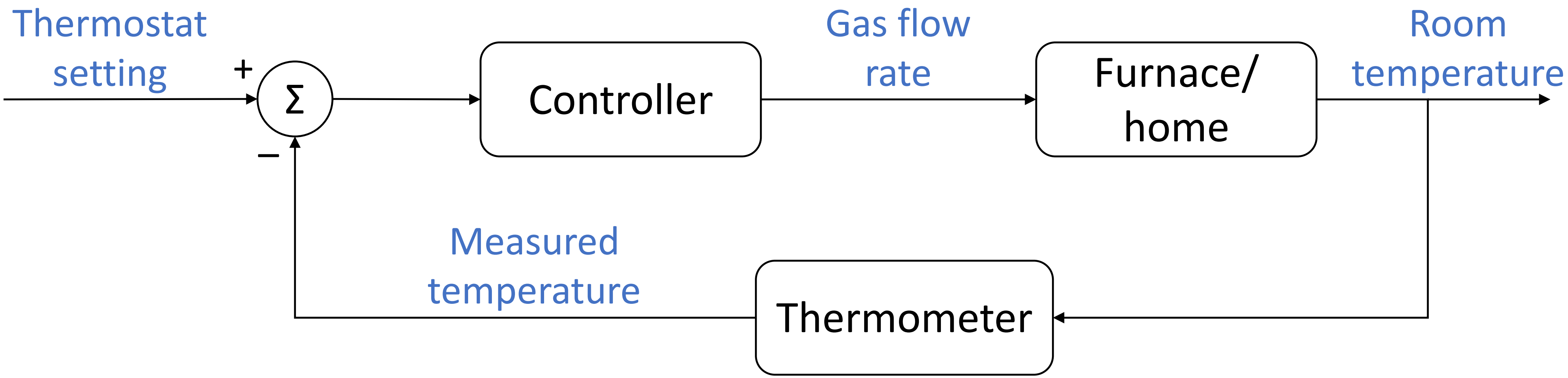
Feedback control

- Tracking a reference signal



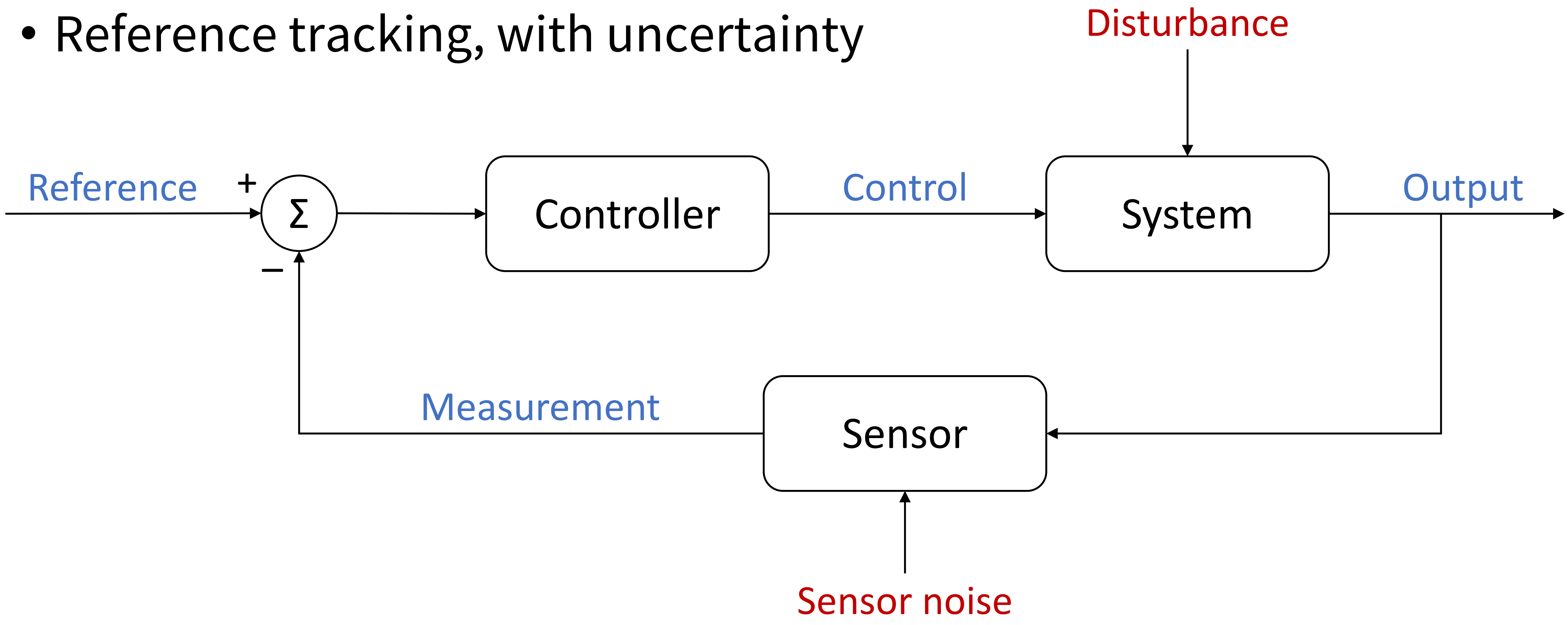
Feedback control

- Tracking a reference signal



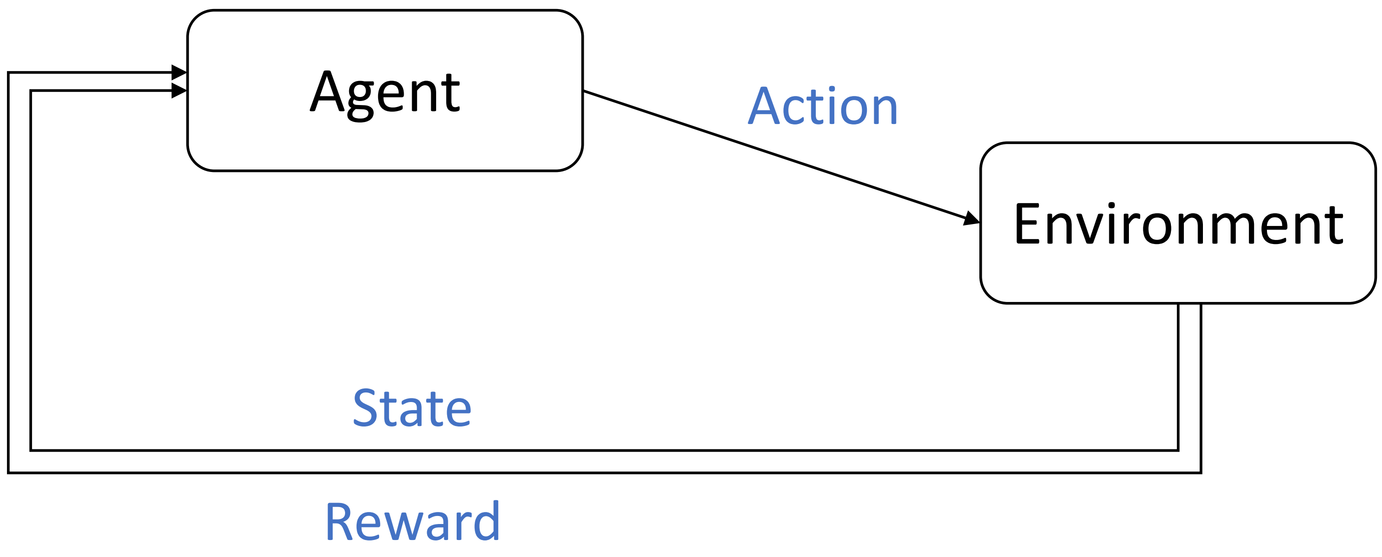
Feedback control

- Reference tracking, with uncertainty



Reinforcement learning

- A brief aside…



Feedback control desiderata

- Stability: multiple notions; loosely system output is “under control”
- Tracking: the output should track the reference “as closely as possible”
- Disturbance rejection: the output should be “as insensitive as possible” to disturbances/noise
- Robustness: controller should still perform well up to “some degree of” model misspecification

