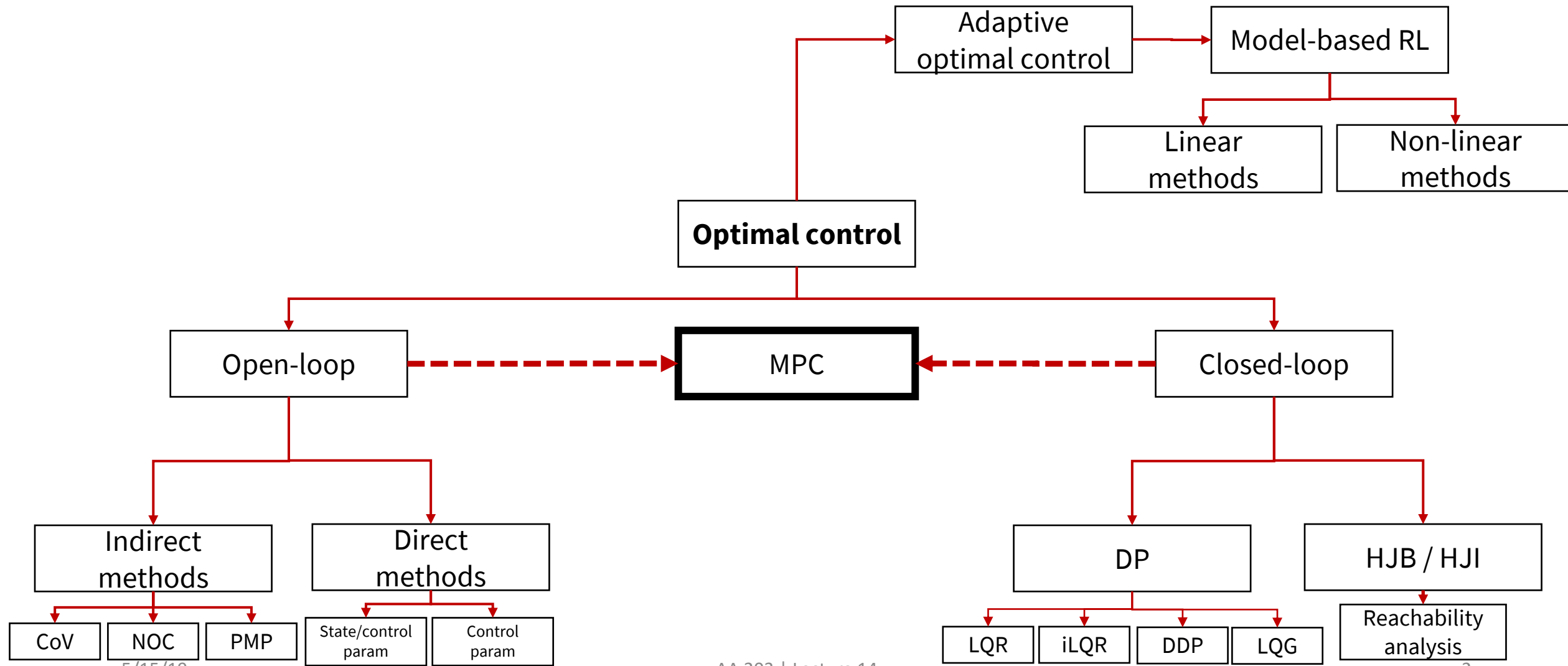


AA203

Optimal and Learning-based Control

Stability of MPC, implementation aspects

Roadmap



Stability of MPC

- Persistent feasibility does not guarantee that the closed-loop trajectories converge towards the desired equilibrium point
- One of the most popular approaches to guarantee persistent feasibility and stability of the MPC law makes use of a control invariant terminal set X_f for feasibility, and of a terminal function $p(\cdot)$ for stability
- To prove stability, we leverage the tool of **Lyapunov stability theory**

Lyapunov stability theory

- **Lyapunov theorem:** Consider the equilibrium point $\mathbf{x} = 0$ for the autonomous system $\{\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)\}$ (with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$). Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded set containing the origin. Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function, continuous at the origin, such that

$$V(\mathbf{0}) = 0 \text{ and } V(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \Omega \setminus \{\mathbf{0}\}$$

$$V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k) < 0 \quad \forall \mathbf{x}_k \in \Omega \setminus \{\mathbf{0}\}$$

