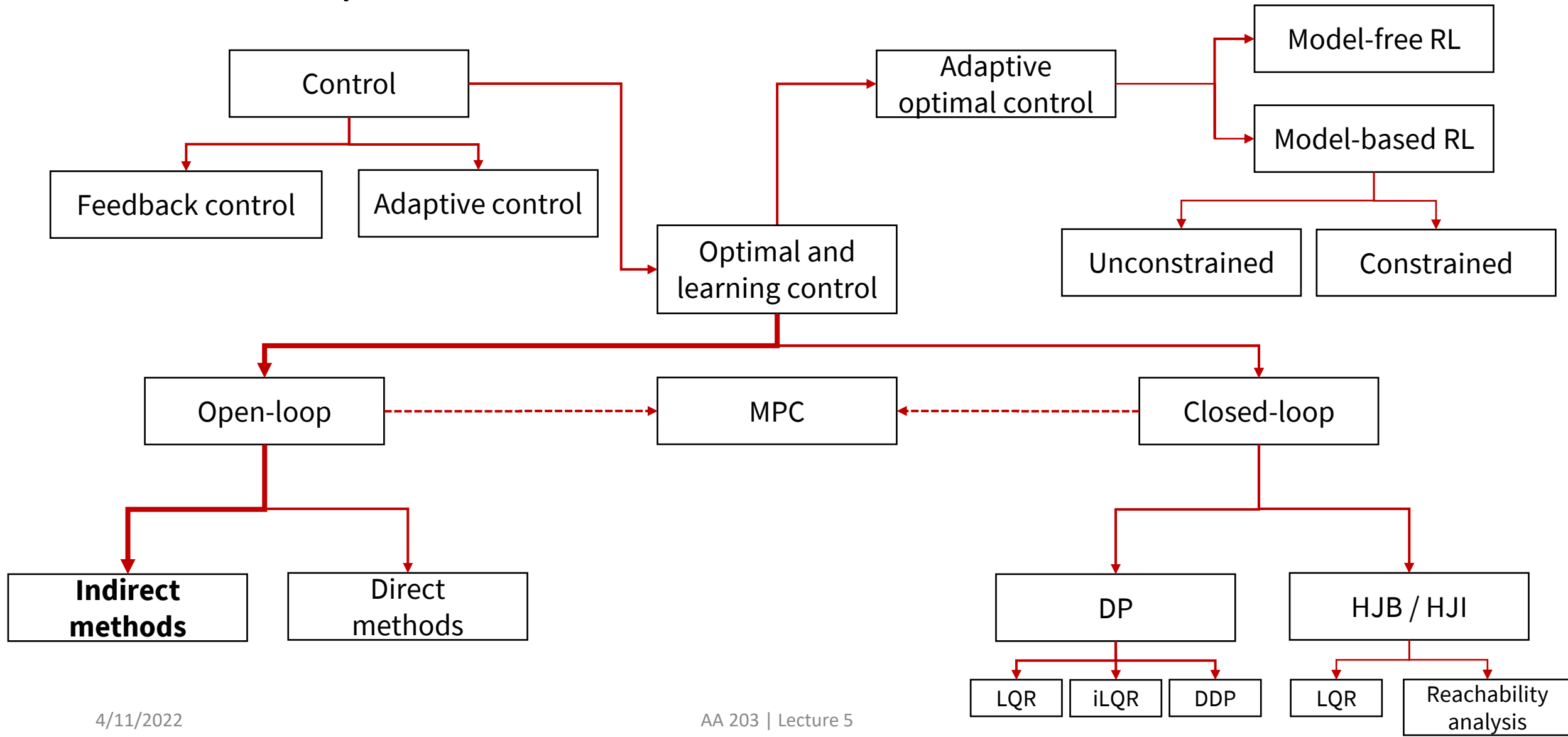# AA203
Optimal and Learning-based Control

CoV extensions, NOC for optimal control

Roadmap

CoV extension I: generalized boundary conditions

- Let $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ be a vector-valued function, where each component x_i is in the class of functions with continuous first derivatives. It is desired to find the function $\mathbf{x}^*$ for which the functional

$$J(\mathbf{x}) = \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) dt$$

  has a relative extremum

- Assumptions:
  - $g \in C^2$
  - t_0 and $\mathbf{x}(0)$ are fixed
  - t_f might be fixed or free, and each component of $\mathbf{x}(t_f)$ might be fixed or free

- Reading:
  - D. E. Kirk. *Optimal Control Theory: An Introduction,* 2004.

