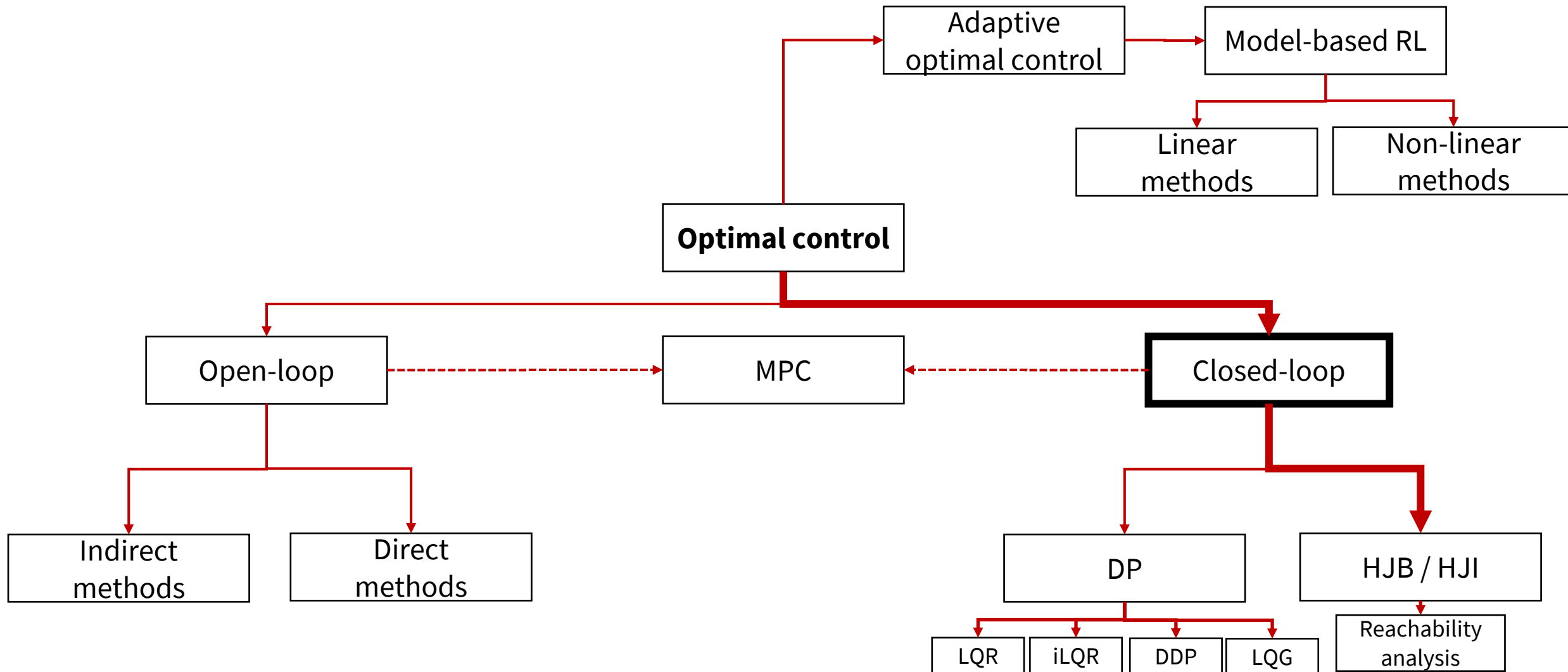


AA203

Optimal and Learning-based Control

HJI Equation and reachability analysis*

Roadmap



Two-person, zero-sum differential games

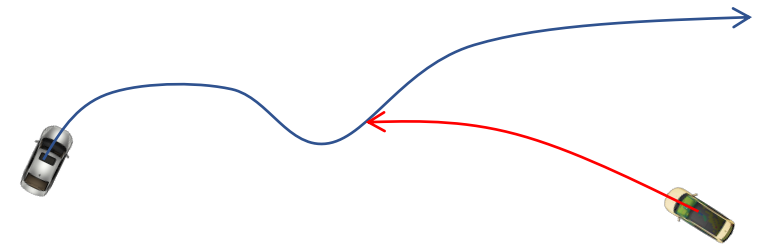
- What if there is another player (e.g., nature) that interferes with the fulfillment of our objective?

Two person differential game:

- Model: $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t))$ (*joint* system dynamics),
- Cost: $J(\mathbf{x}(t)) = h(\mathbf{x}(0)) + \int_t^0 g(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)) d\tau$
- Player 1, with control $\mathbf{u}(\tau)$, will attempt to maximize J , while Player 2, with control $\mathbf{d}(t)$, will aim to minimize J , subject to the *joint* system dynamics
- $\mathbf{x}(\tau)$ is the *joint* system state

Information pattern

- To fully specify the game, we need to specify the *information pattern*
- “Open-loop” strategies
 - Player 1, with control $\mathbf{u}(\tau)$, declares entire plan
 - Player 2, with control $\mathbf{d}(\tau)$, responds optimally
 - Conservative, unrealistic, but computationally cheap
- “Non-anticipative” strategies
 - Other robot acts based on state and control trajectory up to current time
 - Notation: $\mathbf{d}(\cdot) = \Gamma[\mathbf{u}](\cdot)$
 - Disturbance still has the advantage: it gets to react to the control!



