Conditional aggregation-based Choquet integral on discrete space

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Abstract

We derive computational formulas for the generalized Choquet integral based on the novel survival function introduced by M. Boczek et al. [2]. We describe sufficient and necessary conditions under which novel survival functions based on different parameters coincide. This is closely related to the indifference of input vectors (alternatives) in decision-making processes. We demonstrate the usefulness of our results on the Knapsack problem.

Keywords: Choquet integral; conditional aggregation operator; decision making, survival function

1 Introduction

M. Boczek et al. [2], inspired by consumer's problems, aggregation operators, and conditional expectation, introduced a new notion of conditional aggregation operators. Conditional aggregation operators cover many existing aggregations, such as the arithmetic and geometric mean, or plenty of integrals known in the literature [13, 16, 19]. They became the essence of the generalization of survival function (a notion known from [8], from [10] known as the decumulative distribution function or from [3] known as the strict level measure) introduced by M. Boczek et al. [2]. Just as the survival function is the basis of the definition of the famous Choquet integral, the generalized survival function enabled the building of a new integral. The generalized Choquet integral $\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu)$ based on the generalized survival function $\mu_{\mathscr{A}}(\mathbf{x},\alpha)$ has been naturally introduced by

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \int_0^\infty \mu_{\mathscr{A}}(\mathbf{x},\alpha)\,\mathrm{d}\alpha.$$

A discrete form of the famous Choquet integral is of great importance in decision making theory, regarding a finite set $[n]=\{1,\dots,n\}$ as criteria set, a vector $\mathbf{x}\in[0,+\infty)^{[n]}$ as a score vector, and a capacity $\mu\colon 2^{[n]}\to[0,+\infty)$ as the weights of particular sets of criteria. We provide problems, which complement the ones described in [2], and which stress the potential of the novel survival function and integral based on it in decision theory. However, the main aim of the present paper is to provide various computational formulas for $\mu_{\mathscr{A}}(\mathbf{x},\alpha)$, and consequently for $\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu)$, since the computation of $\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu)$ has not yet been thoroughly investigated. Thus, we believe that our computational algorithms improve the practical implementation of the novel concept of aggregation in various problems.

The paper is organized as follows. Section 2 contains necessary terminology on measure and a family of conditional aggregation operators. Moreover, it describes a real-life motivated problem that can be modeled by the generalized survival function. We also present a way to visually handle the formula to identify with it better. This representation will be crucial in further sections. The first series of formulas for computation of generalized survival function is presented in Section 3. The next section reduces the minimization of the reals in the formulas to the minimization of indices, and shows two graphical approaches to obtain the generalized survival function. In addition, it contains a solution to the problem stated in the previous section. The formulas for the generalized Choquet integral computation are listed in Section 5, including simplified fomulas for special types of measures. The study of the conditions such that the novel survival functions based on different parameters coincide is performed in Section 6. Finally, the paper itself contains just proofs of selected results, the remaining proofs are attached in the Appendix.

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2 Background, interpretations and visualizations

Let us introduce the necessary definitions and notations. One of the crucial objects we shall deal with is the generalized survival function. We interpret its formula in the context of the knapsack problem, and we present how to compute the generalized survival function visually. We hope it helps the reader to acquire the concept better.

As we have already mentioned, we shall consider a finite set $[n] := \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, $n \geq 1$. Let us denote $[n]_0 := \{0\} \cup [n]$. By $2^{[n]}$ we mean the power set of $[n]$. A set function $\mu\colon \mathscr{S} \to [0, +\infty)$, $\{\emptyset\} \subseteq \mathscr{S} \subseteq 2^{[n]}$, such that $\mu(E) \leq \mu(F)$ whenever $E \subseteq F$, with $\mu(\emptyset) = 0$, we call *monotone measure* on $\mathscr{S}$. In this sense, a monotone measure is identical to the Šipoš premeasure, [14, 17]. Moreover, if $[n] \in \mathscr{S}$, we assume $\mu([n]) > 0$, and if $\mu([n]) = 1$, the monotone measure μ is called *capacity* or *normalized monotone measure*. Further, we put $\max \emptyset = 0$, $\min \emptyset = +\infty$ and $\sum_{i \in \emptyset} x_i = 0$.

We shall work with nonnegative real-valued vectors, we use the notation $\mathbf{x} = (x_1, \dots, x_n)$, $x_i \in [0, +\infty)$, $i \in [n]$. The family of all nonnegative real-valued vectors on $[n]$ is the set $[0, +\infty)^{[n]}$. By $\mathbf{1}_D$ we shall denote the indicator function of a set $D \subseteq Y$, i.e., $\mathbf{1}_D(x) = 1$ if $x \in D$ and $\mathbf{1}_D(x) = 0$ if $x \notin D$. Especially, $\mathbf{1}_\emptyset(x) = 0$ for each $x \in Y$. We shall work with the indicator function with respect to two different sets. We shall work with $Y = [n]$ when dealing with vectors (i.e. $\mathbf{1}_D$, $D \subseteq [n]$ is the n-tuple such that its i-th component is 1, if $i \in D$, otherwise 0) and $Y = [0, \infty)$ when dealing with survival functions.

Let us consider a set $\mathscr{E} \subseteq 2^{[n]}$. Unless otherwise stated, for application reasons we assume $\{\emptyset, [n]\} \subseteq \mathscr{E}$. We call it the *collection*. Let us denote the number of sets in $\mathscr{E}$ by κ, i.e. $|\mathscr{E}| = \kappa$. Let $\hat{\mathscr{E}} = \{E^c : E \in \mathscr{E}\}$, i.e. $\hat{\mathscr{E}}$ contains the complements of the sets from collection $\mathscr{E}$. The set of all monotone measures on $\hat{\mathscr{E}}$ we shall denote by $\mathbf{M}$.

In the following, we present the definition of the conditional aggregation operator. The inspiration for this concept can be found in probability theory, specifically in conditional expectation. The idea of aggregating data not on the whole, but on a conditional set is expressed in the terms of this definition. The conditional aggregation operator generalizes the classical definition of aggregation operator introduced by Calvo et al. in [5] and forms the basic component of the definition of the generalized survival function.

Definition 2.1. (cf. [2, Definition 4.1.]) Let $\mathscr{E}$ be a collection. A *family of conditional aggregation operators* (FCA for short) is a family

$$\mathscr{A} = \{\mathsf{A}(\cdot|E) : E \in \mathscr{E}\}^1,$$

such that each $\mathsf{A}(\cdot|E)$ is a map $\mathsf{A}(\cdot|E)\colon [0, +\infty)^{[n]} \to [0, +\infty)$ satisfying the following conditions:

(i) $\mathsf{A}(\mathbf{x}|E) \leq \mathsf{A}(\mathbf{y}|E)$ for any $\mathbf{x}, \mathbf{y}$ such that $x_i \leq y_i$ for any $i \in E$, $E \neq \emptyset$;

(ii) $\mathsf{A}(\mathbf{1}_{E^c}|E) = 0$, $E \neq \emptyset$.

If μ is a monotone measure on $\hat{\mathscr{E}} = \{E^c : E \in \mathscr{E}\}$, i.e. $\mu \in \mathbf{M}$, then the *generalized survival function* with respect to $\mathscr{A}$ is defined as

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) := \min\left\{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \leq \alpha,\ E \in \mathscr{E}\right\} \tag{2.1}$$

for any $\alpha \in [0, +\infty)$.

Remark 2.2. *Note that the survival function $\mu(\{\mathbf{x} > \alpha\}) = \mu(\{i \in [n] : x_i > \alpha\})$ can be rewritten as*

$$\begin{aligned}\mu(\{\mathbf{x} > \alpha\}) = \mu([n] \setminus \{\mathbf{x} \leq \alpha\}) &= \min\left\{\mu(E^c) : (\forall i \in E)\ x_i \leq \alpha,\ E \in 2^{[n]}\right\} \\ &= \min\{\mu(E^c) : \max_{i \in E} x_i \leq \alpha, E \in 2^{[n]}\},\end{aligned} \tag{2.2}$$

[1] $\mathscr{A}$ is a family of operators parametrized by a set from $\mathscr{E}$. If $E \neq \emptyset$, then $\mathsf{A}(\cdot|E)$ is called the *conditional aggregation operator w.r.t. E*. Moreover, we consider that each element of FCA satisfies $\mathsf{A}(\cdot|\emptyset) = 0$.

therefore the introduction of generalized survival function consists in the simple idea of replacing $\max_{i\in E} x_i$ *in (2.2) with another functional. Clearly, for* $\mathscr{E} = 2^{[n]}$ *and* $\mathscr{A}^{\max}$ *we get the original strict survival function.*

Example 2.3. Let $\mathbf{x} \in [0, +\infty)^{[n]}$. Typical examples of the FCA are:

(i) $\mathscr{A}^{\max} = \{\mathsf{A}^{\max}(\cdot|E) : E \in \mathscr{E}\}$ with $\mathsf{A}^{\max}(\mathbf{x}|E) = \max_{i\in E} x_i$ for $E \neq \emptyset$,

(ii) $\mathscr{A}^{\text{sum}} = \{\mathsf{A}^{\text{sum}}(\cdot|E) : E \in \mathscr{E}\}$ with $\mathsf{A}^{\text{sum}}(\mathbf{x}|E) = \sum_{i\in E} x_i$ for $E \neq \emptyset$,

(iii) $\mathscr{A}^{\text{WAM}_\mathbf{w}} = \{\mathsf{A}^{\text{WAM}_\mathbf{w}}(\cdot|E) : E \in \mathscr{E}\}$ with $\mathsf{A}^{\text{WAM}_\mathbf{w}}(\mathbf{x}|E) = \sum_{i\in E} w_i x_i$ for $E \neq \emptyset$, $\mathbf{w} \in [0,1]^{[n]}$, $\sum_{i\in E} w_i = 1$.

(iv) $\mathscr{A}^{\text{proj}} = \{\mathsf{A}^{\text{proj}}(\cdot|E) : E \in \{\{1\}, \dots, \{n\}\}\}$ with $\mathsf{A}^{\text{proj}}(\mathbf{x}|\{i\}) = x_i$, $i \in [n]$.

(v) $\mathscr{A}^{\text{Ch}_m} = \{\mathsf{A}^{\text{Ch}_m}(\cdot|E) : E \in \mathscr{E}\}$ with $\mathsf{A}^{\text{Ch}_m}(\mathbf{x}|E) = \sum_{i=1}^{n} x_{\sigma(i)} \left(m(E_{\sigma(i)} \cap E) - m(E_{\sigma(i+1)} \cap E)\right)$, with $\sigma(\cdot)\colon [n] \to [n]$ such that $x_{\sigma(1)} \leq \dots \leq x_{\sigma(n)}$, $E_{\sigma(i)} = \{\sigma(i), \dots, \sigma(n)\}$, $m \in \mathbf{M}$.

Note that the FCA does not have to contain elements of only one type.

Example 2.4. *Let* $\mathbf{x} \in [0, +\infty)^{[3]}$*, and* $\mathscr{E} = \{\emptyset, \{1\}, \{2\}, \{1,3\}, \{1,2,3\}\}$*. We can consider the following FCA*

$$\mathscr{A} = \{\mathsf{A}^{\text{Ch}_m}(\cdot|E) : E \in \{\{1\}, \{2\}, \{1,2,3\}\}\} \cup \{\mathsf{A}^{\min}(\cdot|E) : E \in \{\{1,3\}\}\} \cup \{\mathsf{A}(\cdot|\emptyset)\},$$

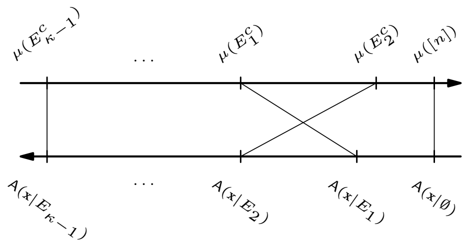
where $\mathsf{A}^{\min}(\mathbf{x}|E) = \min_{i\in E} x_i$ *for* $E \neq \emptyset$*.*

For other examples of FCA we recommend [2]. On several places in this paper, we shall work with the FCA that is *nondecreasing* w.r.t sets, i.e. the map $E \mapsto \mathsf{A}(\cdot|E)$ will be nondecreasing. E.g., the families $\mathscr{A}^{\max}$ and $\mathscr{A}^{\text{sum}}$ in Example 2.3 are nondecreasing w.r.t. sets.

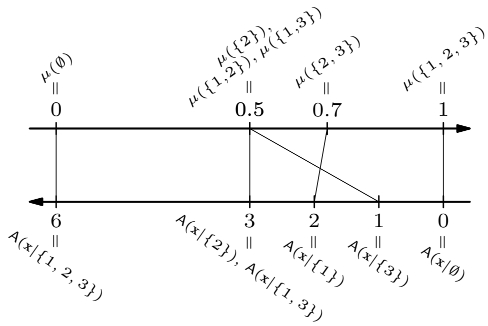
In the following, we present a problem that emphasizes the need for the generalized survival function and the generalized Choquet integral in real situations. We stress that M. Boczek et al. in [2] introduced several problems of this kind. However, we move beyond these examples. The solution to this problem using our results is delayed to Subsection 4.1.

Knapsack problem Let us imagine a person who is preparing for a holiday. The person plans to travel by plane. Therefore while packing the suitcase, he must keep the rules according to which it is allowed to carry less or equal to 1 liter of liquids in the suitcase. Moreover, liquids must be in containers with a volume of up to 100 ml. Let us suppose that the total volume of products the person wants to pack is more than the set limit, so it will be necessary to buy some products abroad. Of course, the person wants to minimize the purchase abroad.

Let $[n]$, $n \geq 1$, be a set of liquid products that the person needs. Then $\mathscr{E} \subseteq 2^{[n]}$ represents all possible combinations of products. Let us consider $\mathbf{x} = (x_1, \dots, x_n) \in [0, +\infty)^{[n]}$, where x_i represents the volume of i-th container and let a monotone measure $\mu \in \mathbf{M}$ represent the price of a package of products. Note that the monotone measure μ need not be additive. It is often possible to buy a package of products that is cheaper than the sum of the prices of the individual products. The task is to choose such a combination $E \in \mathscr{E}$ of products that their volume does not exceed the given limit, i.e. $\sum_{i\in E} x_i \leq 1000$ (milliliters), having in mind that we want to minimize the price of those products that will no longer fit in the suitcase and the person will



(a) The general case of visualization



(b) Visualization for Example 2.5

Figure 1: Visualization of generalized survival function computation

have to buy them during the holiday. In other words, we are faced to solve the optimization problem

$$\min\left\{\mu(E^c) : \sum_{i\in E} x_i \leq 1000,\, E\in\mathscr{E}\right\}.$$

One can observe that the given formula is a special case of the generalized survival function given in (2.1) with the conditional aggregation operator being the sum.

Last, but not least let us point out the essence of the collection $\mathscr{E}$. There are situations when instead of the whole power set one is forced to consider a subcollection $\mathscr{E}\subset 2^{[n]}$. E.g. among the products that one is considering can be shampoo and conditioner separately or shampoo containing conditioner. Of course, these products should not be considered together. Thus all sets consisting of these three products are disqualified and it makes sense to take $\mathscr{E}\subset 2^{[n]}$ instead of the whole power set. The full solution to the problem using our results is accomplished in Subsection 4.1.

The visualization of generalized survival function computation Let us demonstrate the visual representation of the generalized survival function, see formula (2.1). We use the model of two parallel lines oriented reversely, see Figure 1(a). Let us depict the values of conditional aggregation operators on the lower axis and the values of corresponding monotone measures on the upper axis, both ascending in the direction given on the axis. Let us join by a line values $\mathsf{A}(\mathbf{x}|E)$ and $\mu(E^c)$. The visual computation of the generalized survival function we demonstrate in the following example. We shall use the input data below also in other examples in this paper.

Example 2.5. Let us consider the collection $\mathscr{E}=\{\emptyset,\{1\},\{2\},\{3\},\{1,3\},\{1,2,3\}\}$, the family of conditional aggregation operators $\mathscr{A}^{\mathrm{sum}}=\{\mathsf{A}^{\mathrm{sum}}(\cdot|E) : E\in\mathscr{E}\}$, the vector $\mathbf{x}=(2,3,1)$, and the monotone measure μ on $\hat{\mathscr{E}}$ with corresponding values in the table.