CoV extension I: generalized boundary conditions

- Regardless of the boundary conditions, the Euler equations

$$g_{\mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt} g_{\dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) = \mathbf{0}$$

must be satisfied

CoV extension I: generalized boundary conditions

- Regardless of the boundary conditions, the Euler equations

$$g_{\mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt} g_{\dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) = \mathbf{0}$$

  must be satisfied
- The required boundary conditions are found from the equation

$$g_{\dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)^T \delta \mathbf{x}_f + \left[g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) - g_{\dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)^T \dot{\mathbf{x}}^*(t_f) \right] \delta t_f = 0$$

  by making the "appropriate" substitutions for $\delta \mathbf{x}_f$ and δt_f

CoV extension I: generalized boundary conditions

- $\delta \mathbf{x}_f$ and δt_f capture the notion of "allowable" variations at the end point, thus $\delta t_f = 0$ if the final time is fixed, and $\delta x_i(t_f) = 0$ if the end value of state variable $x_i(t_f)$ is fixed

- For example, suppose that t_f is fixed, $x_i(t_f), i = 1, \dots, r$ are fixed, and $x_j(t_f), j = r + 1, \dots, n$ are free. Then the substitutions are:

$$\delta t_f = 0$$
$$\delta x_i(t_f) = 0, \qquad i = 1, \dots, r$$
$$\delta x_j(t_f) \text{ arbitrary}, \qquad j = r + 1, \dots, n$$

CoV extension I: generalized boundary conditions

Problem description	Substitution	Boundary conditions	Remarks
1. $\mathbf{x}(t_f)$, t_f both specified (*Problem 1*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = 0$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
2. $\mathbf{x}(t_f)$ free; t_f specified (*Problem 2*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
3. t_f free; $\mathbf{x}(t_f)$ specified (*Problem 3*)	$\delta\mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $- \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f) = 0$	$(2n+1)$ equations to determine $2n$ constants of integration and t_f
4. t_f, $\mathbf{x}(t_f)$ free and independent (*Problem 4*)	—	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = 0$	$(2n+1)$ equations to determine $2n$ constants of integration and t_f
5. t_f, $\mathbf{x}(t_f)$ free but related by $\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$ (*Problem 4*)	$\delta\mathbf{x}_f = \frac{d\boldsymbol{\theta}}{dt}(t_f)\,\delta t_f$†	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $+ \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f) - \dot{\mathbf{x}}^*(t_f)\right] = 0$†	$(2n+1)$ equations to determine $2n$ constants of integration and t_f

Example

- Determine the smooth curve of smallest length connecting the point $x(0) = 1$ to the line $t = 5$
  - Solution: $x(t) = 1$

CoV extension II: constrained extrema

- Let $\mathbf{w}: \mathbb{R} \to \mathbb{R}^{n+m}$ be a vector-valued function, where each component w_i is in the class of functions with continuous first derivatives. It is desired to find the function $\boldsymbol{w}^*$ for which the functional

$$J(\mathbf{w}) = \int_{t_0}^{t_f} g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) dt$$

  has a relative extremum, subject to the constraints
$$f_i(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) = 0, \qquad i = 1, \dots, n$$

- Assumptions:
  - $g \in C^2$
  - t_0 and $\mathbf{w}(0)$ are fixed

CoV extension II: constrained extrema

- Because of the n differential constraints, only m of the $n + m$ components of $\mathbf{w}$ are independent

- Constraints of this type may represent the state equation constraints in optimal control problems where $\mathbf{w}$ corresponds to the $n + m$ vector $\mathbf{w} = [\mathbf{x}, \mathbf{u}]^{\mathbf{T}}$

- Similar to the case of constrained optimization, define the augmented integrand function
$$g_a(\mathbf{w}(t), \dot{\mathbf{w}}(t), \mathbf{p}(t), t) :=$$
$$g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \boxed{\mathbf{p}(t)^T} \mathbf{f}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)$$

Lagrange multipliers (now functions of time!), the "costate"

CoV extension II: constrained extrema

- A necessary condition for optimality is then

$$\frac{\partial g_a}{\partial \mathbf{w}}\left(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t\right) - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{\mathbf{w}}}\left(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t\right) = \mathbf{0}$$

  along with

$$\mathbf{f}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), t) = \mathbf{0}$$

- That is, to determine the necessary conditions for an extremal we simply form the augmented integrand g_a and write the Euler equations *as if* there were no constraints among the functions $\mathbf{w}(t)$

- Note the similarity with the case of constrained optimization!

