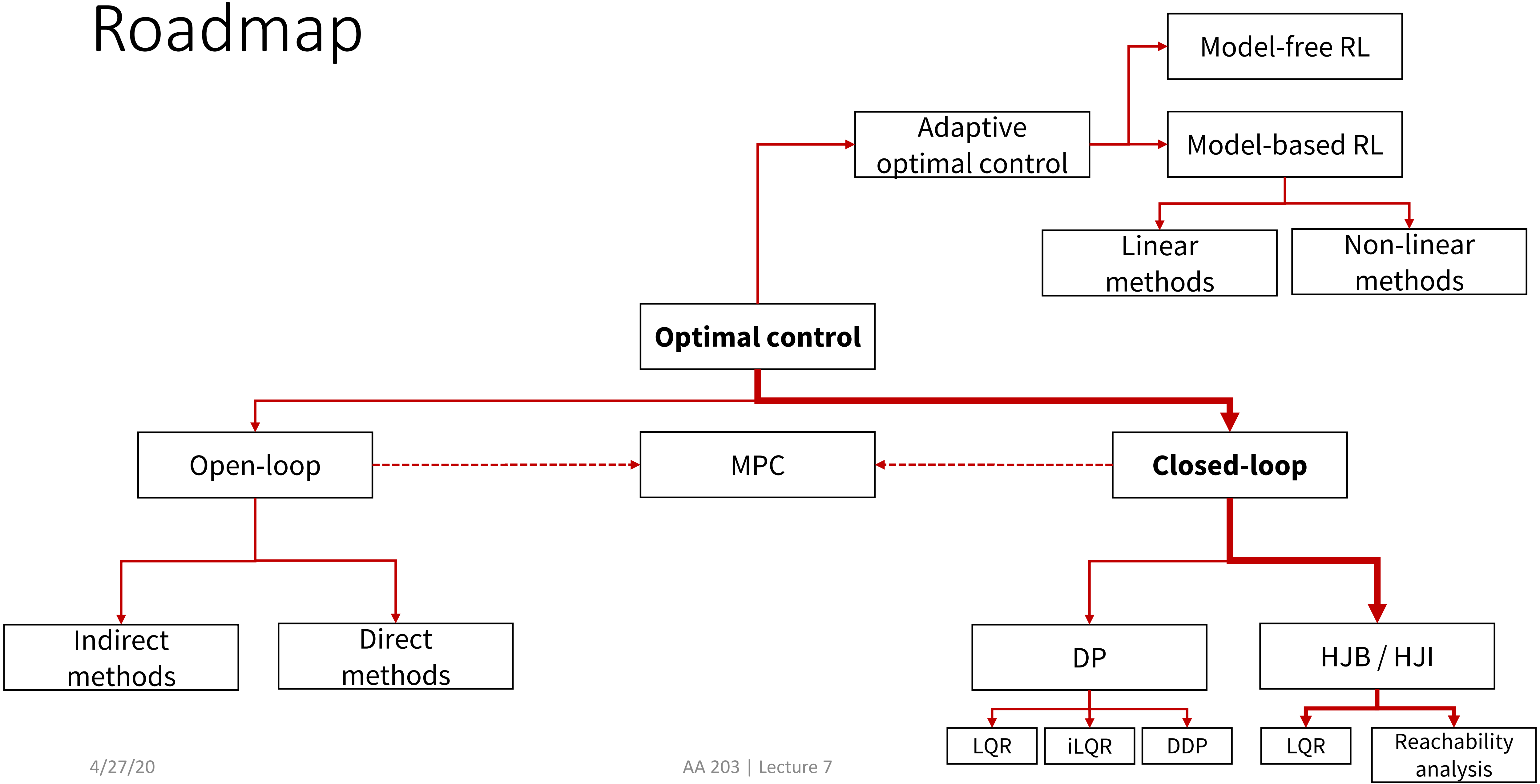
AA203
Optimal and Learning-based Control

HJB, HJI, and reachability analysis

Stanford University

Autonomous Systems Lab

Roadmap


Model-free RL
Adaptive optimal control
Model-based RL
Linear methods
Non-linear methods
Optimal control
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI
LQR
iLQR
DDP
LQR
Reachability analysis

Agenda

Previous lectures: focus on discrete-time setting

This lecture: focus on continuous-time setting

- dynamic programming approach leads to HJB / HJI equation: non-linear partial differential equation
- HJB application: solution to continuous LQR problem
- HJI application: reachability analysis

Readings: lecture notes and references therein, in particular:

- Bansal S., Chen M., Herbert S., Tomlin C. J.” Hamilton-Jacobi reachability: A brief overview and recent advances,” 2017.
- Chen M., Tomlin C. J. “Hamilton–Jacobi reachability: Some recent theoretical advances and applications in unmanned airspace management,” 2018.

Continuous-time model

Last time:

- Model: $\mathbf{x}_{k+1} \ = f(\mathbf{x}_k, \mathbf{u}_k, k),$
- Cost: $J(\mathbf{x}_0) \ = \ h_N(\mathbf{x}_N) + \ \sum_{k=0}^{N-1} g(\mathbf{x}_k, \mathbf{u}_k, k)$

This time:

- Model: $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t),$
- Cost: $J\big(\mathbf{x}(t_0)\big) = h\big(\mathbf{x}(t_f), t_f\big) + \ \int_{t_0}^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau)\, d\tau$

where t_0 and t_f are fixed

Two-person, zero-sum differential games

- What if there is another player (e.g., nature) that interferes with the fulfillment of our objective?

