AA203
Optimal and Learning-based Control

CoV extensions, NOC for optimal control

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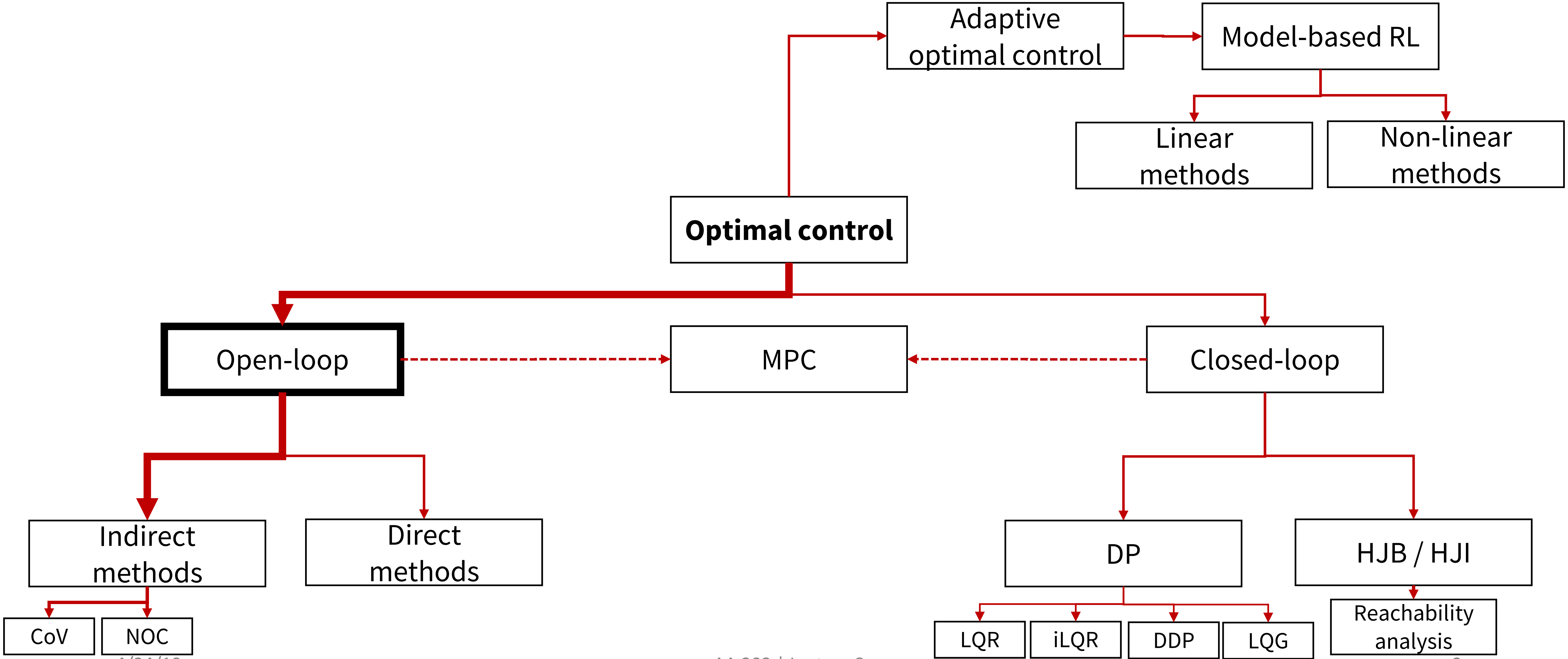
Autonomous Systems Lab

Roadmap


Adaptive
optimal control
Model-based RL
Linear
methods
Non-linear
methods
Optimal control
Open-loop
MPC
Closed-loop
Indirect
methods
Direct
methods
DP
HJB / HJI
CoV
NOC
LQR
iLQR
DDP
LQG
Reachability
analysis

CoV extension I: generalized boundary conditions

- Let $\mathbf{x}$ be a vector function, where each component x_i is in the class of functions with continuous first derivatives. It is desired to find the function $\mathbf{x}^*$ for which the functional

$$J(\mathbf{x}) = \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt$$

  has a relative extremum
- Assumptions:
  - $g \in C^2$
  - t_0 and $\mathbf{x}(0)$ are fixed
  - t_f might be fixed or free, and each component of $\mathbf{x}(t_f)$ might be fixed or free

