On the Problem of Reformulating Systems with Uncertain Dynamics as a Stochastic Differential Equation

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I. Problem Formulation and Error in Literature

Consider a nonlinear system whose state is $x \in \mathbb{R}^n$, control inputs are $u \in \mathbb{R}^m$, such that

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T], \tag{1}$$

where $T > 0$ $x(0) = x_0 \in \mathbb{R}^n$ almost surely, i.e., $x(0)$ is known exactly, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is twice continuously differentiable.

In many applications, f is not known exactly, and prior knowledge is necessary to safely control (1). One such approach consists of assuming that f lies in a known space of functions $\mathcal{H}$, and to impose a prior distribution in this space $\mathbb{P}(\mathcal{H})$. For instance, by assuming that f lies in a bounded reproducing kernel Hilbert space (RKHS), an approach consists of imposing a Gaussian process prior on the uncertain dynamics $f \sim \mathcal{GP}(m, k)$, where $m : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the mean function, and $k : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}^{n \times n}$ is a symmetric positive definite covariance kernel function which uniquely defines $\mathcal{H}$ [1], [2]. An alternative consists of assuming that $f(x, u) = \phi(x, u)\theta^*$, where $\phi : \mathbb{R}^{n+m} \to \mathbb{R}^{n \times p}$ are known basis functions, and $\theta^* \in \mathbb{R}^p$ are unknown parameters. With this approach, one typically sets a prior on θ, e.g. $\theta \sim \mathcal{N}(\bar{\theta}, \Sigma_\theta)$, and updates this belief as further data about the system is gathered.

Given these model assumptions and prior knowledge about f, *safe* learning-based control algorithms often consist of designing a control law u satisfying different specifications, e.g., minimizing fuel consumption $\|u\|$, or satisfying constraints $x(t) \in \mathcal{X}$ $\forall t \in [0, T]$, with $\mathcal{X}$ a set encoding safety and physical constraints.

Next, we describe the issue in [3]–[5], slightly changing notations and assuming a finite-dimensional combination of features for clarity of exposition but without loss of generality. As in [5], consider the problem of safely controlling the uncertain system

$$\dot{x}(t) = \phi(x, u)\theta, \quad \theta \sim \mathcal{N}(\bar{\theta}, \Sigma_\theta), \quad t \in [0, T], \quad x(0) = x_0, \tag{2}$$

where $\bar{\theta} \in \mathbb{R}^p$, and $\Sigma_\theta \in \mathbb{R}^{p \times p}$ is positive definite, with $\Sigma_\theta = B_\theta B_\theta^\top$ its Cholesky decomposition. Note that the case of a Gaussian process with kernel k for f as in [3] and [4] corresponds to the above with $k(x, u) = \phi(x, u)^\top \phi(x, u)$, see [1][1].

[3]–[5] then proceed by introducing the Brownian motion $W(t)$, making the change of variable $\theta \mathrm{d}t = \bar{\theta}\mathrm{d}t + B_\theta \mathrm{d}W$, and reformulating (2) as the following stochastic differential equation (SDE)

$$\mathrm{d}x(t) = \phi(x, u)\bar{\theta}\mathrm{d}t + \phi(x, u)B_\theta \mathrm{d}W(t), \quad x(0) = x_0, \tag{3}$$

with $t \in [0, T]$.

Unfortunately, (3) is not equivalent to (2). Indeed, the solution to (3) is a Markov process, whereas the solution to (2) is not. Intuitively, the increments of the Brownian motion $W(t)$ in (3) are independent, whereas in (2), θ is randomized only once, and its uncertainty is propagated along the entire trajectory. By making this change of variables for $\theta \mathrm{d}t$, the temporal correlation between the trajectory $x(t)$ and the uncertain parameters θ is neglected. To illustrate this inaccuracy, we provide an example in the next section.

[1]For a squared exponential kernel, one needs $p \to \infty$ for this equivalence. Nevertheless, the issue discussed in this paper remains valid for such kernels.

