On permutations dependent operators

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Abstract

In this paper, we introduce permutations dependent operators. The motivation for studying such a concept arises from standard fuzzy integrals, where the permutations play a crucial role. In contrast to the standard fuzzy integrals, our construction allows considering any permutation of the basic set in the formula, rather than limiting it to permutations that reorder the input vector monotonically. We present an approach to integration with respect to sets of permutation pairs, i.e. databases in which each vector has a preselected permutation. This new operator generalizes several concepts known in the literature. We investigate the properties of this new concept and highlight its practical utility in image processing.

Keywords: Choquet like operators; permutations; conditional aggregation operator; image processing

1 Introduction

The standard nonadditive or fuzzy integrals among them the Choquet integral [6], the Sugeno integral [22], or the Shilkret integral [21] are still studied and developed by many researchers. There are several generalizations of them in the literature. Among all, let's mention e.g. $\mathscr{A}$-Choquet functionals [2], eCS operators (the extended Choquet-Sugeno-like operators) [3], both are based on conditional aggregation operators. Further, we can list operators dependent on the sequence of aggregation functions, see [3], the inclusion-exclusion integral based on the interaction operator, [11] or, the (MC)-integral dependent on sublinear means [19], etc.

This paper aims to introduce a new generalization of the aforementioned standard fuzzy integrals, focusing on permutations of a basic set involved in the definition. Our construction revolves around considering any permutation of the basic set in the formula, rather than limiting it to permutations that reorder the input vector monotonically (the idea of the Choquet or the Sugeno integral). This approach finds relevance in real-life decision-making scenarios, where permutations can allow us to establish priorities. Moreover, we consider conditional aggregation operators in the construction and we justify their benefits. Permutations dependent operators cover several concepts e.g. the operator dependent on a sequence of aggregation functions introduced in [3], the MCC-integral, see [13], or the TOWA operator [25]. In this paper, we mainly study Choquet's type of permutations dependent operator. Although this paper primarily focuses on theoretical aspects, we also demonstrate the practical utility of this concept in image processing, particularly for edge detection. Motivated by paper [18], we detect edges in an image and qualitatively and quantitatively compare the results with those obtained by standard methods or concepts presented in [18]. We illustrate the superior performance of our approach.

The paper is organized as follows. In Section 2 we provide basic notations and definitions. We present motivations that led us to construct a new operator. In Section 3 we present the construction of Choquet-like permutations dependent operators. We study some properties of this new operator and we compare it with the $\mathscr{A}$-Choquet operator. In Section 4, we apply a new Choquet-like operator in edge detection as part of image processing. In the last section, we study the permutations dependent operator in more general settings. We also cover the Sugeno and the Shilkret integral. In Conclusion, we describe some suggestions for further research.

2 Basic notations and motivations

For easy orientation and an overview of the entire paper, we present in this section the basic notations and necessary definitions. Considering the potential application of our new aggregations,

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we shall restrict ourselves to discrete settings. We shall consider a finite set

$$[n] := \{1, 2, \dots, n\},$$

$n \in \mathbb{N} = \{1, 2, \dots\}$, and by $2^{[n]}$ we denote the power set of $[n]$. By B^c we mean the complement of a set $B \in 2^{[n]}$, i.e. $B^c = [n] \setminus B$. Under the denotation $B \subset C$, $B, C \in 2^{[n]}$, we mean the proper subset of C, i.e. $B \subseteq C$ and $B \neq C$. The set function $\mu\colon 2^{[n]} \to [0, \infty)$ such that $\mu(\emptyset) = 0$, and $\mu(B) \leq \mu(C)$ for any $B \subseteq C$ is called a *monotone measure*. In addition, we assume that $\mu([n]) > 0$. By $\mathbf{M}$ we denote the set of all monotone measures on the power set $2^{[n]}$. If $\mu([n]) = 1$, the monotone measure $\mu \in \mathbf{M}$ is called a *capacity*. The set of all capacities on $2^{[n]}$ we denote by $\mathbf{M}^1$. A monotone measure $\mu \in \mathbf{M}$ is called *symmetric*, if $\mu(B) = \mu(C)$ whenever $|B| = |C|$. We shall work with permutations of the basic set $[n]$, i.e. bijective mappings $\psi\colon [n] \to [n]$. The set of all permutations of $[n]$ we shall denote by $\mathrm{Perm}([n])$. Under denotation ψ^{-1} we mean inverse mapping (permutation) of $\psi \in \mathrm{Perm}([n])$. Let us point out that throughout the paper, we exclusively use the denotation $E_{\psi(i)}$ in the following sense: For a given $\psi \in \mathrm{Perm}([n])$ we shall denote

$$E_{\psi(i)} := \{\psi(i), \dots, \psi(n)\},$$

$i \in [n]$, and $E_{\psi(n+1)} = \emptyset$ by convention. We shall work with nonnegative real-valued functions, which, because of the domain, can be identified with vectors. We shall use the notation $\mathbf{x} = (x_1, \dots, x_n)$, $x_i \in [0, \infty)$, $i \in [n]$, and the set of all these vectors we denote by $\mathbf{F}$. By $\mathbf{F}^{[0,1]}$ we denote the set of $\mathbf{x} \in \mathbf{F}$ with $x_i \in [0, 1]$, $i \in [n]$. By $\mathbf{1}_B\colon [n] \to \{0, 1\}$ we shall denote the indicator vector of a set $B \subseteq [n]$, i.e., it is the n-tuple such that its i-th component is 1, if $i \in B$, otherwise 0. Further, we shall work with the conditional aggregation operator introduced in [2].

Definition 1. A map $\mathsf{A}(\cdot|B)\colon \mathbf{F} \to [0, \infty)$ is called a *conditional aggregation operator* (CAO for short) with respect to a set $B \in 2^{[n]} \setminus \{\emptyset\}$, if

(i) $\mathsf{A}(\mathbf{x}|B) \leq \mathsf{A}(\mathbf{y}|B)$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{F}$ such that $x_i \leq y_i$ for any $i \in B$,

(ii) $\mathsf{A}(\mathbf{1}_{B^c}|B) = 0$.

For $B = \emptyset$ we consider the convention $\mathsf{A}(\cdot|\emptyset) = 0$. Some examples of CAOs are the *zero* operator $\mathsf{A}^{\mathrm{zero}}(\mathbf{x}|B) = 0$, the *minimum* $\mathsf{A}^{\min}(\mathbf{x}|B) = \min_{i\in B} x_i$, the *maximum* $\mathsf{A}^{\max}(\mathbf{x}|B) = \max_{i\in B} x_i$, also the *sum* $\mathsf{A}^{\mathrm{sum}}(\mathbf{x}|B) = \sum_{i\in B} x_i$, the *arithmetic mean* $\mathsf{A}^{\mathrm{mean}}(\mathbf{x}|B) = \frac{1}{|B|}\sum_{i\in B} x_i$, the *projection* $\mathsf{A}^{\mathrm{proj}}(\mathbf{x}|\{i\}) = x_i$, $i \in [n]$, fuzzy integrals, as the *Choquet* integral

$$\mathsf{A}^{\mathrm{Ch}_m}(\mathbf{x}|B) = \sum_{i=1}^{n} x_{\psi^\uparrow(i)}\big(m(E_{\psi^\uparrow(i)} \cap B) - m(E_{\psi^\uparrow(i+1)} \cap B)\big),$$

$m \in \mathbf{M}$ and $\psi^\uparrow \in \mathrm{Perm}([n])$ such that $x_{\psi^\uparrow(1)} \leq x_{\psi^\uparrow(2)} \leq \dots \leq x_{\psi^\uparrow(n)}$. For other examples of CAOs we recommend [2]. Further, we shall deal with *a sequence of conditional aggregation operators* (SCA for short) $(\mathsf{A}_i)_1^n$, where $\mathsf{A}_i(\cdot) := \mathsf{A}(\cdot|B_i)$ for each $i \in [n]$. Let us point out that as members of SCA, we allow considering different CAOs w.r.t. the same conditional set $B \in 2^{[n]}$, e.g. $\mathsf{A}_i = \mathsf{A}(\cdot|B)$, $\mathsf{A}_j = \widehat{\mathsf{A}}(\cdot|B)$, $i, j \in [n]$, $i \neq j$. For example, let $(\mathsf{A}_i)_1^4$ be SCA such that

$$\mathsf{A}_1 = \mathsf{A}^{\mathrm{sum}}(\cdot|\{1,3\}),\ \mathsf{A}_2 = \mathsf{A}^{\mathrm{proj}}(\cdot|\{3\}),\ \mathsf{A}_3 = \mathsf{A}^{\min}(\cdot|\{1,2,3\}), \text{ and } \mathsf{A}_4 = \mathsf{A}^{\min}(\cdot|\{1,3\}).$$

Let us note that sets B_i have no prescribed form, i.e. B_i is an arbitrary subset of $[n]$. Subscript i expresses order within a SCA and has no other meaning.

Remark 2. *Let us discuss the benefits of CAOs in comparison to standard aggregation functions*[1]*. Many times in real-life situations we aggregate the data not on the whole set, but only on*

[1]A mapping $\mathsf{Ag}\colon \mathbf{F} \to [0, \infty)$ is called an aggregation function, if $\mathsf{Ag}((0, \dots, 0)) = 0$ and $\mathsf{Ag}(\mathbf{x}) \leq \mathsf{Ag}(\mathbf{y})$ for $x_i \leq y_i$ for any $i \in [n]$.

its subset. E.g. we calculate the average salary of women or men, not only the whole population. Thus implementing a conditional set into the construction is appropriate. Let us compare CAOs with similar concepts containing conditional sets:

(i) *It seems natural to compare* $\mathsf{A}(\mathbf{x}|B)$ *and* $\mathsf{Ag}(\mathbf{x}\mathbf{1}_B)$*. These two objects coincide if* 0 *is the neutral element of* Ag*. In general, they do not equal, see the following example*

$$\mathsf{A}^{\min}(\mathbf{x}|B) = \min_{i\in B} x_i \neq \mathsf{Ag}^{\min}(\mathbf{x}\mathbf{1}_B), \quad \mathbf{x}\in\mathbf{F} \text{ such that } x_i > 0 \text{ for any } i \in B \subset [n].$$

(ii) *On the other hand, it is also natural to consider the standard aggregation function with set* B *as a parameter, e.g.* $\mathsf{Ag}_B^{\min}(\mathbf{x}) = \min_{i\in B} x_i$*. In this case, we, however, require to hold monotonicity on the whole set, which is not natural, cf. with Definition 1. Restricting monotonicity to conditional sets will allow us to deeply derive the properties of our new operator, see Subsection 3.1.*

From an implementation point of view, CAOs have the advantage that for aggregating only some components (e.g. from set $B \subset [n]$*) it is not needed to change the dimension of the input vector.*

As we have mentioned, in this paper we shall introduce on permutations dependent operator. In the following, let us present several motivations for introducing such a new concept.

Motivation 1 In the literature, there are several operators in whose construction the permutation of the basic set plays a crucial role. Let $(\mathbf{x},\mu) \in \mathbf{F}\times\mathbf{M}$.

(i) A lot of basic fuzzy integrals, among them the well-known Choquet, Shilkret, or Sugeno integral can be defined (if we restrict to finite space) in the following forms

$$\begin{aligned}
\mathrm{C}(\mathbf{x},\mu) &= \sum_{i=1}^{n} x_{\psi^\uparrow(i)} \cdot \Big(\mu(E_{\psi^\uparrow(i)}) - \mu(E_{\psi^\uparrow(i+1)})\Big) = \sum_{i=1}^{n} (x_{\psi^\uparrow(i)} - x_{\psi^\uparrow(i-1)}) \cdot \mu(E_{\psi^\uparrow(i)}),\\
\mathrm{Sh}(\mathbf{x},\mu) &= \max_{i\in[n]} \Big\{ x_{\psi^\uparrow(i)} \cdot \mu(E_{\psi^\uparrow(i)}) \Big\},\\
\mathrm{Su}(\mathbf{x},\mu) &= \max_{i\in[n]} \min\{x_{\psi^\uparrow(i)}, \mu(E_{\psi^\uparrow(i)})\},
\end{aligned}$$

where $\psi^\uparrow \in \mathrm{Perm}([n])$ such that $x_{\psi^\uparrow(1)} \le \dots \le x_{\psi^\uparrow(n)}$, with the convention $x_{\psi^\uparrow(0)} = 0$.

(ii) Recently, in [13] the authors introduced the maximal chain-based Choquet-like integral, where the permutation is the preselected parameter. Let $\mathscr{C} = \{C_0, C_1, \dots, C_n\}$ be a maximal chain[2]. The MCC integral is given as follows

$$\mathrm{MaxCh}_{\mathscr{C}}^{\otimes}(\mathbf{x},\mu) = \sum_{i=1}^{n} x_{\psi_{\mathscr{C}}(i)} \otimes (\mu(C_i) - \mu(C_{i-1})),$$

where $\psi_{\mathscr{C}} \in \mathrm{Perm}([n])$ is predetermined by a maximal chain given by the formula $\psi_{\mathscr{C}}(i) = c_i$ where c_i is the only element of the difference between two consecutive sets of the chain, i.e. $C_i \setminus C_{i-1} = \{c_i\}$, $i\in[n]$. It is worth noting that this approach relies on the well-established one-to-one correspondence between permutations on $[n]$ and maximal chains on $[n]$. Another instance of an operator with a predetermined permutation is the IOWA operator [24], where the permutation associated with the input vector is derived from another companion vector. And, the permutation remains the same for each input vector.

[2] A chain $\mathscr{C} = \{C_0, C_1, \dots, C_n\}$ is called *maximal* in $2^{[n]}$, if $\emptyset = C_0 \subset C_1 \subset \dots \subset C_n$.

Motivation 2 In the following, we mention further operators that are based on the permutation of the basic set $[n]$. Compared to the operators from Motivation 1, the input vector here is further aggregated.

(i) In [3] the authors investigated a novel concept of an operator dependent on a sequence $(\mathsf{Ag}_i)_1^n$ of aggregation functions. This operator is defined by the formula

$$\mathrm{C}_{\mathsf{Ag}}(\mathbf{x},\mu)=\sum_{i=1}^{n}\mathsf{Ag}_{\psi^{\uparrow}(i)}(\mathbf{x})\cdot(\mu(E_{\psi^{\uparrow}(i)})-\mu(E_{\psi^{\uparrow}(i+1)})), \tag{1}$$

where $\psi^{\uparrow}\in\mathrm{Perm}([n])$ such that $\mathsf{Ag}_{\psi^{\uparrow}(1)}(\mathbf{x})\leq\cdots\leq\mathsf{Ag}_{\psi^{\uparrow}(n)}(\mathbf{x})$. Let us notice that the C_{Ag} operator of $\mathbf{x}\in\mathbf{F}$ is in fact the Choquet integral of the vector $\mathbf{x}_{\mathsf{Ag}}=(\mathsf{Ag}_1(\mathbf{x}),\ldots,\mathsf{Ag}_n(\mathbf{x}))$, i.e. $\mathrm{C}_{\mathsf{Ag}}(\mathbf{x},\mu)=\mathrm{C}(\mathbf{x}_{\mathsf{Ag}},\mu)$.

