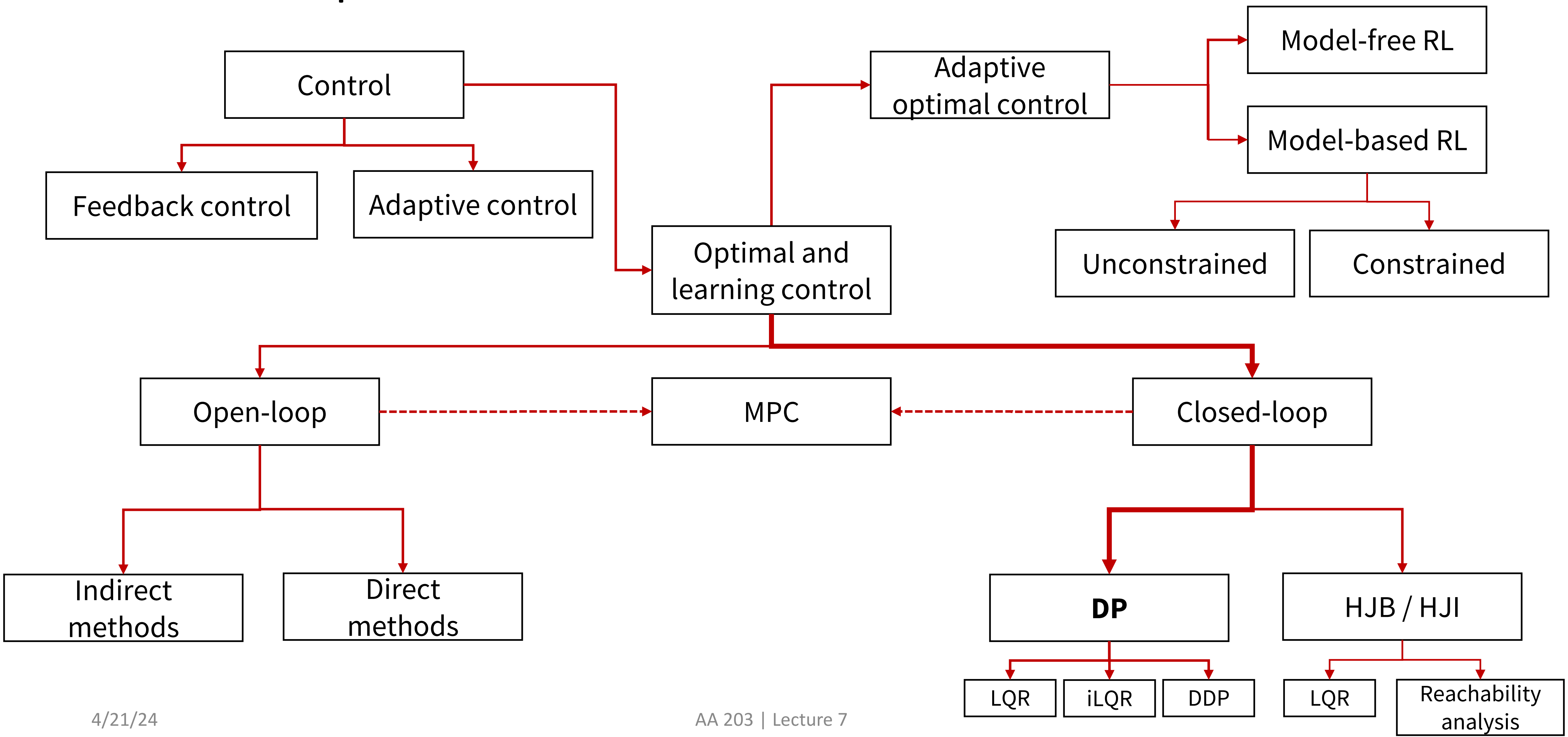
AA203
Optimal and Learning-based Control

Dynamic programming, discrete LQR

Stanford University

Autonomous Systems Lab

Roadmap


Control
Feedback control
Adaptive control
Optimal and learning control
Adaptive optimal control
Model-free RL
Model-based RL
Unconstrained
Constrained
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI
LQR
iLQR
DDP
LQR
Reachability analysis