Two-person differential game:

- Model: $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t))$ (*joint* system dynamics),
- Cost: $J\big(\mathbf{x}(t_0)\big) = h\big(\mathbf{x}(0)\big) + \int_{t_0}^{0} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau))\, d\tau$
- Player 1, with control $\mathbf{u}(\tau)$, will attempt to *maximize* J, while Player 2, with control $\mathbf{d}(t)$, will aim to *minimize* J, subject to the joint system dynamics
- $\mathbf{x}(\tau)$ is the *joint* system state

Information pattern

- To fully specify the game, we need to specify the *information pattern*
- "Open-loop" strategies
  - Player 1, with control $\mathbf{u}(\tau)$, declares entire plan
  - Player 2, with control $\mathbf{d}(\tau)$, responds optimally
  - Conservative, unrealistic, but computationally cheap

- "Non-anticipative" strategies
  - Other robot acts based on state and control trajectory up to current time
  - Notation: $\mathbf{d}(\cdot) = \Gamma[\mathbf{u}](\cdot)$
  - Disturbance still has the advantage: it gets to react to the control!

HJI equation

Key idea: apply principle of optimality

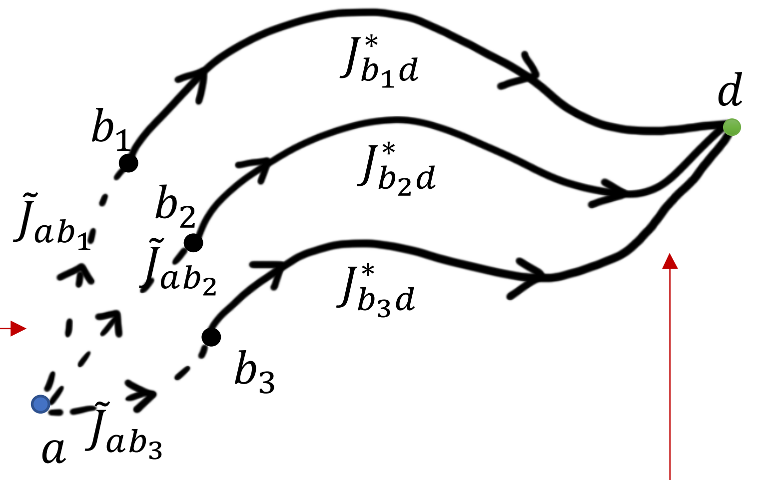
The "truncated" problem is

$$J(\mathbf{x}(t), t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\int_t^0 g\big(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)\big) d\tau + h\big(\mathbf{x}(0)\big) \right]$$

Worst-case disturbance -- does the opposite of the control

HJI equation

- Dynamic programming principle:



$$J(\mathbf{x}(t),t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\int_t^{t+\Delta t} g\big(\mathbf{x}(\tau),\mathbf{u}(\tau),\mathbf{d}(\tau)\big)d\tau + J(\mathbf{x}(t+\Delta t), t+\Delta t) \right]$$

- Approximate integral and Taylor expand $J(\mathbf{x}(t+\Delta t), t+\Delta t)$
- Derive Hamilton-Jacobi-Isaacs partial differential equation (HJI PDE)

HJI equation

- Approximations for small Δt:

$$J(\mathbf{x}(t),t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\underbrace{\int_t^{t+\Delta t} g\big(\mathbf{x}(\tau),\mathbf{u}(\tau),\mathbf{d}(\tau)\big)d\tau}_{g\left(\mathbf{x}(t),\mathbf{u}(t),\mathbf{d}(t)\right)\Delta t} + \underbrace{J(\overbrace{\mathbf{x}(t+\Delta t)}^{\mathbf{x}(t)+\Delta t f(\mathbf{x},\mathbf{u},\mathbf{d})}, t+\Delta t)}_{J(\mathbf{x}(t),t) + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f\left(\mathbf{x}(t),\mathbf{u}(t),\mathbf{d}(t)\right) + \frac{\partial J}{\partial t}\Delta t} \right]$$

  - Omit t dependence…

$$J(\mathbf{x},t) = \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x},\mathbf{u},\mathbf{d})\Delta t + J(\mathbf{x},t) + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f(\mathbf{x},\mathbf{u},\mathbf{d}) + \frac{\partial J}{\partial t}\Delta t \right]$$

    - Assume constant u and d → Optimization over vectors, not functions!
    - Order of max and min reverse (proof given in references)

  - $J(\mathbf{x},t)$ does not depend on $\mathbf{u}$ or $\mathbf{d}$

$$J(\mathbf{x},t) = J(\mathbf{x},t) + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x},\mathbf{u},\mathbf{d})\Delta t + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f(\mathbf{x},\mathbf{u},\mathbf{d}) + \frac{\partial J}{\partial t}\Delta t \right]$$

HJI equation

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HJI equation

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- $J(\mathbf{x},t)$ does not depend on $\mathbf{u}$ or $\mathbf{d}$

$$0 = \frac{\partial J}{\partial t}\cancel{\Delta t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x},\mathbf{u},\mathbf{d})\cancel{\Delta t} + \frac{\partial J}{\partial \mathbf{x}} \cdot \cancel{\Delta t} f(\mathbf{x},\mathbf{u},\mathbf{d}) \right]$$

HJI equation

- Approximations for small Δt:

$$J(\mathbf{x}(t), t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\underbrace{\int_t^{t+\Delta t} g\big(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)\big) d\tau}_{g\left(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t)\right)\Delta t} + \underbrace{J\big(\overbrace{\mathbf{x}(t+\Delta t)}^{\mathbf{x}(t) + \Delta t f(\mathbf{x}, \mathbf{u}, \mathbf{d})}, t+\Delta t\big)}_{J(\mathbf{x}(t), t) + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f\left(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t)\right) + \frac{\partial J}{\partial t}\Delta t} \right]$$