Then $\mathbf{x} = 0$ is asymptotically stable in Ω

- The idea is to show that with appropriate choices of X_f and $p(\cdot)$, J_0^* is a Lyapunov function for the closed-loop system

MPC stability theorem

- **MPC stability theorem** (for quadratic cost): Assume

A0: $Q = Q' > 0, R = R' > 0, P > 0$

A1: Sets X, X_f and U contain the origin in their interior and are closed

A2: $X_f \subseteq X$ is control invariant

A3:
$$\min_{\mathbf{v} \in U, A\mathbf{x} + B\mathbf{v} \in X_f} \left(-p(\mathbf{x}) + q(\mathbf{x}, \mathbf{v}) + p(A\mathbf{x} + B\mathbf{v}) \right) \leq 0, \forall \mathbf{x} \in X_f$$

Then, the origin of the closed-loop system is asymptotically stable with domain of attraction X_0

MPC stability theorem

- Proof:

1. Note that, by assumption A2, persistent feasibility is guaranteed for *any* P, Q, R
2. We want to show that J_0^* is a Lyapunov function for the closed-loop system $\mathbf{x}(t+1) = \mathbf{f}_{cl}(\mathbf{x}(t))$, with respect to the equilibrium $\mathbf{f}_{cl}(\mathbf{0}) = \mathbf{0}$ (the origin is indeed an equilibrium as $0 \in X, 0 \in U$, and the cost is positive for any non-zero control sequence)
3. X_0 is bounded and closed by assumption
4. $J_0^*(\mathbf{0}) = 0$ (for the same previous reasons)

MPC stability theorem

- Proof:

5. $J_0^*(\mathbf{x}) > 0$ for all $\mathbf{x} \in X_0 \setminus \{\mathbf{0}\}$

6. Next we show the decay property. Since the setup is time-invariant, we can study the decay property between $t = 0$ and $t = 1$

- Let $\mathbf{x}(0) \in X_0$, let $U_0^{[0]} = [\mathbf{u}_0^{[0]}, \mathbf{u}_1^{[0]}, \dots, \mathbf{u}_{N-1}^{[0]}]$ be the optimal control sequence, and let $[\mathbf{x}(0), \mathbf{x}_1^{[0]}, \dots, \mathbf{x}_N^{[0]}]$ be the corresponding trajectory
- After applying $\mathbf{u}_0^{[0]}$, one obtains $\mathbf{x}(1) = A\mathbf{x}(0) + B\mathbf{u}_0^{[0]}$
- Consider the sequence of controls $[\mathbf{u}_1^{[0]}, \mathbf{u}_2^{[0]}, \dots, \mathbf{u}_{N-1}^{[0]}, \mathbf{v}]$, where $\mathbf{v} \in U$, and the corresponding state trajectory is $[\mathbf{x}(1), \mathbf{x}_2^{[0]}, \dots, \mathbf{x}_N^{[0]}, A\mathbf{x}_N^{[0]} + B\mathbf{v}]$

MPC stability theorem

- Since $\mathbf{x}_N^{[0]} \in X_f$ (by terminal constraint), and since X_f is control invariant,
$$\exists \bar{\mathbf{v}} \in U \mid A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}} \in X_f$$
- With such a choice of $\bar{\mathbf{v}}$, the sequence $[\mathbf{u}_1^{[0]}, \mathbf{u}_2^{[0]}, \dots, \mathbf{u}_{N-1}^{[0]}, \mathbf{v}]$ is feasible for the MPC optimization problem at time $t = 1$
- Since this sequence is not necessarily optimal

$$J_0^*(\mathbf{x}(1)) \leq p(A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}}) + \sum_{k=1}^{N-1} q(\mathbf{x}_k^{[0]}, \mathbf{u}_k^{[0]}) + q(\mathbf{x}_N^{[0]}, \mathbf{v})$$

MPC stability theorem

- Equivalently

$$J_0^*(\mathbf{x}(1)) \leq p(A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}}) + J_0^*(\mathbf{x}(0)) - p(\mathbf{x}_N^{[0]}) - q(\mathbf{x}(0), \mathbf{u}_0^{[0]}) + q(\mathbf{x}_N^{[0]}, \bar{\mathbf{v}})$$

