

# AA203

# Optimal and Learning-based Control

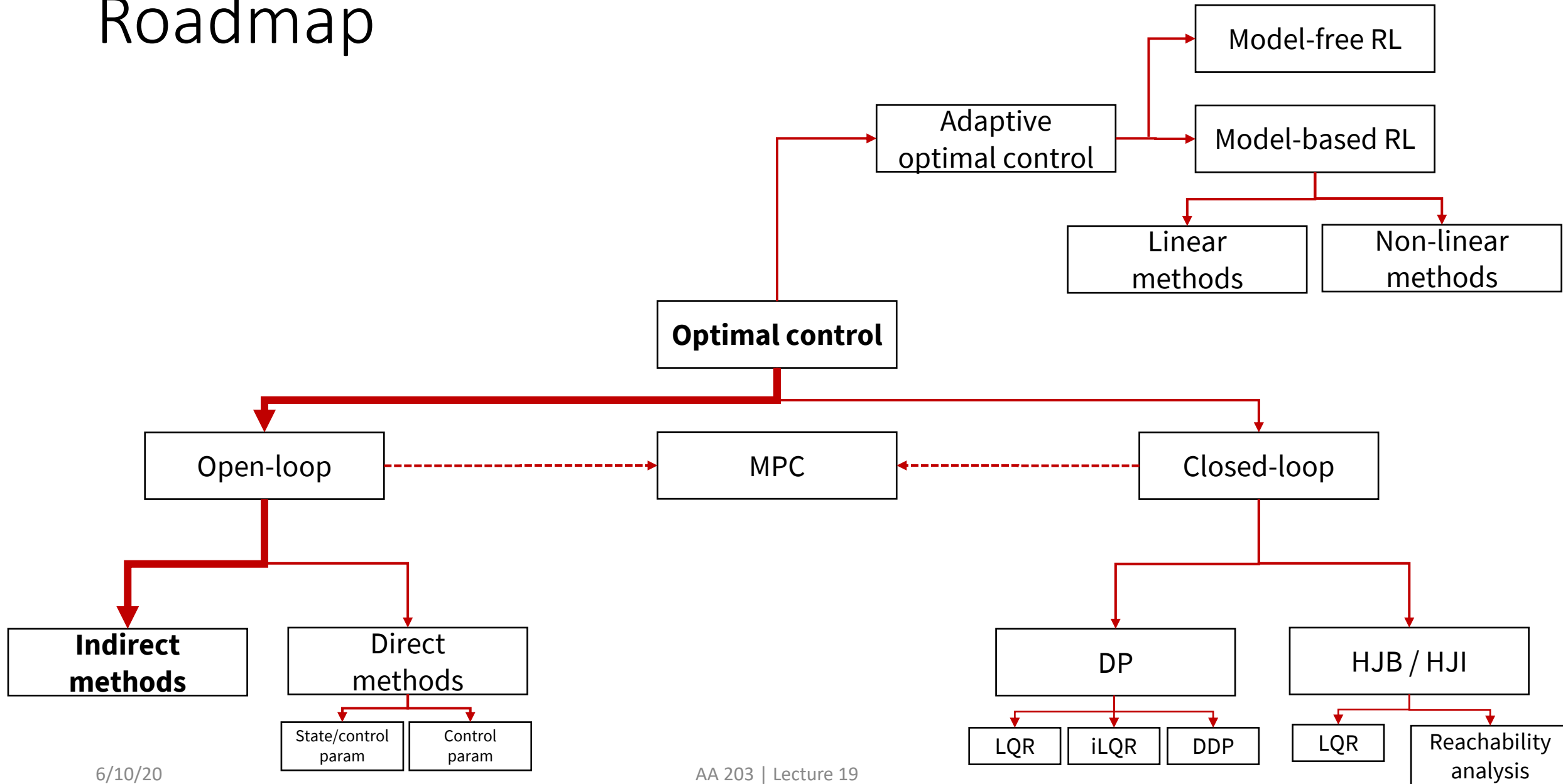
Numerical indirect methods for optimal control\*