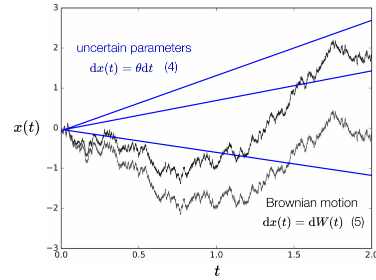

Fig. 1: Counter-example to the reformulation in [3]–[5]: the solutions of (4) and of (5) are distinct. The solution of (5) is not differentiable, whereas the solution of (4) has continuously differentiable sample paths and larger variance for $t > 1$.

II. Counter-Example

Consider the scalar continuous-time linear system

$$\dot{x}(t) = \theta, \quad \theta \sim \mathcal{N}(0, 1), \quad t \in [0, T], \tag{4}$$

where $T > 0$ and $x(0) = 0$ almost surely, i.e., $x(0)$ is known exactly. Solutions to (4) satisfy $x(t) = \theta t$, and are thus centered Gaussian processes satisfying $x(t) \sim \mathcal{N}(0, t^2)$ for any time $t \in [0, T]$[2], with continuously differentiable sample paths.

Following the reasoning from [3]–[5], one makes the change of variables $\theta \mathrm{d}t = \mathrm{d}W(t)$, with $W(t)$ a standard Brownian motion, yielding the following SDE

$$\mathrm{d}x(t) = \mathrm{d}W(t), \quad x(0) = 0 \text{ (a.s.)} \quad t \in [0, T]. \tag{5}$$

The solution of this SDE is a standard Brownian motion $x(t) = W(t)$ started at $W(0) = 0$. This stochastic process satisfies $x(t) \sim \mathcal{N}(0, t)$, $\mathrm{Cov}(x(t), x(s)) = \min(t, s)$, and is not differentiable at any t almost surely. Thus, the solutions to (4) and to (5) are distinct. We illustrate sample paths of these two different stochastic processes in Figure 1.

III. Implications

As (2) and (3) are generally not equivalent, the stability and constraints satisfaction guarantees in [3]–[5] hold for the SDE (3) but not necessarily for (2)[3]. In situations where (2) represents the true system more accurately than (3), this could have implications for safety-critical applications of these algorithms.

IV. Open Problems and Possible Solutions

Although (3) is not equivalent to (2), it is interesting to ask whether (3) is a conservative reformulation of (2) for the purpose of *safe* control. For instance, given a safe set $\mathcal{X} \subset \mathbb{R}^n$, if one opts to encode safety constraints through joint chance constraints of the form

$$\mathbb{P}(x(t) \in \mathcal{X} \ \forall t \in [0, T]) \geq (1 - \delta), \tag{6}$$

where $\delta \in (0, 1)$ is a tolerable probability of failure, there may be settings where a controller u satisfying (6) for the SDE (3) may provably satisfy (6) for the uncertain model (2). Indeed, as solutions of (3) may have unbounded total variation (as in the example presented above), which is not the case for solutions of (2), we make the conjecture that for long horizons T, a standard proportional-derivative-integral (PID) controller may better stabilize (2) than (3), and that similar properties hold for adaptive controllers.

[2]Indeed, $ax(t) + bx(s) = (at + bs)\theta$ follows a Gaussian law, and $\mathbb{E}[ax(t) + bx(s)] = (at + bs)\mathbb{E}[\theta] = 0$, so $x(t) = \theta t$ is a centered Gaussian process. Further, $\mathrm{Cov}(x(t), x(s)) = \mathbb{E}[(\theta t)(\theta s)] = ts$.

[3]For instance, the generator of the Markov process solving the SDE (3) (see [6]) is used to prove stability in [3]–[5]. Unfortunately, (2) does not yield a Markov process. Thus, it would be necessary to adapt the concept of generator to solutions of (2) before concluding the stability of the original system.

Alternatively, approaches which bound the model error through the Bayesian posterior predictive variance [7] or confidence sets holding jointly over time [8], [9] exist. Given these probabilistic bounds, a policy can be synthesized yielding constraints satisfaction guarantees.

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