

AA 203

Optimal and Learning-Based Control

Robust MPC

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Much of this lecture is based on material from the courses “Model predictive control” (2017) and “Advanced MPC” (2022) at ETH Zürich. This material was jointly authored by:

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Thank you!

Agenda

1. Review of model predictive control
2. Robust model predictive control
3. Characterizing the effect of disturbances and robust “open-loop” MPC
4. Robust closed-loop MPC
5. Tube MPC

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Review: Model predictive control

Consider the discrete-time dynamical system $x(t+1) = f(x(t), u(t))$. In *model predictive control (MPC)*, at each time t we solve the optimal control problem

$$\underset{u}{\text{minimize}} \quad V_f(x_{N|t}) + \sum_{k=0}^{N-1} \ell(x_{k|t}, u_{k|t})$$

subject to $x_{0|t} = x(t)$

$$x_{k+1|t} = f(x_{k|t}, u_{k|t}), \quad \forall k \in \{0, 1, \dots, N-1\}$$

$$x_{k|t} \in \mathcal{X}, \quad \forall k \in \{0, 1, \dots, N-1\}$$

$$x_{N|t} \in \mathcal{X}_f$$

$$u_{k|t} \in \mathcal{U}, \quad \forall k \in \{0, 1, \dots, N-1\}$$

for a sequence $\{u_{k|t}^*\}_{k=0}^{N-1}$. The *MPC feedback policy* is then $u(t) = \pi_{\text{MPC}}(x(t)) := u_{0|t}^*$.

The set of feasible initial states is

$$\mathcal{X}_0 := \{x \in \mathbb{R}^n \mid \exists \{u_k\}_{k=0}^{N-1} \subseteq \mathcal{U} : \{x_k\}_{k=0}^{N-1} \in \mathcal{X}, x_N \in \mathcal{X}_f\},$$

where $x_0 = x$ and $x_{k+1} := f(x_k, u_k)$ for all $k \in \{0, 1, \dots, N-1\}$.

Review: Recursive feasibility and stability

Suppose:

- The initial MPC problem is feasible, i.e., $x(0) \in \mathcal{X}_0$.
- There exists a feedback policy $\pi_f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that renders \mathcal{X}_f invariant subject to state and input constraints, i.e.,

$$\mathcal{X}_f \subseteq \mathcal{X}, \quad \pi_f(x) \in \mathcal{U}, \quad f(x, \pi_f(x)) \in \mathcal{X}_f,$$

for all $x \in \mathcal{X}_f$.

- The function f is continuous, and ℓ and V_f are uniformly continuous. Moreover, $f(0, 0) = 0$, $\pi_f(0) = 0$, and ℓ and V_f are positive-definite.
- The sets \mathcal{X} , \mathcal{U} , and \mathcal{X}_f are closed and bounded, and each contain the origin.
- The terminal cost function $V_f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies

$$V_f(f(x, \pi_f(x))) - V_f(x) \leq -\ell(x, \pi_f(x))$$

for all $x \in \mathcal{X}_f$.

Then the MPC feedback policy is *recursively feasible*, and *asymptotically stabilizing* with region of attraction \mathcal{X}_0 . Much of this depends on the *terminal ingredients* $(\mathcal{X}_f, \pi_f, V_f)$.

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Any type of predictive control assumes the system evolves in a predictable fashion, e.g.,

$$x(t+1) = f(x(t), u(t)),$$

with known $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$.

In reality, we often instead have

$$x(t+1) = f(x(t), u(t), w(t)),$$

with a possibly time-varying disturbance $w(t)$ that enters our system with currently unknown structure. The disturbance $w(t)$ can be structured itself, random noise, or some combination thereof.

What can we hope to do in this situation?

Goals of robust constrained control

Consider the uncertain system

$$x(t+1) = f(x(t), u(t), w(t)),$$

where we assume $w(t) \in \mathcal{W}$ for all t and some set \mathcal{W} .

The goal of *robust constrained control* is to design a feedback policy $u(t) = \pi(x(t))$ such that:

- The state and input robustly satisfy constraints, e.g., $x(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ for all t and all possible realizations of $w(t)$.
- The system is robustly stable, e.g., $x(t)$ converges to some bounded neighbourhood of the origin.
- Closed-loop trajectories are “optimal” with respect to some notion of performance, e.g., in expectation or in the worst-case.
- The set of robustly feasible initial states \mathcal{X}_0 is as large as possible.

Achieving these goals requires some knowledge or assumptions about f and \mathcal{W} .

