# AA 203
# Optimal and Learning-Based Control
## Pontryagin's maximum principle and indirect methods

Autonomous Systems Laboratory

Stanford University

April 10, 2023
(last updated May 2, 2023)

# Agenda

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1. Geometry and generalizations of first-order NOCs
2. Weak Pontryagin maximum principle in discrete-time
3. Weak Pontryagin maximum principle in continuous-time
4. Pontryagin maximum principle in continuous-time
5. Indirect methods for optimal control
6. Time-optimal control problems

# Review: First-order NOCs

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \ & f(x) \\ \text{subject to } & h(x) = 0 \\ & g(x) \preceq 0 \end{aligned} \qquad L(x,\lambda,\mu) \coloneqq f(x) + \lambda^\mathsf{T} h(x) + \mu^\mathsf{T} g(x)$$

**Theorem (First-order NOCs)**

*Suppose* $x^* \in \mathbb{R}^n$ *is a local minimum of* $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ *subject to* $h(x^*) = 0$ *and* $g(x^*) \preceq 0$ *with* $h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ *and* $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^r)$. *Moreover, assume*

$$\{\nabla h_i(x^*)\}_{i=1}^m \cup \{\nabla g_j(x^*)\}_{j\in\mathcal{A}_g(x^*)}$$

*are linearly independent. Then there exist unique* $\lambda^* \in \mathbb{R}^m$ *and* $\mu^* \in \mathbb{R}^r$ *such that*

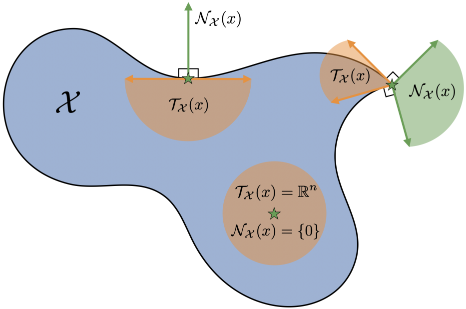
$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0, \qquad \mu^* \succeq 0, \qquad \mu_j^* = 0, \ \forall j \notin \mathcal{A}_g(x^*),$$

The assumption on the constraint gradients is known as the *linear independence constraint qualification (LICQ)*.

## Geometry of first-order NOCs

Tangent cone $\mathcal{T}_\mathcal{X}(x)$ "vectors that stay in $\mathcal{X}$"

Normal cone $\mathcal{N}_\mathcal{X}(x)$ "vectors that leave $\mathcal{X}$"




If $x^*$ is a local minimum of $f$ over $\mathcal{X}$, then $-\nabla f(x^*) \in \mathcal{N}_\mathcal{X}(x^*)$, i.e., there is no feasible component of $-\nabla f(x^*)$ that would allow us to locally decrease $f(x^*)$.

For convenience, we write "$-\nabla f(x^*) \perp_{x^*} \mathcal{X}$". In other literature, you may see "$-\nabla f(x^*) \perp \mathcal{T}_\mathcal{X}(x^*)$".

If $\mathcal{X} = \{x \in \mathbb{R}^n \mid h(x) = 0,\ g(x) \preceq 0\}$ and the LICQ holds at $x^* \in \mathcal{X}$, then

$$\mathcal{T}_\mathcal{X}(x^*) = \left\{ d \in \mathbb{R}^n \mid \frac{\partial h}{\partial x}(x^*) d = 0,\ \nabla g_j(x^*)^\mathsf{T} d \le 0,\ \forall j \in \mathcal{A}_g(x^*) \right\}$$

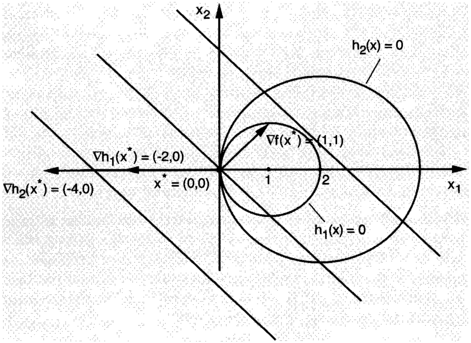
$$\mathcal{N}_\mathcal{X}(x^*) = \left\{ v \in \mathbb{R}^n \mid v = \frac{\partial h}{\partial x}(x^*)^\mathsf{T} \lambda + \frac{\partial g}{\partial x}(x^*)^\mathsf{T} \mu,\ \mu \succeq 0,\ \mu_j = 0, \forall j \notin \mathcal{A}_g(x^*) \right\}$$

