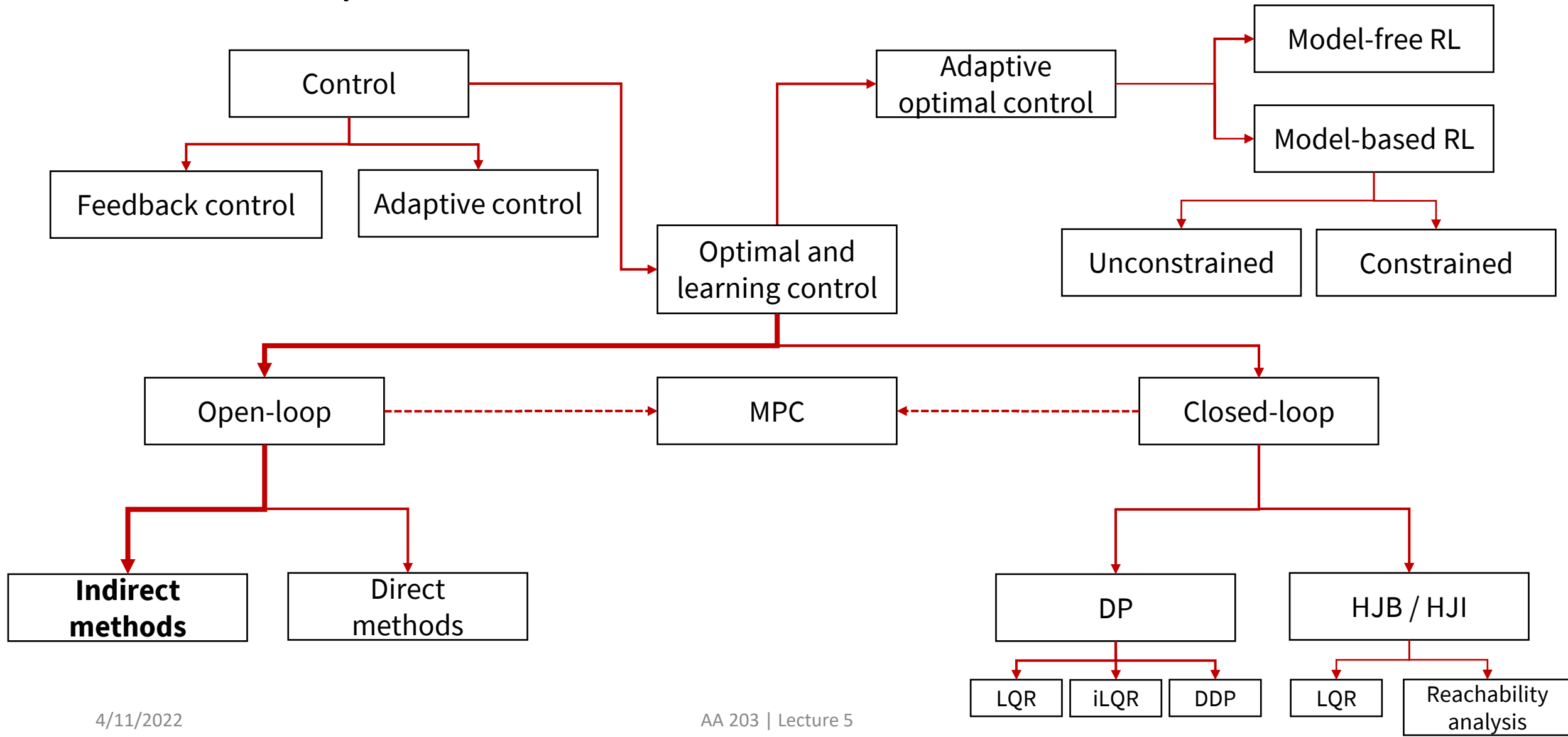


AA203

Optimal and Learning-based Control

CoV extensions, NOC for optimal control

Roadmap



CoV extension I: generalized boundary conditions

- Let $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^n$ be a vector-valued function, where each component x_i is in the class of functions with continuous first derivatives. It is desired to find the function \mathbf{x}^* for which the functional

$$J(\mathbf{x}) = \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) dt$$

has a relative extremum

- Assumptions:
 - $g \in C^2$
 - t_0 and $\mathbf{x}(0)$ are fixed
 - t_f might be fixed or free, and each component of $\mathbf{x}(t_f)$ might be fixed or free
- Reading:
 - D. E. Kirk. *Optimal Control Theory: An Introduction*, 2004.

