Decentralized policies for geometric pattern formation

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Abstract— **This paper presents a decentralized control policy for symmetric formations in multi-agent systems. It is shown that n agents, each one pursuing its leading neighbor along the line of sight rotated by a common offset angle α, eventually converge to a single point, a circle or a logarithmic spiral pattern, depending on the value of α. Simulation results are presented and discussed.**

I. INTRODUCTION

Consider a group of mobile robots and suppose we want them to move in formation; assume, moreover, we allow the robots just to interact with a (small) number of neighbors, without allowing the presence of a leader. This problem is currently of particular interest: in fact, from a theoretical point of view, it is a prototypical distributed control problem and, from a practical point of view, it arises in many applications like distributed sensing using mobile sensor networks, space missions with multiple spacecrafts, and automated parallel delivery of payloads; moreover, decentralized (*i.e.* without a leader) coordination problems are also useful in modelling and understanding biological systems such as animal groups.

Recently, Justh and Krishnaprasad [1] presented a differential geometric setting for the problem of formation control and proposed two strategies to achieve, respectively, rectilinear and circle formation; their approach, however, requires all-to-all communication among agents. Jadbabaie *et al.* [2] formally proved that the nearest neighbor algorithm by Vicsek [3] causes all agents to eventually move in the same direction, despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors change with time as the system evolves. Jeanne *et al.* [4] and Paley *et al.* [5] studied the connections between phase models of coupled oscillators and kinematic models of groups of self-propelled particles and derived control laws for parallel motions and circular motions. Olfati-Saber and Murray [6] and Leonard *et al.* [7] used potential function theory to prescribe flocking behavior. Lin *et al.* [8] exploited cyclic pursuit to achieve alignment among agents, while Marshall *et al.* in [9] and in [10] extended the classic cyclic pursuit to a system of wheeled vehicles, each subject to a single non-

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holonomic constraint, and studied the possible equilibrium formations and their stability.

In this paper, we develop a distributed control policy that allows the robots to achieve different symmetric formations. Our approach is inspired by the cyclic pursuit strategy, where each agent i pursues the next $i + 1$ (modulo n, where n is the number of agents) along the line of sight. Cyclic pursuit is an attractive approach since it is decentralized and requires the minimum number of communication links (n links for n agents) to achieve a formation. It is well known that, under cyclic pursuit, the agents eventually converge to a single point (see, *e.g.*, [11]). Our control policy generalizes the notion of classic cyclic pursuit by letting each of the n mobile agents pursue its leading neighbor along the line of sight *rotated* by a common offset angle α. The key features of the proposed approach are global stability and the possibility to achieve with the same simple control law different formations.

We first assume, in Section III, that each agent is a simple integrator, *i.e.*, a dynamical system with no kinematic constraints on its motion. It is shown that our simple control policy can provide, depending on the value of α, rendez-vous to a point (thus generalizing the classic cyclic pursuit result that is obtained for $\alpha = 0$), evenly spaced circle formation and evenly spaced logarithmic spirals. These results are also illustrative of the intrinsic symmetry that characterizes cyclic pursuit. The study of the proposed control policy relies on an original *parametric* spectral analysis of some special types of circulant matrices.

Then, in Section IV, we extend the above linear scenario to one in which each agent is a non-holonomic mobile robot. The fact that we now consider differentially driven mobile robots complicates the coordination problem, due to the non-holonomic constraints. We address this problem by input-output feedback linearizing the system and by adapting the aforementioned control policy to the linearized input-output dynamics, represented by a double integrator system. The control of non-holonomic wheeled mobile robots by state feedback linearization was introduced in [12] and has been often exploited in the context of formation control, for example in [13]. The closed-loop behaviors are, depending on the value of α, *globally stable* rendez-vous to a point, *globally stable* evenly spaced circle formation and *globally stable* evenly spaced logarithmic spirals. In [10], a formation control law for non-holonomic wheeled robots in cyclic pursuit is introduced and elegantly studied. It is shown that, depending on the value of a gain, *locally stable* rendez-vous to a point, *locally stable* evenly spaced circle formation and *locally stable* evenly spaced logarithmic spirals are achieved.

However, it turns out that these regular formations are not the only stable behaviors. Simulations, reported in [14], indicate that when the vehicles do not converge to a generalized regular polygon formation, they instead fall into a different kind of order: the vehicles "weave" in and out, while the formation as a whole moves along a linear trajectory. Therefore, our globally stable control policy may represent a significant improvement as far as potential applications are concerned.

The paper is organized as follows. In Section II, we discuss some mathematical preliminaries. In Section III, we present and study our control policy, modelling each agent as a simple integrator; then, in Section IV, we extend the above holonomic scenario to one in which each agent is a non-holonomic mobile robot. In Section V, we draw our conclusions.