HJI equation

Key idea: apply principle of optimality

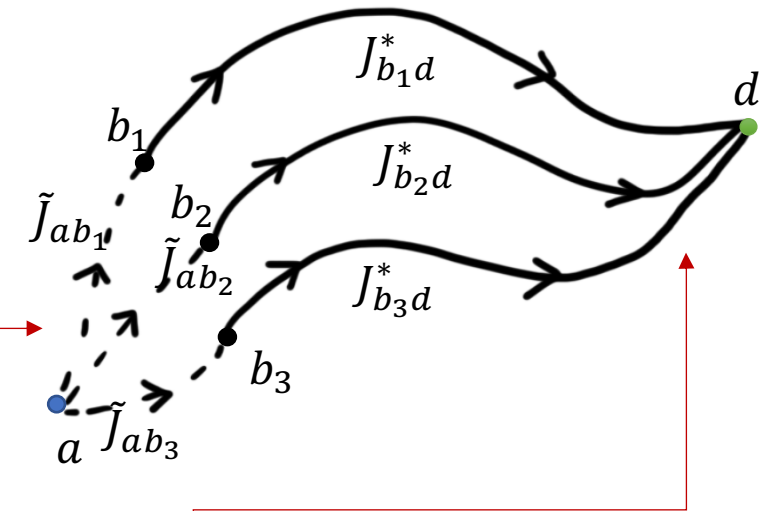
The “truncated” (lower-value) problem is

$$J(\mathbf{x}(t), t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\int_t^0 g(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)) d\tau + h(\mathbf{x}(0)) \right]$$

 Worst-case disturbance -- does the opposite of the control

HJI equation

- Dynamic programming principle:



$$J(\mathbf{x}(t), t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\int_t^{t+\Delta t} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)) d\tau + J(\mathbf{x}(t + \Delta t), t + \Delta t) \right]$$

- Approximate integral and Taylor expand $J(\mathbf{x}(t + \Delta t), t + \Delta t)$
- Derive Hamilton-Jacobi-Isaacs partial differential equation (HJI PDE)

HJI equation

- Approximations for small Δt :

$$J(\mathbf{x}(t), t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\underbrace{\int_t^{t+\Delta t} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)) d\tau}_{g(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t))\Delta t} + \underbrace{J(\mathbf{x}(t + \Delta t), t + \Delta t)}_{J(\mathbf{x}(t), t) + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t)) + \frac{\partial V}{\partial t} \Delta t} \right]$$

- Omit t dependence...

$$J(\mathbf{x}, t) = \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d})\Delta t + J(\mathbf{x}, t) + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial t} \Delta t \right]$$

- Assume constant u and $d \rightarrow$ Optimization over vectors, not functions!
- Order of max and min reverse: disturbance has the advantage

- $J(\mathbf{x}, t)$ does not depend on \mathbf{u} or \mathbf{d}

$$J(\mathbf{x}, t) = J(\mathbf{x}, t) + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d})\Delta t + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial t} \Delta t \right]$$

HJI equation

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HJI equation

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- $J(\mathbf{x}, t)$ does not depend on \mathbf{u} or \mathbf{d}

$$0 = \frac{\partial J}{\partial t} \Delta t + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d})\Delta t + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right]$$

HJI equation

- Approximations for small Δt :

$$J(\mathbf{x}(t), t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\underbrace{\int_t^{t+\Delta t} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)) d\tau}_{g(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t))\Delta t} + \underbrace{J(\mathbf{x}(t + \Delta t), t + \Delta t)}_{J(\mathbf{x}(t), t) + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t)) + \frac{\partial V}{\partial t} \Delta t} \right]$$

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HJI equation

The end result is the Hamilton-Jacobi-Isaacs (HJI) equation

$$0 = \frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right]$$

with boundary condition

$$J(\mathbf{x}, 0) = h(\mathbf{x})$$

- Given the cost-to-go function, the optimal control for Player 1 is

$$\mathbf{u}^*(\mathbf{x}, t) = \arg \max_{\mathbf{u}} \min_{\mathbf{d}} g(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

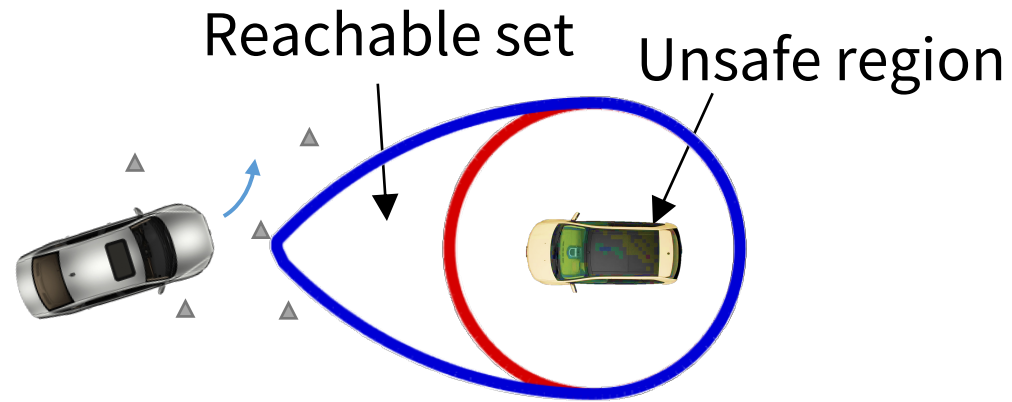
Applications of differential games

- Pursuit-evasion games
 - homicidal chauffeur problem
 - the lady in the lake
- Reachability analysis
- And many more (e.g., in economics)