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HJI equation

The end result is the Hamilton-Jacobi-Isaacs (HJI) equation

$$\boxed{0 = \frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right]}$$

with boundary condition

$$\boxed{J(\mathbf{x}, 0) = h(\mathbf{x})}$$

- Given the cost-to-go function, the optimal control for Player 1 is

$$\mathbf{u}^*(\mathbf{x}, t) = \arg\max_{\mathbf{u}} \min_{\mathbf{d}} g(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

HJB equation

In case there is no disturbance, end result is the Hamilton-Jacobi-Bellman (HJB) equation

Without a disturbance, $\mathbf{u}$ is usually selected to minimize cost

$$\boxed{0 = \frac{\partial J}{\partial t} + \min_{\mathbf{u}} \left[g(\mathbf{x}, \mathbf{u}, t) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, t) \right]}$$

with boundary condition $\boxed{J(\mathbf{x}(t_f), t_f) = h(\mathbf{x}(t_f), t_f)}$

- Given the cost-to-go function, the optimal control is

$$\mathbf{u}^*(\mathbf{x}, t) = \arg\min_{\mathbf{u}} g(\mathbf{x}, \mathbf{u}, t) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, t)$$

Continuous LQR

Continuous LQR: select control inputs to minimize

$$J(\mathbf{x}_0) = \frac{1}{2}\mathbf{x}(t_f)' H \mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}(t)' Q(t) \mathbf{x}(t) + \mathbf{u}(t)' R(t) \mathbf{u}(t)\right] dt$$

subject to the dynamics

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

Assumptions:

- $H = H' \geq 0,\ Q(t) = Q(t)' \geq 0, R(t) = R(t)' > 0$
- t_0 and t_f specified
- $\mathbf{x}(t)$ and $\mathbf{u}(t)$ unconstrained

Continuous LQR

- The HJB equation reduces to a set of differential equation (the Riccati equation):

$$-\dot{K}(t) = Q(t) - K(t)B(t)R(t)^{-1}B(t)'K(t) + K(t)A(t) + A(t)'K(t)$$

- Riccati equation is integrated backwards, with boundary condition $K(t_f) = H$
- Once we find $K(t)$, the control policy is

$$\mathbf{u}^*(t) = -R(t)^{-1}B(t)'K(t)\mathbf{x}(t)$$

- Analogously to the discrete case, under some additional assumptions, $K(t) \rightarrow$ constant in the infinite horizon setting
- LQG generalization is presented in the lecture notes

Applications of differential games

- Pursuit-evasion games
  - homicidal chauffeur problem
  - the lady in the lake
- Reachability analysis
- And many more (e.g., in economics)

Applications of differential games

- Pursuit-evasion games
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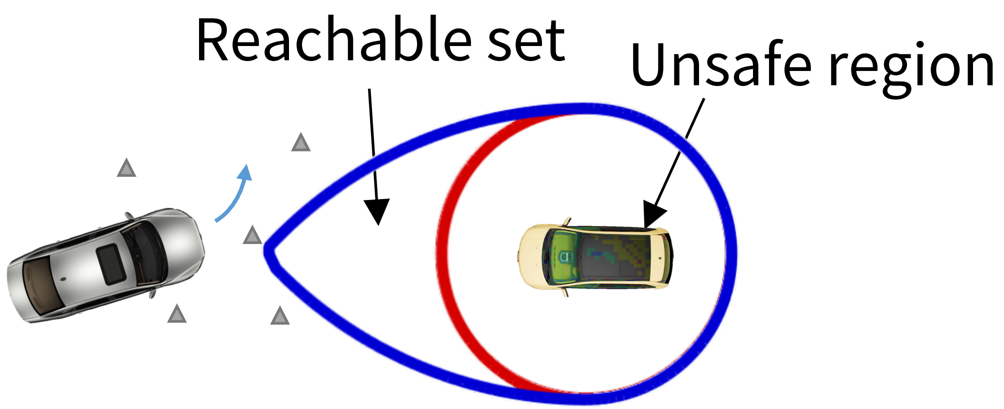
Reachability analysis: avoidance