CoV extension I: generalized boundary conditions

- Regardless of the boundary conditions, the Euler equations

$$g_{\mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt} g_{\dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) = 0$$

  must be satisfied
- The required boundary conditions are found from the equation

$$g_{\dot{\mathbf{x}}}\left(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f\right)' \delta \mathbf{x}_f + \Big[g\left(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f\right) - g_{\dot{\mathbf{x}}}\left(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f\right)' \dot{\mathbf{x}}(t_f)\Big] \delta t_f = 0$$

  by making the "appropriate" substitutions for $\delta \mathbf{x}_f$ and δt_f

CoV extension I: generalized boundary conditions

- $\delta \mathbf{x}_f$ and δt_f capture the notion of "allowable" variations at the end point, thus $\delta t_f = 0$ if the final time is fixed, and $\delta x_i(t_f) = 0$ if the end value of state variable $x_i(t_f)$ is fixed
- For example, suppose that δt_f is fixed, $x_i(t_f), i = 1, \ldots, r$ are fixed, and $x_j(t_f), j = r+1, \ldots, n$ are free. The, substitutions are:

$$\delta t_f = 0$$

$$\delta x_i(t_f) = 0, \qquad i = 1, \ldots, r$$

$$\delta x_j(t_f) \text{ arbitrary}, \qquad j = r+1, \ldots, n$$

CoV extension I: generalized boundary conditions

Problem description	*Substitution*	*Boundary conditions*	*Remarks*
1. $\mathbf{x}(t_f)$, t_f both specified (*Problem 1*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
2. $\mathbf{x}(t_f)$ free; t_f specified (*Problem 2*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
3. t_f free; $\mathbf{x}(t_f)$ specified (*Problem 3*)	$\delta\mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $- \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f) = 0$	$(2n+1)$ equations to determine $2n$ constants of integration and t_f
4. t_f, $\mathbf{x}(t_f)$ free and independent (*Problem 4*)	—	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = 0$	$(2n+1)$ equations to determine $2n$ constants of integration and t_f
5. t_f, $\mathbf{x}(t_f)$ free but related by $\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$ (*Problem 4*)	$\delta\mathbf{x}_f = \frac{d\boldsymbol{\theta}}{dt}(t_f)\, \delta t_f$†	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $+ \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f) - \dot{\mathbf{x}}^*(t_f)\right] = 0$†	$(2n+1)$ equations to determine $2n$ constants of integration and t_f

From Kirk, Section 4.3

Example

- Determine the smooth curve of smallest length connecting the point $x(0) = 1$ to the line $t = 5$
    - Solution: $x(t) = 1$

CoV extension II: constrained extrema

- Let $\mathbf{w} \in \mathbb{R}^{n+m}$ be a vector function, where each component w_i is in the class of functions with continuous first derivatives. It is desired to find the function $\boldsymbol{w}^*$ for which the functional

$$J(\mathbf{w}) = \int_{t_0}^{t_f} g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)dt$$

has a relative extremum, subject to the constraints

$$f_i(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) = 0, \qquad i = 1, \dots, n$$

- Assumptions:
  - $g \in C^2$
  - t_0 and $\mathbf{x}(0)$ are fixed

CoV extension II: constrained extrema

- Because of the n differential constraints, only m of the $n+m$ components of $\mathbf{w}$ are independent
- Constraints of this type may represent the state equation constraints in optimal control problems where $\mathbf{w}$ corresponds to the $n+m$ vector $\mathbf{w} = [\mathbf{x}, \mathbf{u}]'$
- Similarly to the case of constrained optimization, define the augmented integrand function

$$g_a(\mathbf{w}(t), \dot{\mathbf{w}}(t), \mathbf{p}(t), t) := g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \boxed{\mathbf{p}(t)}'\mathbf{f}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)$$

Lagrange multipliers (now functions of time!)

CoV extension II: constrained extrema

- A necessary condition for optimality is then

$$\frac{\partial g_a}{\partial \mathbf{w}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t) - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{\mathbf{w}}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t) = 0$$

  along with

$$\mathbf{f}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), t) = 0$$

- That is, to determine the necessary conditions for an extremal we simply form the augmented function g_a and write the Euler equations *as if* there were no constraints among the functions $\mathbf{w}(t)$
- Note the similarity with the case of constrained optimization!

Special case: Beltrami identity

- Consider the case when $x \in C^1$ and

No *explicit* dependence on t

$$J(x) = \int_{t_0}^{t_f} g\big(x(t), \dot{x}(t)\big) dt$$

- Then, the Euler–Lagrange equation reduces to the Beltrami identity (usually much simpler to solve!)

$$g(x^*(t), \dot{x}^*(t)) - \dot{x}^*(t)\, g_{\dot{x}}\big(x^*(t), \dot{x}^*(t)\big) = c$$

- Proof: Kirk, Appendix 3

Example

- Brachistochrone Problem: find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time
  - Solution: parametric equations of a cycloid

The variational approach to optimal control

Roadmap:

1. We will first derive necessary conditions for optimal control assuming that the admissible controls are not bounded
2. Next, we will heuristically introduce the Pontryagin's minimum principle as a generalization of the fundamental theorem of CoV
3. Finally, we will consider special cases of problems with bounded controls and state variables

Necessary conditions for optimal control
(with unbounded controls)