Applications of differential games

- Pursuit-evasion games
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Reachability analysis: avoidance



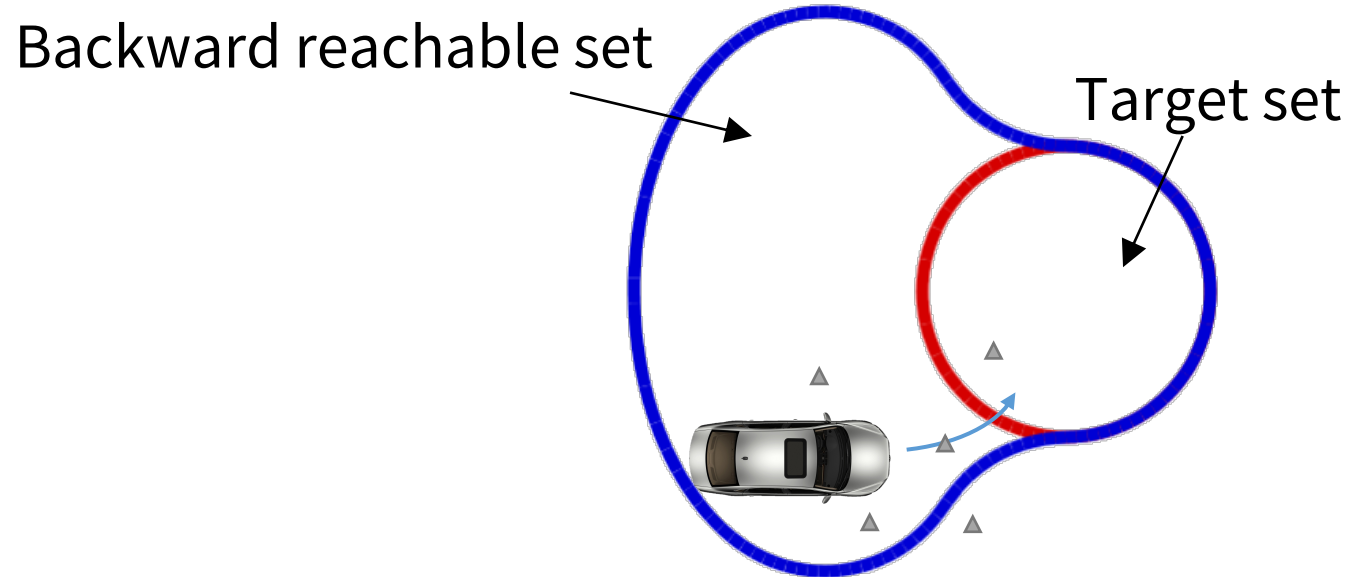
Assumptions:

- Model of robot
- Unsafe region: e.g., obstacle

Control policy

Backward reachable set
(States leading to danger)

Reachability analysis: goal reaching



- Model of robot
- Goal region



Control policy

Backward reachable set
(States leading to goal)

Reachability analysis

- Model of robot
- Unsafe region



$$\mathcal{A}(t) = \{\bar{\mathbf{x}}: \exists \Gamma[\mathbf{u}](\cdot), \forall \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$$

Backward reachable set (states leading to danger)

Control policy

- $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d})$
- \mathcal{T}

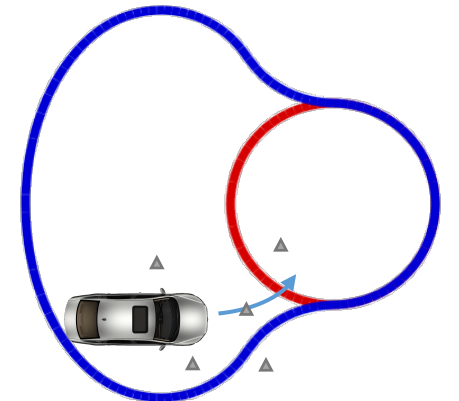
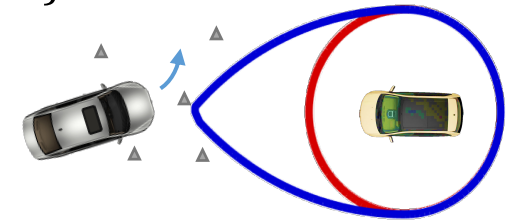
$$\mathbf{u}^*(\mathbf{x}, t)$$

Control policy

Backward reachable set (states leading to goal)

$$\mathcal{R}(t) = \{\bar{\mathbf{x}}: \forall \Gamma[\mathbf{u}](\cdot), \exists \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$$

