Linear Algebra & Linear Dynamical Systems

AA 203 Recitation #1

April 10th, 2020

Overview

1. Linear Algebra
2. Linear Dynamical Systems (LDS)
   a. Discrete time LDS.
   b. Continuous time LDS.

Overview (cont.)

1. Linear Algebra
    a. Vector Spaces.
    b. Matrices and linear functions.
    d. Matrix Multiplication and Matrix Inverse.
    e. Singular Value Decomposition.
    f. Eigenvalue Decomposition.

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   a. Vector Spaces.
   b. Matrices and linear functions.
   d. Matrix Multiplication and Matrix Inverse.
   e. Singular Value Decomposition.
   f. Eigenvalue Decomposition.
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   a. Stabilization and tracking.
   b. Constraint structure.
   c. Operator norm & stability via linear feedback.

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    e. Singular Value Decomposition.
    f. Eigenvalue Decomposition.
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    a. Stabilization and tracking.
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3. Continuous time Linear Dynamical Systems
    a. Stabilization and tracking.
    b. State evolution.
    c. Diagonalization & stability via linear feedback.

1. Linear Algebra

1. Linear Algebra - Vector Spaces

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- Vector addition is **entrywise**: $x + y = z$ means $x_i + y_i = z_i$ for all $1 \leq i \leq n$. **Example**:

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$$

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$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$$

- Scalar multiplication is **entrywise**: $cx = y$ means $cx_i = y_i$ for all $1 \le i \le n$. **Example**:

$$2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

1. Linear Algebra - Linear Combinations

Definition (Linear Combination)

A linear combination of the vectors $x_1, x_2, ..., x_m$ is any vector of the form

$$\sum_{i=1}^{m} c_i x_i \text{ where } c_1, ..., c_m \in \mathbb{R}.$$

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Definition (Span)

The span of a collection of vectors $x_1, ..., x_m$, denoted by $\text{span}(x_1, ..., x_m)$ is the collection of all possible linear combinations.

$$\text{span}(x_1, ..., x_m) := \left\{ y : y = \sum_{i=1}^{m} c_i x_i \text{ for some } c_1, ..., c_m \in \mathbb{R} \right\}$$

1. Linear Algebra - Linear Independence

Definition (Linear Independence)

A collection of vectors $x_1, \ldots, x_m$ is linearly independent if no member can be written as a linear combination of the other members.

Equivalently, the vectors $x_1, \ldots, x_m$ are independent if

$$\sum_{i=1}^{m} c_i x_i = 0 \implies c_i = 0 \text{ for all } 1 \leq i \leq m.$$

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A set $V \subset \mathbb{R}^n$ is a linear subspace if it is

1. Closed under scalar multiplication: $v \in V \implies cv \in V$ for all $c \in \mathbb{R}$.
2. Closed under vector addition: $u, v \in V \implies u + v \in V$.

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Example 2: For any set of vectors V_0, $\text{span}(V_0)$ is a linear subspace.

Example 3: $\{x \in \mathbb{R}^n : x_1 = 0\}$ is a linear subspace.

1. Linear Algebra - Spanning sets, Basis, Dimension

Definition (Spanning Set)

Given a linear subspace V, a collection of vectors $x_1, \ldots, x_m$ is a spanning set if $\text{span}(x_1, \ldots, x_m) = V$.

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Definition (Dimension)

Every basis of a subspace V contains the same number of vectors. This number is the dimension of V.

Definition

Given $x, y \in \mathbb{R}^n$, their dot product $\langle x, y \rangle$ is defined as:

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i .$$

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Vectors are also matrices! A vector $x \in \mathbb{R}^n$ is a $n \times 1$ matrix.