II. MATHEMATICAL BACKGROUND

In this section, we provide some basic definitions and results concerning the theory of block circulant matrices, which will be later applied to analyze the proposed control strategy.

Let $A_1, A_2, \ldots, A_n$ be square matrices each of order m. A *block circulant* matrix of type (m, n) is a matrix of order mn of the form:

$$\hat{A} = \begin{pmatrix} A_1 & A_2 & \ldots & A_n \\ A_n & A_1 & \ldots & A_{n-1} \\ \vdots & \vdots & & \vdots \\ A_2 & A_3 & \ldots & A_1 \end{pmatrix}.$$

Note that $\hat{A}$ is not necessarily circulant (only *block* circulant). The entire matrix is determined by the first block row and we can write:

$$\hat{A} = \mathrm{circ}[A_1, A_2, \ldots, A_n].$$

The Fourier matrix provide a diagonalization formula in the special case where all submatrices A_i are themselves circulant [15]. In the general case, a block circulant matrix can only be *block* diagonalized by means of the Fourier matrix (see [15] for details). Define $\varphi_i = e^{2(i-1)\pi j/n}$ where $j = \sqrt{-1}$; each diagonal block has the following structure:

$$D_i = A_1 + \varphi_i A_2 + \varphi_i^2 A_3 + \ldots + \varphi_i^{n-1} A_n, \qquad i = 1, 2, \ldots, n. \tag{1}$$

Since the eigenvalues of a block diagonal matrix are the collection of all the eigenvalues of each diagonal block, the block diagonalization formula greatly simplify the study of the eigenvalues of a block circulant matrix.

III. FORMATION CONTROL: HOLONOMIC ROBOTS

Consider in the plane n mobile agents (uniquely labelled by an integer $i \in \{1, 2, \ldots, n\}$), where agent i pursues the next $i + 1$, modulo n. Let $\boldsymbol{\xi}_i(t) = [x_i(t), y_i(t)]^T \in \mathbb{R}^2$ be the position at time $t \geq 0$ of the ith agent, where $i \in 1, 2, \ldots, n$. The kinematics of each agent is described by a simple integrator:

$$\dot{\boldsymbol{\xi}}_i = \mathbf{u}_i. \tag{2}$$

In the classic cyclic pursuit strategy the control input is:

$$\mathbf{u}_i = k(\boldsymbol{\xi}_{i+1} - \boldsymbol{\xi}_i), \quad k \in \mathbb{R}^+. \tag{3}$$

Thus, agent i pursues its leading neighbor, agent $i+1$, along the line of sight, with a velocity proportional to the vector from agent i to agent $i + 1$. The overall system dynamics is in compact form:

$$\dot{\boldsymbol{\xi}} = C\boldsymbol{\xi}, \tag{4}$$

where $\boldsymbol{\xi} = [\boldsymbol{\xi}_1^T, \boldsymbol{\xi}_2^T, \ldots, \boldsymbol{\xi}_n^T]^T$ and C is the block circulant matrix $C = \mathrm{circ}[-I_{2\mathrm{x}2}, I_{2\mathrm{x}2}, 0_{2\mathrm{x}2}, \ldots, 0_{2\mathrm{x}2}]$, where $I_{2\mathrm{x}2}$ is the identity matrix. Since the blocks are themselves circulant, the matrix C can be diagonalized by means of the Fourier matrix.

Suppose, now, we extend the aforementioned classic linear cyclic pursuit scenario to one in which each agent pursues the leading neighbor along the line of sight *rotated* by an offset angle $\alpha \in [-\pi, \quad \pi]$. Thus, the control input for each agent i is given by:

$$\mathbf{u}_i = kR(\alpha)(\boldsymbol{\xi}_{i+1} - \boldsymbol{\xi}_i), \tag{5}$$

where $R(\alpha)$ is the rotation matrix:

$$R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

The overall dynamics of the n agents is:

$$\dot{\boldsymbol{\xi}} = k\hat{A}\boldsymbol{\xi}, \tag{6}$$

where $\boldsymbol{\xi} = [\boldsymbol{\xi}_1^T, \boldsymbol{\xi}_2^T, \ldots, \boldsymbol{\xi}_n^T]^T$ and $\hat{A}$ is now the block circulant matrix $\hat{A} = \mathrm{circ}[-R(\alpha), R(\alpha), 0_{2\mathrm{x}2}, \ldots, 0_{2\mathrm{x}2}]$. Since its blocks are not themselves circulant, $\hat{A}$ is only *block* diagonalized by means of the Fourier matrix.

It is easy to see that the centroid of the agents is stationary. In the remainder of the paper, we will assume, without loss of generality, that $k = 1$.