- Model of robot
- Goal region



Reachability analysis

States at time t satisfying the following:

there exists a disturbance such that for all control, system enters target set at $t = 0$

$$\mathcal{A}(t) = \{\bar{\mathbf{x}}: \exists \Gamma[\mathbf{u}](\cdot), \forall \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$$

- Model of robot
- Unsafe region



Backward reachable set (States leading to danger)

Control policy

- $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d})$
- \mathcal{T}

$$\mathbf{u}^*(\mathbf{x}, t)$$

- Model of robot
- Goal region



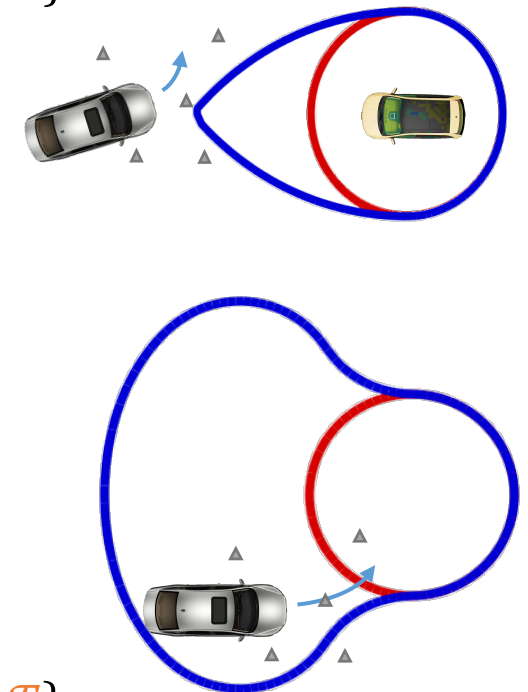
Control policy

Backward reachable set (States leading to goal)

$$\mathcal{R}(t) = \{\bar{\mathbf{x}}: \forall \Gamma[\mathbf{u}](\cdot), \exists \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$$

States at time t satisfying the following:

for all disturbances, there exists a control such that system enters target set at $t = 0$



From HJI to reachability analysis

- Computation of the BRS entails solving a differential *game of kind*, where the outcome is Boolean (the system either reaches the target set or not)
- One can “encode” this Boolean outcome by (1) removing the running cost and (2) picking the final cost intelligently

From HJI to reachability analysis

- Hamilton-Jacobi Equation

- $0 = \frac{\partial J}{\partial t} + \max_{\mathbf{d}} \min_{\mathbf{u}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right], J(\mathbf{x}, 0) = h(\mathbf{x})$

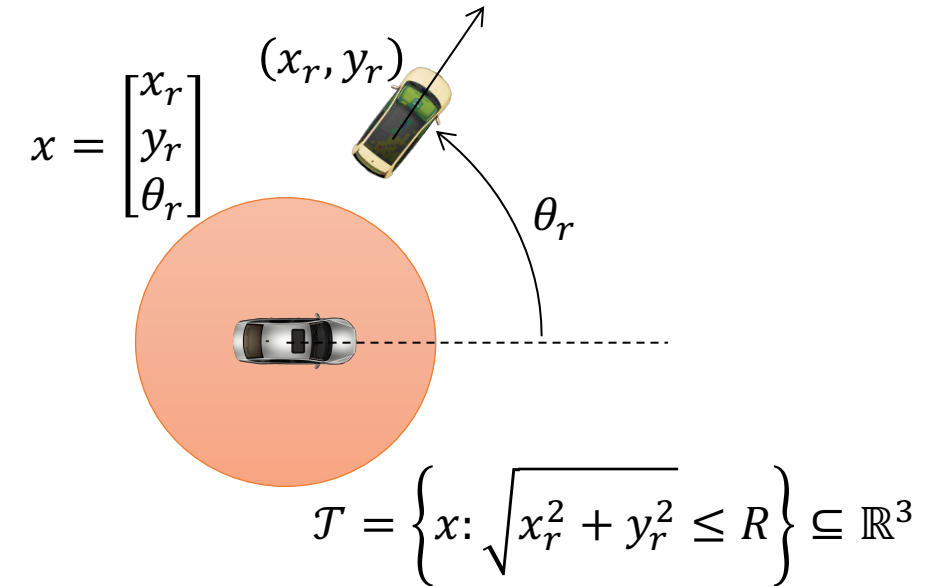
- Remove running cost

- $0 = \frac{\partial J}{\partial t} + \max_{\mathbf{d}} \min_{\mathbf{u}} \left[\frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right], J(\mathbf{x}, 0) = h(\mathbf{x})$

