Spring 2019

Stanford
AA 203: Optimal and Learning-based Control
Problem Set 8, due on May 29

Please remember to attach your code to your problem set.

Problem 1: Consider the discrete-time system

$$x(t+1) = ax(t) + bx(t-1) + w(t) \tag{1}$$

This *autoregressive (AR) model* represents a stochastic process where the next state depends linearly on previous states and some process noise $w(t)$.

a) Suppose we have measurements $\{x(t)\}_{t=0}^{N}$, but we do not know the true parameter values (a, b). Formulate the linear least-squares problem for determining an estimate of (a, b); i.e., identify the regressors and all vectors that appear in the least-squares objective for this data set.

b) Generate such data sets for $N \in \{10, 100, 1000\}$ by simulating the system above with $(a, b) = (1, -0.1)$, $x(0) = 1$, $x(1) = 0.5$, and $w(t) \sim \mathcal{N}(0, 1)$. For each different data set size N, compute the least-squares estimate of (a, b) averaged over 100 trials. Report the mean and standard deviation of your estimates for each N, and briefly comment on any variation.

c) Repeat part (b) with $w(0) \sim \mathcal{N}(0, 1)$ and $w(t) \sim \mathcal{N}(0, 2\,|w(t-1)|)$ for $t > 0$. How do the mean and standard deviation of your parameter estimates compare to part (b)? Why do you think this is the case?

d) Now consider the nonlinear system

$$x(t+1) = cx(t)x(t-1) + w(t) \tag{2}$$

with the unknown parameter c. The term $x(t)x(t-1)$ is nonlinear, yet why can you still apply the linear least-squares procedure to this system? Repeat parts (a) and (b) for this system. Use $c = 0.1$, $x(0) = 1$, $x(1) = 0.5$, and $w(t) \sim \mathcal{N}(0, 1)$.

Problem 2: Consider the continuous-time system

$$\dot{y}(t) + ay(t) = bu(t) \tag{3}$$

We want to control this system, but we do not know the true plant parameters (a, b). You will use *model-reference adaptive control (MRAC)* to match the behaviour of the true plant with that of the reference model

$$\dot{y}_m(t) + a_m y_m(t) = b_m r(t) \tag{4}$$

where (a_m, b_m) are *known* constant parameters, and $r(t)$ is a chosen bounded external reference signal.

a) Consider the control law

$$u(t) = k_r(t)r(t) + k_y(t)y(t) \tag{5}$$

where $k_r(t)$ and $k_y(t)$ are time-varying feedback gains. Write out the differential equation for the resulting closed-loop dynamics. Use this to verify that, if we knew (a, b), the following constant control gains

$$\begin{aligned} k_r^* &:= \frac{b_m}{b} \\ k_y^* &:= \frac{a - a_m}{b} \end{aligned} \tag{6}$$

would make the true plant dynamics perfectly match the reference model.

b) When we do not know (a, b), we need to adaptively update our controller over time. Specifically, we want an *adaptation law* for $k_r(t)$ and $k_y(t)$ to make $y(t)$ tend towards $y_m(t)$ asymptotically. For this, we define the tracking error $e(t) := y(t) - y_m(t)$ and the parameter errors

$$\begin{aligned} \delta_r(t) &:= k_r(t) - k_r^* \\ \delta_y(t) &:= k_y(t) - k_y^* \end{aligned} \tag{7}$$

Determine the differential equation governing the dynamics of $e(t)$, in terms of e, any of its derivatives, y, r, δ_y, δ_r, and suitable constants.

We will consider the adaptation law for k_r and k_y described by

$$\begin{aligned} \dot{k}_r(t) &= -\operatorname{sign}(b)\gamma e(t)r(t) \\ \dot{k}_y(t) &= -\operatorname{sign}(b)\gamma e(t)y(t) \end{aligned} \tag{8}$$

where $\gamma \in \mathbb{R}_{>0}$ is a chosen constant *adaptation gain*. For this adaptation law, we must at least know the sign of b, which indicates in what direction the input $u(t)$ "pushes" the output $y(t)$ in **??**. For example, when modeling a car, you could reasonably assume that an increased braking force slows down the car.

To show that tracking error and parameter errors are stabilized by our chosen control law and adaptation law, we will use Lyapunov theory. While previously we applied Lyapunov theory to discrete-time systems in the context of MPC, we are now dealing with continuous-time systems.

Theorem 1 (Lyapunov): *Consider the continuous-time system $\dot{x} = f(x,t)$, where $x = 0$ is an equilibrium point, i.e., $f(0,t) \equiv 0$. If there exists a continuously differentiable scalar function $V(x,t)$ such that*

- *V is positive definite in x, and*
- *$\dot{V}$ is negative semi-definite in x,*

then $x = 0$ is a stable point in the sense of Lyapunov, i.e., $\|x(t)\|$ remains bounded as long as $\|x(0)\|$ is bounded.

c) Now, consider the state $x \coloneqq (e, \delta_r, \delta_y)$ and the Lyapunov function candidate

$$V(x) = \frac{1}{2}e^2 + \frac{|b|}{2\gamma}(\delta_r^2 + \delta_y^2) \tag{9}$$

Show that $\dot{V} = -a_m e^2$. Based on Lyapunov theory, what can you say about $e(t)$, $\delta_r(t)$, and $\delta_y(t)$ for all $t \in [0, \infty)$ if $a_m > 0$? Furthermore, apply Barbalat's lemma to $\dot{V}$ to make a stronger statement about $e(t)$ than you originally did with just Lyapunov theory.

Theorem 2 (Barbalat's Lemma): *If a differentiable function $g(t)$ has a finite limit as $t \to \infty$, and if $\dot{g}(t)$ is uniformly continuous, then $\dot{g}(t) \to 0$ as $t \to \infty$.*

Hint: To prove that a function is uniformly continuous, it suffices to show that its derivative is bounded. Lipschitz continuity and thus uniform continuity follow from this.

With a given control law and adaptation law, MRAC proceeds as follows. First, we choose a reference signal $r(t)$ to excite the reference output $y_m(t)$ and construct the input signal $u(t)$, which is used to excite the true model. The output $y(t)$ is then observed and fed back into the control law, and the tracking error $e(t)$ is fed into the adaptation law.

d) Apply MRAC to the unstable plant

$$\dot{y}(t) - y(t) = 3u(t) \tag{10}$$

That is, simulate an adaptive controller for this system that does not have access to the true model parameters $(a, b) = (-1, 3)$. The desired reference model is

$$\dot{y}_m(t) + 4y_m(t) = 4r(t) \tag{11}$$

i.e., $(a_m, b_m) = (4, 4)$. Use an adaptation gain of $\gamma = 2$, and zero initial conditions for y, y_m, k_r, and k_y. Plot both $y(t)$ and $y_m(t)$ over time in one figure, and $k_r(t)$, k_r^*, $k_y(t)$, and k_y^* over time in another figure for $r(t) = 4$. Then repeat this for $r(t) = 4\sin(3t)$. That is, you should have four figures in total. What do you notice about the trends for different reference signals? Why do you think this occurs?

Hint: To do the simulation in MATLAB, form a system of ODEs for either (y, y_m, k_r, k_y) or $(y, e, \delta_r, \delta_y)$, then use `ode45()`.

Learning goals for this problem set:

Problem 1: To understand the utility and limitations of linear least-squares regression for system identification.

Problem 2: To explore the theoretical underpinnings of MRAC, and observe its behaviour on an example system in simulation.