Convex Optimization

AA 203 Recitation #4

May 1st, 2020

Agenda

Preliminaries

- Why study Convex Optimization?
- Convex Sets & Convex Functions
- Convex Programming
- Linear Matrix Inequalities

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Optimization Models and Tools

- Solvers (i.e. CPLEX, CVX).
- Linear Programming
- Quadratic Programming

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- Linear Programming
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Algorithms

- Simplex Method
- Cutting Plane Methods (Ellipsoid Method)
- Interior Point Method

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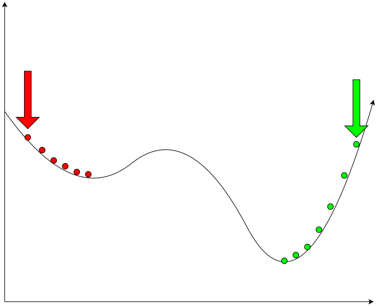

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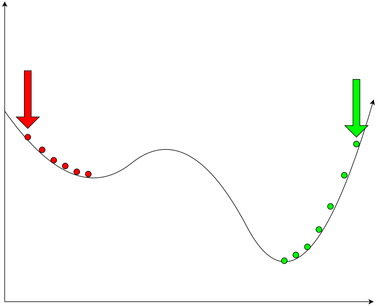

Observation 3: This issue doesn't show up for convex problems. For convex optimization problems, every locally optimal solution is also globally optimal.

Convex Sets

Definition (Convex Set)

A set $S \subset \mathbb{R}^d$ is convex if and only if: for any $x, y \in S$ and any $\alpha \in [0, 1]$, we also have $\alpha x + (1 - \alpha)y \in S$.

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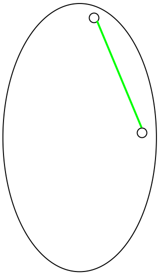

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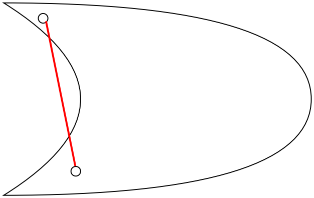

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A function $f : S \to \mathbb{R}$ over a convex set $S \subset \mathbb{R}^d$ is convex if the set

$$\text{epigraph}(f) := \left\{(x, y) \in \mathbb{R}^{d+1} : x \in S, y \in \mathbb{R} \text{ and } y \geq f(x)\right\} \text{ is convex.}$$

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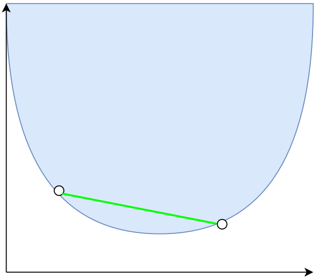

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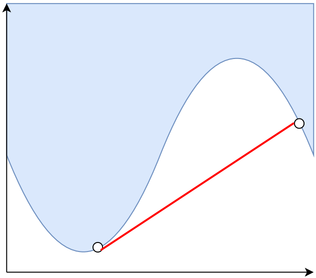

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A convex program (aka convex optimization problem) is a minimization problem of a convex function over a convex set:

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Definition (Local Minimum)

For an optimization problem $\min_{x \in S} f(x)$, a point x^* is a local minimum if there exists some $\epsilon > 0$ so that for every $x \in S$ with $||x - x^*||_2 \leq \epsilon$, $f(x^*) \leq f(x)$.

Convex Program: Local Optima are Global Optima

Theorem (Equivalence of Local and Global Optima)

Let $\min_{x \in S} f(x)$ *be a convex program. If* x^* *is a local minimum, then* $f(x^*) \leq f(x)$ *for every* $x \in S$*. In other words,* x^* *is a global minimum.*

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This contradicts the fact that x^* is a local minimum. $\square$

Linear Matrix Inequalities (LMI)

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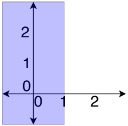
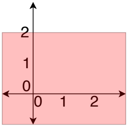
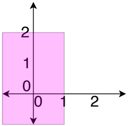
Example: $\left\{ x : x \preceq \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$







$x_1 \leq 1$ $\qquad$ $x_2 \leq 2$ $\qquad$ $x \preceq (1, 2)^\top$

Matrix Inequalities

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Applications of SDPs: Sum of Squares Programming, Lyapunov Stability analysis, approximation algorithms for combinatorial optimization.

Optimization Models and Tools

Optimization Software

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- Linear Programming (LP).
- Quadratic Programming (QP).
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A linear programming instance is specified by
$c \in \mathbb{R}^n, b \in \mathbb{R}^p, A \in \mathbb{R}^{p \times n}, b_{eq} \in \mathbb{R}^q, A_{eq} \in \mathbb{R}^{q \times n}$.

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Software:

CPLEX: `x = cplexlp(c, A, b, Aeq, beq)`.

MATLAB: `x = linprog(c, A,b, Aeq, beq)`.

LP Example - Marketplace Efficiency

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Objective: Match buyers to sellers to maximize the total utility of the marketplace.