CoV extension I: generalized boundary conditions

- Regardless of the boundary conditions, the Euler equations

$$g_{\mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt} g_{\dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) = \mathbf{0}$$

must be satisfied

CoV extension I: generalized boundary conditions

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$$g_{\mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt} g_{\dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) = \mathbf{0}$$

must be satisfied

- The required boundary conditions are found from the equation

$$g_{\dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)^T \delta \mathbf{x}_f + \left[g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) - g_{\dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)^T \dot{\mathbf{x}}^*(t_f) \right] \delta t_f = 0$$

by making the “appropriate” substitutions for $\delta \mathbf{x}_f$ and δt_f

CoV extension I: generalized boundary conditions

- $\delta \mathbf{x}_f$ and δt_f capture the notion of “allowable” variations at the end point, thus $\delta t_f = 0$ if the final time is fixed, and $\delta x_i(t_f) = 0$ if the end value of state variable $x_i(t_f)$ is fixed
- For example, suppose that t_f is fixed, $x_i(t_f)$, $i = 1, \dots, r$ are fixed, and $x_j(t_f)$, $j = r + 1, \dots, n$ are free. Then the substitutions are:

$$\begin{aligned}\delta t_f &= 0 \\ \delta x_i(t_f) &= 0, \quad i = 1, \dots, r \\ \delta x_j(t_f) &\text{arbitrary}, \quad j = r + 1, \dots, n\end{aligned}$$

CoV extension I: generalized boundary conditions

<i>Problem description</i>	<i>Substitution</i>	<i>Boundary conditions</i>	<i>Remarks</i>
1. $\mathbf{x}(t_f)$, t_f both specified (Problem 1)	$\delta \mathbf{x}_f = \delta \mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
2. $\mathbf{x}(t_f)$ free; t_f specified (Problem 2)	$\delta \mathbf{x}_f = \delta \mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
3. t_f free; $\mathbf{x}(t_f)$ specified (Problem 3)	$\delta \mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $-\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
4. t_f , $\mathbf{x}(t_f)$ free and independent (Problem 4)	—	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
5. t_f , $\mathbf{x}(t_f)$ free but related by $\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$ (Problem 4)	$\delta \mathbf{x}_f = \frac{d\boldsymbol{\theta}}{dt}(t_f) \delta t_f^\dagger$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $+ \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f) - \dot{\mathbf{x}}^*(t_f)\right] = 0^\dagger$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f

Example

- Determine the smooth curve of smallest length connecting the point $x(0) = 1$ to the line $t = 5$
 - Solution: $x(t) = 1$

CoV extension II: constrained extrema

- Let $\mathbf{w} : \mathbb{R} \rightarrow \mathbb{R}^{n+m}$ be a vector-valued function, where each component w_i is in the class of functions with continuous first derivatives. It is desired to find the function \mathbf{w}^* for which the functional

$$J(\mathbf{w}) = \int_{t_0}^{t_f} g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) dt$$

has a relative extremum, subject to the constraints

$$f_i(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) = 0, \quad i = 1, \dots, n$$

- Assumptions:
 - $g \in \mathcal{C}^2$
 - t_0 and $\mathbf{w}(0)$ are fixed

CoV extension II: constrained extrema

- Because of the n differential constraints, only m of the $n + m$ components of \mathbf{w} are independent
- Constraints of this type may represent the state equation constraints in optimal control problems where \mathbf{w} corresponds to the $n + m$ vector $\mathbf{w} = [\mathbf{x}, \mathbf{u}]^T$
- Similar to the case of constrained optimization, define the augmented integrand function

$$g_a(\mathbf{w}(t), \dot{\mathbf{w}}(t), \mathbf{p}(t), t) :=$$

$$g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \mathbf{p}(t)^T \mathbf{f}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)$$