# Example: A problem with linearly dependent constraints

$$\begin{aligned} \underset{x\in\mathbb{R}^2}{\text{minimize}} \quad & f(x) := x_1 + x_2 \\ \text{subject to} \quad & h_1(x) := (x_1-1)^2 + x_2^2 - 1 = 0 \\ & h_2(x) := (x_1-2)^2 + x_2^2 - 4 = 0 \end{aligned}$$

At the only feasible point $x^* = 0$, we have

$$\nabla f(x^*) = (1,1)$$
$$\nabla h_1(x^*) = (-2,0), \;\; \nabla h_2(x^*) = (-4,0)$$




The constraint gradients are linearly dependent (i.e., the LICQ does not hold), so we cannot write $\nabla f(x^*) + \lambda_1^* \nabla h_1(x^*) + \lambda_2^* \nabla h_2(x^*) = 0$.

In essence, the constraints "pinch together" so that just one $x^*$ is feasible, regardless of the objective value.

# Fritz John first-order NOCs

**Theorem (Fritz John first-order NOCs)**

*Let* $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, $h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$, *and* $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^r)$. *Suppose* $x^* \in \mathbb{R}^n$ *is a local minimum of the problem*

$$\begin{aligned} \underset{x \in \mathcal{S}}{\text{minimize}} \; & f(x) \\ \text{subject to } & h(x) = 0 \cdot \\ & g(x) \preceq 0 \end{aligned}$$

*Then there exist* $(\eta, \lambda^*, \mu^*) \in \{0, 1\} \times \mathbb{R}^m \times \mathbb{R}^r$ *such that*

$$\begin{aligned} (\eta, \lambda^*, \mu^*) &\neq 0 && \textit{non-triviality} \\ -\nabla_x L_\eta(x^*, \lambda^*, \mu^*) &\perp_{x^*} \mathcal{S} && \textit{stationarity} \\ \mu_j^* \geq 0, \;\; \mu_j^* g_j(x^*) &= 0, \;\; \forall j \in \{1, 2, \ldots, r\} && \textit{complementarity} \end{aligned}$$

*where* $L_\eta(x, \lambda, \mu)$ *is the partial Lagrangian*

$$L_\eta(x, \lambda, \mu) := \eta f(x) + \lambda^\mathsf{T} h(x) + \mu^\mathsf{T} g(x).$$

# Fritz John first-order NOCs

**Theorem (Fritz John first-order NOCs)**

*If $x^*$ is a local minimum, there exist $(\eta, \lambda^*, \mu^*) \in \{0,1\} \times \mathbb{R}^m \times \mathbb{R}^r$ such that*

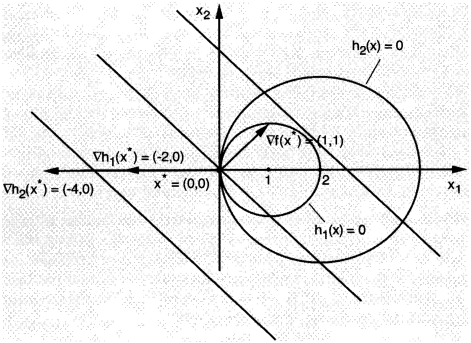
$$(\eta, \lambda^*, \mu^*) \neq 0$$

$$-\nabla_x L_\eta(x^*, \lambda^*, \mu^*) \perp_{x^*} \mathcal{S}$$

$$\mu_j^* \geq 0, \;\; \mu_j^* g_j(x^*) = 0, \;\; \forall j \in \{1, 2, \ldots, r\}$$

*where $L_\eta(x, \lambda, \mu)$ is the partial Lagrangian*

$$L_\eta(x, \lambda, \mu) := \eta f(x) + \lambda^\mathsf{T} h(x) + \mu^\mathsf{T} g(x).$$