In this lecture, we will focus on linear systems with additive bounded disturbances, i.e.,

$$x(t+1) = Ax(t) + Bu(t) + w(t),$$

where (A, B) are known, and $w(t) \in \mathcal{W}$ with known, bounded \mathcal{W} .

The nominal dynamics are linear and time-invariant, but are impacted by a random, bounded disturbance at each time step.

Despite using a linear model, this setup can be used to handle nonlinear systems, albeit in a conservative fashion.

The disturbance $w(t)$ is *persistent* in the sense that it does not converge to zero in the limit.

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Uncertain linear system evolution

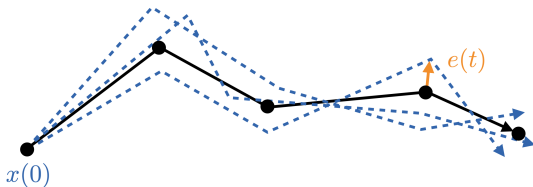
Consider the nominal system $\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t)$ alongside the actual system $x(t+1) = Ax(t) + Bu(t) + w(t)$, and define the error $e(t) := x(t) - \bar{x}(t)$. Then

$$e(t+1) = Ae(t) + B(u(t) - \bar{u}(t)) + w(t).$$

If $u(t) = \bar{u}(t)$, then

$$e(t+1) = Ae(t) + w(t)$$

$$\implies e(t) = A^t e(0) + \sum_{k=0}^{t-1} A^k w(t-1-k)$$

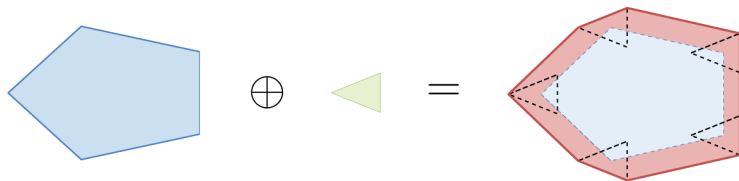


Define the *linear transform* $A\mathcal{W} := \{Aw, \forall w \in \mathcal{W}\}$. Then the component of the error at time t corresponding to the disturbance $w(t-1-k)$ lies in $A^k\mathcal{W}$.

The Minkowski sum and disturbance reachable sets

To characterize the cumulative effect of past disturbances on the error at time t , we need the *Minkowski sum*. For two sets \mathcal{X} and \mathcal{Y} , it is defined by

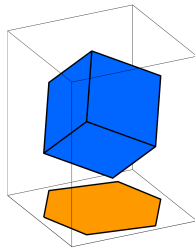
$$\mathcal{X} \oplus \mathcal{Y} := \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}.$$



For polytopes $\mathcal{X} = \{x \mid Ax \preceq b\}$ and $\mathcal{Y} = \{y \mid Cy \preceq d\}$, we have

$$\begin{aligned}\mathcal{X} \oplus \mathcal{Y} &= \{x + y \mid Ax \preceq b, Cy \preceq d\} \\ &= \{z \mid \exists y : A(z - y) \preceq b, Cy \preceq d\} \\ &= \left\{z \mid \exists y : \begin{bmatrix} A & -A \\ 0 & C \end{bmatrix} \begin{pmatrix} z \\ y \end{pmatrix} \preceq \begin{pmatrix} b \\ d \end{pmatrix}\right\}\end{aligned}$$

This is a *projection* of a polytope from (z, y) onto z .



The Minkowski sum and disturbance reachable sets

For the nominal state $\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t)$ and actual state $x(t+1) = Ax(t) + Bu(t) + w(t)$, the error $e(t) := x(t) - \bar{x}(t)$ with $u(t) = \bar{u}(t)$ satisfies

$$\begin{aligned} e(t+1) &= Ae(t) + w(t) \\ \implies e(t) &= A^t e(0) + \sum_{k=0}^{t-1} A^k w(t-1-k) \end{aligned}$$

The error at time t lies in the *disturbance reachable set (DRS)*

$$\begin{aligned} \mathcal{E}_t &:= \{A^t e(0)\} \oplus \left(\bigoplus_{k=0}^{t-1} A^k \mathcal{W} \right) = \mathcal{W} \oplus A\mathcal{W} \oplus A^2\mathcal{W} \oplus \dots \oplus A^{t-1}\mathcal{W} \oplus \{A^t e(0)\} \\ \implies \mathcal{E}_{t+1} &= A\mathcal{E}_t \oplus \mathcal{W}, \quad \mathcal{E}_0 := \{e(0)\} \end{aligned}$$

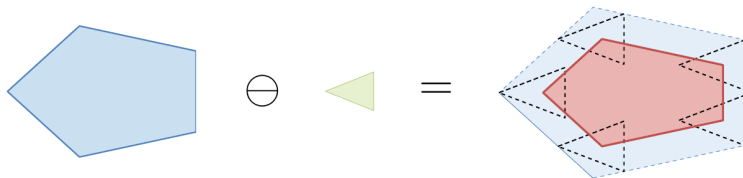
where the recursion follows from *distributivity* of linear transforms over Minkowski sums. Generally we will set $e(0) = 0$.

