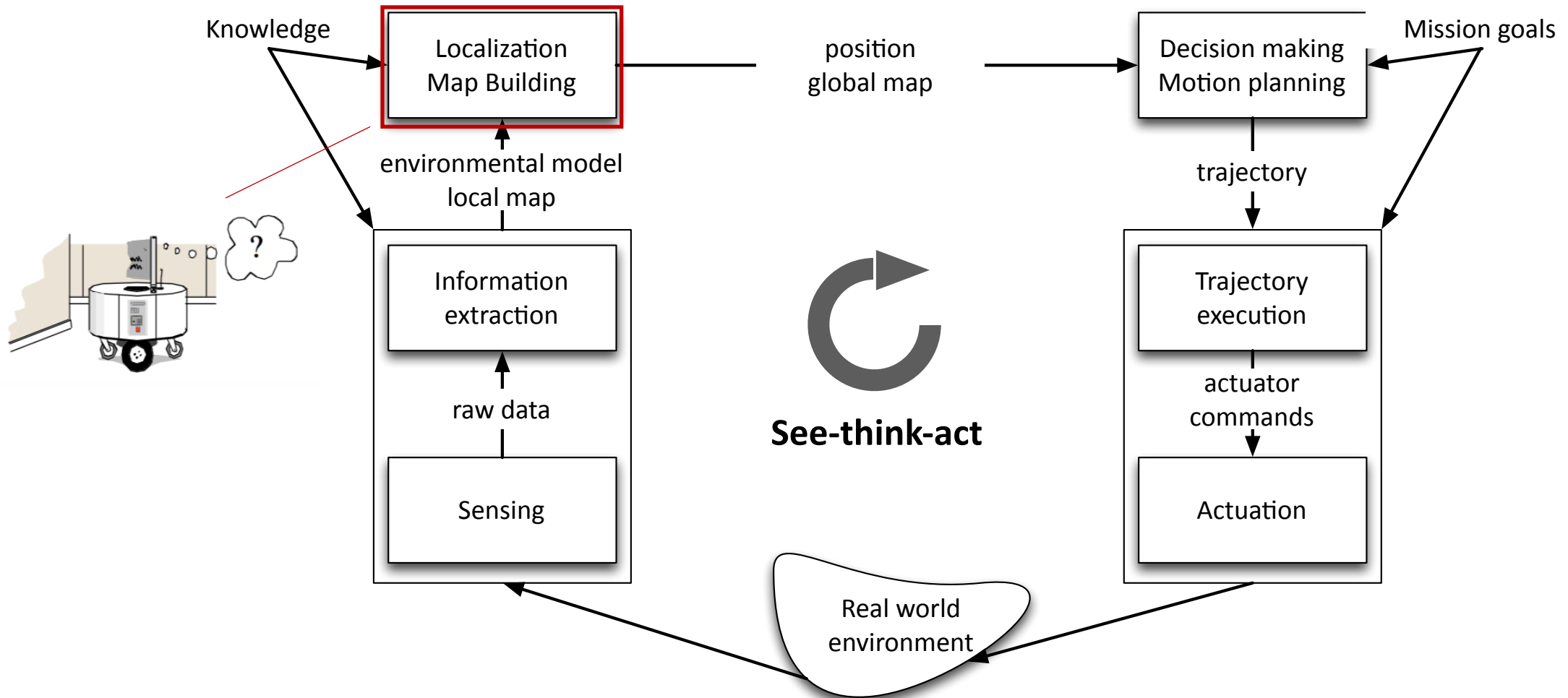


AA 274

Principles of Robotic Autonomy

Localization I: fundamental probabilistic concepts and filtering theory

Module 3



Today's lecture

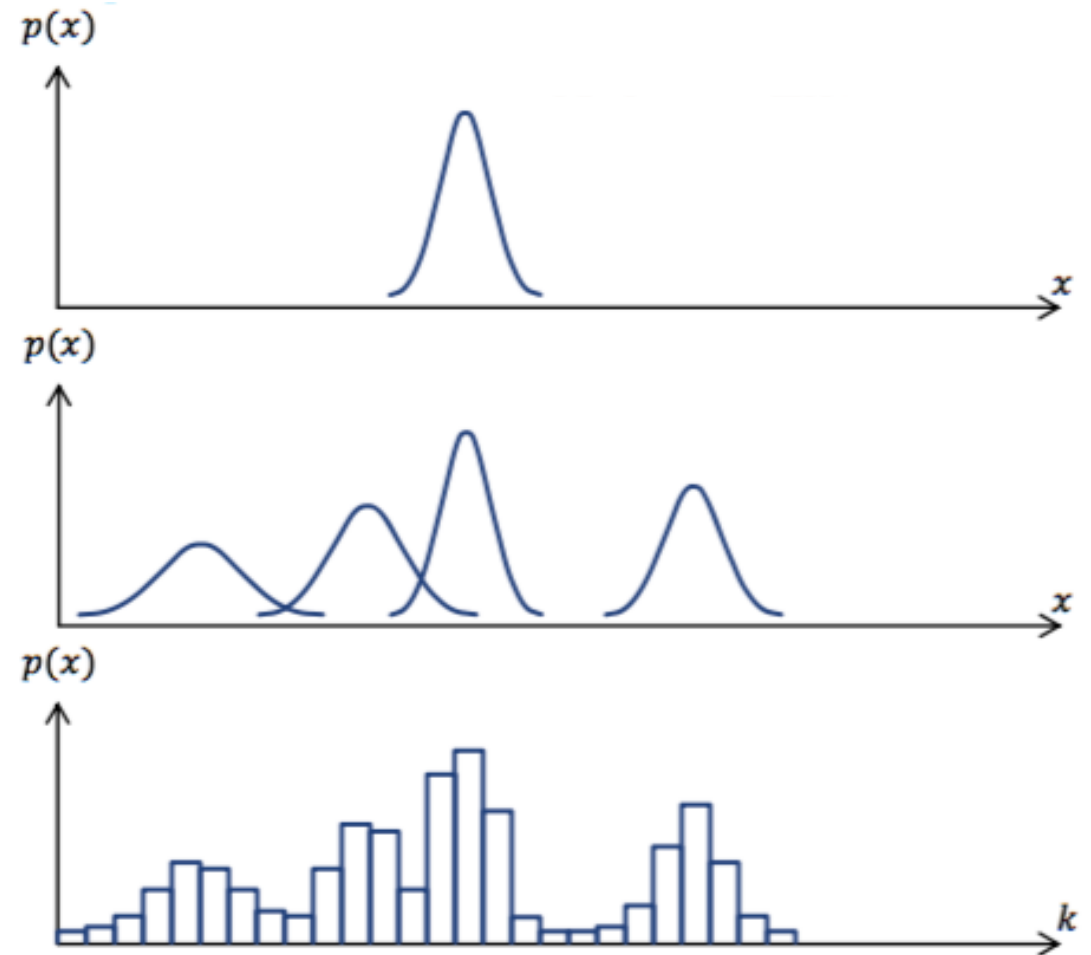
- Aim
 - Learn basic concepts about Bayesian filtering
- Readings
 - S. Thrun, W. Burgard, and D. Fox. Probabilistic robotics. MIT press, 2005. Chapter 2

Localization

- Two main approaches:
 1. **Behavioral approach**: design a set of behaviors that together result in the desired robot motion (no need for a map)
 2. **Map-based approaches**: robot *explicitly* attempts to localize by collecting sensor data, then updating belief about its position with respect to a map
- We will focus on **map-based approaches**; two main aspects:
 - Map representation: how to represent the environment?
 - Belief representation: how to model the belief regarding the position within the map?

Probabilistic map-based localization

- **Key idea:** represent belief as a probability distribution
 1. Encodes sense of position
 2. Maintains notion of robot's uncertainty
- Belief representation:
 1. Single-hypothesis vs. multiple hypothesis
 2. Continuous vs. discretized
- Today we will overview basic concepts in **Bayesian filtering**



Basic concepts in probability

- **Key idea:** quantities such as sensor measurements, states of a robot, and its environment are modeled as **random variables (RVs)**
- **Discrete RV:** the space of all the values that a random variable X can take on is *discrete*; characterized by probability mass function (pmf)

$$p(\overset{\text{Random variable}}{X} = \overset{\text{Specific value}}{x}) \quad (\text{or } p(x)), \quad \sum_x p(X = x) = 1$$

- **Continuous RV:** the space of all the values that a random variable X can take on is *continuous*; characterized by probability density function (pdf)

$$P(a \leq X \leq b) = \int_a^b p(x) dx, \quad \int_{-\infty}^{\infty} p(x) dx = 1$$