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# Roadmap



# Optimal Control Problem

$$\min h(\mathbf{x}(t_f)) + \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad , \quad t \in [0, t_f]$$

**(OCP)**

$$\mathbf{x}(0) = \mathbf{x}_0 \quad , \quad \mathbf{x}(t_f) \in M_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m \quad , \quad t \in [0, t_f]$$

For simplicity:

- $h$  does not explicitly depend on  $t$
- We assume  $t_0 = 0$

- Indirect Methods:

1. Apply necessary conditions for optimality to **(OCP)**
2. Solve a two-point boundary value problem

- Direct Methods:

1. Transcribe **(OCP)** into a nonlinear, constrained optimization problem
2. Solve the optimization problem via nonlinear programming

# 1) Free Final Time and Fixed Final Point

## Pontryagin's Minimum Principle

$$\min h(\mathbf{x}(t_f)) + \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad t \in [0, t_f]$$

**(OCP)**

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \quad t \in [0, t_f]$$

1. Define the Hamiltonian:

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) = \mathbf{p}' \mathbf{f}(\mathbf{x}, \mathbf{u}, t) + g(\mathbf{x}, \mathbf{u}, t)$$

2. Find candidates for optimal control solutions:

$$\mathbf{u}^*(\mathbf{x}, \mathbf{p}, t) = \operatorname{argmin}_{\mathbf{u} \in U} H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t)$$

3. Define the system of differential equations:

$$\begin{cases} \dot{\mathbf{x}}(t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \\ \dot{\mathbf{p}}(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \end{cases}$$

with conditions:  $\mathbf{x}(0) = \mathbf{x}_0, \mathbf{p}(0) = \mathbf{p}_0, \mathbf{x}(t_f) = \mathbf{x}_f$  and

$$H(\mathbf{x}(t_f), \mathbf{p}(t_f), \mathbf{u}(t_f), t_f) = 0$$

# 1) Free Final Time and Fixed Final Point

## Two-Point Boundary Value Problem

- Define the new variable and dynamics:

$$\mathbf{z} = (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n},$$

$$\mathbf{R}(\mathbf{z}, t) = \left( \frac{\partial H}{\partial \mathbf{p}}(\mathbf{z}, \mathbf{u}^*(\mathbf{z}, t), t), -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{z}, \mathbf{u}^*(\mathbf{z}, t), t) \right)$$

- For every value  $\mathbf{p}_0 \in \mathbb{R}^n$  and final time  $t_f$ , denote  $\mathbf{z}_{\mathbf{p}_0}^{t_f}(\cdot)$  the solution of:

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{R}(\mathbf{z}(t), t), & t \in [0, t_f] \\ \mathbf{z}(0) = (\mathbf{x}_0, \mathbf{p}_0) \end{cases}$$

- Denote  $\text{proj}_{\mathbf{x}}(\mathbf{x}, \mathbf{p}) = \mathbf{x}$  and define the function:

$$\begin{aligned} S : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^{n+1} \\ (\mathbf{p}_0, t_f) &\mapsto \left( \text{proj}_{\mathbf{x}} \left( \mathbf{z}_{\mathbf{p}_0}^{t_f}(t_f) \right) - \mathbf{x}_f, H(t_f) \right) \end{aligned}$$

## Pontryagin's Minimum Principle

- Define the Hamiltonian:

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) = \mathbf{p}' \mathbf{f}(\mathbf{x}, \mathbf{u}, t) + g(\mathbf{x}, \mathbf{u}, t)$$

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$$\mathbf{u}^*(\mathbf{x}, \mathbf{p}, t) = \operatorname{argmin}_{\mathbf{u} \in U} H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t)$$