E	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,3\}$	$\{1,2,3\}$
$\mathsf{A}^{\mathrm{sum}}(\mathbf{x}\|E)$	0	2	3	1	3	6
E^c	$\{1,2,3\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2\}$	$\{2\}$	$\emptyset$
$\mu(E^c)$	1	0.7	0.5	0.5	0.5	0

Having the diagram of the generalized survival function constructed according to the above-mentioned rules, see the Figure 1(b), we can derive its formula: E.g. the value of the generalized survival function at 2.5 is 0.5 what is the minimum of the values

$$1\leftrightarrow\mu(\{1,2,3\}),\ 0.5\leftrightarrow\mu(\{1,2\}),\ 0.7\leftrightarrow\mu(\{2,3\})$$

that correspond to values of conditional aggregation operators of sets to the right of the observed value, i.e., $0 \leftrightarrow \mathsf{A}(\mathbf{x}|\emptyset)$, $1 \leftrightarrow \mathsf{A}(\mathbf{x}|\{3\})$, $2 \leftrightarrow \mathsf{A}(\mathbf{x}|\{1\})$. Repeating this procedure for each nonnegative real number we get the formula of the generalized survival function

$$\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mathbf{1}_{[0,1)}(\alpha) + 0.5\cdot\mathbf{1}_{[1,6)}(\alpha), \quad \alpha\in[0,+\infty).$$

We shall also deal with the visualization of the generalized survival function later in Section 4, however, of a modified version handling with indices.

3 Computational formulas

3.1 Basic denotation to computation

When working with a family of conditional aggregation operators $\mathscr{A}$, an input vector $\mathbf{x}$ is aggregated on sets E from $\mathscr{E}\subseteq 2^{[n]}$ and we obtain $\kappa = |\mathscr{E}|$ values of the family of conditional aggregation operators $\mathscr{A}$. Let us denote by $\mathfrak{e}$ the bijection $\mathfrak{e}\colon [\kappa-1]_0 \to \mathscr{E}$ such that

$$0 = \mathsf{A}_0 \le \mathsf{A}_1 \le \dots \le \mathsf{A}_{\kappa-2} \le \mathsf{A}_{\kappa-1} < +\infty, \tag{3.1}$$

where

$$\mathsf{A}_i = \mathsf{A}(\mathbf{x}|E_i) \tag{3.2}$$

for any $i\in[\kappa]_0$ with the convention $\mathsf{A}_\kappa = +\infty$. While we shall denote sets in collection $\mathscr{E}$ by E_i, $i\in[\kappa-1]_0$, their complements in $\hat{\mathscr{E}}$ we shall denote by F_j, $j\in[\kappa-1]_0$, because of technical details. And, for the same reasons also the second bijection will be useful for us, i.e. $\mathfrak{f}\colon [\kappa-1]_0\to\hat{\mathscr{E}}$ such that denoting $\mu_j = \mu(F_j)$ we have:

$$0 = \mu_0 \le \mu_1 \le \dots \le \mu_{\kappa-2} \le \mu_{\kappa-1} < +\infty. \tag{3.3}$$

One can notice that maps $\mathfrak{e}$ and $\mathfrak{f}$ need not be unique (they are unique just in case they are injective on $[\kappa-1]_0$), but this has no influence on presented results.

Let us recall that $\hat{\mathscr{E}}$ is the collection of complements of $\mathscr{E}$, i.e. for every $F_j\in\hat{\mathscr{E}}$ there exists $E_i\in\mathscr{E}$ such that $F_j = E_i^c$. So, it is seen that indices corresponding to conditional aggregation operators and indices corresponding to monotone measures are connected by some mapping that we formalize in the following.

Let us consider the map $(\cdot)\colon [\kappa-1]_0 \to [\kappa-1]_0$ connecting the conditional aggregation operator with its corresponding monotone measure with respect to these sets defined by

$$(i) = j \text{ whenever } E_i = F_j^c,$$

i.e., $F_{(i)} = E_i^c$. Let us point out that the connection process can be seen also reciprocally, i.e., let us define $\langle\cdot\rangle\colon [\kappa-1]_0\to[\kappa-1]_0$

$$\langle j\rangle = i \text{ whenever } E_i^c = F_j,$$

i.e., $E_{\langle j\rangle} = F_j^c$. It is easy to check that $\langle\cdot\rangle = (\cdot)^{-1}$. In the whole paper for ease of writing, we shall use a shortcut notation

$$\mathsf{A}_{\langle j\rangle} := \mathsf{A}(\mathbf{x}|E_{\langle j\rangle}) = \mathsf{A}(\mathbf{x}|F_j^c) \quad\text{and}\quad \mu_{(i)} := \mu(F_{(i)}) = \mu(E_i^c).$$

In the following example, we only demonstrate the introduction of bijection $\mathfrak{e}$, $\mathfrak{f}$, and maps $(\cdot)$, $\langle\cdot\rangle$, respectively.

Example 3.1. Let us consider the same inputs as in Example 2.5. Using maps $\mathfrak{e}$ and $\mathfrak{f}$ we obtain the following arrangement:

i, j	0	1	2	3	4	5
E_i	$\emptyset$	$\{3\}$	$\{1\}$	$\{1,3\}$	$\{2\}$	$\{1,2,3\}$
A_i	0	1	2	3	3	6
F_j	$\emptyset$	$\{2\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
μ_j	0	0.5	0.5	0.5	0.7	1

To calculate the value (0) we need the set $E_0 = \emptyset$. Its complement is $\{1,2,3\}$, which has index 5 in the enumeration $\mathfrak{f}$, i.e., $F_5 = \{1,2,3\}$. Thus $(0) = 5$. Similarly, to compute (1), we see that $E_1 = \{3\}$ and $E_1^c = \{1,2\} = F_2$. Thus $(1) = 2$. Continuing in a similar fashion, we obtain all the values,

$$(2) = 4,\ (3) = 1,\ (4) = 3,\ (5) = 0.$$

For completeness,

$$\langle 0\rangle = 5, \langle 1\rangle = 3, \langle 2\rangle = 1,\ \langle 3\rangle = 4,\ \langle 4\rangle = 2,\ \langle 5\rangle = 0.$$

3.2 Two approaches to computation

The first approach to computing generalized survival function values is based on an (ascending) arrangement (3.1) of conditional aggregation operator values. Let us note that this approach is directly derived from Definition 2.1. In the computation process using this approach, we look for generalized survival function values that are achieved at intervals $[\mathsf{A}_i, \mathsf{A}_{i+1})$ with the convention $\mathsf{A}_\kappa = +\infty$.

Theorem 3.2. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$. Then*

(i) *for any $i \in [\kappa - 1]_0$ it holds that*

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \min_{k \le i} \mu_{(k)} \quad \text{for any } \alpha \in [\mathsf{A}_i, \mathsf{A}_{i+1});$$

(ii)
$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \sum_{i=0}^{\kappa-1} \min_{k \le i} \mu_{(k)} \mathbf{1}_{[\mathsf{A}_i, \mathsf{A}_{i+1})}(\alpha) \quad \text{for any } \alpha \in [0,+\infty). \tag{3.4}$$

Proof. (i) Let us consider an arbitrary (fixed) $i \in [\kappa - 1]_0$ such that $\mathsf{A}_i < \mathsf{A}_{i+1}$, and let us take $\alpha \in [\mathsf{A}_i, \mathsf{A}_{i+1})$. Since $0 = \mathsf{A}_0 \le \mathsf{A}_1 \le \dots \le \mathsf{A}_i \le \alpha$, then

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \min\{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \le \alpha,\ E \in \mathscr{E}\} = \min\{\mu_{(k)} : k \in \{0, \dots, i\}\} = \min_{k \le i} \mu_{(k)}.$$

The case (ii) immediately follows from (i). $\square$

Remark 3.3. *Let $(x_1, \dots, x_n) \in [0,+\infty)^{[n]}$. Let us denote $\mathbf{x} = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$, with $\sigma\colon [n] \to [n]$ being a permutation such that $x_{\sigma(1)} \le x_{\sigma(2)} \le \dots \le x_{\sigma(n)}$ with the convention $x_{\sigma(0)} = 0$. Also let $\mathscr{A}^{\max}$ be a FCA with the collection $\mathscr{E} = \{G^c_{\sigma(i+1)} : i \in [n]_0\}$ where $G_{\sigma(i)} = \{\sigma(i), \dots, \sigma(n)\}$ for $i \in [n]$ and $G_{\sigma(n+1)} = \emptyset$. Then from the previous proposition we get the standard formula of the survival function [12]:*

$$\mu_{\mathscr{A}^{\max}}(\mathbf{x}, \alpha) = \sum_{i=0}^{n-1} \mu(G_{\sigma(i+1)}) \mathbf{1}_{[x_{\sigma(i)}, x_{\sigma(i+1)})}(\alpha) \quad \text{for any } \alpha \in [0,+\infty).$$

Indeed, $\mathsf{A}^{\max}(\mathbf{x}|G^c_{\sigma(i+1)}) = x_{\sigma(i)}$ for any $i \in [n]_0$. Moreover, if $x_{\sigma(i)} = \mathsf{A}^{\max}_{l_i}$, $l_i \in [\kappa - 1]_0$, then $\min_{k \le l_i} \mu_{(k)} = \min_{k \le i} \mu(G_{\sigma(k)}) = \mu(G_{\sigma(i)})$ because of the monotonicity of μ.

Remark 3.4. Since $[n] \in \mathscr{E}$, then the definition of generalized survival function guarantees to achieve zero value. According to arrangement in (3.1), for any $\alpha \geq \mathsf{A}_{\kappa-1} \geq \mathsf{A}(\mathbf{x}|[n])$ it holds

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \min\{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \leq \alpha,\ E \in \mathscr{E}\} \leq \mu([n]^c) = \mu(\emptyset) = 0.$$

On the other hand $\mu_{\mathscr{A}}(\mathbf{x},\alpha) \geq 0$. Thus, the generalized survival function achieves zero value on the interval $[\mathsf{A}(\mathbf{x}|[n]),+\infty) \supseteq [\mathsf{A}_{\kappa-1},+\infty)$.

Example 3.5. Let us consider the same inputs as in Example 2.5. Using Theorem 3.2(i) let us compute the generalized survival function for any $\alpha \in [0,+\infty)$.

- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_{(5)} = 0$ for any $\alpha \in [\mathsf{A}_5,\mathsf{A}_6) = [6,+\infty)$, which corresponds to Remark 3.4.
- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_{(4)} = 0.5$ for any $\alpha \in [\mathsf{A}_4,\mathsf{A}_5) = [3,6)$.
- $[\mathsf{A}_3,\mathsf{A}_4) = [3,3) = \emptyset$.
- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_{(1)} = 0.5$ for any $\alpha \in [\mathsf{A}_2,\mathsf{A}_3) = [2,3)$.
- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_{(1)} = 0.5$ for any $\alpha \in [\mathsf{A}_1,\mathsf{A}_2) = [1,2)$.
- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_{(0)} = 1$ for any $\alpha \in [\mathsf{A}_0,\mathsf{A}_1) = [0,1)$.

So we get

$$\begin{aligned}\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) &= \mathbf{1}_{[0,1)}(\alpha) + 0.5\cdot\mathbf{1}_{[1,2)}(\alpha) + 0.5\cdot\mathbf{1}_{[2,3)}(\alpha) + 0.5\cdot\mathbf{1}_{[3,6)}(\alpha)\\ &= \mathbf{1}_{[0,1)}(\alpha) + 0.5\cdot\mathbf{1}_{[1,6)}(\alpha),\end{aligned}$$

$\alpha \in [0,+\infty)$.

The second approach to computing generalized survival function values is based on an (ascending) arrangement (3.3) of measure values. Similarly, as in Remark 3.4, let us point out first where the generalized survival function achieves zero value.

Remark 3.6. *From arrangement of* μ, *see* (3.3), *we have that* $\mathsf{A}_{\langle 0\rangle} \in \{\mathsf{A}_{\langle j\rangle} : \mu_j = 0,\ j \in [\kappa-1]_0\}$. *Thus, for any* $\alpha \in [\mathsf{A}_{\langle 0\rangle},+\infty)$ *it holds* $\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \mu_0 = 0$.

Compared to the first approach presented in Theorem 3.2, here in the second approach we look for intervals at which monotone measure values $\mu(F_j)$ (i.e. generalized survival function values) are achieved. It is worth noting that the first approach is appropriate to use if the number of aggregation operator values is much smaller than the number of measure values. In the opposite case, it is more effective to use the second approach.

Theorem 3.7. *Let* $\mathscr{A}$ *be a FCA,* $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$. *Then*

(i) *for any* $j \in [\kappa-1]_0$ *it holds that*

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \mu_j \quad \textit{for any } \alpha \in \Big[\min_{k\leq j}\mathsf{A}_{\langle k\rangle}, \min_{k<j}\mathsf{A}_{\langle k\rangle}\Big);$$

(ii)
$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \sum_{j=0}^{\kappa-1}\mu_j\mathbf{1}_{\big[\min\limits_{k\leq j}\mathsf{A}_{\langle k\rangle},\min\limits_{k<j}\mathsf{A}_{\langle k\rangle}\big)}(\alpha) \quad \textit{for any } \alpha \in [0,+\infty). \tag{3.5}$$

Proof. For $j = 0$ we have, that $\Big[\min\limits_{k\leq 0}\mathsf{A}_{\langle k\rangle}, \min\limits_{k<0}\mathsf{A}_{\langle k\rangle}\Big) = [\mathsf{A}_{\langle 0\rangle},+\infty)$. Then, from Remark 3.6 is easy to see, that for any $\alpha \in [\mathsf{A}_{\langle 0\rangle},+\infty)$ it holds $\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \mu_0$. Let us take an arbitrary (fixed) $j \in [\kappa-1]$. The case when $\min\limits_{k\leq j}\mathsf{A}_{\langle k\rangle} = \min\limits_{k<j}\mathsf{A}_{\langle k\rangle}$ is trivial. Let us suppose that $\min\limits_{k\leq j}\mathsf{A}_{\langle k\rangle} \neq \min\limits_{k<j}\mathsf{A}_{\langle k\rangle}$. Then from the fact that

$$\min\Big\{\min_{k<j}\mathsf{A}_{\langle k\rangle}, \mathsf{A}_{\langle j\rangle}\Big\} = \min_{k\leq j}\mathsf{A}_{\langle k\rangle} < \min_{k<j}\mathsf{A}_{\langle k\rangle},$$

we have $\min\limits_{k\leq j} \mathsf{A}_{\langle k\rangle} = \mathsf{A}_{\langle j\rangle}$. Further, $\min\limits_{k<j} \mathsf{A}_{\langle k\rangle} \leq \mathsf{A}_{\langle l\rangle}$ for each $l \in [j-1]$. Then for any α such that

$$\mathsf{A}_{\langle j\rangle} = \min_{k\leq j} \mathsf{A}_{\langle k\rangle} \leq \alpha < \min_{k<j} \mathsf{A}_{\langle k\rangle} \leq \mathsf{A}_{\langle l\rangle}$$

with $l \in [j-1]$ we have

- $\mathsf{A}_{\langle j\rangle} \in \{\mathsf{A}(\mathbf{x}|E) \leq \alpha,\, E \in \mathscr{E}\}$, therefore $\mu_j \in \{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \leq \alpha,\, E \in \mathscr{E}\}$. Thus,

$$\min\{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \leq \alpha,\, E \in \mathscr{E}\} \leq \mu_j.$$

- $\mathsf{A}_{\langle l\rangle} \notin \{\mathsf{A}(\mathbf{x}|E) \leq \alpha,\, E \in \mathscr{E}\}$, therefore $\mu_l \notin \{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \leq \alpha,\, E \in \mathscr{E}\}$ and from the ordering (3.3) we have $\mu(E^c) \geq \mu_j$ for each $E \in \mathscr{E}$ such that $\mathsf{A}(\mathbf{x}|E) \leq \alpha$, therefore

$$\min\{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \leq \alpha,\, E \in \mathscr{E}\} \geq \mu_j.$$

The formula in (ii) directly follows from (i). □

Example 3.8. Let us consider the same inputs as in Example 2.5. According to Remark 3.6, $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = 0$ for any $\alpha \in [\mathsf{A}_{\langle 0\rangle}, +\infty) = [6,+\infty)$.

- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_1 = 0.5.$ for any $\alpha \in \big[\mathsf{A}_{\langle 1\rangle}, \mathsf{A}_{\langle 0\rangle}\big) = [3,6)$.
- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_2 = 0.5$ for any $\alpha \in \big[\mathsf{A}_{\langle 2\rangle}, \mathsf{A}_{\langle 1\rangle}\big) = [1,3)$.
- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_3 = 0.5$ for any $\alpha \in \big[\mathsf{A}_{\langle 2\rangle}, \mathsf{A}_{\langle 2\rangle}\big) = \emptyset$.
- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_4 = 0.7$ for any $\alpha \in \big[\mathsf{A}_{\langle 2\rangle}, \mathsf{A}_{\langle 2\rangle}\big) = \emptyset$.
- $\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_5 = 1$ for any $\alpha \in \big[\mathsf{A}_{\langle 5\rangle}, \mathsf{A}_{\langle 2\rangle}\big) = [0,1)$.

Therefore, the generalized survival function has the form

$$\begin{aligned}\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) &= 0.5\cdot \mathbf{1}_{[3,6)}(\alpha) + 0.5\cdot \mathbf{1}_{[1,3)}(\alpha) + \mathbf{1}_{[0,1)}(\alpha)\\ &= \mathbf{1}_{[0,1)}(\alpha) + 0.5\cdot \mathbf{1}_{[1,6)}(\alpha),\end{aligned}$$

$\alpha \in [0,+\infty)$, compare with Example 3.5.

The following expressions of generalized survival functions are w.r.t. special measures. Their detailed proofs are in Appendix.

Corollary 3.9. *Let $\mathbf{x} \in [0,+\infty)^{[n]}$.*

(i) *Let $\mathscr{A}$ be FCA and $\bar{\mu}$ be the greatest monotone capacity, i.e., $\bar{\mu}(F) = 0$, if $F = \emptyset$ and $\bar{\mu}(F) = 1$, otherwise. Then the generalized survival function w.r.t. $\bar{\mu}$ takes the form*

$$\bar{\mu}_{\mathscr{A}}(\mathbf{x},\alpha) = \mathbf{1}_{[0,\mathsf{A}(\mathbf{x}|[n]))}(\alpha).$$

(ii) *Let $\mathscr{A}$ be FCA and $\underline{\mu}$ be the weakest monotone capacity, i.e., $\underline{\mu}(F) = 1$, if $F = [n]$ and $\underline{\mu}(F) = 0$, otherwise. Then the generalized survival function w.r.t. $\underline{\mu}$ takes the form*

$$\underline{\mu}_{\mathscr{A}}(\mathbf{x},\alpha) = \mathbf{1}_{\big[0,\min\limits_{E\neq\emptyset} \mathsf{A}(\mathbf{x}|E)\big)}(\alpha).$$

(iii) *Let $\mathscr{A}$ be FCA nondecreasing w.r.t. sets with $\mathscr{E} = 2^{[n]}$. Let μ be a symmetric measure, i.e., $\mu(F) = \mu(G)$ for $F, G \in 2^{[n]}$ such that $|F| = |G|$ (see e.g. [15]). Let us set $\mu^i := \mu(F)$, if $|F| = i$, $i \in [n]_0$. Then the generalized survival function w.r.t. μ takes the form*

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \sum_{i=0}^{n} \mu^i \mathbf{1}_{\big[\min\limits_{|E|=n-i} \mathsf{A}(\mathbf{x}|E),\, \min\limits_{|E|=n-i+1} \mathsf{A}(\mathbf{x}|E)\big)}(\alpha).$$

(iv) *Let $\mathscr{A}$ be FCA nondecreasing w.r.t. sets with $\mathscr{E} = 2^{[n]}$. Let Π be a possibility measure given for any $F \subseteq [n]$ as $\Pi(F) = \max_{i\in F} \pi(i)$. The function $\pi\colon [n] \to [0,1]$, $\pi(i) = \Pi(\{i\})$ is called a possibility distribution (of Π), see e.g. [21]. Let $\sigma\colon [n] \to [n]$ be a permutation such that $0 = \pi(\sigma(0)) \le \pi(\sigma(1)) \le \cdots \le \pi(\sigma(n)) = 1$ and $G_{\sigma(i)} = \{\sigma(i), \dots, \sigma(n)\}$ for $i \in \{1, \dots, n\}$ with $G_{\sigma(n+1)} = \emptyset$ and $\mathsf{A}(\mathbf{x}|G_{\sigma(0)}) = \infty$. Then the generalized survival function w.r.t. Π takes the form*

$$\Pi_{\mathscr{A}}(\mathbf{x}, \alpha) = \sum_{i=0}^{n} \pi(\sigma(i)) \mathbf{1}_{\left[\mathsf{A}(\mathbf{x}|G_{\sigma(i+1)}), \mathsf{A}(\mathbf{x}|G_{\sigma(i)})\right)}(\alpha).$$

(v) *Let $\mathscr{A}$ be FCA nondecreasing w.r.t. sets with $\mathscr{E} = 2^{[n]}$. Let N be a necessity measure given for any $F \subseteq [n]$ as $\mathrm{N}(F) = 1 - \max_{i \notin F} \pi(i)$ with the convention that the maximum of empty set is 0. Let $\sigma\colon [n] \to [n]$ be a permutation given as in the previous case. Then the generalized survival function w.r.t. N takes the form*

$$N_{\mathscr{A}}(\mathbf{x}, \alpha) = \sum_{i=0}^{n} \big(1 - \pi(\sigma(i))\big) \mathbf{1}_{\left[\min\limits_{k \ge i} \mathsf{A}(\mathbf{x}|\{\sigma(k)\}), \min\limits_{k>i} \mathsf{A}(\mathbf{x}|\{\sigma(k)\})\right)}(\alpha),$$

where we need the convention $\min_{k\ge 0} \mathsf{A}(\mathbf{x}|\{\sigma(k)\}) = 0$.

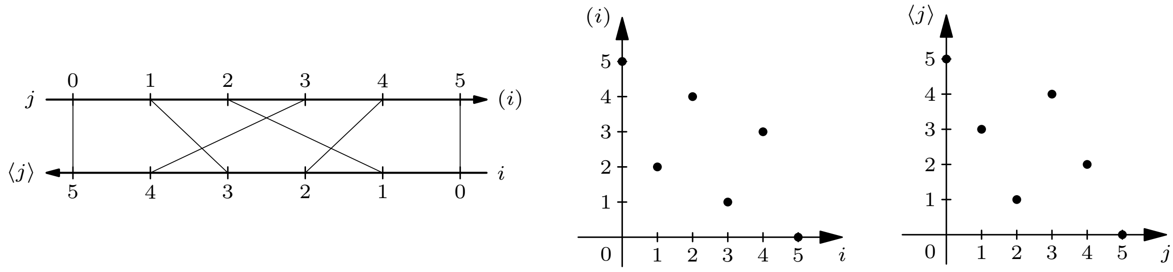
Let us compare both approaches to calculate the generalized survival function. Comparing Example 3.5 and Example 3.8 one can observe that the domain partition of the generalized survival function (the interval $[0, \infty)$) in Example 3.5 is a superset of the domain partition of the same generalized survival function in Example 3.8. This observation, we shall show, holds in general. As a consequence, we get that the number of nonzero summands in expression (3.5) is less than in expression (3.4). The proof can be found in Appendix.

Proposition 3.10. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0, +\infty)^{[n]}$. For each $i \in [\kappa - 1]_0$ there exists j such that $[\mathsf{A}_i, \mathsf{A}_{i+1}) \subseteq [\min_{k\le j} \mathsf{A}_{\langle k\rangle}, \min_{k<j} \mathsf{A}_{\langle k\rangle})$.*

4 Permutations and visualization

Both formulas in Section 3, presented in Theorem 3.2, 3.7 ask to minimize the values of either monotone measure or aggregation operator. The main aim of the present section is to show that the computation of the generalized survival function (and the minimization process) may be accomplished just on a set of integer indices of the values of monotone measure and aggregation operator. Moreover, the whole procedure may be visualized, so the computation of the generalized survival function becomes easily accessible. The main tools are several functions on a set of indices $[\kappa - 1]_0$.

Let us start this section with the visualization of maps $(\cdot)$ and $\langle\cdot\rangle$ that we introduced in the previous section. It will be beneficial for better understanding of presented results. Let us follow the idea of visualization presented in Section 2, however, let us depict corresponding indices on both axis instead of values of conditional aggregation operators and the monotone measure, see Figure 2(a). The maps $(\cdot)$ and $\langle\cdot\rangle$ can also be represented in the Cartesian coordinate system, see Figures 2(b) and 2(c). Each of these representations has certain advantages. In diagram, the domain of $(\cdot)$ is the lower axis, while the codomain is the upper one. Thus the indices of the aggregation operator are in the lower axis, and the indices in the upper axis correspond to the values of the monotone measure. From a practical point of view, the axes are in reversed order. The visualization in the coordinate system we shall discuss in Section 4.3 in more detail.



(a) Diagram of $(\cdot)$, resp. $\langle\cdot\rangle = (\cdot)^{-1}$ (b) Graph of $(\cdot)$ (c) Graph of $\langle\cdot\rangle = (\cdot)^{-1}$

Figure 2: Diagram and graphs of functions $(\cdot)$, $\langle\cdot\rangle$ for data from Example 2.5

4.1 Computation via indices

Once comparing Definition 2.1 of the generalized survival function and the map $(\cdot)$, one can see that the assignment $(\cdot)$ is a part of the formula defining the generalized survival function. More precisely, we have

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \min\left\{\mu_{(i)} : \mathsf{A}_i \leq \alpha,\, i \in [\kappa-1]_0\right\}, \quad \alpha \in [0,+\infty).$$

Then the whole computation in the previous formula may be visualised via Figure 2(a). Indeed, once we need to compute $\mu_{\mathscr{A}}(\mathbf{x},\alpha)$ for a given α, we can find the largest index i such that $\mathsf{A}_i \leq \alpha$. Afterwards we need to consider all the indices on the upper axis in Figure 2(a) adjacent with the indices greater than or equal to i (the right-hand side indices with respect to i on the lower axis). Finally, the minimization the values of monotone measure of sets with the selected indices on the upper axis leads to the value $\mu_{\mathscr{A}}(\mathbf{x},\alpha)$. However, the application of the assignment $(\cdot)$ goes far beyond the latter observations.

To understand the real contribution of $(\cdot)$, we need to analyse crossing-overs in Figure 2(a). The crossing-overs can be in only two cases:

- if $\min_{k\leq i}\mu_{(k)} < \mu_{(i)}$, then the value $\mu_{(i)}$ is not achieved by the generalized survival function because of Theorem 3.2;
- if $\mu_{(k)} = \mu_{(l)}$, $k,l \in [\kappa-1]_0$, where $k < l$ and $(k) < (l)$, which corresponds to the ambiguity of the arrangement (3.1), or (3.3).

These connections can be removed or redefined in an appropriate manner without changing the formula of generalized survival functions. Thus let us redefine the mapping $(\cdot)$ in a manner that will be beneficial for us. Let us define a map $\mathbf{i}\colon [\kappa-1]_0 \to [\kappa-1]_0$ by

$$\mathbf{i}(i) = \min\{(0),\ldots,(i)\}. \tag{4.1}$$

The previous mapping will shorten a lot of further expressions, and calculations. It will be useful in Section 5 in deriving the generalized Choquet integral formulas.

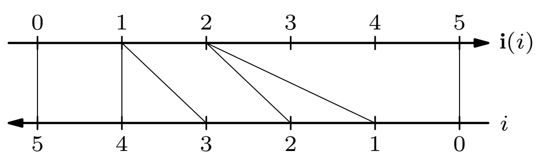


Figure 3: Diagram of the map $\mathbf{i}$ for data from Example 2.5

Figure 3 contains a diagram describing function $\mathbf{i}$ computed for data from Example 2.5. Although one can use formula (4.1) directly and compute $\mathbf{i}$ algebraically, it can be easily obtained also from the diagram in Figure 2(a). Indeed, since (4) = 3 and (3) = 1, the corresponding edges in Figure 2(a) are crossed-over. It is just enough to eliminate this crossing-over by re-defining $\mathbf{i}(4) = 1$. Similarly for crossing-over, which corresponds to values (1) and (2). After its elimination we obtain $\mathbf{i}$.

Lemma 4.1. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$, and $\mathbf{i}$ be a map given in (4.1). Then for any $i \in [\kappa-1]_0$ it holds*

$$\min_{k \le i} \mu_{(k)} = \mu_{\mathbf{i}(i)}.$$

Proof. Because of the definition of $(\cdot)$ and from the arrangement (3.3) we get equalities

$$\min_{k \le i} \mu_{(k)} = \min\{\mu_l : l = (k), k \le i\} = \mu_{\min\{(0),\dots,(i)\}} = \mu_{\mathbf{i}(i)}.$$

Thus we get the required result. □

Via the assignment $\mathbf{i}$, the formula of the generalized survival function can be described directly, without the need to minimize, compare the following corollary with Theorem 3.2.

Corollary 4.2. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$, and $\mathbf{i}$ be a map given in (4.1). Then*

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \sum_{i=0}^{\kappa-1} \mu_{\mathbf{i}(i)} \mathbf{1}_{[\mathsf{A}_i,\mathsf{A}_{i+1})}(\alpha) \tag{4.2}$$

for any $\alpha \in [0,+\infty)$.

Proof. The required result follows from Lemma 4.1 and Theorem 3.2. □

The difference between Theorem 3.2 and Theorem 3.7 is in the object being minimized. In Theorem 3.2 are minimized the values of monotone measure, while in Theorem 3.7 are minimized the values of aggregation operator. Thus Corollary 4.2 is the counterpart of Theorem 3.2 and naturally, the need to introduce a counterpart of Theorem 3.7 arises. Similarly, as the formula for the generalized survival function can be rewritten via permutation $(\cdot)$, it can be also rewritten via permutation $\langle\cdot\rangle$ as follows

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \min\left\{\mu_j : \mathsf{A}_{\langle j\rangle} \le \alpha,\, j \in [\kappa-1]_0\right\}, \quad \alpha \in [0,+\infty).$$

Let us recall that $\langle\cdot\rangle = (\cdot)^{-1} \colon [\kappa-1]_0 \to [\kappa-1]_0$, i.e., $\langle(i)\rangle = i$, $(\langle j\rangle) = j$, and $E_{\langle j\rangle} = F_j^c$, i.e., $F_j = E_{\langle j\rangle}^c$. The permutation $\langle\cdot\rangle$ is in accordance with the previously adopted notation since $\mathsf{A}_{\langle j\rangle} = \mathsf{A}(\mathbf{x}|E_{\langle j\rangle}) = \mathsf{A}(\mathbf{x}|F_j^c)$. Let us define a map $\mathbf{j} \colon [\kappa-1]_0 \to [\kappa-1]_0$ as follows

$$\mathbf{j}(j) = \min\{\langle 0\rangle, \dots, \langle j\rangle\}. \tag{4.3}$$

Remark 4.3. *It is easy to see that for any $k \in [\kappa-1]_0$ it holds $\mathbf{i}(k) \le (k)$ and $\mathbf{j}(k) \le \langle k\rangle$. Moreover, the maps $\mathbf{i}, \mathbf{j}$ are nonincreasing.*

Similarly as in the case of the map $\mathbf{i}$ we can shorten a lot of expressions, and calculations with the map $\mathbf{j}$. The corresponding diagram of $\mathbf{j}$ based on data from Example 2.5 is depicted in Figure 4. Note that similarly to the diagram in Figure 3 it does not contain any crossing-over, and may be analogously created either directly using formula (4.3) or by eliminating crossings-over in the diagram in Figure 2.

Lemma 4.4. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$, and $\mathbf{j}$ be a map given as in (4.3). Then for any $j \in [\kappa-1]_0$ it holds*

$$\min_{k \le j} \mathsf{A}_{\langle k\rangle} = \mathsf{A}_{\mathbf{j}(j)}.$$

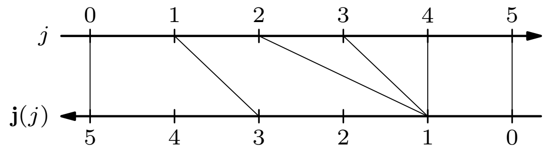


Figure 4: Diagram of the map $\mathbf{j}$ for data from Example 2.5.