Joint distribution, independence, and conditioning

- Joint distribution of two random variables X and Y is denoted as

$$p(x, y) := p(X = x \text{ and } Y = y)$$

- If X and Y are independent

$$p(x, y) = p(x)p(y)$$

- Suppose we know that $Y = y$ (with $p(y) > 0$); conditioned on this fact, the probability that the X 's value is x is given by

$$p(x | y) := \frac{p(x, y)}{p(y)}$$

Conditional probability

Note: if X and Y are independent

$$p(x | y) := p(x)!$$

Theorem of total probability

- For discrete RVs:

$$p(x) = \sum_y p(x, y) = \sum_y p(x | y)p(y)$$

- For continuous RVs:

$$p(x) = \int p(x, y)dy = \int p(x | y)p(y)dy$$

- Note: if $p(y) = 0$, define the product $p(x | y)p(y) = 0$

Bayes' rule

- Key relation between $p(x | y)$ and its “inverse,” $p(y | x)$
- For discrete RVs:

$$p(x | y) = \frac{p(y | x)p(x)}{p(y)} = \frac{p(y | x)p(x)}{\sum_{x'} p(y | x')p(x')}$$

- For continuous RVs:

$$p(x | y) = \frac{p(y | x)p(x)}{p(y)} = \frac{p(y | x)p(x)}{\int p(y | x')p(x') dx'}$$

Bayes' rule and probabilistic inference

- Assume x is a quantity we would like to infer from y
- Bayes rule allows us to do so through the inverse probability, which specifies the probability of data y assuming that x was the cause

Posterior probability distribution

Prior probability distribution

$$p(x | y) = \frac{p(y | x)p(x)}{\int p(y | x')p(x') dx'}$$

Data

Normalizer, does not depend on $x := \eta^{-1}$

- Notational simplification

$$p(x | y) = \eta p(y | x)p(x)$$

More on Bayes' rule and independence

- Extension of Bayes rule: conditioning Bayes rule on $Z=z$ gives

$$p(x | y, z) = \frac{p(y | x, z)p(x | z)}{p(y | z)}$$

- Extension of independence: *conditional independence*

$$p(x, y | z) = p(x | z)p(y | z), \quad \text{equivalent to } \begin{cases} p(x | z) = p(x | z, y) \\ p(y | z) = p(y | z, x) \end{cases}$$

- Note: in general

$$p(x, y | z) = p(x | z)p(y | z) \not\Rightarrow p(x, y) = p(x)p(y)$$

$$p(x, y) = p(x)p(y) \not\Rightarrow p(x, y | z) = p(x | z)p(y | z)$$

Expectation of a RV

- Expectation for discrete RVs: $E[X] = \sum_x x p(x)$
- Expectation for continuous RVs: $E[X] = \int x p(x) dx$
- Expectation is a linear operator: $E[aX + b] = a E[X] + b$
- Expectation of a vector of RVs is simply the vector of expectations
- Covariance

$$\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])^T] = E[XY^T] - E[X]E[Y]^T$$

Model for robot-environment interaction

- Two fundamental types of robot-environment interactions: the robot can influence **the state** of its environment through **control actions**, and gather information about the **state** through **measurements**
- **State x_t** : collection at time t of all aspects of the robot and its environment that can impact the future
 - Robot pose (e.g., robot location and orientation)
 - Robot velocity
 - Location and features of surrounding objects in the environment, etc.
- Useful notation: $x_{t_1:t_2} := x_{t_1}, x_{t_1+1}, x_{t_1+2}, \dots, x_{t_2}$
- A state x_t is called *complete* if no variables prior to x_t can influence the evolution of future states -> **Markov property**

Measurement and control data

- **Measurement data** z_t : information about state of the environment at time t ; useful notation

$$z_{t_1:t_2} := z_{t_1}, z_{t_1+1}, z_{t_1+2}, \dots, z_{t_2}$$

- **Control data** u_t : information about the change of state at time t ; useful notation

$$u_{t_1:t_2} := u_{t_1}, u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2}$$

- Key difference: measurement data tends to increase robot's knowledge, while control actions tend to induce a loss of knowledge

State equation

- General probabilistic generative model

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})$$

Convention: first take control action and then take measurement

- **Key assumption:** state is complete (i.e., the Markov property holds)

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

State transition probability

- In other words, we assume *conditional independence*, with respect to conditioning on x_{t-1}