- Since $\mathbf{x}_N^{[0]} \in X_f$, by assumption A3, we can select $\bar{\mathbf{v}}$ such that

$$J_0^*(\mathbf{x}(1)) \leq J_0^*(\mathbf{x}(0)) - q(\mathbf{x}(0), \mathbf{u}_0^{[0]})$$

- Since $q(\mathbf{x}(0), \mathbf{u}_0^{[0]}) > 0$ for all $\mathbf{x}(0) \in X_0 \setminus \{0\}$,

$$J_0^*(\mathbf{x}(1)) - J_0^*(\mathbf{x}(0)) < 0$$

- The last step is to prove continuity; details are omitted and can be found in Borrelli, Bemporad, Morari, 2017
- Note: A2 is used to guarantee persistent feasibility; this assumption can be replaced with an assumption on the horizon N

How to choose X_f and P ?

- Case 1: assume A is asymptotically stable
 - Set X_f as the maximally positive invariant set O_∞ for system $\mathbf{x}(t + 1) = A\mathbf{x}(t)$, $\mathbf{x}(t) \in X$
 - X_f is a control invariant set for system $\mathbf{x}(t + 1) = A\mathbf{x}(t) + B\mathbf{u}(t)$, as $\mathbf{u} = 0$ is a feasible control
 - As for stability, $\mathbf{u} = 0$ is feasible and $A\mathbf{x} \in X_f$ if $\mathbf{x} \in X_f$, thus assumption A3 becomes

$$-\mathbf{x}'P\mathbf{x} + \mathbf{x}'Q\mathbf{x} + \mathbf{x}'A'PA\mathbf{x} \leq 0, \text{ for all } \mathbf{x} \in X_f,$$

which is true since, due to the fact that A is asymptotically stable,

$$\exists P > 0 \mid -P + Q + A'PA = 0$$

How to choose X_f and P ?

- Case 2: general case
 - Let F_∞ be the optimal gain for the infinite-horizon LQR controller
 - Set X_f as the maximal positive invariant set for system $\{\mathbf{x}(t+1) = (A + BF_\infty)\mathbf{x}(t)\}$ (with constraints $\mathbf{x}(t) \in X$, and $F_\infty\mathbf{x}(t) \in U$)
 - Set P as the solution P_∞ to the discrete-time Riccati equation

Explicit MPC

- In some cases, the MPC law can be *pre-computed* → no need for online optimization
- Important case: constrained LQR

$$\begin{aligned} J_0^*(\mathbf{x}) = \min_{\mathbf{u}_0, \dots, \mathbf{u}_{N-1}} \quad & \mathbf{x}'_N P \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}'_k Q \mathbf{x}_k + \mathbf{u}'_k R \mathbf{u}_k \\ \text{subject to} \quad & \mathbf{x}_{k+1} = A \mathbf{x}_k + B \mathbf{u}_k, \quad k = 0, \dots, N-1 \\ & \mathbf{x}_k \in X, \quad \mathbf{u}_k \in U, \quad k = 0, \dots, N-1 \\ & \mathbf{x}_N \in X_f \\ & \mathbf{x}_0 = \mathbf{x} \end{aligned}$$

Explicit MPC

- The solution to the constrained LQR problem is a control \mathbf{u}^* which is a continuous piecewise affine function on polyhedral partition of the state space X , that is $\mathbf{u}^* = \pi(\mathbf{x})$ where

$$\pi(\mathbf{x}) = F^j \mathbf{x} + g^j \quad \text{if } H^j \mathbf{x} \leq K^j, \quad j = 1, \dots, N^r$$

- Thus, online, one has to locate in which cell of the polyhedral partition the state \mathbf{x} lies, and then one obtains the optimal control via a look-up table query

Tuning and Practical Use

- At present there is no other technique to design controllers for general large linear multivariable systems with input and output constraints with a stability guarantee
- Objective function: The squared 2-norm is employed more often as an indicator of control quality than the 1- or ∞ -norm
- Design approach:
 - Choose horizon length N and the control invariant target set X_f
 - Control invariant target set X_f should be as large as possible for performance
 - Choose the parameters Q and R freely to affect the control performance
 - Adjust P as per the stability theorem
 - Useful toolbox: <https://www.mpt3.org/>

Next time

- Applications of MPC, introduction to adaptive optimal control