Basic problem – discrete-time setting

- System: $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k), \quad k = 0, \dots, N-1$
- Control constraints: $\mathbf{u}_k \in U(\mathbf{x}_k)$
- Cost:

$$J(\mathbf{x}_0; \boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1}) = h_N(\mathbf{x}_N) + \sum_{k=0}^{N-1} g(\mathbf{x}_k, \mathbf{u}_k, k)$$

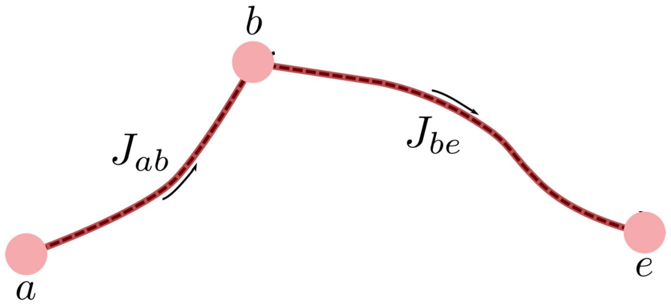
- Focus is now on finding optimal closed-loop policies:

$$\mathbf{u}_k^* = \pi^*(\mathbf{x}_k, k) \text{ (or } \pi_k^*(\mathbf{x}_k))$$

Principle of optimality

The key concept behind the dynamic programming approach is the principle of optimality

Suppose optimal path for a multi-stage decision-making problem is



- first decision yields segment $a - b$ with cost J_{ab}
- remaining decisions yield segments $b - e$ with cost J_{be}
- optimal cost is then $J_{ae}^{*} = J_{ab} + J_{be}$

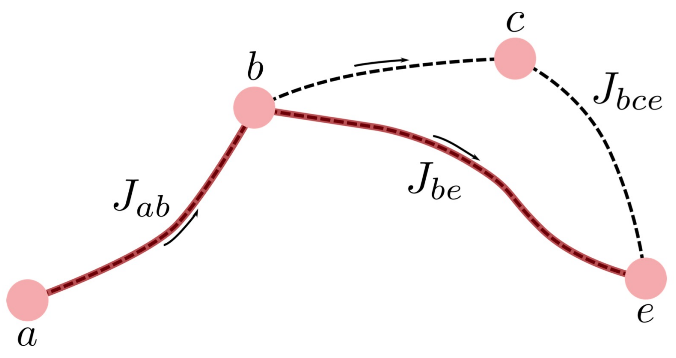
Principle of optimality

- Claim: If $a - b - e$ is optimal path from a to e, then $b - e$ is optimal path from b to e
- *Proof*: Suppose $b - c - e$ is the optimal path from b to e. Then

$$J_{bce} < J_{be}$$

and

$$J_{ab} + J_{bce} < J_{ab} + J_{be} = J_{ae}^{*}$$



Contradiction!

Principle of optimality

Principle of optimality: Let $\{\mathbf{u}_0^*, \mathbf{u}_1^*, \dots, \mathbf{u}_{N-1}^*\}$ be an optimal control sequence, which together with $\mathbf{x}_0^*$ determines the corresponding state sequence $\{\mathbf{x}_0^*, \mathbf{x}_1^*, \dots, \mathbf{x}_N^*\}$. Consider the subproblem whereby we are at $\mathbf{x}_k^*$ at time k and we wish to minimize the cost-to-go from time k to time N, i.e.,