Proof. Because of the definition of $(\cdot)$ and from the arrangement (3.1) we immediately have

$$\min_{k \leq j} \mathsf{A}_{\langle k \rangle} = \min\{\mathsf{A}_l : l = \langle k \rangle, k \leq j\} = \mathsf{A}_{\min\{\langle 0 \rangle, \dots, \langle j \rangle\}} = \mathsf{A}_{\mathbf{j}(j)},$$

thus we get the required result. $\square$

From the formula (3.5) we get the new expression of the generalized survival function in the following form.

Corollary 4.5. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0, +\infty)^{[n]}$, and $\mathbf{j}$ be a map given as in (4.3). Then*

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \sum_{j=0}^{\kappa-1} \mu_j \mathbf{1}_{\left[\mathsf{A}_{\mathbf{j}(j)}, \mathsf{A}_{\mathbf{j}(j-1)}\right)}(\alpha) \tag{4.4}$$

for any $\alpha \in [0, +\infty)$ with the convention $\mathsf{A}_{\mathbf{j}(-1)} = +\infty$.

Proof. The required result follows from a Lemma 4.4 and Theorem 3.7. $\square$

Let us conclude this subsection with the observation that if $(\cdot)$ is decreasing, then formulas (3.4), (3.5) and (4.2), (4.4) will be simplified.

Corollary 4.6. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0, +\infty)^{[n]}$. If the mapping $(\cdot)$ is decreasing, then*

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \sum_{i=0}^{\kappa-1} \mu_{\kappa-1-i} \cdot \mathbf{1}_{\left[\mathsf{A}_i, \mathsf{A}_{i+1}\right)}(\alpha) = \sum_{j=0}^{\kappa-1} \mu_j \mathbf{1}_{\left[\mathsf{A}_{\kappa-1-j}, \mathsf{A}_{\kappa-j}\right)}(\alpha)$$

for any $\alpha \in [0, +\infty)$.

Proof. If the map $(\cdot)$ is decreasing, then $\langle \cdot \rangle$ is also decreasing and it holds $(\cdot) = \mathbf{i}$, $\langle \cdot \rangle = \mathbf{j}$. Then expressions (4.2), (4.4) are simplified

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \sum_{i=0}^{\kappa-1} \mu_{(i)} \mathbf{1}_{[\mathsf{A}_i, \mathsf{A}_{i+1})}(\alpha) = \sum_{j=0}^{\kappa-1} \mu_j \mathbf{1}_{\left[\mathsf{A}_{\langle j \rangle}, \mathsf{A}_{\langle j-1 \rangle}\right)}(\alpha).$$

Further, the map $(\cdot)$ is decreasing if and only if for any $i \in [\kappa - 1]_0$ it holds $(i) = \kappa - 1 - i$, or equivalently for any $j \in [\kappa - 1]_0$ it holds $\langle j \rangle = \kappa - 1 - j$, thus $(\cdot) = \langle \cdot \rangle$. This completes the proof. $\square$

Example 4.7. *Let us consider the collection $\mathscr{E} = \{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$, the family of conditional aggregation operators $\mathscr{A}^{\text{sum}} = \{\mathsf{A}^{\text{sum}}(\cdot|E) : E \in \mathscr{E}\}$, the vector $\mathbf{x} = (2, 3, 4)$, and the monotone measure μ on $\hat{\mathscr{E}}$ with values $\mu(\emptyset) = 0$, $\mu(\{3\}) = 0.3$, $\mu(\{1, 2\}) = 0.5$, $\mu(\{2, 3\}) = 0.8$, $\mu(\{1, 2, 3\}) = 1$.*

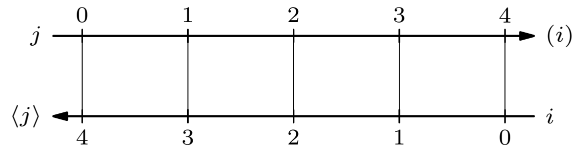


Figure 5: Diagram of $(\cdot)$ and $\langle\cdot\rangle$ from Example 4.7

i, j	0	1	2	3	4
E_i	$\emptyset$	$\{1\}$	$\{3\}$	$\{1,2\}$	$\{1,2,3\}$
A_i	0	2	4	5	9
F_j	$\emptyset$	$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
μ_j	0	0.3	0.5	0.8	1

The diagram of functions $(\cdot)$ and $\langle\cdot\rangle$ can be seen in the Figure 5. It is easy to verify that the assumptions of Corollary 4.6 are satisfied. According to this result, the generalized survival function has the form

$$\mu_{\mathscr{A}^{\text{sum}}}(\mathbf{x}, \alpha) = 1 \cdot \mathbf{1}_{[0,2)}(\alpha) + 0.8 \cdot \mathbf{1}_{[2,4)}(\alpha) + 0.5 \cdot \mathbf{1}_{[4,5)}(\alpha) + 0.3 \cdot \mathbf{1}_{[5,9)}(\alpha)$$

for any $\alpha \in [0, +\infty)$.

From the previous one can see that the assumption of Corollary 4.6 can be graphically interpreted in the way that there are no crossing-overs in the diagrams, see Figure 5.

4.2 Solution of the Knapsack problem

In the following, we apply the obtained formulas to the solution of the Knapsack problem described in Section 2.

Let us assume that a total of 800 ml of liquids are already packed in the knapsack. Thus, it is still possible to pack 200 ml of liquids. Let us choose from the products listed in Table 1. In the table, except for the volume of the products also their price is indicated.

product	a	b	c	d
volume	80	75	55	65
price	1.2	1	0.6	0.8

Table 1: List of products

As we have pointed out in Section 2, our goal is to minimize purchase costs for products that we shall need to buy at the holiday destination. Moreover, we have to keep a limit of 200 ml of liquids we can carry, i.e., this is the limit for the total volume of products, we shall pack at home. We shall solve the optimization problem

$$\min\{\mu(E^c) : \mathsf{A}^{\text{sum}}(\mathbf{x}|E) \leq 200,\ E \in \mathscr{E}\},$$

with $\mathbf{x} = (80, 75, 55, 65)$, the collection $\mathscr{E}$ consists of all possible combinations of elements a, b, c, d, and the monotone measure μ represents the price of products. The values of μ together with the values of the conditional aggregation operator A^{sum} can be seen in Table 2.

μ_j	$\mu(E^c)$	E^c	E	$\mathsf{A}^{\mathrm{sum}}$	A_i	μ_j	$\mu(E^c)$	E^c	E	$\mathsf{A}^{\mathrm{sum}}$	A_i
μ_{15}	3.6	$\{a,b,c,d\}$	$\emptyset$	0	A_0	μ_9	1.8	$\{a,d\}$	$\{b,c\}$	130	A_6
μ_{11}	2.4	$\{b,c,d\}$	$\{a\}$	80	A_4	μ_7	1.8	$\{a,c\}$	$\{b,d\}$	140	A_8
μ_{13}	2.5	$\{a,c,d\}$	$\{b\}$	75	A_3	μ_{10}	2.2	$\{a,b\}$	$\{c,d\}$	120	A_5
μ_{14}	3.0	$\{a,b,d\}$	$\{c\}$	55	A_1	μ_2	0.8	$\{d\}$	$\{a,b,c\}$	210	A_{13}
μ_{12}	2.5	$\{a,b,c\}$	$\{d\}$	65	A_2	μ_1	0.6	$\{c\}$	$\{a,b,d\}$	220	A_{14}
μ_5	1.4	$\{c,d\}$	$\{a,b\}$	155	A_{10}	μ_3	1.0	$\{b\}$	$\{a,c,d\}$	200	A_{12}
μ_8	1.8	$\{b,d\}$	$\{a,c\}$	135	A_7	μ_4	1.2	$\{a\}$	$\{b,c,d\}$	195	A_{11}
μ_6	1.4	$\{b,c\}$	$\{a,d\}$	145	A_9	μ_0	0	$\emptyset$	$\{a,b,c,d\}$	275	A_{15}

Table 2: Values of the conditional aggregation operator and corresponding monotone measure

We can notice that in Table 2 there are 15 different values of the conditional aggregation operator and 12 different values of the (nonadditive) monotone measure. To solve the knapsack problem we have to find the value of the generalized survival function at point 200. This can be easily done using formula (3.4) or (4.2). The value of 200 lies in the interval $[\mathsf{A}_{12}, \mathsf{A}_{13})$, where the generalized survival function takes the value

$$\min_{k\le 12} \mu_{(k)} = \mu_{\mathbf{i}(12)} = \mu_3 = 1.$$

The same result we get when we determine the whole formula of the generalized survival function using formula (6.2)

$$\begin{aligned}\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) &= 3.6\cdot \mathbf{1}_{[0;55)}(\alpha) + 3\cdot \mathbf{1}_{[55;65)}(\alpha) + 2.5\cdot \mathbf{1}_{[65;80)}(\alpha) + 2.4\cdot \mathbf{1}_{[80;120)}(\alpha)\\ &\quad + 2.2\cdot \mathbf{1}_{[120;130)}(\alpha) + 1.8\cdot \mathbf{1}_{[130;145)}(\alpha) + 1.4\cdot \mathbf{1}_{[145;195)}(\alpha) + 1.2\cdot \mathbf{1}_{[195;200)}(\alpha)\\ &\quad + \mathbf{1}_{[200;210)}(\alpha) + 0.8\cdot \mathbf{1}_{[210;220)}(\alpha) + 0.6\cdot \mathbf{1}_{[220;275)}(\alpha).\end{aligned}$$

As we can see, the result coincides with the result from the previous calculation. So, a traveler should pack products a, c, d at home and product b buy at the destination.

4.3 How to draw a graph of a generalized survival function

In the previous section, we have presented visualizations of maps $(\cdot)$ and $\langle\cdot\rangle$ using both a diagram and a graph in the Cartesian coordinate system, see demonstratory Figure 2 based on inputs from Example 2.5. We have already pointed out the advantages of the first visualization, and in the following we deal with the second one. In fact, we shall show how the graph of $(\cdot)$ or $\langle\cdot\rangle$ can be transformed to the plot of the generalized survival function.

The first step is to transform the graphs of $(\cdot)$ or $\langle\cdot\rangle$, see Figure 2(b)(c), into the graphs of the mappings $\mathbf{i}$ or $\mathbf{j}$, see Figure 6, decreasing some of the values of $(\cdot)$ and $\langle\cdot\rangle$ to produce nonincreasing functions $\mathbf{i}$ and $\mathbf{j}$, respectively. More precisely, the formulas (4.1) and (4.3) are used to compute $\mathbf{i}$ and $\mathbf{j}$ from $(\cdot)$ and $\langle\cdot\rangle$, respectively.

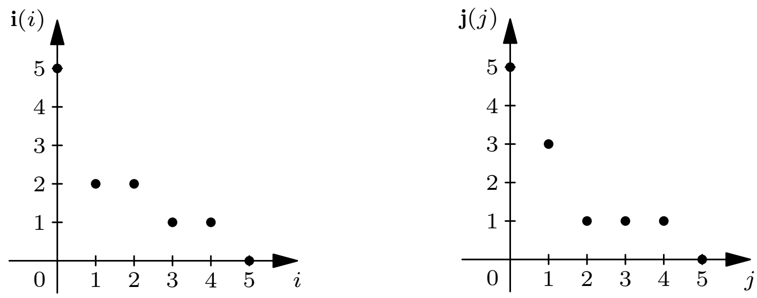
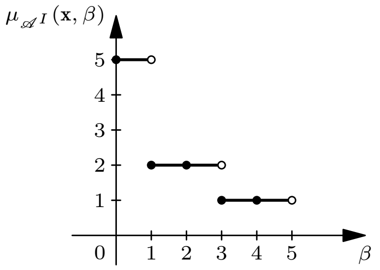
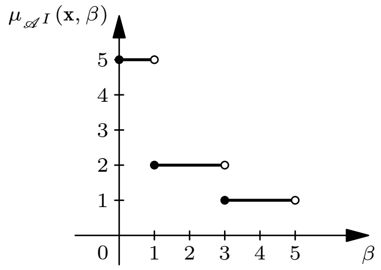


Figure 6: Graphs of $\mathbf{i}$ and $\mathbf{j}$.

The second step is to extend the domain of $\mathbf{i}$ from $[\kappa-1]_0$ to $[0,+\infty)$, and obtain the graph of the *indexed generalized survival function* $\mu_{\mathscr{A}^I}(\mathbf{x},\beta)$. It is enough to naturally define $\mu_{\mathscr{A}^I}(\mathbf{x},\beta)=\mathbf{i}(\lfloor\beta\rfloor)$ for $\beta<\kappa-1$ and $\mu_{\mathscr{A}^I}(\mathbf{x},\beta)=\mathbf{i}(\kappa-1)$ otherwise, as is done in Figure 7(a). There is similar way to obtain the indexed generalized survival function from the graph of $\mathbf{j}$, depicted in Figure 7(b), which we describe later.



(a) Indexed generalized survival function (using $\mathbf{i}$)



(b) Indexed generalized survival function (using $\mathbf{j}$)

Figure 7: Indexed generalized survival functions deriving by using maps $\mathbf{i}$ and $\mathbf{j}$

Before we proceed to the last step, let us stress that Figure 7 of $\mu_{\mathscr{A}^I}(\mathbf{x},\beta)$ is very close to how the real generalized survival function $\mu_{\mathscr{A}}(\mathbf{x},\alpha)$ depicted in Figure 8 looks like. Vaguely speaking, the only difference is that the graph of $\mu_{\mathscr{A}}(\mathbf{x},\alpha)$ has values $\mathsf{A}_0,\mathsf{A}_1,\dots,\mathsf{A}_{\kappa-1}$ on the horizontal axis instead of values $0,1,\dots,\kappa-1$, and values $\mu_0,\mu_1,\dots,\mu_{\kappa-1}$ on the vertical axis again instead of values $0,1,\dots,\kappa-1$. Moreover, some values $\mathsf{A}_i,\mathsf{A}_j$ and μ_i,μ_j may coincide even for different i,j.

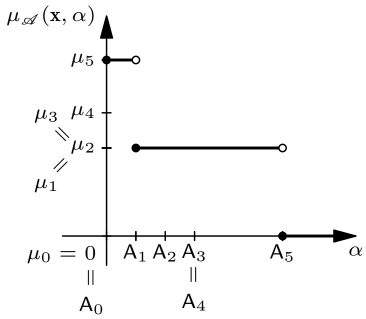


Figure 8: Generalized survival function

More formally, the similarities between $\mu_{\mathscr{A}^I}(\mathbf{x},\beta)$ and $\mu_{\mathscr{A}}(\mathbf{x},\alpha)$ may be understood once we represent the latter one via the formula (4.2), and the first one in a similar way, namely

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha)=\sum_{i=0}^{\kappa-1}\mu_{\mathbf{i}(i)}\mathbf{1}_{[\mathsf{A}_i,\mathsf{A}_{i+1})}(\alpha),\qquad \mu_{\mathscr{A}^I}(\mathbf{x},\beta)=\sum_{i=0}^{\kappa-2}\mathbf{i}(i)\mathbf{1}_{[i,\,i+1)}(\beta). \tag{4.5}$$

The equalities (4.5) are based on function $\mathbf{i}$, but we may proceed similarly with function $\mathbf{j}$ using the equality (4.4):

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha)=\sum_{i=0}^{\kappa-1}\mu_{\mathbf{i}(i)}\mathbf{1}_{[\mathsf{A}_i,\mathsf{A}_{i+1})}(\alpha),\qquad \mu_{\mathscr{A}^I}(\mathbf{x},\beta)=\sum_{j=1}^{\kappa-1}j\mathbf{1}_{[\mathbf{j}(j),\mathbf{j}(j-1))}(\beta). \tag{4.6}$$

The equality (4.6) gives clues how to obtain the graph of the indexed generalized survival function $\mu_{\mathscr{A}^I}(\mathbf{x},\beta)$ from the graph of the map $\mathbf{j}$. There is necessary one more step, namely, one has to draw a graph of the generalized inverse $\mathbf{j}^-$ from the graph of $\mathbf{j}$ first. Then, similarly to the case of $\mathbf{i}$, define $\mu_{\mathscr{A}^I}(\mathbf{x},\beta)=\mathbf{j}^-(\lfloor\beta\rfloor)$ for $\beta<\kappa-1$ and $\mu_{\mathscr{A}^I}(\mathbf{x},\beta)=\mathbf{j}^-(\kappa-1)$ otherwise, see Figure 7(b).

