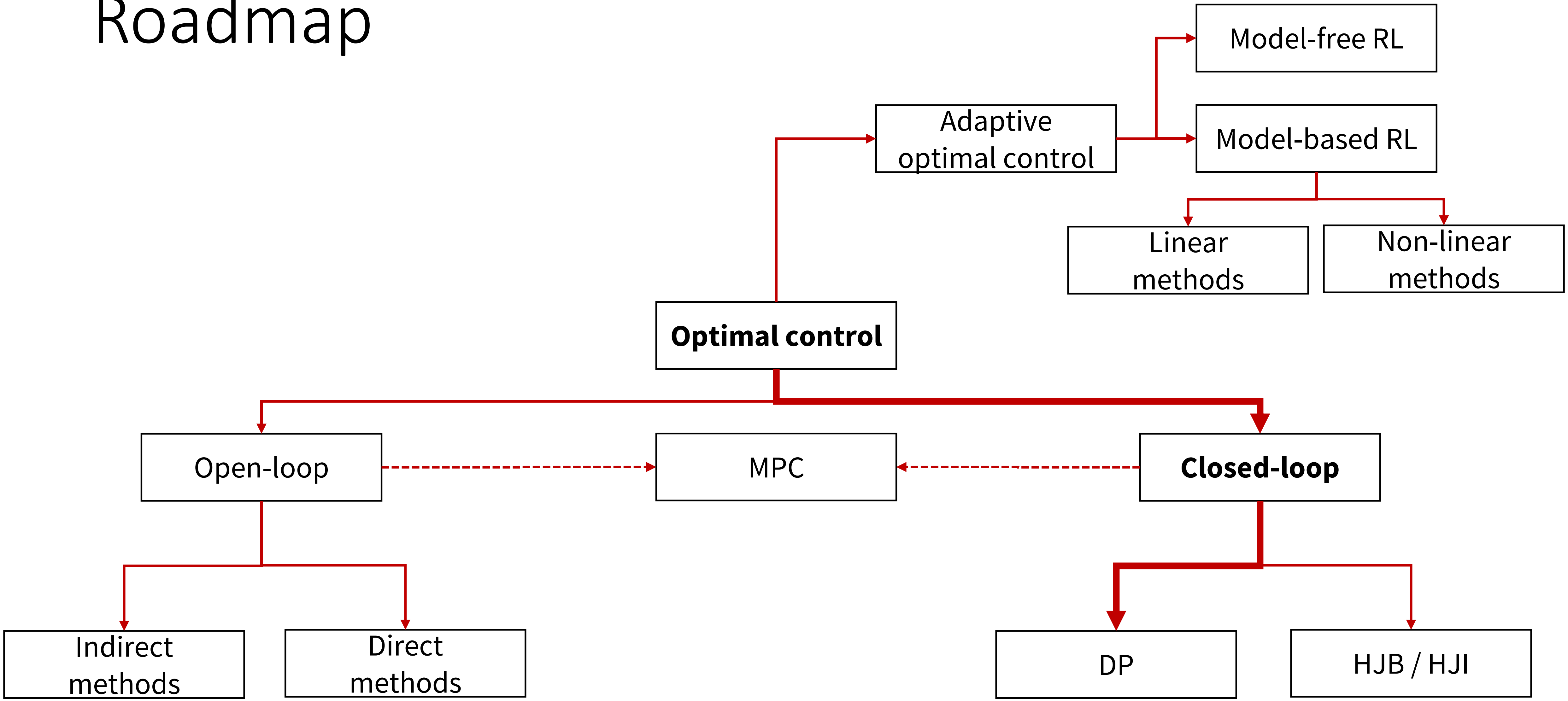
AA203
Optimal and Learning-based Control

Stochastic DP, value iteration, policy iteration

Roadmap


Model-free RL
Adaptive optimal control
Model-based RL
Linear methods
Non-linear methods
Optimal control
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI

Today's lecture

- Aim
  - Provide intro to stochastic DP

References:

- Bertsekas, *Reinforcement Learning and Optimal Control*

Stochastic optimal control problem (MDPs)