- Define the system of differential equations:

$$\begin{cases} \dot{\mathbf{x}}(t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \\ \dot{\mathbf{p}}(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \end{cases}$$

with conditions:  $\mathbf{x}(0) = \mathbf{x}_0, \mathbf{p}(0) = \mathbf{p}_0, \mathbf{x}(t_f) = \mathbf{x}_f$  and

$$H(\mathbf{x}(t_f), \mathbf{p}(t_f), \mathbf{u}(t_f), t_f) = 0$$

Find the value of  $\mathbf{p}_0$  that satisfies the system above

# 1) Free Final Time and Fixed Final Point

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- Denote  $\text{proj}_{\mathbf{x}}(\mathbf{x}, \mathbf{p}) = \mathbf{x}$  and define the function:

$$S : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$$

$$(\mathbf{p}_0, t_f) \mapsto \left( \text{proj}_{\mathbf{x}} \left( \mathbf{z}_{\mathbf{p}_0}^{t_f}(t_f) \right) - \mathbf{x}_f, H(t_f) \right)$$

## Example: Zermelo's Problem

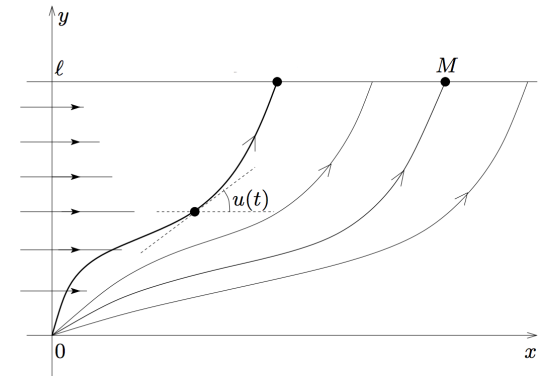
$$\min \int_0^{t_f} 1 \, dt$$

$$\dot{x}(t) = v \cos(u(t)) + fl(y(t)), \quad t \in [0, t_f]$$

$$\dot{y}(t) = v \sin(u(t)), \quad t \in [0, t_f]$$

$$(x, y)(0) = 0, \quad (x, y)(t_f) = (M, \ell)$$

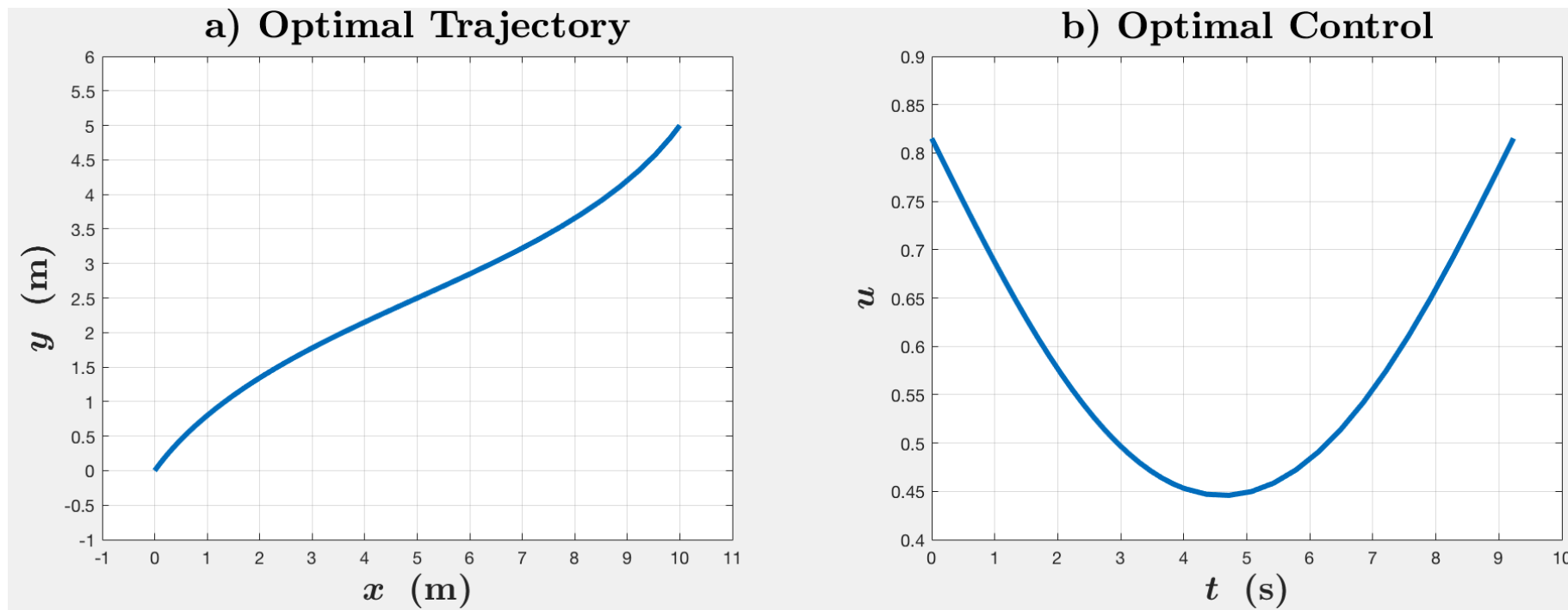
$$u(t) \in \mathbb{R}, \quad t \in [0, t_f]$$