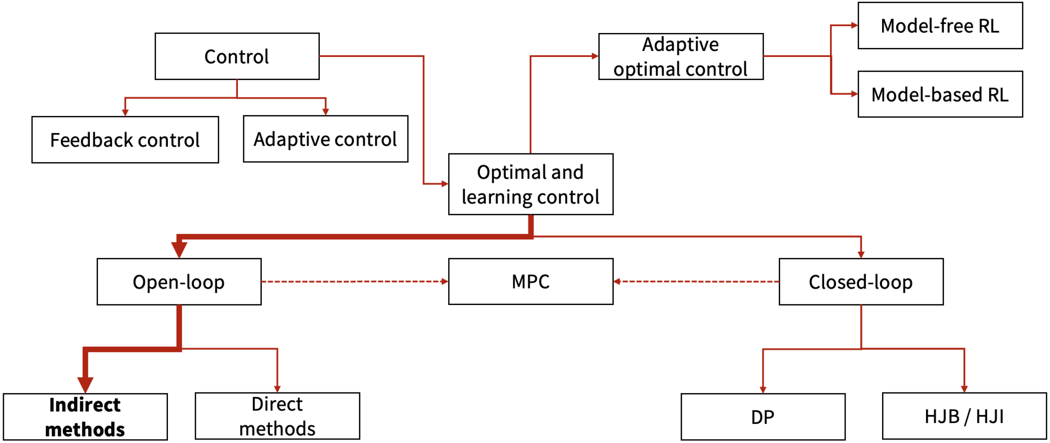

The "abnormal case" $\eta = 0$ yields necessary conditions independent of the objective $f$.

**Corollary**

*If $\mathcal{S} = \mathbb{R}^n$ and the LICQ holds, then $\eta = 1$ and $\nabla_x L_1(x^*, \lambda^*, \mu^*) = 0$.*

# Agenda

# Course overview



# Optimal control problem (discrete-time)

Consider the discrete-time optimal control problem (OCP)

$$
\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & \ell_T(x_T) + \sum_{t=0}^{T-1} \ell(t, x_t, u_t) && \text{cost (terminal + stage)} \\
\text{subject to} \quad & x_{t+1} = f(t, x_t, u_t), \ \forall t \in \{0, 1, \dots, T-1\} && \text{dynamical feasibility} \\
& x_0 = \bar{x}_0 && \text{initial condition} \\
& x_T \in \mathcal{X}_T && \text{terminal condition} \\
& u_t \in \mathcal{U}, \ \forall t \in \{0, 1, \dots, T-1\} && \text{input constraints}
\end{aligned}
$$

An optimal control $u^* = \{u_t^*\}_{t=0}^{T-1}$ for a specific initial state $\bar{x}_0$ is an *open-loop* input.

An optimal control of the form $u_t^* = \pi^*(t, x_t)$ is a *closed-loop* input.

# Lagrangian, Hamiltonian, and the adjoint equation (discrete-time)

The partial Lagrangian is

$$\begin{aligned} L_\eta(x,u,p) &= \eta\ell_T(x_T) + \underbrace{p_0^\mathsf{T}(x_0 - \bar{x}_0)}_{\text{initial condition}} + \sum_{t=0}^{T-1}\Big(\eta\ell(t,x_t,u_t) + \underbrace{p_{t+1}^\mathsf{T}(x_{t+1} - f(t,x_t,u_t))}_{\text{dynamical feasibility}}\Big) \\ &= \eta\ell_T(x_T) + p_0^\mathsf{T}(x_0 - \bar{x}_0) + \sum_{t=0}^{T-1}\big(p_{t+1}^\mathsf{T}x_{t+1} - H_\eta(t,x_t,u_t,p_{t+1})\big) \end{aligned},$$

with normality $\eta \in \{0,1\}$, Lagrange multipliers $\{p_t\}_{t=0}^T \subset \mathbb{R}^n$, and *Hamiltonian*

$$H_\eta(t,x,u,p) \coloneqq p^\mathsf{T} f(t,x,u) - \eta\ell(t,x,u).$$

Setting $\nabla_{x_t} L(x^*,u^*) = 0$ for $t \in \{0,1,\ldots,T-1\}$ yields

$$p_t^* = \nabla_x H_\eta(t,x_t^*,u_t^*,p_{t+1}^*),\ \forall t \in \{0,1,\ldots,T-1\},$$

which is a *backwards recursion* for the *adjoint* or *co-state* $p_t^*$.

# Transversality and the maximum condition (discrete-time)

The partial Lagrangian is

$$L_\eta(x,u,p) = \eta\ell_T(x_T) + p_0^\mathsf{T}(x_0 - \bar{x}_0) + \sum_{t=0}^{T-1}\left(p_{t+1}^\mathsf{T}x_{t+1} - H_\eta(t,x_t,u_t,p_{t+1})\right)$$

where we left out $x_T \in \mathcal{X}_T$ and $u_t \in \mathcal{U}$. Setting $-\nabla_{x_T} L_\eta(x^*,u^*) \perp_{x_T^*} \mathcal{X}_T$ yields the *transversality condition*

$$-p_T^* - \eta\,\nabla\ell_T(x_T^*) \perp_{x_T^*} \mathcal{X}_T,$$

and setting $-\nabla_{u_t} L(x^*,u^*) \perp_{u_t^*} \mathcal{U}$ yields the *weak maximum condition*

$$\nabla_u H_\eta(t,x_t^*,u_t^*,p_{t+1}^*) \perp_{u_t^*} \mathcal{U},\ \forall t \in \{0,1,\ldots,T-1\}.$$

We refer to this condition as “weak” since it is a necessary, but not sufficient condition for a solution of the problem

$$\operatorname*{maximize}_{u\in\mathcal{U}} H_\eta(t,x_t^*,u,p_{t+1}^*).$$

# Pontryagin maximum principle (discrete-time)

Collect these necessary conditions together to get the Pontryagin maximum principle (PMP).

