AA203
Optimal and Learning-based Control

Direct methods for optimal control, sequential convex programming (SCP)

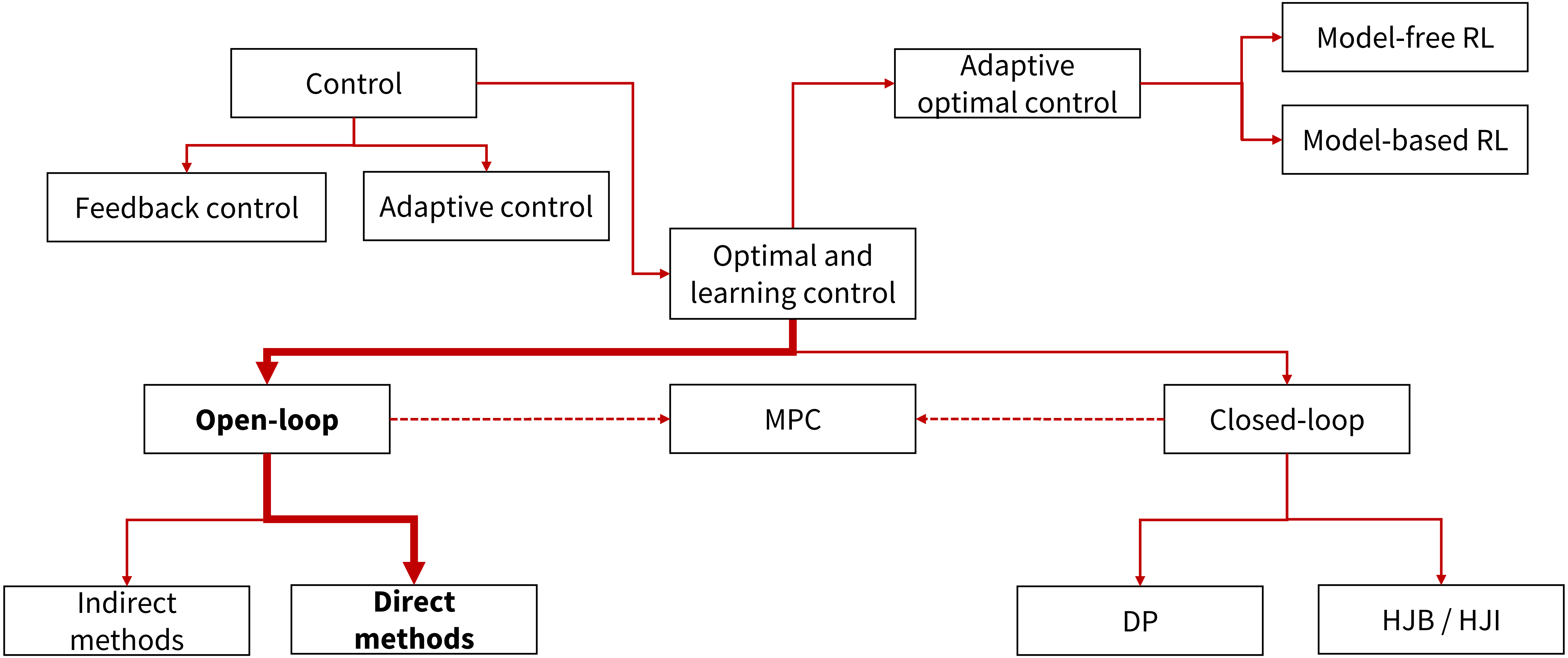
Stanford University

Autonomous Systems Lab

Last time: iLQR and DDP

- Trajectory optimization with a linear feedback tracking policy as a bonus
  - Interpretation as variants of Newton's method in Nm dimensions
- Drawbacks
  - Output policy applies only locally
  - Dependent on *feasible* initial trajectory
    - (see also Jur van den Berg, "Extended LQR," 2013.)
  - Other than dynamics, only soft-constraints may be incorporated

Roadmap


Control
Feedback control
Adaptive control
Optimal and learning control
Adaptive optimal control
Model-free RL
Model-based RL
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI

Optimal control problem

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) \; dt$$

(**OCP**)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \;\; t \in [0, t_f]$$

$$\mathbf{x}(0) = \; \mathbf{x}_0$$
$$\mathbf{x}\left(t_f\right) \in M_f = \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = 0\}$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \;\; t \in [0, t_f]$$