Recall that:

- $H(x, y, p_x, p_y, u) = (p_x, p_y) \cdot f(x, y, u) + 1$
- $u^*(x, y, p_x, p_y) = \operatorname{argmin}_{u \in \mathbb{R}} H(x, y, p_x, p_y, u)$

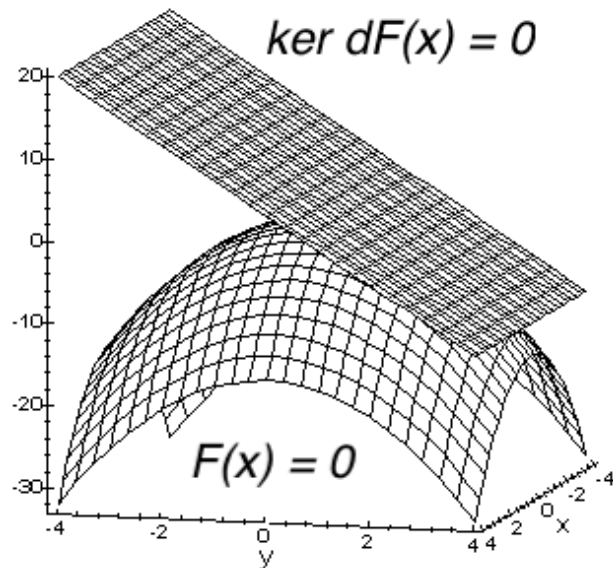
# 1) Free Final Time and Fixed Final Point



## 2) Free Final Time and Free Final Point

$$\min h(\mathbf{x}(t_f)) + \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

**(OCP)**  $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad t \in [0, t_f]$   
 $\mathbf{x}(0) = \mathbf{x}_0, \mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \quad t \in [0, t_f]$   
 $\mathbf{x}(t_f) \in M_f := \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = 0\}$



### Pontryagin's Minimum Principle

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2. Find candidates for optimal control solutions:

$$\mathbf{u}^*(\mathbf{x}, \mathbf{p}, t) = \operatorname{argmin}_{\mathbf{u} \in U} H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t)$$

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with conditions:  $\mathbf{x}(0) = \mathbf{x}_0, \mathbf{p}(0) = \mathbf{p}_0, \mathbf{x}(t_f) = \mathbf{x}_f$ ,  
 $H(\mathbf{x}(t_f), \mathbf{p}(t_f), \mathbf{u}(t_f), t_f) = 0$  and:

$$\mathbf{p}(t_f) - \nabla h(\mathbf{x}(t_f)) \perp \ker dF(\mathbf{x}(t_f))$$



## 2) Free Final Time and Free Final Point

### Two-Point Boundary Value Problem

- Define the new variable and dynamics:

$$\mathbf{z} = (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n},$$

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- For every value  $\mathbf{p}_0 \in \mathbb{R}^n$  and final time  $t_f$ , denote  $\mathbf{z}_{\mathbf{p}_0}^{t_f}(\cdot)$  the solution of:

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- Denote  $\text{proj}_{\mathbf{p}}(\mathbf{x}, \mathbf{p}) = \mathbf{p}$  and  $\mathbf{x}(t_f) = \text{proj}_{\mathbf{x}}(\mathbf{z}_{\mathbf{p}_0}^{t_f}(t_f))$ .