- The problem is to find an *admissible control* $\mathbf{u}^*$ which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

to follow an *admissible trajectory* $\mathbf{x}^*$ that minimizes the *functional*

$$J(\mathbf{u}) = h\big(\mathbf{x}(t_f), t_f\big) + \int_{t_0}^{t_f} g\,(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

- Assumptions: $h \in C^2$, state and control regions are unbounded, t_0 and $\mathbf{x}(0)$ are fixed, $\mathbf{x}$ is $n\times 1$ and $\mathbf{u}$ is $m\times 1$

Necessary conditions for optimal control
(with unbounded controls)

- Define the Hamiltonian

$$H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \coloneqq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}(t)' \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

- The necessary conditions are

$$\left.\begin{aligned} \dot{\mathbf{x}}^*(t) &= \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ 0 &= \frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \end{aligned}\right\} \text{for all } t \in [t_0, t_f]$$

Necessary conditions for optimal control
(with unbounded controls)

with boundary conditions

$$\left[\frac{\partial h}{\partial \mathbf{x}}\left(\mathbf{x}^*(t_f), t_f\right) - \mathbf{p}^*(t_f)\right]' \delta \mathbf{x}_f$$
$$+ \left[H\left(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f\right) + \frac{\partial h}{\partial t}\left(\mathbf{x}^*(t_f), t_f\right)\right] \delta t_f = 0$$

Necessary conditions for optimal control

(with unbounded controls)

- Necessary conditions consist of a set of $2n$, *first-order*, differential equations (state and co-state equations), and a set of m algebraic equations (control equations)
- The solution to the state and co-state equations will contain $2n$ constants of integration
- To pinpoint the constants, we use the n equations $\mathbf{x}(t_0) = \mathbf{x}_0$, and an additional set of n (or $n + 1$) equations from the boundary conditions
- We are again confronted by a *2-point boundary value problem*
- To determine the boundary conditions, one has to make the "appropriate" substitutions

Necessary conditions for optimal control

(with unbounded controls)

Problem	*Description*	*Substitution in Eq. (5.1-18)*	*Boundary-condition equations*	*Remarks*
t_f fixed	1. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
	2. $\mathbf{x}(t_f)$ free	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
	3. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$	$(2n + k)$ equations to determine the $2n$ constants of integration and the variables $d_1, \ldots, d_k$
t_f free	4. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	5. $\mathbf{x}(t_f)$ free		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f

From Kirk, Section 5.1

Example

Find optimal control $u(t)$ to steer the system

$$\ddot{x}(t) = u(t)$$

from $x(0) = 10, \dot{x}(0) = 0$ to the origin $x(t_f) = 0, \dot{x}(t_f) = 0$, and to minimize

$$J = \frac{1}{2}\alpha t_f^2 + \frac{1}{2}\int_{t_0}^{t_f} b\, u^2(t)dt\,, \quad \alpha, b > 0$$

- Solution: optimal time is

$$t_f = \left(\frac{1800b}{\alpha}\right)^{1/5}$$

Proof of NOC

- For simplicity, assume that the terminal penalty is equal to zero, and that t_f and $\mathbf{x}(t_f)$ are fixed and given
- Consider the augmented cost function
  $$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) := g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}(t)'[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)]$$
  where the $\{p_i(t)\}$'s are Lagrange multipliers
- Note that we have simply added zero to the cost function!
- The augmented cost function is then
  $$J_a(\mathbf{u}) = \int_{t_0}^{t_f} g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) \; dt$$

Proof of NOC

On an extremal, by applying the fundamental theorem of the CoV

By the CoV theorem

$$
\begin{aligned}
0 = \delta J_a(\mathbf{u}) = \int_{t_0}^{t_f} \Bigg(\Bigg[& \overbrace{\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)}^{= \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)' \mathbf{p}(t)} - \overbrace{\frac{d}{dt} \frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)}^{= -\frac{d}{dt}(-\mathbf{p}^*(t))} \Bigg]' \delta \mathbf{x}(t) \\
+ & \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right] \delta \mathbf{u}(t) + \underbrace{\left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]'}_{= \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)} \delta \mathbf{p}(t) \Bigg) dt
\end{aligned}
$$

Proof of NOC

Considering each term in sequence,

- $\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t) = 0$, on an extremal
- The Lagrange multipliers are arbitrary, so we can select them to make the coefficient of $\delta\mathbf{x}(t)$ equal to zero, that is
$$\dot{\mathbf{p}}^*(t) = -\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)' \mathbf{p}^*(t)$$
- The remaining variation $\delta\mathbf{u}(t)$, is independent, so its coefficient must be zero; thus
$$\frac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)' \mathbf{p}(t) = 0$$

By using the Hamiltonian formalism, one obtains the claim

Next time

- Derivation of LQR (again!)
- Pontryagin's minimum principle
- Special cases