- Pick final cost such that

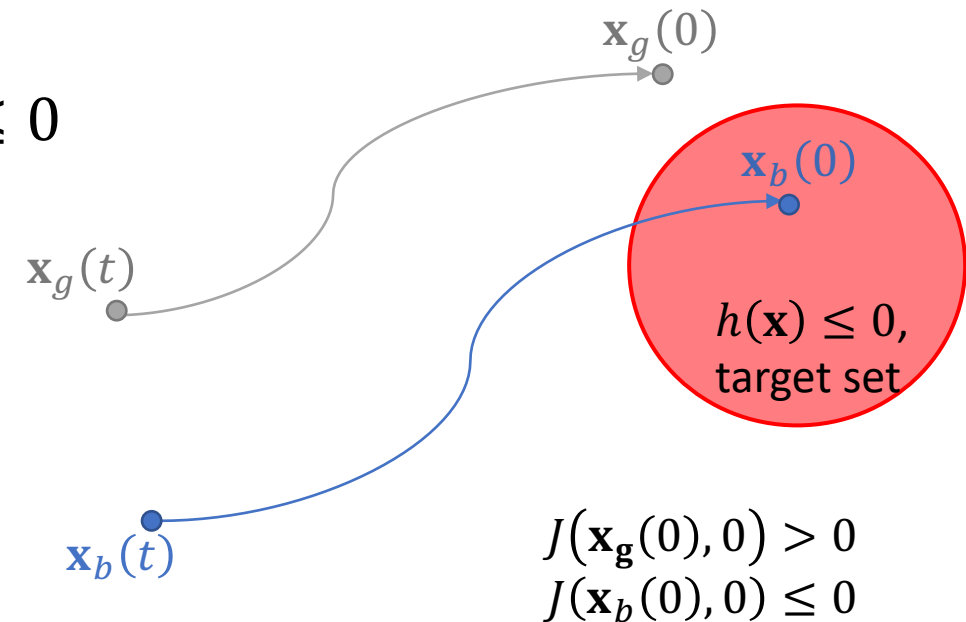
- $\mathbf{x} \in \mathcal{T} \Leftrightarrow h(\mathbf{x}) \leq 0$

- Example: If $\mathcal{T} = \{\mathbf{x}: \sqrt{x_r^2 + y_r^2} \leq R\} \subseteq \mathbb{R}^3$, we can pick $h(x_r, y_r, \theta_r) = \sqrt{x_r^2 + y_r^2} - R$



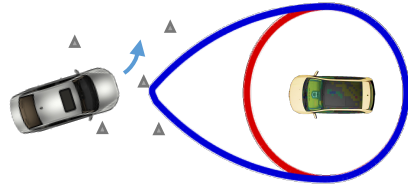
Pick Final Cost

- Pick final cost such that
 - $x \in \mathcal{T} \Leftrightarrow h(\mathbf{x}) \leq 0$
 - If $\mathcal{T} = \{x: \sqrt{x_r^2 + y_r^2} \leq R\} \subseteq \mathbb{R}^3$, we can pick $h(x_r, y_r, \theta_r) = \sqrt{x_r^2 + y_r^2} - R$
- Why is this correct?
 - Final state $\mathbf{x}(0)$ is in \mathcal{T} if and only if $h(\mathbf{x}(0)) \leq 0$
 - To avoid \mathcal{T} , control should maximize $h(\mathbf{x}(0))$
 - Worst-case disturbance would minimize
 - $J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} h(\mathbf{x}(0))$



Reaching vs. Avoiding

- Avoiding danger



- BRS definition
 $\mathcal{A}(t) = \{\bar{\mathbf{x}}: \exists \Gamma[\mathbf{u}](\cdot), \forall \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$

- Value function

$$J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} h(\mathbf{x}(0))$$

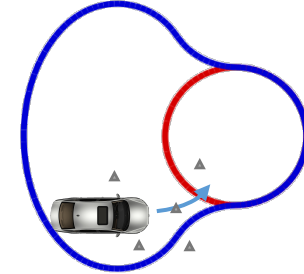
- HJI

$$\frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}} \right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right] = 0$$

- Optimal control

$$\mathbf{u}^* = \arg \max_{\mathbf{u}} \min_{\mathbf{d}} \left(\frac{\partial J}{\partial \mathbf{x}} \right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

- Reaching a goal



- BRS definition
 $\mathcal{R}(t) = \{\bar{\mathbf{x}}: \forall \Gamma[\mathbf{u}](\cdot), \exists \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$

- Value function

$$J(\mathbf{x}, t) = \max_{\Gamma[\mathbf{u}]} \min_{\mathbf{u}} h(\mathbf{x}(0))$$