The key idea in robust MPC is to *pre-compute* DRSs and use them in *constraint tightening* to account for all possible disturbance realizations.

The Pontryagin difference and constraint tightening

To tighten constraint sets, we need the *Pontryagin difference*. For two sets \mathcal{X} and \mathcal{Y} , it is defined by

$$\mathcal{X} \ominus \mathcal{Y} := \{x \mid x + y \in \mathcal{X}, \forall y \in \mathcal{Y}\}.$$



For polytopes $\mathcal{X} = \{x \mid Ax \preceq b\}$ and $\mathcal{Y} = \{y \mid Cy \preceq d\}$, we have

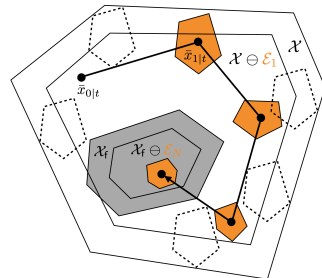
$$\mathcal{X} \ominus \mathcal{Y} = \{x \mid A(x + y) \preceq b, \forall y : Cy \preceq d\} = \left\{x \mid Ax \preceq b - \max_{y: Cy \preceq d} Ay\right\}$$

where $\max_{y: Cy \preceq d} Ay$ (with the maximum applied element-wise) is the *support vector*. It can be computed by solving a linear program for each row of A .

The Pontryagin difference and constraint tightening

We can use *pre-computed* DRSs to tighten constraints and pose the new online MPC problem

$$\begin{aligned} & \underset{u}{\text{minimize}} && V_f(\bar{x}_{N|t}) + \sum_{k=0}^{N-1} \ell(\bar{x}_{k|t}, u_{k|t}) \\ & \text{subject to} && \bar{x}_{0|t} = x(t) \\ & && \bar{x}_{k+1|t} = A\bar{x}_{k|t} + Bu_{k|t}, \quad \forall k \in \{0, 1, \dots, N-1\} \\ & && \bar{x}_{k|t} \in \mathcal{X} \ominus \mathcal{E}_k, \quad \forall k \in \{0, 1, \dots, N-1\} \\ & && \bar{x}_{N|t} \in \mathcal{X}_f \ominus \mathcal{E}_N \\ & && u_{k|t} \in \mathcal{U}, \quad \forall k \in \{0, 1, \dots, N-1\} \end{aligned}$$



which predicts forward in time using the *nominal state*, and relies on the tightened constraints to ensure the actual state satisfies $x_{k|t} \in \mathcal{X}$ and $x_{N|t} \in \mathcal{X}_f$.

We consider the *nominal case* (i.e., with $w \equiv 0$) in our objective. We could also consider the *expected case* (e.g., if we knew the distribution of w over \mathcal{W}), or the *worst case* (e.g., if we can evaluate the maximum over \mathcal{W}).

However, we are not done. We need to ensure the state will remain in \mathcal{X}_f , regardless of future disturbances.

A set \mathcal{X}_f is said to be *robust positive invariant (RPI)* for $x(t+1) = f(x(t), \pi_f(x(t)), w(t))$ with $w(t) \in \mathcal{W}$ if

$$f(x, \pi_f(x), w) \in \mathcal{X}_f, \quad \forall (x, w) \in \mathcal{X}_f \times \mathcal{W}.$$

Given a set \mathcal{X}_f and closed-loop dynamics $x(t+1) = f(x(t), \pi_f(x(t)), w(t))$ with $w(t) \in \mathcal{W}$, the *robust pre-set* of \mathcal{X}_f is

$$\text{pre}(\mathcal{X}_f; \mathcal{W}) := \{x \mid f(x, \pi_f(x), w) \in \mathcal{X}_f, \quad \forall w \in \mathcal{W}\},$$

i.e., the set of states that evolve into \mathcal{X}_f after one step regardless of the disturbance realization.

A set \mathcal{X}_f is RPI if and only if $\mathcal{X}_f \subseteq \text{pre}(\mathcal{X}_f; \mathcal{W})$.

