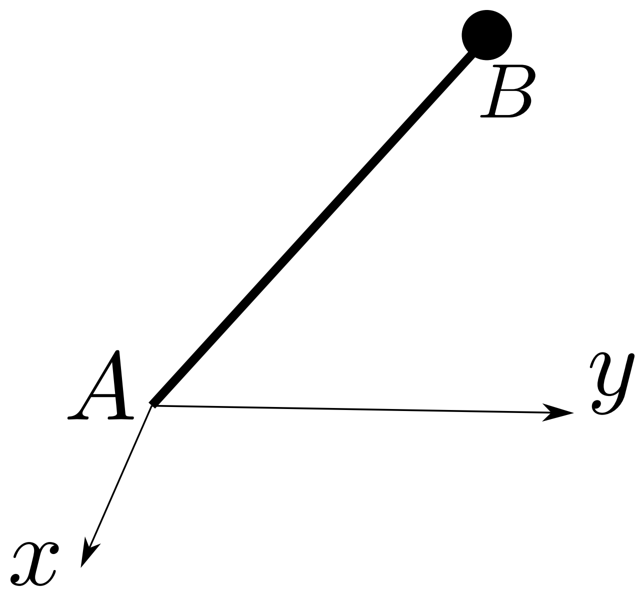

Consider a massless pole with a sphere attached to one end, with sphere mass being m_s. The other end of the pole (point A) is attached to the end-effector with mass m_e at point A. The state of the system is the velocity of the end-effector point A $(\dot{x}_A, \dot{y}_A, \dot{z}_A)$, the delta position between the sphere and the end-effector in the horizontal plane $x_{AB} = x_B - x_A, y_{AB} = y_B - y_A$, together with its time derivative $\dot{x}_{AB}, \dot{y}_{AB}$.

The position of the mass is

$$\begin{bmatrix} x_A + x_{AB} \\ y_A + y_{AB} \\ z_A + \sqrt{l^2 - x_{AB}^2 - y_{AB}^2} \end{bmatrix} \tag{1}$$

The velocity of the mass is

$$\begin{bmatrix} \dot{x}_A + \dot{x}_{AB} \\ \dot{y}_A + \dot{y}_{AB} \\ \dot{z}_A - \frac{l x_{AB}\dot{x}_{AB} + l y_{AB}\dot{y}_{AB}}{\sqrt{l^2 - x_{AB}^2 - y_{AB}^2}} \end{bmatrix} \tag{2}$$

The total kinetic energy of the system is

$$\begin{aligned} T = 0.5 m_e(\dot{x}_A^2 + \dot{y}_A^2 + \dot{z}_A^2) + 0.5 m_s(\dot{x}_A^2 + \dot{y}_A^2 + \dot{z}_A^2 + \dot{x}_{AB}^2 + \dot{y}_{AB}^2 + \frac{l^2(x_{AB}^2\dot{x}_{AB}^2 + y_{AB}^2\dot{y}_{AB}^2 + 2x_{AB}y_{AB}\dot{x}_{AB}\dot{y}_{AB})}{l^2 - x_{AB}^2 - y_{AB}^2} \\ + 2\dot{x}_A\dot{x}_{AB} + 2\dot{y}_A\dot{y}_{AB} - \frac{2l\dot{z}_A(x_{AB}\dot{x}_{AB} + y_{AB}\dot{y}_{AB})}{\sqrt{l^2 - x_{AB}^2 - y_{AB}^2}}) \end{aligned} \tag{3}$$

The total potential energy is

$$V = m_e g z_A + m_s g(z_A + \sqrt{l^2 - x_{AB}^2 - y_{AB}^2}) \tag{4}$$

Using Lagrangian $L = T - V$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Bu$, we have

$$(m_e + m_s)\ddot{x}_A + m_s\ddot{x}_{AB} = u_x \tag{5}$$

$$(m_e + m_s)\ddot{y}_A + m_s\ddot{y}_{AB} = u_y \tag{6}$$

$$(m_e+m_s)(\ddot{z}_A+g)-m_s\left(\dot{x}_{AB}^2\frac{l^2-y_{AB}^2}{z_{AB}^3}+\frac{x_{AB}}{z_{AB}}\ddot{x}_{AB}+\dot{y}_{AB}^2\frac{l^2-x_{AB}^2}{z_{AB}^3}+\frac{y_{AB}}{z_{AB}}\ddot{y}_{AB}-2\frac{x_{AB}y_{AB}\dot{x}_{AB}\dot{y}_{AB}}{z_{AB}^3}\right)=u_z \tag{7}$$