(ii) Let $\mathbf{w}=(w_1,\ldots,w_n)\in\mathbf{F}^{[0,1]}$ such that $\sum\limits_{i=1}^{n}w_i=1$, and T be a t-norm[3]. The TOWA operator of $\mathbf{x}\in\mathbf{F}^{[0,1]}$ introduced in [25] is defined as

$$\mathrm{TOWA}_{\mathbf{w}}(\mathbf{x})=\sum_{i=1}^{n}w_i\mathop{\mathtt{T}}_{k=1}^{i}x_{\phi^{\downarrow}(k)},$$

where $\mathop{\mathtt{T}}\limits_{k=1}^{i}$ is the extension of T to i-ary operation[4], $i\in[n]$, and $\phi^{\downarrow}\in\mathrm{Perm}([n])$ such that $x_{\phi^{\downarrow}(1)}\geq\cdots\geq x_{\phi^{\downarrow}(n)}$.

Summarizing the aforementioned ideas, we aim to offer a unified framework encompassing recent approaches. Our primary focus is on Choquet-like permutation dependent operators. We emphasize the essence of our approach:

(i) From Motivation 1 we can observe that the permutation of the basic set that is used in the computational formula can be derived from the input vector, e.g. it reorders vector components in nondecreasing order. Or, the permutation does not have to be derived from the vector, it can be chosen arbitrarily in advance. This is the fundamental idea underlying the creation of sets of permutation pairs.

(ii) The permutation of the basic set need not be solely related to the input vector but can pertain to the aggregation of its components.

(iii) Consequently, a sequence of conditional aggregation operators is incorporated into the construction of the proposed operator.

In the next section, we introduce a new concept, on permutation dependent operator, where we formalize these observations.

3 On permutation dependent Choquet-like operator

In this section, we aim to introduce a new operator, by which we will be able to cover all the mentioned operators from the previous section. The main idea of the construction is the establishment of "databases" in which each vector is associated with its own one (arbitrarily

[3] t-norm is a binary operation on $[0,1]$, i.e. $T\colon[0,1]^2\to[0,1]$, which is commutative, associative, monotone and satisfies the boundary condition $T(x,1)=x$.

[4] The extension is meant as $\mathop{\mathtt{T}}\limits_{k=1}^{i}y_k=T\left(\mathop{\mathtt{T}}\limits_{k=1}^{i-1}y_k,y_i\right)$, $i\in[n]\setminus\{1\}$ with the convention $\mathop{\mathtt{T}}\limits_{k=1}^{1}y_k=y_1$, see [14].

Ψ^{con}			
$\mathbf{z}\in\mathbf{F}$	$\psi^{\text{con}}_{\mathbf{z}}\in\text{Perm}([3])$		
	$\psi^{\text{con}}_{\mathbf{z}}(1)$	$\psi^{\text{con}}_{\mathbf{z}}(2)$	$\psi^{\text{con}}_{\mathbf{z}}(3)$
$(0,0,1)$	3	1	2
$(0,0,2)$	3	1	2
$\vdots$		$\vdots$	
$(8,3,7)$	3	1	2
$(8,3,8)$	3	1	2
$(8,3,9)$	3	1	2
$\vdots$		$\vdots$	

$\Psi^{\uparrow}$			
$\mathbf{z}\in\mathbf{F}$	$\psi^{\uparrow}_{\mathbf{z}}\in\text{Perm}([3])$		
	$\psi^{\uparrow}_{\mathbf{z}}(1)$	$\psi^{\uparrow}_{\mathbf{z}}(2)$	$\psi^{\uparrow}_{\mathbf{z}}(3)$
$(0,0,1)$	1	2	3
$(0,0,2)$	1	2	3
$\vdots$		$\vdots$	
$(8,3,7)$	2	3	1
$(8,3,8)$	2	1	3
$(8,3,9)$	2	1	3
$\vdots$		$\vdots$	

Ψ			
$\mathbf{z}\in\mathbf{F}$	$\psi_{\mathbf{z}}\in\text{Perm}([3])$		
	$\psi_{\mathbf{z}}(1)$	$\psi_{\mathbf{z}}(2)$	$\psi_{\mathbf{z}}(3)$
$(0,0,1)$	2	3	1
$(0,0,2)$	3	1	2
$\vdots$		$\vdots$	
$(8,3,7)$	2	1	3
$(8,3,8)$	1	2	3
$(8,3,9)$	2	3	1
$\vdots$		$\vdots$	

Table 1: Examples of sets of permutation pairs

chosen) preselected permutation. Formally, let $\mathbf{z}\in\mathbf{F}$ and $\psi_{\mathbf{z}}\in\text{Perm}([n])$ be its associated permutation. Under a *set of permutation pairs* we mean

$$\Psi=\{\langle\mathbf{z},\psi_{\mathbf{z}}\rangle:\mathbf{z}\in\mathbf{F}\}.$$

The set of all such sets we shall denote by $\mathscr{P}_{\text{pair}}$. Some examples of sets of permutation pairs are given in Table 1. Note that a given vector may have a different associated permutation in different sets of permutation pairs. In accordance with the denotation we have used until now, by $\psi^{\uparrow}_{\mathbf{z}}$ we denote a permutation of $[n]$ with the property that it reorders the components of vector $\mathbf{z}$ in nondecreasing order. The set of permutation pairs $\langle\mathbf{z},\psi^{\uparrow}_{\mathbf{z}}\rangle$ we shall denote by $\Psi^{\uparrow}$, i.e. $\Psi^{\uparrow}=\{\langle\mathbf{z},\psi^{\uparrow}_{\mathbf{z}}\rangle:\mathbf{z}\in\mathbf{F}\}$. Analogously we mean $\Psi^{\downarrow}$. The set of permutation pairs with the same permutation for each vector we shall call *constant*, Ψ^{con}. An example of a constant set of permutation pairs is $\Psi^{\text{id}}=\{\langle\mathbf{z},\psi^{\text{id}}_{\mathbf{z}}\rangle:\mathbf{z}\in\mathbf{F}\}$ where $\psi^{\text{id}}_{\mathbf{z}}$ is the identity.

Remark 3. *In real-life decision-making scenarios, the permutations can allow us to establish priorities. For example, for a printing company, it can be important to print orders first for high-value customers and then for others. Moreover, the priority can change depending on the value of the order (i.e. depending on the input vector). Sets of permutation pairs are a suitable tool for modeling this situation, i.e. allow us to model the preferences depending on the "levels". For example, let us consider $\Psi\in\mathscr{P}_{\text{pair}}$ such that $\psi_{\mathbf{z}}$ changes with respect to "levels", i.e. the values of components of vectors:*

Ψ	
$\mathbf{z}\in\mathbf{F}$	$\psi_{\mathbf{z}}\in\text{Perm}([n])$
$\mathbf{z}:\max\limits_{i\in[n]} z_i\leq 100$	$\psi^{\text{id}}_{\mathbf{z}}$
$\mathbf{z}:\max\limits_{i\in[n]} z_i> 100$	$\psi^{\downarrow}_{\mathbf{z}}$

The idea of constructing a new operator depending on levels appeared in [10], where, however, the monotone measure changes w.r.t. levels.

Remark 4. *Let us note that when dealing with $\langle\mathbf{z},\psi^{\uparrow}_{\mathbf{z}}\rangle\in\Psi^{\uparrow}$ then there can exist several permutations that reorder the input vector in nondecreasing order (in case of ties among components of a vector, see $\langle(8,3,8),\psi^{\uparrow}\rangle$ in Table 1). However, in the database $\Psi^{\uparrow}$ we have to choose one. Conditions under which this ambiguity does not affect the result we shall discuss in Proposition 10.*

Let $\langle\mathbf{z},\psi_{\mathbf{z}}\rangle\in\Psi$. Following the denotation in [9], the rearranged vector according to a given permutation we shall denote by $\mathbf{z}_{\psi_{\mathbf{z}}}$, or $\mathbf{z}_{\psi}$ if no confusion may arise, i.e.

$$\mathbf{z}_{\psi}:=(z_{\psi(1)},z_{\psi(2)},\dots,z_{\psi(n)}).$$

The set of permutation pairs Ψ do not always maintains the monotonicity property, i.e. if $\mathbf{x} \leq \mathbf{y}$, $\mathbf{x}, \mathbf{y} \in \mathbf{F}$, then $\mathbf{x}_{\psi_{\mathbf{x}}} \leq \mathbf{y}_{\psi_{\mathbf{y}}}$ does not generally hold. Indeed, let us consider $\langle (8,3,8), \psi_{(8,3,8)} \rangle$ and $\langle (8,3,9), \psi_{(8,3,9)} \rangle$ in Table 1. The set of permutation pairs that maintains the monotonicity property we shall denote by Ψ^{mon}.

In view of previous ideas we can define a new operator. We shall use two (possibly different) sets of permutation pairs. One of them relates to the input vectors $\mathbf{x} \in \mathbf{F}$, and the second one relates to vectors modified by aggregations, see Motivation 2 (i). Moreover, in the following construction, we use the sequence of CAOs. Using CAOs has several advantages we have discussed in Remark 2 deeply.

Definition 5. Let $(\mathsf{A}_i)_1^n$ be a SCA, $\Psi, \Phi \in \mathscr{P}_{\mathrm{pair}}$, and $\otimes$ be a binary operation on $[0,\infty)$. Then the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator of $\mathbf{x} \in \mathbf{F}$ w.r.t. $\mu \in \mathbf{M}$ is defined as

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x}, \mu) = \sum_{i=1}^{n} \mathsf{A}_{\psi(i)}(\mathbf{x}_{\phi}) \otimes \left(\mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)})\right), \tag{2}$$

with $\langle \mathbf{x}, \phi \rangle \in \Phi$, $\langle \mathbf{x}_{\mathsf{A},\phi}, \psi \rangle \in \Psi$, $\mathbf{x}_{\mathsf{A},\phi} = (\mathsf{A}_1(\mathbf{x}_\phi), \dots, \mathsf{A}_n(\mathbf{x}_\phi))$, and $E_{\psi(i)} = \{\psi(i), \psi(i+1), \dots, \psi(n)\}$ for each $i \in [n]$ with the convention $E_{\psi(n+1)} = \emptyset$.

Remark 6. *By set of permutation pairs Φ in (2) we can effectively generate new aggregations considering only one initial aggregation. Let us consider the following CAO:*

$$\mathsf{A}(\mathbf{y}|B) = \frac{1}{|B|} \sum_{i \in B} y_i, \qquad \mathbf{y} \in \mathbf{F},\ B \subseteq [n].$$

Let $\langle \mathbf{x}, \phi^{\uparrow}_{\mathbf{x}} \rangle \in \Phi^{\uparrow}$. For $B = \{1\}$ we get the minimum of components of $\mathbf{x}$, if $B = \{n\}$ the maximum. Further, for $B = \{\frac{n}{2}\}$, if n is odd, or $B = \{\frac{n}{2}, \frac{n}{2}+1\}$, if n is even, we get the median. Of course, if $B = [n]$ we get the arithmetic mean regardless of the choice of permutation pair.

Let us demonstrate the calculation of the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator for a better understanding of this concept. In the first case, we point out how the choice of the sets of permutation pairs affects the calculation of the operator for the same input vector and monotone measure. In the second case, we point out the calculation for different vectors.

Example 7. *Let $\otimes$ be a binary operation on $[0,\infty)$, $(\mathsf{A}_i)_1^3$ be the SCA such that*

$$\mathsf{A}(\mathbf{z}|B_1) = \max_{i \in B_1} z_i, \qquad \mathsf{A}(\mathbf{z}|B_2) = \sum_{i \in B_2} \tfrac{1}{1+i} z_i, \qquad \mathsf{A}(\mathbf{z}|B_3) = \sum_{i \in B_3} z_i,$$

with $B_1 = \{1,2\}$, $B_2 = \{1,2,3\}$, $B_3 = \{2,3\}$.

(i) *Let us consider $\Phi^{\mathrm{con}}, \Psi^{\mathrm{con}}, \widehat{\Psi}^{\mathrm{con}} \in \mathscr{P}_{\mathrm{pair}}$ given in the following table:*

$\mathbf{z} \in \mathbf{F}$	Φ^{con}			Ψ^{con}			$\widehat{\Psi}^{\mathrm{con}}$		
	$\phi^{\mathrm{con}}_{\mathbf{z}}(1)$	$\phi^{\mathrm{con}}_{\mathbf{z}}(2)$	$\phi^{\mathrm{con}}_{\mathbf{z}}(3)$	$\psi^{\mathrm{con}}_{\mathbf{z}}(1)$	$\psi^{\mathrm{con}}_{\mathbf{z}}(2)$	$\psi^{\mathrm{con}}_{\mathbf{z}}(3)$	$\widehat{\psi}^{\mathrm{con}}_{\mathbf{z}}(1)$	$\widehat{\psi}^{\mathrm{con}}_{\mathbf{z}}(2)$	$\widehat{\psi}^{\mathrm{con}}_{\mathbf{z}}(3)$
(z_1, z_2, z_3)	2	3	1	3	1	2	1	3	2

Then $\mathbf{x}_{\phi^{\mathrm{con}}} = (x_2, x_3, x_1)$, $\mathbf{x}_{\mathsf{A},\phi^{\mathrm{con}}} = (\mathsf{A}_1(\mathbf{x}_{\phi^{\mathrm{con}}}), \mathsf{A}_2(\mathbf{x}_{\phi^{\mathrm{con}}}), \mathsf{A}_3(\mathbf{x}_{\phi^{\mathrm{con}}}))$, and we have

$$\begin{aligned}
\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\mathrm{con}},\Phi^{\mathrm{con}}}(\mathbf{x}, \mu) &= \mathsf{A}((x_2,x_3,x_1)|B_3) \otimes (\mu(\{1,2,3\}) - \mu(\{1,2\})) \\
&\quad + \mathsf{A}((x_2,x_3,x_1)|B_1) \otimes (\mu(\{1,2\}) - \mu(\{2\})) + \mathsf{A}((x_2,x_3,x_1)|B_2) \otimes \mu(\{2\}), \\
\mathrm{C}^{\otimes}_{\mathsf{A},\widehat{\Psi}^{\mathrm{con}},\Phi^{\mathrm{con}}}(\mathbf{x}, \mu) &= \mathsf{A}((x_2,x_3,x_1)|B_1) \otimes (\mu(\{1,2,3\}) - \mu(\{2,3\})) \\
&\quad + \mathsf{A}((x_2,x_3,x_1)|B_3) \otimes (\mu(\{2,3\}) - \mu(\{2\})) + \mathsf{A}((x_2,x_3,x_1)|B_2) \otimes \mu(\{2\}).
\end{aligned}$$