For the autonomous system $x(t+1) = f(x(t)) + w(t)$ with additive disturbance $w(t) \in \mathcal{W}$, the robust pre-set is

$$\text{pre}(\mathcal{X}_f; \mathcal{W}) = \{x \mid f(x) + w \in \mathcal{X}_f, \forall w \in \mathcal{W}\} = \{x \mid f(x) \in \mathcal{X}_f \ominus \mathcal{W}\} = \text{pre}(\mathcal{X}_f \ominus \mathcal{W}),$$

where $\text{pre}(\cdot)$ is the non-robust pre-set operator.

If \mathcal{X}_f and \mathcal{W} are polytopic, then we can compute $\mathcal{X}_f \ominus \mathcal{W} = \{x \mid Cx \preceq d\}$ via linear programming.

If additionally $f(x) = Ax$, then $\text{pre}(\mathcal{X}_f; \mathcal{W}) = \{x \mid CAx \preceq d\}$.

Computing robust pre-sets for linear systems with polytopic constraints

Consider now the closed-loop system $x(t+1) = f(x(t), \pi_f(x(t))) + w(t)$ with control constraint set \mathcal{U} . Then the robust pre-set we are interested in is

$$\begin{aligned}\text{pre}(\mathcal{X}_f; \mathcal{U}, \mathcal{W}) &= \left\{ x \mid \begin{pmatrix} f(x, \pi_f(x)) + w \\ \pi_f(x) \end{pmatrix} \in \mathcal{X}_f \times \mathcal{U}, \forall w \in \mathcal{W} \right\} \\ &= \left\{ x \mid \begin{pmatrix} f(x, \pi_f(x)) \\ \pi_f(x) \end{pmatrix} \in (\mathcal{X}_f \ominus \mathcal{W}) \times \mathcal{U} \right\}.\end{aligned}$$

If \mathcal{X}_f and \mathcal{W} are polytopic, then we can compute $\mathcal{X}_f \ominus \mathcal{W} = \{x \mid Cx \preceq d\}$ via linear programming.

If additionally $f(x, u) = Ax + Bu$, $\pi_f(x) = K_f x$, and $\mathcal{U} = \{u \mid Gu \leq h\}$, then

$$\text{pre}(\mathcal{X}_f; \mathcal{U}, \mathcal{W}) = \left\{ x \mid \begin{pmatrix} C(A + BK_f) \\ GK_f \end{pmatrix} x \preceq \begin{pmatrix} d \\ h \end{pmatrix} \right\}.$$

Computing robust invariant sets for linear systems with polytopic constraints

Variants of recursive feasibility and stability carry over to the robust setting as long as \mathcal{X}_f is an RPI. Generally, we want the *maximal RPI (MRPI)*, i.e., an RPI containing all other RPIs.

We use the fact that \mathcal{X}_f is RPI if and only if $\mathcal{X}_f \subseteq \text{pre}(\mathcal{X}_f; \mathcal{U}, \mathcal{W})$ to develop the following conceptual algorithm for finding an MRPI.

Input: dynamics f , controller π_f , state set \mathcal{X} , control set \mathcal{U} , disturbance set \mathcal{W}

Output: maximal RPI $\mathcal{X}_f \subseteq \mathcal{X}$

initialize $\mathcal{X}_f^{\text{prev}} = \mathcal{X}$, $\mathcal{X}_f = \text{pre}(\mathcal{X}_f^{\text{prev}}; \mathcal{U}, \mathcal{W}) \cap \mathcal{X}_f^{\text{prev}}$

while $\mathcal{X}_f \neq \mathcal{X}_f^{\text{prev}}$

$\mathcal{X}_f^{\text{prev}} \leftarrow \mathcal{X}_f$

$\mathcal{X}_f \leftarrow \text{pre}(\mathcal{X}_f^{\text{prev}}; \mathcal{U}, \mathcal{W}) \cap \mathcal{X}_f^{\text{prev}}$

return \mathcal{X}_f

If the dynamics and controller are linear, and all the sets are polytopic, then we can compute the robust pre-set at each iteration using the method described previously.

