AA203
Optimal and Learning-based Control

Proof of NOC, Pontryagin's minimum principle

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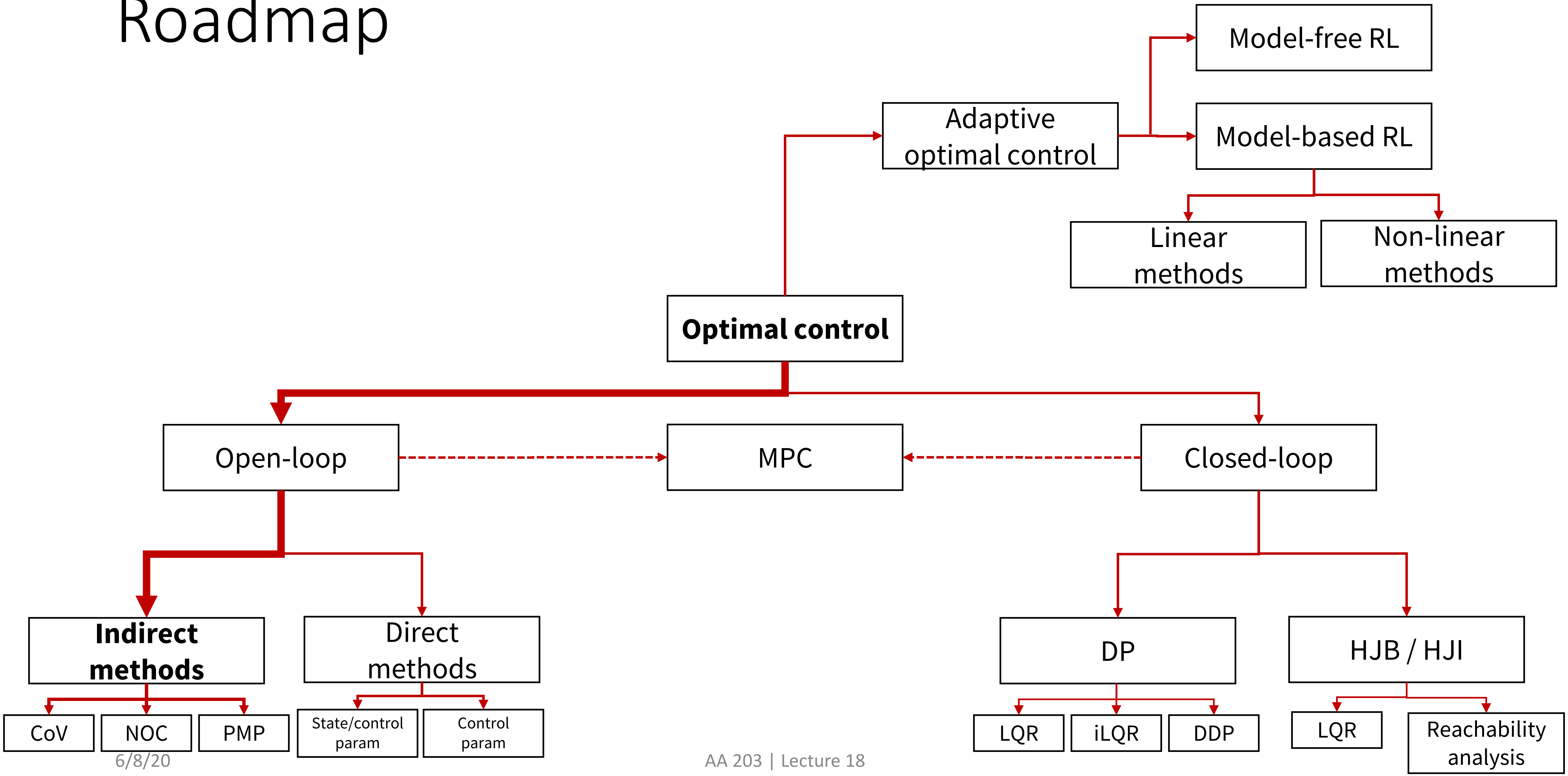
Roadmap


Model-free RL
Adaptive optimal control
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Open-loop
MPC
Closed-loop
Indirect methods
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DP
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DDP
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Reachability analysis

