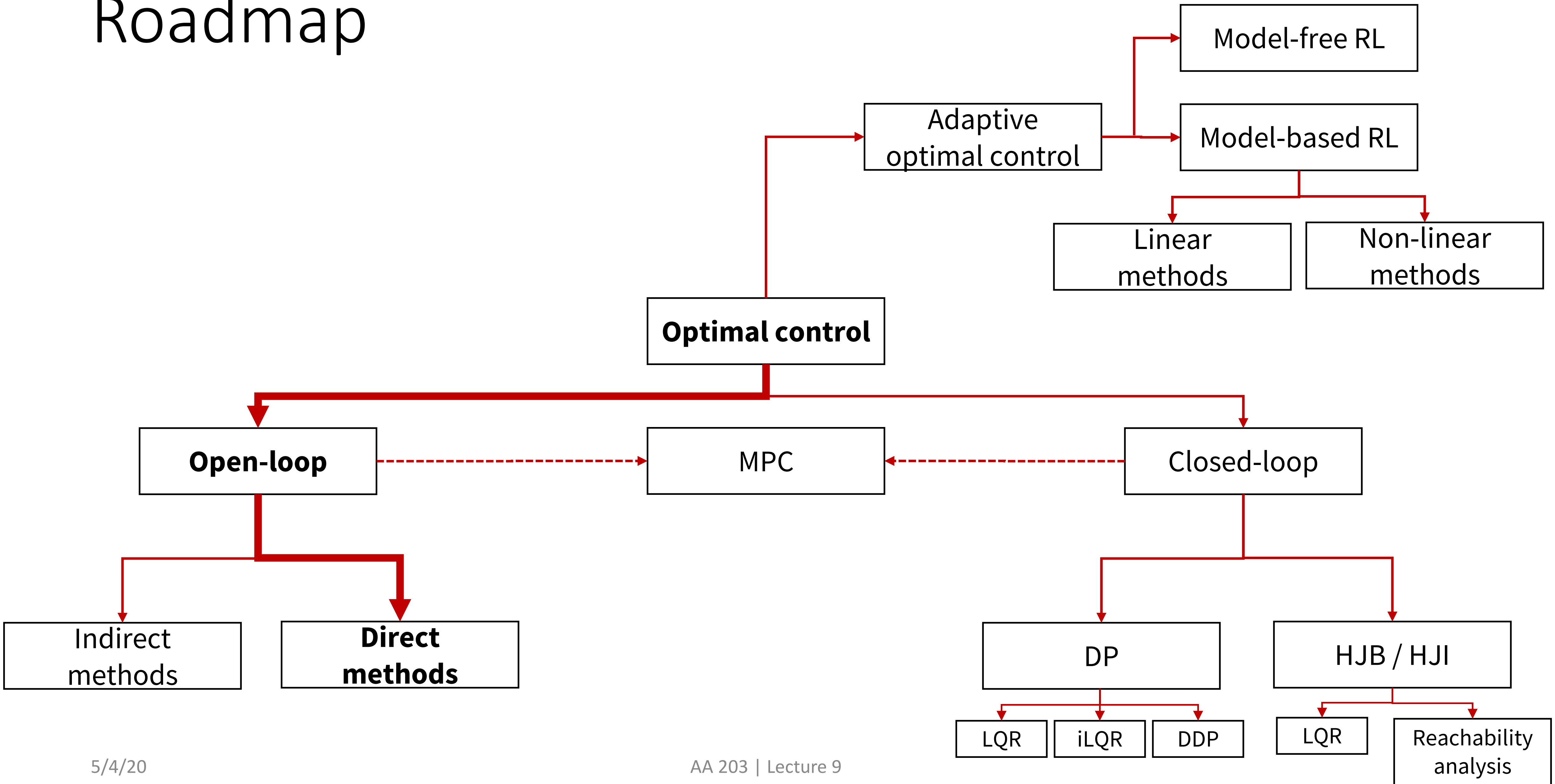
# AA203
# Optimal and Learning-based Control

## Direct methods for optimal control: direct collocation and SCP*

*Slides prepared by Riccardo Bonalli,
rbonalli@stanford.edu

# Roadmap



Model-free RL
Adaptive optimal control
Model-based RL
Linear methods
Non-linear methods
Optimal control
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI
LQR
iLQR
DDP
LQR
Reachability analysis

# Agenda

- Direct Collocation Methods
- Sequential Convex Programming (SCP)

# Specific direct methods

- Direct Methods:
    1. Transcribe (OCP) into a nonlinear, constrained optimization problem
    2. Solve the optimization problem via nonlinear programming

- Specific Methods – natural improvements to direct shooting :
    1. High Order Direct Methods, i.e., Direct Collocation Methods: approximate the solution through interpolating polynomials. We will focus on the Hermite-Simpson method, the most classical one.