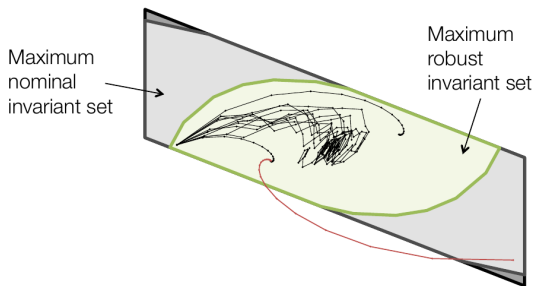
Example: MRPI for a stabilized double-integrator

Consider the dynamics

$$x(t+1) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} K \right) x(t) + w(t),$$

where $w(t) \in \mathcal{W}$ with $\mathcal{W} = \{w \in \mathbb{R}^2 \mid \|w\|_\infty \leq 0.3\}$, and $K \in \mathbb{R}^{1 \times 2}$ is the infinite-horizon LQR gain for $Q = 0.1I$ and $R = 1$. Moreover,

$$\begin{aligned} \mathcal{X} &= \{x \mid \|x\|_\infty \leq 5\} \\ \mathcal{U} &= \{u \mid \|u\|_\infty \leq 1\} \\ \Rightarrow \mathcal{X}_f &\subseteq \underbrace{\{x \mid \|x\|_\infty \leq 5, \|Kx\|_\infty \leq 1\}}_{\text{"}\mathcal{X}\text{" for initialization}} \end{aligned}$$



Robust “open-loop” MPC via constraint tightening

With *pre-computed* DRSs for constraint tightening and an RPI set \mathcal{X}_f for $x(t+1) = (A + BK_f)x(t)$ with some stabilizing $K_f \in \mathbb{R}^{m \times n}$, we have

$$\underset{u}{\text{minimize}} \quad V_f(\bar{x}_{N|t}) + \sum_{k=0}^{N-1} \ell(\bar{x}_{k|t}, u_{k|t})$$

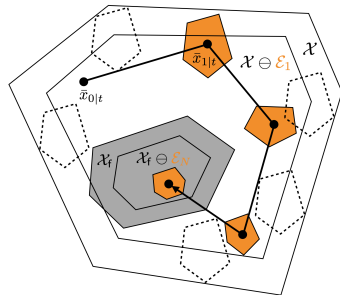
subject to $\bar{x}_{0|t} = x(t)$

$$\bar{x}_{k+1|t} = A\bar{x}_{k|t} + Bu_{k|t}, \quad \forall k \in \{0, 1, \dots, N-1\}$$

$$\bar{x}_{k|t} \in \mathcal{X} \ominus \mathcal{E}_k, \quad \forall k \in \{0, 1, \dots, N-1\}$$

$$\bar{x}_{N|t} \in \mathcal{X}_f \ominus \mathcal{E}_N$$

$$u_{k|t} \in \mathcal{U}, \quad \forall k \in \{0, 1, \dots, N-1\}$$



The robust “open-loop” MPC policy is $u(t) = \pi_{\text{MPC}}(x(t)) := u_{0|t}^*$, which is *recursively feasible*. That is, if $x(t) \in \mathcal{X}_0$, then $Ax(t) + B\pi_{\text{MPC}}(x(t)) + w(t) \in \mathcal{X}_0$ for all $w \in \mathcal{W}$.

Robust “open-loop” MPC potentially has a *very* small region of attraction, particularly for systems where A is unstable.

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Closed-loop predictions

$$\underset{u}{\text{minimize}} \quad V_f(\bar{x}_{N|t}) + \sum_{k=0}^{N-1} \ell(\bar{x}_{k|t}, u_{k|t})$$

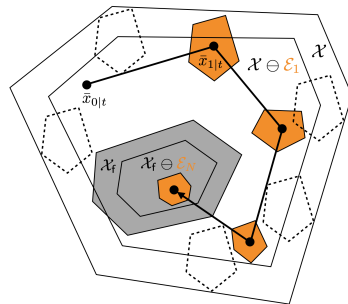
$$\text{subject to } \bar{x}_{0|t} = x(t)$$

$$\bar{x}_{k+1|t} = A\bar{x}_{k|t} + Bu_{k|t}, \quad \forall k \in \{0, 1, \dots, N-1\}$$

$$\bar{x}_{k|t} \in \mathcal{X} \ominus \mathcal{E}_k, \quad \forall k \in \{0, 1, \dots, N-1\}$$

$$\bar{x}_{N|t} \in \mathcal{X}_f \ominus \mathcal{E}_N$$

$$u_{k|t} \in \mathcal{U}, \quad \forall k \in \{0, 1, \dots, N-1\}$$



Notice that

$$x(t) = A^t x(0) + \sum_{k=0}^{t-1} A^k B u(t-1-k) + \sum_{k=0}^{t-1} A^k w(t-1-k)$$

In the “open-loop” MPC problem, we assume $u_{k|t} = \bar{u}_{k|t}$, i.e., that it is chosen based on the nominal state $\bar{x}_{k|t}$ rather than the actual state $x_{k|t} = \bar{x}_{k|t} + r_{k|t}$.