For the vector $\mathbf{x} = (18, 16, 10)$ we get

$$\begin{aligned}
\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\mathrm{con}},\Phi^{\mathrm{con}}}(\mathbf{x}, \mu) &= 28 \otimes (\mu(\{1,2,3\}) - \mu(\{1,2\})) + 16 \otimes (\mu(\{1,2\}) - \mu(\{2\})) + \tfrac{95}{6} \otimes \mu(\{2\}), \\
\mathrm{C}^{\otimes}_{\mathsf{A},\widehat{\Psi}^{\mathrm{con}},\Phi^{\mathrm{con}}}(\mathbf{x}, \mu) &= 16 \otimes (\mu(\{1,2,3\}) - \mu(\{2,3\})) + 28 \otimes (\mu(\{2,3\}) - \mu(\{2\})) + \tfrac{95}{6} \otimes \mu(\{2\}).
\end{aligned}$$

(ii) *Let us consider* $\Phi^{\mathrm{id}}, \Psi \in \mathscr{P}_{\mathrm{pair}}$ *such that*

$\mathbf{z} \in \mathbf{F}$	Ψ		
	$\psi_{\mathbf{z}}(1)$	$\psi_{\mathbf{z}}(2)$	$\psi_{\mathbf{z}}(3)$
$(18, \frac{101}{6}, 26)$	3	1	2
$(7, \frac{13}{3}, 9)$	2	1	3
$\vdots$		$\vdots$	

Let $\mathbf{x} = (18, 16, 10)$*,* $\mathbf{y} = (3, 7, 2)$*. Then* $\mathbf{x}_{\mathsf{A},\phi^{\mathrm{id}}} = (18, \frac{101}{6}, 26)$*, and* $\mathbf{y}_{\mathsf{A},\phi^{\mathrm{id}}} = (7, \frac{13}{3}, 9)$*. Then*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi^{\mathrm{id}}}(\mathbf{x}, \mu) = 26 \otimes (\mu(\{1,2,3\}) - \mu(\{1,2\})) + 18 \otimes (\mu(\{1,2\}) - \mu(\{2\})) + \tfrac{101}{6} \otimes \mu(\{2\}),$$
$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi^{\mathrm{id}}}(\mathbf{y}, \mu) = \tfrac{13}{3} \otimes (\mu(\{1,2,3\}) - \mu(\{1,3\})) + 7 \otimes (\mu(\{1,3\}) - \mu(\{3\})) + 9 \otimes \mu(\{3\}).$$

In the previous example, the conditional aggregation operator $\mathsf{A}(\cdot|B_2)$ is not symmetric, and $\mathsf{A}(\cdot|B_3)$ is not an averaging behavior operator. Thus, it can be expected that even the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator does not acquire these properties in general. The properties of $\mathrm{C}^{\otimes}_{\mathsf{A},\Phi,\Psi}$ operator we shall study in the Subsection 3.1.

Remark 8. *As we have pointed out in Introduction, the* $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ *operator covers several concepts known in the literature:*

(i) *Taking* $(\mathsf{A}_i(\cdot|[n]))_1^n$*,* $\otimes = \mathrm{Prod}$*, and* $\Phi^{\mathrm{id}}, \Psi^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$*, we get the* C_{Ag} *operator, see formula (1). Consequently,* $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ *operator covers the Choquet integral, the two-fold integral [23], the (MC)-integral [19], the Choquet-Shilkret integral [8], see [3].*

(ii) *Taking SCA such that* $\mathsf{A}(\cdot|B_i) = x_i$ *with* $B_i \ni i$ *for any* $i \in [n]$*, and* $\Phi^{\mathrm{id}}, \Psi^{\mathrm{con}} = \{\langle \mathbf{z}, \psi^{\mathrm{con}}_{\mathbf{z}} \rangle : \psi^{\mathrm{con}}_{\mathbf{z}} = \psi_{\mathscr{C}}(n-i+1), \mathbf{z} \in \mathbf{F}\}$ *with* $\psi_{\mathscr{C}}$ *described in Motivation 1 (ii), we get the* $\mathrm{MaxCh}^{\otimes}_{\mathscr{C}}$*-integral w.r.t. the maximal chain* $\mathscr{C}$*.*

(iii) *For* $\mathsf{A}_i(\mathbf{y}) = \mathsf{A}(\mathbf{y}|\{1,\dots,i\}) = \mathop{\mathrm{T}}_{k=1}^{i} y_k$ *for any* $\mathbf{y} \in \mathbf{F}^{[0,1]}$*, and* $\mu_{\mathbf{w}} \in \mathbf{M}$ *such that* $\mu_{\mathbf{w}}(E) = \sum_{i=n-|E|+1}^{n} w_i$ *for given weighted vector* $\mathbf{w} \in \mathbf{F}^{[0,1]}$ *we get*

$$\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^{\mathrm{id}},\Phi^{\downarrow}}(\mathbf{x}, \mu_{\mathbf{w}}) = \mathrm{TOWA}_{\mathbf{w}}(\mathbf{x})$$

for any $\mathbf{x} \in \mathbf{F}$*.*

The $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator is well defined. The value of $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator of a vector $\mathbf{x} \in \mathbf{F}$ w.r.t. $\mu \in \mathbf{M}$ is unique. This is thanks to the fact that SCA and sets of permutation pairs are chosen in advance. Even, the set $\Psi^{\uparrow}$ has to be chosen in advance, i.e. if several permutations of $[n]$ reorder the components of a vector in nondecreasing order, we have to choose one. Indeed, there exists $\widehat{\Psi}^{\uparrow}, \widetilde{\Psi}^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$ such that $\mathrm{C}^{\otimes}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi} \neq \mathrm{C}^{\otimes}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi}$ as the following example shows.

Example 9. *Let* $(\mathsf{A}^{\mathrm{proj}}_i)_1^3$ *be SCA,* $\widehat{\Psi}^{\uparrow}, \widetilde{\Psi}^{\uparrow}, \Phi^{\mathrm{id}} \in \mathscr{P}_{\mathrm{pair}}$ *such that*

$\mathbf{z} \in \mathbf{F}$	$\widehat{\Psi}^{\uparrow}$			$\widetilde{\Psi}^{\uparrow}$		
	$\widehat{\psi}^{\uparrow}_{\mathbf{z}}(1)$	$\widehat{\psi}^{\uparrow}_{\mathbf{z}}(2)$	$\widehat{\psi}^{\uparrow}_{\mathbf{z}}(3)$	$\widetilde{\psi}^{\uparrow}_{\mathbf{z}}(1)$	$\widetilde{\psi}^{\uparrow}_{\mathbf{z}}(2)$	$\widetilde{\psi}^{\uparrow}_{\mathbf{z}}(3)$
$(0.8, 0.6, 0.6)$	3	2	1	2	3	1
$\vdots$		$\vdots$			$\vdots$	

Let $\mathbf{x} = (0.8, 0.6, 0.6)$ *and* $\mu \in \mathbf{M}$ *such that* $\mu(\emptyset) = 0$, $\mu(\{1\}) = 0.1$, $\mu(\{1,2\}) = 0.8$, $\mu(\{1,3\}) = 0.6$, *and* $\mu(\{1,2,3\}) = 1$. *Then* $\mathsf{A}_i(\mathbf{x}_{\phi^{\mathrm{id}}}) = x_i$ *for any* $i \in [3]$, *therefore* $\mathbf{x}_{\mathsf{A},\phi} = (0.8, 0.6, 0.6)$, *and*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi^{\mathrm{id}}}(\mathbf{x},\mu) = 0.6 \otimes (\mu(\{1,2,3\}) - \mu(\{1,2\})) + 0.6 \otimes (\mu(\{1,2\}) - \mu(\{1\})) + 0.8 \otimes \mu(\{1\}),$$
$$\mathrm{C}^{\otimes}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi^{\mathrm{id}}}(\mathbf{x},\mu) = 0.6 \otimes (\mu(\{1,2,3\}) - \mu(\{1,3\})) + 0.6 \otimes (\mu(\{1,3\}) - \mu(\{1\})) + 0.8 \otimes \mu(\{1\}).$$

For $\otimes = \max$, *we get*

$$\mathrm{C}^{\max}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi^{\mathrm{id}}}(\mathbf{x},\mu) = 0.6 + 0.7 + 0.8 = 2.1,$$
$$\mathrm{C}^{\max}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi^{\mathrm{id}}}(\mathbf{x},\mu) = 0.6 + 0.6 + 0.8 = 2,$$

thus $\mathrm{C}^{\max}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi^{\mathrm{id}}}(\mathbf{x},\mu) \neq \mathrm{C}^{\max}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi^{\mathrm{id}}}(\mathbf{x},\mu)$.

In the case of basic fuzzy integrals (the Choquet, the Shilkret, the Sugeno integral), each permutation of $[n]$ that reorders the components of the input vector monotonically leads to the same value of the integral. Let $(\mathsf{A}_i)_1^n$ be a SCA, $\Phi, \widehat{\Psi}^{\uparrow}, \widetilde{\Psi}^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$. It is easy to see, that $\mathrm{C}^{\otimes}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu) = \mathrm{C}^{\otimes}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu)$ for each $\mathbf{x} \in \mathbf{F}$ and each symmetric monotone measure $\mu \in \mathbf{M}$. Indeed,

$$\mathsf{A}_{\widehat{\psi}^{\uparrow}(i)}(\mathbf{x}_{\phi}) = \mathsf{A}_{\widetilde{\psi}^{\uparrow}(i)}(\mathbf{x}_{\phi}) \text{ and } \mu(E_{\widehat{\psi}^{\uparrow}(i)}) - \mu(E_{\widehat{\psi}^{\uparrow}(i+1)}) = \mu(E_{\widetilde{\psi}^{\uparrow}(i)}) - \mu(E_{\widetilde{\psi}^{\uparrow}(i+1)})$$

for each $i \in [n]$. However, more interesting is to find conditions under which does the operator $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\uparrow},\Phi}$ not depend on the choice of $\Psi^{\uparrow}$ for any $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}$, i.e.

$$\mathrm{C}^{\otimes}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi} = \mathrm{C}^{\otimes}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi} \text{ for any } \widehat{\Psi}^{\uparrow}, \widetilde{\Psi}^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}.$$

In the following, we shall use the denotation

$$S_{\mathsf{A},\Phi} := \{\mathsf{A}_i(\mathbf{x}_{\phi}) : \mathsf{A}_{\psi^{\uparrow}(i)}(\mathbf{x}_{\phi}) = \mathsf{A}_{\psi^{\uparrow}(i+1)}(\mathbf{x}_{\phi}),\, \mathbf{x} \in \mathbf{F}, i \in [n-1]\}$$

with $\langle \mathbf{x}, \phi \rangle \in \Phi$ and $\langle \mathbf{x}_{\mathsf{A},\phi}, \psi^{\uparrow} \rangle \in \Psi^{\uparrow}$, $\Phi, \Psi^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$. Note that $\Psi^{\uparrow}$ does not appear in the denotation $S_{\mathsf{A},\Phi}$, since the choice of $\Psi^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$ does not affect the set of $S_{\mathsf{A},\Phi}$.

Proposition 10. *Let* $(\mathsf{A}_i)_1^n$ *be a SCA,* $\Phi \in \mathscr{P}_{\mathrm{pair}}$, *and* $\otimes\colon [0,\infty) \times [0,\infty) \to [0,\infty)$. *Then the following assertions are equivalent*

(i) $a \otimes (b - c) = a \otimes b - a \otimes c + a \otimes 0$ *for each* $(a,b,c) \in S_{\mathsf{A},\Phi} \times [0,\infty)^2$, $b \geq c$.

(ii) $\mathrm{C}^{\otimes}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu) = \mathrm{C}^{\otimes}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu)$ *for each* $\widehat{\Psi}^{\uparrow}, \widetilde{\Psi}^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$ *and for each* $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}$.

Proof. For simplicity, let us denote

$$y_{\widetilde{\psi}^{\uparrow}(i)} := \mathsf{A}_{\widetilde{\psi}^{\uparrow}(i)}(\mathbf{x}_{\phi}), \quad y_{\widehat{\psi}^{\uparrow}(i)} := \mathsf{A}_{\widehat{\psi}^{\uparrow}(i)}(\mathbf{x}_{\phi}), \quad z_{\widetilde{\psi}^{\uparrow}(i)} := \mu(E_{\widetilde{\psi}^{\uparrow}(i)}), \quad z_{\widehat{\psi}^{\uparrow}(i)} := \mu(E_{\widehat{\psi}^{\uparrow}(i)}),$$

$i \in [n]$. It is easy to see that $y_{\widetilde{\psi}^{\uparrow}(i)} = y_{\widehat{\psi}^{\uparrow}(i)} := y_{\psi^{\uparrow}(i)}$ for any $i \in [n]$. In addition, $z_{\widetilde{\psi}^{\uparrow}(i)} \geq z_{\widetilde{\psi}^{\uparrow}(i+1)}$ and $z_{\widehat{\psi}^{\uparrow}(i)} \geq z_{\widehat{\psi}^{\uparrow}(i+1)}$ for any $i \in [n]$. Moreover, for $i = 1, n+1$, and for each $i \in [n] \setminus \{1\}$ such that $y_{\psi^{\uparrow}(i-1)} < y_{\psi^{\uparrow}(i)}$ it holds

$$z_{\widetilde{\psi}^{\uparrow}(i)} = z_{\widehat{\psi}^{\uparrow}(i)}.$$

Indeed, $\{\widetilde{\psi}(1), \widetilde{\psi}(2), \ldots, \widetilde{\psi}(n)\} = \{\widehat{\psi}(1), \widehat{\psi}(2), \ldots, \widehat{\psi}(n)\}$, therefore $z_{\widetilde{\psi}^{\uparrow}(1)} = z_{\widehat{\psi}^{\uparrow}(1)}$. Further, if $y_{\psi^{\uparrow}(i-1)} < y_{\psi^{\uparrow}(i)}$ for some $i \in [n] \setminus \{1\}$, then $E_{\widetilde{\psi}^{\uparrow}(i)} = E_{\widehat{\psi}^{\uparrow}(i)}$, thence $\mu(E_{\widetilde{\psi}^{\uparrow}(i)}) = \mu(E_{\widehat{\psi}^{\uparrow}(i)})$.
Let us prove the implication $\Rightarrow$. We show that part (i) of this assertion implies the equality

$$\sum_{i \in [n]} y_{\psi^{\uparrow}(i)} \otimes (z_{\widetilde{\psi}^{\uparrow}(i)} - z_{\widetilde{\psi}^{\uparrow}(i+1)}) = \sum_{i \in [n]} y_{\psi^{\uparrow}(i)} \otimes (z_{\widehat{\psi}^{\uparrow}(i)} - z_{\widehat{\psi}^{\uparrow}(i+1)}).$$