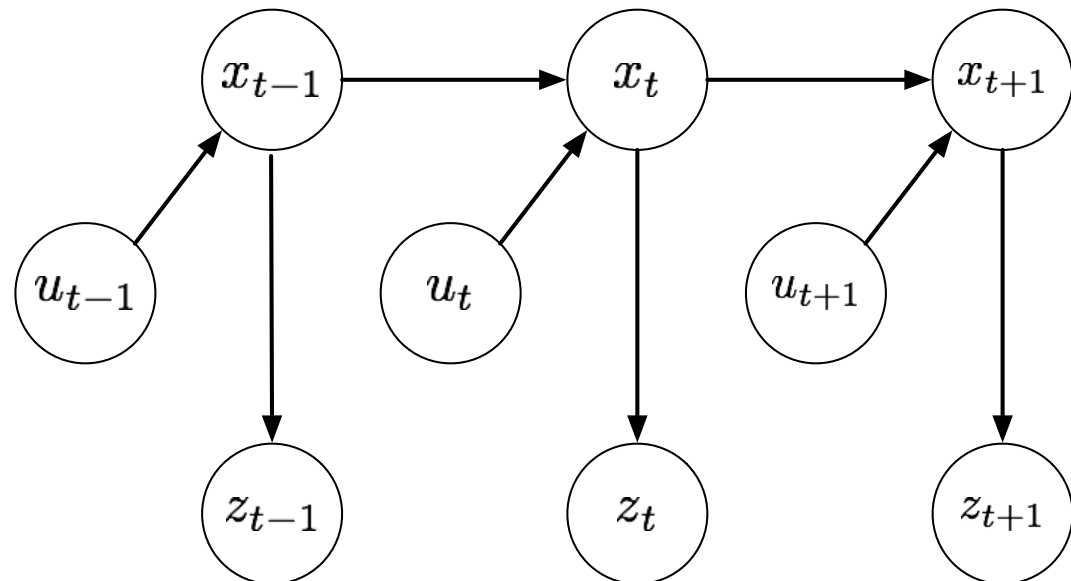
Measurement equation and overall stochastic model

- Assuming x_t is complete

→ $p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$

Measurement probability

- Overall dynamic Bayes network model (also referred to as hidden Markov model)



Belief distribution

- **Belief distribution**: reflects internal knowledge about the state
- A belief distribution assigns a probability to each possible hypothesis with regard to the true state
- Formally, belief distributions are posterior probabilities over state variables conditioned on the available data

$$bel(x_t) := p(x_t \mid z_{1:t}, u_{1:t})$$

- Similarly, the *prediction* distribution is defined as

$$\overline{bel}(x_t) := p(x_t \mid \textcolor{red}{z}_{1:t-1}, u_{1:t})$$

- Calculating $bel(x_t)$ from $\overline{bel}(x_t)$ is called correction or measurement update

Bayes filter algorithm

- **Bayes' filter algorithm**: most general algorithm for calculating beliefs
- **Key assumption**: state is complete

- Recursive algorithm
 - Step 1 (prediction):
compute $\overline{bel}(x_t)$
 - Step 2 (measurement update):
compute $bel(x_t)$
- Algorithm initialized with $bel(x_0)$
(e.g., uniform or points mass)

Data: $bel(x_{t-1}), u_t, z_t$

Result: $bel(x_t)$

foreach x_t **do**

$\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1};$
 $bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t);$

end

Return $bel(x_t)$

Update rule



Derivation: measurement update

$$\begin{aligned} \text{bel}(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\ &= \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t})}{\underbrace{p(z_t \mid z_{1:t-1}, u_{1:t})}_{:= \eta^{-1}}} && \text{Bayes rule} \\ &= \eta p(z_t \mid x_t) \underbrace{p(x_t \mid z_{1:t-1}, u_{1:t})}_{=\overline{\text{bel}(x_t)}} && \text{Markov property} \end{aligned}$$

Derivation: correction update

$$\begin{aligned}\overline{bel}(x_t) &= p(x_t \mid z_{1:t-1}, u_{1:t}) \\&= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1} && \text{Total probability} \\&= \int p(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1} && \text{Markov} \\&= \int p(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, \textcolor{red}{u}_{1:t-1}) dx_{t-1} && \text{For general output feedback policies, } u_t \text{ does not provide additional information on } x_{t-1} \\&= \int p(x_t \mid x_{t-1}, u_t) bel(x_{t-1}) dx_{t-1}\end{aligned}$$

Discrete Bayes' filter

- **Discrete Bayes' filter algorithm:** applies to problems with *finite* state spaces

- Belief $bel(x_t)$
represented as pmf
 $\{p_{k,t}\}$

Data: $\{p_{k,t-1}\}, u_t, z_t$

Result: $\{p_{k,t}\}$

foreach k **do**

$\bar{p}_{k,t} = \sum_i p(X_t = x_k \mid u_t, X_{t-1} = x_i) p_{i,t-1};$
 $p_{k,t} = \eta p(z_t \mid X_t = x_k) \bar{p}_{k,t};$

end

Return $\{p_{k,t}\}$

Next time