- System: $\boldsymbol{x}_{k+1} = f_k(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k), k = 0, \ldots, N-1$
- Control constraints: $\boldsymbol{u}_k \in U(\boldsymbol{x}_k)$
- Probability distribution: $P_k(\cdot \mid \boldsymbol{x}_k, \boldsymbol{u}_k)$ of $\boldsymbol{w}_k$
- Policies: $\pi = \{\pi_0 \ldots, \pi_{N-1}\}$, where $\boldsymbol{u}_k = \pi_k(\boldsymbol{x}_k)$
- Expected Cost:

$$J_\pi(\boldsymbol{x}_0) = E_{\boldsymbol{w}_k, k=0,\ldots,N-1}\left[g_N(\boldsymbol{x}_N) + \sum_{k=0}^{N-1} g_k(\boldsymbol{x}_k, \pi_k(\boldsymbol{x}_k), \boldsymbol{w}_k)\right]$$

- Stochastic optimal control problem

$$J^*(x_0) = \min_\pi J_\pi(\boldsymbol{x}_0)$$

Key points

- Discrete-time model
- Markovian model
- Objective: find optimal closed-loop
- Additive cost (central assumption)
- Risk-neutral formulation

Other communities use different notation: Powell, W. B. *AI, OR and control theory: A Rosetta Stone for stochastic optimization*. Princeton University, 2012.

http://castlelab.princeton.edu/Papers/AIOR_July2012.pdf

Principle of optimality

- Let $\pi^* = \{\pi_0^*, \pi_1^*, \dots, \pi_{N-1}^*\}$ be an optimal policy
- Consider tail subproblem

$$E\left[g_N(\boldsymbol{x}_N) + \sum_{k=i}^{N-1} g_k(\boldsymbol{x}_k, \pi_k(\boldsymbol{x}_k), \boldsymbol{w}_k)\right]$$

and the tail policy $\{\pi_i^*, \dots, \pi_{N-1}^*\}$

Principle of optimality: The tail policy is optimal for the tail subproblem

The DP algorithm (stochastic case)

Intuition

- DP first solves ALL tail subproblems at the final stage
- At generic step, it solves ALL tail subproblems of a given time length, using solution of tail subproblems of shorter length

The DP algorithm (stochastic case)

The DP algorithm

- Start with

$$J_N(\boldsymbol{x}_N) = g_N(\boldsymbol{x}_N)$$

and go backwards using

$$J_k(x_k) = \min_{\boldsymbol{u}_k \in U(\boldsymbol{x}_k)} E_{w_k}\left[g_k(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k) + J_{k+1}\left(f(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k)\right)\right]$$

for $k = 0, 1, \ldots, N-1$

- Then $J^*(\boldsymbol{x}_0) = J_0(\boldsymbol{x}_0)$ and optimal policy is constructed by setting $\pi_k^*(\boldsymbol{x}_k) = \boldsymbol{u}_k^*$

Example: Inventory Control Problem (1/3)

- Stock available $x_k \in \mathbb{N}$, inventory $u_k \in \mathbb{N}$, and demand $w_k \in \mathbb{N}$
- Dynamics: $x_{k+1} = \max(0, x_k + u_k - w_k)$
- Constraints: $x_k + u_k \leq 2$
- Probabilistic structure: $p(w_k = 0) = 0.1$, $p(w_k = 1) = 0.7$, and $p(w_k = 2) = 0.2$
- Cost

$$E\left[\underbrace{0}_{g_3(x_3)} + \sum_{k=0}^{2} \underbrace{\left(u_k + (x_k + u_k - w_k)^2 \right)}_{g_k(x_k, u_k, w_k)} \right]$$

Example: Inventory Control Problem (2/3)

- Algorithm takes form

$$J_k(x_k) = \min_{0 \leq u_k \leq 2 - x_k} E_{w_k}[u_k + (x_k + u_k - w_k)^2$$
$$+ J_{k+1}(\max(0, \; x_k + u_k - w_k))]$$

for $k = 0,1,2$

- For example

$$J_2(0) = \min_{u_2 = 0,1,2} E_{w_2}[u_2 + (u_2 - w_2)^2] =$$
$$\min_{u_2 = 0,1,2} u_2 + 0.1(u_2)^2 + 0.7(u_2 - 1)^2 + 0.2(u_2 - 2)^2$$

which yields $J_2(0) = 1.3$, and $\pi_2^*(0) = 1$

Example: Inventory Control Problem (3/3)