Lagrange multipliers (now functions of time!), the “costate”

CoV extension II: constrained extrema

- A necessary condition for optimality is then

$$\frac{\partial g_a}{\partial \mathbf{w}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t) - \frac{d}{dt} \frac{\partial g_a}{\partial \dot{\mathbf{w}}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t) = \mathbf{0}$$

along with

$$\mathbf{f}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), t) = \mathbf{0}$$

- That is, to determine the necessary conditions for an extremal we simply form the augmented integrand g_a and write the Euler equations *as if* there were no constraints among the functions $\mathbf{w}(t)$
- Note the similarity with the case of constrained optimization!

The variational approach to optimal control

Roadmap:

1. We will first derive necessary conditions for optimal control assuming that the admissible controls are not bounded
2. Next, we will heuristically introduce Pontryagin's Minimum Principle as a generalization of the fundamental theorem of CoV
3. Finally, we will consider special cases of problems with bounded controls and state variables

Necessary conditions for optimal control (with unbounded controls)

- The problem is to find an *admissible control* \mathbf{u}^* which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

to follow an *admissible trajectory* \mathbf{x}^* that minimizes the *functional*

$$J(\mathbf{u}) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

- Assumptions: $h \in C^2$, state and control regions are unbounded, t_0 and $\mathbf{x}(0)$ are fixed

Necessary conditions for optimal control (with unbounded controls)

- Define the Hamiltonian

$$H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) := g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}(t)^T \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

- The necessary conditions for optimality (proof to follow) are

$$\left. \begin{aligned} \dot{\mathbf{x}}^*(t) &= \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \mathbf{0} &= \frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \end{aligned} \right\} \text{ for all } t \in [t_0, t_f]$$

with boundary conditions

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) \right]^T \delta \mathbf{x}_f + \left[H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) \right] \delta t_f = 0$$

Necessary conditions for optimal control (with unbounded controls)

<i>Problem</i>	<i>Description</i>	<i>Substitution in Eq. (5.1-18)</i>	<i>Boundary-condition equations</i>	<i>Remarks</i>
t_f fixed	1. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta \mathbf{x}_f = \delta \mathbf{x}(t_f) = 0$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
	2. $\mathbf{x}(t_f)$ free	$\delta \mathbf{x}_f = \delta \mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = 0$	$2n$ equations to determine $2n$ constants of integration
	3. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = 0$	$\delta \mathbf{x}_f = \delta \mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \sum_{i=1}^k d_i \left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) \right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = 0$	$(2n + k)$ equations to deter- mine the $2n$ constants of integration and the variables d_1, \dots, d_k
t_f free	4. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta \mathbf{x}_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $\mathcal{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to deter- mine the $2n$ constants of integration and t_f
	5. $\mathbf{x}(t_f)$ free		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = 0$ $\mathcal{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to deter- mine the $2n$ constants of integration and t_f

Interlude: Pontryagin's minimum principle (with bounded controls)

- Assuming **bounded controls** $\mathbf{u} \in U$, the necessary optimality conditions are (H is the Hamiltonian)

$$\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)$$

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \leq H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t), \text{ for all } \mathbf{u}(t) \in U$$

for all
 $t \in [t_0, t_f]$

along with the boundary conditions:

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) \right]^T \delta \mathbf{x}_f + \left[H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) \right] \delta t_f = 0$$

Necessary conditions for optimal control (with unbounded controls)

- Necessary conditions consist of a set of $2n$, *first-order*, differential equations (state and costate equations), and a set of m algebraic equations (control equations)
- The solution to the state and costate equations will contain $2n$ constants of integration
- To obtain values for the constants, we use the n equations $\mathbf{x}(t_0) = \mathbf{x}_0$, and an additional set of n (or $n + 1$) equations from the boundary conditions
- Once again: *2-point boundary value problem*