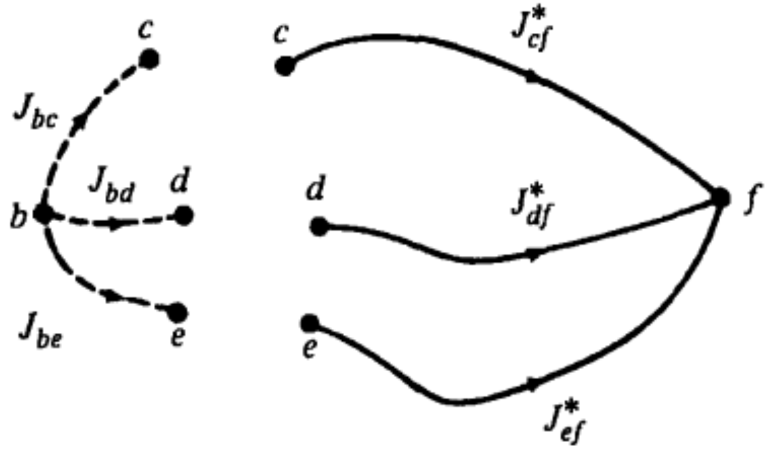
$$g_k(\mathbf{x}_k^*, \mathbf{u}_k) + \sum_{m=k+1}^{N-1} g_m(\mathbf{x}_m, \mathbf{u}_m) + h_N(\mathbf{x}_N)$$

Then the truncated optimal sequence $\{\mathbf{u}_k^*, \mathbf{u}_{k+1}^*, \dots, \mathbf{u}_{N-1}^*\}$ is optimal for the subproblem

- Tail of optimal sequences optimal for tail subproblems

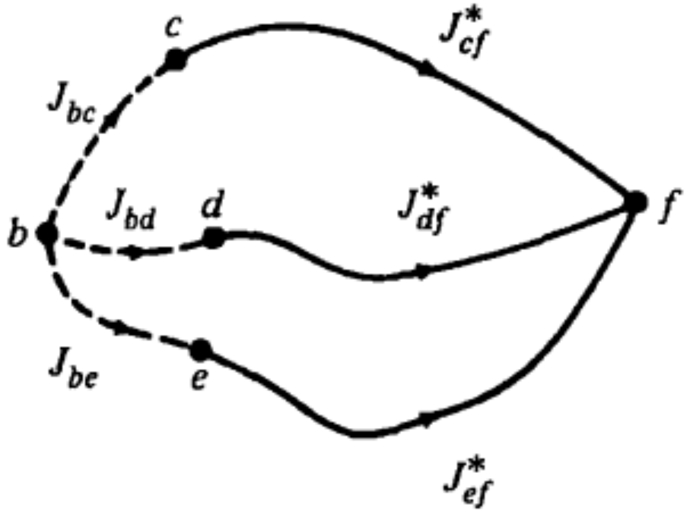
Applying the principle of optimality

Principle of optimality: if $b - c$ is the initial segment of the optimal path from b to f, then $c - f$ is the terminal segment of this path



Hence, the optimal trajectory is found by comparing:

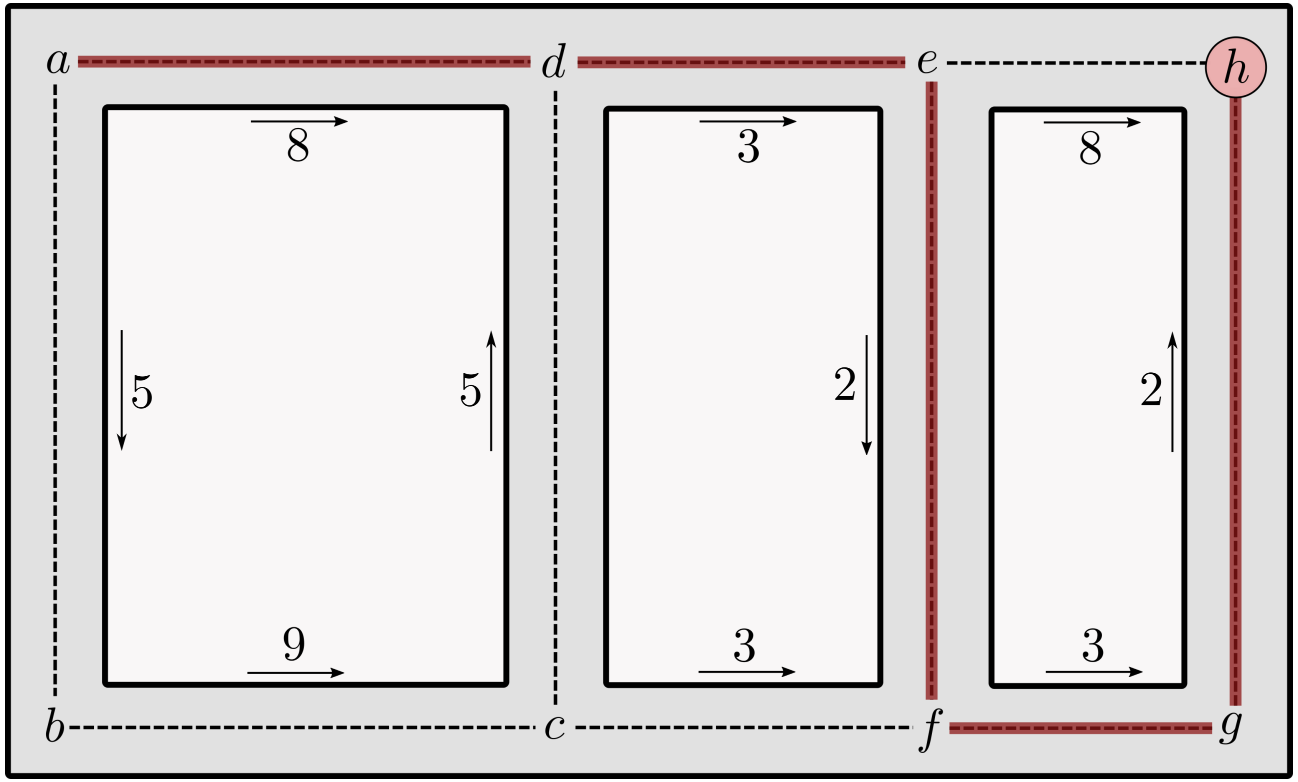
$$C_{bcf} = J_{bc} + J^*_{cf}$$
$$C_{bdf} = J_{bd} + J^*_{df}$$
$$C_{bef} = J_{be} + J^*_{ef}$$



Applying the principle of optimality

- need only to compare the concatenations of immediate decisions and optimal decisions → significant decrease in computation / possibilities
- in practice: carry out this procedure backward in time

Example



Optimal cost: 18

Optimal path: $a \rightarrow\ d \rightarrow\ e \rightarrow\ f \rightarrow\ g \rightarrow\ h$

DP Algorithm

- Model: $\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, k), \quad \mathbf{u}_k \in U(\mathbf{x}_k)$
- Cost: $J(\mathbf{x}_0) = h_N(\mathbf{x}_N) + \sum_{k=0}^{N-1} g(\mathbf{x}_k, \pi_k(\mathbf{x}_k), k)$

DP Algorithm: For every initial state $\mathbf{x}_0$, the optimal cost $J^*(\mathbf{x}_0)$ is equal to $J_0(\mathbf{x}_0)$, given by the last step of the following algorithm, which proceeds backward in time from stage $N-1$ to stage 0:

$$J_N(\mathbf{x}_N) = h_N(\mathbf{x}_N)$$

$$J_k(\mathbf{x}_k) = \min_{\mathbf{u}_k \in U(\mathbf{x}_k)} g(\mathbf{x}_k, \mathbf{u}_k, k) + J_{k+1}\big(f(\mathbf{x}_k, \mathbf{u}_k, k)\big), \qquad k = 0, \dots, N-1$$

Furthermore, if $\mathbf{u}_k^* = \pi_k^*(\mathbf{x}_k)$ minimizes the right-hand side of the above equation for each $\mathbf{x}_k$ and k, the policy $\{\pi_0^*, \pi_1^*, \dots, \pi_{N-1}^*\}$ is optimal

Comments

- discretization (from differential equations to difference equations)
- quantization (from continuous to discrete state variables / controls)
- global minimum
- constraints, in general, simplify the numerical procedure
- optimal control in closed-loop form
- curse of dimensionality

Example: discrete LQR

- In most cases, DP algorithm needs to be performed numerically
- A few cases can be solved analytically