Let us consider an arbitrary, fixed $i \in [n-1]$, and let us distinguish two cases

- If $y_{\psi^\uparrow(i)} \notin S_{\mathsf{A},\Phi}$, then $y_{\psi^\uparrow(i-1)} < y_{\psi^\uparrow(i)} < y_{\psi^\uparrow(i+1)}$ (specifically, for $i = 1$ we have only $y_{\psi^\uparrow(1)} < y_{\psi^\uparrow(2)}$). Therefore $z_{\widetilde{\psi}^\uparrow(i)} = z_{\widehat{\psi}^\uparrow(i)}$, $z_{\widetilde{\psi}^\uparrow(i+1)} = z_{\widehat{\psi}^\uparrow(i+1)}$ and we have $y_{\psi^\uparrow(i)} \otimes (z_{\widetilde{\psi}^\uparrow(i)} - z_{\widetilde{\psi}^\uparrow(i+1)}) = y_{\psi^\uparrow(i)} \otimes (z_{\widehat{\psi}^\uparrow(i)} - z_{\widehat{\psi}^\uparrow(i+1)})$.
- Let $y_{\psi^\uparrow(i)} \in S_{\mathsf{A},\Phi}$. Let us denote

$$\gamma := \{j \in [n] : y_{\psi^\uparrow(j)} = y_{\psi^\uparrow(i)}\}.$$

Further, let us denote $i_* := \min \gamma$, $i^* := \max \gamma$. It is easy to see that $\gamma = \{i_*, i_*+1, \dots, i^*\}$ and we have

$$\begin{aligned}
\sum_{i\in\gamma} y_{\psi^\uparrow(i)} \otimes (z_{\widetilde{\psi}^\uparrow(i)} - z_{\widetilde{\psi}^\uparrow(i+1)}) &= \sum_{i\in\gamma} y_{\psi^\uparrow(i)} \otimes z_{\widetilde{\psi}^\uparrow(i)} - y_{\psi^\uparrow(i)} \otimes z_{\widetilde{\psi}^\uparrow(i+1)} + y_{\psi^\uparrow(i)} \otimes 0\\
&= y_{\psi^\uparrow(i_*)} \otimes z_{\widetilde{\psi}^\uparrow(i_*)} - y_{\psi^\uparrow(i^*)} \otimes z_{\widetilde{\psi}^\uparrow(i^*+1)} + |\gamma| \cdot (y_{\psi^\uparrow(i^*)} \otimes 0)\\
\sum_{i\in\gamma} y_{\psi^\uparrow(i)} \otimes (z_{\widehat{\psi}^\uparrow(i)} - z_{\widehat{\psi}^\uparrow(i+1)}) &= \sum_{i\in\gamma} y_{\psi^\uparrow(i)} \otimes z_{\widehat{\psi}^\uparrow(i)} - y_{\psi^\uparrow(i)} \otimes z_{\widehat{\psi}^\uparrow(i+1)} + y_{\psi^\uparrow(i)} \otimes 0\\
&= y_{\psi^\uparrow(i_*)} \otimes z_{\widehat{\psi}^\uparrow(i_*)} - y_{\psi^\uparrow(i^*)} \otimes z_{\widehat{\psi}^\uparrow(i^*+1)} + |\gamma| \cdot (y_{\psi^\uparrow(i^*)} \otimes 0).
\end{aligned}$$

However, both expressions coincide: Indeed, for $i_* = 1$ we trivially have $z_{\widetilde{\psi}^\uparrow(1)} = z_{\widehat{\psi}^\uparrow(1)}$. Further for $i_* \neq 1$: Since $y_{\psi^\uparrow(i_*-1)} < y_{\psi^\uparrow(i_*)}$, then $z_{\widetilde{\psi}^\uparrow(i_*)} = z_{\widehat{\psi}^\uparrow(i_*)}$. Moreover, for $i^* = n$ we trivially have $z_{\widetilde{\psi}^\uparrow(n+1)} = z_{\widehat{\psi}^\uparrow(n+1)}$. Further for $i_* \neq n$: Since $y_{\psi^\uparrow(i^*)} < y_{\psi^\uparrow(i^*+1)}$, then $z_{\widetilde{\psi}^\uparrow(i^*+1)} = z_{\widehat{\psi}^\uparrow(i^*+1)}$.

$\Leftarrow$ Let us consider $i \in [n-1]$ such that $y_{\psi^\uparrow(i)} = y_{\psi^\uparrow(i+1)} := a$. Since part (ii) holds for each $\mu \in \mathbf{M}$, then it also holds for

$$z_{\widetilde{\psi}^\uparrow(j)} = b,\ z_{\widehat{\psi}^\uparrow(j)} = c \text{ for } j \leq i,\ z_{\widetilde{\psi}^\uparrow(i+1)} = z_{\widehat{\psi}^\uparrow(i+1)} = c, \text{ and } z_{\widetilde{\psi}^\uparrow(j)} = z_{\widehat{\psi}^\uparrow(j)} = 0 \text{ for } j > i+1,$$

$b \geq c$. So, we have

$$a \otimes (b - c) = a \otimes (c - c). \tag{3}$$

On the other hand, the equality holds for

$$z_{\widetilde{\psi}^\uparrow(j)} = c,\ z_{\widehat{\psi}^\uparrow(j)} = b \text{ for } j \leq i, \text{ and } z_{\widetilde{\psi}^\uparrow(j)} = z_{\widehat{\psi}^\uparrow(j)} = 0 \text{ for } j \geq i+1,$$

$b \geq c$. Then we have

$$a \otimes (c - 0) = a \otimes (b - 0). \tag{4}$$

Summing up the equations (3), (4), we have

$$a \otimes (b - c) + a \otimes c = a \otimes b + a \otimes 0,$$

which is the required result. □

Remark 11. *If* 0 *is the right zero element of* $\otimes$*, then the condition (i) in the previous proposition simplifies to the left distributivity of* $\otimes$ *over* $-$ *on the corresponding set. Moreover, the previous proposition is the alternative to Theorem 3.2 from [3] to prove that some generalizations of the Choquet integral are well defined. E.g. for* $\mathsf{A}_i(\cdot) = \mathsf{A}^{\mathrm{proj}}(\cdot|\{i\})$ *for any* $i \in [n]$*,* Φ^{id}*,* $\Psi^\uparrow$ *and* $\otimes = \circ$ *we get the operator from [12]. In this case* $S_{\mathsf{A},\Phi^{\mathrm{id}}} = [0,\infty)$*.*

To complete this discussion, let us mention that for each $\widehat{\Phi}^\uparrow, \widetilde{\Phi}^\uparrow \in \mathscr{P}_{\mathrm{pair}}$ it always holds $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\widehat{\Phi}^\uparrow}(\mathbf{x},\mu) = \mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\widetilde{\Phi}^\uparrow}(\mathbf{x},\mu)$ for any $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}$. If the binary operation $\otimes$ satisfies the condition (i) of Proposition 10, the value of the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^\uparrow,\Phi}$ operator does not depend on $\Psi^\uparrow$. Thus, one can use permutations $\psi^\uparrow$ in formula (2) and simply write $\mathrm{C}^{\otimes}_{\mathsf{A},\Phi}$.

3.1 On some properties of the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator

In Remark 3 we pointed out the possible application of the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator in real situations, and we will see its benefits also in Section 5. Many applications using data aggregation require certain properties from the aggregation. For example, in image processing, the aggregation of multiple colors (new colors) must be within the color scale. This is the averaging behavior property, which is for monotone aggregations equivalent to the idempotency.

Definition 12. Let $\otimes, \circ$ be binary operations on $[0,\infty)$, i.e. $\otimes, \circ\colon [0,\infty)\times[0,\infty)\to[0,\infty)$.

- (i) $\otimes$ is *left (right) distributive* over $\circ$, if $a\otimes(b\circ c) = (a\otimes b)\circ(a\otimes c)$ $[(b\circ c)\otimes a = (b\otimes a)\circ(c\otimes a)]$ for any $a,b,c\in[0,\infty)$.
- (ii) $\otimes$ is *left (right) nondecreasing* if $a\otimes c\leq b\otimes c$ $[c\otimes a\leq c\otimes b]$ for any $a,b,c\in[0,\infty)$ such that $a\leq b$.
- (iii) the element $z\in[0,\infty)$ is *left (right) zero element*, if $z\otimes a = z$ $[a\otimes z = z]$ for any $a\in[0,\infty)$,
- (iv) the element $e\in[0,\infty)$ is *left (right) identity* , if $e\otimes a = a$ $[a\otimes e = a]$ for any $a\in[0,\infty)$,
- (v) the element $a^{-1}\in[0,\infty)$ is *left (right) inverse* element to element $a\in[0,\infty)$, if $a^{-1}\otimes a = e$ $[a\otimes a^{-1} = e]$, where $e\in[0,\infty)$ is the identity element.

If the binary operation $\otimes$ is left and right distributive we shall simply call it *distributive*. Similarly with other properties.

The solution to the following problem will help us to derive some properties. As it is known the Choquet integral can be expressed by several equivalent formulas, among them formulas mentioned in Section 2, part Motivation. Thus, we are motivated to ask under which conditions the following equality holds:

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) \stackrel{?}{=} \sum_{i=1}^{n} \max\{0, \mathsf{A}_{\psi(i)}(\mathbf{x}_{\phi}) - \mathsf{A}_{\psi(i-1)}(\mathbf{x}_{\phi})\}\otimes\mu(E_{\psi(i)}),$$

with the convention $\mathsf{A}_{\psi(0)}(\cdot) = 0$. In contrast to the standard Choquet integral, the previous equality does not have to hold in general.

Example 13. *Let us consider the SCA from the Example 7, the binary operation* $\otimes = \mathrm{Prod}$, *and* $\Psi^{\mathrm{id}},\Phi^{\mathrm{id}}\in\mathscr{P}_{\mathrm{pair}}$. *Thus,* $\mathbf{x}_{\phi^{\mathrm{id}}} = \mathbf{x} = (18,12,16)$, $\mathbf{x}_{\mathsf{A},\phi^{\mathrm{id}}} = (\mathsf{A}_1(\mathbf{x}_{\phi^{\mathrm{id}}}), \mathsf{A}_2(\mathbf{x}_{\phi^{\mathrm{id}}}), \mathsf{A}_3(\mathbf{x}_{\phi^{\mathrm{id}}}))$ *with*

$$\mathsf{A}_1(\mathbf{x}_{\phi^{\mathrm{id}}}) = \max\{18,12\} = 18,\quad \mathsf{A}_2(\mathbf{x}_{\phi^{\mathrm{id}}}) = \tfrac{1}{2}\cdot 18 + \tfrac{1}{3}\cdot 12 + \tfrac{1}{4}\cdot 16 = 17,\quad \mathsf{A}_3(\mathbf{x}_{\phi^{\mathrm{id}}}) = 12 + 16 = 28.$$

Then, $\sum_{i=1}^{n} \max\{0, \mathsf{A}_{\psi^{\mathrm{id}}(i)}(\mathbf{x}_{\phi^{\mathrm{id}}}) - \mathsf{A}_{\psi^{\mathrm{id}}(i-1)}(\mathbf{x}_{\phi^{\mathrm{id}}})\}\otimes\mu(E_{\psi^{\mathrm{id}}(i)})$, $\mu\in\mathbf{M}$, *is equal to*

$$\begin{aligned}\max\{0,18\}\cdot\mu(\{1,2,3\}) + \max\{0,-1\}\cdot\mu(\{2,3\}) + \max\{0,11\}\cdot\mu(\{3\})\\ = 18\cdot\mu(\{1,2,3\}) + 11\cdot\mu(\{3\}),\end{aligned}$$

and $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\mathrm{id}},\Phi^{\mathrm{id}}}(\mathbf{x},\mu) = 18\cdot\mu(\{1,2,3\}) - 1\cdot\mu(\{2,3\}) + 11\cdot\mu(\{3\})$. *Thus, we see that* $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\mathrm{id}},\Phi^{\mathrm{id}}}(\mathbf{x},\mu) \neq \sum_{i=1}^{n} \max\{0, \mathsf{A}_{\psi^{\mathrm{id}}(i)}(\mathbf{x}_{\phi^{\mathrm{id}}}) - \mathsf{A}_{\psi^{\mathrm{id}}(i-1)}(\mathbf{x}_{\phi^{\mathrm{id}}})\}\otimes\mu(E_{\psi^{\mathrm{id}}(i)})$ *in general (if* $\mu(\{2,3\})\neq 0$*).*

In the following, we shall use the denotation

$$S^{*}_{\mathsf{A},\Psi,\Phi} := \{(\mathsf{A}_{\psi(i)}(\mathbf{x}_{\phi}), \mathsf{A}_{\psi(i+1)}(\mathbf{x}_{\phi})) : \mathbf{x}\in\mathbf{F},\ i\in[n-1]\}.$$

Proposition 14. *Let $(\mathsf{A}_i)_1^n$ be a SCA, $\Psi, \Phi \in \mathscr{P}_{\mathrm{pair}}$, and $\otimes$ be a binary operation on $[0,\infty)$ with 0 being right zero element. The following assertions are equivalent:*

(i) *For any $(a,d) \in S^*_{\mathsf{A},\Psi,\Phi}$, $b \geq c \geq 0$ it holds*

$$a \otimes (b-c) + d \otimes c = a \otimes b + \max\{0, d-a\} \otimes c.$$

(ii) $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) = \sum_{i=1}^{n} \max\{0, \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) - \mathsf{A}_{\psi(i-1)}(\mathbf{x}_\phi)\} \otimes \mu(E_{\psi(i)})$ *for any $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}$.*

Proof. Let us prove the implication (i) $\Rightarrow$ (ii). For simplicity, let us denote

$$\mathrm{C}^{\otimes,2-\mathrm{type}}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) := \sum_{i=1}^{n} \max\{0, \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) - \mathsf{A}_{\psi(i-1)}(\mathbf{x}_\phi)\} \otimes \mu(E_{\psi(i)}).$$

We show that

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) + \sum_{i=2}^{n} \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(i)}) = \mathrm{C}^{\otimes,2-\mathrm{type}}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) + \sum_{i=2}^{n} \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(i)}). \tag{5}$$

This will imply the desired result. Indeed,

$$\begin{aligned}
&\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) + \sum_{i=2}^{n} \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(i)}) \\
&= \mathsf{A}_{\psi(n)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(n)}) + \sum_{i=1}^{n-1} \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) \otimes \big(\mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)})\big) + \mathsf{A}_{\psi(i+1)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(i+1)}) \\
&= \mathsf{A}_{\psi(n)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(n)}) + \sum_{i=1}^{n-1} \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(i)}) + \max\{0, \mathsf{A}_{\psi(i+1)}(\mathbf{x}_\phi) - \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi)\} \otimes \mu(E_{\psi(i+1)}) \\
&= \sum_{i=2}^{n} \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(i)}) + \mathsf{A}_{\psi(1)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(1)}) + \sum_{i=1}^{n-1} \max\{0, \mathsf{A}_{\psi(i+1)}(\mathbf{x}_\phi) - \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi)\} \otimes \mu(E_{\psi(i+1)}) \\
&= \sum_{i=2}^{n} \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(i)}) + \sum_{i=1}^{n} \max\{0, \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) - \mathsf{A}_{\psi(i-1)}(\mathbf{x}_\phi)\} \otimes \mu(E_{\psi(i)}) \\
&= \mathrm{C}^{\otimes,2-\mathrm{type}}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) + \sum_{i=2}^{n} \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) \otimes \mu(E_{\psi(i)}),
\end{aligned}$$

where the second equality holds because of (i) with $a = \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi)$, $b = \mu(E_{\psi(i)})$, $c = \mu(E_{\psi(i+1)})$, $d = \mathsf{A}_{\psi(i+1)}(\mathbf{x}_\phi)$ and the fourth equality holds because $\mathsf{A}_{\psi(1)}(\mathbf{x}_\phi) = \max\{0, \mathsf{A}_{\psi(1)}(\mathbf{x}_\phi) - \mathsf{A}_{\psi(0)}(\mathbf{x}_\phi)\}$.