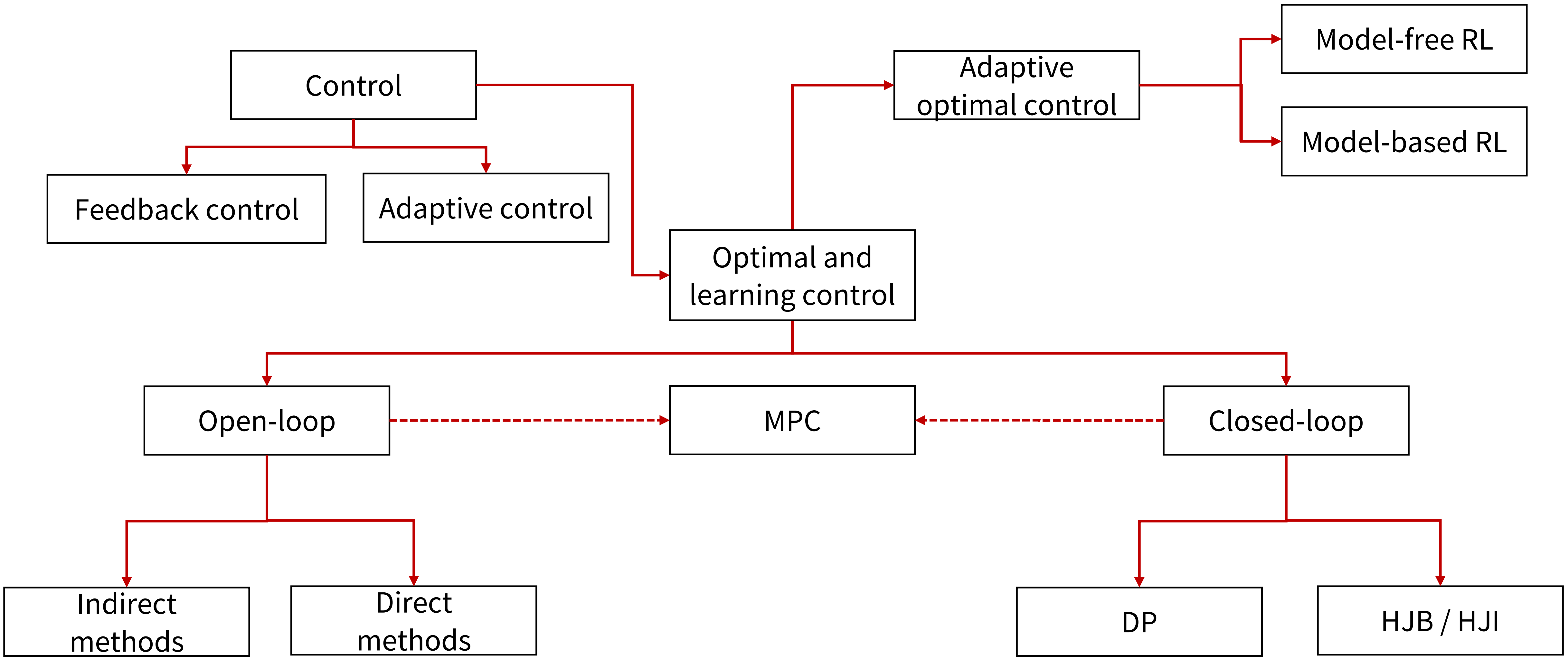
What’s missing?