In order to analyze the different geometric patterns achievable by system (6) for the various values of α, we start by studying the eigenvalues and eigenvectors of $\hat{A}$.

A. Eigenvalues of $\hat{A}$

Given the block diagonalization formula (1), the eigenvalues of $\hat{A}$ can be readily determined.

Theorem 3.1: The eigenvalues of $\hat{A}$ are the collection of:

$$\lambda_i^\pm = (\varphi_i - 1)e^{\pm j\alpha}, \tag{7}$$

for $i = 1, 2, \ldots, n$

Proof: As stated in the previous section, $\hat{A}$ is *block* diagonalized by means of the Fourier matrix and each diagonal block has the structure (1) with $A_1 = -R(\alpha)$, $A_2 = R(\alpha)$ and $A_i = 0_{2\mathrm{x}2}$ for $i = 3, 4, \ldots, n$:

$$D_i = -R(\alpha) + \varphi_i R(\alpha), \qquad i = 1, 2, \ldots, n. \tag{8}$$

The characteristic polynomial of D_i is:

$$p_{D_i}(\lambda) = \lambda^2 - 2\lambda\cos\alpha(\varphi_i - 1) + (\varphi_i - 1)^2.$$

Therefore, we get:

$$\lambda_i^\pm = \cos\alpha(\varphi_i - 1) \pm \sqrt{(\cos^2\alpha - 1)(\varphi_i - 1)^2}.$$

Thus we obtain $\lambda_i^{\pm} = (\varphi_i - 1)e^{\pm j\alpha}$. Since the eigenvalues of a block diagonal matrix are the collection of all the eigenvalues of each diagonal block, we get the claim. ∎

B. Analysis of the Eigenvalues of $\hat{A}$

The eigenvalues of $\hat{A}$ show the following property.

Lemma 3.1:

$$\lambda_i^+ = \overline{\lambda_{n-i+2}^-} \quad i = 2, 3, \ldots, n. \tag{9}$$

Proof: The proof is straightforward:

$$\overline{\lambda_{n-i+2}^-} = \overline{(\varphi_{n-i+2} - 1)e^{-j\alpha}} = (\overline{\varphi_{n-i+2}} - 1)e^{j\alpha} =$$
$$= (e^{2\pi j[(i-1)-n]/n} - 1)e^{j\alpha} = (e^{2\pi j(i-1)} - 1)e^{j\alpha} = \lambda_i^+.$$
∎

Define the set:

$$\mathcal{H} = \{\lambda_i^{\pm}, \quad i = 1, 2, \ldots, n\}. \tag{10}$$

Let us now be more explicit with the structure of the eigenvalues; define:

$$\theta_i^{\pm} = \left(\frac{i-1}{n}\pi \pm \alpha\right). \tag{11}$$

After some algebraic manipulations we can write the eigenvalues $\lambda_i^{\pm}$ as:

$$\lambda_i^{\pm} = 2\sin\left(\frac{i-1}{n}\pi\right)(-\sin\theta_i^{\pm} + j\cos\theta_i^{\pm}). \tag{12}$$

Lemma 3.2: The block circulant matrix $\hat{A}$ has a zero eigenvalue with algebraic multiplicity $m_{\lambda=0} = 2$; the two zero-eigenvalues in $\mathcal{H}$ are $\lambda_1^{\pm}$.

Proof: If $i = 1$ clearly $\lambda_1^{\pm} = 0 \quad \forall\alpha$. If $i > 1$ it must be $\|\lambda_i^{\pm}\| \neq 0$ since the functions $\sin(\cdot)$ and $\cos(\cdot)$ can not be zero at the same time. ∎

Lemma 3.3: The eigenvalues of $\hat{A}$ can have algebraic multiplicity at most $m_\lambda = 2$. In particular the non zero-eigenvalues that can separately collapse are $(\lambda_i^+, \lambda_i^-)$, $(\lambda_i^+, \lambda_{n-(i-2)}^-)$, $(\lambda_{n-(i-2)}^+, \lambda_i^-)$ and $(\lambda_{n-(i-2)}^+, \lambda_{n-(i-2)}^-)$ $i = 2, 3, \ldots, n$.

