Spring 2017

# Stanford
# AA 203: Introduction to Optimal Control and Dynamic Optimization

# Midterm

**Instructions**:

- Time allowed: 75 minutes.
- Closed book, you are allowed one side of a page for notes.
- Calculators, laptops and smartphones are not allowed.
- Please read all questions carefully before answering. Make sure that you provide answers to all questions asked. Partial credit will be available if sufficient detail is provided.
- The exam consists of **four** problems. Good luck!

**Problem 1 (Nonlinear optimization – 25 points):** Let $\alpha_1, \ldots, \alpha_n$ be positive scalars with $\sum_{i=1}^n \alpha_i = 1$. Use the Lagrange multiplier method to solve the problem

$$\begin{array}{rl} \text{minimize} & \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \\ \text{subject to} & x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = 1, \qquad x_i > 0, \quad i = 1, \ldots, n. \end{array}$$

Use this result to establish the arithmetic-geometric mean inequality

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \leq \alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n,$$

for a set of positive numbers $x_i$, $i = 1, \ldots, n$. *Hints: Use the change of variables $y_i = \ln x_i$ and assume that the optimization problem has a global minimum (this can be rigorously proven by using a generalized version of the Weierstrass's theorem).*

**Problem 2 (Dynamic programming – 25 points):** A certain material is passed through a sequence of two ovens. Denote by

- $x_0$: the initial temperature of the material,
- $x_k$, $k = 1, 2$: the temperature of the material at the exit of oven $k$,
- $u_k$, $k = 0, 1$: prevailing temperature in oven $k$.

The ovens are modeled as

$$x_{k+1} = (1-a)\,x_k + a\,u_k, \quad k = 0, 1,$$

where $a$ is a known scalar from the interval $(0, 1)$. The objective is to get the final temperature $x_2$ close to a given target $T$, while expending relatively little energy. This is expressed by a cost function of the form

$$r\,(x_2 - T)^2 + u_0^2 + u_1^2,$$

where $r > 0$ is a given scalar. For simplicity, assume no constraints on $u_k$. Solve the problem using the DP algorithm assuming $a = 1/2$, $T = 0$, and $r = 1$. Specifically, determine the optimal control policies $u_0^*(x_0)$ and $u_1^*(x_1)$, and find the optimal cost function $J^*(x_0)$ as a function of the initial condition $x_0$.

**Problem 3 (Calculus of variations – 25 points):** Under appropriate assumptions, the total drag experienced by a slender body of revolution is approximated by

$$J(x) = 4\pi\rho v^2 \int_0^L x(t)\,\dot{x}(t)^3\,dt,$$

where $\rho$ and $v$ are free stream density and velocity, respectively. The boundary conditions are $x(0) = 0$ and $x(L) = R$. Find the optimal shape for the body, i.e., the shape minimizing drag. *Hint: the solution to the differential equation:*

$$\dot{x}(t) = \alpha\,x(t)^{-1/3}$$

*is* $x(t) = \left(\frac{4}{3}\alpha t + c\right)^{3/4}$, *where $c$ is a constant of integration.*

**Problem 4 (Optimal control – 25 points):** Determine the control function that solves the optimal control problem:

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x(t) + u^2(t)\right) dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t) + 1, \\ & x(0) = 0, \end{aligned}$$

where the final time is *fixed* and the final state is *free*.