For simplicity:

- We assume the terminal cost h is equal to 0
- We assume $t_0 = 0$

- Direct Methods:
  1. Transcribe (**OCP**) into a nonlinear, constrained optimization problem
  2. Solve the optimization problem via nonlinear programming
- Indirect Methods:
  1. Apply necessary conditions for optimality to (**OCP**)
  2. Solve a two-point boundary value problem

Direct methods

Resources:

- Notes Chapter 5 and references therein, and also:
  - Rao A. V., “A survey of numerical methods for optimal control,” 2009.
  - Kelly, M., “An Introduction to Trajectory Optimization,” 2017.

Transcription methods

Optimization: what are the decision variables?

1. State and control parameterization methods
   - “Collocation”/“simultaneous”

2. Control parameterization methods
   - “Shooting”

Transcription into nonlinear programming
(state and control parametrization method)

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

(OCP)
$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t),\ \ t \in [0, t_f]$$
$$\mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{x}(t_f) \in M_f = \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = 0\}$$
$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m,\ \ t \in [0, t_f]$$

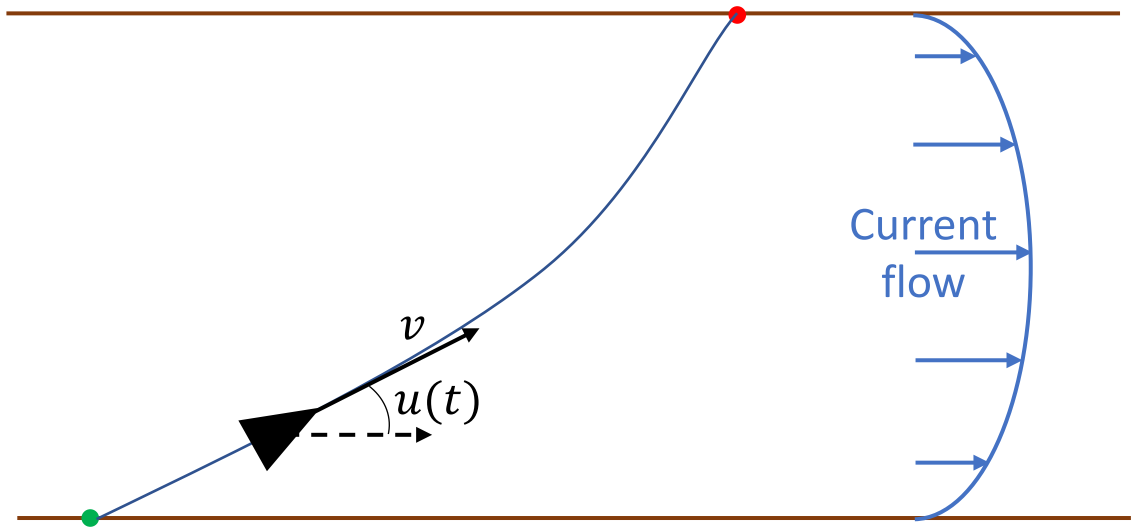
(NLOP)
$$\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i, t_i)$$
$$\mathbf{x}_{i+1} = \mathbf{x}_i + h_i \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, t_i), \qquad i = 0, \dots, N-1$$
$$\mathbf{u}_i \in U, i = 0, \dots, N-1, \qquad F(\mathbf{x}_N) = 0$$

Forward Euler time discretization

1. Select a discretization $0 = t_0 < t_1 < \cdots < t_N = t_f$ for the interval $[0, t_f]$ and, for every $i = 0, \dots, N-1$, define $\mathbf{x}_i \sim \mathbf{x}(t)$, $\mathbf{u}_i \sim \mathbf{u}(t)$, $t \in [t_i, t_{i+1})$ and $\mathbf{x}_0 \sim \mathbf{x}(0)$

2. By denoting $h_i = t_{i+1} - t_i$, (**OCP**) is transcribed into the following nonlinear, constrained optimization problem

Illustrative example: Zermelo's Problem



(OCP)

$$\min \int_0^{t_f} u(t)^2 \, dt$$

$$\dot{x}(t) = v \cos\big(u(t)\big) + \text{flow}\big(y(t)\big), t \in [0, t_f]$$

$$\dot{y}(t) = v \sin\big(u(t)\big), \; t \in [0, t_f]$$

$$(x, y)(0) = 0, \; (x, y)(t_f) = (M, \ell)$$

$$|u(t)| \leq u_{max}, \; t \in [0, t_f]$$

Example: Zermelo's Problem

State and control parameterization method

- Transcribe optimal control problem into a non-linear program, and solve it via `fmincon` (MATLAB), `scipy.optimize.minimize` (python), etc.