**Theorem (Pontryagin maximum principle (discrete-time))**

*Let $(x^*, u^*)$ be a local minimum of the discrete-time OCP with terminal set $\mathcal{X}_T$ and control set $\mathcal{U}$. Then $\eta \in \{0,1\}$ and $\{p_t^*\}_{t=0}^T \subset \mathbb{R}^n$ exist such that*

$$(\eta, p_0^*, p_1^*, \dots, p_T^*) \neq 0 \qquad \textit{non-triviality}$$

$$p_t^* = \nabla_x H_\eta(t, x_t^*, u_t^*, p_{t+1}^*), \ \forall t \in \{0, 1, \dots, T-1\} \qquad \textit{adjoint equation}$$

$$-p_T^* - \eta \nabla \ell_T(x_T^*) \perp_{x_T^*} \mathcal{X}_T \qquad \textit{transversality}$$

$$\nabla_u H_\eta(t, x_t^*, u_t^*, p_{t+1}^*) \perp_{u_t^*} \mathcal{U}, \ \forall t \in \{0, 1, \dots, T-1\} \qquad \textit{maximum condition (weak)}$$

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# Optimal control problem (continuous-time)

Consider the continuous-time optimal control problem (OCP)

$$\begin{array}{lll} \underset{x,u}{\text{minimize}} & \ell_T(x(T)) + \displaystyle\int_0^T \ell(t, x(t), u(t))\, dt & \text{cost (terminal + stage)} \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t)), \ \forall t \in [0, T] & \text{dynamical feasibility} \\ & x(0) = x_0 & \text{initial condition} \\ & x(T) \in \mathcal{X}_T & \text{terminal condition} \\ & u(t) \in \mathcal{U}, \ \forall t \in [0, T] & \text{input constraints} \end{array}$$

An optimal control $u^*(t)$ for a specific initial state $x_0$ is an *open-loop* input.

An optimal control of the form $u^*(t) = \pi^*(t, x(t))$ is a *closed-loop* input.

# Discretized OCPs

Consider piecewise continuous trajectories such that $x(t) = x(t_k)$ and $u(t) = u(t_k)$ for $t \in [t_k, t_{k+1})$, with $k \in \{0, 1, \ldots, N-1\}$, $t_0 = 0$ and $t_N = T$.

Define $\Delta t_k \coloneqq t_{k+1} - t_k$ such that $\Delta t_k > 0$ for all $k \in \{0, 1, \ldots, N-1\}$.

Consider the discretized OCP

$$
\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & \ell_T(x(t_N)) + \sum_{k=0}^{N-1} \Delta t_k \ell(t_k, x(t_k), u(t_k)) \\
\text{subject to} \quad & x(t_{k+1}) = x(t_k) + \Delta t_k f(t_k, x(t_k), u(t_k)), \ \forall k \in \{0, 1, \ldots, N-1\} \\
& x(t_0) = x_0 \\
& x(t_N) \in \mathcal{X}_T \\
& u(t_k) \in \mathcal{U}, \ \forall k \in \{0, 1, \ldots, N-1\}
\end{aligned}
$$

# Discrete-time PMP as a heuristic for continuous-time OCPs

Use the discrete-time PMP on a local minimum $(x^*, u^*)$ of the discretized OCP to get

$$(\eta, p(t_0), p(t_1), \ldots, p(t_N)) \neq 0$$

$$-\frac{(p^*(t_{k+1}) - p^*(t_k))}{\Delta t_k} = \nabla_x H_\eta(t_k, x^*(t_k), u^*(t_k), p^*(t_{k+1})), \ \forall k \in \{0, 1, \ldots, N-1\}$$

$$-p^*(t_N) - \eta \nabla \ell_T(x^*(t_N)) \perp_{x^*(t_N)} \mathcal{X}_T$$

$$\nabla_u H_\eta(t_k, x^*(t_k), u^*(t_k), p^*(t_{k+1})) \perp_{u_t^*} \mathcal{U}, \ \forall k \in \{0, 1, \ldots, N-1\}$$

where we use the *continuous-time Hamiltonian*

$$H_\eta(t, x, u, p) \coloneqq p^\mathsf{T} f(t, x, u) - \eta \ell(t, x, u).$$

# Pontryagin maximum principle (continuous-time, weak)

The above conditions suggest the following continuous-time PMP as $\Delta t_k \to 0$.

