



1 Pole/end-effector sub system

Consider a massless pole with a sphere attached to one end, with sphere mass being m_s . The other end of the pole (point A) is attached to the end-effector with mass m_e at point A . The state of the system is the velocity of the end-effector point A ($\dot{x}_A, \dot{y}_A, \dot{z}_A$), the delta position between the sphere and the end-effector in the horizontal plane $x_{AB} = x_B - x_A, y_{AB} = y_B - y_A$, together with its time derivative $\dot{x}_{AB}, \dot{y}_{AB}$.

The position of the mass is

$$\begin{bmatrix} x_A + x_{AB} \\ y_A + y_{AB} \\ z_A + \sqrt{l^2 - x_{AB}^2 - y_{AB}^2} \end{bmatrix} \quad (1)$$

The velocity of the mass is

$$\begin{bmatrix} \dot{x}_A + \dot{x}_{AB} \\ \dot{y}_A + \dot{y}_{AB} \\ \dot{z}_A - \frac{x_{AB}\dot{x}_{AB} + y_{AB}\dot{y}_{AB}}{\sqrt{l^2 - x_{AB}^2 - y_{AB}^2}} \end{bmatrix} \quad (2)$$

The total kinetic energy of the system is

$$\begin{aligned} T = 0.5m_e(\dot{x}_A^2 + \dot{y}_A^2 + \dot{z}_A^2) + 0.5m_s(\dot{x}_A^2 + \dot{y}_A^2 + \dot{z}_A^2 + \dot{x}_{AB}^2 + \dot{y}_{AB}^2 + \frac{(x_{AB}^2\dot{x}_{AB}^2 + y_{AB}^2\dot{y}_{AB}^2 + 2x_{AB}y_{AB}\dot{x}_{AB}\dot{y}_{AB})}{l^2 - x_{AB}^2 - y_{AB}^2} \\ + 2\dot{x}_A\dot{x}_{AB} + 2\dot{y}_A\dot{y}_{AB} - \frac{2\dot{z}_A(x_{AB}\dot{x}_{AB} + y_{AB}\dot{y}_{AB})}{\sqrt{l^2 - x_{AB}^2 - y_{AB}^2}}) \end{aligned} \quad (3)$$

The total potential energy is

$$V = m_e g z_A + m_s g (z_A + \sqrt{l^2 - x_{AB}^2 - y_{AB}^2}) \quad (4)$$

Using Lagrangian $L = T - V$ and $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Bu$, we have

$$(m_e + m_s)\ddot{x}_A + m_s\ddot{x}_{AB} = f_x \quad (5)$$

$$(m_e + m_s)\ddot{y}_A + m_s\ddot{y}_{AB} = f_y \quad (6)$$

$$(m_e + m_s)(\ddot{z}_A + g) - m_s \left(\dot{x}_{AB}^2 \frac{l^2 - y_{AB}^2}{z_{AB}^3} + \frac{x_{AB}}{z_{AB}} \ddot{x}_{AB} + \dot{y}_{AB}^2 \frac{l^2 - x_{AB}^2}{z_{AB}^3} + \frac{y_{AB}}{z_{AB}} \ddot{y}_{AB} - 2 \frac{x_{AB} y_{AB} \dot{x}_{AB} \dot{y}_{AB}}{z_{AB}^3} \right) = f_z \quad (7)$$

$$m_s(\ddot{x}_A + \ddot{x}_{AB}) - m_s x_{AB}(g + \ddot{z}_A)/z_{AB} + m_s(x_{AB}^2 \ddot{x}_{AB} + x_{AB} \dot{x}_{AB}^2 + x_{AB} y_{AB} \ddot{y}_{AB} + x_{AB} \dot{y}_{AB}^2)/z_{AB}^2 + m_s(x_{AB}^3 \dot{x}_{AB}^2 + 2x_{AB}^2 y_{AB} \dot{x}_{AB} \dot{y}_{AB} + x_{AB} y_{AB}^2 \dot{y}_{AB}^2)/z_{AB}^4 = 0 \quad (8)$$

$$m_s(\ddot{y}_A + \ddot{y}_{AB}) - m_s y_{AB}(g + \ddot{z}_A)/z_{AB} + m_s(y_{AB}^2 \ddot{y}_{AB} + y_{AB} \dot{y}_{AB}^2 + y_{AB} x_{AB} \ddot{x}_{AB} + y_{AB} \dot{x}_{AB}^2)/z_{AB}^2 + m_s(y_{AB}^3 \dot{y}_{AB}^2 + 2y_{AB}^2 x_{AB} \dot{y}_{AB} \dot{x}_{AB} + y_{AB} x_{AB}^2 \dot{x}_{AB}^2)/z_{AB}^4 = 0 \quad (9)$$