Let us summarize the calculation of the indexed generalized survival function. First, we construct the bijections $\mathfrak{e}$ a $\mathfrak{f}$ as it shown in (3.1) a (3.3). Then, using the permutation $(\cdot)\colon [\kappa-1]_0 \to [\kappa-1]_0$ described above, we obtain the values of $\mathbf{i}(i)$ and $\mathbf{j}(i)$ for each $i \in [\kappa-1]_0$, and thus we can construct the indexed generalized survival function. Finally, for each index $k \in [\kappa-1]_0$ on the horizotal, resp. vertical, axes we assign the value A_k, resp. μ_k. This entire process can be described by the following Algorithm 1 and Algorithm 2. Note that these algorithms assume the fact that the values of map $(\cdot)$, resp. $\langle\cdot\rangle$ are known to the user, or for completeness, we present algorithms for its calculation in Appendix.

Note that although the maps $\mathbf{i}$ and $\mathbf{j}$ is not appear explicitly in the algorithms, their generation is hidden in their for loops. In this part of algorithms, the indexed generalized survival function is also generated, while its standard version is immediately assigned to it. A zero value of generalized survival function can also be marked in the graph (Figure 8), which corresponds to Remark 3.4 and Remark 3.6, but it is not necessary, since this value does not bring any new information.

Method *the-graph-of-GSF*$(\mathscr{E},\mu,\mathbf{x},\mathscr{A})$
  $(0),\dots,(\kappa-1) \leftarrow$
   the-$(\cdot)$-map$(\mathscr{E},\mu,\mathbf{x},\mathscr{A})$;
  $GSF := 0$;
  for $(i=0,\ i<\kappa-1,\ i{+}{+})$ **do**
    if $(i+1) > (i)$ **then**
      $(i+1) := (i)$;
    end
    $GSF := GSF + \mu_{(i)} \cdot \mathbf{1}_{[\mathsf{A}_i,\mathsf{A}_{i+1})}$;
  end
  return *GSF;*
end

Algorithm 1: Calculation of generalized survival function (GSF) using the map $\mathbf{i}$

Method *the-graph-of-GSF*$(\mathscr{E},\mu,\mathbf{x},\mathscr{A})$
  $\langle 0\rangle,\dots,\langle\kappa-1\rangle \leftarrow$
   the-$\langle\cdot\rangle$-map$(\mathscr{E},\mu,\mathbf{x},\mathscr{A})$;
  $GSF := 0$;
  for $(j=1,\ j<\kappa,\ j{+}{+})$ **do**
    if $\langle j-1\rangle < \langle j\rangle$ **then**
      $\langle j\rangle := \langle j-1\rangle$;
    end
    $GSF := GSF + \mu_j \cdot \mathbf{1}_{[\mathsf{A}_{\langle j\rangle},\mathsf{A}_{\langle j-1\rangle})}$;
  end
  return *GSF;*
end

Algorithm 2: Calculation of generalized survival function (GSF) using the map $\mathbf{j}$

Remark 4.8. *The following relationships can be observed from formulas* (4.5) *and* (4.6)*. Let* $I = \{\mathbf{i}(i) : i \in [\kappa-1]_0\}$ *and* $J = \{\mathbf{j}(j) : j \in [\kappa-1]_0\}$, $j \in I$ *and* $i \in J$, *then*

$$\min\{k \in [\kappa-1]_0 : \mathbf{j}(k) = i\} = \mathbf{i}(i),$$
$$\min\{k \in [\kappa-1]_0 : \mathbf{i}(k) = j\} = \mathbf{j}(j).$$

These equalities hold because of

$$\begin{aligned}\min\{k \in [\kappa-1]_0 : \mathbf{j}(k) = i\} &= \min\{k \in [\kappa-1]_0 : \min\{\langle 0\rangle,\dots,\langle k\rangle\} = i\}\\ &= \min\{k \in [\kappa-1]_0 : \langle k\rangle = i\} = \min\{k \in [\kappa-1]_0 : k = (i)\}\\ &= \min\{(0),\dots,(i)\} = \mathbf{i}(i).\end{aligned}$$

The penultimate equality follows from the fact that $(\cdot)$ *is nonincreasing, see Remark 4.3. Similarly for the second formula.*

Remark 4.9. *It is also possible to use* (4.5) *and* (4.6) *with analogy to the formula* (2.1) *for calculating the indexed generalized survival function as follows*

$$\begin{aligned}\mu_{\mathscr{A}^I}(\mathbf{x},\beta) &= \min\{(i) : i \le \beta,\ i \in [\kappa-1]_0\} = \mathbf{i}(\beta)\\ &= \min\{j : \langle j\rangle \le \beta,\ j \in [\kappa-1]_0\} = \mathbf{j}^-(\beta)\end{aligned}$$

for any $\beta \in [0,+\infty)$ *with respect to extended domain of* $\mathbf{i}$ *and* $\mathbf{j}^-$, *i.e.* $[0,+\infty)$, *described above.*

5 Generalized Choquet integral computation

In the literature there exists a lot of integrals whose construction is based on the survival function, e.g., the Choquet integral, the Sugeno integral, the Shilkret integral, the seminormed integral [4], regular fuzzy integrals [18] or universal integrals [13], etc. In this paper, we focus on the Choquet integral that is defined for $\mathbf{x} \in [0,\infty)^{[n]}$ w.r.t $\mu \in \mathbf{M}$ as follows[2]

$$\mathrm{C}(\mathbf{x},\mu) = \int_0^\infty \mu(\{\mathbf{x} > \alpha\})\,\mathrm{d}\alpha.$$

On discrete space we can write

$$\mathrm{C}(\mathbf{x},\mu) = \sum_{i=1}^{n} \mu(G_{\sigma(i)})(x_{\sigma(i)} - x_{\sigma(i-1)}) = \sum_{i=1}^{n} x_{\sigma(i)}(\mu(G_{\sigma(i)}) - \mu(G_{\sigma(i+1)})), \tag{5.1}$$

where $\mathbf{x} = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$, with $\sigma\colon [n] \to [n]$ being a permutation such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$ with the convention $x_{\sigma(0)} = 0$, $G_{\sigma(i)} = \{\sigma(i), \dots, \sigma(n)\}$ for $i \in [n]$, and $G_{\sigma(n+1)} = \emptyset$.

Based on generalized survival function, a new concept generalizing the Choquet integral naturally arises. In this section, we aim to provide formulas for discrete $\mathscr{A}$-Choquet integral.

Definition 5.1. (cf. [2, Definition 5.4.]) Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$ and $\mathbf{x} \in [0,+\infty)^{[n]}$. The Choquet integral with respect to $\mathscr{A}$ and μ ($\mathscr{A}$-Choquet integral, for short) of $\mathbf{x}$ is defined as

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \int_0^\infty \mu_{\mathscr{A}}(\mathbf{x},\alpha)\,\mathrm{d}\alpha. \tag{5.2}$$

As we have obtained the generalized survival function expressions, see Sections 3, 4, the computation of the $\mathscr{A}$-Choquet integral becomes a trivial matter. Note that the formulas in the first two lines are obtained by means of mappings $\mathbf{i}$ and $\mathbf{j}$, respectively. By introducing these mappings, we managed to obtain expressions similar to the original one in (5.1).

Theorem 5.2. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$ and $\mathbf{x} \in [0,+\infty)^{[n]}$. Then*

(i) $\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \sum_{i=0}^{\kappa-2} \mu_{\mathbf{i}(i)}(\mathsf{A}_{i+1} - \mathsf{A}_i) = \sum_{i=0}^{\kappa-2} \mathsf{A}_{i+1}\big(\mu_{\mathbf{i}(i)} - \mu_{\mathbf{i}(i+1)}\big),$ [cf. Corollary 4.2]

(ii) $\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \sum_{i=1}^{\kappa-1} \mu_i\big(\mathsf{A}_{\mathbf{j}(i-1)} - \mathsf{A}_{\mathbf{j}(i)}\big) = \sum_{i=1}^{\kappa-1} \mathsf{A}_{\mathbf{j}(i-1)}(\mu_i - \mu_{i-1}),$ [cf. Corollary 4.5]

(iii) $\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \sum_{i=0}^{\kappa-2} \min_{k\leq i} \mu_{(k)}(\mathsf{A}_{i+1} - \mathsf{A}_i) = \sum_{i=0}^{\kappa-2} \mathsf{A}_{i+1}\big(\min_{k\leq i} \mu_{(k)} - \min_{k\leq i+1} \mu_{(k)}\big),$ [cf. Theorem 3.2]

(iv) $\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \sum_{i=1}^{\kappa-1} \mu_i\big(\min_{k<i} \mathsf{A}_{\langle k\rangle} - \min_{k\leq i} \mathsf{A}_{\langle k\rangle}\big) = \sum_{i=1}^{\kappa-1} \min_{k<i} \mathsf{A}_{\langle k\rangle}(\mu_i - \mu_{i-1}).$ [cf. Theorem 3.7]

Let us recall that in Corollary 3.9 we listed computational formulas for the generalized survival function for special measures. Thus, Corollary 3.9 helps us to substantially improve the formulas presented in Theorem 5.2 for special measures.

Corollary 5.3. *Let $\mathbf{x} \in [0,+\infty)^{[n]}$.*

[2] $\mu(\{\mathbf{x} > \alpha\}) = \mu(\{i \in [n] : x_i > \alpha\})$

(i) *Let $\mathscr{A}$ be FCA and $\bar{\mu}$ be the greatest monotone capacity. Then*

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x}, \underline{\mu}) = \mathsf{A}(\mathbf{x}|[n]).$$

(ii) *Let $\mathscr{A}$ be FCA. Let $\underline{\mu}$ be the weakest monotone capacity. Then*

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x}, \bar{\mu}) = \min_{E \neq \emptyset} \mathsf{A}(\mathbf{x}|E).$$

(iii) *Let $\mathscr{A}$ be FCA nondecreasing w.r.t. sets with $\mathscr{E} = 2^{[n]}$. Let μ be a symmetric measure and set $\mu^i := \mu(F)$, if $|F| = i$, $i \in [n] \cup \{0\}$. Then*

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x}, \mu) = \sum_{i=1}^{n} (\mu^i - \mu^{i-1}) \min_{|E|=n-i+1} \mathsf{A}(\mathbf{x}|E).$$

(iv) *Let $\mathscr{A}$ be FCA nondecreasing w.r.t. sets with $\mathscr{E} = 2^{[n]}$. Let us consider settings as in Corollary 3.9 (iv). Then*

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x}, \Pi) = \sum_{i=1}^{n} (\pi(\sigma(i)) - \pi(\sigma(i-1))) \, \mathsf{A}(\mathbf{x}|G_{\sigma(i)}).$$

(v) *Let $\mathscr{A}$ be FCA nondecreasing w.r.t. sets with $\mathscr{E} = 2^{[n]}$. Let us consider settings as in Corollary 3.9 (v). Then*

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x}, N) = \sum_{i=1}^{n} \big(\pi(\sigma(i)) - \pi(\sigma(i-1))\big) \min_{k \geq i} \mathsf{A}(\mathbf{x}|\{\sigma(k)\}).$$

Remark 5.4. *In the following, we aim to emphasize that the $\mathscr{A}$-Choquet integral covers the famous Choquet integral. Let us consider a special family $\mathscr{A}^{\max}$. Taking symmetric measures we obtain*

$$\mathrm{C}_{\mathscr{A}^{\max}}(\mathbf{x}, \mu) = \sum_{i=1}^{n} (\mu^i - \mu^{i-1}) x_{\sigma(n-i+1)} = \sum_{i=1}^{n} (\mu^{n-i+1} - \mu^{n-i}) x_{\sigma(i)},$$

where $\sigma\colon [n] \to [n]$ is a permutation such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$ with the convention $x_{\sigma(0)} = 0$. This is a formula of Yager's ordered weighted averaging (OWA) operator [20] as Grabisch has shown, see [11]. Therefore, the generalized $\mathscr{A}$-Choquet integral w.r.t. symmetric measures can be seen as a new type of the OWA operator.

The formula for possibility measure in Corollary 5.3 (iv) simplifies to the form

$$\mathrm{C}_{\mathscr{A}^{\max}}(\mathbf{x}, \Pi) = \sum_{i=1}^{n} (\pi(\sigma(i)) - \pi(\sigma(i-1))) \max_{k \geq i} x_{\sigma(k)} = \sum_{i=1}^{n} (\pi(\sigma(i)) - \pi(\sigma(i-1))) \max_{\pi(k) \geq \pi(\sigma(i))} x_k,$$

which is the famous Choquet integral with respect to possibility measure, cf. [7], and permutation σ as in Corollary 3.9. Similarly, taking necessity measure in Corollary 5.3 (v) we obtain

$$\mathrm{C}_{\mathscr{A}^{\max}}(\mathbf{x}, \mathrm{N}) = \sum_{i=1}^{n} (\pi(\sigma(i)) - \pi(\sigma(i-1))) \min_{k \geq i} x_{\sigma(k)} = \sum_{i=1}^{n} (\pi(\sigma(i)) - \pi(\sigma(i-1))) \min_{\pi(k) \geq \pi(\sigma(i))} x_k.$$

6 Searching optimal intervals, and indistinguishibility

The Choquet integral is a basic tool for multicriteria decision making and modeling of decision under risk and uncertainty. In [7], Dubois and Rico studied the equality conditions of Choquet integrals of particular input vectors. They considered Choquet integrals with respect to possibility and necessity measures. In [6], Chen et al. continued their research with a view to a wider class of so-called universal integrals [13]. Universal integrals form one class of utility functions in multicriteria decision making.

In this section, we formulate the equality conditions of generalized survival functions considering arbitrary measures. Naturally, if generalized survival functions coincide, then their Choquet integrals equal. In order to obtain the equality conditions, we study the greatest possible intervals on which the generalized survival function takes its possible values. For this purpose, for any $j \in [\kappa-1]_0$ let us set

$$\varphi_*(j) = \min\{k \in [\kappa-1]_0 : \mu_k = \mu_j\} \quad \text{and} \quad \varphi^*(j) := \max\{k \in [\kappa-1]_0 : \mu_k = \mu_j\}. \tag{6.1}$$

The following proposition summarizes the basic properties of φ_* and φ^*.

Proposition 6.1. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,\infty)^{[n]}$, φ_* and φ^* are defined as in* (6.1) *and $a, b, j \in [\kappa-1]_0$. Then the following properties hold.*

(i) $\varphi_*(j) \leq j \leq \varphi^*(j)$,

(ii) $\mu_{\varphi^*(j)} = \mu_k = \mu_{\varphi_*(j)}$ *for any integer* $k \in \{\varphi_*(j), \dots, \varphi^*(j)\}$,

(iii) $\displaystyle\bigcup_{\varphi_*(j) \leq l \leq \varphi^*(j)} \Big[\min_{k \leq l} \mathsf{A}_{\langle k\rangle}, \min_{k<l} \mathsf{A}_{\langle k\rangle}\Big) = \Big[\min_{k \leq \varphi^*(j)} \mathsf{A}_{\langle k\rangle}, \min_{k<\varphi_*(j)} \mathsf{A}_{\langle k\rangle}\Big)$,

(iv) φ_* *and* φ^* *are nondecreasing.*

Proof. The statements (i) and (ii) follow directly from definitions of φ_*, φ^*, and arrangement (3.3). The validity of the statement (iii) follows from the fact that $\min\limits_{k<l} \mathsf{A}_{\langle k\rangle} = \min\limits_{k \leq l-1} \mathsf{A}_{\langle k\rangle}$ for $l \in \{\varphi_*(j), \dots, \varphi^*(j)\}$, $l \neq 0$, and because of

$$\min_{k \leq \varphi^*(j)} \mathsf{A}_{\langle k\rangle} \leq \min_{k<\varphi^*(j)} \mathsf{A}_{\langle k\rangle} = \min_{k \leq \varphi^*(j)-1} \mathsf{A}_{\langle k\rangle} \leq \cdots \leq \min_{k<\varphi_*(j)+1} \mathsf{A}_{\langle k\rangle} = \min_{k \leq \varphi_*(j)} \mathsf{A}_{\langle k\rangle} \leq \min_{k<\varphi_*(j)} \mathsf{A}_{\langle k\rangle}$$

with the convention already stated in this paper $\min\limits_{k<0} \mathsf{A}_{\langle k\rangle} = \min \emptyset = +\infty$. Now we show (iv). If $a \leq b$, then from arrangement (3.3) we have $\mu_a \leq \mu_b$ and thus

$$\max\{k \in [\kappa-1]_0 : \mu_a = \mu_k\} \leq \max\{k \in [\kappa-1]_0 : \mu_b = \mu_k\},$$

hence $\varphi^*(a) \leq \varphi^*(b)$. The second part of the statement (iv) can be shown analogously. □

By means of φ^* and φ_* we can determine the greatest possible intervals corresponding to given monotone measure. Simultaneously, we show under what conditions the value of a given monotone measure is not achieved by the generalized survival function.