Discrete LQR: select control inputs to minimize

$$J(\mathbf{x}_0) = \frac{1}{2}\mathbf{x}_N' \, H \, \mathbf{x}_N + \frac{1}{2}\sum_{k=0}^{N-1}[\mathbf{x}_k' \, Q \, \mathbf{x}_k + \mathbf{u}_k' \, R \, \mathbf{u}_\mathbf{k}]$$

subject to the dynamics

$$\mathbf{x}_{k+1} = A_k \mathbf{x}_k + B_k \, \mathbf{u}_k$$

Assumption: $H = H' \geq 0, \; Q = Q' \geq 0, R = R' > 0$

Example: discrete LQR

First step:

$$J_N^*(\mathbf{x}_N) = \frac{1}{2}\mathbf{x}_N' H \mathbf{x}_N := \frac{1}{2}\mathbf{x}_N' P_N \mathbf{x}_N$$

Going backward

$$J_{N-1}(\mathbf{x}_{N-1}) = \min_{\mathbf{u}_{N-1}} \ \frac{1}{2}\left\{\mathbf{x}_{N-1}' Q \mathbf{x}_{N-1} + \mathbf{u}_{N-1}' R \mathbf{u}_{N-1} + \mathbf{x}_N' H \mathbf{x}_N\right\}$$

$$\min_{\mathbf{u}_{N-1}} \ \frac{1}{2}\Big\{\mathbf{x}_{N-1}' Q \mathbf{x}_{N-1} + \mathbf{u}_{N-1}' R \mathbf{u}_{N-1} +$$

$$(A_{N-1}\mathbf{x}_{N-1} + B_{N-1}\mathbf{u}_{N-1})' H (A_{N-1}\mathbf{x}_{N-1} + B_{N-1}\mathbf{u}_{N-1})\Big\}$$

Example: discrete LQR

Taking derivative

$$\frac{\partial J^*_{N-1}(\mathbf{x}_{N-1})}{\partial \mathbf{u}_{N-1}} = R\mathbf{u}_{N-1} + B'_{N-1}H(A_{N-1}\mathbf{x}_{N-1} + B_{N-1}\mathbf{u}_{N-1}) = 0$$

and

$$\frac{\partial^2 J^*_{N-1}(\mathbf{x}_{N-1})}{\partial \mathbf{u}^2_{N-1}} = R + B'_{N-1}HB_{N-1} > 0$$

DP for discrete LQR

Hence, the optimizer satisfies

$$(R + B'_{N-1} H B_{N-1})\mathbf{u}^*_{N-1} + B'_{N-1} H A_{N-1} \mathbf{x}_{N-1} = 0$$

so

$$\mathbf{u}^*_{N-1} = -(R + B'_{N-1} H B_{N-1})^{-1} B'_{N-1} H A_{N-1} \mathbf{x}_{N-1} := F_{N-1} \mathbf{x}_{N-1}$$

DP for discrete LQR

Plugging in

$$
\begin{aligned}
J_{N-1}(\mathbf{x}_{N-1}) =& \frac{1}{2}\mathbf{x}'_{N-1}\Big\{Q + F'_{N-1}RF_{N-1} + \\
& (A_{N-1} + B_{N-1}F_{N-1})'H(A_{N-1} + B_{N-1}F_{N-1})\Big\}\mathbf{x}_{N-1} \\
:=& \mathbf{x}'_{N-1}P_{N-1}\mathbf{x}_{N-1} \\
F_{N-1} =& -(R + B'_{N-1}P_NB_{N-1})^{-1}B'_{N-1}P_NA_{N-1}
\end{aligned}
$$

DP for discrete LQR

Proceeding by induction, the solution is given by

1. $J_N(\mathbf{x}_N) = \frac{1}{2}\mathbf{x}_N' P_N \mathbf{x}_N$, where $P_N = H$
2. $\mathbf{u}_k^* = F_k \mathbf{x}_k$, where $F_k = -(R + B_k' P_{k+1} B_k)^{-1} B_k' P_{k+1} A_k$
3. $J_k(\mathbf{x}_k) = \frac{1}{2}\mathbf{x}_k' P_k \mathbf{x}_k$, where

$$P_k = Q + F_k' R F_k + (A_k + B_k F_k)' P_{k+1} (A_k + B_k F_k)$$

At the end, $J_0(\mathbf{x}_0) = \frac{1}{2}\mathbf{x}_0' P_0 \mathbf{x}_0$

Next time

- Stochastic DP, value iteration, policy iteration