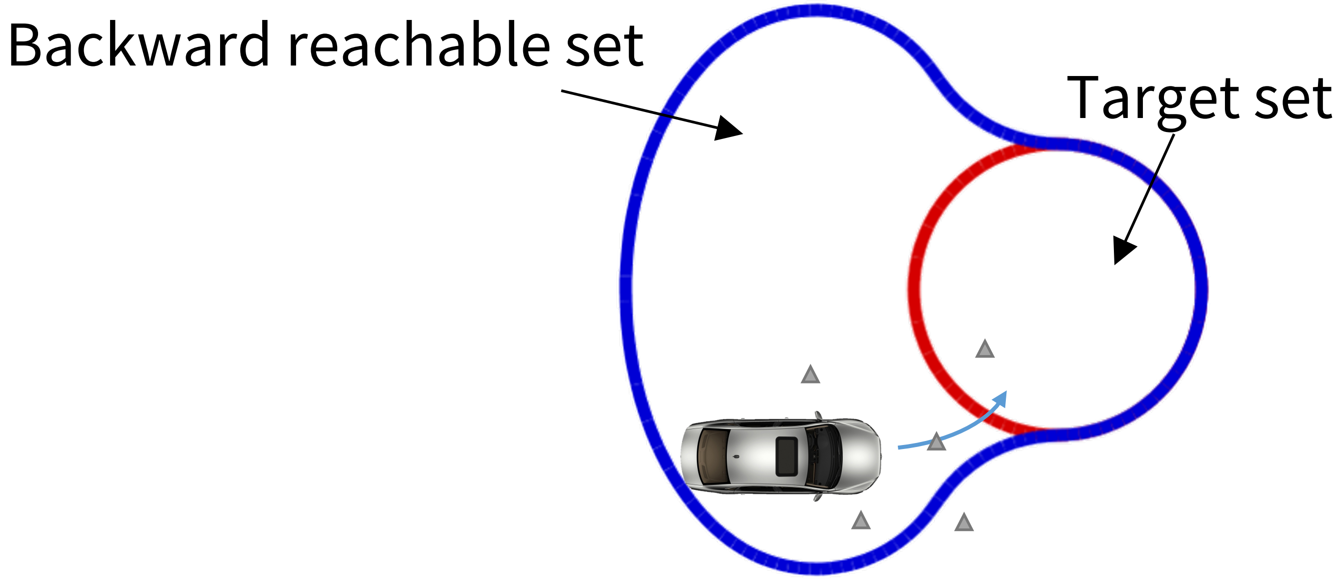


Assumptions:
- Model of robot
- Unsafe region: e.g., obstacle

Control policy

Backward reachable set
(States leading to danger)

Reachability analysis: goal reaching



- Model of robot
- Goal region

Control policy

Backward reachable set
(States leading to goal)

Reachability analysis

- Model of robot
- Unsafe region

- $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d})$
- $\mathcal{T}$

- Model of robot
- Goal region

- $\mathcal{A}(t) = \{\bar{\mathbf{x}}: \exists \Gamma[\boldsymbol{u}](\cdot), \forall \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$

Backward reachable set (states leading to danger)

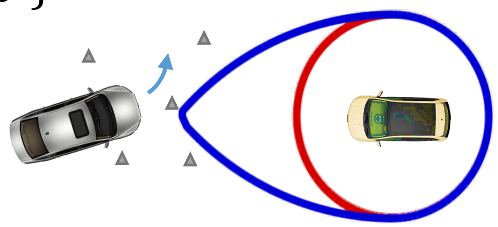
Control policy

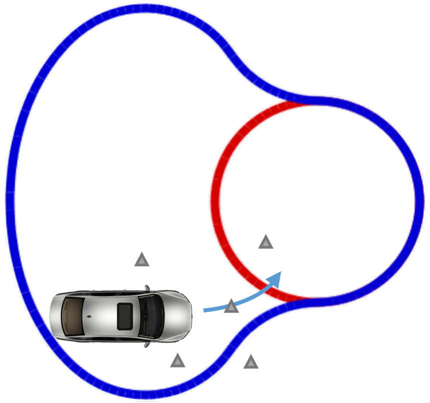
- $\mathbf{u}^*(\mathbf{x}, t)$

Control policy

Backward reachable set (states leading to goal)

- $\mathcal{R}(t) = \{\bar{x}: \forall \Gamma[\mathbf{u}](\cdot), \exists \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$





Reachability analysis

States at time t satisfying the following:

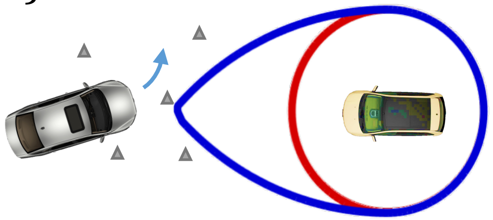
there exists a disturbance such that for all control, system enters target set at $t = 0$

- $\mathcal{A}(t) = \{\bar{\mathbf{x}}: \exists \Gamma[\mathbf{u}](\cdot), \forall \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$

- Model of robot
- Unsafe region

Backward reachable set (States leading to danger)

Control policy



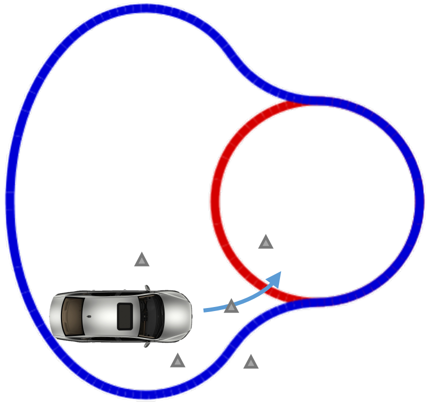
- $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d})$
- $\mathcal{T}$

- $\mathbf{u}^*(\mathbf{x}, t)$

- Model of robot
- Goal region

Control policy

Backward reachable set (States leading to goal)



- $\mathcal{R}(t) = \{\bar{\mathbf{x}}: \forall \Gamma[\mathbf{u}](\cdot), \exists \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), x(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$

States at time t satisfying the following:

for all disturbances, there exists a control such that system enters target set at $t = 0$

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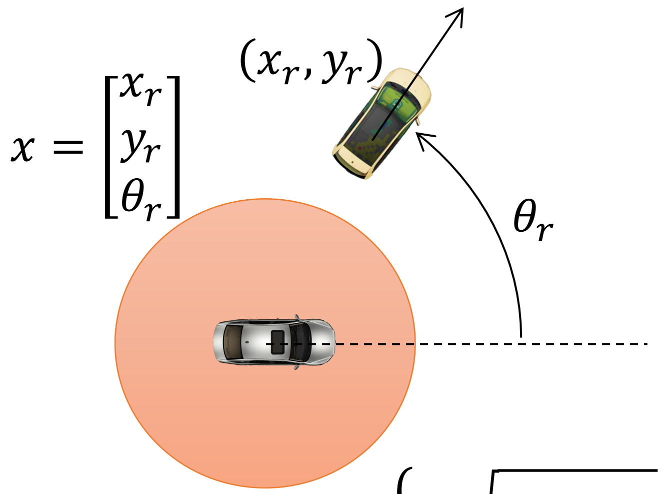
From HJI to reachability analysis