- HJI

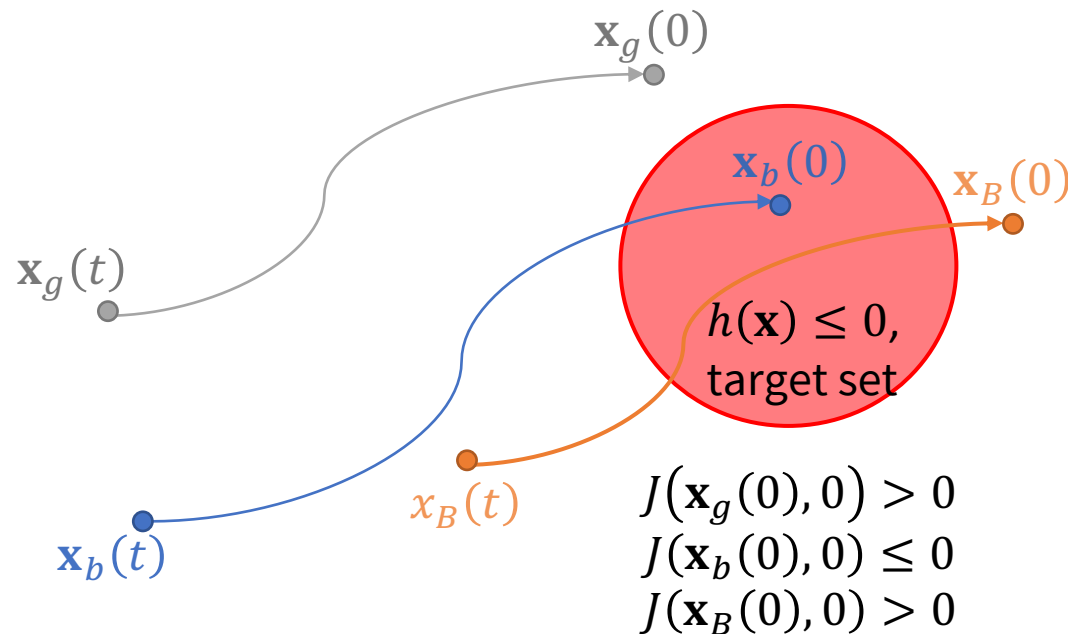
$$\frac{\partial J}{\partial t} + \min_{\mathbf{u}} \max_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}} \right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right] = 0$$

- Optimal control

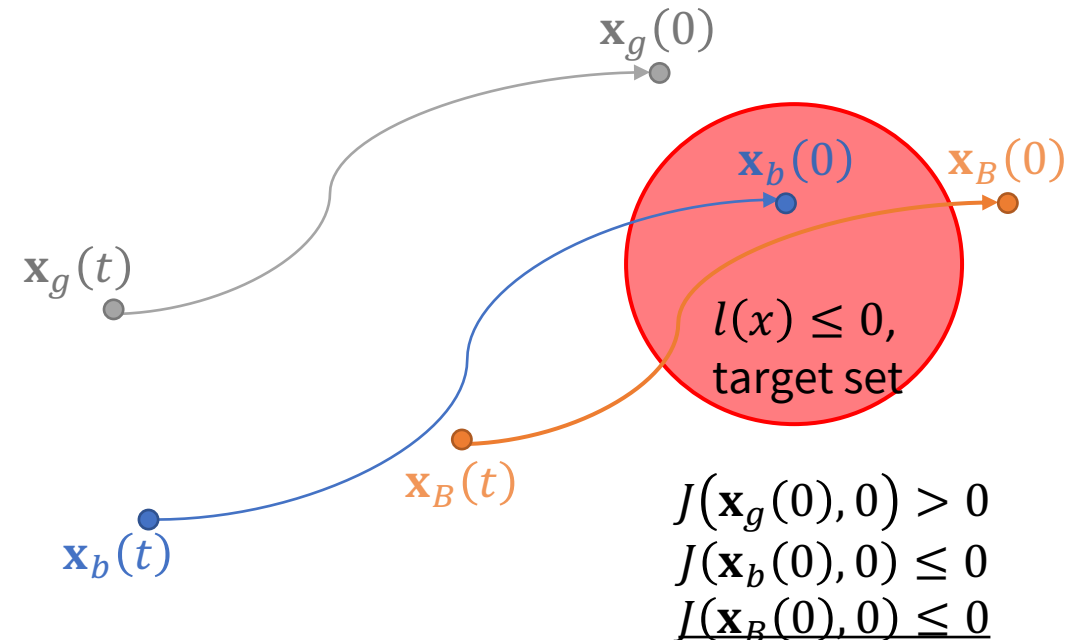
$$\mathbf{u}^* = \arg \min_{\mathbf{u}} \max_{\mathbf{d}} \left(\frac{\partial J}{\partial \mathbf{x}} \right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

“Sets” vs. “Tubes”

- Backward reachable set (BRS)
 - Only final time matters
 - Initial states that passing through target are not necessarily in BRS
 - Not ideal for safety