Notice that

$$x(t) = A^t x(0) + \sum_{k=0}^{t-1} A^k B u(t-1-k) + \sum_{k=0}^{t-1} A^k w(t-1-k)$$

In the “open-loop” MPC problem, we assume $u_{k|t} = \bar{u}_{k|t}$, i.e., that it is chosen based on the nominal state $\bar{x}_{k|t}$ rather than the actual state $x_{k|t} = \bar{x}_{k|t} + r_{k|t}$.

It would be preferable to optimize for a time-varying policy $\pi_t : \mathbb{N}_{\geq 0} \times \mathbb{R}^n$, such that

$$\begin{aligned} u_{0|t} &= \pi_t(0, x_{0|t}) = \pi_t(0, x(t)) \\ u_{k|t} &= \pi_t(k, x_{k|t}) = \pi_t\left(k, A^k x(t) + \sum_{i=0}^{k-1} A^i B u_{k-1-i|t} + \sum_{i=0}^{k-1} A^i w_{k-1-i|t}\right), \end{aligned}$$

then apply the first optimal control input $u_{0|t}^* = \pi_t^*(0, x(t))$. However, we cannot tractably optimize over an arbitrary policy π_t !

Closed-loop MPC

In *closed-loop MPC*, we assume some structure for $\pi_t : \mathbb{N}_{\geq 0} \times \mathbb{R}^n$ that allows us to optimize for it directly. Some options are:

Pre-stabilization Choose $\pi_t(k, x) = \bar{u}_{k|t} + Kx_{k|t}$ with a fixed $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is stable, and optimize over $\bar{u}_{k|t}$ for each k . This is a simple, often conservative choice.

Linear feedback Choose $\pi_t(k, x) = \bar{u}_{k|t} + K_{k|t}x_{k|t}$ and optimize over $\bar{u}_{k|t}$ and $K_{k|t}$ for each k . This yields a difficult non-convex problem.

Disturbance feedback Choose $\pi_t(k, x) = \bar{u}_{k|t} + \sum_{i=0}^{k-1} K_{ki|t}w_{i|t}$ and optimize over $\bar{u}_{k|t}$ and $\{K_{ki|t}\}_{i=0}^{k-1}$ for each k . This is equivalent to linear feedback and yields a convex problem, yet it can be computationally intensive ([Goulart et al., 2006](#)).

Tube MPC Choose $\pi_t(k, x) = \bar{u}_{k|t} + K(x - \bar{x}_{k|t})$ with a fixed $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is stable, and optimize over $\bar{u}_{k|t}$ and $\bar{x}_{k|t}$ for each k . This yields a convex problem and is usually quite effective ([Mayne et al., 2005](#)).

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We will focus on tube MPC, where the key idea is to separate control authority into

- a part $\bar{u}(t)$ that steers the nominal system $\bar{x}(t+1) = Ax(t) + B\bar{u}(t)$ to the origin, and
- a part that compensates for deviations from this system, such that

$$u(t) = \bar{u}(t) + K(x(t) - \bar{x}(t)) = \bar{u}(t) + Ke(t)$$

for some pre-computed gain $K \in \mathbb{R}^{m \times n}$ that stabilizes the nominal system.

The dynamics of the error $e(t) := x(t) - \bar{x}(t)$ are given by

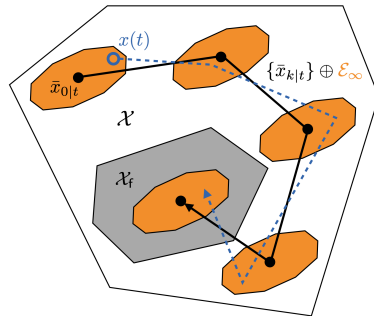
$$\begin{aligned} e(t+1) &= (A + BK)e(t) + w(t) \\ \implies e(t) &= (A + BK)^t e(0) + \sum_{k=0}^{t-1} (A + BK)^k w(t-1-k) \end{aligned}$$

where $w(t) \in \mathcal{W}$. Since $A + BK$ is stable and \mathcal{W} is assumed to be bounded, there must be some set \mathcal{E}_∞ such that $e(t) \in \mathcal{E}_\infty$ for all $t \geq 0$.

Tube MPC computes a nominal pair $\bar{x}_{\cdot|t} = \{\bar{x}_{k|t}\}_{k=0}^N$ and $\bar{u}_{\cdot|t} = \{\bar{u}_{k|t}\}_{k=0}^{N-1}$, while *planning* to apply the policy

$$u_{k|t} = \bar{u}_{k|t} + K(x_{k|t} - \bar{x}_{k|t})$$

to account for future information gain.