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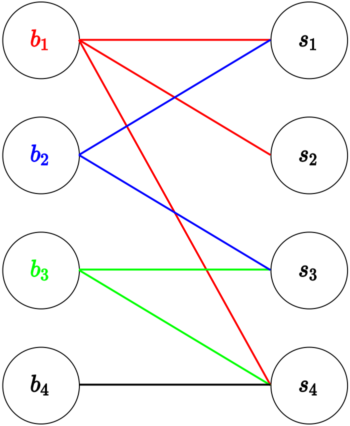

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\begin{aligned}
\underset{x \in \mathbb{R}^{mn}}{\text{maximize}} \quad & \sum_{i=1}^{n} \sum_{j=1}^{m} u_{ij} x_{ij} && (1) \\
\text{subject to} \quad & \sum_{j=1}^{m} x_{ij} \leq 1 \text{ for all } 1 \leq i \leq n && (2) \\
& \sum_{i=1}^{n} x_{ij} \leq 1 \text{ for all } 1 \leq j \leq m && (3) \\
& x \succeq 0. && (4)
\end{aligned}
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(2) ensures each buyer buys at most one item, (3) ensures each seller sells at most one item.

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If $x^*_{ij} = 1$, have buyer b_i buy from seller s_j.

Remark: If we look at the KKT conditions of LP (1), the dual variables of the constraints can be used as prices with the following property:

> If the sellers s_j lists their item for a price λ_j,
>
> Buyer i chooses to buy the item $\arg\max_j u_{ij} - \lambda_j$,

then the resulting allocation will be x^*!

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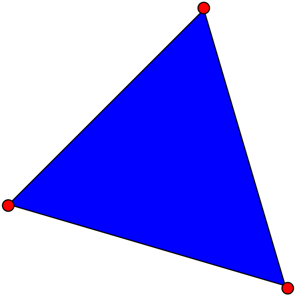

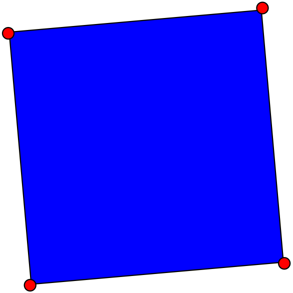

Linear Programming - Properties

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Since x^* is a minimizer, x' must also be a minimizer.

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However, this fact motivated the first implementation of an LP solver.

LP Algorithm: Simplex Method

Key Idea: Visit extreme points of the feasible region until you find the optimal solution.

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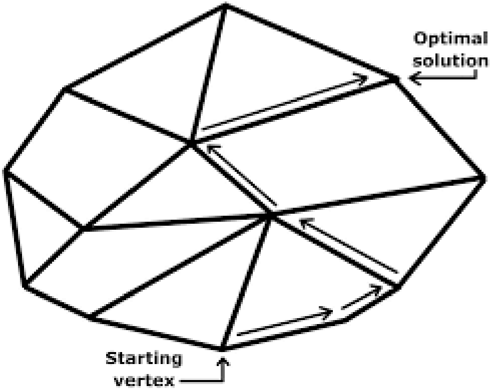

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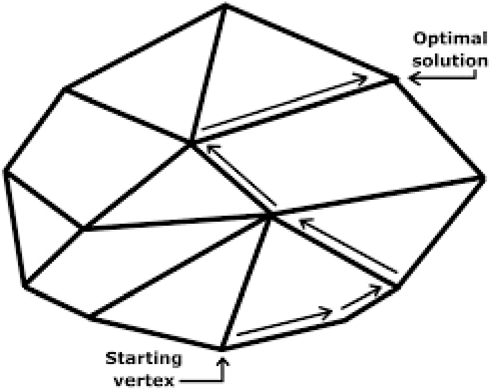

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Remark: Worst case running time is exponential in the size of the input, but the set of "bad instances" is very small. In practice, the algorithm works well.

Quadratic Programming

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A quadratic programming instance is specified by
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Software:

CPLEX: `x = cplexqp(H, f, A, b, Aeq, beq)`.

MATLAB: `x = quadprog(H, f, A,b, Aeq, beq)`.

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\text{subject to } & x_{t+1} = Ax_t + Bu_t \text{ for all } 0 \le t \le T-1 && (6)\\
& x_0 = \text{initial condition} && (7)\\
& u_{LB} \preceq u_t \preceq u_{UB} \text{ for all } 0 \le t \le T-1. && (8)
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Optimization Algorithms

Algorithms for Convex Optimization

Simplex Method (Recap)

- Can solve linear programs.
- Was the first proposed algorithm to solve linear programs.
- Has a worst-case running time that is exponential in the input size.
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Interior Point Methods

- Can solve convex problems.
- Based on sequential convex programming and warm-starts.
- Currently the best algorithm for general linear programming.

Ellipsoid Method - Cutting Plane Methods

Suppose we want to solve

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Thus the global minimum cannot be in the set $\left\{y : \nabla f(x)^\top (y - x) > 0\right\}$. Therefore we can eliminate/"cut away" this set from our search space. Hence the name, "cutting plane method".

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Similarly, if $g_i(x) > 0$, then by convexity of g_i, we know

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In particular, all points in $\left\{y : \nabla g_i(x)^\top (y - x) > 0\right\}$ are infeasible! So we can prune away this set.