       Pros: higher robustness than classical direct multi-shooting methods; solutions are smooth functions of the time.
       Cons: consistent amount of memory might be required (preferred for offline computations); theoretical guarantees are usually difficult to achieve for complex problems.

    2. Sequential Convex Programming (SCP): find the solution to the non-convex problem by solving a sequence of convex subproblems. We will focus on Linear SCP, whose convex subproblems are linear.

       Pros: fast convergence; theoretical guarantees stem from the definition of the method.
       Cons: if not correctly set up, it might converge to infeasible solutions.

# Optimal control problem

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

**(OCP)**

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t),\ t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0,$$

$$\mathbf{x}(t_f) \in M_f = \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = 0\}$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m,\ \ t \in [0, t_f]$$

For simplicity:

- We assume the terminal cost $h$ is equal to 0
- We assume $t_0 = 0$

- Direct Methods:
  1. Transcribe (OCP) into a nonlinear, constrained optimization problem
  2. Solve the optimization problem via nonlinear programming

- Specific Methods:
  1. High Order Direct Methods, i.e., Direct Collocation Methods
  2. Sequential Convex Programming (SCP)

# Direct Collocation: High-Order Direct Method

## Hermite-Simpson Method

- Select a discretization $0 = t_0 < t_1 < \cdots < t_N = t_f$ for the interval $[0, t_f]$ and denote $h_i = t_{i+1} - t_i$
- In every subinterval $[t_i, t_{i+1}]$, approximate $\mathbf{x}(t)$ with the cubic polynomial $\mathbf{x}(t) = \mathbf{c}_0^i + \mathbf{c}_1^i(t - t_i) + \mathbf{c}_2^i(t - t_i)^2 + \mathbf{c}_3^i(t - t_i)^3$ so that its derivative is given by $\dot{\mathbf{x}}(t) = \mathbf{c}_1^i + 2\mathbf{c}_2^i(t - t_i) + 3\mathbf{c}_3^i(t - t_i)^2$
- By denoting $\mathbf{x}_i = \mathbf{x}(t_i), \mathbf{x}_{i+1} = \mathbf{x}(t_{i+1}), \dot{\mathbf{x}}_i = \dot{\mathbf{x}}(t_i)$ and $\dot{\mathbf{x}}_{i+1} = \dot{\mathbf{x}}(t_{i+1})$, the relations above give us the coefficients

$$\begin{bmatrix} \mathbf{x}_i \\ \dot{\mathbf{x}}_i \\ \mathbf{x}_{i+1} \\ \dot{\mathbf{x}}_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & h_i & h_i^2 & h_i^3 \\ 0 & 1 & 2h_i & 3h_i^2 \end{bmatrix} \begin{bmatrix} \mathbf{c}_0^i \\ \mathbf{c}_1^i \\ \mathbf{c}_2^i \\ \mathbf{c}_3^i \end{bmatrix} \Longrightarrow \begin{bmatrix} \mathbf{c}_0^i \\ \mathbf{c}_1^i \\ \mathbf{c}_2^i \\ \mathbf{c}_3^i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{h_i^2} & -\frac{2}{h_i} & \frac{3}{h_i^2} & -\frac{1}{h_i} \\ \frac{2}{h_i^2} & \frac{1}{h_i^2} & -\frac{2}{h_i^3} & \frac{1}{h_i^2} \end{bmatrix} \begin{bmatrix} \mathbf{x}_i \\ \dot{\mathbf{x}}_i \\ \mathbf{x}_{i+1} \\ \dot{\mathbf{x}}_{i+1} \end{bmatrix}$$