Necessary conditions for optimal control
(with unbounded controls)

We want to prove that, with unbounded controls, the necessary optimality conditions are (H is the Hamiltonian)

$$\left.\begin{aligned}\dot{\mathbf{x}}^*(t) &= \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\\ \mathbf{0} &= \frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\end{aligned}\right\} \text{for all } t \in [t_0, t_f]$$

along with the boundary conditions:

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]' \delta \mathbf{x}_f + \left[H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0$$

Proof of NOC

- For simplicity, assume that the terminal penalty is equal to zero, and that t_f and $\mathbf{x}(t_f)$ are fixed and given
- Consider the augmented cost function
  $$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) := g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}(t)'[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)]$$
  where the $\{p_i(t)\}$'s are Lagrange multipliers
- Note that we have simply added zero to the cost function!
- The augmented cost function is then
$$J_a(\mathbf{u}) = \int_{t_0}^{t_f} g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) \; dt$$

Proof of NOC

On an extremal, by applying the fundamental theorem of the CoV

By the CoV theorem

$$
\begin{aligned}
0 = \delta J_a(\mathbf{u}) = \int_{t_0}^{t_f} \Bigg(\Bigg[& \overbrace{\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)}^{= \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)' \mathbf{p}^*(t)} - \overbrace{\frac{d}{dt} \frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)}^{= -\frac{d}{dt}(-\mathbf{p}^*(t))} \Bigg]' \delta \mathbf{x}(t) \\
& + \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]' \delta \mathbf{u}(t) + \underbrace{\left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \right]'}_{= \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)} \delta \mathbf{p}(t) \Bigg) dt
\end{aligned}
$$

Proof of NOC

Considering each term in sequence,

- $\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t) = \mathbf{0}$, on an extremal
- The Lagrange multipliers are arbitrary, so we can select them to make the coefficient of $\delta\mathbf{x}(t)$ equal to zero, that is
$$\dot{\mathbf{p}}^*(t) = -\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)' \mathbf{p}^*(t)$$
- The remaining variation $\delta\mathbf{u}(t)$, is independent, so its coefficient must be zero; thus
$$\frac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)' \mathbf{p}^*(t) = \mathbf{0}$$

By using the Hamiltonian formalism, one obtains the claim

Necessary conditions for optimal control
(with bounded controls)

- So far, we have assumed that the admissible controls and states are not constrained by any boundaries
- However, in realistic systems, such constraints do commonly occur
  - control constraints often occur due to actuation limits
  - state constraints often occur due to safety considerations
- We will now consider the case with control constraints, which will lead to the statement of the Pontryagin's minimum principle

Why control constraints complicate the analysis?

- By definition, the control $\mathbf{u}^*$ causes the functional J to have a relative minimum if

$$J(\mathbf{u}) - J(\mathbf{u}^*) = \Delta J \geq 0$$

for all admissible controls "close" to $\mathbf{u}^*$

- If we let $\mathbf{u} = \mathbf{u}^* + \delta\mathbf{u}$, the increment in J can be expressed as

$$\Delta J(\mathbf{u}^*, \delta\mathbf{u}) = \delta J(\mathbf{u}^*, \delta\mathbf{u}) + \text{higher order terms}$$

- The variation $\delta\mathbf{u}$ is arbitrary *only if* the extremal control is strictly within the boundary for all time in the interval $[t_0, t_f]$
- In general, however, an extremal control lies on a boundary during at least one subinterval of the interval $[t_0, t_f]$

Why control constraints complicate the analysis?

- As a consequence, admissible control variations $\delta \mathbf{u}$ exist whose negatives $(-\delta \mathbf{u})$ are not admissible
- This implies that a necessary condition *for* $\mathbf{u}^*$ to minimize J is
$$\delta J(\mathbf{u}^*, \delta \mathbf{u}) \geq 0$$
for all admissible variations with $||\delta \mathbf{u}||$ small enough
- The reason why the equality $(= 0)$ in the fundamental theorem of CoV (where we assumed *no* constraints) is replaced with an inequality (≥ 0) is the presence of the control constraints
- This result has an analog in calculus, where the necessary condition for a scalar function f to have a relative minimum at an end point is that the differential df is ≥ 0