Proposition 6.2. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$, φ_* and φ^* be given as in (6.1), and $j \in [\kappa-1]_0$. Then*

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \mu_j \text{ for any } \alpha \in \Big[\min_{k \leq \varphi^*(j)} \mathsf{A}_{\langle k\rangle}, \min_{k<\varphi_*(j)} \mathsf{A}_{\langle k\rangle}\Big)$$

with the convention $\min\limits_{k<0} \mathsf{A}_{\langle k\rangle} = \min \emptyset = +\infty$ and this interval is the greatest possible.

Proof. According to Theorem 3.7, Proposition 6.1(i), (ii), (iii) the generalized survival function $\mu_{\mathscr{A}}(\mathbf{x},\alpha)$ achieves the value μ_j on interval $\Big[\min\limits_{k\leq\varphi^*(j)}\mathsf{A}_{\langle k\rangle},\min\limits_{k<\varphi_*(j)}\mathsf{A}_{\langle k\rangle}\Big)$. However, it is not clear that this interval is the greatest possible. By contradiction: So let exist $\tilde{j}\in[\kappa-1]_0$ such that

$$\min_{k\leq\varphi^*(\tilde{j})}\mathsf{A}_{\langle k\rangle}<\min_{k\leq\varphi^*(j)}\mathsf{A}_{\langle k\rangle}\quad\text{or}\quad\min_{k<\varphi_*(j)}\mathsf{A}_{\langle k\rangle}<\min_{k<\varphi_*(\tilde{j})}\mathsf{A}_{\langle k\rangle},$$

and $\mu_{\mathscr{A}}(\mathbf{x},\alpha)=\mu_j$ for $\alpha\in\Big[\min\limits_{k\leq\varphi^*(\tilde{j})}\mathsf{A}_{\langle k\rangle},\min\limits_{k\leq\varphi^*(j)}\mathsf{A}_{\langle k\rangle}\Big)$ or $\alpha\in\Big[\min\limits_{k<\varphi_*(j)}\mathsf{A}_{\langle k\rangle},\min\limits_{k<\varphi_*(\tilde{j})}\mathsf{A}_{\langle k\rangle}\Big)$. Let us discuss these cases.

- By Theorem 3.7(i) for $\alpha=\min\limits_{k\leq\varphi^*(\tilde{j})}\mathsf{A}_{\langle k\rangle}$ it holds $\mu_{\mathscr{A}}(\mathbf{x},\alpha)=\mu_{\varphi^*(\tilde{j})}=\mu_{\tilde{j}}$, where the last equality holds because of Proposition 6.1(i), (ii). However, $\mu_{\tilde{j}}\neq\mu_j$, otherwise we get a contradiction. Indeed, if $\mu_{\tilde{j}}=\mu_j$, then $\varphi^*(j)=\varphi^*(\tilde{j})$ and thus $\min\limits_{k\leq\varphi^*(\tilde{j})}\mathsf{A}_{\langle k\rangle}=\min\limits_{k\leq\varphi^*(j)}\mathsf{A}_{\langle k\rangle}$, which is in conflict with choice of $\tilde{j}$.
- Using the same arguments as in the previous case, for $\alpha=\min\limits_{k<\varphi_*(j)}\mathsf{A}_{\langle k\rangle}=\min\limits_{k\leq\varphi_*(j)-1}\mathsf{A}_{\langle k\rangle}$, we have $\mu_{\mathscr{A}}(\mathbf{x},\alpha)=\mu_{\varphi_*(j)-1}$. Further, $\mu_{\varphi_*(j)-1}<\mu_{\varphi_*(j)}=\mu_j$, thus $\mu_{\varphi_*(j)-1}\neq\mu_j$.

This completes the proof. □

From the above proposition, we immediately obtain the following result.

Corollary 6.3. *Let $\mathscr{A}$ be a FCA, $\mu\in\mathbf{M}$, $\mathbf{x}\in[0,+\infty)^{[n]}$, φ_* and φ^* be given as in (6.1), and $j\in[\kappa-1]_0$. The value μ_j is not achieved if and only if $\min\limits_{k\leq\varphi^*(j)}\mathsf{A}_{\langle k\rangle}=\min\limits_{k<\varphi_*(j)}\mathsf{A}_{\langle k\rangle}$.*

Lemma 6.4. *Let $\mathscr{A}$ be a FCA, $\mu\in\mathbf{M}$, $\mathbf{x}\in[0,+\infty)^{[n]}$, φ_* and φ^* be given as in (6.1).*

(i) *Let $j\in[\kappa-1]$. Then it holds*

  (i1) *If $\mu_j>\mu_{j-1}$, then $\varphi_*(j)-1=\varphi^*(j-1)$ and μ_j is achieved on $\big[\min\limits_{k\leq\varphi^*(j)}\mathsf{A}_{\langle k\rangle},\min\limits_{k\leq\varphi^*(j-1)}\mathsf{A}_{\langle k\rangle}\big)$. Moreover, this interval is the greatest possible.*

  (i2) *If $\mu_j=\mu_{j-1}$, then $\varphi^*(j)=\varphi^*(j-1)$.*

(ii) *Let $j\in[\kappa-1]_0$. Then it holds*

  (ii1) *If $\mu_j<\mu_{j+1}$, then $\varphi^*(j)+1=\varphi_*(j+1)$ and μ_j is achieved on $\Big[\min\limits_{k<\varphi_*(j+1)}\mathsf{A}_{\langle k\rangle},\min\limits_{k<\varphi_*(j)}\mathsf{A}_{\langle k\rangle}\Big)$, with $\min\limits_{k<\varphi_*(\kappa)}\mathsf{A}_{\langle k\rangle}=0$ by convention. Moreover, this interval is the greatest possible.*

  (ii2) *If $\mu_j=\mu_{j+1}$, then $\varphi_*(j)=\varphi_*(j+1)$.*

Proof. Let us prove part (i).

(i1) If $\mu_j>\mu_{j-1}$, then directly from definitions of φ_* and φ^* we have $\varphi_*(j)=j$ and $\varphi^*(j-1)=j-1$, thus $\varphi^*(j-1)=\varphi_*(j)-1$. Further, according to Proposition 6.2, we have

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha)=\mu_j$$

for each $\alpha\in\big[\min\limits_{k\leq\varphi^*(j)}\mathsf{A}_{\langle k\rangle},\min\limits_{k\leq\varphi_*(j)-1}\mathsf{A}_{\langle k\rangle}\big)$ and this interval is the greatest possible. Using the above equality we have

$$\big[\min_{k\leq\varphi^*(j)}\mathsf{A}_{\langle k\rangle},\min_{k\leq\varphi_*(j)-1}\mathsf{A}_{\langle k\rangle}\big)=\big[\min_{k\leq\varphi^*(j)}\mathsf{A}_{\langle k\rangle},\min_{k\leq\varphi^*(j-1)}\mathsf{A}_{\langle k\rangle}\big).$$

(i2) It holds trivially.

Part (ii) can be proved analogously. $\square$

Let us notice that Proposition 6.2 and its corollaries are crucial to state the sufficient conditions for indistinguishability of generalized Choquet integral equivalent pairs, see Section 5. Using the above results, one can obtain the improvement of formula (3.5).

Proposition 6.5. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$, φ_* and φ^* be given as in (6.1). Then*

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \sum_{j=1}^{\kappa-1} \mu_j \mathbf{1}_{\left[\min\limits_{k\leq\varphi^*(j)} \mathsf{A}_{\langle k\rangle}, \min\limits_{k\leq\varphi^*(j-1)} \mathsf{A}_{\langle k\rangle}\right)}(\alpha) = \sum_{j=1}^{\kappa-1} \mu_j \mathbf{1}_{\left[\min\limits_{k<\varphi_*(j+1)} \mathsf{A}_{\langle k\rangle}, \min\limits_{k<\varphi_*(j)} \mathsf{A}_{\langle k\rangle}\right)}(\alpha) \tag{6.2}$$

for any $\alpha \in [0,\infty)$, with the conventions $\varphi_*(\kappa) := \kappa$ (thus $\min\limits_{k<\varphi_*(\kappa)} \mathsf{A}_{\langle k\rangle} = 0$), $\min\limits_{k<0} \mathsf{A}_{\langle k\rangle} = \min\emptyset = +\infty$.

Proof. Let us consider an arbitrary (fixed) $j \in [\kappa-1]$. If $\mu_j > \mu_{j-1}$, then according to Lemma 6.4 (i1) μ_j is achieved on $\left[\min\limits_{k\leq\varphi^*(j)} \mathsf{A}_{\langle k\rangle}, \min\limits_{k\leq\varphi^*(j-1)} \mathsf{A}_{\langle k\rangle}\right)$. Moreover, this interval is the greatest possible. If $\mu_j = \mu_{j-1}$, then $\varphi^*(j) = \varphi^*(j-1)$, see Lemma 6.4(i2), thus $\left[\min\limits_{k\leq\varphi^*(j)} \mathsf{A}_{\langle k\rangle}, \min\limits_{k\leq\varphi^*(j-1)} \mathsf{A}_{\langle k\rangle}\right) = \emptyset$. This demonstrates that each value of generalized survival function is included just once in the sum and the first formula is right. The second formula can be proved analogously. $\square$

If we use the mapping $\mathbf{j}$ defined by (4.3), then formulas in (6.2) can be rewritten similarly as in Corollary 4.5, i.e.,

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \sum_{j=1}^{\kappa-1} \mu_j \mathbf{1}_{\left[\mathsf{A}_{\mathbf{j}(\varphi^*(j))}, \mathsf{A}_{\mathbf{j}(\varphi^*(j-1))}\right)}(\alpha) = \sum_{j=1}^{\kappa-1} \mu_j \mathbf{1}_{\left[\mathsf{A}_{\mathbf{j}(\varphi_*(j+1)-1)}, \mathsf{A}_{\mathbf{j}(\varphi_*(j)-1)}\right)}(\alpha).$$

The use of the approach described in the previous proposition is shown in the following example.

Example 6.6. Let us consider the same inputs as in Example 2.5. Let us use the last formula given in (6.2) to calculate the generalized survival function.

- For $j = 1$ we get $\mu_1 \cdot \mathbf{1}_{[\mathsf{A}_{\mathbf{j}(0)}, \mathsf{A}_{\mathbf{j}(0)})} = 0.5 \cdot \mathbf{1}_{\emptyset}$.
- For $j = 2$ we get $\mu_2 \cdot \mathbf{1}_{[\mathsf{A}_{\mathbf{j}(0)}, \mathsf{A}_{\mathbf{j}(0)})} = 0.5 \cdot \mathbf{1}_{\emptyset}$.
- For $j = 3$ we get $\mu_3 \cdot \mathbf{1}_{[\mathsf{A}_{\mathbf{j}(3)}, \mathsf{A}_{\mathbf{j}(0)})} = \mu_3 \cdot \mathbf{1}_{[\mathsf{A}_1, \mathsf{A}_5)} = 0.5 \cdot \mathbf{1}_{[1,6)}$.
- For $j = 4$ we get $\mu_4 \cdot \mathbf{1}_{[\mathsf{A}_{\mathbf{j}(4)}, \mathsf{A}_{\mathbf{j}(3)})} = \mu_4 \cdot \mathbf{1}_{[\mathsf{A}_1, \mathsf{A}_1)} = 0.7 \cdot \mathbf{1}_{\emptyset}$.
- For $j = 5$ we get $\mu_5 \cdot \mathbf{1}_{[\mathsf{A}_{\mathbf{j}(5)}, \mathsf{A}_{\mathbf{j}(4)})} = \mu_5 \cdot \mathbf{1}_{[\mathsf{A}_0, \mathsf{A}_1)} = 1 \cdot \mathbf{1}_{[0,1)}$.

Therefore, the generalized survival function has the form

$$\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mathbf{1}_{[0,1)}(\alpha) + 0.5 \cdot \mathbf{1}_{[1,6)}(\alpha),$$

$\alpha \in [0,+\infty)$, compare with Example 3.8.

As in the whole paper let us introduce the greatest possible intervals on which a value of monotone measure is achieved using bijection $\mathfrak{e}\colon [\kappa-1]_0 \to \mathscr{E}$, see (3.2). Since the strategy of deriving it is analogous, let us postpone it to the Appendix and let us present here the main result. Let $i \in [\kappa-1]_0$, let us define

$$\begin{aligned} \psi_*(i) &:= \min\{l \in [\kappa-1]_0 : \min_{k\leq l} \mu_{(k)} = \min_{k\leq i} \mu_{(k)}\}, \\ \text{and} \quad \psi^*(i) &:= \max\{l \in [\kappa-1]_0 : \min_{k\leq l} \mu_{(k)} = \min_{k\leq i} \mu_{(k)}\}. \end{aligned} \tag{6.3}$$

Proposition 6.7. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$, ψ_* and ψ^* be given as in (6.3), and $i \in [\kappa-1]_0$. Then*

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \min_{k\le i} \mu_{(k)} \text{ for any } \alpha \in \left[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi^*(i)+1}\right)$$

and this interval is the greatest possible.

Proof. See Appendix. □

Proposition 6.8. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$, ψ_* and ψ^* be given as in (6.3), and $i \in [\kappa-1]_0$. Then the formula of generalized survival function is as follows*

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \sum_{i=0}^{\kappa-2} \mu_{(\psi_*(i))} \mathbf{1}_{\left[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi_*(i+1)}\right)}(\alpha) = \sum_{i=0}^{\kappa-2} \mu_{(\psi_*(i))} \mathbf{1}_{\left[\mathsf{A}_{\psi^*(i-1)+1}, \mathsf{A}_{\psi^*(i)+1}\right)}(\alpha) \tag{6.4}$$

for any $\alpha \in [0,+\infty)$ with the convention $\psi^*(-1) = 0$.

Proof. See Appendix. □

If we use the mapping $\mathbf{i}$ defined by (4.1), then formulas in (6.4) can be rewritten similarly as in Corollary 4.2, i.e.,

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \sum_{i=0}^{\kappa-2} \mu_{\mathbf{i}(i)} \mathbf{1}_{\left[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi_*(i+1)}\right)}(\alpha) = \sum_{i=0}^{\kappa-2} \mu_{\mathbf{i}(i)} \mathbf{1}_{\left[\mathsf{A}_{\psi^*(i-1)+1}, \mathsf{A}_{\psi^*(i)+1}\right)}(\alpha).$$

Remark 6.9. *The software implementation of the generalized survival function computation and the computation of conditional aggregation-based Choquet integral on discrete space using the maps φ^* and ψ_* can be accessible in the repository at the link*

HTTPS://GITHUB.COM/STANISLAV-B/CAO_CHOQUET_INTEGRAL_COMPUTATION.GIT

Let us note that the presented algorithms for the generalized Choquet integral computation makes its applicability in various areas time efficient. For example in [1], we used this concept in image processing, namely edge detection, where with standard dimensions (in pixels) of the image, it is necessary to perform millions of computations.

As we have already mentioned, searching for optimal intervals will be helpful for studying the indistinguishability of generalized survival functions. In the following, we state sufficient and necessary conditions under which the generalized survival functions coincide. This is applicable to decision-making problems. In fact, if the generalized survival functions of two alternatives (e.g. two offers of accommodation) are the same, then their overall score will be the same. Both alternatives will be in the same place in the ranking.