- Choose intermediate times $t_i^c = t_i + \frac{h_i}{2}$, i.e., collocation points, and define interpolated controls $\mathbf{u}_i^c = \frac{\mathbf{u}_i + \mathbf{u}_{i+1}}{2}$. From above:

$$\mathbf{x}_i^c := \mathbf{x}\left(t_i + \frac{h_i}{2}\right) = \frac{1}{2}(\mathbf{x}_i + \mathbf{x}_{i+1}) + \frac{h_i}{8}(\mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, t_i) - \mathbf{f}(\mathbf{x}_{i+1}, \mathbf{u}_{i+1}, t_{i+1}))$$

$$\dot{\mathbf{x}}_i^c := \dot{\mathbf{x}}\left(t_i + \frac{h_i}{2}\right) = -\frac{3}{2h_i}(\mathbf{x}_i - \mathbf{x}_{i+1}) - \frac{1}{4}(\mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, t_i) + \mathbf{f}(\mathbf{x}_{i+1}, \mathbf{u}_{i+1}, t_{i+1}))$$

# Direct Collocation: High-Order Direct Method

Hermite-Simpson Method

**(OCP)**

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) \; dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \; t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0,$$

$$\mathbf{x}(t_f) \in M_f = \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = 0\}$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \; t \in [0, t_f]$$

**(NLOP)**

$$\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(t_i, \mathbf{x}_i, \mathbf{u}_i)$$

$$\mathbf{u}_i \in U, i = 0, \dots, N-1, \qquad F(\mathbf{x}_N) = 0$$

$$\dot{\mathbf{x}}_i^c - \mathbf{f}(t_i^c, \mathbf{x}_i^c, \mathbf{u}_i^c) = 0, i = 0, \dots, N-1$$

Where, from the previous computations:

$$\dot{\mathbf{x}}_i^c - \mathbf{f}(t_i^c, \mathbf{x}_i^c, \mathbf{u}_i^c) =$$

$$\mathbf{x}_i - \mathbf{x}_{i+1} + \frac{h_i}{6}\left(\mathbf{f}(t_i, \mathbf{x}_i, \mathbf{u}_i) + 4\mathbf{f}(t_i^c, \mathbf{x}_i^c, \mathbf{u}_i^c) + \mathbf{f}(t_{i+1}, \mathbf{x}_{i+1}, \mathbf{u}_{i+1})\right)$$

**Implicit constraints that improve robustness!**

# Practical Example

Designing Hermite-Simpson Method in Matlab

Solve the following optimal control problem via the Hermite-Simpson method, using the Matlab function "**fmincon**"

Modified Zermelo Problem

**(OCP)**

$$\min \int_0^{t_f} u(t)^2 \, dt$$

$$\dot{x}(t) = v\cos\big(u(t)\big) + \mathrm{flow}\big(y(t)\big),\ t \in [0, t_f]$$

$$\dot{y}(t) = v\sin\big(u(t)\big),\ t \in [0, t_f]$$

$$(x, y)(0) = 0\ ,\ (x, y)(t_f) = (M, \ell)$$

$$|u(t)| \leq u_{\max},\ t \in [0, t_f]$$

$\Rightarrow$

**(NLOP)**

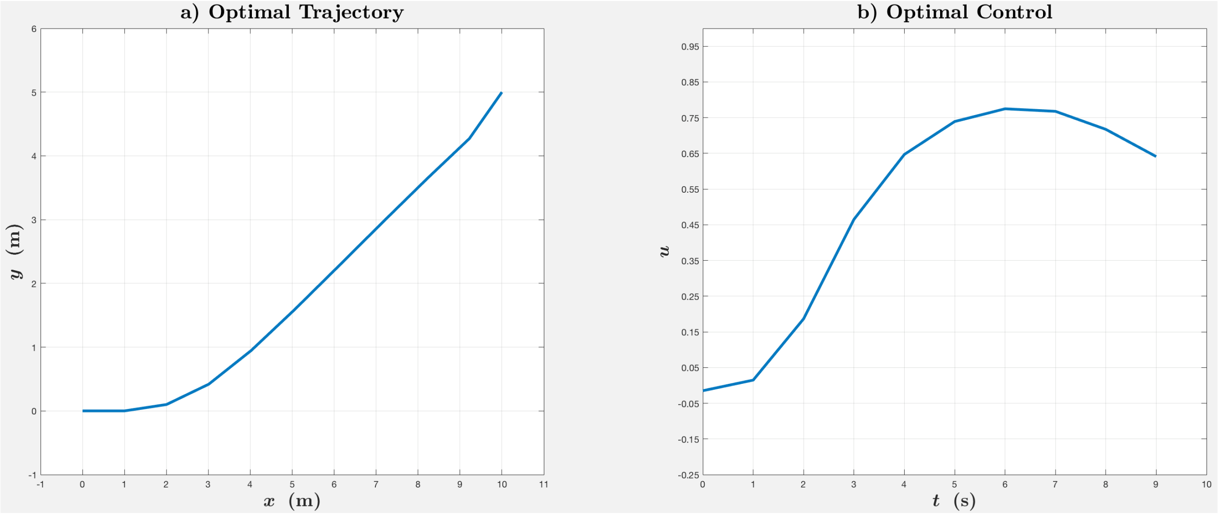
$$\min_{(x_i, y_i, u_i)} \sum_{i=0}^{N-1} u_i^2$$