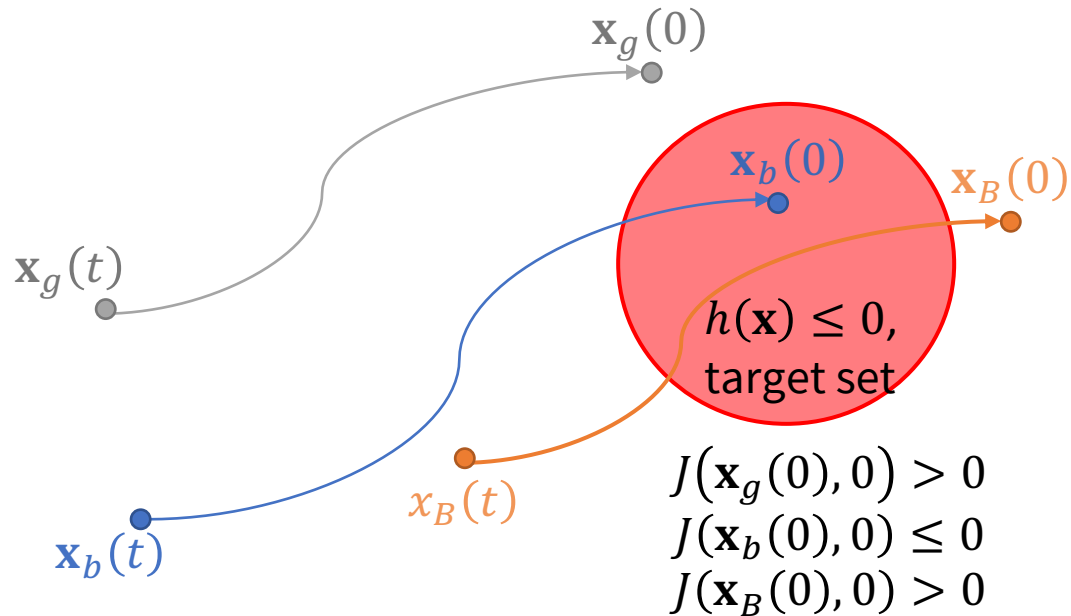


- Backward reachable tube (BRT)
 - Keep track of entire time duration
 - Initial states that pass through target are in BRT
 - Used to make safety guarantees



“Sets” vs. “Tubes”

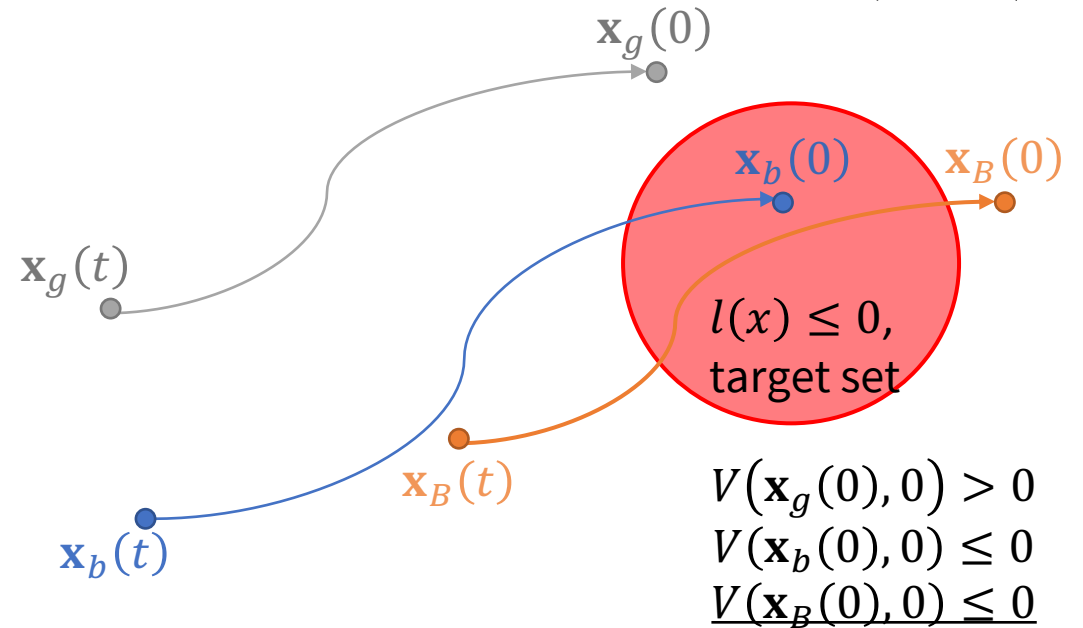
- Backward reachable set (BRS)



- Value function definition
 - $J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} h(\mathbf{x}(0))$
- Value function obtained from

$$\frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}} \right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right] = 0$$

- Backward reachable tube (BRT)



- Value function definition
 - $J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} \min_{\tau \in [t, 0]} h(\mathbf{x}(\tau))$
- Value function obtained from

$$\min \left\{ \frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}} \right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right], h(\mathbf{x}) - J(\mathbf{x}, t) \right\} = 0$$