Pontryagin's minimum principle

- Assuming bounded controls $\mathbf{u} \in U$, the necessary optimality conditions are (H is the Hamiltonian)

$$\left.\begin{aligned} \dot{\mathbf{x}}^*(t) &= \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ H(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) &\le H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t), \text{ for all } \mathbf{u}(t) \in U \end{aligned}\right\} \text{for all } t \in [t_0, t_f]$$

along with the boundary conditions:

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]' \delta \mathbf{x}_f + \left[H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0$$

Pontryagin's minimum principle

- $\mathbf{u}^*(t)$ is a control that causes $H(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t)$ to assume its *global* minimum
- Harder condition in general to analyze
- Example: consider the system having state equations:

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = -x_2(t) + u(t);$$

it is desired to minimize the functional

$$J = \int_{t_0}^{t_f} \frac{1}{2}[x_1^2(t) + u^2(t)]dt$$

with t_f fixed and the final state free

Pontryagin's minimum principle

Solution:

- If the control is unconstrained,
$$u^*(t) = -p_2^*(t)$$
- If the control is constrained as $|u(t)| \leq 1$, then
$$u^*(t) = \begin{cases} -1 & \text{for } 1 < p_2^*(t) \\ -p_2^*(t), & -1 \leq p_2^*(t) \leq 1 \\ +1 & \text{for } p_2^*(t) < -1 \end{cases}$$
- To determine $u^*(t)$ explicitly, the state and co-state equations must be solved (more on this in the next lecture)

Additional necessary conditions

1. If the final time is fixed and the Hamiltonian does not depend explicitly on time, then

$$H\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)\right) = c \quad \text{for all } t \in \left[t_0, t_f\right]$$

2. If the final time is free and the Hamiltonian does not depend explicitly on time, then

$$H\left(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t)\right) = 0 \quad \text{for all } t \in [t_0, t_f]$$

Minimum time problems

- Find the control input sequence

$$M_i^- \leq u_i(t) \leq M_i^+ \text{ for } i = 1, \dots, m$$

that drives the control affine system

$$\dot{\mathbf{x}} = A(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}(t)$$

from an arbitrary state $\mathbf{x}_0$ to the origin, and minimizes time

$$J = \int_{t_0}^{t_f} dt$$

Minimum time problems

- Form the Hamiltonian

$$H = 1 + \mathbf{p}(t)'\{A(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u}(t)\}$$

$$= 1 + \mathbf{p}(t)'\{A(\mathbf{x},t) + [\mathbf{b}_1(\mathbf{x},t)\ \ \mathbf{b}_2(\mathbf{x},t) \cdots \mathbf{b}_m(\mathbf{x},t)]\mathbf{u}(t)\}$$

$$= 1 + \mathbf{p}(t)'A(\mathbf{x},t) + \sum_{i=1}^{m} \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t)u_i(t)$$

- By the PMP, select $u_i(t)$ to minimize H, which gives

$$u_i^*(t) = \begin{cases} M_i^+ \text{ if } \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t) < 0 \\ M_i^- \text{ if } \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t) > 0 \end{cases}$$

Bang bang control

- Then solve for the co-state equations (more on this in the next lecture)

Minimum time problems

- Note: we showed what to do when $\mathbf{p}(t)'\mathbf{b}_i(\mathbf{x}, t) \neq 0$
- Not obvious what to do if $\mathbf{p}(t)'\mathbf{b}_i(\mathbf{x}, t) = 0$
- If $\mathbf{p}(t)'\mathbf{b}_i(\mathbf{x}, t) = 0$ for some finite time interval, then the coefficient of $u_i(t)$ in the Hamiltonian is zero, so the PMP provides no information on how to select $u_i(t)$
- The treatment of such a *singular condition* requires a more sophisticated analysis
- The analysis in the linear case is significantly easier, see Kirk Sec. 5.4

Minimum fuel problems

- Find the control input sequence

$$M_i^- \le u_i(t) \le M_i^+ \text{ for } i = 1, \dots, m$$

that drives the control affine system

$$\dot{\mathbf{x}} = A(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}(t)$$

from an arbitrary state $\mathbf{x}_0$ to the origin in a fixed time, and minimizes