$$|u_i| \leq u_{\max},\ (x_0, y_0) = 0,\ (x_N, y_N) = (M, \ell)$$

$$\dot{x}_i^c - v\cos(u_i^c) - \mathrm{flow}(y_i^c) = 0$$

$$\dot{y}_i^c - v\sin(u_i^c) = 0$$

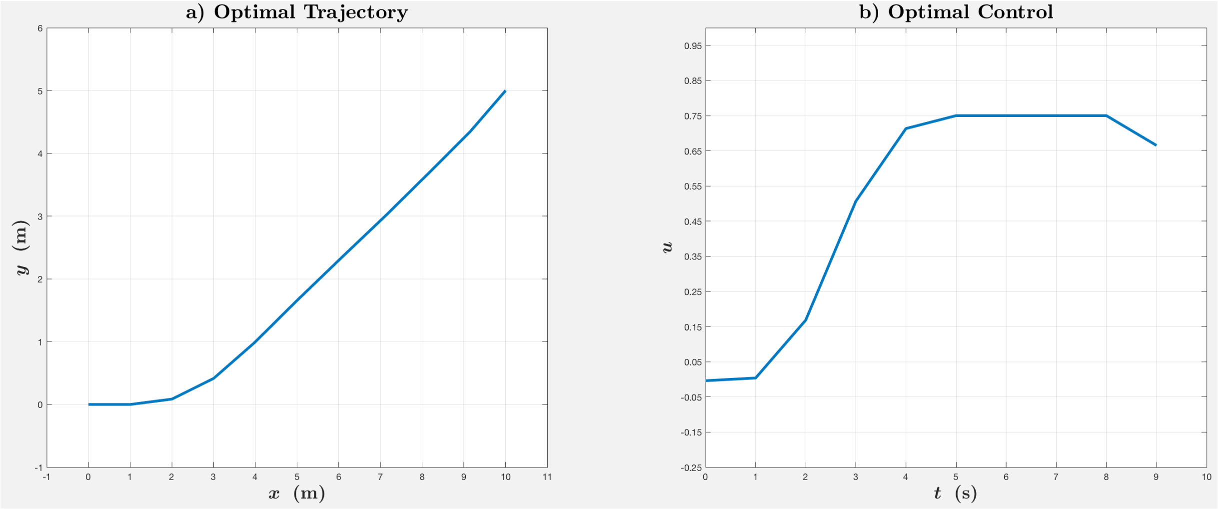
# Practical Example




$|u(t)| \leq 1$

N = 10 (20)

12 iterations (28)




$|u(t)| \leq 0.75$

N = 10 (20)

12 iterations (23)

# Sequential Convex Programming

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

**(OCP)**

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t),\ \ t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m,\ \ t \in [0, t_f]$$

The sources of nonconvexities are the dynamics and (possibly) the cost. Idea: linearizing them around nominal curves!

1. Assume that $g$ is convex. Let $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ be a nominal tuple of trajectory and control. $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ does not need to be feasible!

# Sequential Convex Programming

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t),\ \ t \in [0, t_f]$$

**(OCP)**

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m,\ \ t \in [0, t_f]$$

The sources of nonconvexities are the dynamics and (possibly) the cost. Idea: linearizing them around nominal curves!

1. Assume that $g$ is convex. Let $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ be a nominal tuple of trajectory and control. $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ does not need to be feasible!