- Computation of the BRS entails solving a differential *game of kind*, where the outcome is Boolean (the system either reaches the target set or not)
- One can "encode" this Boolean outcome by (1) removing the running cost and (2) picking the final cost intelligently

From HJI to reachability analysis

- Hamilton-Jacobi Equation
  - $0 = \frac{\partial J}{\partial t} + \max_{\mathbf{d}} \min_{\mathbf{u}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right],\ J(\mathbf{x}, 0) = h(\mathbf{x})$

- Remove running cost
  - $0 = \frac{\partial J}{\partial t} + \max_{\mathbf{d}} \min_{\mathbf{u}} \left[\frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right],\ J(\mathbf{x}, 0) = h(\mathbf{x})$



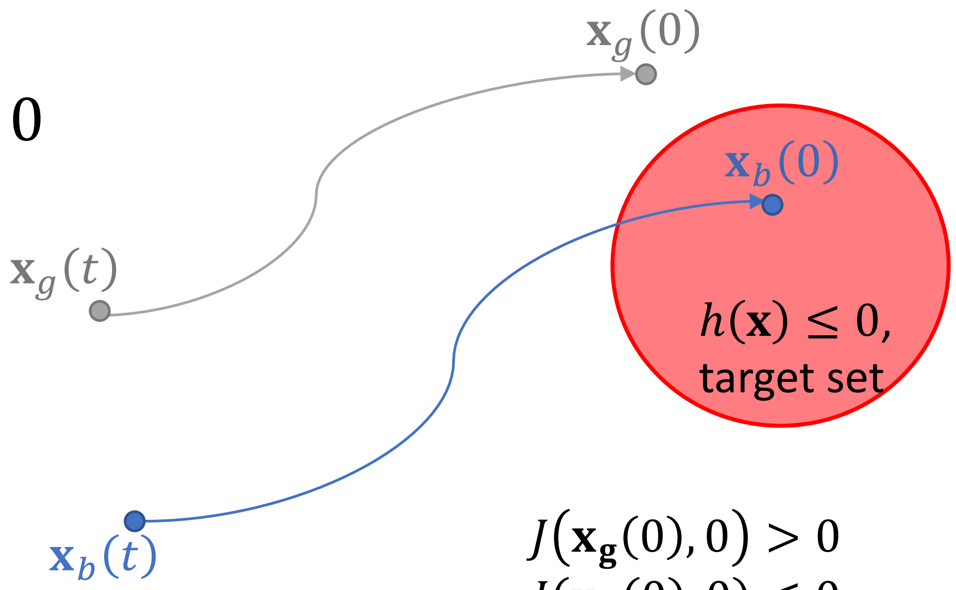
$\mathcal{T} = \left\{ x\colon \sqrt{x_r^2 + y_r^2} \leq R \right\} \subseteq \mathbb{R}^3$

- Pick final cost such that
  - $\mathbf{x} \in \mathcal{T} \Leftrightarrow h(\mathbf{x}) \leq 0$
  - Example: If $\mathcal{T} = \left\{ \mathbf{x}\colon \sqrt{x_r^2 + y_r^2} \leq R \right\} \subseteq \mathbb{R}^3$, we can pick $h(x_r, y_r, \theta_r) = \sqrt{x_r^2 + y_r^2} - R$

Pick Final Cost

- Pick final cost such that
  - $x \in \mathcal{T} \Leftrightarrow h(\mathbf{x}) \leq 0$
  - If $\mathcal{T} = \left\{x: \sqrt{x_r^2 + y_r^2} \leq R\right\} \subseteq \mathbb{R}^3$, we can pick $h(x_r, y_r, \theta_r) = \sqrt{x_r^2 + y_r^2} - R$

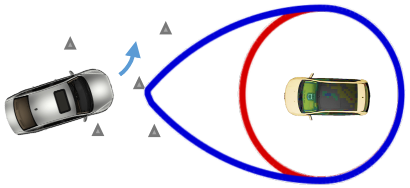
- Why is this correct?
  - Final state $\mathbf{x}(0)$ is in $\mathcal{T}$ if and only if $h\big(\mathbf{x}(0)\big) \leq 0$
  - To avoid $\mathcal{T}$, control should maximize $h\big(\mathbf{x}(0)\big)$
    - Worst-case disturbance would minimize
  - $J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} h\big(\mathbf{x}(0)\big)$



$J\big(\mathbf{x_g}(0), 0\big) > 0$
$J\big(\mathbf{x}_b(0), 0\big) \leq 0$

Reaching vs. Avoiding

- Avoiding danger



  - BRS definition

$$\mathcal{A}(t) = \{\bar{\mathbf{x}}: \exists\Gamma[\mathbf{u}](\cdot), \forall\mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$$

  - Value function

$$J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} h\big(\mathbf{x}(0)\big)$$

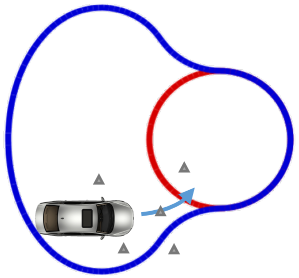
  - HJI

$$\frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}}\right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d})\right] = 0$$