**Theorem (Pontryagin maximum principle (continuous-time, weak))**

*Let $(x^*, u^*)$ be a local minimum of the continuous-time optimal control problem with terminal set $\mathcal{X}_T$ and control set $\mathcal{U}$. Then $\eta \in \{0, 1\}$ and $p^* : [0, T] \to \mathbb{R}^n$ exist such that*

$$
\begin{aligned}
(\eta, p(t)) &\not\equiv 0 && \textit{non-triviality} \\
-\dot{p}^*(t) &= \nabla_x H_\eta(t, x^*(t), u^*(t), p^*(t)),\ \forall t \in [0, T] && \textit{adjoint equation} \\
-p^*(T) - \eta \nabla \ell_T(x^*(T)) &\perp_{x^*(T)} \mathcal{X}_T && \textit{transversality} \\
\nabla H_\eta(t, x^*(t), u^*(t), p^*(t)) &\perp_{u^*(t)} \mathcal{U},\ \forall t \in [0, T] && \textit{maximum condition (weak)}
\end{aligned}
$$

"$(\eta, p(t)) \not\equiv 0$" means there exists at least one $t \in [0, T]$ such that $(\eta, p(t)) \neq 0$.

# Agenda

1. Geometry and generalizations of first-order NOCs
2. Weak Pontryagin maximum principle in discrete-time
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# Norms in function spaces

Recall that $(x^*, u^*)$ is a *local minimum* of $J(x^*, u^*)$ if there exists $\varepsilon > 0$ such that $J(x^*, u^*) \leq J(x, u)$ for all $(x, u)$ in the $\varepsilon$-sized norm ball around $(x^*, u^*)$.

In using the discrete-time PMP as a heuristic to obtain the continuous-time PMP, we are implicitly using the $\mathcal{C}^0$*-norm* for both $x^*$ and $u^*$, i.e.,

$$\|x - x^*\|_{\mathcal{C}^0} \coloneqq \max_{t \in [0,T]} \|x(t) - x^*(t)\|, \quad \|u - u^*\|_{\mathcal{C}^0} \coloneqq \max_{t \in [0,T]} \|u(t) - u^*(t)\|.$$

We can strengthen the continuous-time PMP if we use the $\mathcal{C}^0$-norm for $x^*$ and the $\mathcal{L}^1$-norm for $u^*$, i.e.,

$$\|x - x^*\|_{\mathcal{C}^0} \coloneqq \max_{t \in [0,T]} \|x(t) - x^*(t)\|, \quad \|u - u^*\|_{\mathcal{L}^1} \coloneqq \int_0^T \|u(t) - u^*(t)\| \, dt.$$

## Strengthening the maximum condition via needle perturbations

In general, the $\mathcal{L}^1$-norm ball for $u^*$ allows for *large pointwise variations* at each time $t$. Suppose the control set $\mathcal{U}$ is bounded, i.e., $\|u - v\| \leq c$ for all $u, v \in \mathcal{U}$ and some $c > 0$.

Given some $u^* : [0, T] \to \mathcal{U}$, any $\tau \in [0, T)$ and $\varepsilon > 0$ such that $[\tau, \tau + \varepsilon) \subset [0, T]$, and any $v \in \mathcal{U}$, define

$$u(t) = \begin{cases} v, & t \in [\tau, \tau + \varepsilon) \\ u^*(t), & t \in [0, \tau) \cup [\tau + \varepsilon, T] \end{cases}$$

This is a *spatial needle perturbation* of $u^*(t)$. Then it can be shown that

$$\|u - u^*\|_{\mathcal{L}^1} \coloneqq \int_0^T \|u(t) - u^*(t)\| \, dt = \int_\tau^{\tau+\varepsilon} \|v - u^*(t)\| \, dt \leq \int_\tau^{\tau+\varepsilon} c \, dt = \varepsilon c.$$

$$x(T) \approx x^*(T) + \varepsilon d, \; d \in \mathcal{T}_{\mathcal{X}_T}(x^*(T))$$

for small enough $\varepsilon$. Overall, a large spatial perturbation in $u^*(t)$ can correspond to small feasible perturbations to both $x^*$ and $u^*$.