Definition 6.10. *The triples $(\mu, \mathscr{A}, \mathbf{x})$ and $(\mu', \mathscr{A}', \mathbf{x}')$, where μ, μ' are monotone measures, $\mathscr{A}, \mathscr{A}'$ are FCA and $\mathbf{x}, \mathbf{x}'$ are vectors, are called integral equivalent, if*

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \mu'_{\mathscr{A}'}(\mathbf{x}',\alpha).$$

Proposition 6.11. *Let $\mathscr{A}$ and $\mathscr{A}'$ be a FCA, $\mu, \mu' \in \mathbf{M}$, $\mathbf{x}, \mathbf{x}' \in [0,+\infty)^{[n]}$ and φ_*, φ^* be given as in (6.1). Then the following assertions are equivalent:*

(i) *$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \mu'_{\mathscr{A}'}(\mathbf{x}',\alpha)$ for any $\alpha \in [0,\infty)$;*

(ii) *for each $j \in [\kappa-1]_0$ with $\min\limits_{k\le\varphi^*(j)} \mathsf{A}(\mathbf{x}|F_k^c) < \min\limits_{k<\varphi_*(j)} \mathsf{A}(\mathbf{x}|F_k^c)$ there exists $j' \in [\kappa'-1]_0$ such that $\mu_j = \mu'_{j'}$, $\min\limits_{k\le\varphi^*(j)} \mathsf{A}(\mathbf{x}|F_k^c) = \min\limits_{k\le\varphi^*(j')} \mathsf{A}'(\mathbf{x}'|F_k^c)$ and $\min\limits_{k<\varphi_*(j)} \mathsf{A}_{\langle k\rangle} = \min\limits_{k<\varphi_*(j')} \mathsf{A}'_{\langle k\rangle}$.*

Proof. It follows from Proposition 6.2. □

Remark 6.12. *Following Proposition 6.7, condition (ii) in the previous proposition can be equivalently formulated as follows: for each* $i \in [\kappa - 1]_0$ *with* $\mathsf{A}_{\psi_*(i)} < \mathsf{A}_{\psi^*(i)+1}$ *there exists* $i' \in [\kappa' - 1]$ *such that* $\mu_{\psi_*(i)} = \mu'_{\psi_*(i')}$, $\mathsf{A}(\mathbf{x}|E_{\psi_*(i)}) = \mathsf{A}'(\mathbf{x}'|E_{\psi_*(i')})$ *and* $\mathsf{A}(\mathbf{x}|E_{\psi^*(i)+1}) = \mathsf{A}'(\mathbf{x}'|E_{\psi^*(i')+1})$.

Fixing a collection $\mathscr{E}$ and a monotone measure μ we obtain the following sufficient and necessary condition for integral equivalence of triples $(\mu, \mathsf{A}, \mathbf{x})$ and $(\mu, \mathsf{A}', \mathbf{x}')$.

Corollary 6.13. *Let* $\mathscr{A} = \{\mathsf{A}(\cdot|E) : E \in \mathscr{E}\}$ *and* $\mathscr{A}' = \{\mathsf{A}'(\cdot|E) : E \in \mathscr{E}\}$ *be a FCA,* $\mu \in \mathbf{M}$, $\mathbf{x}, \mathbf{x}' \in [0, +\infty)^{[n]}$, *and* φ^*, ψ_* *be given as in (6.1) and (6.3), respectively. Then the following assertions are equivalent:*

(i) $\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \mu_{\mathscr{A}'}(\mathbf{x}', \alpha)$ *for any* $\alpha \in [0, +\infty)$;

(ii) $\min\limits_{k \leq \varphi^*(j)} \mathsf{A}(\mathbf{x}|F_k^c) = \min\limits_{k \leq \varphi^*(j)} \mathsf{A}'(\mathbf{x}'|F_k^c)$ *for any* $j \in [\kappa - 1]_0$
(or equivalently, $\mathsf{A}(\mathbf{x}|E_{\psi_*(i)}) = \mathsf{A}'(\mathbf{x}'|E_{\psi_*(i)})$ *for each* $i \in [\kappa - 1]_0$*).*

Remark 6.14. *In [6], the authors derived the necessary and sufficient condition of equality* $\mu(\{\mathbf{x} \geq \alpha\}) = \mu(\{\mathbf{y} \geq \alpha\})$ *with* μ *being a possibility and necessity measure, respectively. Our result includes equality* $\mu(\{\mathbf{x} > \alpha\}) = \mu(\{\mathbf{y} > \alpha\})$ *with* μ *being an arbitrary monotone measure.*

Example 6.15. *Let us consider the collection* $\mathscr{E} = \{\emptyset, \{1\}, \{1, 2, 3\}\}$, *the families of conditional aggregation operators* $\mathscr{A}^{\max} = \{\mathsf{A}^{\max}(\cdot|E) : E \in \mathscr{E}\}$ *and* $\mathscr{A}^{\mathrm{sum}} = \{\mathsf{A}^{\mathrm{sum}}(\cdot|E) : E \in \mathscr{E}\}$, *the vectors* $\mathbf{x} = (2, 5, 9)$ *and* $\mathbf{x}' = (2, 3, 4)$, *and the monotone measure* $\mu \in \mathbf{M}$ *with corresponding values in table.*

j	0	1	2
F_j	$\emptyset$	$\{2,3\}$	$\{1,2,3\}$
μ_j	0	0.5	1
$\mathsf{A}^{\max}(\mathbf{x}\vert F_j^c) = \mathsf{A}^{\mathrm{sum}}(\mathbf{x}'\vert F_j^c)$	9	2	0

It is easy to see that for any $j \in [2]_0$ *we have* $\varphi^*(j) = j$ *and* $\min\limits_{k \leq j} \mathsf{A}^{\max}(\mathbf{x}|F_k^c) = \min\limits_{k \leq j} \mathsf{A}^{\mathrm{sum}}(\mathbf{x}'|F_k^c)$. *Then according to previous proposition one can expect the equality of corresponding generalized survival functions. And, it is:*

$$\mu_{\mathscr{A}^{\max}}(\mathbf{x}, \alpha) = \mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x}', \alpha) = 1 \cdot \mathbf{1}_{[0,2)} + 0.5 \cdot \mathbf{1}_{[2,9)}$$

for any $\alpha \in [0, +\infty)$.

The following example demonstrates a standard situation in decision-making processes. Different alternatives can be evaluated with the same score, thus it is not possible to decide which one is better. This situation is sometimes undesirable, mainly when a tool (some aggregation, e.g. integral) does not distinguish a lot of input vectors. However, sometimes indistinguishibility is expected as we are showing it in the following example presenting an accommodation option problem. This example also demonstrates the potential of the use of generalized Choquet integral in decision-making problems. In the literature, nonadditive integrals are used to model many decision-making problems because via monotone measure they are able to model interactions among criteria (e.g. usually with a higher review score one can expect a higher price), see [9].

Example 6.16. *A person is going to holiday and he is looking for accommodation in the destination. Let us consider two options that were offered by the search engine and were evaluated by the vacationer on the scale of* 1–10 *based on three criteria: distance from the beach, price of accommodation, and reviews, respectively:* $\mathbf{x} = (2, 2, 5)$ *and* $\mathbf{x}' = (7, 5, 2)$*. Let us compare these two options by the generalized Choquet integral. Let us choose* $\mathscr{E} = 2^{[3]}$ *because we intuitively want to take into account all possible interactions among criteria. Further, it is natural to sum up points on conditional sets, therefore let us choose the family* $\mathscr{A}^{\mathrm{sum}}$*. The person does not emphasize the distance from the beach, i.e.* $\mu(\{1\}) = 0$*. Criteria* $2, 3$ *have the same importance, therefore their corresponding monotone measure is the same,* $\mu(\{2\}) = \mu(\{3\}) = 0.5$*. And, due to interactions let us choose the values of the rest of the sets as it is given in the following table:*

j	0	1	2	3	4	5	6	7
F_j	$\emptyset$	$\{1\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{2\}$	$\{2,3\}$	$\{1,2,3\}$
μ_j	0	0	0.5	0.5	0.5	0.5	0.8	1
$\mathsf{A}^{\mathrm{sum}}(\mathbf{x}\vert F_k^c)$	9	7	7	4	5	2	2	0
$\mathsf{A}^{\mathrm{sum}}(\mathbf{x}'\vert F_k^c)$	14	7	9	12	2	5	7	0

It is easy to see that $\varphi^*(0) = \varphi^*(1) = 1$*,* $\varphi^*(2) = \varphi^*(3) = \varphi^*(4) = \varphi^*(5) = 5$*,* $\varphi^*(6) = 6$*,* $\varphi^*(7) = 7$ *and*

$$\begin{aligned}
\min_{k\leq 1} \mathsf{A}^{\mathrm{sum}}(\mathbf{x}|F_k^c) &= 7 = \min_{k\leq 1} \mathsf{A}^{\mathrm{sum}}(\mathbf{x}'|F_k^c),\\
\min_{k\leq 5} \mathsf{A}^{\mathrm{sum}}(\mathbf{x}|F_k^c) &= 2 = \min_{k\leq 5} \mathsf{A}^{\mathrm{sum}}(\mathbf{x}'|F_k^c),\\
\min_{k\leq 6} \mathsf{A}^{\mathrm{sum}}(\mathbf{x}|F_k^c) &= 2 = \min_{k\leq 6} \mathsf{A}^{\mathrm{sum}}(\mathbf{x}'|F_k^c),\\
\min_{k\leq 7} \mathsf{A}^{\mathrm{sum}}(\mathbf{x}|F_k^c) &= 0 = \min_{k\leq 7} \mathsf{A}^{\mathrm{sum}}(\mathbf{x}'|F_k^c).
\end{aligned}$$

Then according to previous proposition one can expect the equality of corresponding generalized survival functions. And, it is:

$$\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) = \mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x}',\alpha) = 1\cdot\mathbf{1}_{[0,2)} + 0.5\cdot\mathbf{1}_{[2,7)}$$

for any $\alpha \in [0,+\infty)$*. The indistinguishability of given vectors is desirable. Indeed, since the first criterion has no impact on the decision it does not matter what are the first components of vectors. Further, because of the symmetry of monotone measure with respect to sets* $\{2\}$ *and* $\{3\}$ *the order of the second and the third component of the input vector is not important. We could also choose the alternative calculation of indistinguishability using the map* ψ_**.*

Immediately, we get the sufficient condition for equality of generalized Choquet integrals defined w.r.t. different FCA and vectors.

Corollary 6.17. *Let* $\mathscr{A}$ *and* $\mathscr{A}'$ *be a FCA,* $\mu\colon \hat{\mathscr{E}} \to [0,+\infty)$*,* $\mathbf{x},\mathbf{x}' \in [0,+\infty)^{[n]}$*,* φ^**,* ψ_* *be given as in (6.1) and (6.3). If* $\min\limits_{k\leq\varphi^*(j)} \mathsf{A}(\mathbf{x}|F_k^c) = \min\limits_{k\leq\varphi^*(j)} \mathsf{A}'(\mathbf{x}'|F_k^c)$ *for any* $j \in [\kappa-1]_0$ *(or equivalently,* $\mathsf{A}(\mathbf{x}|E_{\psi_*(i)}) = \mathsf{A}'(\mathbf{x}'|E_{\psi_*(i)})$ *for each* $i \in [\kappa-1]_0$*), then*

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \mathrm{C}_{\mathscr{A}'}(\mathbf{x}',\mu).$$

Conclusion

In the paper, we dealt with the concept of the generalized survival function and the generalized Choquet integral related to it. Considering their applications, see Section 2, we were mainly interested in their computational formulas on discrete space. The derivation of these formulas required the introduction of new notations whose idea and practical meaning can be visually interpreted, as we described in Sections 3 and 4. Interesting results are Proposition 3.2, Proposition 3.7, Corollary 4.2 and Corollary 4.5. In Section 4 we also pointed out the direct and efficient construction of the graph of the generalized survival function.

Motivated by applications we solved the indistinguishability of generalized survival functions. This question is interesting, especially in decision-making processes, because the indistinguishability means that both alternatives are equally good, i.e. we do not know which one is better (it is not always the desired result). The interesting results given in Proposition 6.5, Proposition 6.8 and Proposition 6.11 are related to this.

In this paper, we also mentioned the formulas for calculating the generalized Choquet integral with respect to special types of monotone measure, and we solved the introductory problems serving as a motivation to study mentioned concepts.

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Appendix

Proof of Corollary 3.9

(i) Clearly, in accordance with (3.3) we have $0 = \mu_0 = \mu(\emptyset)$ and according to Theorem 3.7 it is achieved on $\left[\min_{k\leq 0} \mathsf{A}_{\langle k\rangle}, +\infty\right) = [\mathsf{A}_{\langle 0\rangle}, +\infty) = [\mathsf{A}(\mathbf{x}|[n]), +\infty)$. Then $1 = \mu_1 = \cdots = \mu_{\kappa-1}$ is achieved on $[0, \mathsf{A}(\mathbf{x}|[n]))$.

(ii) From Theorem 3.7, the value $1 = \mu_{\kappa-1}$ is achieved on $\left[0, \min_{k<\kappa-1} \mathsf{A}_{\langle k\rangle}\right) = \left[0, \min_{E\neq\emptyset} \mathsf{A}(\mathbf{x}|E)\right)$.

(iii) Let us consider the bijection F_* given in (3.3) such that $|F_k| < |F_l|$ implies $k < l$ and for any $i \in [n]_0$ let us set

$$\omega^*(i) = \max\{j \in [\kappa-1]_0 : |F_j| = i\},\ \omega_*(i) = \min\{j \in [\kappa-1]_0 : |F_j| = i\}.$$

It is easy to see that $\omega_*(i) - 1 = \omega^*(i-1)$ with the convention $\omega^*(-1) = -1$. Further, because of monotonicity of μ and FCA for any $i \in [n]_0$ we have

$$\min_{k\leq\omega^*(i)} \mathsf{A}_{\langle k\rangle} = \min_{k\leq\omega^*(i)} \mathsf{A}(\mathbf{x}|F_k^c) = \min_{k\in[\omega_*(i),\omega^*(i)]} \mathsf{A}(\mathbf{x}|F_k^c) = \min_{|E|=n-i} \mathsf{A}(\mathbf{x}|E).$$

Indeed, for any $k < \omega_*(i)$ since $\mathscr{E} = 2^{[n]}$ there exists $\tilde{k} \in [\omega_*(i), \omega^*(i)]$ such that $F_{\tilde{k}} \supset F_k$. Then $\mathsf{A}(\mathbf{x}|F^c_{\tilde{k}}) \leq \mathsf{A}(\mathbf{x}|F^c_k)$. According to Theorem 3.7 we have that μ^i is achieved on

$$\bigcup_{j\in\{\omega_*(i),\dots,\omega^*(i)\}} \Big[\min_{k\leq j} \mathsf{A}_{\langle k\rangle}, \min_{k<j} \mathsf{A}_{\langle k\rangle}\Big) = \Big[\min_{k\leq \omega^*(i)} \mathsf{A}_{\langle k\rangle}, \min_{k<\omega_*(i)} \mathsf{A}_{\langle k\rangle}\Big)$$
$$= \Big[\min_{k\leq \omega^*(i)} \mathsf{A}_{\langle k\rangle}, \min_{k\leq\omega^*(i-1)} \mathsf{A}_{\langle k\rangle}\Big)$$
$$= \Big[\min_{E=n-i} \mathsf{A}_{\langle k\rangle}, \min_{E=n-i+1} \mathsf{A}_{\langle k\rangle}\Big).$$

(iv) Let us consider the bijection F_* given in (3.3) such that $F_0 = \emptyset$ and for any $i \in \{2, 3, \dots, n\}$

$$\{\sigma(i-1)\} \subseteq F_k \subseteq G^c_{\sigma(i)},\ \{\sigma(i)\} \subseteq F_l \subseteq G^c_{\sigma(i+1)},\ \text{implies } k < l.$$