Let us prove the implication (ii) $\Rightarrow$ (i). Let $i \in [n-1]$ be arbitrary, but fixed. If the equality holds for each monotone measure, then it holds also for $\mu, \mu^* \in \mathbf{M}$ such that

$$\begin{gathered}
\mu(E_{\psi(j)}) = b \text{ for } j \leq i,\ \mu(E_{\psi(i+1)}) = c \text{ and } \mu(E_{\psi(j)}) = 0 \text{ for } j > i+1,\ b \geq c \geq 0, \\
\mu^*(E_{\psi(j)}) = b \text{ for } j \leq i,\ \mu^*(E_{\psi(i+1)}) = 0 \text{ and } \mu^*(E_{\psi(j)}) = 0 \text{ for } j > i+1,\ b \geq c \geq 0.
\end{gathered}$$

Moreover, the equality holds for each $\mathbf{x} \in \mathbf{F}$, therefore it holds for $a := \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi)$, $d := \mathsf{A}_{\psi(i+1)}(\mathbf{x}_\phi)$. Further, from (ii) for μ and μ^* we get two equations. By subtracting both equations and because 0 is the right zero element of $\otimes$ we have:

$$a \otimes (b-c) + d \otimes c = a \otimes b + \max\{0, d-a\} \otimes c.$$

$\square$

Remark 15. *Condition (i) of Proposition 14 is satisfied e.g. for*

(i) $\otimes = \otimes_{\mathrm{zero}}$, *and any $(\mathsf{A}_i)_1^n$, $\Psi, \Phi \in \mathscr{P}_{\mathrm{pair}}$, $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}$,*

(ii) $(\mathsf{A}_i)_1^n$ *such that* $\mathsf{A}_i = \mathsf{A}_i^{\text{zero}}$, $\otimes$ *with* 0 *being zero element, and any* $\Psi, \Phi \in \mathscr{P}_{\text{pair}}$, $(\mathbf{x}, \mu) \in \mathbf{F} \times \mathbf{M}$,

(iii) $\Psi = \Psi^{\uparrow}$, $\otimes$ *distributive over* $-$, *and any* $(\mathsf{A}_i)_1^n$, $\Phi \in \mathscr{P}_{\text{pair}}$, $(\mathbf{x}, \mu) \in \mathbf{F} \times \mathbf{M}$.

In the following we shall investigate the monotonicity property of the $\mathrm{C}_{\mathsf{A},\Psi,\Phi}$ operator w.r.t. monotone measures as well as w.r.t. vectors. These properties are not satisfied in general, see Example 16.

Example 16. *Let* $(\mathsf{A}_i)_1^3$ *be a SCA with* $\mathsf{A}^{\text{proj}}(\cdot|\{i\})$ *for any* $i \in [3]$. *Let* $\mathbf{x} = (2,3,4), \mathbf{y} = (2,6,5)$. *Thus,* $\mathbf{x} \leq \mathbf{y}$. *Let* $\mu, \nu \in \mathbf{M}$ *are such that*

B	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(B)$	0	0.1	0.4	0.2	0.5	0.7	0.9	1
$\nu(B)$	0	0.1	0.3	0.1	0.4	0.6	0.8	1

So, $\nu \leq \mu$. *Let* $\otimes$ *be a binary operation on* $[0,\infty)$ *such that* $a \otimes b = \max\{a, 12b\}$, *and* $\Phi, \Psi^{\downarrow} \in \mathscr{P}_{\text{pair}}$ *such that*

$\mathbf{z} \in \mathbf{F}$	Φ			$\Psi^{\downarrow}$		
	$\phi_{\mathbf{z}}(1)$	$\phi_{\mathbf{z}}(2)$	$\phi_{\mathbf{z}}(3)$	$\psi^{\downarrow}_{\mathbf{z}}(1)$	$\psi^{\downarrow}_{\mathbf{z}}(2)$	$\psi^{\downarrow}_{\mathbf{z}}(3)$
$(2,3,4)$	2	1	3	3	2	1
$(2,6,5)$	1	3	2	2	3	1
$(3,2,4)$	2	3	1	3	1	2
$(2,5,6)$	3	1	2	3	2	1
$\vdots$		$\vdots$			$\vdots$	

Then $\mathbf{x}_{\phi_{\mathbf{x}}} = \mathbf{x}_{\mathsf{A},\phi_{\mathbf{x}}} = (3,2,4)$, *and* $\mathbf{y}_{\phi_{\mathbf{y}}} = \mathbf{y}_{\mathsf{A},\phi_{\mathbf{y}}} = (2,5,6)$. *Therefore*

$$\begin{aligned}
\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\downarrow},\Phi}(\mathbf{x},\mu) &= \sum_{i=1}^{3} \mathsf{A}_{\psi^{\downarrow}(i)}(\mathbf{x}_{\phi_{\mathbf{x}}}) \otimes \Big(\mu(E_{\psi^{\downarrow}(i)}) - \mu(E_{\psi^{\downarrow}(i+1)})\Big) = \\
&= \max\{4, 12\cdot(1-0.5)\} + \max\{3, 12\cdot(0.5-0.4)\} + \max\{2, 12\cdot 0.4\} = 13.8, \\
\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\downarrow},\Phi}(\mathbf{y},\mu) &= \sum_{i=1}^{3} \mathsf{A}_{\psi^{\downarrow}(i)}(\mathbf{y}_{\phi_{\mathbf{y}}}) \otimes \Big(\mu(E_{\psi^{\downarrow}(i)}) - \mu(E_{\psi^{\downarrow}(i+1)})\Big) = \\
&= \max\{6, 12\cdot(1-0.5)\} + \max\{5, 12\cdot(0.5-0.1)\} + \max\{2, 12\cdot 0.1\} = 13, \\
\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\downarrow},\Phi}(\mathbf{y},\nu) &= \sum_{i=1}^{3} \mathsf{A}_{\psi^{\downarrow}(i)}(\mathbf{y}_{\phi_{\mathbf{y}}}) \otimes \Big(\nu(E_{\psi^{\downarrow}(i)}) - \nu(E_{\psi^{\downarrow}(i+1)})\Big) = \\
&= \max\{6, 12\cdot(1-0.4)\} + \max\{5, 12\cdot(0.4-0.1)\} + \max\{2, 12\cdot 0.1\} = 14.2.
\end{aligned}$$

Thus, $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\downarrow},\Phi}(\mathbf{x},\mu) \geq \mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\downarrow},\Phi}(\mathbf{y},\mu)$, *and* $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\downarrow},\Phi}(\mathbf{y},\nu) \geq \mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\downarrow},\Phi}(\mathbf{y},\mu)$. *In other words, the* $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ *operator is not monotone neither w.r.t. vectors nor monotone measures.*

Trivially, the monotonicity property holds for $\otimes = \otimes_{\text{con}}$[5] or if $\mathsf{A}_i = \mathsf{A}_i^{\text{zero}}$ for any $i \in [n]$ and 0 is left zero element of $\otimes$. But, in the following, we try to state nontrivial sufficient conditions to hold the monotonicity. We discuss not only monotonicity w.r.t. vectors and monotone measures but also w.r.t. binary operations and SCAs.

[5]Under the denotation $\otimes_{\text{con}}$ we mean the binary operation on $[0,\infty)$ for which there exists $\alpha \in [0,\infty)$ such that $a \otimes_{\text{con}} b = \alpha$ for any $a, b \in [0,\infty)$.

Proposition 17. *Let $\otimes, \otimes_1, \otimes_2$ be a binary operations on $[0,\infty)$.*

(i) *Let $\otimes$ be left nondecreasing. If $\mathbf{x} \leq \mathbf{y}$, then $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\mathrm{con}},\Phi^{\mathrm{mon}}}(\mathbf{x},\mu) \leq \mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\mathrm{con}},\Phi^{\mathrm{mon}}}(\mathbf{y},\mu)$ for each $(\mathsf{A}_i)_1^n$, and for each $\mu \in \mathbf{M}$.*

(ii) *Let $\otimes$ be right nondecreasing with 0 as the right zero element and*

$$a \otimes (b-c) + d \otimes c = a \otimes b + \max\{0, d-a\} \otimes c \text{ for any } a,b,c,d \in [0,\infty),\ b \geq c.$$

If $\mu \leq \nu$, then $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) \leq \mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\nu)$ for each $\mathbf{x} \in \mathbf{F}$, $(\mathsf{A}_i)_1^n$, $\Psi, \Phi \in \mathscr{P}_{\mathrm{pair}}$.

(iii) *Let $\otimes$ be left nondecreasing. If $(\mathsf{A}_i)_1^n$ and $(\widehat{\mathsf{A}}_i)_1^n$ are SCAs such that $\mathsf{A}_i(\cdot) \leq \widehat{\mathsf{A}}_i(\cdot)$ for any $i \in [n]$, then $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\mathrm{con}},\Phi} \leq \mathrm{C}^{\otimes}_{\widehat{\mathsf{A}},\Psi^{\mathrm{con}},\Phi}$ for each $\Psi, \Phi \in \mathscr{P}_{\mathrm{pair}}$.*

(iv) *If $\otimes_1 \leq \otimes_2$*[6]*, then $\mathrm{C}^{\otimes_1}_{\mathsf{A},\Psi,\Phi} \leq \mathrm{C}^{\otimes_2}_{\mathsf{A},\Psi,\Phi}$ for each $(\mathsf{A}_i)_1^n$, $\Psi, \Phi \in \mathscr{P}_{\mathrm{pair}}$.*

Proof. Parts (i), (iii), (iv) follow from the definition, part (ii) is based on Proposition 14 and the monotonicity of $\otimes$. □

Natural required properties from the application point of view are averaging behavior and idempotency of the investigated operators. In the following proposition, we consider a conditional aggregation operator $\mathsf{A}(\cdot|B)$ w.r.t. a set $B \in 2^{[n]}$ that is *idempotent*, i.e. $\mathsf{A}(c\mathbf{1}_{[n]}|B) = c$ for any $c \in [0,\infty)$.

Proposition 18. *Let $(\mathsf{A}_i)_1^n$ be a SCA, $\otimes\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ left distributive over $-$, 1 is right identity, 0 is right zero element. Then*

(i) *If $\otimes$ is left nondecreasing and $\mathsf{A}_i^{\min} \leq \mathsf{A}_i \leq \mathsf{A}_i^{\max}$ for any $i \in [n]$, then*

$$\min_{i \in [n]} x_i \leq \mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) \leq \max_{i \in [n]} x_i$$

for any $\Psi, \Phi \in \mathscr{P}_{\mathrm{pair}}$, $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}^1$.

(ii) *If A_i is idempotent for any $i \in [n]$, then*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}((c,\dots,c),\mu) = c$$

for any $\Psi, \Phi \in \mathscr{P}_{\mathrm{pair}}$, $c \in [0,\infty)$, $\mu \in \mathbf{M}^1$.

Proof. Let us prove part (i). Because of the assumptions we have

$$\begin{aligned}
\sum_{i=1}^{n} \mathsf{A}_{\psi(i)}(\mathbf{x}_\phi) \otimes \big(\mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)})\big) &\leq \sum_{i=1}^{n} \max_{j \in [n]} x_j \otimes \big(\mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)})\big) \\
&= \sum_{i=1}^{n} \max_{j \in [n]} x_j \otimes \mu(E_{\psi(i)}) - \max_{j \in [n]} x_j \otimes \mu(E_{\psi(i+1)}) \\
&= \max_{j \in [n]} x_j \otimes \mu([n]) = \max_{j \in [n]} x_j \otimes 1 = \max_{j \in [n]} x_j .
\end{aligned}$$

Analogously for boundary from below. Part (ii) follows directly from the definition. □

For specific permutations, we can derive further results that are not included in the previous propositions.

[6]Let $\otimes_1, \otimes_2\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ be binary operations. Under denotation $\otimes_1 \leq \otimes_2$ we mean $a \otimes_1 b \leq a \otimes_2 b$ for any $a, b \in [0,\infty)$.

Proposition 19. *Let* $(\mathsf{A}_i)_1^n$ *be a SCA,* $\otimes\colon [0,\infty)\times[0,\infty)\to[0,\infty)$.

(i) $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi^{\uparrow}}$ *and* $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi^{\downarrow}}$ *are symmetric operators*[7] *for any* $\Psi,\Phi^{\uparrow},\Phi^{\downarrow}\in\mathscr{P}_{\mathrm{pair}}$.

(ii) *If* $\big(\widehat{\mathsf{A}}_i\big)_1^n$ *be the SCA such that* $\widehat{\mathsf{A}}_i(\cdot)=\mathsf{A}(\cdot|\widehat{B}_i)$ *with* $\widehat{B}_i=\{(\phi^{\mathrm{con}})^{-1}(i): i\in B_i\}$, $i\in[n]$, Φ^{con}, $\Phi^{\mathrm{id}}\in\mathscr{P}_{\mathrm{pair}}$, *then* $\mathrm{C}^{\otimes}_{\widehat{\mathsf{A}},\Psi,\Phi^{\mathrm{con}}}(\mathbf{x},\mu)=\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi^{\mathrm{id}}}(\mathbf{x},\mu)$ *for any* $\Psi\in\mathscr{P}_{\mathrm{pair}}$, *and any* $(\mathbf{x},\mu)\in\mathbf{F}\times\mathbf{M}$.