If $A \in \mathbb{R}^{m \times n}$, then its transpose is $A' \in \mathbb{R}^{n \times m}$ and

$$A_{ij} = A'_{ji}.$$

1. Linear Algebra - Matrix Transpose

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Example:

$$A = \begin{bmatrix} 1 & 6 \\ 0 & 2 \\ 5 & 3 \end{bmatrix}, A' = \begin{bmatrix} 1 & 0 & 5 \\ 6 & 2 & 3 \end{bmatrix}$$

1. Linear Algebra - Matrix Multiplication

Matrix Multiplication: For $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, AB exists if and only if $n = p$. In this case, AB is a $m \times q$ matrix.

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Dot product as matrix multiplication: For $x, y \in \mathbb{R}^n$,
$\langle x, y \rangle = x'y = y'x$.

1. Linear Algebra - Linear functions

Definition (Linear functions)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if for any $x, y \in \mathbb{R}^n$ and any $c_1, c_2 \in \mathbb{R}$,

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Example: Any matrix $A \in \mathbb{R}^{m \times n}$ defines a linear function $f(x) := Ax$.

$$\begin{aligned} f(c_1 x + c_2 y) &= A(c_1 x + c_2 y) = A(c_1 x) + A(c_2 y) = c_1 Ax + c_2 Ay \\ &= c_1 f(x) + c_2 f(y). \end{aligned}$$

All linear functions are matrices!

1. Linear Algebra - Linear Representation

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Theorem

For every linear $f : \mathbb{R}^n \to \mathbb{R}^m$*, there exists a matrix* A *so that* $f(x) = Ax$ *for every* $x \in \mathbb{R}^n$*.*

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Idea: Let $e_1, e_2, ..., e_n$ be a basis of $\mathbb{R}^n$. By linearity,

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i).$$

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A matrix A specifies the image of the standard basis vectors.

e_i is mapped to the ith column of A.

For $A \in \mathbb{R}^{m \times n}$, let

1. $R \subset \mathbb{R}^n$ be the set of rows of A.
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The row space of A is denoted $\text{row}(A) := \text{span}(R)$.

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Definition

The kernel of A is denoted $\ker(A) := \{x \in \mathbb{R}^n : Ax = 0\}$.

$\text{row}(A), \text{col}(A), \ker(A)$ are all subspaces.

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A is full rank if its rank is $\min(n, m)$.

Rank-Nullity: $\text{rank}(A) + \dim(\ker(A)) = n$.

1. Linear Algebra - Identity Matrix

$I_n \in \mathbb{R}^{n \times n}$ is the identity matrix:

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

I_n is a **multiplicative identity**: $AI_n = A$ and $I_n B = B$ whenever A, B have compatible dimensions to multiply with I_n.

1. Linear Algebra - Matrix Inverse & Orthogonal Matrices

Definition (Matrix Inverse)

The inverse of a matrix $A \in \mathbb{R}^{n \times n}$ is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$.

Remark: Not all matrices have inverses. A matrix has an inverse if and only if it is full rank.

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Definition (Orthogonal Matrix)

A matrix $A \in \mathbb{R}^{n \times n}$ is orthogonal if and only if $A' = A^{-1}$.

Orthogonal Matrices are Isometries

Theorem (Orthogonal Matrices preserve length)

If $U \in \mathbb{R}^{n \times n}$ is orthogonal, then for every $x \in \mathbb{R}^n$,

$$||Ux||_2 = ||x||_2 \, .$$

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Theorem (Orthogonal Matrices preserve length)

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Proof:

$$||x||_2^2 = x'x = x'I_n x = x'U'Ux = (Ux)'(Ux) = ||Ux||_2^2 \, .$$

Take square roots of both sides.

For any matrix $A \in \mathbb{R}^{m \times n}$, we can factorize

$$A = U \Sigma V'$$

where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices,

$\Sigma \in \mathbb{R}^{m \times n}$ is diagonal:

- $\Sigma_{ij} = 0$ if $i \neq j$.
- $\Sigma_{ii} \geq 0$.

1. Linear Algebra - Eigenvalue Decomposition

For any symmetric matrix $A \in \mathbb{R}^{n \times n}$, we can factorize

$$A = U \Lambda U'$$

where $U \in \mathbb{R}^{n \times n}$ is orthogonal and $\Lambda \in \mathbb{R}^{n \times n}$ is a diagonal matrix.