Example

Find optimal control $u(t)$ to steer the system

$$\ddot{x}(t) = u(t)$$

from $x(0) = 10, \dot{x}(0) = 0$ to the origin
 $x(t_f) = 0, \dot{x}(t_f) = 0$, and to minimize

$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_{t_0}^{t_f} b u^2(t)dt, \quad \alpha, b > 0$$

(note: the final time t_f is free)

Example

Find optimal control $u(t)$ to steer the system

$$\ddot{x}(t) = u(t)$$

from $x(0) = 10, \dot{x}(0) = 0$ to the origin
 $x(t_f) = 0, \dot{x}(t_f) = 0$, and to minimize

$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_{t_0}^{t_f} b u^2(t)dt, \quad \alpha, b > 0$$

- Solution: optimal time is

$$t_f = \left(\frac{1800b}{\alpha} \right)^{1/5}$$

Necessary conditions for optimal control (with unbounded controls)

We want to prove that, with unbounded controls, the necessary optimality conditions are ($H = g + \mathbf{p}^T \mathbf{f}$ is the Hamiltonian)

$$\left. \begin{aligned} \dot{\mathbf{x}}^*(t) &= \frac{\partial H}{\partial \mathbf{p}} (\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial H}{\partial \mathbf{x}} (\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \mathbf{0} &= \frac{\partial H}{\partial \mathbf{u}} (\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \end{aligned} \right\} \text{ for all } t \in [t_0, t_f]$$

along with the boundary conditions:

$$\left[\frac{\partial h}{\partial \mathbf{x}} (\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) \right]^T \delta \mathbf{x}_f + \left[H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t} (\mathbf{x}^*(t_f), t_f) \right] \delta t_f = 0$$

Proof sketch of NOC


- For simplicity, assume that the terminal penalty is equal to zero, and that t_f and $\mathbf{x}(t_f)$ are fixed and given
- Consider the augmented cost function
$$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) := g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}(t)^T [\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)]$$
where the $\{p_i(t)\}$'s are Lagrange multipliers
- Note that we have simply added zero to the cost function!
- The augmented cost function is then

$$J_a(\mathbf{u}) = \int_{t_0}^{t_f} g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) dt$$

Proof sketch of NOC

On an extremal, by applying the fundamental theorem of the CoV

By the CoV
theorem


$$0 = \delta J_a(\mathbf{u}) = \int_{t_0}^{t_f} \left(\left[\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) - \frac{d}{dt} \frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{x}(t) \right. \\ \left. + \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{u}(t) + \left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{p}(t) \right) dt$$

Proof sketch of NOC

On an extremal, by applying the fundamental theorem of the CoV

By the CoV
theorem

$$\begin{aligned}
 &= \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)^T \mathbf{p}^*(t) &&= -\frac{d}{dt}(-\mathbf{p}^*(t)) \\
 0 = \delta J_a(\mathbf{u}) &= \int_{t_0}^{t_f} \left(\underbrace{\left[\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) - \frac{d}{dt} \frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T}_{= \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)} \delta \mathbf{x}(t) \right. \\
 &\quad \left. + \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T \delta \mathbf{u}(t) + \underbrace{\left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]^T}_{= \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)} \delta \mathbf{p}(t) \right) dt \\
 & &&= \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)
 \end{aligned}$$

Proof sketch of NOC

Considering each term in sequence,

- $\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t) = \mathbf{0}$, on an extremal
- The Lagrange multipliers are arbitrary, so we can select them to make the coefficient of $\delta \mathbf{x}(t)$ equal to zero, that is

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)^T \mathbf{p}^*(t)$$

- The remaining variation $\delta \mathbf{u}(t)$, is independent, so its coefficient must be zero; thus

$$\frac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)^T \mathbf{p}^*(t) = \mathbf{0}$$

By using the Hamiltonian formalism, one obtains the claim

Next time

- Pontryagin's Minimum Principle
- Intro to dynamic programming