2. Linearize $\mathbf{f}$ around $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$:

$$\mathbf{f}_1(\mathbf{x}, \mathbf{u}, t)$$
$$= \mathbf{f}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{x} - \mathbf{x}_0(t)) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{u} - \mathbf{u}_0(t))$$

# Sequential Convex Programming

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

**(LOCP)$_1$**

$$\dot{\mathbf{x}}(t) = \mathbf{f}_1(\mathbf{x}(t), \mathbf{u}(t), t), \ \ t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \ \ t \in [0, t_f]$$

The sources of nonconvexities are the dynamics and (possibly) the cost. Idea: linearizing them around nominal curves!

1. Assume that $g$ is convex. Let $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ be a nominal tuple of trajectory and control. $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ does not need to be feasible!

2. Linearize $\mathbf{f}$ around $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$:
$$\mathbf{f}_1(\mathbf{x}, \mathbf{u}, t)$$
$$= \mathbf{f}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{x} - \mathbf{x}_0(t)) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{u} - \mathbf{u}_0(t))$$

3. Solve the new problem **(LOCP)$_1$** for $(\mathbf{x}_1(\cdot), \mathbf{u}_1(\cdot))$

# Sequential Convex Programming

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) \; dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}_{k+1}(\mathbf{x}(t), \mathbf{u}(t), t), \;\; t \in [0, t_f]$$

**(LOCP)**$_{k+1}$

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \;\; t \in [0, t_f]$$

**$\mathbf{f}_{k+1}$ is linear in $\mathbf{x}$ and $\mathbf{u}$ !**

The sources of nonconvexities are the dynamics and (possibly) the cost. Idea: linearizing them around nominal curves!

1. Assume that $g$ is convex. Let $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ be a nominal tuple of trajectory and control. $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$ does not need to be feasible!

2. Linearize $\mathbf{f}$ around $(\mathbf{x}_0(\cdot), \mathbf{u}_0(\cdot))$:
$$\mathbf{f}_1(\mathbf{x}, \mathbf{u}, t) = \mathbf{f}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{x} - \mathbf{x}_0(t)) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)(\mathbf{u} - \mathbf{u}_0(t))$$

3. Solve the new problem **(LOCP)**$_1$ for $(\mathbf{x}_1(\cdot), \mathbf{u}_1(\cdot))$

4. Iterate this procedure until convergence is achieved: linearize $\mathbf{f}$ around the solution $(\mathbf{x}_k(\cdot), \mathbf{u}_k(\cdot))$ at iteration $k$:
$$\mathbf{f}_{k+1}(\mathbf{x}, \mathbf{u}, t) = \mathbf{f}(\mathbf{x}_k(t), \mathbf{u}_k(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_k(t), \mathbf{u}_k(t), t)(\mathbf{x} - \mathbf{x}_k(t)) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_k(t), \mathbf{u}_k(t), t)(\mathbf{u} - \mathbf{u}_k(t))$$
and solve the problem **(LOCP)**$_{k+1}$ for $(\mathbf{x}_{k+1}(\cdot), \mathbf{u}_{k+1}(\cdot))$

# Sequential Convex Programming

$(\textbf{LOCP})_{k+1}$

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}_{k+1}(\mathbf{x}(t), \mathbf{u}(t), t), \ \ t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \ \ t \in [0, t_f]$$

**$\mathbf{f}_{k+1}$ is linear in $\mathbf{x}$ and $\mathbf{u}$ !**

Discretize and Solve a Convex Problem at Each Iteration

1. Select a discretization $0 = t_0 < t_1 < \cdots < t_N = t_f$ for the interval $[0, t_f]$ and, for every $i = 0, \dots, N-1$, define $\mathbf{x}_{i+1} \sim \mathbf{x}(t)$, $\mathbf{u}_i \sim \mathbf{u}(t)$, $t \in (t_i, t_{i+1}]$ and $\mathbf{x}_0 \sim \mathbf{x}(0)$
2. By denoting $h_i = t_{i+1} - t_i$, $(\textbf{LOCP})_{k+1}$ is transcribed into the following convex optimization problem

$(\textbf{DLOCP})_{k+1}$

$$\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i, t_i)$$

$$\mathbf{x}_{i+1} = \mathbf{x}_i + h_i \mathbf{f}_{k+1}(\mathbf{x}_i, \mathbf{u}_i, t_i), \ i = 0, \dots, N-1$$

$$\mathbf{u}_i \in U, \ i = 0, \dots, N-1, \qquad \mathbf{x}_N = \mathbf{x}_f$$

# Sequential Convex Programming

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) \; dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}_{k+1}(\mathbf{x}(t), \mathbf{u}(t), t), \;\; t \in [0, t_f]$$