Proof: The statement is true for the zero eigenvalue, obtained for $i = 1$. Consider therefore $i > 1$. As evident from equation (12), as α is varied, the eigenvalues move on circles with radius $r_i = 2\sin(\pi(i-1)/n)$; clearly, only the eigenvalues that move on the same circles can collapse. Since:

$$0 < \frac{i-1}{n}\pi < \pi \quad i = 2, 3, \ldots, n,$$

for symmetry we deduce that the eigenvalues that move on the same circle are:

$$\lambda_i^+ \quad \lambda_{n-(i-2)}^+ \quad \lambda_i^- \quad \lambda_{n-(i-2)}^-, \quad i = 2, 3, \ldots, n;$$

in fact, $\sin\left(\pi\frac{(n-(i-2))-1}{n}\right) = \sin\left(\pi\frac{i-1}{n}\right)$. Therefore just 4 eigenvalues move on the same circle (2 in the case that n is even and $i = n/2 + 1$; in particular these eigenvalues move on the most external circle since the argument of $\sin(\cdot)$ is $\pi/2$).

As it can be inferred from (12), as α is increased, the eigenvalues λ_i^+ and $\lambda_{n-(i-2)}^+$ move counter-clockwise with

fixed phase difference $\delta = \pi(n - 2i + 2)/n$, while the eigenvalues λ_i^- and $\lambda_{n-(i-2)}^-$ move clockwise with the same fixed phase difference δ. Therefore at most two eigenvalues can collapse, in particular the eigenvalue λ_i^+ can collapse with the eigenvalues λ_i^- or $\lambda_{n-(i-2)}^-$; similarly for the other eigenvalues. ∎
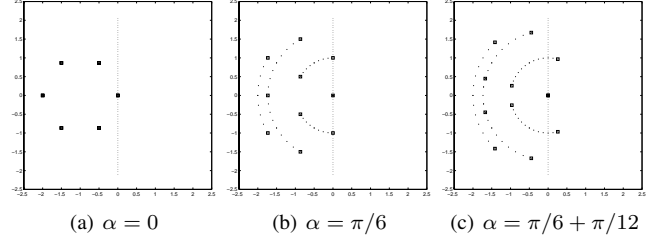

Fig. 1. Eigenvalue placement for various α and $n = 6$. On the most external circle there are just two eigenvalues.

(a) $\alpha = 0$ (b) $\alpha = \pi/6$ (c) $\alpha = \pi/6 + \pi/12$

In the classic cyclic pursuit we have $\alpha = 0$; in this case it is straightforward to notice that the $2n - 2$ non-zero eigenvalues in $\mathcal{H}$ lie in the open left half complex plane. Let us now consider the case $\alpha \neq 0$.

Theorem 3.2: Suppose:

$$k\pi/n < |\alpha| \leq (k+1)\pi/n, \tag{13}$$

where $k \in \mathbb{N}^0$, $0 \leq k \leq n - 1$. The following statements hold:

(i)   among the $2n - 2$ non-zero eigenvalues $\in \mathcal{H}$, $2k$ eigenvalues lie in the open right half complex plane;

(ii) among the $2n - 2$ non-zero eigenvalues $\in \mathcal{H}$, two eigenvalues lies on the imaginary axis if $|\alpha| = (k+1)\pi/n$ with $k = 0, 1, \ldots, n - 2$.

Proof: Consider the case $\alpha \in]0, \pi]$; the proof with $\alpha \in [-\pi, 0[$ is analogous. Let us prove fact (i). From the previous considerations, the non-zero eigenvalues are obtained for $i > 1$. Let us firstly consider λ_i^-. Since $i > 1$ we have:

$$-\pi < \theta_i^- < \pi(n-1)/n \quad i = 2, 3, \ldots, n. \tag{14}$$

Therefore, we can just compare θ_i^- with zero to check for which indices i the condition $\sin(\theta_i^-) < 0$ holds.

For α chosen as in (13), θ_i^- is negative for $i = 2, 3, \ldots, k + 1$, while for $i = k + 2, k + 3, \ldots, n$ we have that $\theta_i^- \geq 0$. Therefore, we can conclude that exactly k eigenvalues among the λ_i^- eigenvalues lie in the open right half complex plane. With similar arguments (or recalling Lemma 3.1) we can state that also k eigenvalues among the λ_i^+ eigenvalues lie in the open right half complex plane.

Next we consider statement (ii). From (14), a non-zero eigenvalue among the λ_i^- eigenvalues lies on the imaginary axis if and only if $\theta_i^- = 0$: this happens if $\alpha = (i-1)\pi/n$ for some $i = 2, 3, \ldots, n$, *i.e.* if $\alpha = (k+1)\pi/n$ with $k = 0, 1, \ldots, n - 2$. The proof is completed considering Lemma 3.1. ∎

C. Eigenvectors of $\hat{A}$

To study the possible polygons of pursuit, we have to study the eigenvectors of $\hat{A}$. Here we present our basic results; a more in-depth discussion can be found in [17].