- Performance: mathematical quantification of the above desiderata, and providing a control that best realizes the tradeoffs between them

- Planning: providing an appropriate reference trajectory for the controller to track (particularly nontrivial, e.g., when controlling mobile robots)

- Learning: a controller that adapts to an initially unknown, or possibly time-varying system

Course overview



Course goals

To learn the *theoretical* and *implementation* aspects of main techniques in optimal and learning-based control

To provide a *unified framework and context* for understanding and relating these techniques to each other

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Mathematical model

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t) \\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t) \\
&\ \,\vdots \\
\dot{x}_n(t) &= f_n(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t)
\end{aligned}
$$

Where

- $x_1(t), x_2(t), \ldots, x_n(t)$ are the state variables
- $u_1(t), u_2(t), \ldots, u_m(t)$ are the control inputs

Mathematical model

In compact form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

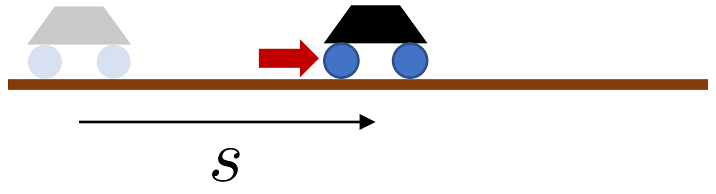
- a history of control input values during the interval $[t_0, t_f]$ is called a *control history* and is denoted by **u**
- a history of state values during the interval $[t_0, t_f]$ is called a *state trajectory* and is denoted by **x**

Illustrative example

- Double integrator: point mass under controlled acceleration

$$\ddot{s}(t) = a(t)$$

$$\begin{bmatrix} \dot{s} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ a \end{bmatrix}$$



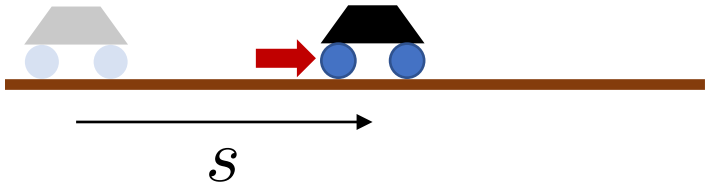
Example system

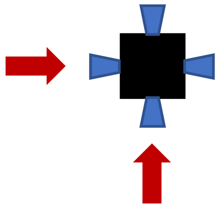
- Double integrator: point mass under controlled acceleration

$$\begin{bmatrix} \dot{s} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [a]$$

$$\dot{\mathbf{x}}(t) = A \quad \mathbf{x}(t) + B \quad \mathbf{u}(t)$$

$$\begin{bmatrix} \dot{\mathbf{s}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} [\mathbf{a}]$$

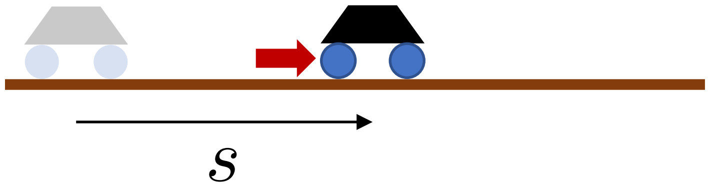




Example controller

Let's drive from $[5, 0]^T$ to $[0, 0]^T$.

Proposal: use a linear feedback control law.



$$a = -k_p s - k_d v$$

$$\begin{bmatrix} \dot{s} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_p & k_d \end{bmatrix} \begin{bmatrix} s \\ v \end{bmatrix}$$

$$\begin{bmatrix} \dot{s} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} \begin{bmatrix} s \\ v \end{bmatrix} \qquad \left(\dot{\mathbf{x}}(t) = (A - BK)\mathbf{x}(t) \right)$$