$(\textbf{LOCP})_{k+1}$

$$\mathbf{x}(0) = \; \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \quad t \in [0, t_f]$$

**$\mathbf{f}_{k+1}$ is linear in x and u !**

Methodological Scheme for SCP

At each iteration $k$:

Linearize **f** around the solution $(\mathbf{x}_k(\cdot), \mathbf{u}_k(\cdot))$

Define the continuous time problem $(\textbf{LOCP})_{k+1}$

Discretize $(\textbf{LOCP})_{k+1}$ in time and define the convex optimization problem $(\textbf{DLOCP})_{k+1}$

Solve $(\textbf{DLOCP})_{k+1}$ via convex programming for a discretized version of $(\mathbf{x}_{k+1}(\cdot), \mathbf{u}_{k+1}(\cdot))$

$$\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i, t_i)$$

$(\textbf{DLOCP})_{k+1}$

$$\mathbf{x}_{i+1} = \mathbf{x}_i + h_i \mathbf{f}_{k+1}(\mathbf{x}_i, \mathbf{u}_i, t_i), \; i = 0, \dots, N-1$$

$$\mathbf{u}_i \in U \,, i = 0, \dots, N-1, \qquad \mathbf{x}_N = \mathbf{x}_f$$

# Practical Example

## Designing SCP Method in Matlab

Solve the following optimal control problem via SCP, using the Matlab framework “**Cvx**”

## Modified Zermelo Problem

**(OCP)**

$$\min \int_0^{t_f} u(t)^2 \, dt$$

$$\dot{x}(t) = v\cos\big(u(t)\big) + \text{flow}\big(y(t)\big), \; t \in [0, t_f]$$

$$\dot{y}(t) = v\sin\big(u(t)\big), \; t \in [0, t_f]$$

$$(x,y)(0) = 0\,, \; (x,y)(t_f) = (M, \ell)$$

$$|u(t)| \le u_{\max}, \; t \in [0, t_f]$$

**(DLOCP)**$_{k+1}$

$$\min_{(x_i, y_i, u_i)} \sum_{i=0}^{N-1} u_i^2$$

$$|u_i| \le u_{\max}, \; (x_0, y_0) = 0, \; (x_N, y_N) = (M, \ell)$$

$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} + h_i \mathbf{f}_{k+1}(x_i, y_i, u_i, t_i)$$

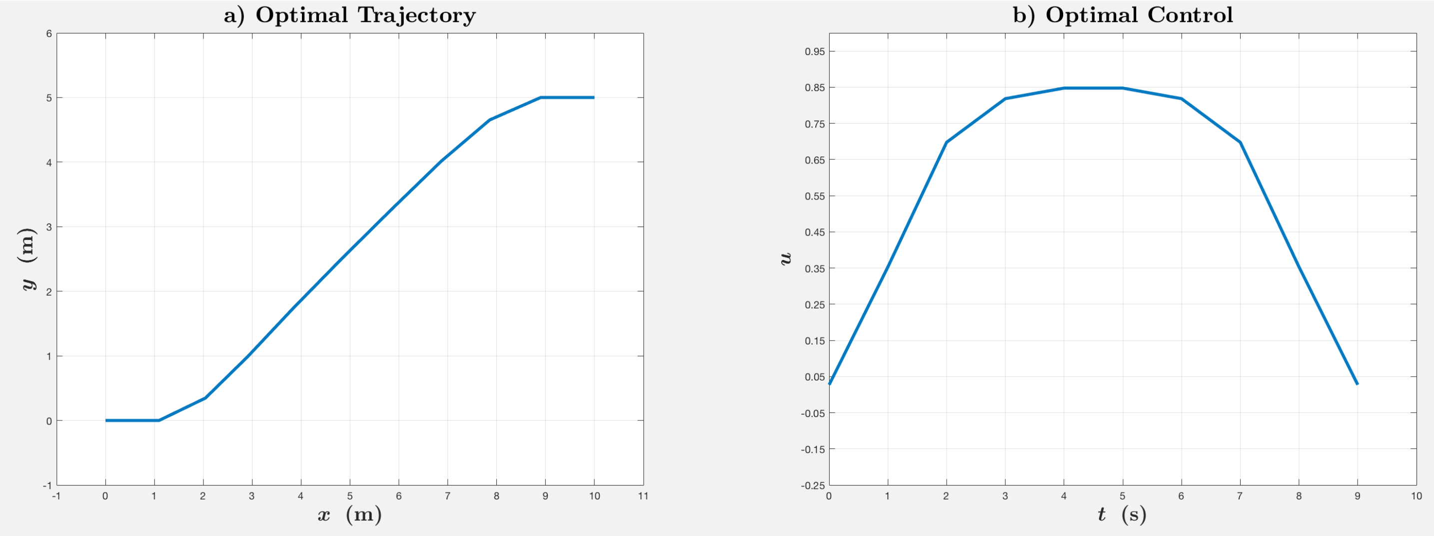
$$\frac{\partial \mathbf{f}}{\partial (x,y)}(x,y,u) = \begin{pmatrix} 0 & \frac{\partial \text{flow}}{\partial y}(y) \\ 0 & 0 \end{pmatrix}, \quad \frac{\partial \mathbf{f}}{\partial u}(x,y,u) = \begin{pmatrix} -v\sin u \\ v\cos u \end{pmatrix}$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}_{k+1}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$= \left( \mathbf{f}(\mathbf{x}_k(t), \mathbf{u}_k(t), t) - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_k(t), \mathbf{u}_k(t), t)\mathbf{x}_k(t) - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_k(t), \mathbf{u}_k(t), t)\mathbf{u}_k(t) \right)$$