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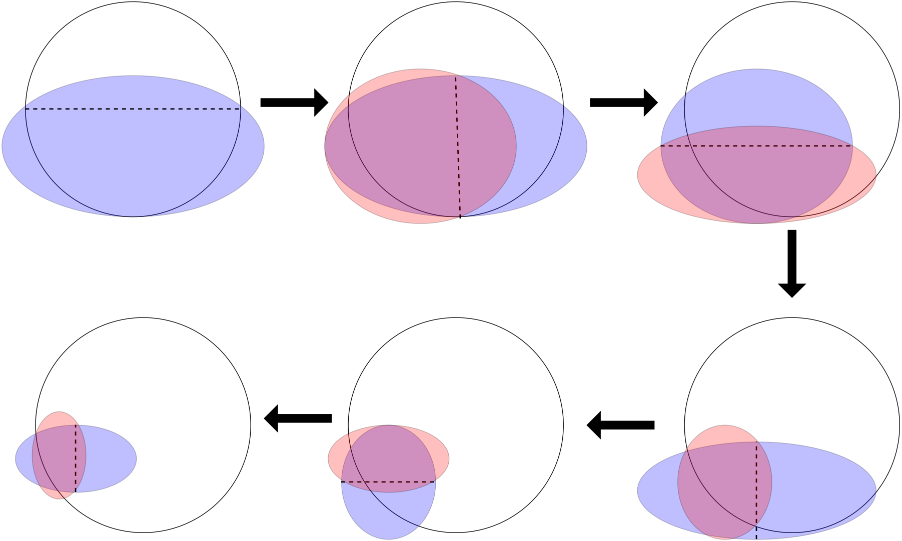
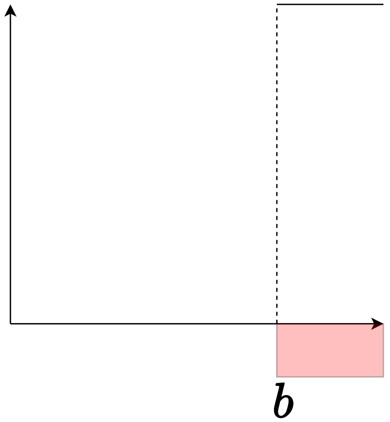
One can show that $\text{vol}(\mathcal{E})$ decreases fast enough in each iteration so that the algorithm will terminate in polynomial time with a high quality solution.

Ellipsoid Method - Visualization



Key Idea: Convert a constrained problem into an unconstrained problem using penalty functions.

Example: Consider the constraint $x \leq b$.



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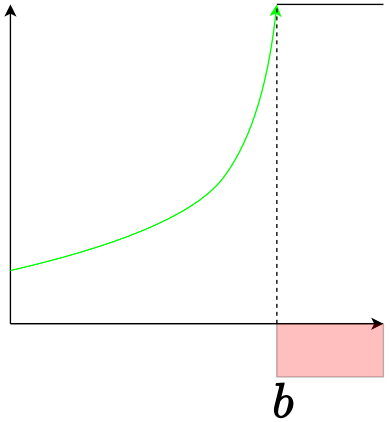

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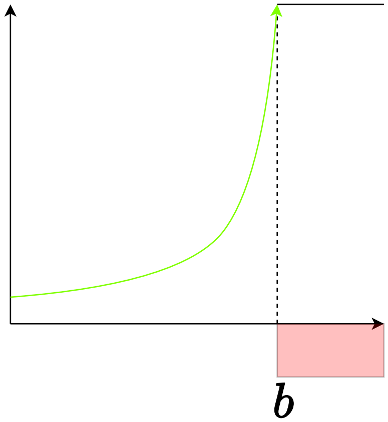

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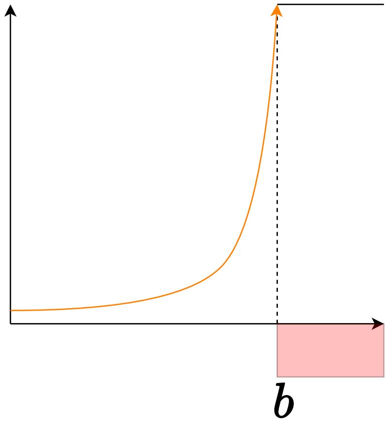

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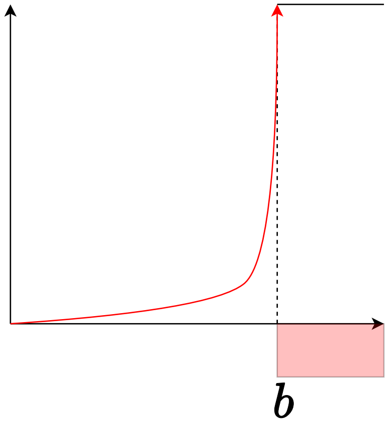

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- Provide a warm start to the problem with a steep barrier to overcome poor conditioning.
- Obtain a warm start point by solving the problem with a shallower barrier.
- Continue using each solution to warm start a problem with a steeper barrier until convergence.