The variational approach to optimal control

Roadmap:

1. We will first derive necessary conditions for optimal control assuming that the admissible controls are not bounded

2. Next, we will heuristically introduce Pontryagin's Minimum Principle as a generalization of the fundamental theorem of CoV

3. Finally, we will consider special cases of problems with bounded controls and state variables

Necessary conditions for optimal control
(with unbounded controls)

- The problem is to find an *admissible control* $\mathbf{u}^*$ which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

  to follow an *admissible trajectory* $\mathbf{x}^*$ that minimizes the *functional*

$$J(\mathbf{u}) = h\big(\mathbf{x}(t_f), t_f\big) + \int_{t_0}^{t_f} g\left(\mathbf{x}(t), \mathbf{u}(t), t\right) dt$$

- Assumptions: $h \in C^2$, state and control regions are unbounded, t_0 and $\mathbf{x}(0)$ are fixed

Necessary conditions for optimal control
(with unbounded controls)

- Define the Hamiltonian

$$H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) := g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}(t)^T \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

- The necessary conditions for optimality (proof to follow) are

$$\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$
for all $t \in [t_0, t_f]$

$$\mathbf{0} = \frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

with boundary conditions

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta \mathbf{x}_f + \left[H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0$$

Necessary conditions for optimal control
(with unbounded controls)

Problem	Description	Substitution in Eq. (5.1-18)	Boundary-condition equations	Remarks
t_f fixed	1. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = 0$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
	2. $\mathbf{x}(t_f)$ free	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = 0$	$2n$ equations to determine $2n$ constants of integration
	3. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = 0$	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = 0$	$(2n + k)$ equations to determine the $2n$ constants of integration and the variables $d_1, \ldots, d_k$
t_f free	4. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	5. $\mathbf{x}(t_f)$ free		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = 0$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f

From Kirk, Section 5.1

Interlude: Pontryagin's minimum principle
(with bounded controls)

- Assuming bounded controls $\mathbf{u} \in U$, the necessary optimality conditions are (H is the Hamiltonian)

$$\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \leq H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t), \text{ for all } \mathbf{u}(t) \in U$$

for all $t \in [t_0, t_f]$

along with the boundary conditions:

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta \mathbf{x}_f + \left[H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0$$

Necessary conditions for optimal control
(with unbounded controls)

- Necessary conditions consist of a set of $2n$, *first-order*, differential equations (state and costate equations), and a set of m algebraic equations (control equations)

- The solution to the state and costate equations will contain $2n$ constants of integration

- To obtain values for the constants, we use the n equations $\mathbf{x}(t_0) = \mathbf{x}_0$, and an additional set of n (or $n+1$) equations from the boundary conditions

- Once again: *2-point boundary value problem*

Example

Find optimal control $u(t)$ to steer the system

$$\ddot{x}(t) = u(t)$$

from $x(0) = 10, \dot{x}(0) = 0$ to the origin
$x(t_f) = 0, \dot{x}(t_f) = 0$, and to minimize

$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_{t_0}^{t_f} b\, u^2(t)dt, \quad \alpha, b > 0$$

(note: the final time t_f is free)

Example

Find optimal control $u(t)$ to steer the system

$$\ddot{x}(t) = u(t)$$

from $x(0) = 10, \dot{x}(0) = 0$ to the origin
$x(t_f) = 0, \dot{x}(t_f) = 0$, and to minimize

$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_{t_0}^{t_f} b\, u^2(t)dt\,, \quad \alpha, b > 0$$

- Solution: optimal time is

$$t_f = \left(\frac{1800b}{\alpha}\right)^{1/5}$$

Necessary conditions for optimal control
(with unbounded controls)

We want to prove that, with unbounded controls, the necessary optimality conditions are ($H = g + \mathbf{p}^{\mathrm{T}}\mathbf{f}$ is the Hamiltonian)

$$\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \mathbf{p}}\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t\right)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}}\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t\right) \qquad \text{for all } t \in [t_0, t_f]$$

$$\mathbf{0} = \frac{\partial H}{\partial \mathbf{u}}\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t\right)$$

along with the boundary conditions:

$$\left[\frac{\partial h}{\partial \mathbf{x}}\left(\mathbf{x}^*(t_f), t_f\right) - \mathbf{p}^*(t_f)\right]^{T} \delta \mathbf{x}_f + \left[H\left(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f\right) + \frac{\partial h}{\partial t}\left(\mathbf{x}^*(t_f), t_f\right)\right]\delta t_f = 0$$

Proof sketch of NOC

- For simplicity, assume that the terminal penalty is equal to zero, and that t_f and $\mathbf{x}(t_f)$ are fixed and given

- Consider the augmented cost function
$$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) := g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}(t)^T [\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)]$$

  where the $\{p_i(t)\}$'s are Lagrange multipliers

- Note that we have simply added zero to the cost function!