Theorem 3.3: The eigenvectors corresponding, respectively, to the eigenvalues λ_i^+ and λ_i^- are:

$$\mathbf{w}_i^+ = \left[1, j, \varphi_i, j\varphi_i, \ldots, \varphi_i^{(n-1)}, j\varphi_i^{(n-1)}\right]^T,$$
$$\mathbf{w}_i^- = \left[1, -j, \varphi_i, -j\varphi_i, \ldots, \varphi_i^{(n-1)}, -j\varphi_i^{(n-1)}\right]^T. \quad (15)$$

Proof: Let us just make a direct verification. Consider firstly $\mathbf{w}_i^+$ and write shortly $\mathbf{w}_i^+ = \mathbf{w}$ and $\lambda_i^+ = \lambda$. Let us partition $\mathbf{w} \in \mathbb{C}^{2n}$ into n components $\chi := [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n]$. Expanding the eigenvalue equation for each component $\mathbf{w}_k$ we get:

$$-R(\alpha)\mathbf{w}_k + R(\alpha)\mathbf{w}_{k+1} = \lambda \mathbf{w}_k.$$

Further expanding, we obtain the identity:

$$\begin{pmatrix} -e^{j\alpha}\varphi_i^{(k-1)} + e^{j\alpha}\varphi_i^k \\ -je^{j\alpha}\varphi_i^{(k-1)} + je^{j\alpha}\varphi_i^k \end{pmatrix} = (\varphi_i - 1)\,e^{j\alpha} \begin{pmatrix} \varphi_i^{(k-1)} \\ j\varphi_i^{(k-1)} \end{pmatrix}.$$

The proof for $\mathbf{w}_i^-$ is analogous. ∎

Remark 3.1: The eigenvectors of $\hat{A}$ do not depend on the offset angle α.

An important question to be addressed is if $\hat{A}$ is defective for some α.

Lemma 3.4: Two eigenvectors $\mathbf{w}_p^+$ and $\mathbf{w}_q^-$ are linearly independent $\forall p, q \in \{1, 2, \ldots, n\}$.

Proof: It is sufficient to observe the first two components of each eigenvector. ∎

Theorem 3.4: The block circulant matrix $\hat{A}$ is diagonalizable $\forall \alpha \in [-\pi, \ \pi]$.

Proof: From Theorem 3.2, an eigenvalue of $\hat{A}$ can have at most algebraic multiplicity $m_\lambda = 2$. In particular the couples that can collapse, beside the eigenvalues $\lambda_1^\pm$, are: $(\lambda_i^+, \lambda_i^-)$, $(\lambda_i^+, \lambda_{n-(i-2)}^-)$, $(\lambda_{n-(i-2)}^+, \lambda_i^-)$ and $(\lambda_{n-(i-2)}^+, \lambda_{n-(i-2)}^-)$ with $i = 2, 3, \ldots, n$. A consequence of Lemma 3.4 is that the corresponding eigenvectors are linearly independent. The other eigenvectors are independent for the eigenspace independence theorem. Therefore the matrix $\hat{A}$ is diagonalizable. ∎

D. Pattern formation

We are now ready to show the achievable geometric patterns. Since the block circulant matrix $\hat{A}$ is diagonalizable, the general solution has the form:

$$\boldsymbol{\xi}(t) = \sum_{k=1}^{q} \sum_{j=1}^{m_{\lambda_k}} \alpha_{kj} e^{t\lambda_k} \mathbf{w}_{kj}, \quad (16)$$

where q is the number of distinct eigenvalues, the α_{kj} are constants and the $\mathbf{w}_{kj}$ are the eigenvectors corresponding to the kth eigenvalue.

We can rewrite (16) by picking out the two eigenvectors corresponding to the zero eigenvalues $\lambda_1^\pm$, *i.e.* $\mathbf{w}_1^\pm$, and replacing them with the two eigenvectors:

$$\mathbf{w}_G^1 = \frac{1}{2} \left[\mathbf{w}_1^+ + \mathbf{w}_1^-\right] = [1, 0, 1, 0, \ldots, 1, 0],$$
$$\mathbf{w}_G^2 = \frac{1}{2j} \left[\mathbf{w}_1^+ - \mathbf{w}_1^-\right] = [0, 1, 0, 1, \ldots, 0, 1],$$

eq. (16) becomes:

$$\boldsymbol{\xi}(t) = \sum_{k=2}^{q} \sum_{j=1}^{m_{\lambda_k}} \alpha_{kj} e^{t\lambda_k} \mathbf{w}_{kj} + x_G \mathbf{w}_G^1 + y_G \mathbf{w}_G^2, \quad (17)$$

where x_G and y_G are the coordinates of the center of mass. We are now ready to show how different geometric patterns can be achieved by modulating the angle α.