Define the function:

$$S : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$$

$$(\mathbf{p}_0, t_f) \mapsto \left( \left( \text{proj}_{\mathbf{p}}(\mathbf{z}_{\mathbf{p}_0}^{t_f}(t_f)) - \nabla h(\mathbf{x}(t_f)) \right) \cdot \ker dF(\mathbf{x}(t_f)), F(\mathbf{x}(t_f)), H(t_f) \right)$$

### Pontryagin's Minimum Principle

- Define the Hamiltonian:

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) = \mathbf{p}' \mathbf{f}(\mathbf{x}, \mathbf{u}, t) + g(\mathbf{x}, \mathbf{u}, t)$$

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$$\mathbf{p}(t_f) - \nabla h(\mathbf{x}(t_f)) \perp \ker dF(\mathbf{x}(t_f))$$

Find the value of  $\mathbf{p}_0$  that satisfies the system above

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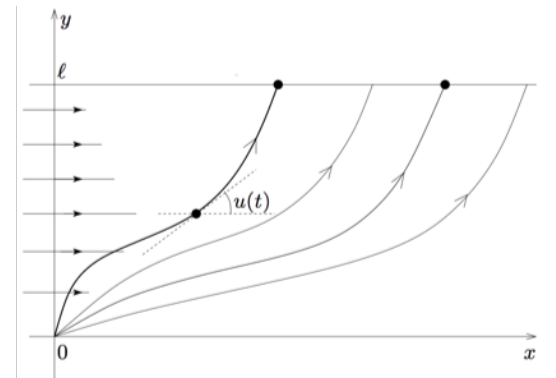
$$\min \int_0^{t_f} 1 \, dt$$

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$$\dot{y}(t) = v \sin(u(t)), \quad t \in [0, t_f]$$

$$(x, y)(0) = 0, u(t) \in \mathbb{R}, \quad t \in [0, t_f]$$

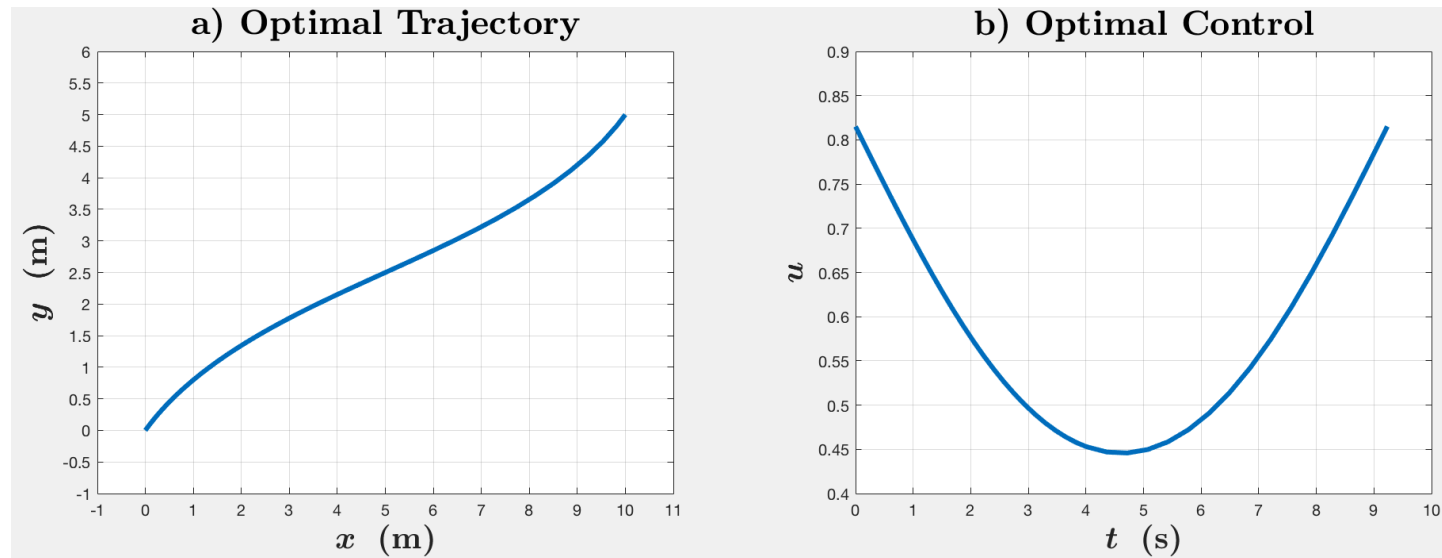
$$(x, y)(t_f) \in M_f = \{ (x, y) \in \mathbb{R}^2 : F(x, y) = y - \ell = 0 \}$$