$$+ \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_k(t), \mathbf{u}_k(t), t)\mathbf{x}(t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_k(t), \mathbf{u}_k(t), t)\mathbf{u}(t)$$

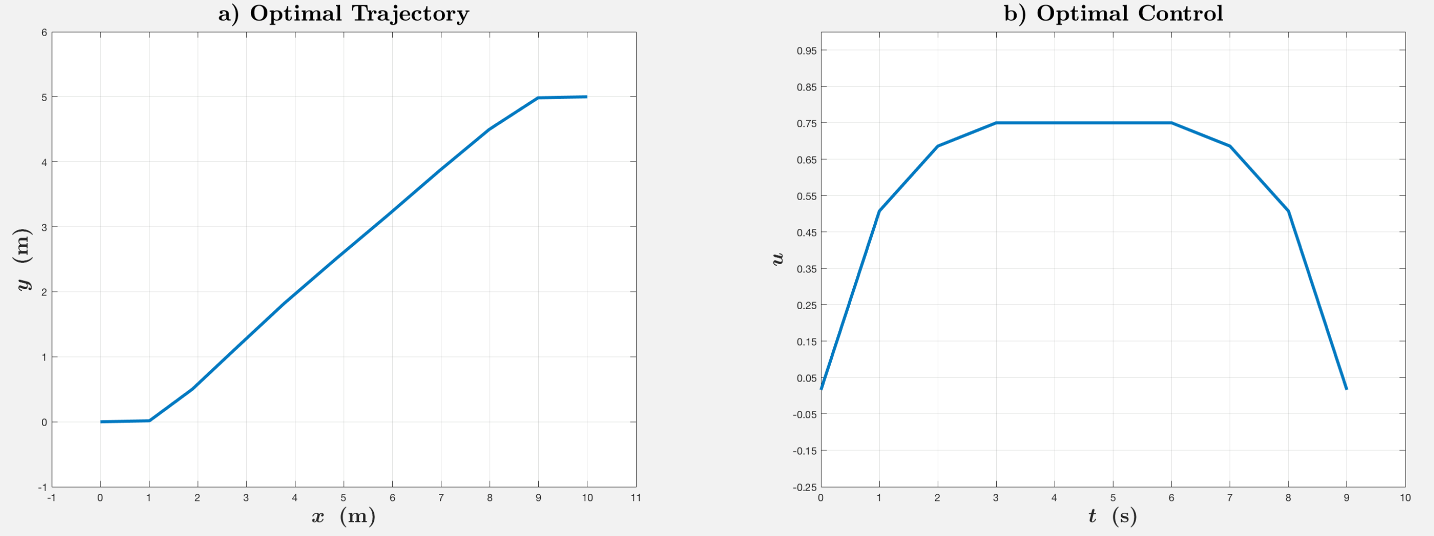
# Practical Example




$$|u(t)| \leq 1$$

N = 10

2 SCP iterations (12)




$$|u(t)| \leq 0.75$$

N = 10

3 SCP iterations (12)

# Next time

- MPC