  - Optimal control

$$\mathbf{u}^* = \arg\max_{\mathbf{u}} \min_{\mathbf{d}} \left(\frac{\partial J}{\partial \mathbf{x}}\right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

- Reaching a goal



  - BRS definition

$$\mathcal{R}(t) = \{\bar{\mathbf{x}}: \forall\Gamma[\mathbf{u}](\cdot), \exists\mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$$

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$$\frac{\partial J}{\partial t} + \min_{\mathbf{u}} \max_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}}\right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d})\right] = 0$$

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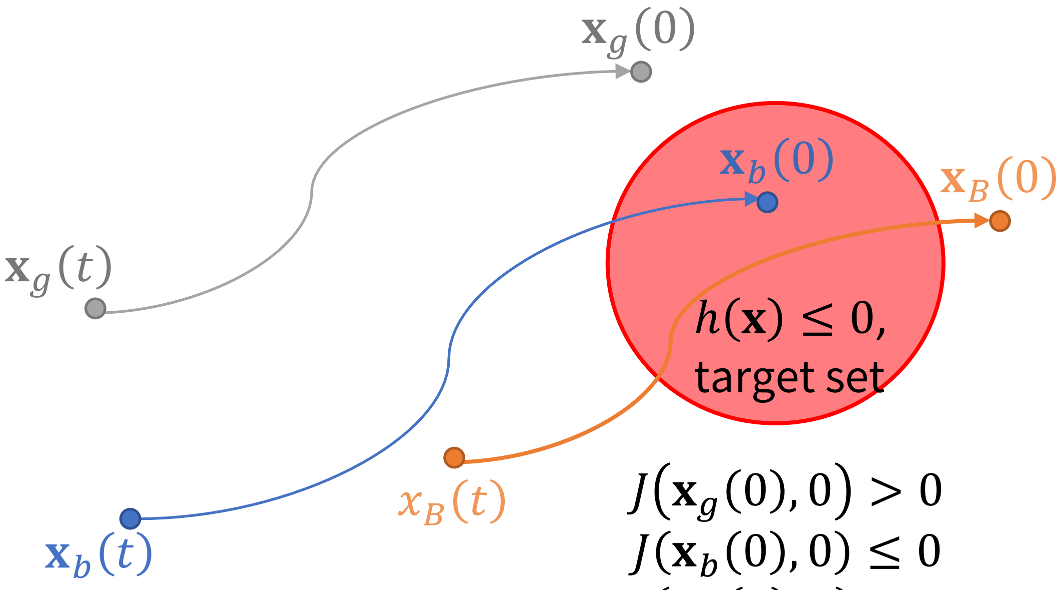
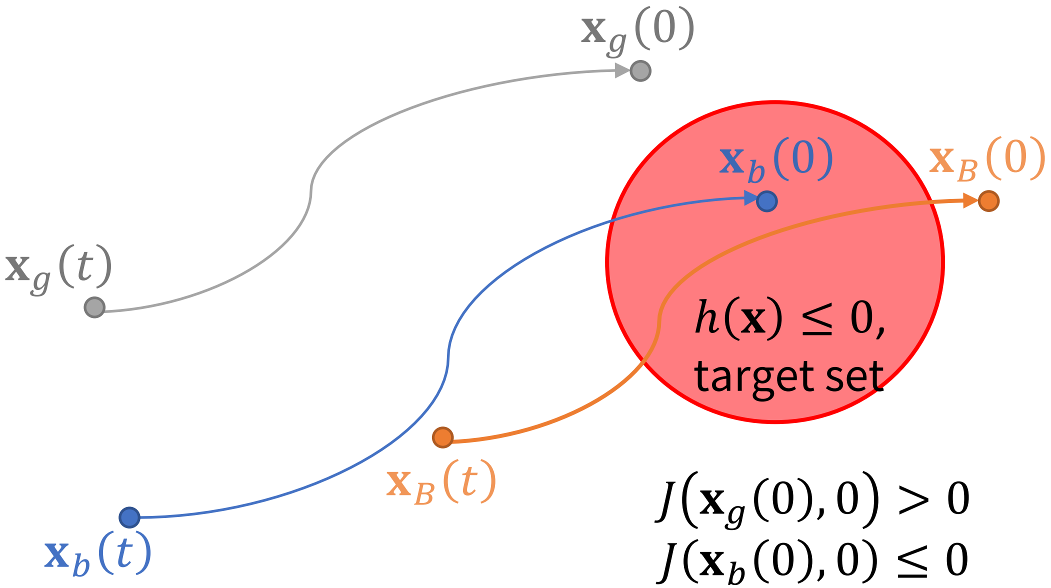
$$\mathbf{u}^* = \arg\min_{\mathbf{u}} \max_{\mathbf{d}} \left(\frac{\partial J}{\partial \mathbf{x}}\right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

“Sets” vs. “Tubes”

- Backward reachable set (BRS)
  - Only final time matters
  - Initial states that pass through target are not necessarily in BRS
  - Not ideal for safety
- Backward reachable tube (BRT)
  - Keep track of entire time duration
  - Initial states that pass through target are in BRT
  - Used to make safety guarantees