1) Rendez-vous to a point: Consider $|\alpha| < \pi/n$; from Theorem 3.2, all the non-zero eigenvalues lie in the open left half complex plane. Therefore we have:

$$\lim_{t \to +\infty} \boldsymbol{\xi}(t) = x_G \mathbf{w}_G^1 + y_G \mathbf{w}_G^2.$$

Therefore, for any initial condition, all agents exponentially converge to a single limit point: their initial center of mass. This result extends the classic cyclic pursuit result that is obtained when $\alpha = 0$.

2) Evenly spaced circle formation: Let us consider $\alpha = \pi/n$. From Theorem 3.2, besides the two zero eigenvalues, there are two non-zero eigenvalues lying on the imaginary axis, while all the other eigenvalues lie in the open left half complex plane. The imaginary eigenvalues are λ_n^+ and λ_2^- and are equal to $\mp j2\sin(\pi/n) = \mp j\gamma$.

We can rewrite (17) by picking out also the two eigenvectors corresponding to the imaginary eigenvalues, i.e. $\mathbf{w}_n^+$ and $\mathbf{w}_2^-$:

$$\boldsymbol{\xi}(t) = \sum_{k=4}^{q} \sum_{j=1}^{m_{\lambda_k}} \alpha_{kj} e^{t\lambda_k} \mathbf{w}_{kj} + d_1 e^{-jt\gamma} \mathbf{w}_n^+ + d_2 e^{jt\gamma} \mathbf{w}_2^- +$$
$$+ x_G v_G^1 + y_G v_G^2, \quad (18)$$

where d_1 and d_2 are two constants. We replace the two complex eigenfunctions with two real independent eigenfunctions obtained as follows:

$$\mathbf{w}_{\text{dom}}^1 = \frac{1}{2} \left[e^{jt\gamma} \mathbf{w}_2^- + e^{-jt\gamma} \mathbf{w}_n^+\right],$$
$$\mathbf{w}_{\text{dom}}^2 = \frac{1}{2j} \left[e^{jt\gamma} \mathbf{w}_2^- - e^{-jt\gamma} \mathbf{w}_n^+\right].$$

Thus we get, defining $\delta_i = \frac{2\pi(i-1)}{n} \quad i = 1, 2, \ldots, n$:

$$\mathbf{w}_{\text{dom}}^1 = [\cos(\delta_1 + \gamma t), \sin(\delta_1 + \gamma t), \ldots,$$
$$\cos(\delta_n + \gamma t), \sin(\delta_n + \gamma t)]^T$$
$$\mathbf{w}_{\text{dom}}^2 = [\sin(\delta_1 + \gamma t), -\cos(\delta_1 + \gamma t), \ldots,$$
$$\sin(\delta_n + \gamma t), -\cos(\delta_n + \gamma t)]^T.$$

We have for $t \to +\infty$:

$$\boldsymbol{\xi}(t) = c_1 \mathbf{w}_{\text{dom}}^1 + c_2 \mathbf{w}_{\text{dom}}^2 + x_G v_G^1 + y_G v_G^2.$$

Therefore, after a transient, the trajectory of each agent is (assuming $c_1 \neq 0$ and $c_2 \neq 0$):

$$x_i(t) \simeq c_1 \cos(\delta_i + \gamma t) + c_2 \sin(\delta_i + \gamma t) + x_G,$$
$$y_i(t) \simeq c_1 \sin(\delta_i + \gamma t) - c_2 \cos(\delta_i + \gamma t) + y_G,$$

i.e. a circular motion with center (x_G, y_G), radius $r = \sqrt{c_1^2 + c_2^2}$ and period $2\pi/\gamma$. The agents are evenly spaced since the angular distance between agent i and agent $i + 1$ is $2\pi/n$.

If we take $\alpha = -\pi/n$, we have again, asymptotically, a circular motion, but the direction of motion is reversed.

3) Evenly spaced logarithmic spiral formation: Let us consider $\alpha \in]\pi/n, \quad 2\pi/n[$. From Theorem 3.2, besides the two zero eigenvalues, there are two complex positive eigenvalues: λ_n^+ and λ_2^- equal, shortly, to $\beta \mp j\gamma$. Therefore, with similar arguments, we obtain that the trajectory of each agent is for t quite large (assuming $c_1 \neq 0$ and $c_2 \neq 0$):

$$x_i(t) \simeq e^{\beta t}[c_1 \cos(\delta_i + \gamma t) + c_2 \sin(\delta_i + \gamma t)] + x_G,$$
$$y_i(t) \simeq e^{\beta t}[c_1 \sin(\delta_i + \gamma t) - c_2 \cos(\delta_i + \gamma t)] + y_G,$$

i.e. a logarithmic spiral with growing rate β. The agents are evenly spaced since the angular distance between agent i and agent $i + 1$ is $2\pi/n$.