Final solution:

- $J_0(0) \ = \ 3.7$,
- $J_0(1) \ = \ 2.7$, and
- $J_0(2) \ = \ 2.818$

Problems with imperfect state information

- Now the controller, instead of having perfect knowledge of the state, has access to observations $\boldsymbol{z}_k$ of the form
$$\boldsymbol{z}_0 = h_0(\boldsymbol{x}_0, \boldsymbol{v}_0), \qquad \boldsymbol{z}_k = h_k(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{v}_k), \qquad k = 1,2, \ldots, N-1$$

- The random observation disturbance is characterized by a given probability distribution
$$P_{\boldsymbol{v}_k}(\cdot \mid \boldsymbol{x}_k, \ldots, \boldsymbol{x}_0, \boldsymbol{u}_{k-1}, \ldots, \boldsymbol{u}_0, \boldsymbol{w}_{k-1}, \ldots, \boldsymbol{w}_0, \boldsymbol{v}_{k-1}, \ldots, \boldsymbol{v}_0)$$

- The initial state $\boldsymbol{x}_0$ is also random and characterized by given $P_{\boldsymbol{x}_0}$

Control policies

- Define the *information vector* as

$$\boldsymbol{I}_k = (\boldsymbol{z}_0, \dots, \boldsymbol{z}_k, \boldsymbol{u}_0, \dots, \boldsymbol{u}_{k-1}), \qquad \boldsymbol{I}_0 = \boldsymbol{z}_0$$

- Focus is now on *admissible* policies $\pi_k(\boldsymbol{I}_k) \in U_k$

- We want then to find an admissible policy that minimizes

$$J_\pi = E_{\substack{\boldsymbol{x}_0, \boldsymbol{w}_k, \boldsymbol{v}_k \\ k=0,\dots,N-1}} \left[g_N(\boldsymbol{x}_N) + \sum_{k=0}^{N-1} g_k(\boldsymbol{x}_k, \pi_k(\boldsymbol{I}_k), \boldsymbol{w}_k) \right]$$

Solution strategies

1. Reformulation as a perfect state information problem (main idea: make the information vector the state of the system)
   - Main drawback: state has *expanding* dimension!
2. Reason in terms of sufficient statistics, i.e., quantities that ideally are smaller than $\boldsymbol{I}_k$ and yet summarize all its essential content
   - Main example: conditional probability distribution $P_{\boldsymbol{x}_k|\boldsymbol{I}_k}$ (assuming $\boldsymbol{v}_k \sim P_{\boldsymbol{v}_k}(\cdot\,|\boldsymbol{x}_{k-1}, \boldsymbol{u}_{k-1}, \boldsymbol{w}_{k-1}))$
   - Condition probability distribution leads to a decomposition of the optimal controller in two parts:

$$\pi_k\left(P_{\boldsymbol{x}_k|\boldsymbol{I}_k}\right)$$

Actuator

Estimator

LQG

Discrete LQG: find admissible control policy that minimizes

$$E\left[\boldsymbol{x}_N' Q \boldsymbol{x}_N + \sum_{k=0}^{N-1} (\boldsymbol{x}_k' Q_k \boldsymbol{x}_k + \boldsymbol{u}_k' R_k \boldsymbol{u}_k)\right]$$

subject to

- the dynamics $\boldsymbol{x}_{k+1} = A_k \boldsymbol{x}_k + B_k \boldsymbol{u}_k + \boldsymbol{w}_k$
- the measurement equation $\boldsymbol{z}_k = C_k \boldsymbol{x}_k + \boldsymbol{v}_k$

and with $\boldsymbol{x}_0, \{\boldsymbol{w}_k\}, \{\boldsymbol{v}_k\}$, independent and Gaussian vectors (and in addition $\{\boldsymbol{w}_k\}, \{\boldsymbol{v}_k\}$ zero mean)

LQG – solution

Let

- $M_k := E[\boldsymbol{w}_k \boldsymbol{w}_k']$
- $N_k := E[\boldsymbol{v}_k \boldsymbol{v}_k']$
- $S := E[(\boldsymbol{x}_0 - E[\boldsymbol{x}_0])(\boldsymbol{x}_0 - E[\boldsymbol{x}_0])']$