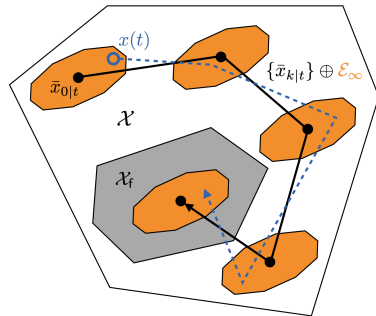
We must ensure all possible trajectories satisfy the constraints, i.e., that $\{\bar{x}_{k|t}\} \oplus \mathcal{E}_{\infty} \subseteq \mathcal{X}$ and $\{\bar{u}_{k|t}\} \oplus K\mathcal{E}_{\infty} \subseteq \mathcal{U}$.

We usually just apply the first input $u_{0|t}^* = \bar{u}_{0|t}^* + K(x(t) - \bar{x}_{0|t}^*)$, but the planned feedback law can serve as a valid sub-optimal plan (e.g., in case we do not have time to solve another MPC problem right away).

Tube MPC computes a nominal pair $\bar{x}_{\cdot|t} = \{\bar{x}_{k|t}\}_{k=0}^N$ and $\bar{u}_{\cdot|t} = \{\bar{u}_{k|t}\}_{k=0}^{N-1}$, while *planning* to apply the policy

$$u_{k|t} = \bar{u}_{k|t} + K(x_{k|t} - \bar{x}_{k|t})$$

to account for future information gain.



Overall, we need to:

- Compute the set \mathcal{E}_{∞} that the error will remain inside.
- Modify the constraints so $\{\bar{x}_{k|t}\} \oplus \mathcal{E}_{\infty} \subseteq \mathcal{X}$ and $\{\bar{u}_{k|t}\} \oplus K\mathcal{E}_{\infty} \subseteq \mathcal{U}$.
- Formulate the tube MPC problem as a convex optimization.

We can then prove that the constraints are robustly satisfied, the tube MPC problem is recursively feasible, and the closed-loop system is robustly stable.

Computing the limiting DRS

The set \mathcal{E}_∞ is precisely the limiting DRS, i.e.,

$$\mathcal{E}_\infty := \lim_{t \rightarrow \infty} \mathcal{E}_t = \lim_{t \rightarrow \infty} \left(\{(A + BK)^t e(0)\} \oplus \left(\bigoplus_{k=0}^{t-1} (A + BK)^k \mathcal{W} \right) \right)$$

Input: initial error $e(0)$, stable matrix $A + BK$, disturbance set \mathcal{W}

Output: limiting DRS (i.e., mRPI) \mathcal{E}_∞

initialize $\mathcal{E}_\infty^{\text{prev}} = \{e(0)\}$, $\mathcal{E}_\infty = \{A\mathcal{E}_\infty^{\text{prev}} \oplus \mathcal{W}\}$

while $\mathcal{E}_\infty \neq \mathcal{E}_\infty^{\text{prev}}$

$\mathcal{E}_\infty^{\text{prev}} \leftarrow \mathcal{E}_\infty$

$\mathcal{E}_\infty \leftarrow \{A\mathcal{E}_\infty^{\text{prev}} \oplus \mathcal{W}\}$

return \mathcal{E}_∞

Generally we use $e(0) := x(0) - \bar{x}(0) = 0$. A finite t such that $\mathcal{E}_{t+1} = \mathcal{E}_t$ does not always exist, but usually a large number of iterations is good enough.

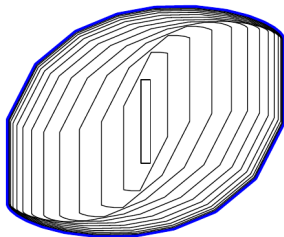
The set \mathcal{E}_∞ is referred to as the *minimal RPI (mRPI)* for $e(t+1) = (A + BK)e(t) + w(t)$, since it is the smallest set that the error $e(t)$ remains in despite the disturbance $w(t) \in \mathcal{W}$.

Example: Limiting DRS for a stabilized double-integrator

Consider the dynamics

$$e(t+1) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} K \right) e(t) + w(t),$$

where $w(t) \in \mathcal{W}$ with $\mathcal{W} = \{(w_1, w_2) \in \mathbb{R}^2 \mid |w_1| \leq 0.01, |w_2| \leq 0.1\}$, and $K \in \mathbb{R}^{1 \times 2}$ is the infinite-horizon LQR gain for $Q = I$ and $R = 10$.



The recursion $\mathcal{E}_{t+1} = A\mathcal{E}_t \oplus \mathcal{W}$ converges to \mathcal{E}_∞ as $t \rightarrow \infty$.