In the matrix form, we have

$$M \begin{bmatrix} \ddot{x}_A \\ \ddot{y}_A \\ \ddot{z}_A \\ \ddot{x}_{AB} \\ \ddot{y}_{AB} \end{bmatrix} + C = \begin{bmatrix} f_x \\ f_y \\ f_z \\ 0 \\ 0 \end{bmatrix} \quad (10)$$

where

$$M = \begin{bmatrix} m_e + m_s & 0 & 0 & m_s & 0 \\ 0 & m_e + m_s & 0 & 0 & m_s \\ 0 & 0 & m_e + m_s & -m_s \frac{x_{AB}}{z_{AB}^2} & -m_s \frac{y_{AB}}{z_{AB}} \\ m_s & 0 & -m_s \frac{x_{AB}}{z_{AB}} & m_s + m_s \frac{x_{AB}^2}{z_{AB}^2} & m_s \frac{x_{AB} y_{AB}}{z_{AB}^2} \\ 0 & m_s & -m_s \frac{y_{AB}}{z_{AB}} & m_s \frac{x_{AB} y_{AB}}{z_{AB}^2} & m_s + m_s \frac{y_{AB}^2}{z_{AB}^2} \end{bmatrix} \quad (11)$$

$$C = \begin{bmatrix} 0 \\ 0 \\ (m_e + m_s)g - m_s \left(\dot{x}_{AB}^2 \frac{l^2 - y_{AB}^2}{z_{AB}^3} + \dot{y}_{AB}^2 \frac{l^2 - x_{AB}^2}{z_{AB}^3} - 2 \frac{x_{AB} y_{AB} \dot{x}_{AB} \dot{y}_{AB}}{z_{AB}^3} \right) \\ -m_s g \frac{x_{AB}}{z_{AB}} + m_s x_{AB} \left(\frac{\dot{x}_{AB}^2 + \dot{y}_{AB}^2}{z_{AB}^2} + \frac{(x_{AB} \dot{x}_{AB} + y_{AB} \dot{y}_{AB})^2}{z_{AB}^4} \right) \\ -m_s g \frac{y_{AB}}{z_{AB}} + m_s y_{AB} \left(\frac{\dot{x}_{AB}^2 + \dot{y}_{AB}^2}{z_{AB}^2} + \frac{(x_{AB} \dot{x}_{AB} + y_{AB} \dot{y}_{AB})^2}{z_{AB}^4} \right) \end{bmatrix} \quad (12)$$

2 Whole system

Assuming that we construct a controller, that given the current state of the pole/end-effector system, this controller computes the force f_x, f_y, f_z applied from the robot to the end-effector, now we want to compute the robot joint torque τ to apply that force.

We first apply the force f_x, f_y, f_z computed from the controller as input to the pole/end-effector system, from the dynamics equation (10) we can compute the acceleration of the end effector $\ddot{x}_A, \ddot{y}_A, \ddot{z}_A$, together with the acceleration of the pole $\ddot{x}_B, \ddot{y}_B, \ddot{z}_B$ (where we use both $\ddot{x}_A, \ddot{y}_A, \ddot{z}_A$ and $\ddot{x}_{AB}, \ddot{y}_{AB}, \ddot{z}_{AB}$). We know that for the pole to achieve this acceleration, the end-effector has to apply a force

$$f_B = m_s \begin{bmatrix} \ddot{x}_B \\ \ddot{y}_B \\ \ddot{z}_B + g \end{bmatrix} \quad (13)$$

onto the pole, applied at where the pole makes contact with the end effector. Based on Newton's third law, there is an equal and opposite force $-f_B$ applied on the end-effector at the contact point P between the

end-effector and the pole. Hence our goal is to compute the joint torque of the robot arm, such that the end-effector can achieve the desired acceleration $\ddot{x}_A, \ddot{y}_A, \ddot{z}_A$ under the external force $-f_B$.

We can write the manipulator equation for the IIWA arm (together with the end-effector welded to the wrist joint)

$$M_{iiwa}\ddot{q}_{iiwa} + C = g(q_{iiwa}) + \tau - ({}^W J^P)^T f_B \quad (14)$$

where ${}^W J^P$ is the Jacobian of the contact point P written in the world frame. And we also have the constraints on the end-effector acceleration

$${}^W J^E \ddot{q}_{iiwa} + {}^W \dot{J}^E \dot{q}_{iiwa} = {}^W a_{des}^E \quad (15)$$

where ${}^W J^E$ is the Jacobian of the end-effector written in the world frame. ${}^W a_{des}^E$ is the desired acceleration of the IIWA end-effector frame E in the world frame W . This acceleration includes both the linear acceleration $\ddot{x}_A, \ddot{y}_A, \ddot{z}_A$, together with the desired angular acceleration. The desired acceleration can be set in two ways

- If our goal is to keep the end-effector to be horizontal, then we can compute the desired angular acceleration using a PD law depending on the orientation error.
- We can also set the desired end-effector acceleration to be zero. Note that this won't handle any orientation drift. Namely if currently the end-effector angular velocity is non-zero, then the orientation will continue to drift.

Combining the equations (14) and (15) we have unknown variable $\tau \in \mathbb{R}^7$. The problem is under-constrained, we can also impose the cost as $\min \tau^T \tau$ to get a unique (and optimal) τ . If we ignore the joint torque limit, then this ends up being an equality-constrained QP which can be solved very efficiently in the closed form.