(OCP)

$$\min \int_0^{t_f} u(t)^2\, dt$$

$$\dot{x}(t) = v\cos\big(u(t)\big) + \text{flow}\big(y(t)\big),\ t \in [0, t_f]$$

$$\dot{y}(t) = v\sin\big(u(t)\big),\ t \in [0, t_f]$$

$$(x, y)(0) = 0,\ (x, y)(t_f) = (M, \ell)$$

$$|u(t)| \le u_{max},\ t \in [0, t_f]$$

(NLOP)

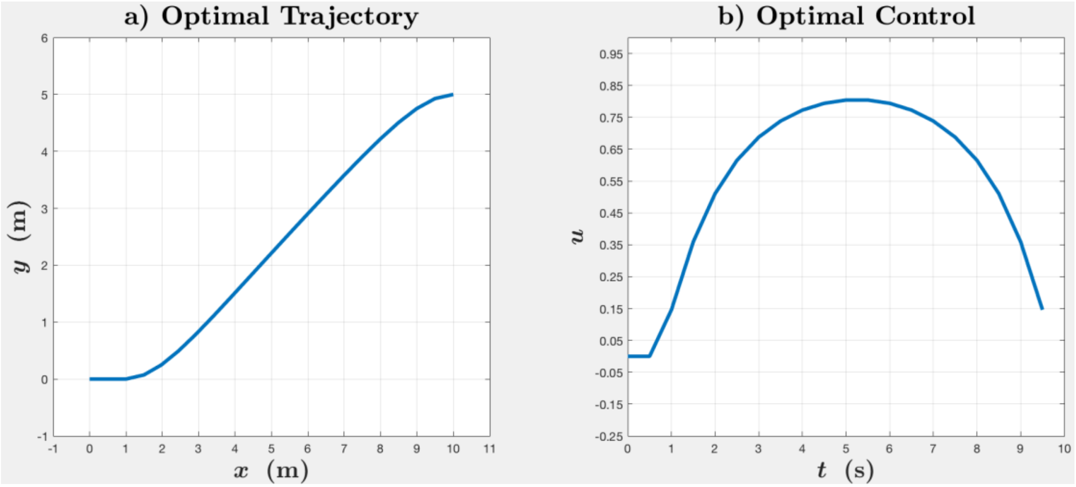
$$\min_{(x_i, u_i)} \sum_{i=0}^{N-1} h\, u_i^2$$

$$x_{i+1} = x_i + h\big(v\cos(u_i) + \text{flow}(y_i)\big)$$

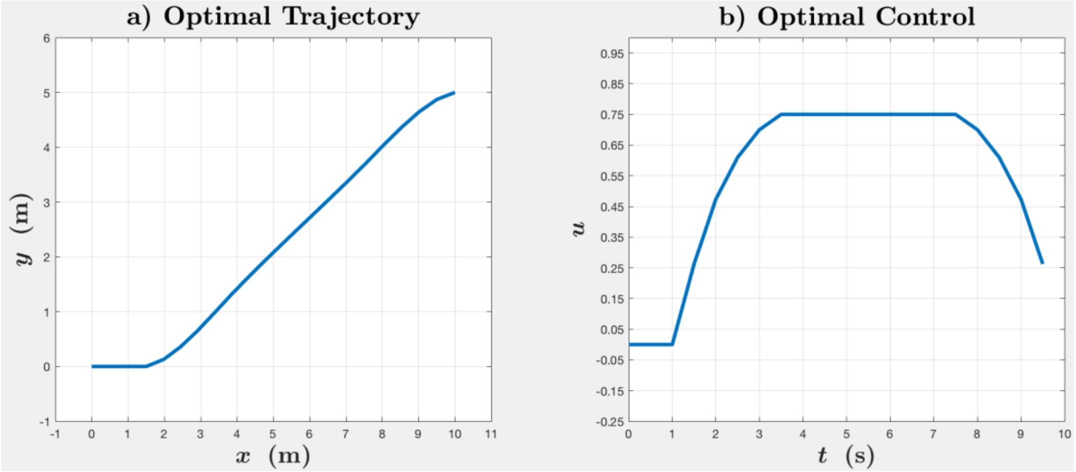
$$y_{i+1} = y_i + h\, v\sin(u_i)\,,\ |u_i| \le u_{max}$$