- The augmented cost function is then
$$J_a(\mathbf{u}) = \int_{t_0}^{t_f} g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) \; dt$$

Proof sketch of NOC

On an extremal, by applying the fundamental theorem of the CoV

By the CoV
theorem

$$0 = \delta J_a(\mathbf{u}) = \int_{t_0}^{t_f} \left(\left[\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) - \frac{d}{dt} \frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{x}(t) \right.$$

$$\left. + \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{u}(t) + \left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{p}(t) \right) dt$$

Proof sketch of NOC

On an extremal, by applying the fundamental theorem of the CoV

By the CoV
theorem

$$= \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)^T \mathbf{p}^*(t) \qquad = -\frac{d}{dt}(-\mathbf{p}^*(t))$$

$$0 = \delta J_a(\mathbf{u}) = \int_{t_0}^{t_f} \left(\left[\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) - \frac{d}{dt} \frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{x}(t) \right.$$

$$\left. + \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{u}(t) + \left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{p}(t) \right) dt$$

$$= \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)$$

Proof sketch of NOC

Considering each term in sequence,

- $\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t) = \mathbf{0}$, on an extremal

- The Lagrange multipliers are arbitrary, so we can select them to make the coefficient of $\delta\mathbf{x}(t)$ equal to zero, that is

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)^T \mathbf{p}^*(t)$$

- The remaining variation $\delta\mathbf{u}(t)$, is independent, so its coefficient must be zero; thus

$$\frac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)^T \mathbf{p}^*(t) = \mathbf{0}$$

By using the Hamiltonian formalism, one obtains the claim

Necessary conditions for optimal control
(with bounded controls)

- So far, we have assumed that the admissible controls and states are not constrained by any boundaries

- However, in realistic systems, such constraints do commonly occur
  - control constraints often occur due to actuation limits
  - state constraints often occur due to safety considerations

- We will now consider the case with control constraints, which will lead to the statement of the Pontryagin's minimum principle

Why do control constraints complicate the analysis?

- By definition, the control $\mathbf{u}^*$ causes the functional J to have a relative minimum if

$$J(\mathbf{u}) - J(\mathbf{u}^*) = \Delta J \geq 0$$

for all admissible controls "close" to $\mathbf{u}^*$

- If we let $\mathbf{u} = \mathbf{u}^* + \delta\mathbf{u}$, the increment in J can be expressed as

$$\Delta J(\mathbf{u}^*, \delta\mathbf{u}) = \delta J(\mathbf{u}^*, \delta\mathbf{u}) + \text{higher order terms}$$

- The variation $\delta\mathbf{u}$ is arbitrary *only if* the extremal control is strictly within the boundary for all time in the interval $[t_0, t_f]$

- In general, however, an extremal control lies on a boundary during at least one subinterval of the interval $[t_0, t_f]$

Why do control constraints complicate the analysis?

- As a consequence, admissible control variations $\delta\mathbf{u}$ exist whose negatives ($-\delta\mathbf{u}$) are not admissible

- This implies that a necessary condition *for* $\mathbf{u}^*$ to minimize J is
$$\delta J(\mathbf{u}^*, \delta\mathbf{u}) \geq 0$$

  for all admissible variations with $\|\delta\mathbf{u}\|$ small enough

Pontryagin's minimum principle

- Assuming bounded controls $\mathbf{u} \in U$, the necessary optimality conditions are (H is the Hamiltonian)

$$\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \leq H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t), \text{ for all } \mathbf{u}(t) \in U$$

for all $t \in [t_0, t_f]$

along with the boundary conditions:

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}_f + \left[H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]\delta t_f = 0$$

Additional necessary conditions

1. If the final time is fixed and the Hamiltonian does not depend explicitly on time, then

$$H\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)\right) = c \quad \text{for all } t \in \left[t_0, t_f\right]$$

2. If the final time is free and the Hamiltonian does not depend explicitly on time, then

$$H\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)\right) = 0 \quad \text{for all } t \in \left[t_0, t_f\right]$$

Next time

- Notable applications of PMP
- Intro to dynamic programming