Analyzing stability

$$\begin{bmatrix} \dot{s} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} \begin{bmatrix} s \\ v \end{bmatrix}$$

$$\begin{bmatrix} s(t) \\ v(t) \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} t \right) \begin{bmatrix} s(0) \\ v(0) \end{bmatrix}$$

$$\begin{bmatrix} s(t) \\ v(t) \end{bmatrix} = V^{-1} \begin{bmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{bmatrix} V \begin{bmatrix} s(0) \\ v(0) \end{bmatrix}$$

$$\text{where } \lambda_\pm = \left(-k_d \pm \sqrt{k_d^2 - 4k_p} \right) / 2$$

or

$$\begin{bmatrix} s(t) \\ v(t) \end{bmatrix} = e^{\lambda t} V^{-1} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} V \begin{bmatrix} s(0) \\ v(0) \end{bmatrix}$$

$$\text{where } \lambda = -k_d/2, \text{ if } k_d^2 - 4k_p = 0$$

Analyzing stability

$$\begin{bmatrix} s(t) \\ v(t) \end{bmatrix} = V^{-1} \begin{bmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{bmatrix} V \begin{bmatrix} s(0) \\ v(0) \end{bmatrix}$$

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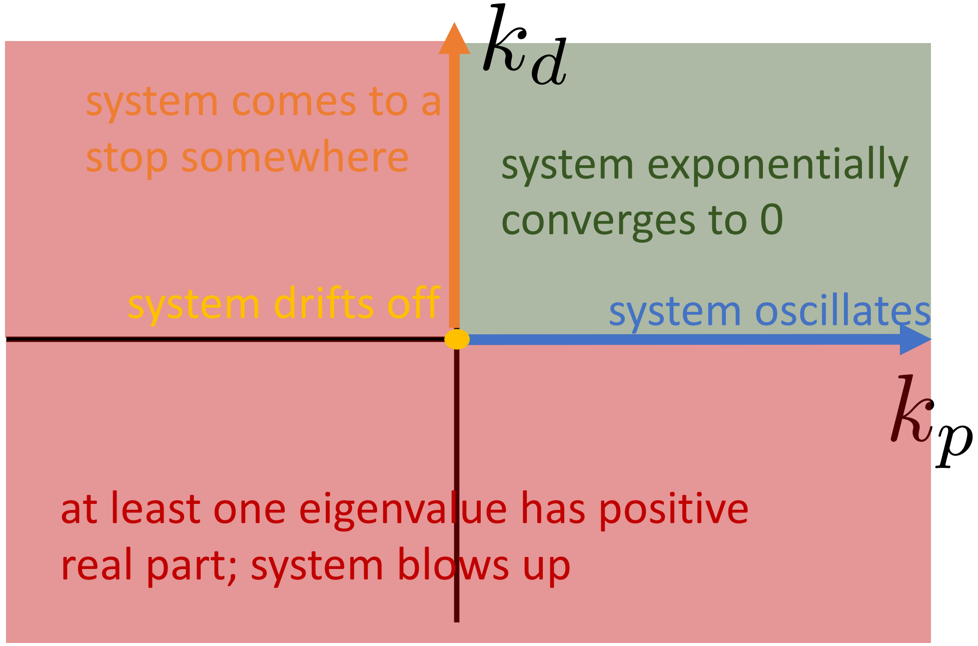
or

$$\begin{bmatrix} s(t) \\ v(t) \end{bmatrix} = e^{\lambda t} V^{-1} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} V \begin{bmatrix} s(0) \\ v(0) \end{bmatrix}$$

$$\text{where } \lambda = -k_d/2, \text{ if } k_d^2 - 4k_p = 0$$

$\mathrm{Re}(\lambda)$ → exponential growth (> 0), exponential decay (< 0), or constant (=0)

$\mathrm{Im}(\lambda)$ → sinusoidal oscillation



Mathematical definitions of stability

Many notions:

- Asymptotic stability
  - Global: all trajectories converge to the equilibrium
  - Local: all trajectories starting near the equilibrium converge to the equilibrium
- Exponential stability
  - Same as asymptotic stability, but with exponential rate
- Marginal stability
- Bounded-input, bounded-output stability
- Lyapunov stability