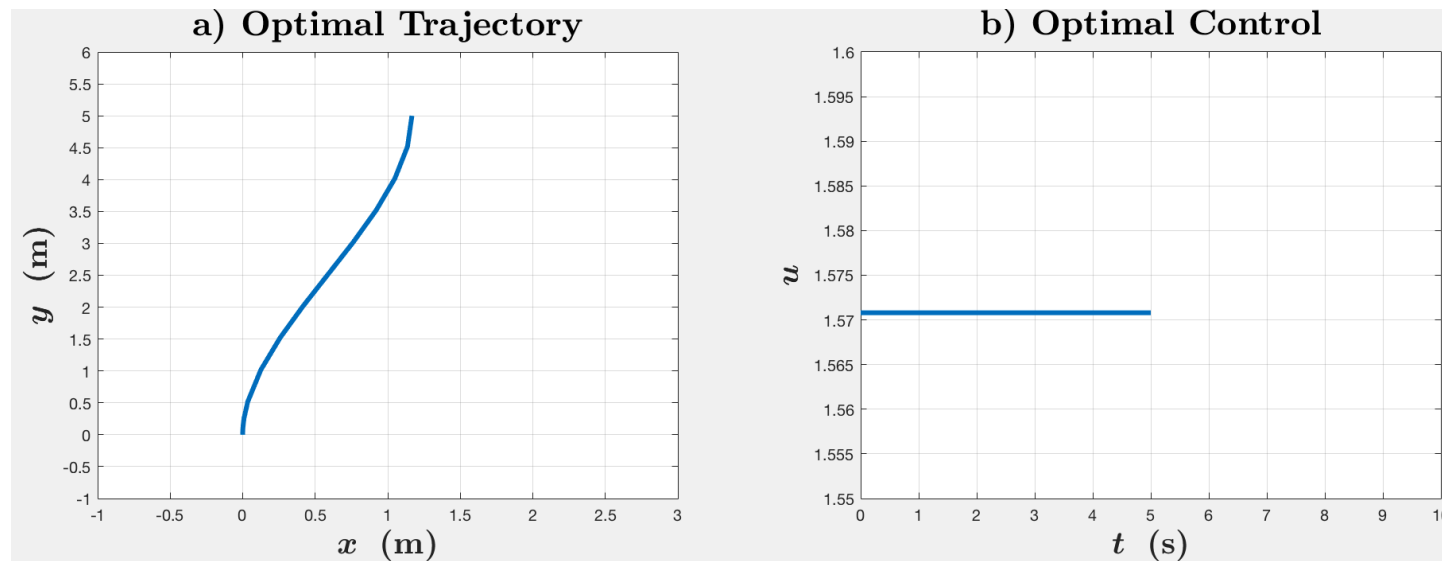
Recall that:

- $H(x, y, p_x, p_y, u) = (p_x, p_y) \cdot f(x, y, u) + 1$
- $u(x, y, p_x, p_y) = \operatorname{argmin}_{u \in \mathbb{R}} H(x, y, p_x, p_y, u)$

## 2) Free Final Time and Free Final Point



Fixed Final Point



Free Final Point

Thank you!