# Pontryagin maximum principle (continuous-time)

The possibility of large spatial control perturbations still corresponding to "feasible neighbours" of $(x^*, u^*)$ suggests the following strengthened PMP.

**Theorem (Pontryagin maximum principle (continuous-time))**

*Let $(x^*, u^*)$ be a local minimum (using the $\mathcal{C}^0$-norm and $\mathcal{L}^1$-norm, respectively) of the continuous-time OCP with terminal set $\mathcal{X}_T$ and bounded control set $\mathcal{U}$. Then $\eta \in \{0, 1\}$ and $p^* : [0, T] \to \mathbb{R}^n$ exist such that*

$$(\eta, p^*(t)) \not\equiv 0 \qquad \textit{non-triviality}$$

$$-\dot{p}^*(t) = \nabla_x H_\eta(t, x^*(t), u^*(t), p^*(t)), \ \forall t \in [0, T] \qquad \textit{adjoint equation}$$

$$-p^*(T) - \eta \nabla \ell_T(x^*(T)) \perp_{x^*(T)} \mathcal{X}_T \qquad \textit{transversality}$$

$$H_\eta(t, x^*(t), u^*(t), p^*(t)) = \sup_{u \in \mathcal{U}} H_\eta(t, x^*(t), u, p^*(t)), \ \forall t \in [0, T] \qquad \textit{maximum condition}$$

A rigorous proof relies on variational calculus (Liberzon, 2012; Clarke, 2013).

# Example: Minimum fuel for a control-affine system

Consider the continuous-time OCP

$$
\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & \int_0^T \sum_{j=1}^m \alpha_j |u_j(t)| \, dt \\
\text{subject to} \quad & \dot{x}(t) = a(t, x(t)) + \sum_{j=1}^m u_j(t) b_j(t, x(t)), \ \forall t \in [0, T] \\
& x(0) = x_0 \\
& x(T) = 0 \\
& -\bar{u} \preceq u(t) \preceq \bar{u}, \ \forall t \in [0, T]
\end{aligned}
$$

where $\bar{u} \succ 0$. The Hamiltonian is

$$
H_\eta(t, x, u, p) = p^\mathsf{T} \left( a(t, x) + \sum_{j=1}^m u_j b_j(t, x) \right) - \eta \sum_{j=1}^m \alpha_j |u_j|
$$

# Example: Minimum fuel for a control-affine system

The Hamiltonian is

$$H_\eta(t,x,u,p) = a(t,x)^\mathsf{T} p + \sum_{j=1}^{m} \big(u_j b_j(t,x)^\mathsf{T} p - \eta\alpha_j |u_j|\big)$$

The adjoint equation is

$$\dot{p}^* = -\nabla_x H_\eta(t,x^*,u^*,p^*) = -\frac{\partial a}{\partial x}(t,x^*)p^* - \sum_{j=1}^{m} u_j^* \frac{\partial b_j}{\partial x}(t,x^*)p^*$$

The maximum condition is

$$u_j^* = \operatorname*{arg\,max}_{u_j \in [-\bar{u}_j, \bar{u}_j]} \big(u_j b_j(t,x^*)^\mathsf{T} p^* - \eta\alpha_j |u_j|\big) = \begin{cases} -\bar{u}_j, & b_j(t,x^*)^\mathsf{T} p^* < -\eta\alpha_j \\ 0, & b_j(t,x^*)^\mathsf{T} p^* \in [-\eta\alpha_j, \eta\alpha_j] \\ \bar{u}_j, & b_j(t,x^*)^\mathsf{T} p^* > \eta\alpha_j \end{cases},$$

which for $\eta = 1$ is an example of "bang-off-bang" control.

# Example: Minimum fuel for a control-affine system

Assume $\eta = 1$, i.e., the "normal" case. Altogether, we have the *boundary value problem (BVP)*

$$\begin{pmatrix} \dot{x}^* \\ \dot{p}^* \end{pmatrix} = \begin{pmatrix} a(t, x^*) + \sum_{j=1}^m u_j^* b_j(t, x^*) \\ -\frac{\partial a}{\partial x}(t, x^*) p^* - \sum_{j=1}^m u_j^* \frac{\partial b_j}{\partial x}(t, x^*) p^* \end{pmatrix}, \quad u_j^* = \begin{cases} -\bar{u}_j, & b_j(t, x^*)^\mathsf{T} p^* < -\alpha_j \\ 0, & b_j(t, x^*)^\mathsf{T} p^* \in [-\alpha_j, \alpha_j] \\ \bar{u}_j, & b_j(t, x^*)^\mathsf{T} p^* > \alpha_j \end{cases},$$

with boundary conditions $x^*(0) = x_0$ and $x^*(T) = 0$.