LQG – solution

The optimal controller is $\boldsymbol{u}_k = F_k \hat{\boldsymbol{x}}_k$, where

- F_k is the LQR gain
- $\hat{\boldsymbol{x}}_{k+1} = A_k \hat{\boldsymbol{x}}_k + B_k \boldsymbol{u}_k + \Sigma_{k+1|k+1} C'_{k+1} N_{k+1}^{-1} (\boldsymbol{z}_{k+1} - C_{k+1}(A_k \hat{\boldsymbol{x}}_k + B_k \boldsymbol{u}_k))$
- $\hat{\boldsymbol{x}}_0 = E[\boldsymbol{x}_0] + \Sigma_{0|0} C'_0 N_0^{-1} (\boldsymbol{z}_0 - C_0 E[\boldsymbol{x}_0])$
- and matrices $\Sigma_{k|k}$ are *precomputable* (given in the lecture notes)

- Key property: the estimation portion of the optimal controller is an optimal solution of the problem of estimating the state $\boldsymbol{x}_k$ assuming no control takes place, while the actuator portion is an optimal solution of the control problem assuming perfect state information → separation principle

Infinite Horizon MDPs

State: $x \in \mathcal{X}$ (often $s \in \mathcal{S}$)

Action: $u \in \mathcal{U}$ (often $a \in \mathcal{A}$)

Transition Function: $T(x_t \,|x_{t-1}\,, u_{t-1}) = p(x_t | x_{t-1}, u_{t-1})$

Reward Function: $r_t = \; R(x_t, u_t)$

Discount Factor: γ

MDP: $\mathcal{M} = (\mathcal{X}, \mathcal{U}, T, R, \gamma)$

Infinite Horizon MDPs

MDP: $\mathcal{M} = (\mathcal{X}, \mathcal{U}, T, R, \gamma)$

Stationary policy: $u_t = \pi(x_t)$

Goal: Choose policy that **maximizes cumulative reward**

$$\pi^* = \arg\max_{\pi} E\left[\sum_{t\geq 0} \gamma^t R\big(x_t, \pi(x_t)\big)\right]$$

Infinite Horizon MDPs

- The optimal reward $V^*(x)$ satisfies Bellman's equation

$$V^*(x) = \max_u \left(R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u) \, V^*(x') \right)$$

- For any stationary policy π, the reward $V_\pi(x)$ is the unique solution to the equation

$$V_\pi(x) = R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u) \, V_\pi(x')$$

Solving infinite-horizon MDPs

If you know the model, use DP-ideas

- Value Iteration / Policy Iteration

RL: Learning from interaction

- Model-Based
- Model-free
  - Value based
  - Policy based

Value Iteration

- Initialize $V_0 \ (x) = 0$ for all states x
- Loop until finite horizon / convergence:

$$V_{k+1}(x) = \max_u \left(R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u)\, V_k \ (x') \right)$$

Q functions

$$V^*(x) = \max_u \left(R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u)\, V^*(x') \right)$$

$$V^*(x) = \max_u Q^*(x,u)$$

- VI for Q functions

$$Q_{k+1}(x,u) = R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u) \max_u Q_k(x',u)$$

Policy Iteration

Suppose we have a policy $\pi_k(x)$

We can use VI to compute $V_{\pi_k}(x)$

Define $\pi_{k+1}(x) = \arg\max_u \left(R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u)\, V_{\pi_k}(x') \right)$

Proposition: $V_{\pi_{k+1}}(x) \geq V_{\pi_k}(x)\ \forall\ x \in \mathcal{X}$

Inequality is strict if π_k is suboptimal

Use this procedure to iteratively improve policy until convergence

Recap

- Value Iteration
  - Estimate optimal value function
  - Compute optimal policy from optimal value function
- Policy Iteration
  - Start with random policy
  - Iteratively improve it until convergence to optimal policy
- Require **model of MDP** to work!

Next time

- Iterative LQR/ LQG, DDP

$$\delta \boldsymbol{x}_k := \boldsymbol{x}_k - \bar{\boldsymbol{x}}_k \text{ and } \delta \boldsymbol{u}_k := \boldsymbol{u}_k - \bar{\boldsymbol{u}}_k$$