Computational aspects

- Computational complexity

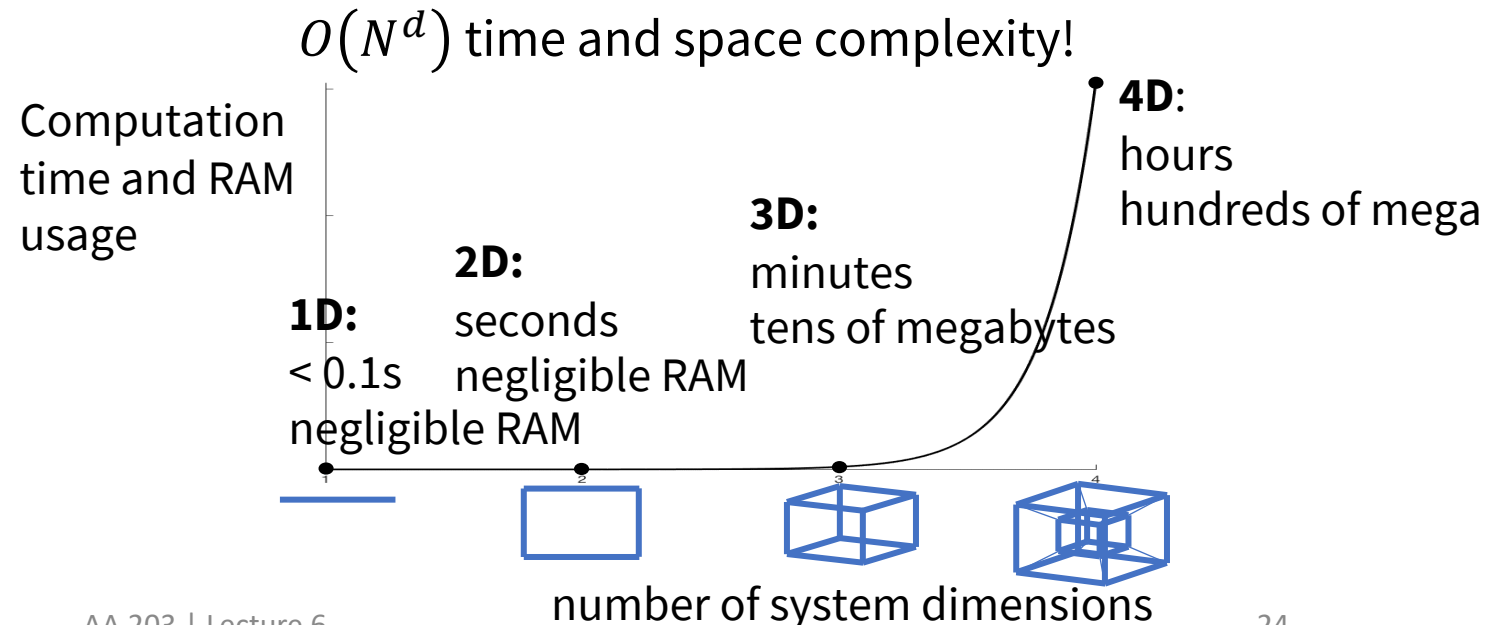
- $J(\mathbf{x}, t)$ is computed on an $(n + 1)$ -dimensional grid
- Currently, $n \leq 5$ is possible. GPU acceleration under-way
- Dimensionality reduction methods sometimes help

- Related approaches

- Sacrifice global optimality
- Give up guarantees
- Sampling-based methods
- Reinforcement learning

6D:
intractable!

5D:
days
gigabytes



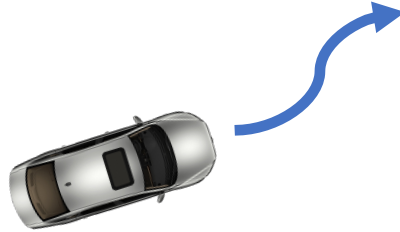
Numerical toolboxes

- helperOC Matlab toolbox
 - <https://github.com/HJReachability/helperOC.git>
 - Reachability wrapper around the level set toolbox
 - Requires level set toolbox
 - Hamilton-Jacobi PDE solver by Ian Mitchell, UBC
 - https://bitbucket.org/ian_mitchell/toolboxls
- C++ and CUDA version in development, beta also available
 - C++: 5+ times faster than Matlab
 - CUDA: Up to 100 times faster than Matlab
 - <https://github.com/HJReachability/beacsl>

Example – waypoint reaching with Dubins Car

Dubins Car Model

$$\begin{cases} \dot{x} = v \cos \theta + \mathbf{d}_x \\ \dot{y} = v \sin \theta \\ \dot{\theta} = k \mathbf{u} \end{cases}$$



Control: \mathbf{u}

Disturbance: \mathbf{d}_x

- Target set:

$$\mathcal{T} = \{(x, y, \theta) \in \mathbb{R}^3 : h(x, y, \theta) := ((x, y, \theta) - (x_{max}, y_{max}, \theta_{max}), (x_{min}, y_{min}, \theta_{min}) - (x, y, \theta)) \leq 0\}$$

- HJI equation:

$$\frac{\partial J}{\partial t}(x, y, \theta, t) + \min_{|u| \leq u_{max}} \max_{|d_x| \leq d_{max}^x} \nabla J(x, y, \theta, t)' f(x, y, \theta, u, d) = 0$$

- Optimal quantities:

$$u^*(x, y, \theta, t) = \arg \min_{|u| \leq u_{max}} \max_{|d_x| \leq d_{max}^x} \nabla J(x, y, \theta, t)^\top f(x, y, \theta, u, d)$$

$$d^*(x, y, \theta, t) = \arg \max_{|d_x| \leq d_{max}^x} \nabla J(x, y, \theta, t)^\top f(x, y, \theta, u^*, d)$$



$$\mathbf{u} = -u_{max} \operatorname{sign}\left(\frac{\partial J}{\partial \theta}\right)$$

$$\mathbf{d}_x = d_{max}^x \operatorname{sign}\left(\frac{\partial J}{\partial x}\right)$$

Next time

- Calculus of variations