$$\begin{aligned}m_s(\ddot{x}_A+\ddot{x}_{AB})-m_s x_{AB}(g+\ddot{z}_A)/z_{AB}+m_s(x_{AB}^2\ddot{x}_{AB}+x_{AB}\dot{x}_{AB}^2+x_{AB}y_{AB}\ddot{y}_{AB}+x_{AB}\dot{y}_{AB}^2)/z_{AB}^2\\+m_s(x_{AB}^3\dot{x}_{AB}^2+2x_{AB}^2y_{AB}\dot{x}_{AB}\dot{y}_{AB}+x_{AB}y_{AB}^2\dot{y}_{AB}^2)/z_{AB}^4=0\end{aligned} \tag{8}$$

$$\begin{aligned}m_s(\ddot{y}_A+\ddot{y}_{AB})-m_s y_{AB}(g+\ddot{z}_A)/z_{AB}+m_s(y_{AB}^2\ddot{y}_{AB}+y_{AB}\dot{y}_{AB}^2+y_{AB}x_{AB}\ddot{x}_{AB}+y_{AB}\dot{x}_{AB}^2)/z_{AB}^2\\+m_s(y_{AB}^3\dot{y}_{AB}^2+2y_{AB}^2x_{AB}\dot{y}_{AB}\dot{x}_{AB}+y_{AB}x_{AB}^2\dot{x}_{AB}^2)/z_{AB}^4=0\end{aligned} \tag{9}$$

In the matrix form, we have

$$M\begin{bmatrix}\ddot{x}_A\\ \ddot{y}_A\\ \ddot{z}_A\\ \ddot{x}_{AB}\\ \ddot{y}_{AB}\end{bmatrix}+C=\begin{bmatrix}u_x\\ u_y\\ u_z\\ 0\\ 0\end{bmatrix} \tag{10}$$

where

$$M=\begin{bmatrix}m_e+m_s & 0 & 0 & m_s & 0\\ 0 & m_e+m_s & 0 & 0 & m_s\\ 0 & 0 & m_e+m_s & -m_s\frac{x_{AB}}{z_{AB}} & -m_s\frac{y_{AB}}{z_{AB}}\\ m_s & 0 & -m_s\frac{x_{AB}}{z_{AB}} & m_s+m_s\frac{x_{AB}^2}{z_{AB}^2} & m_s\frac{x_{AB}y_{AB}}{z_{AB}^2}\\ 0 & m_s & -m_s\frac{y_{AB}}{z_{AB}} & m_s\frac{x_{AB}y_{AB}}{z_{AB}^2} & m_s+m_s\frac{y_{AB}^2}{z_{AB}^2}\end{bmatrix} \tag{11}$$

$$C=\begin{bmatrix}0\\ 0\\ (m_e+m_s)g-m_s\left(\dot{x}_{AB}^2\frac{l^2-y_{AB}^2}{z_{AB}^3}+\dot{y}_{AB}^2\frac{l^2-x_{AB}^2}{z_{AB}^3}-2\frac{x_{AB}y_{AB}\dot{x}_{AB}\dot{y}_{AB}}{z_{AB}^3}\right)\\ -m_s g\frac{x_{AB}}{z_{AB}}+m_s x_{AB}\left(\frac{\dot{x}_{AB}^2+\dot{y}_{AB}^2}{z_{AB}^2}+\frac{(x_{AB}\dot{x}_{AB}+y_{AB}\dot{y}_{AB})^2}{z_{AB}^4}\right)\\ -m_s g\frac{y_{AB}}{z_{AB}}+m_s y_{AB}\left(\frac{\dot{x}_{AB}^2+\dot{y}_{AB}^2}{z_{AB}^2}+\frac{(x_{AB}\dot{x}_{AB}+y_{AB}\dot{y}_{AB})^2}{z_{AB}^4}\right)\end{bmatrix} \tag{12}$$