2. Discrete Linear Dynamical Systems

Dynamics:

$$x_{k+1} = Ax_k + Bu_k$$

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- $u_k \in \mathbb{R}^m$ is the control/action that is applied at timestep k.
- $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$.

2. Discrete Linear Dynamical Systems - Stabilizing

Stabilization Objective: Try to get x_k to converge to the origin as fast as possible.

$$
\begin{aligned}
\underset{\{u_k\}_{k=1}^{T-1}}{\text{minimize}} \quad & \sum_{k=1}^{T} ||x_k||_2^2 \\
\text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k \text{ for } 1 \leq k \leq T-1
\end{aligned}
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$$\begin{aligned} \underset{\{u_k\}_{k=1}^{T-1}}{\text{minimize}} \; & \sum_{k=1}^{T} x_k' R x_k + u_k' Q u_k \\ \text{s.t.} \; & x_{k+1} = Ax_k + Bu_k \text{ for } 1 \le k \le T-1 \end{aligned}$$

where $R, Q \succeq 0$.

2. Discrete Linear Dynamical Systems - Optimization

The relation between x_{k+1}, x_k, u_k is **linear**. We can incorporate the dynamics constrains as one large matrix equation: $\mathcal{A}z = 0$.

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The relation between x_{k+1}, x_k, u_k is **linear**. We can incorporate the dynamics constrains as one large matrix equation: $\mathcal{A}z = 0$.

$$\underbrace{\begin{bmatrix} A & B & -I_n & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & A & B & -I_n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & A & B & -I_n \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} x_1 \\ u_1 \\ x_2 \\ u_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ u_{T-1} \\ x_T \end{bmatrix}}_{z} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus the stabilization and tracking problems can be solved by optimizing an objective function subject to linear equality constraints:

$$\begin{aligned} \underset{z}{\text{minimize}} \ & J(z) \\ \text{s.t.} \ & \mathcal{A}z = 0. \end{aligned}$$

2. Discrete Linear Dynamical Systems - Control Synthesis

How do we choose u so that $\lim_{k \to \infty} x_k = 0$?

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$$\begin{aligned} x_{k+1} &= (A - BK)x_k \\ \iff x_{k+1} &= U\Lambda U' x_k \\ \implies U' x_{k+1} &= \underbrace{U'U}_{I_n} \Lambda U' x_k \end{aligned}$$

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Iterate: $y_{k,i} = (\Lambda_{ii})^k y_{0,i}$.

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3. Continuous Linear Dynamical Systems

Dynamics:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

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- $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

3. Continuous Linear Dynamical Systems - Stabilizing

Stabilizing Objective: Guide $x(t)$ to zero.

$$\begin{aligned} \underset{u:[0,T]\to\mathbb{R}^m}{\text{minimize}} & \int_0^T x'(t)Rx(t) + u'(t)Qu(t)dt \\ \text{s.t. } & \dot{x}(t) = Ax(t) + Bu(t). \end{aligned}$$

where $R, Q \succeq 0$.

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If $A - BK$ is symmetric, then $A - BK = U\Lambda U'$ for some diagonal Λ and orthogonal U. Thus

$$\begin{aligned} \dot{x}(t) &= U\Lambda U' x(t) \\ \implies U'\dot{x}(t) &= \Lambda U' x(t). \end{aligned}$$

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Since Λ is diagonal, $\dot{y}_i(t) = \Lambda_{ii}y(t)$ for all i. Solve this ODE!

$$\begin{aligned} \frac{\dot{y}_i(t)}{y(t)} &= \Lambda_{ii} \\ \implies \ln\left(y_i(t)\right) &= \Lambda_{ii}t + C \\ \implies y_i(t) &= e^C e^{\Lambda_{ii}t} = y_i(0)e^{\Lambda_{ii}t}. \end{aligned}$$

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