Transversality did not factor into this problem, since the normal cone of the singleton $\mathcal{X}_T = \{0\}$ is just $\mathbb{R}^n$ (i.e., any direction "leaves" the terminal set).

# Agenda

# Indirect methods for optimal control

An *indirect method* generally focuses on solving the BVP

$$\begin{pmatrix} \dot{x}^* \\ \dot{p}^* \end{pmatrix} = \begin{pmatrix} f(t, x^*, u^*) \\ -\nabla_x H_\eta(t, x^*, u^*(t, x^*, p^*), p^*) \end{pmatrix}, \quad x^*(0) = x_0, \quad h(x^*(T), p^*(T)) = 0.$$

where $h(x^*(T), p^*(T)) \in \mathbb{R}^n$. The *open-loop* optimal control candidate $u^*(t, x^*(t), p^*(t))$ is then extracted.

The boundary condition $h(x^*(T), p^*(T)) = 0$ is determined by the terminal set constraint $x^*(T) \in \mathcal{X}_T$ and the transversality condition $-p^*(T) - \eta \nabla \ell_T(x^*(T)) \perp_{x^*(T)} \mathcal{X}_T$.

We are implicitly assuming an optimal control exists. Even then, there may be multiple local optima.

# Shooting methods

To solve the BVP

$$\begin{pmatrix} \dot{x}^* \\ \dot{p}^* \end{pmatrix} = \begin{pmatrix} f(t, x^*, u^*) \\ -\nabla_x H_\eta(t, x^*, u^*(t, x^*, p^*), p^*) \end{pmatrix}, \quad x^*(0) = x_0, \quad h(x^*(T), p^*(T)) = 0,$$

we consider the associated initial value problem (IVP)

$$\begin{pmatrix} \dot{x}^* \\ \dot{p}^* \end{pmatrix} = \begin{pmatrix} f(t, x^*, u^*) \\ -\nabla_x H_\eta(t, x^*, u^*(t, x^*, p^*), p^*) \end{pmatrix}, \quad x^*(0) = x_0, \quad p^*(0) = p_0.$$

We can integrate the IVP forward in time to get $x^*(T; p_0)$ and $p^*(T; p_0)$, which are parameterized by $p_0$.

We can use a root-finding method (e.g., bisection search, Newton-Raphson method) to find $p_0$ such that $h(x^*(T; p_0), p^*(T; p_0)) = 0$. This is called *single shooting* and gives us a solution of the BVP.

## Agenda

Consider the continuous-time OCP

$$\begin{aligned} \underset{x,u,T\geq 0}{\text{minimize}}\ & \ell_T(T,x(T)) + \int_0^T \ell(t,x(t),u(t))\,dt && \text{cost (terminal + stage)} \\ \text{subject to}\ & \dot{x}(t) = f(t,x(t),u(t)),\ \forall t \in [0,T] && \text{dynamical feasibility} \\ & x(0) = x_0 && \text{initial condition} \\ & x(T) \in \mathcal{X}_T && \text{terminal condition} \\ & u(t) \in \mathcal{U},\ \forall t \in [0,T] && \text{input constraints} \end{aligned}$$

The final time $T$ is now a *free variable* (subject to $T \geq 0$).

# Time-optimal control problems

Use the change of variables $t(s) = Ts$ with $s \in [0, 1]$ to get

$$\begin{array}{lll}
\underset{(x,t),(u,T)}{\text{minimize}} & \ell_T(t(1), x(1)) + T \int_0^1 \ell(t(s), x(s), u(s))\, ds & \text{cost (terminal + stage)} \\
\text{subject to} & \dot{x}(s) = T f(t(s), x(s), u(s)),\ \dot{t}(s) = T,\ \forall s \in [0, 1] & \text{dynamical feasibility} \\
& x(0) = x_0,\ t(0) = 0 & \text{initial condition} \\
& x(1) \in \mathcal{X}_T & \text{terminal condition} \\
& u(s) \in \mathcal{U},\ T \in [0, \infty),\ \forall s \in [0, 1] & \text{input constraints}
\end{array}$$

To derive a new form of the PMP for time-optimal problems, we apply the fixed final time PMP to the problem above, where we treat $t$ and $T$ as a new state and input, respectively.