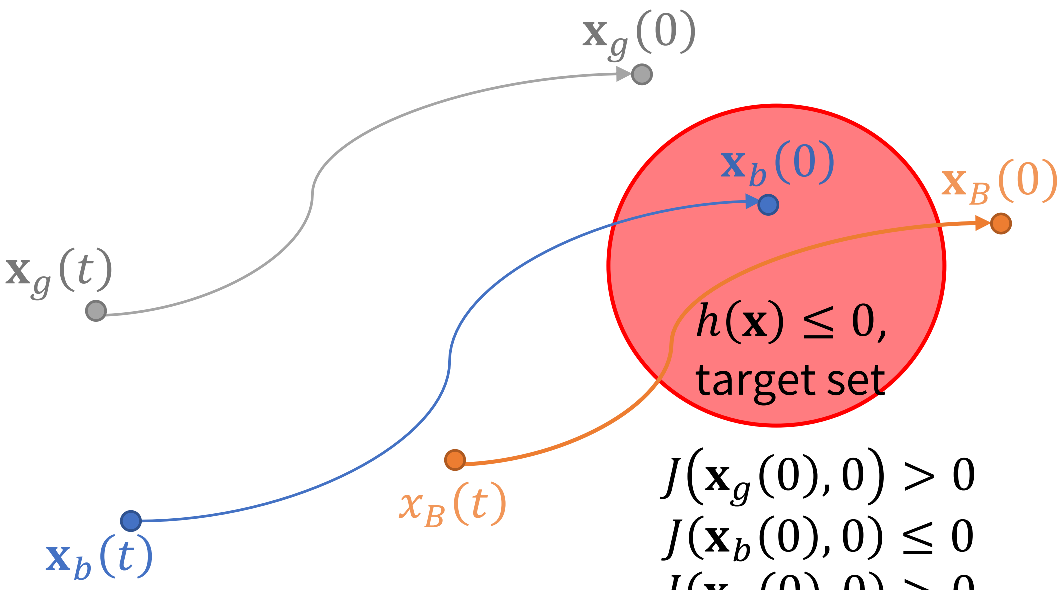
$$J\big(\mathbf{x}_g(0), 0\big) > 0$$
$$J(\mathbf{x}_b(0), 0) \leq 0$$
$$J(\mathbf{x}_B(0), 0) > 0$$



$$J\big(\mathbf{x}_g(0), 0\big) > 0$$
$$J(\mathbf{x}_b(0), 0) \leq 0$$
$$\underline{J(\mathbf{x}_B(0), 0) \leq 0}$$

"Sets" vs. "Tubes"

- Backward reachable set (BRS)

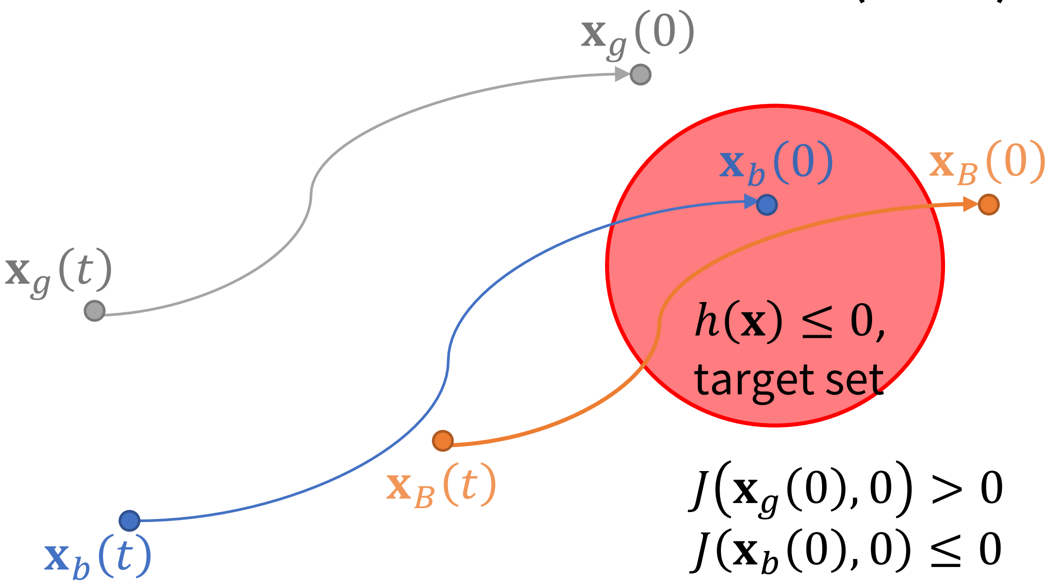


$$J(\mathbf{x}_g(0), 0) > 0$$
$$J(\mathbf{x}_b(0), 0) \le 0$$
$$J(\mathbf{x}_B(0), 0) > 0$$

- Value function definition
  - $J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} h\big(\mathbf{x}(0)\big)$
- Value function obtained from

$$\frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}} \right)' f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right] = 0$$

- Backward reachable tube (BRT)



$$J(\mathbf{x}_g(0), 0) > 0$$
$$J(\mathbf{x}_b(0), 0) \le 0$$
$$\underline{J(\mathbf{x}_B(0), 0) \le 0}$$

- Value function definition
  - $J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} \min_{\tau \in [t,0]} h\big(\mathbf{x}(\tau)\big)$
- Value function obtained from

$$\min \left\{ \frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}} \right)^\top f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right], h(\mathbf{x}) - J(\mathbf{x}, t) \right\} = 0$$