(iii) $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\widehat{\Phi}}((c,\dots,c),\mu)=\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\widetilde{\Phi}}((c,\dots,c),\mu)$ *for any* $\Psi,\widehat{\Phi},\widetilde{\Phi}\in\mathscr{P}_{\mathrm{pair}}$, $c\in[0,\infty)$ *and* $\mu\in\mathbf{M}$.

(iv) *If* $\mathbf{x}\le\mathbf{y}$, $\mathbf{x},\mathbf{y}\in\mathbf{F}$, *then* $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^{\uparrow},\Phi^{\mathrm{mon}}}(\mathbf{x},\mu)\le\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^{\downarrow},\Phi^{\mathrm{mon}}}(\mathbf{y},\mu)$ *for any* $\Psi^{\uparrow},\Psi^{\downarrow},\Phi^{\mathrm{mon}}\in\mathscr{P}_{\mathrm{pair}}$ *and* $\mu\in\mathbf{M}$.

3.2 Formulas for specific inputs

The following proposition contains formulas of the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator in special cases. In other words, considering special parameters, the formula (2) can be simplified.

Proposition 20. *Let* $(\mathsf{A}_i)_1^n$ *be a SCA, and* $\Psi,\Phi\in\mathscr{P}_{\mathrm{pair}}$.

(i) *If* μ *is the weakest monotone capacity, i.e.* $\mu(B)=1$, *if* $B=[n]$, *and* $\mu(B)=0$, *otherwise, and* 0 *is the right zero element of* $\otimes$, *then*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu)=\mathsf{A}_{\psi(1)}(\mathbf{x}_{\phi})\otimes 1.$$

(ii) *If* μ *is the greatest monotone capacity, i.e.* $\mu(B)=0$, *if* $B=\emptyset$, *and* $\mu(B)=1$, *otherwise, and* 0 *is the right zero element of* $\otimes$, *then*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu)=\mathsf{A}_{\psi(n)}(\mathbf{x}_{\phi})\otimes 1.$$

(iii) *If* μ *is symmetric monotone measure, then*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu)=\sum_{i=1}^{n}\mathsf{A}_{\psi(i)}(\mathbf{x}_{\phi})\otimes(c_{n-i+1}-c_{n-i}),$$

with $\mu(E)=c_j$ *for each* $|E|=j$, $j\in[n]$ *and* $c_0=0$.

In the following proposition, we point out the fact that the $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator generalizes several basic aggregation functions.

Proposition 21. *Let* $(\mathsf{A}_i)_1^n$ *be a SCA, and* $\otimes$ *be a binary operation on* $[0,\infty)$ *with* 0 *as zero element and* 1 *as right identity.*

(i) *If* $\mathsf{A}_i=\mathsf{A}^{\mathrm{proj}}(\{i\})$ *for any* $i\in[n]$, $\Psi^{\mathrm{id}},\Phi^{\mathrm{id}}\in\mathscr{P}_{\mathrm{pair}}$, *and* μ *is the weakest monotone capacity, then for any* $\Psi^{\uparrow},\Psi^{\downarrow},\Phi^{\uparrow},\Phi^{\downarrow}\in\mathscr{P}_{\mathrm{pair}}$, $\mathbf{x}\in\mathbf{F}$ *it holds*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\mathrm{id}},\Phi^{\uparrow}}(\mathbf{x},\mu)=\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\uparrow},\Phi^{\mathrm{id}}}(\mathbf{x},\mu)=\min_{i\in[n]}x_i,$$
$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\mathrm{id}},\Phi^{\downarrow}}(\mathbf{x},\mu)=\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\downarrow},\Phi^{\mathrm{id}}}(\mathbf{x},\mu)=\max_{i\in[n]}x_i.$$

(ii) *If* $\mathsf{A}_i=\frac{1}{n}\mathsf{A}^{\mathrm{proj}}(\{i\})$ *for any* $i\in[n]$, *and* $\mu\in\mathbf{M}$ *such that* $\mu(B)=|B|$, *then for any* $\Psi,\Phi\in\mathscr{P}_{\mathrm{pair}}$, $\mathbf{x}\in\mathbf{F}$ *it holds*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu)=\frac{1}{n}\sum_{i=1}^{n}x_i.$$

[7] $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator is called *symmetric* if for any $\mathbf{x}\in\mathbf{F}$, any $\sigma\in\mathrm{Perm}([n])$ it holds $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu)=\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x}_{\sigma},\mu)$.

(iii) *If* $\mathsf{A}_1 = \frac{1}{|B_1|}\mathsf{A}_1^{\text{sum}}$ *with* $B_1 = \{\frac{n}{2}, \frac{n}{2}+1\}$ *for n even, and* $B_1 = \{\frac{n+1}{2}\}$ *for n odd, and* $B_i = \emptyset$ *(or* $\mathsf{A}_i = \mathsf{A}^{\text{zero}}$*) for* $i \in [n] \setminus \{1\}$*, then for any* $\mathbf{x} \in \mathbf{F}$ *and the weakest monotone capacity it holds*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\text{id}},\Phi^{\uparrow}}(\mathbf{x},\mu) = \mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\downarrow},\Phi^{\uparrow}}(\mathbf{x},\mu) = \operatorname{med}(\mathbf{x}),$$

(iv) *If* $\mathsf{A}_i = \mathsf{A}^{\text{proj}}(\{i\})$ *for any* $i \in [n]$*, and μ is a symmetric monotone measure, then for any* $\mathbf{x} \in \mathbf{F}$ *it holds*

$$\mathrm{C}^{\text{Prod}}_{\mathsf{A},\Psi^{\text{id}},\Phi^{\uparrow}}(\mathbf{x},\mu) = \mathrm{OWA}_{\mathbf{w}}(\mathbf{x}),$$

where $\mathbf{w} \in \mathbf{F}$ *with* $w_i = \mu(E_{\psi(i)}) - \mu(E_{\psi(i+1)})$ *for any* $i \in [n]$.

Remark 22. *Another way how to get minimum, or maximum is to consider* $\mathsf{A}_1 = \mathsf{A}^{\text{proj}}(\cdot|\{1\})$*, and* $B_i = \emptyset$ *(or* $\mathsf{A}_i = \mathsf{A}^{\text{zero}}$*) for* $i \in [n] \setminus \{1\}$*, then for any* $\mathbf{x} \in \mathbf{F}$ *and the weakest monotone capacity it holds*

$$\mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\text{id}},\Phi^{\uparrow}}(\mathbf{x},\mu) = \min_{i\in[n]} x_i \quad \textit{and} \quad \mathrm{C}^{\otimes}_{\mathsf{A},\Psi^{\text{id}},\Phi^{\downarrow}}(\mathbf{x},\mu) = \max_{i\in[n]} x_i.$$

3.3 The $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator versus the $\mathrm{C}_{\mathscr{A}}$ operator

The idea of aggregating by CAOs comes from [2]. In this paper, the authors defined the $\mathscr{A}$-Choquet operator with $\mathscr{A} = \{\mathsf{A}(\cdot|B) : B \in \mathscr{E} \supseteq \{\emptyset, [n]\}\}$ being a *family of conditional aggregation operators* (FCA for short)[8]:

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) := \int_0^{\infty} \mu_{\mathscr{A}}(\mathbf{x},\alpha)\,\mathrm{d}\alpha, \tag{6}$$

where $\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \min\{\mu(B^c) : \mathsf{A}(\mathbf{x}|B) \le \alpha,\ B \in \mathscr{E} \supseteq \{\emptyset,[n]\}\}$, $\alpha \in [0,\infty)$. In [4] it was proved that the $\mathrm{C}_{\mathscr{A}}$ operator is in fact the Choquet integral, but on hyperspace, i.e.

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \mathrm{C}(T_{\mathscr{A},\mathbf{x}}, \mathtt{N}_{\mu}), \tag{7}$$

where $T_{\mathscr{A},\mathbf{x}} \colon \mathscr{E} \to [0,\infty)$ such that $T_{\mathscr{A},\mathbf{x}}(B) := \mathsf{A}(\mathbf{x}|B)$ and $\mathtt{N}_{\mu} \colon 2^{\mathscr{E}} \to [0,\infty)$ such that $\mathtt{N}_{\mu}(\mathscr{G}) := \min\{\mu(B^c) : B \in \mathscr{E} \setminus \mathscr{G}\}$. In discrete space, using formula (7), it is possible to express the $\mathrm{C}_{\mathscr{A}}$ operator (see Proposition 5.5 in [4]) as

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \sum_{i=1}^{p} \mathsf{A}(\mathbf{x}|B_i)\left(\min_{j\in\{0,1,\dots,i-1\}} \mu(B_j^c) - \min_{j\in\{0,1,\dots,i\}} \mu(B_j^c)\right), \tag{8}$$

where $0 = \mathsf{A}(\mathbf{x}|B_0) \le \mathsf{A}(\mathbf{x}|B_1) \le \dots \le \mathsf{A}(\mathbf{x}|B_p)$, $p = |\mathscr{E}| - 1$.

Since the operator $\mathrm{C}^{\text{Prod}}_{\mathsf{A},\Psi,\Phi}$ as well as the $\mathrm{C}_{\mathscr{A}}$ operator are of the Choquet type (based on the same operations: the sum, the product), it is natural to compare them. In the following, we shall use the concept of *dominance* of aggregation functions, see [9]. We say that the SCA $(\mathsf{A}_i)_1^n$ is *dominance σ-arrangeable*, if there exists permutation $\sigma \in \mathrm{Perm}([n])$ such that $\mathsf{A}_{\sigma(1)}(\cdot) \le \dots \le \mathsf{A}_{\sigma(n)}(\cdot)$. We shall use the denotation $\mathbf{F}_{\Phi} = \{\mathbf{x}_{\phi} : \langle \mathbf{x}, \phi\rangle \in \Phi, \mathbf{x} \in \mathbf{F}\}$.

Proposition 23. *If* $(\mathsf{A}_i)_1^n$ *is a dominance* $\psi^{\uparrow}$*-arrangeable SCA on* $\mathbf{F}_{\Phi}$*, and*

$$\mathscr{A} = \{\widehat{\mathsf{A}}(\mathbf{x}|E^c_{\psi^{\uparrow}(i+1)}) : \widehat{\mathsf{A}}(\mathbf{x}|E^c_{\psi^{\uparrow}(i+1)}) = \mathsf{A}_{\psi^{\uparrow}(i)}(\mathbf{x}_{\phi})\,\textit{for each}\ \mathbf{x} \in \mathbf{F}, i \in [n] \cup \{0\}\}$$

with the convention $\mathsf{A}_{\psi^{\uparrow}(0)} = \mathsf{A}^{\text{zero}}$ *is the FCA, then* $\mathrm{C}^{\text{Prod}}_{\mathsf{A},\Psi^{\uparrow},\Phi}(\mathbf{x},\mu) = \mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu)$ *for each* $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}$.

[8] $\mathscr{A}$ is a family of operators parametrized by sets from $\mathscr{E}$. For example, $\mathscr{A}^{\min} = \{\mathsf{A}^{\min}(\cdot|B) : B \in \mathscr{E}\}$, $\mathscr{A}^{\text{sum}} = \{\mathsf{A}^{\text{sum}}(\cdot|B) : B \in \mathscr{E}\}$.

(a) grayscale (b) smoothing (SM1) (c) smoothing (SM2) (d) smoothing (SM3)

Figure 1: Image 118035 from BSDS500 dataset

Proof. Since $\widehat{\mathsf{A}}(\mathbf{x}|E^c_{\psi^\uparrow(i+1)}) = \mathsf{A}_{\psi^\uparrow(i)}(\mathbf{x}_\phi)$, and $(\mathsf{A}_i)_1^n$ is dominance $\psi^\uparrow$-arrangeable, then

$$\widehat{\mathsf{A}}(\mathbf{x}|E^c_{\psi^\uparrow(1)}) \leq \dots \leq \widehat{\mathsf{A}}(\mathbf{x}|E^c_{\psi^\uparrow(i+1)}) \leq \dots \leq \widehat{\mathsf{A}}(\mathbf{x}|E^c_{\psi^\uparrow(n+1)}).$$

Thus, $B_i = E^c_{\psi^\uparrow(i+1)}$, $i \in [n] \cup \{0\}$. Moreover, $\min\limits_{j\in\{0,1,\dots,i\}} \mu(B^c_j) = \mu(B^c_i) = \mu(E_{\psi^\uparrow(i+1)})$. ☐

The consequence of the previous proposition is that the (MC)-integral, see [19], with $(\mathsf{M}_i)_1^n$, $\mathsf{M}_i(\mathbf{x}) := \mathsf{M}(\mathbf{x}|\{1,\dots,i\})$[9] being dominance $\psi^\uparrow$-arrangeable SCA is a special case of the $\mathrm{C}_{\mathscr{A}}$ operator.

Corollary 24. *If* $(\mathsf{M}_i)_1^n$ *is dominance* $\psi^\uparrow$*-arrangeable on* $\mathbf{F}$*, then* $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{M},\Psi^\uparrow,\Phi^{\mathrm{id}}}(\mathbf{x},\mu) = \mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu)$ *for each* $(\mathbf{x},\mu) \in \mathbf{F}\times\mathbf{M}$ *with* $\mathscr{A} = \{\mathsf{A}(\cdot\,|E^c_{\psi^\uparrow(i+1)}) = \mathsf{M}_{\psi^\uparrow(i)}(\cdot) : i \in [n]\cup\{0\}\}$.

4 Experimental study

In [18] the authors presented a fuzzy-logic-based method for image edge detection. As the authors stated, this approach achieves better results in many cases than gradient-based methods. The main idea of this new approach is to simulate neural processes in the human visual system by combining a fuzzy approach and a computer vision scheme described in the early 1980s. This scheme consists of four phases: conditioning, feature extraction, blending, and scaling of an image. We follow the ideas and algorithm proposed in [18], but in the suitable phase we apply

$$\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi_\downarrow,\Phi^\uparrow},\ \mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^\uparrow,\Phi_\downarrow},\ \mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi_\downarrow,\Phi^{\mathrm{id}}},\ \text{and}\ \mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi^\uparrow,\Phi^{\mathrm{id}}}$$

operators with respect to the *power measure* $\mu \in \mathbf{M}$ defined as

$$\mu(B) = \left(\frac{|B|}{n}\right)^q,$$

where $B \in 2^{[n]}$, $q \in (0,\infty)$. In this experimental study, we use $q = 0.9$.

An image is a function $I\colon R\times C \to L$, where $R = \{1,\dots,r\}$ and $C = \{1,\dots,c\}$, $r,c\in\mathbb{N}$ are rows and columns of the image pixels. If an image is in color, then $L = \{0,\dots,255\}^3$ (RGB scale), for a greyscale image $L = \{0,\dots,255\}$, and for a binary image $L = \{0,1\}$. Let $I(x,y) \in L$ denote the color of the pixel at position $[x,y]$. To compare the results, we select the input image 118035 from BSDS500 [1] for edge detection, just like in [18]. Database BSDS500 (Berkeley Segmentation Dataset and Benchmark) contains 200 images together with their over 1000 ground truth labelings, which is essential for statistical results of edge detection.