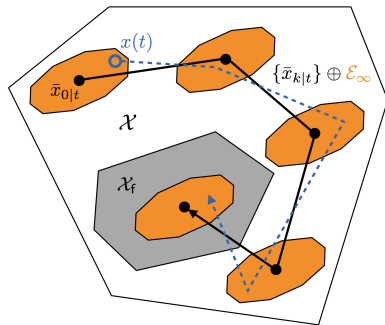


The error $e(t)$ remains inside \mathcal{E}_∞ for all t and all realizations of $w(t) \in \mathcal{W}$.

We fix K such that $A + BK$ is stable and compute \mathcal{E}_∞ offline. We want $\{\bar{x}_{k|t}\} \oplus \mathcal{E}_\infty \subseteq \mathcal{X}$ and $\{\bar{u}_{k|t}\} \oplus K\mathcal{E}_\infty \subseteq \mathcal{U}$ to hold, for which $\bar{x}_{k|t} \in \mathcal{X} \ominus \mathcal{E}_\infty$ and $\bar{u}_{k|t} \in \mathcal{U} \ominus K\mathcal{E}_\infty$, respectively, are sufficient conditions.

The tube MPC problem is then

$$\begin{aligned}
 & \underset{\bar{x}, \bar{u}}{\text{minimize}} && V_f(\bar{x}_{N|t}) + \sum_{k=0}^{N-1} \ell(\bar{x}_{k|t}, \bar{u}_{k|t}) \\
 & \text{subject to} && x(t) \in \{\bar{x}_{0|t}\} \oplus \mathcal{E}_\infty \\
 & && \bar{x}_{k+1|t} = A\bar{x}_{k|t} + B\bar{u}_{k|t}, \quad \forall k \in \{0, 1, \dots, N-1\} \\
 & && \bar{x}_{k|t} \in \mathcal{X} \ominus \mathcal{E}_\infty, \quad \forall k \in \{0, 1, \dots, N-1\} \\
 & && \bar{x}_{N|t} \in \mathcal{X}_f \\
 & && \bar{u}_{k|t} \in \mathcal{U} \ominus K\mathcal{E}_\infty, \quad \forall k \in \{0, 1, \dots, N-1\}
 \end{aligned}$$



The tube MPC policy is $u(t) = \pi_{\text{tube}}(x(t)) := \bar{u}_{0|t}^* + K(x(t) - \bar{x}_{0|t})$.

Terminal ingredients for tube MPC

Suppose:

- The initial tube MPC problem is feasible, i.e., $x(0) \in \mathcal{X}_0$.
- There exists a feedback policy $\pi_f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that renders \mathcal{X}_f invariant subject to *tightened* state and input constraints, i.e.,

$$\mathcal{X}_f \subseteq \mathcal{X} \ominus \mathcal{E}_\infty, \quad \pi_f(\bar{x}) \in \mathcal{U} \ominus K\mathcal{E}_\infty, \quad A\bar{x} + B\pi_f(\bar{x}) \in \mathcal{X}_f,$$

for all $\bar{x} \in \mathcal{X}_f$.

- The functions ℓ and V_f are uniformly continuous. Moreover, $\pi_f(0) = 0$, and ℓ and V_f are positive-definite.
- The sets \mathcal{X} , \mathcal{U} , and \mathcal{X}_f are closed and bounded, and each contain the origin.
- The terminal cost function $V_f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies

$$V_f(A\bar{x} + B\pi_f(\bar{x})) - V_f(\bar{x}) \leq -\ell(\bar{x}, \pi_f(\bar{x}))$$

for all $\bar{x} \in \mathcal{X}_f$.

Then the tube MPC feedback policy is *recursively feasible*. Moreover, for any $x(0) \in \mathcal{X}_0$, we have $\lim_{t \rightarrow \infty} \min_{e \in \mathcal{E}_\infty} \|x(t) - e\| = 0$, i.e., “ $x(t) \rightarrow \mathcal{E}_\infty$ ”.

The key idea behind tube MPC is to separate control authority into a part that steers the nominal dynamics, and a part that compensates for deviations. We optimize the nominal trajectory, and tighten constraints to ensure deviations do not cause them to be violated.

Tube MPC is less conservative than robust “open-loop” MPC, since we are planning for future information gain. Tube MPC also works better for open-loop unstable systems. The optimization problem is convex and simple to solve.

However, tube MPC is still sub-optimal and has a reduced feasible set when compared to nominal MPC. Moreover, we need to be able to characterize \mathcal{W} and \mathcal{E}_∞ .

Adaptive and learning-based MPC

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