If we take $\alpha \in]-2\pi/n, \quad -\pi/n[$, we have, asymptotically, the same pattern, except that the direction of motion is reversed.

(a) Circle formation with $\alpha = \pi/6$
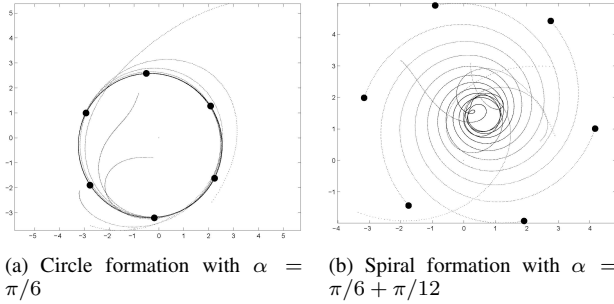
(b) Spiral formation with $\alpha = \pi/6 + \pi/12$

Fig. 2. Pattern formation for $n = 6$ agents.

IV. FORMATION CONTROL: NON-HOLONOMIC ROBOTS

We now extend the previous formation control law to the non-holonomic case. Consider, for instance, that each agent is a Hilare-type mobile robot with nonlinear state model [13]:

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \\ \dot{v}_i \\ \dot{\omega}_i \end{pmatrix} = \begin{pmatrix} v_i \cos\theta_i \\ v_i \sin\theta_i \\ \omega_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/J \end{pmatrix} \begin{pmatrix} F_i \\ \tau_i \end{pmatrix}, \tag{19}$$

where $r_i = (x_i, y_i)^T$ is the inertial position of the ith robot, θ_i is the orientation, v_i is the linear speed, ω_i is the angular speed, m_i is the mass, J_i is the moment of inertia, F_i is the force input and τ_i is the torque input. Let $\boldsymbol{u}_i = (F_i, \tau_i)^T$. In the model it is evident the non-holonomic Pfaffian constraint: $\dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i = 0$. Assume that we

want to maintain in formation a point off the wheel axis of the agents. Specifically, let us define, as in [13], the "hand" position of an agent to be the point $h = (h_x, h_y)$ that lies a distance $L \neq 0$ along the line that is normal to the wheel axis and intersects the wheel axis at the center point. It is possible to show [13] that a Hilare-type robot can be feedback linearized about the hand position, the diffeomorphism that transforms the system in normal form is *globally* invertible and the corresponding zero dynamics are *stable*. Since we can reasonably consider sensors located at the hand, this approach does not indeed represent a significant limitation.

The hand position dynamics is given by $\ddot{\mathbf{h}}_i = \boldsymbol{\nu}_i$ and the output feedback linearizing control is given by:

$$\mathbf{u}_i = \begin{pmatrix} \frac{1}{m}\cos\theta_i & -\frac{L}{J}\sin\theta_i \\ \frac{1}{m}\sin\theta_i & \frac{L}{J}\cos\theta_i \end{pmatrix}^{-1} \times$$
$$\times \left[\boldsymbol{\nu}_i - \begin{pmatrix} -v_i\omega_i \sin\theta_i - L\omega_i^2 \cos\theta_i \\ v_i\omega_i \cos\theta_i - L\omega_i^2 \sin\theta_i \end{pmatrix} \right]. \tag{20}$$

Let us focus, therefore, on the control law for the hand position. To have stable circle and spiral formations, the hand position velocity should be as in Eq. (5). Thus, a possible approach is to consider a velocity tracking problem.

Let $\mathbf{v}_i = \dot{\mathbf{h}}_i$ and $\mathbf{v}_i^d = R(\alpha)(\mathbf{h}_{i+1} - \mathbf{h}_i)$ be the velocity and the desired velocity of the hand position, respectively. If we choose:

$$\boldsymbol{\nu}_i = -K\mathbf{v}_i + \mathbf{r}_i,$$

we get the first order system:

$$\dot{\mathbf{v}}_i + K\mathbf{v}_i = \mathbf{r}_i.$$

Choose now:

$$\mathbf{r}_i = \dot{\mathbf{v}}_i^d + K\mathbf{v}_i^d = R(\alpha)(\dot{\mathbf{h}}_{i+1} - \dot{\mathbf{h}}_i) + KR(\alpha)(\mathbf{h}_{i+1} - \mathbf{h}_i),$$

we obtain the following homogeneous first order differential equation:

$$\dot{\widetilde{\mathbf{v}}}_i + K\widetilde{\mathbf{v}}_i = \mathbf{0}, \tag{21}$$

where $\widetilde{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{v}_i^d$. Eq. (21) describes the dynamics of the velocity error. If K is positive definite, the error dynamics is asymptotically stable.