Computational aspects

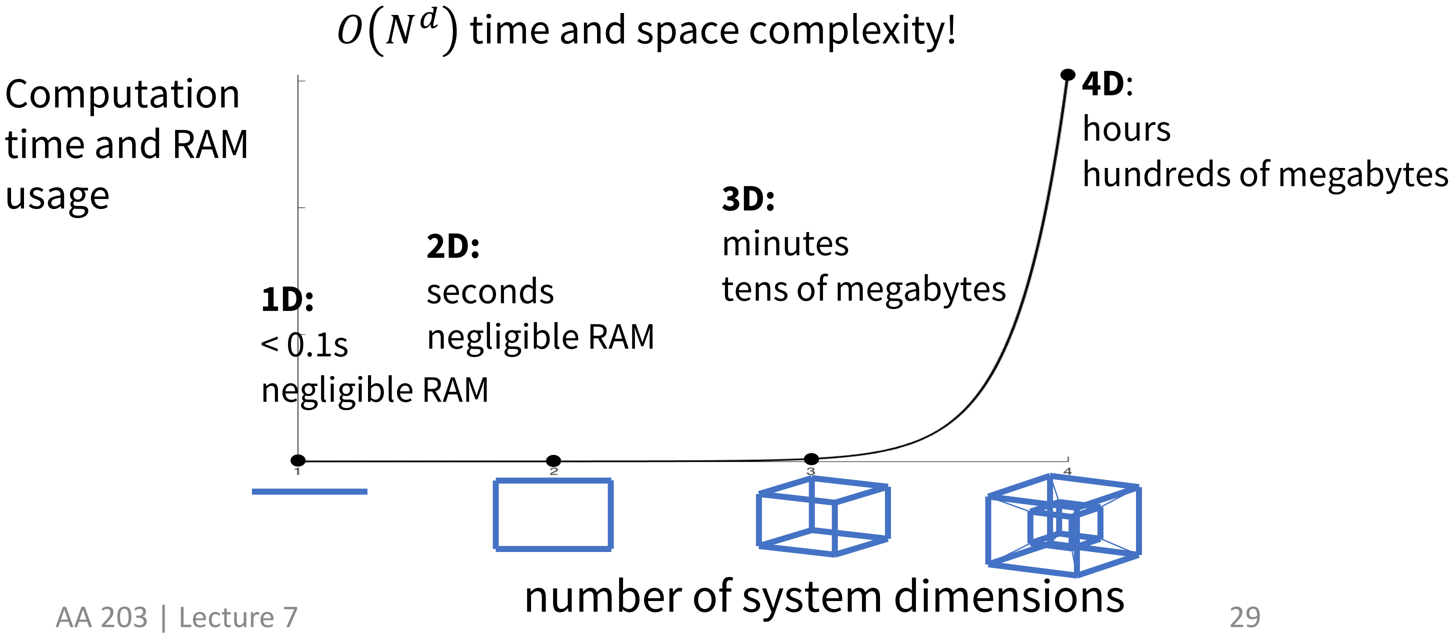
- Computational complexity
  - $J(\mathbf{x}, t)$ is computed on an $(n+1)$-dimensional grid
  - Currently, $n \leq 5$ is possible. GPU acceleration under-way
  - Dimensionality reduction methods sometimes help

- Related approaches
  - Sacrifice global optimality
  - Give up guarantees

  - Sampling-based methods
  - Reinforcement learning

6D:
intractable!

5D:
days
gigabytes



Numerical toolboxes

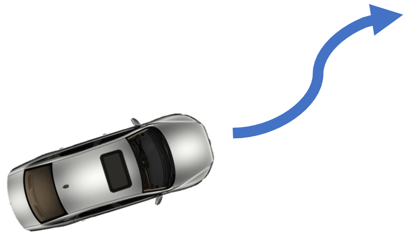
- helperOC Matlab toolbox
  - https://github.com/HJReachability/helperOC.git
  - Reachability wrapper around the level set toolbox
  - Requires level set toolbox
    - Hamilton-Jacobi PDE solver by Ian Mitchell, UBC
    - https://bitbucket.org/ian_mitchell/toolboxls

- C++ and CUDA version in development, beta also available
  - C++: 5+ times faster than Matlab
  - CUDA: Up to 100 times faster than Matlab
  - https://github.com/HJReachability/beacls

Example – waypoint reaching with Dubins Car

Dubins Car Model

$$\begin{cases} \dot{x} = v\cos\theta + \boldsymbol{d_x} \\ \dot{y} = v\sin\theta \\ \dot{\theta} = k\,\boldsymbol{u} \end{cases}$$



Control: $\boldsymbol{u}$

Disturbance: $\boldsymbol{d_x}$

- Target set:

$$\mathcal{T} = \{(x, y, \theta) \in \mathbb{R}^3 : h(x, y, \theta) \coloneqq \max\,[(x - x_{max}), (y - y_{max}), (\theta - \theta_{max}), (x_{min} - x), (y_{min} - y), (\theta_{min} - \theta)] \le 0\}$$

- HJI equation:

$$\frac{\partial J}{\partial t}(x, y, \theta, t) + \min_{|u| \le u_{max}} \max_{|d_x| \le d^x_{max}} \nabla J(x, y, \theta, t)' f(x, y, \theta, u, d) = 0$$

- Optimal quantities:

$$u^*(x, y, \theta, t) = \arg \min_{|u| \le u_{max}} \max_{|d_x| \le d^x_{max}} \nabla J(x, y, \theta, t)^\top f(x, y, \theta, u, d)$$

$$d^*(x, y, \theta, t) = \arg \max_{|d_x| \le d^x_{max}} \nabla J(x, y, \theta, t)^\top f(x, y, \theta, u^*, d)$$

$$u = -u_{max}\,\mathrm{sign}\left(\frac{\partial J}{\partial \theta}\right)$$

$$d_x = d^x_{max}\,\mathrm{sign}\left(\frac{\partial J}{\partial x}\right)$$

Next time

- Direct methods for optimal control