Quantifying performance

$$
\begin{aligned}
\min \int_0^{t_f} & \|\mathbf{x}(t)\|_2 + \|\mathbf{u}(t)\|_2 dt \\
\text{s.t.} \quad & \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\
& \mathbf{x}(0) = \mathbf{x_0}
\end{aligned}
$$

Quantifying performance

$$
\begin{aligned}
\min \int_0^{t_f} & \|\mathbf{x}(t)\|_2 + \|\mathbf{u}(t)\|_2 dt \\
\text{s.t.} \quad & \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\
& \mathbf{x}(0) = \mathbf{x_0},\ \mathbf{x}(t_f) = \mathbf{x_f}
\end{aligned}
$$

Quantifying performance

$$\min \int_0^{t_f} \mathbf{x}(t)^T Q \mathbf{x}(t) + \mathbf{u}(t)^T R \mathbf{u}(t) dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

$$\mathbf{x}(0) = \mathbf{x_0}, \; \mathbf{x}(t_f) = \mathbf{x_f}$$

Quantifying performance

$$\min \int_0^{t_f} \mathbf{x}(t)^T Q \mathbf{x}(t) + \|\mathbf{u}(t)\|_1 dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

$$\mathbf{x}(0) = \mathbf{x_0}, \ \mathbf{x}(t_f) = \mathbf{x_f}$$

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Problem formulation

- Mathematical description of the system to be controlled
- Statement of the constraints
- Specification of a performance criterion

Performance measure

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

- h and g are scalar functions
- t_f may be specified or free

Constraints

- initial and final conditions (boundary conditions)

$$\mathbf{x}(t_0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f$$

- constraints on state trajectories

$$\underline{X} \leq \mathbf{x}(t) \leq \overline{X}$$

- control authority

$$\underline{U} \leq \mathbf{u}(t) \leq \overline{U}$$

- and many more...

Constraints

- A control history which satisfies the control constraints during the entire time interval $[t_0, t_f]$ is called an admissible control
- A state trajectory which satisfies the state variable constraints during the entire time interval $[t_0, t_f]$ is called an admissible trajectory

Optimal control problem

Find an *admissible control* $\mathbf{u}^*$ which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

to follow an *admissible trajectory* $\mathbf{x}^*$ that minimizes the performance measure

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

Optimal control problem

Comments:

- minimizer $(\mathbf{x}^*, \mathbf{u}^*)$ called optimal trajectory-control pair
- existence: in general, not guaranteed
- uniqueness: optimal control may not be unique
- minimality: we are seeking a global minimum
- for maximization, we rewrite the problem as $\min_{\mathbf{u}} -J$

Form of optimal control

1. if $\mathbf{u}^* = \pi(\mathbf{x}(t), t)$, then π is called optimal control law or optimal policy (*closed-loop*)
   - important example: $\pi(\mathbf{x}(t), t) = F\,\mathbf{x}(t)$

2. if $\mathbf{u}^* = e(\mathbf{x}(t_0), t)$, then the optimal control is *open-loop*
   - optimal *only* for a particular initial state value

Discrete-time formulation

- System: $\mathbf{x}_{k+1} \;= \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k), \quad k = 0, \dots, N-1$
- Control constraints: $\mathbf{u}_k \in U$
- Cost:

$$J(\mathbf{x}_0; \boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1}) \;=\; h_N(\mathbf{x}_N) + \sum_{k=0}^{N-1} g_k(\mathbf{x}_k, \mathbf{u}_k, k)$$

- Decision-making problem:

$$J^*(\mathbf{x}_0) \;= \min_{\mathbf{u}_k \in U,\, k=0,\dots,N-1} J(\mathbf{x}_0; \boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1})$$

<u>Extension to stochastic setting will be covered later in the course</u>

Next class

Introduction to learning;

System identification and adaptive control