$$(x_0, y_0) = 0\,,\ (x_N, y_N) = (M, \ell)$$

Results



$|u(t)| \leq 1$
(effectively, no control constraint)



$|u(t)| \leq 0.75$

Transcription into nonlinear programming
(control parametrization method)

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

(OCP)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t),\ \ t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

$$\mathbf{x}(t_f) \in M_f = \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = 0\}$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m,\ \ t \in [0, t_f]$$

Time and control discretization

1. Select a discretization $0 = t_0 < t_1 < \cdots < t_N = t_f$ for the interval $[0, t_f]$ and, for every $i = 0, \ldots, N-1$, define
   $\mathbf{u}_i \sim \mathbf{u}(t)\ ,\ \ t \in [t_i, t_{i+1})$
2. By denoting $h_i = t_{i+1} - t_i$, (**OCP**) is transcribed into the following nonlinear, constrained optimization problem

(NLOP-C)

$$\min_{\mathbf{u}_i} \sum_{i=0}^{N-1} h_i g(\mathbf{x}(t_i), \mathbf{u}_i, t_i)$$

$$\mathbf{u}_i \in U\ , i = 0, \ldots, N-1\ , \qquad F(\mathbf{x}(t_N)) = 0$$

where each $\mathbf{x}(t_i)$ is recursively computed via

$$\mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + h_i \mathbf{f}(\mathbf{x}(t_i), \mathbf{u}_i, t_i), i = 0, \ldots, N-1$$

Example: Zermelo's Problem

Control parameterization method

- Transcribe optimal control problem into a non-linear program, and solve it via `fmincon` (MATLAB), `scipy.optimize.minimize` (python), etc.

$$
\textbf{(OCP)}\quad
\begin{aligned}
&\min \int_0^{t_f} u(t)^2\, dt \\
&\dot{x}(t) = v\cos\big(u(t)\big) + \mathrm{flow}\big(y(t)\big),\ t \in [0, t_f] \\
&\dot{y}(t) = v\sin\big(u(t)\big),\ t \in [0, t_f] \\
&(x, y)(0) = 0,\ (x, y)(t_f) = (M, \ell) \\
&|u(t)| \le u_{max},\ t \in [0, t_f]
\end{aligned}
$$

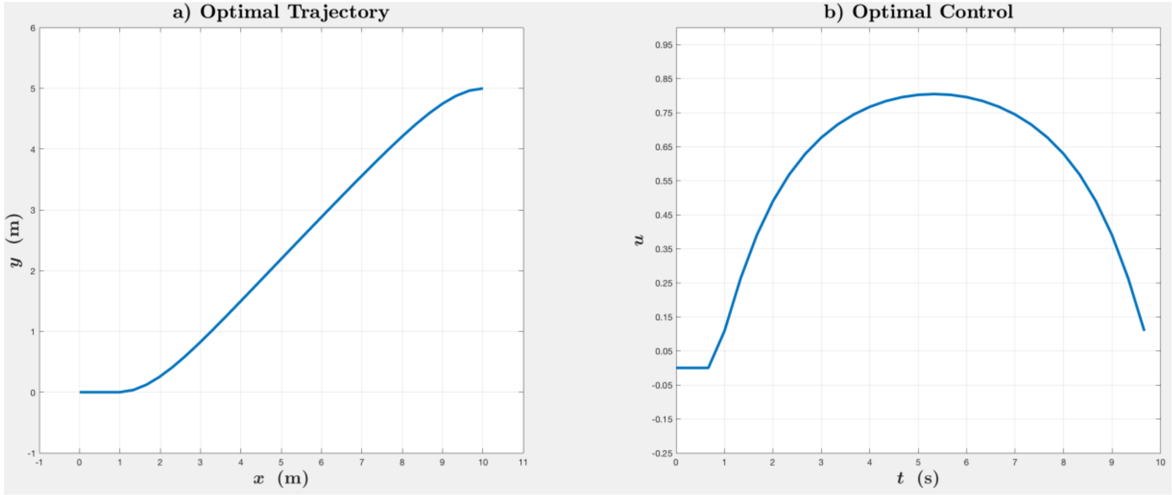
$$
\min_{u_i} \sum_{i=0}^{N-1} h\, u_i^2 \qquad \textbf{(NLOP-C)}
$$