Thus, the following control law applied to the hand position:

$$\boldsymbol{\nu}_i = K\left(R(\alpha)(\mathbf{h}_{i+1} - \mathbf{h}_i) - \dot{\mathbf{h}}_i\right) + R(\alpha)(\dot{\mathbf{h}}_{i+1} - \dot{\mathbf{h}}_i), \tag{22}$$

where K is positive definite, asymptotically provides the desired hand velocity. This implies that control law (22), together with (20), guarantees, depending on the value of α, *globally stable* rendez-vous to a point, *globally stable* evenly spaced circle formation and *globally stable* evenly spaced logarithmic spirals.

Remark 4.1: The proposed control policy requires that each robot knows its orientation θ_i, its velocity $\dot{\mathbf{h}}_i$, the relative position $(\mathbf{h}_{i+1} - \mathbf{h}_i)$, the relative velocity $(\dot{\mathbf{h}}_{i+1} - \dot{\mathbf{h}}_i)$ and the total number of agents n. Note that agents do not recall past actions and observations (*i.e.*, they are oblivious). Oblivious algorithms are, by definition, self-stabilizing in the

(a) Circle formation for $n = 8$ non-holonomic agents - detail

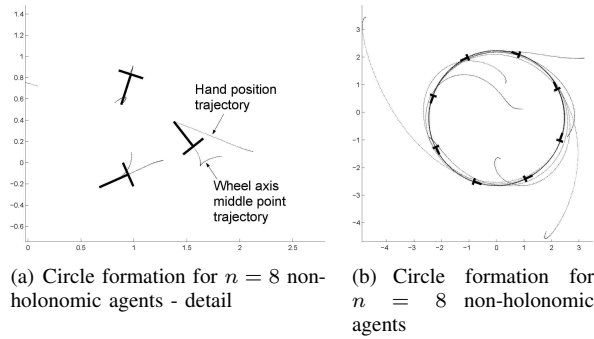(b) Circle formation for $n = 8$ non-holonomic agents

Fig. 3. Formations of $n = 8$ non-holonomic agents.

sense that they achieve their goal even in the presence of a finite number of sensor and control errors [16]. Moreover, no agent has to measure the absolute positions of other agents or its own. Communication among agents is needed only to exchange the identifier information.

Remark 4.2: It is interesting to compare our control policy with the control policy, similar in spirit, proposed by Marshall and coauthors in [10]. Marshall's approach can be summarized as follows. Consider a kinematic version of the dynamic model (23):

$$\left(\begin{array}{c} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{array} \right) = \left(\begin{array}{cc} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{array} \right) \left(\begin{array}{c} v_i \\ \omega_i \end{array} \right). \qquad (23)$$

Let r_i denote the distance between unicycles numbered i and $i+1$, and let α_i be the angle from the ith unicycle's heading to the heading that would take it directly towards unicycle $i + 1$. The control law they consider is to assign unicycle i's linear speed v_i in proportion to r_i, while assigning its angular speed ω_i in proportion to α_i, that is:

$$v_i = k_r r_i \quad \text{and} \quad \omega_i = k_\alpha \alpha_i. \qquad (24)$$

It is shown that, depending on the value of the ratio k_r/k_α and the total number of agents n, *locally stable* rendez-vous to a point, *locally stable* evenly spaced circle formation and *locally stable* evenly spaced logarithmic spirals are achieved. However, it turns out that these regular formations are not the only stable behaviors. Simulations, reported in [14], indicate that when the vehicles do not converge to a generalized regular polygon formation, they instead fall into a different kind of order: the vehicles "weave" in and out, while the formation as a whole moves along a linear trajectory [14]. Therefore, our globally stable control policy may represent a significant improvement, as far as applications are concerned. Moreover, since we consider a dynamic model, the physical implementation is simplified.

V. CONCLUSION

In this paper we have proposed a decentralized strategy aimed at achieving symmetric formations. It was shown that n agents, each one pursuing its leading neighbor along the line of sight rotated by a common offset angle, eventually converge to a single point, a circle or a logarithmic spiral

pattern, depending on the value of the angle. The equilibrium formations are globally stable also when non-holonomic robots are considered.

There are many possible directions for future research. Preliminary studies show that it is possible to achieve more complicated geometric formations by making the offset angle α a function of locally available information. In particular, we are currently studying Archimedes spiral formations (an Archimedes spiral is a curve defined by a polar equation of the form: $\rho(\varphi) = a\varphi$), since they are of interest in coverage path planning problems, where the objective is to ensure that at least one agent eventually moves to within a given distance from any point in the target environment. (See [17] for a more detailed discussion and a preliminary mathematical analysis.) Other possible directions for future research include (1) stability in presence of actuator saturation, (2) stabilization to a circle of desired radius, (3) a strategy to make the agents anonymous.

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