In the conditioning phase, the input image is converted to grayscale, see Figure 1(a). Instead of three components of RGB, it is sufficient to consider only one representative value, while the information about the occurrence of the edge is preserved. Subsequently, the image is necessary

[9]Sublinear mean is a mean $M\colon \mathbf{F}\times 2^{[n]} \to [0,\infty)$ that is for every set $B\in 2^{[n]}$ monotone, homogeneous, subadditive, see [19].

i	$\mathsf{A}_i/\widehat{\mathsf{A}}_i$	B_i	$\widehat{B}_i$	i	$\mathsf{A}_i/\widehat{\mathsf{A}}_i$	B_i	$\widehat{B}_i$
1	$\mathsf{A}^{\mathrm{mean}}$	$[8]$	$[8]$	5	$\mathsf{A}^{\mathrm{Ch}_\mu}$	$[8]$	$[8]$
2	$\mathsf{A}^{\mathrm{mean}}$	$\{4,5\}$	$\{2,4,5,7\}$	6	$\mathsf{A}^{\mathrm{Ch}_\mu}$	$\{1,2,7,8\}$	$\{1,3,6,8\}$
3	$\mathsf{A}^{\mathrm{mean}}$	$\{2,3,6,7\}$	$\{1,3,6,8\}$	7	$\mathsf{A}^{\mathrm{Ch}_\mu}$	$\{2,3,6,7\}$	$\{2,4,5,7\}$
4	$\mathsf{A}^{\mathrm{mean}}$	$\{1,8\}$	$\{4,5\}$	8	$\mathsf{A}^{\mathrm{Ch}_\mu}$	$\{1,2,3,6,7,8\}$	$\{2,7\}$

Table 2: The SCA for $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi_\downarrow,\Phi^\uparrow}$, $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^\uparrow,\Phi_\downarrow}$ and $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi_\downarrow,\Phi^{\mathrm{id}}}$ and $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi^\uparrow,\Phi^{\mathrm{id}}}$

to blur, thereby reducing spurious artifacts and noise that could affect edge detection. For blurring we use the Gaussian smoothing (SM1) with $\sigma = 2$, see Figure 1(b), and the Gravitational smoothing [17] with parameters

(SM2) $G = 0.05$, $cF = 20$, $t = 30$, (SM3) $G = 0.05$, $cF = 70$, $t = 30$,

see Figure 1(c) and 1(d), respectively. The feature extraction phase simulates the layers of the visual cortex in the human visual system. For each pixel $[x, y] \in R \times C$ of an image I is created the vector $\mathbf{x} = (x_1, \dots, x_8) \in \mathbf{F}$ whose components are the absolute values of the color differences of the pixel $[x, y]$ and its neighbors, i.e.

$$x_k = |I(x, y) - I(x + i, y + j)|$$

for $((i, j)_k)_{k=1}^{8} = ((-1,-1),(0,-1),(1,-1),(-1,0),(1,0),(-1,1),(0,1),(1,1))$. Values of x_k close to the value 255 represent an edge (adjacent pixels have different colors). The components of the vector $\mathbf{x}$ are further aggregated to one resulting value representing the edge (its intensity). For aggregation, we use variations of $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi,\Phi}$ operator mentioned above with the following two types of conditional aggregation operators:

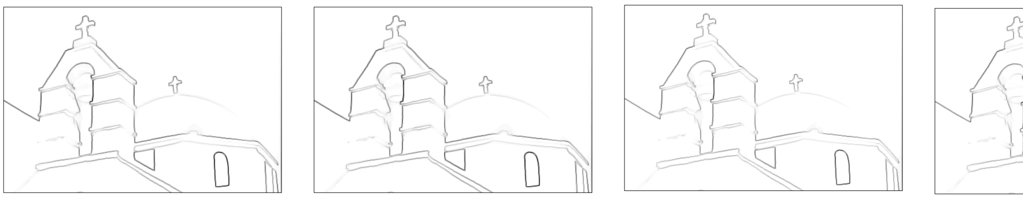
$$\mathsf{A}^{\mathrm{mean}}(\cdot|B) \qquad \text{and} \qquad \mathsf{A}^{\mathrm{Ch}_m}(\cdot|B),$$

see examples after Definition 1, where $m \in \mathbf{M}$ is the power measure with parameter $q = 0.9$ (see above). For $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi_\downarrow,\Phi^\uparrow}$ and $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^\uparrow,\Phi_\downarrow}$ operators we use $\mathbf{x}$ as the input vector. The input for $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi_\downarrow,\Phi^{\mathrm{id}}}$ and $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi^\uparrow,\Phi^{\mathrm{id}}}$ operators is the vector $\widehat{\mathbf{x}} = \mathbf{x} \odot \mathbf{w} = (x_1 w_1, \dots, x_n w_n)$, where

$$\mathbf{w} = \left(\tfrac{1}{\sqrt{2}}, 1, \tfrac{1}{\sqrt{2}}, 1, 1, \tfrac{1}{\sqrt{2}}, 1, \tfrac{1}{\sqrt{2}}\right)$$

expresses appropriate weights that take into account the Euclidean distance of neighboring pixels of the central pixel. We use two SCAs, $(\mathsf{A}_i)_1^8 = (\mathsf{A}(\cdot|B_i))_1^8$ for $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi_\downarrow,\Phi^\uparrow}$ and $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^\uparrow,\Phi_\downarrow}$ operators, and $(\widehat{\mathsf{A}}_i)_1^8 = (\widehat{\mathsf{A}}(\cdot|\widehat{B}_i))_1^8$ for $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi_\downarrow,\Phi^{\mathrm{id}}}$ and $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi^\uparrow,\Phi^{\mathrm{id}}}$ operators, see Table 2. The choice of these sequences of conditional aggregation operators is natural. Operators A_1 and $\widehat{\mathsf{A}}_1$ express the mean of all vector components, A_2 their median, $\widehat{\mathsf{A}}_2$ the mean of pixels in the nearest neighborhood, and A_3 the mean of the first and third quartile, or $\widehat{\mathsf{A}}_3$ the mean of pixels in the furthest neighborhood. Operator A_4 expresses the mean of the most outlying values, and operator $\widehat{\mathsf{A}}_4$ the mean of horizontal neighbors of central pixel. Operators A_5 to A_8, or $\widehat{\mathsf{A}}_5$ to $\widehat{\mathsf{A}}_8$, respectively, describe the same ideas, but in the context of the Choquet integral.

The aggregated values by above mentioned $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi,\Phi}$ operators must be within the RGB scale. This is guaranteed by the averaging behavior of the mentioned operators. Indeed, since the power measure is the capacity, and A_i for each $i \in [8]$ satisfy assumptions of Proposition 18, the $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi_\downarrow,\Phi^\uparrow}$, $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^\uparrow,\Phi_\downarrow}$, $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi_\downarrow,\Phi^{\mathrm{id}}}$ and $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi^\uparrow,\Phi^{\mathrm{id}}}$ are an averaging behavior and idempotent operators. The output of aggregations (feature extraction phase) is the image with featured edges, see Figure 2, with respect to smoothing (SM2). In the scaling phase, the image from the previous phase is transformed to a binary image with thin lines using non-maxima suppression [20] and hysteresis. We get the final detection of the edges of the input image, see Figure 3.



(a) $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi_{\downarrow},\Phi^{\uparrow}}$ (b) $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^{\uparrow},\Phi_{\downarrow}}$ (c) $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi_{\downarrow},\Phi^{\mathrm{id}}}$ (d) $\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi^{\uparrow},\Phi^{\mathrm{id}}}$

Figure 2: Feature extraction by $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi,\Phi}$ operators w.r.t. (SM2)

We can see that the resulting edge detection using the $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi,\Phi}$ operators corresponds to what we would expect. However, we can also assess the result quantitatively. As we mentioned above, the BSDS500 dataset also contains several truth labelings by humans based on their perception of edges. Quantitative evaluation of edge detection can be taken as a binary classification problem. To compute the displacement-tolerant correspondence of edges, the standard procedure Estrada and Jepson [7]. The spatial tolerance is 2.5 % of the length of the image diagonal. Quantitative evaluation of hand-labelled edges is based on $\mathrm{F}_{0.5}$ measure, see [15],

$$\mathrm{F}_{\alpha} = \frac{\mathrm{Prec}\cdot\mathrm{Rec}}{\alpha\cdot\mathrm{Prec}+(1-\alpha)\cdot\mathrm{Rec}},$$

where

$$\mathrm{Prec} = \frac{\mathrm{TP}}{\mathrm{TP}+\mathrm{FP}} \quad \text{and} \quad \mathrm{Rec} = \frac{\mathrm{TP}}{\mathrm{TP}+\mathrm{FN}}.$$

For input image, the quantitative results in form $(\mathrm{Prec},\mathrm{Rec},\mathrm{F}_{0.5})$ are follows:

	$\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi_{\downarrow},\Phi^{\uparrow}}$			$\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^{\uparrow},\Phi_{\downarrow}}$			$\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi_{\downarrow},\Phi^{\mathrm{id}}}$			$\mathrm{C}^{\mathrm{Prod}}_{\widehat{\mathsf{A}},\Psi^{\uparrow},\Phi^{\mathrm{id}}}$		
	Prec	Rec	$\mathrm{F}_{0.5}$	Prec	Rec	$\mathrm{F}_{0.5}$	Prec	Rec	$\mathrm{F}_{0.5}$	Prec	Rec	$\mathrm{F}_{0.5}$
(SM1)	0.640	0.921	0.742	0.625	0.921	0.732	0.659	0.920	0.755	0.652	0.921	0.750
(SM2)	0.717	0.894	0.784	0.716	0.893	0.784	0.723	0.897	0.789	0.722	0.896	0.788
(SM3)	0.665	0.923	0.760	0.664	0.923	0.759	0.675	0.924	0.767	0.675	0.923	0.767

Table 3: Quantitative comparison of $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi,\Phi}$ operators in edge detection

As we can see statistically, the results regardless of the given smoothness are almost the same, but the best agreement with human perception of the edge occurred for the $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi^{\uparrow},\Phi_{\downarrow}}$ operator with respect to (SM2). We note, that these results are better than results presented in [18], and in addition, better than results obtained with standard methods such as Canny, Marr-Hildreth, Robert cross, Sobel, or Prewitt edge detector (see statistical results in cited paper).

5 Permutation dependent operator in more general construction

In analogy to Definition 5 we can introduce Sugeno's type or Shilkret's type of permutations dependent operator as follows: Let $(\mathsf{A}_i)_1^n$ be a SCA, $\Psi,\Phi\in\mathscr{P}_{\mathrm{pair}}$,

$$\mathrm{Sh}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) = \max_{i\in[n]}\left(\mathsf{A}_{\psi(i)}(\mathbf{x}_{\phi})\cdot\mu(E_{\psi(i)})\right), \quad \mathrm{Su}^{\otimes}_{\mathsf{A},\Psi,\Phi}(\mathbf{x},\mu) = \max_{i\in[n]}\min\{\mathsf{A}_{\psi(i)}(\mathbf{x}_{\phi}),\mu(E_{\psi(i)})\}.$$

However, all these operators, together with $\mathrm{C}^{\otimes}_{\mathsf{A},\Psi,\Phi}$ operator, can be covered under one formula. Further, let $\langle\mathbf{x},\phi\rangle\in\Phi$, and $\langle\mathbf{x}_{\mathsf{A},\phi},\psi\rangle\in\Psi$, with $\mathbf{x}_{\mathsf{A},\phi}=(\mathsf{A}_1(\mathbf{x}_{\phi}),\dots,\mathsf{A}_n(\mathbf{x}_{\phi}))$. One can observe that all the above-mentioned operators are based on mixing the components of the following vectors:

$$(\mathbf{x}_{\mathsf{A},\phi})_{\psi} = (\mathsf{A}_{\psi(1)}(\mathbf{x}_{\phi}),\dots,\mathsf{A}_{\psi(n)}(\mathbf{x}_{\phi})), \quad \mathbf{h}_{\mu,\psi} := (\mu(E_{\psi(1)}),\dots,\mu(E_{\psi(n)}))$$

with $E_{\psi(i)}=\{\psi(i),\psi(i+1),\dots,\psi(n)\}$ for each $i\in[n]$.

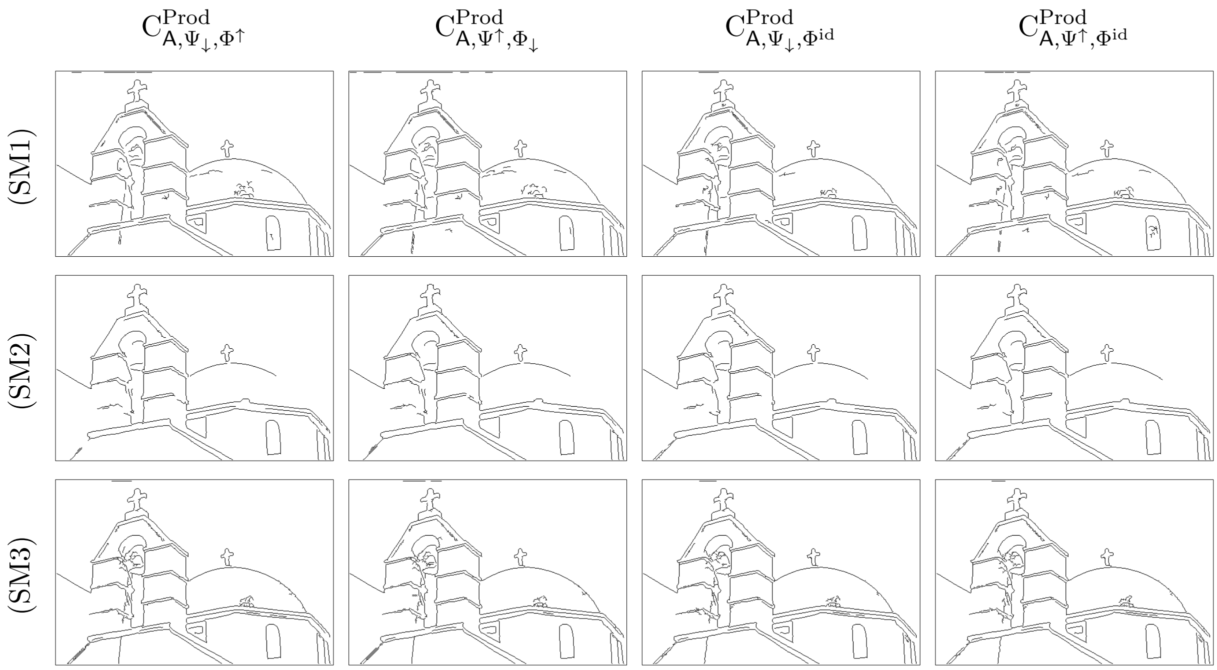


Figure 3: Final edge detection by $\mathrm{C}^{\mathrm{Prod}}_{\mathsf{A},\Psi,\Phi}$ operators

Definition 25. Let $(\mathsf{A}_i)_1^n$ be a SCA, $\Psi, \Phi \in \mathscr{P}_{\mathrm{pair}}$, and $\mathrm{L}\colon \mathbf{F} \times \mathbf{F} \to [0, \infty)$. The $\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi,\Phi}$ operator of a vector $\mathbf{x} \in \mathbf{F}$ w.r.t. $\mu \in \mathbf{M}$ is defined as

$$\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi,\Phi}(\mathbf{x}, \mu) = \mathrm{L}((\mathbf{x}_{\mathsf{A},\phi})_{\psi}, \mathbf{h}_{\mu,\psi}), \tag{9}$$

where $\langle \mathbf{x}, \phi \rangle \in \Phi$, $\langle \mathbf{x}_{\mathsf{A},\phi}, \psi \rangle \in \Psi$, $\mathbf{x}_{\mathsf{A},\phi} = (\mathsf{A}_1(\mathbf{x}_{\phi}), \dots, \mathsf{A}_n(\mathbf{x}_{\phi}))$, and $\mathbf{h}_{\mu,\psi} = (\mu(E_{\psi(1)}), \dots, \mu(E_{\psi(n)}))$ with $E_{\psi(i)} = \{\psi(i), \psi(i+1), \dots, \psi(n)\}$ for each $i \in [n]$.