$$
(x, y)(t_N) = (M, \ell), \qquad |u_i| \le u_{max}
$$

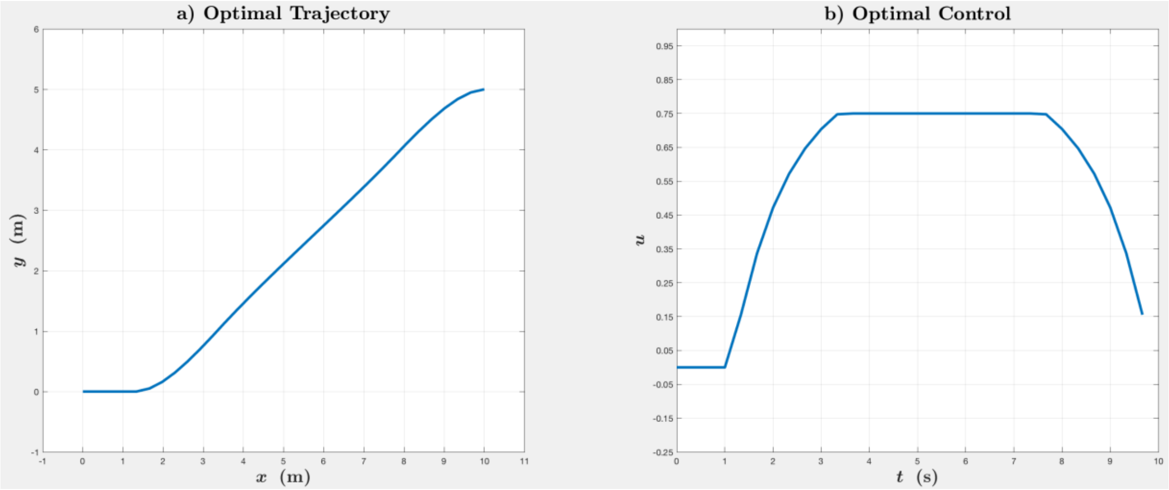
where, recursively:

$$
x_N = x_0 + h \sum_{i=0}^{N-1} \big(v\cos(u_i) + \mathrm{flow}(y_i)\big), \qquad y_i = y_0 + h \sum_{j=0}^{i} v\sin(u_j)
$$

Results



$|u(t)| \leq 1$
(effectively, no control constraint)



$|u(t)| \leq 0.75$

Example: Zermelo's Problem

(OCP)

$$\min \int_0^{t_f} u(t)^2\, dt$$
$$\dot{x}(t) = v\cos\big(u(t)\big) + \text{flow}\big(y(t)\big), t \in [0, t_f]$$
$$\dot{y}(t) = v\sin\big(u(t)\big),\ t \in [0, t_f]$$
$$(x,y)(0) = 0,\ (x,y)(t_f) = (M,\ell)$$
$$|u(t)| \le u_{max},\ t \in [0, t_f]$$

(NLOP)

$$\min_{(x_i,u_i)} \sum_{i=0}^{N-1} h\, u_i^2$$
$$x_{i+1} = x_i + h\big(v\cos(u_i) + \text{flow}(y_i)\big)$$
$$y_{i+1} = y_i + h\, v\sin(u_i)\ ,\ |u_i| \le u_{max}$$
$$(x_0,y_0) = 0\ ,\ (x_N,y_N) = (M,\ell)$$

Direct Transcription

(NLOP-C)

$$\min_{u_i} \sum_{i=0}^{N-1} h\, u_i^2$$
$$(x,y)(t_N) = (M,\ell), \qquad |u_i| \le u_{max}$$

where, recursively:

$$x_N = x_0 + h\sum_{i=0}^{N-1}\big(v\cos(u_i) + \text{flow}(y_i)\big)$$
$$y_i = y_0 + h\sum_{j=0}^{i} v\sin(u_j)$$