# Deriving the time-optimal PMP

Applying the fixed final time PMP gives us the Hamiltonian

$$\tilde{H}_\eta(s, x, t, u, T, p, \lambda) = T(H(t, x, u, p) + \lambda),$$

where $H(t, x, u, p)$ is the usual Hamiltonian, and $\lambda$ is the adjoint for the new "state" $t(s) = Ts$. Taking derivatives with respect to $(x, t)$ yields the adjoint equations

$$\frac{dp^*}{ds} = -T^* \nabla_x H(t, x^*, u^*, p^*), \quad \frac{d\lambda^*}{ds} = -T^* \frac{\partial H}{\partial t}(t, x^*, u^*, p^*),$$

which by the chain rule with $\frac{dt}{ds} = T$ become

$$\dot{p}^* = -\nabla_x H(t, x^*, u^*, p^*), \quad \dot{\lambda}^* = -\frac{\partial H}{\partial t}(t, x^*, u^*, p^*).$$

Since $t$ has no terminal constraint, we have the transversality conditions

$$-p^*(1) - \eta \nabla_x \ell_T(t(1), x^*(1)) \perp_{x^*(1)} \mathcal{X}_T, \quad -\lambda^*(1) - \eta \nabla_T \ell_T(t(1), x^*(1)) = 0.$$

which after using $t = sT$ gives us

$$-p^*(T) - \eta \nabla_x \ell_T(T^*, x^*(T)) \perp_{x^*(T)} \mathcal{X}_T, \quad -\lambda^*(T^*) = \eta \nabla_T \ell_T(T^*, x^*(T)).$$

# Deriving the time-optimal PMP

Applying the fixed final time PMP gives us the Hamiltonian

$$\tilde{H}_\eta(s, x, t, u, T, p, \lambda) = T(H(t, x, u, p) + \lambda),$$

where $H(t, x, u, p)$ is the usual Hamiltonian, and $\lambda$ is the adjoint for the new "state" $t(s) = Ts$

We are considering the absolute value norm for $T$, and $[0, \infty)$ is unbounded. So we use the maximum condition for $u^*$ and the weak maximum condition for $T^*$ to get

$$\nabla_T \tilde{H}_\eta(t, x^*, u^*, p^*) \perp_{T^*} [0, \infty) \implies H(t, x^*, u^*, p^*) + \lambda^* = 0,$$

where we have assumed $T^* > 0$ to get that the normal cone is just $\{0\}$. Evaluating this condition at $t = T^*$ gives us

$$H(T^*, x^*(T^*), u^*(T^*), p^*(T^*)) = -\lambda^*(T^*) = \eta \nabla_t \ell_T(T^*, x^*(T)),$$

which is the additional boundary condition we need for free final time $T^*$.

# Time-optimal PMP

Collecting all of the conditions we derived above gives us the free final time PMP.

**Theorem (Pontryagin maximum principle (continuous-time, free final time))**

*Let $(x^*, u^*, T^*)$ be a local minimum (using the $\mathcal{C}^0$-norm, $\mathcal{L}^1$-norm, and absolute value, respectively) of the continuous-time OCP with terminal set $\mathcal{X}_T$, bounded control set $\mathcal{U}$, and free final time $T \geq 0$. Then $\eta \in \{0, 1\}$ and $p^* : [0, T^*] \to \mathbb{R}^n$ exist such that*

$$(\eta, p^*(t)) \not\equiv 0 \qquad \textit{non-triviality}$$

$$-\dot{p}^*(t) = \nabla_x H_\eta(t, x^*(t), u^*(t), p^*(t)), \ \forall t \in [0, T^*] \qquad \textit{adjoint equation}$$

$$-p^*(T^*) - \eta \nabla \ell_T(T^*, x^*(T^*)) \perp_{x^*(T)} \mathcal{X}_T \qquad \textit{transversality}$$

$$H_\eta(t, x^*(t), u^*(t), p^*(t)) = \sup_{u \in \mathcal{U}} H_\eta(t, x^*(t), u, p^*(t)), \ \forall t \in [0, T^*] \qquad \textit{maximum condition}$$

$$H_\eta(T^*, x^*(T^*), u^*(T^*), p^*(T^*)) = \eta \frac{\partial \ell_T}{\partial T}(T^*, x^*(T^*)) \qquad \textit{maximum condition (boundary)}$$

# Next class

Direct methods for optimal control
(i.e., solving discretized optimal control problems directly)

# References


F. Clarke. *Functional Analysis, Calculus of Variations and Optimal Control*. Springer, 2013.

D. Liberzon. *Calculus of Variations and Optimal Control Theory: A Concise Introduction*. Princeton University Press, 2012.