Remark 26. *In a special case, from* (9) *we get all basic fuzzy integrals. For* $\mathsf{A}_i(\cdot) = \mathsf{A}^{\mathrm{proj}}(\cdot|\{i\})$ *for any* $i \in [n]$*,* $\Phi^{\mathrm{id}}, \Psi^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$*, and*

- $\mathrm{L}(\mathbf{y}, \mathbf{z}) = \max\limits_{i \in [n]} \min\{y_i, z_i\}$ *we get the Sugeno integral,*
- $\mathrm{L}(\mathbf{y}, \mathbf{z}) = \max\limits_{i \in [n]} (y_i \cdot z_i)$ *we get the Shilkret integral,*
- $\mathrm{L}(\mathbf{y}, \mathbf{z}) = \sum\limits_{i \in [n]} (y_i \cdot \max\{0, z_i - z_{i+1}\})$ *with the convention* $z_{n+1} = 0$*, or for* $\mathrm{L}(\mathbf{y}, \mathbf{z}) = \sum\limits_{i \in [n]} (\max\{0, y_i - y_{i-1}\} \cdot z_i)$ *with the convention* $y_0 = 0$*, we get the Choquet integral.*

Further, for $\mathrm{L}(\mathbf{y}, \mathbf{z}) = \sum\limits_{i \in [n]} y_i \circ \max\{0, z_i - z_{i+1}\}$ *we get the operator from [12], for* $\mathrm{L}(\mathbf{y}, \mathbf{z}) = \sum\limits_{i \in [n]} \delta(y_i, y_{i-1}) \circ z_i$ *we get the operator from [5], and for* $\mathrm{L}(\mathbf{y}, \mathbf{z}) = \sum\limits_{i \in [n]} (C(y_i, z_i) - C(y_{i-1}, z_i))$ *we get the operator from [16].*

The $\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi,\Phi}$ operator is well defined. The value of $\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi,\Phi}$ operator of a vector $\mathbf{x} \in \mathbf{F}$ w.r.t. $\mu \in \mathbf{M}$ is unique. This is thanks to the fact that SCA and sets of permutation pairs are chosen in advance. Similarly, as we have discussed in Section 3, the set $\Psi^{\uparrow}$ has to be chosen in advance, i.e. if several permutations of $[n]$ reorder the components of a vector $\mathbf{z} \in \mathbf{F}$ in nondecreasing order, we have to assign this vector only one of them. Indeed, $\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi^{\uparrow}\Phi}$, $\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi}$ are not the same in general, as the following example shows.

Example 27. *Let us consider* $\mathrm{L}(\mathbf{y},\mathbf{z}) = y_1 + z_2$, $(\mathsf{A}_i^{\mathrm{proj}})_1^n$, $n \geq 2$, *with* $\mathsf{A}_i(\mathbf{x}) = \mathsf{A}^{\mathrm{proj}}(\mathbf{x}|\{i\})$, *the input vector* $\mathbf{x} = (a, a, b, \dots, b)$ *with* $0 \leq a < b$. *Let us take* $\Psi^{\uparrow}, \widehat{\Psi}^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$ *such that*

$\mathbf{z} \in \mathbf{F}$	$\Psi^{\uparrow}$						$\widehat{\Psi}^{\uparrow}$					
	$\psi_{\mathbf{z}}^{\uparrow}(1)$	$\psi_{\mathbf{z}}^{\uparrow}(2)$	$\psi_{\mathbf{z}}^{\uparrow}(3)$	$\psi_{\mathbf{z}}^{\uparrow}(4)$	$\cdots$	$\psi_{\mathbf{z}}^{\uparrow}(n)$	$\widehat{\psi}_{\mathbf{z}}^{\uparrow}(1)$	$\widehat{\psi}_{\mathbf{z}}^{\uparrow}(2)$	$\widehat{\psi}_{\mathbf{z}}^{\uparrow}(3)$	$\widehat{\psi}_{\mathbf{z}}^{\uparrow}(4)$	$\cdots$	$\widehat{\psi}_{\mathbf{z}}^{\uparrow}(n)$
$(a,a,b,\dots,b)$	1	2	3	4	$\cdots$	n	2	1	3	4	$\cdots$	n
$\vdots$			$\vdots$						$\vdots$			

Then

$$\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi^{\uparrow},\Phi}(\mathbf{x},\mu) = a + \mu(\{2,3,\dots,n\}) \neq a + \mu(\{1,3,\dots,n\}) = \mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu)$$

for $\mu \in \mathbf{M}$ *such that* $\mu(\{2,3,\dots,n\}) \neq \mu(\{1,3,\dots,n\})$.

Similarly as in Section 3, we shall ask when the $\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi^{\uparrow},\Phi}$ operator does not depend on the choice of $\Psi^{\uparrow}$ for any $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}$, i.e.

$$\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi^{\uparrow},\Phi} = \mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi^{\uparrow},\Phi} \text{ for any } \widehat{\Psi}^{\uparrow}, \widetilde{\Psi}^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}.$$

The consequence of such a result is that we will be able to justify the correctness of several constructions of integrals known in the literature. In the following, we shall use the denotations

$$\mathbf{F}^{\downarrow} = \{\mathbf{y} \in \mathbf{F} : y_1 \geq y_2 \geq \dots \geq y_n\}, \quad \mathbf{F}^{\uparrow} = \{\mathbf{y} \in \mathbf{F} : y_1 \leq y_2 \leq \dots \leq y_n\},$$

$$\mathbf{F}^{\uparrow}_{\mathsf{A},\Phi} = \{(\mathbf{x}_{\mathsf{A},\phi})_{\psi} \in \mathbf{F}^{\uparrow} : \mathbf{x} \in \mathbf{F}\}$$

with $\langle \mathbf{x}_{\mathsf{A},\phi}, \psi^{\uparrow}\rangle \in \Psi^{\uparrow}$, $\Psi^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$. Note that $\Psi^{\uparrow}$ does not appear in the denotation $\mathbf{F}^{\uparrow}_{\mathsf{A},\Phi}$, since the choice of $\Psi^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$ does not affect the set of $\mathbf{F}^{\uparrow}_{\mathsf{A},\Phi}$.

Proposition 28. *Let* $(\mathsf{A}_i)_1^n$ *be a SCA,* $\Phi \in \mathscr{P}_{\mathrm{pair}}$, *and* $\mathrm{L}\colon \mathbf{F} \times \mathbf{F} \to [0,\infty)$. *Then the following assertions are equivalent:*

(i) $\mathrm{L}(\mathbf{y},\widetilde{\mathbf{z}}) = \mathrm{L}(\mathbf{y},\widehat{\mathbf{z}})$ *for each* $(\mathbf{y},\widetilde{\mathbf{z}},\widehat{\mathbf{z}}) \in \mathbf{F}^{\uparrow}_{\mathsf{A},\Phi} \times \mathbf{F}^{\downarrow} \times \mathbf{F}^{\downarrow}$ *such that* $\widetilde{z}_1 = \widehat{z}_1$, *and* $\widetilde{z}_i = \widehat{z}_i$ *for* $i \in [n] \setminus \{1\}$ *if it holds* $y_{i-1} < y_i$.

(ii) $\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu) = \mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu)$ *for each* $\widetilde{\Psi}^{\uparrow}, \widehat{\Psi}^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$ *and for each* $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}$.

Proof. Let us denote

$$(\mathbf{x}_{\mathsf{A},\phi})_{\widetilde{\psi}^{\uparrow}} := \widetilde{\mathbf{y}},\ (\mathbf{x}_{\mathsf{A},\phi})_{\widehat{\psi}^{\uparrow}} := \widehat{\mathbf{y}},\ \mathbf{h}_{\mu,\widetilde{\psi}^{\uparrow}} := \widetilde{\mathbf{z}},\ \mathbf{h}_{\mu,\widehat{\psi}^{\uparrow}} := \widehat{\mathbf{z}}.$$

It is easy to verify that $\mathbf{F}^{\uparrow}_{\mathsf{A},\Phi} \ni \widetilde{\mathbf{y}} = \widehat{\mathbf{y}} := \mathbf{y}$, and $\widetilde{\mathbf{z}}, \widehat{\mathbf{z}} \in \mathbf{F}^{\downarrow}$. Moreover, we show that

$$\widetilde{z}_i = \widehat{z}_i$$

for $i = 1$ and for each $i \in [n] \setminus \{1\}$ such that $y_{i-1} < y_i$. Indeed, $\{\widetilde{\psi}(1), \widetilde{\psi}(2), \dots, \widetilde{\psi}(n)\} = \{\widehat{\psi}(1), \widehat{\psi}(2), \dots, \widehat{\psi}(n)\}$, therefore $\widetilde{z}_1 = \widehat{z}_1$. Further, if $y_{i-1} < y_i$ for some $i \in [n] \setminus \{1\}$, then $E_{\widetilde{\psi}^{\uparrow}(i)} = E_{\widehat{\psi}^{\uparrow}(i)}$, thence $\mu(E_{\widetilde{\psi}^{\uparrow}(i)}) = \mu(E_{\widehat{\psi}^{\uparrow}(i)})$.

(i) $\Rightarrow$ (ii) Let us consider arbitrary $\widetilde{\psi}^{\uparrow}, \widehat{\psi}^{\uparrow} \in \mathscr{P}_{\mathrm{pair}}$, $(\mathbf{x},\mu) \in \mathbf{F} \times \mathbf{M}$. From the general observation mentioned at the beginning of this proof, vectors $(\mathbf{x}_{\mathsf{A},\phi})_{\widetilde{\psi}^{\uparrow}}$, $(\mathbf{x}_{\mathsf{A},\phi})_{\widehat{\psi}^{\uparrow}}$, $\mathbf{h}_{\mu,\widetilde{\psi}^{\uparrow}}$, $\mathbf{h}_{\mu,\widehat{\psi}^{\uparrow}}$ satisfy the properties of $\mathbf{y}, \widetilde{\mathbf{z}}, \widehat{\mathbf{z}}$, respectively, from part (i). Therefore it holds

$$\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu) = \mathrm{L}((\mathbf{x}_{\mathsf{A},\phi})_{\widetilde{\psi}^{\uparrow}}, \mathbf{h}_{\mu,\widetilde{\psi}^{\uparrow}}) = \mathrm{L}((\mathbf{x}_{\mathsf{A},\phi})_{\widehat{\psi}^{\uparrow}}, \mathbf{h}_{\mu,\widehat{\psi}^{\uparrow}}) = \mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu).$$

(ii) $\Rightarrow$ (i) Let (ii) holds, then

$$\mathrm{L}((\mathbf{x}_{\mathsf{A},\phi})_{\widetilde{\psi}^{\uparrow}}, \mathbf{h}_{\mu,\widetilde{\psi}^{\uparrow}}) = \mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\widetilde{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu) = \mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\widehat{\Psi}^{\uparrow},\Phi}(\mathbf{x},\mu) = \mathrm{L}((\mathbf{x}_{\mathsf{A},\phi})_{\widehat{\psi}^{\uparrow}}, \mathbf{h}_{\mu,\widehat{\psi}^{\uparrow}})$$

for each $\widetilde{\Psi}^{\uparrow}, \widehat{\Psi}^{\uparrow} \in \mathscr{P}_{\text{pair}}$, and $(\mathbf{x}, \mu) \in \mathbf{F} \times \mathbf{M}$. Because of the general observation mentioned at the beginning of the proof and w.r.t. the introduced denotation, we have

$$\mathrm{L}(\mathbf{y}, \widetilde{\mathbf{z}}) = \mathrm{L}(\mathbf{y}, \widehat{\mathbf{z}})$$

for each $(\mathbf{y}, \widetilde{\mathbf{z}}, \widehat{\mathbf{z}}) \in \mathbf{F}^{\uparrow}_{\mathsf{A},\Phi} \times \mathbf{F}^{\downarrow} \times \mathbf{F}^{\downarrow}$ such that $\widetilde{z}_1 = \widehat{z}_1$, and $\widetilde{z}_i = \widehat{z}_i$ for $i \in [n] \setminus \{1\}$ if it holds $y_{i-1} < y_i$. □

Remark 29. *The previous proposition is the alternative to Theorem 3.2 from [3] to prove that some generalizations of the Choquet integral are well defined, among them the constructions mentioned in Remark 26. Moreover, let us mention that relations from Remark 26 corresponding to the Choquet, Shilkret, Sugeno integral satisfy condition (i) of Proposition 28 trivially.*

Since the value of the $\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Psi^{\uparrow},\Phi}$ operator with L satisfying condition (i) of Proposition 28 does not depend on $\Psi^{\uparrow}$, one can use permutations $\psi^{\uparrow}$ in formula (9) and simply write $\mathrm{CS}^{\mathrm{L}}_{\mathsf{A},\Phi}$.

Conclusion

In this paper, we have introduced new operators. We have studied their properties and connections to existing operators. A lot of propositions are in the form of the necessary and sufficient condition, see e.g. Proposition 14, Proposition 10. So we have provided a complete solution to the studied problems. The construction of this new operator is closely related to OWA operators. Thus, in the future we plan to explore deeper connections of our new operator to existing generalizations of OWA operators.

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