Direct Shooting

Transcription methods: extensions

- Multiple shooting
  - Hybrid of simultaneous / (single) shooting methods
- Alternative trajectory parameterizations
  - Euler integration (above): piecewise linear effective state trajectory (C^0), zero-order hold control trajectory
  - Hermite-Simpson collocation (see Notes §5.2.1): piecewise cubic effective state trajectory (C^1), first-order hold control trajectory
    - Dynamics constraint is enforced at “collocation points,” exact form is derived by implicit integration
  - Pseudospectral methods: global polynomial basis functions (instead of piecewise polynomials)
  - Shooting methods: higher-order integration schemes (e.g., RK4)
    - Dynamics constraint is enforced by explicit integration

Sequential Convex Programming

$$
\textbf{(OCP)} \quad \begin{aligned}
&\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) \; dt \\
&\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \;\; t \in [0, t_f] \\
&\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f \\
&\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \;\; t \in [0, t_f]
\end{aligned}
$$

The sources of nonconvexities are the dynamics and (possibly) the cost. Idea: linearize (and convexify) them around nominal trajectories!

Sequential Convex Programming

$$
\text{(OCP)} \quad
\begin{aligned}
&\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) \; dt \\
&\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \;\; t \in [0, t_f] \\
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&\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \;\; t \in [0, t_f]
\end{aligned}
$$

The sources of nonconvexities are the dynamics and (possibly) the cost. Idea: linearize (and convexify) them around nominal trajectories!

1. Assume that g is convex. Let $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ be a nominal tuple of trajectory and control. $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ does not need to be feasible!

Sequential Convex Programming

$$
\text{(OCP)} \quad \begin{aligned} \min \quad & \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) \; dt \\ & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \;\; t \in [0, t_f] \\ & \mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f \\ & \mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \;\; t \in [0, t_f] \end{aligned}
$$

The sources of nonconvexities are the dynamics and (possibly) the cost. Idea: linearize (and convexify) them around nominal trajectories!

1. Assume that g is convex. Let $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ be a nominal tuple of trajectory and control. $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ does not need to be feasible!

2. Linearize $\mathbf{f}$ around $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$:

$$
\begin{aligned} & \mathbf{f}_1(\mathbf{x}, \mathbf{u}, t) \\ & = \mathbf{f}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{x} - \mathbf{x}_0(t)) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{u} - \mathbf{u}_0(t)) \end{aligned}
$$

Sequential Convex Programming

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

$$(\mathbf{LOCP})_1 \quad \begin{aligned} &\dot{\mathbf{x}}(t) = \mathbf{f}_1(\mathbf{x}(t), \mathbf{u}(t), t), \ t \in [0, t_f] \\ &\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f \\ &\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \ \ t \in [0, t_f] \end{aligned}$$

The sources of nonconvexities are the dynamics and (possibly) the cost. Idea: linearize (and convexify) them around nominal trajectories!

1. Assume that g is convex. Let $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ be a nominal tuple of trajectory and control. $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ does not need to be feasible!

2. Linearize $\mathbf{f}$ around $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$:

$$\mathbf{f}_1(\mathbf{x}, \mathbf{u}, t) = \mathbf{f}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{x} - \mathbf{x}_0(t)) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{u} - \mathbf{u}_0(t))$$

3. Solve the new problem $(\mathbf{LOCP})_1$ for $(\mathbf{x}_1(\cdot), \mathbf{u}_1(\cdot))$

Sequential Convex Programming

$$
(\mathbf{LOCP})_{k+1} \quad
\begin{aligned}
&\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt \\
&\dot{\mathbf{x}}(t) = \mathbf{f}_{k+1}(\mathbf{x}(t), \mathbf{u}(t), t),\ \ t \in [0, t_f] \\
&\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f \\
&\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \ \ t \in [0, t_f]
\end{aligned}
$$

The sources of nonconvexities are the dynamics and (possibly) the cost. Idea: linearize (and convexify) them around nominal trajectories!