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} c_i \, |u_i(t)| \, dt$$

Minimum fuel problems

- Form the Hamiltonian

$$H = \sum_{i=1}^{m} c_i\,|u_i(t)| + \mathbf{p}(t)'\{A(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u}(t)\}$$
$$= \sum_{i=1}^{m} c_i\,|u_i(t)| + \mathbf{p}(t)'A(\mathbf{x},t) + \sum_{i=1}^{m} \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t)u_i(t)$$
$$= \sum_{i=1}^{m} [c_i\,|u_i(t)| + \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t)u_i(t)] + \mathbf{p}(t)'A(\mathbf{x},t)$$

- By the PMP, select $u_i(t)$ to minimize H, that is

$$\sum_{i=1}^{m} [c_i\,|u_i^*(t)| + \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t)u_i^*(t)] \leq \sum_{i=1}^{m} [c_i\,|u_i(t)| + \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t)u_i(t)]$$

Minimum fuel problems

- Since the components of $\mathbf{u}(t)$ are independent, then one can just look at
$$c_i\,|u_i^*(t)| + \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t)u_i^*(t) \leq c_i\,|u_i(t)| + \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t)u_i(t)$$
- The resulting control law is

$$u_i^*(t) = \begin{cases} M_i^- & \text{if } c_i < \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t) \\ 0 & \text{if } -c_i < \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t) < c_i \\ M_i^+ & \text{if } \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x},t) < -c_i \end{cases}$$

Minimum energy problems

- Find the control input sequence

$$M_i^- \le u_i(t) \le M_i^+ \text{ for } i = 1, \dots, m$$

that drives the control affine system

$$\dot{\mathbf{x}} = A(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}(t)$$

from an arbitrary state $\mathbf{x}_0$ to the origin in a fixed time, and minimizes

$$J = \frac{1}{2}\int_{t_0}^{t_f} \mathbf{u}(t)' R \mathbf{u}(t) dt, \quad \text{where } R > 0 \text{ and diagonal}$$

Minimum energy problems

- Form the Hamiltonian

$$H = \frac{1}{2}\mathbf{u}(t)' R\mathbf{u}(t) + \mathbf{p}(t)'\{A(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}(t)\}$$

$$= \frac{1}{2}\mathbf{u}(t)' R\mathbf{u}(t) + \mathbf{p}(t)' B(\mathbf{x}, t)\mathbf{u}(t) + \mathbf{p}(t)' A(\mathbf{x}, t)$$

- By the PMP, we need to solve

$$\mathbf{u}^*(t) = \arg\min_{\mathbf{u}(t)\in U}\left[\sum_{i=1}^{m}\frac{1}{2}R_{ii}u_i(t)^2 + \mathbf{p}(t)'\mathbf{b}_i(\mathbf{x}, t)u_i(t)\right]$$

Minimum energy problems

- In the unconstrained case, the optimal solution for each component of $\mathbf{u}(t)$ would be

$$\hat{u}_i(t) = -R_{ii}^{-1}\,\mathbf{p}(t)'\mathbf{b}_i(\mathbf{x}, t)$$

- Considering the input constraints, the resulting control law is

$$u^*(t) = \begin{cases} M_i^- & \text{if} \quad \hat{u}_i(t) < M_i^- \\ \hat{u}_i(t) & \text{if} \quad M_i^- < \hat{u}_i(t) < M_i^+ \\ M_i^+ & \text{if} \quad M_i^+ < \hat{u}_i(t) \end{cases}$$

Uniqueness and existence

- Note: uniqueness and existence are not in general guaranteed!

- Example 1 (non uniqueness): find a control sequence $u(t)$ to transfer the system $\dot{x}(t) = u(t)$ from an arbitrary initial state x_0 to the origin, and such that the functional $J = \int_0^{t_f} |u(t)| dt$ is minimized. The final time is free, and the admissible controls are $|u(t)| \leq 1$

- Example 2 (non existence): find a control sequence $u(t)$ to transfer the system $\dot{x}(t) = -x(t) + \ u(t)$ from an arbitrary initial state x_0 to the origin, and such that the functional $J = \int_{t_0}^{t_f} |u(t)| dt$ is minimized. The final time is free, and the admissible controls are $|u(t)| \leq 1$

Next time

- Numerical methods for indirect optimal control