4. Iterate this procedure until convergence is achieved: linearize $\mathbf{f}$ around the solution $\left(\mathbf{x}_k(\cdot), \mathbf{u}_k(\cdot)\right)$ at iteration k:

$$
\begin{aligned}
&\mathbf{f}_{k+1}(\mathbf{x}, \mathbf{u}, t) \\
&= \mathbf{f}(\mathbf{x}_k(t), \mathbf{u}_k(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_k(t), \mathbf{u}_k(t), t)(\mathbf{x} - \mathbf{x}_k(t)) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_k(t), \mathbf{u}_k(t), t)(\mathbf{u} - \mathbf{u}_k(t))
\end{aligned}
$$

and solve the problem $(\mathbf{LOCP})_{k+1}$ for $\left(\mathbf{x}_{k+1}(\cdot), \mathbf{u}_{k+1}(\cdot)\right)$

Sequential Convex Programming

$$
(\mathbf{LOCP})_{k+1} \quad
\begin{aligned}
&\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt \\
&\dot{\mathbf{x}}(t) = \mathbf{f}_{k+1}(\mathbf{x}(t), \mathbf{u}(t), t),\ t \in [0, t_f] \\
&\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f \\
&\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \quad t \in [0, t_f]
\end{aligned}
$$

Discretize and Solve a Convex Problem at Each Iteration

1. Select a discretization $0 = t_0 < t_1 < \cdots < t_N = t_f$ for the interval $[0, t_f]$ and, for every $i = 0, \ldots, N-1$, define $\mathbf{x}_{i+1} \sim \mathbf{x}(t),\ \mathbf{u}_i \sim \mathbf{u}(t),$ $t \in (t_i, t_{i+1}]$ and $\mathbf{x}_0 \sim \mathbf{x}(0)$

2. By denoting $h_i = t_{i+1} - t_i$, $(\mathbf{LOCP})_{k+1}$ is transcribed into the following convex optimization problem

$$
(\mathbf{DLOCP})_{k+1} \quad
\begin{aligned}
&\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i, t_i) \\
&\mathbf{x}_{i+1} = \mathbf{x}_i + h_i \mathbf{f}_{k+1}(\mathbf{x}_i, \mathbf{u}_i, t_i),\ i = 0, \ldots, N-1 \\
&\mathbf{u}_i \in U,\ i = 0, \ldots, N-1, \qquad \mathbf{x}_N = \mathbf{x}_f
\end{aligned}
$$

Sequential Convex Programming

$$
(\textbf{LOCP})_{k+1} \quad
\begin{aligned}
&\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt \\
&\dot{\mathbf{x}}(t) = \mathbf{f}_{k+1}(\mathbf{x}(t), \mathbf{u}(t), t),\ t \in [0, t_f] \\
&\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f \\
&\mathbf{u}(t) \in U \subseteq \mathbb{R}^m,\ \ t \in [0, t_f]
\end{aligned}
$$

$$
(\textbf{DLOCP})_{k+1} \quad
\begin{aligned}
&\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i, t_i) \\
&\mathbf{x}_{i+1} = \mathbf{x}_i + h_i \mathbf{f}_{k+1}(\mathbf{x}_i, \mathbf{u}_i, t_i),\ i = 0, \dots, N-1 \\
&\mathbf{u}_i \in U,\ i = 0, \dots, N-1, \qquad \mathbf{x}_N = \mathbf{x}_f
\end{aligned}
$$

SCP Methodology: at each iteration k,

Linearize $\mathbf{f}$ around the solution $(\mathbf{x}_k(\cdot), \mathbf{u}_k(\cdot))$ → Define the continuous time problem $(\textbf{LOCP})_{k+1}$ → Discretize $(\textbf{LOCP})_{k+1}$ in time and define the convex optimization problem $(\textbf{DLOCP})_{k+1}$ → Solve $(\textbf{DLOCP})_{k+1}$ via convex programming for a discretized version of $(\mathbf{x}_{k+1}(\cdot), \mathbf{u}_{k+1}(\cdot))$

Direct Methods in Practice

"As you begin to play with these algorithms on your own problems, you might feel like you're on an emotional roller-coaster." – Russ Tedrake

- Better initial guess trajectories ("warm-starting" the optimization, as seen in `zermelo_simultaneous`)

- Cost function/constraint tuning (as seen in `zermelo_scp`)
  - Penalty methods; augmented Lagrangian-based solvers

Next time

- Dynamic programming in continuous time