According to Theorem 3.7 it is clear that $\pi(\sigma(0))$ is achieved on $\Big[\mathsf{A}(\mathbf{x}|G_{\sigma(1)}), +\infty\Big)$. Further, for any $i \in [n]$ let us set

$$\tau^*(\sigma(i)) = \max\{j \in [\kappa-1]_0 : \Pi(F_j) = \pi(\sigma(i)) \text{ and } \{\sigma(i)\} \subseteq F_j \subseteq G^c_{\sigma(i+1)}\},$$
$$\tau_*(\sigma(i)) = \min\{j \in [\kappa-1]_0 : \Pi(F_j) = \pi(\sigma(i)) \text{ and } \{\sigma(i)\} \subseteq F_j \subseteq G^c_{\sigma(i+1)}\}.$$

It is easy to see that $\tau_*(\sigma(i)) - 1 = \tau^*(\sigma(i-1))$ with the convention $\tau^*(0) = 0$. Further, because of monotonicity of μ and FCA for any $i \in [n]_0$ we have

$$\min_{k\leq\tau^*(\sigma(i))} \mathsf{A}_{\langle k\rangle} = \min_{k\leq\tau^*(\sigma(i))} \mathsf{A}(\mathbf{x}|F^c_k) = \min_{k\in\{\tau_*(\sigma(i)),\dots,\tau^*(\sigma(i))\}} \mathsf{A}(\mathbf{x}|F^c_k) = \mathsf{A}(\mathbf{x}|G_{\sigma(i+1)}).$$

Indeed, for any $k < \tau_*(\sigma(i))$ we have $F_k \subseteq G^c_{\sigma(i+1)}$, where $G^c_{\sigma(i+1)} = F_{\tilde{k}}$ for some $\tilde{k} \in [\tau_*(\sigma(i)), \tau^*(\sigma(i))]$, thus explaining the second equality. Further, it holds that for any $k \in [\tau_*(\sigma(i)), \tau^*(\sigma(i))]$

$$F_k \subseteq G^c_{\sigma(i+1)}$$

therefore $\mathsf{A}(\mathbf{x}|G_{\sigma(i+1)}) \leq \mathsf{A}(\mathbf{x}|F^c_k)$. According to Theorem 3.7 we have that $\pi(\sigma(i))$ is achieved on

$$\bigcup_{j\in\{\tau_*(\sigma(i)),\dots,\tau^*(\sigma(i))\}} \Big[\min_{k\leq j} \mathsf{A}_{\langle k\rangle}, \min_{k<j} \mathsf{A}_{\langle k\rangle}\Big) = \Big[\min_{k\leq \tau^*(\sigma(i))} \mathsf{A}_{\langle k\rangle}, \min_{k<\tau_*(\sigma(i))} \mathsf{A}_{\langle k\rangle}\Big)$$
$$= \Big[\min_{k\leq \tau^*(\sigma(i))} \mathsf{A}_{\langle k\rangle}, \min_{k\leq\tau^*(\sigma(i-1))} \mathsf{A}_{\langle k\rangle}\Big)$$
$$= \Big[\mathsf{A}(\mathbf{x}|G_{\sigma(i+1)}), \mathsf{A}(\mathbf{x}|G_{\sigma(i)})\Big).$$

(v) It is clear that $N([n]) = 1 - \Pi(\emptyset) = 1$ and for each $\{\sigma(i)\} \subseteq F^c \subseteq G^c_{\sigma(i+1)}$

$$N(F) = 1 - \Pi(F^c) = 1 - \pi(\sigma(i)),$$

with $i \in [n]$ and

$$0 = 1 - \pi(\sigma(n)) \leq \cdots \leq 1 - \pi(\sigma(i)) \leq \cdots \leq 1 - \pi(\sigma(0)) = 1.$$

Let us consider the bijection F_* given in (3.3) such that $F_{\kappa-1} = \emptyset$ and for any $i \in \{2, 3, \dots, n\}$

$$\{\sigma(i-1)\} \subseteq F_k \subseteq G^c_{\sigma(i)},\ \{\sigma(i)\} \subseteq F_l \subseteq G^c_{\sigma(i+1)},\ \text{implies } k > l.$$

According to Theorem 3.7 it is clear that $0 = 1-\pi(\sigma(n))$ is achieved on $\Big[\mathsf{A}(\mathbf{x}|\{\sigma(n)\}), +\infty\Big)$. Further, for any $i \in [n-1]$ let us set

$$\rho^*(\sigma(i)) = \max\{j \in [\kappa-1]_0 : N(F_j) = 1-\pi(\sigma(i)) \text{ and } \{\sigma(i)\} \subseteq F_j^c \subseteq G_{\sigma(i+1)}^c\}$$
$$\rho_*(\sigma(i)) = \min\{j \in [\kappa-1]_0 : N(F_j) = 1-\pi(\sigma(i)) \text{ and } \{\sigma(i)\} \subseteq F_j^c \subseteq G_{\sigma(i+1)}^c\}.$$

It is easy to see that $\rho^*(\sigma(i)) \geq \rho_*(\sigma(i)) > 0$ and $\rho_*(\sigma(i)) - 1 = \rho^*(\sigma(i+1))$. Further, because of monotonicity of μ and FCA for any $i \in [n-1]_0$ we have

$$\min_{k \leq \rho^*(\sigma(i))} \mathsf{A}_{\langle k \rangle} = \min_{k \leq \rho^*(\sigma(i))} \mathsf{A}(\mathbf{x}|F_k^c) = \min_{k \geq i} \mathsf{A}(\mathbf{x}|\{\sigma(k)\}).$$

Indeed, for any $k \leq \rho^*(\sigma(i))$ there exists $\tilde{k} \geq i$ such that $N(F_k) = 1-\pi(\sigma(\tilde{k})) \leq 1-\pi(\sigma(i))$. Then $F_k^c \supseteq \{\sigma(\tilde{k})\}$. According to Theorem 3.7 we have that $1-\pi(\sigma(i))$ is achieved on

$$\bigcup_{j \in \{\rho_*(\sigma(i)),\ldots,\rho^*(\sigma(i))\}} \Big[\min_{k \leq j} \mathsf{A}_{\langle k \rangle}, \min_{k < j} \mathsf{A}_{\langle k \rangle}\Big) = \Big[\min_{k \leq \rho^*(\sigma(i))} \mathsf{A}_{\langle k \rangle}, \min_{k < \rho_*(\sigma(i))} \mathsf{A}_{\langle k \rangle}\Big)$$
$$= \Big[\min_{k \leq \rho^*(\sigma(i))} \mathsf{A}_{\langle k \rangle}, \min_{k \leq \rho^*(\sigma(i+1))} \mathsf{A}_{\langle k \rangle}\Big) = \Big[\min_{k \geq i} \mathsf{A}(\mathbf{x}|\{\sigma(k)\}), \min_{k \geq i+1} \mathsf{A}(\mathbf{x}|\{\sigma(k)\})\Big). \qquad \square$$

Proof of Proposition 3.10 Let us consider an arbitrary, fixed $i \in [\kappa-1]_0$ such that $\mathsf{A}_i \neq \mathsf{A}_{i+1}$ (otherwise it is trivial). Let us denote

$$j = \min\{l : \mathsf{A}_{\langle l \rangle} \leq \mathsf{A}_i \text{ and } \min_{k \leq i} \mu_{(k)} = \mu_l\}.$$

The above-mentioned set is nonempty. Indeed, for each l such that $\min_{k \leq i} \mu_{(k)} = \mu_l$ w.r.t. denotations from the beginning of this section there exists $i_l \leq i$ such that $F_l = E_{i_l}^c$ and $\mu_l = \mu_{(i_l)}$. Further, it is clear that $\mathsf{A}_{\langle l \rangle} = \mathsf{A}_{i_l} \leq \mathsf{A}_i$.

From the definition of j we immediately have $\min_{k \leq j} \mathsf{A}_{\langle k \rangle} \leq \mathsf{A}_{\langle j \rangle} \leq \mathsf{A}_i$. Moreover, it also holds that $\min_{k<j} \mathsf{A}_{\langle k \rangle} \geq \mathsf{A}_{i+1}$. By contradiction: Let $\min_{k<j} \mathsf{A}_{\langle k \rangle} < \mathsf{A}_{i+1}$. Then there exists $k^* < j$ such that $\mathsf{A}_{\langle k^* \rangle} < \mathsf{A}_{i+1}$. Because of (3.1) we get $\mathsf{A}_{\langle k^* \rangle} \leq \mathsf{A}_i$. Further, $\mu_{k^*} < \mu_j$: From the fact that $k^* < j$ we have $\mu_{k^*} \leq \mu_j$, however, the equality can not happen because of the definition of j. Further for $\alpha \in [\mathsf{A}_i, \mathsf{A}_{i+1})$ because of formula (3.4) and because of the definition of j we have $\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \mu_j$. However, since $\mathsf{A}_{\langle k^* \rangle} \leq \mathsf{A}_i$ for $\alpha \in [\mathsf{A}_i, \mathsf{A}_{i+1})$ we get

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \min\big\{\mu_j : \mathsf{A}_{\langle j \rangle} \leq \alpha\big\} \leq \mu_{k^*} < \mu_j.$$

This is a contradiction. $\square$

Auxiliary pseudocodes used in Algorithms 1 and Algorithms 2.

The first pseudocode, namely method *find-sets-E_i-and-F_i*, describe the calculation of permutation listed in (3.1) and (3.3). Second and third algorithms describe determination of values of $(\cdot)$, $\langle \cdot \rangle$.

```
Method find-sets-E_i-and-F_i(𝓔, μ, x, 𝓐)
    i := 0, 𝓔' := 𝓔;
    do
        E_i: E_i ∈ 𝓔 and A(x|E_i) = min{A(x|E) : E ∈ 𝓔};
        F_i: F_i ∈ 𝓔' and μ(F_i) = min{μ(F) : F ∈ 𝓔'};
        𝓔 := 𝓔 \ {E_i};
        𝓔' := 𝓔' \ {F_i};
        i := i + 1;
    while 𝓔 ≠ ∅;
    return E_0, ..., E_{κ−1}, F_0, ..., F_{κ−1};
end
```

Algorithm 3: Determination of sets E_i and F_i, $i \in [\kappa-1]_0$

```
Method the-(·)-map(𝓔, μ, x, 𝓐)
    E_0, ..., E_{κ−1}, F_0, ..., F_{κ−1} ←
      find-sets-E_i-and-F_i(𝓔, μ, x, 𝓐);
    for (i = 0, i < κ, i++) do
        for (j = 0, j < κ, j++) do
            if (E_i = F_j^c) then
                (i) := j;
            end
        end
    end
    return (0), ..., (κ − 1);
end
```

Algorithm 4: Determination of values of a map $(\cdot)$

```
Method the-⟨·⟩-map(𝓔, μ, x, 𝓐)
    E_0, ..., E_{κ−1}, F_0, ..., F_{κ−1} ←
      find-sets-E_i-and-F_i(𝓔, μ, x, 𝓐);
    for (i = 0, i < κ, i++) do
        for (j = 0, j < κ, j++) do
            if (E_i = F_j^c) then
                ⟨j⟩ := i;
            end
        end
    end
    return ⟨0⟩, ..., ⟨κ − 1⟩;
end
```

Algorithm 5: Determination of values of a map $\langle\cdot\rangle$

Proposition. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,\infty)^{[n]}$, ψ^*, ψ_* be defined as in (6.3) and $a, b, i \in [\kappa-1]_0$. Then the following properties hold.*

(i) $\psi_*(i) = \min\{l \in [\kappa-1]_0 : \mu_{(l)} = \min_{k\le i} \mu_{(k)}\}$.

(ii) $\psi_*(0) = 0$, $\psi_*(i) \le i \le \psi^*(i)$.

(iii) *For any integer $u \in \{\psi_*(i), \dots, \psi^*(i)\}$ it holds* $\min_{k\le u} \mu_{(k)} = \min_{k\le i} \mu_{(k)} = \mu_{(\psi_*(i))} = \mu_{\mathbf{i}(i)}$.

(iv) $\min_{k\le a} \mu_{(k)} = \min_{k\le b} \mu_{(k)}$ *if only if* $\psi_*(a) = \psi_*(b)$ *if only if* $\psi^*(a) = \psi^*(b)$.

(v) $\min_{k\le a} \mu_{(k)} > \min_{k\le b} \mu_{(k)}$ *if only if* $\psi_*(a) < \psi_*(b)$ *if only if* $\psi^*(a) < \psi^*(b)$.

(vi) *ψ_* and ψ^* are nondecreasing.*

Proof. (i) Let us denote

$$i^\star = \min\{l \in [\kappa-1]_0 : \mu_{(l)} = \min_{k\le i} \mu_{(k)}\}.$$

It is clear that $i^\star \le i$, $\min_{k\le i} \mu_{(k)} = \mu_{(i^\star)}$. Further,

- Since $\mu_{(i^\star)} \geq \min_{k \leq i^\star} \mu_{(k)} \geq \min_{k \leq i} \mu_{(k)} = \mu_{(i^\star)}$, then $\min_{k \leq i^\star} \mu_{(k)} = \min_{k \leq i} \mu_{(k)}$. Therefore $\psi_*(i) \leq i^\star$.
- For any $l < i^\star$ it holds
$$\min_{k \leq l} \mu_{(k)} > \min_{k \leq i} \mu_{(k)}.$$
Indeed, $\min_{k \leq l} \mu_{(k)} \geq \min_{k \leq i^\star} \mu_{(k)} = \mu_{(i^\star)}$. However, $\min_{k \leq l} \mu_{(k)} \neq \min_{k \leq i^\star} \mu_{(k)}$. By contradiction: Let $\min_{k \leq l} \mu_{(k)} = \mu_{(i^\star)}$. Then there exist $k^\star \leq l < i^\star$ such that $\mu_{(k^\star)} = \mu_{(i^\star)}$. This is a contradiction with the definition of $j^\star$. Therefore $\psi_*(i) > l$.

From the previous it follows that $\psi_*(i) = i^\star$.

(ii) It follows directly from definitions of ψ^*, ψ_*.

(iii) It follows from the equalities $\min_{k \leq \psi_*(i)} \mu_{(k)} = \min_{k \leq i} \mu_{(k)} = \min_{k \leq \psi^*(i)} \mu_{(k)}$. Moreover, from part (i) and because of Lemma 4.1 we have $\mu(\psi_*(i)) = \min_{k \leq i} \mu_{(k)} = \mathbf{i}(i)$.

(iv) It follows directly from definitions of ψ_*, ψ^* and from the fact that if $\psi_*(a) = \psi_*(b)$ ($\psi^*(a) = \psi^*(b)$), then there is $\tilde{l} \in [\kappa - 1]_0$ such that $\min_{k \leq a} \mu_{(k)} = \min_{k \leq \tilde{l}} \mu_{(k)} = \min_{k \leq b} \mu_{(k)}$.

(v) It follows from the proof of (iv) and from the fact that if $\psi_*(a) < \psi_*(b)$, then $\min_{k \leq \psi_*(a)} \mu_{(k)} > \min_{k \leq \psi_*(b)} \mu_{(k)}$ (equality does not occur because of statement (v)). Then we get
$$\min_{k \leq a} \mu_{(k)} = \min_{k \leq \psi_*(a)} \mu_{(k)} > \min_{k \leq \psi_*(b)} \mu_{(k)} = \min_{k \leq b} \mu_{(k)},$$
where the first and the last equality hold because of definition ψ_*. The same for $\psi^*(a) < \psi^*(b)$.

(vi) Let us denote
$$M_1 = \{l_1 \in [\kappa - 1]_0 : \min_{k \leq l_1} \mu_{(k)} = \min_{k \leq a} \mu_{(k)}\} \quad M_2 = \{l_2 \in [\kappa - 1]_0 : \min_{k \leq l_2} \mu_{(k)} = \min_{k \leq b} \mu_{(k)}\}.$$
Since $a \leq b$, then $\min_{k \leq a} \mu_{(k)} \geq \min_{k \leq b} \mu_{(k)}$. If $\min_{k \leq a} \mu_{(k)} = \min_{k \leq b} \mu_{(k)}$, then directly from definition ψ_*, resp. ψ^*, we have $\psi_*(a) = \psi_*(b)$, resp. $\psi^*(a) = \psi^*(b)$. Further, let us suppose that $\min_{k \leq a} \mu_{(k)} > \min_{k \leq b} \mu_{(k)}$. Then for any $l_1 \in M_1$ and any $l_2 \in M_2$ it holds
$$\min_{k \leq l_1} \mu_{(k)} = \min_{k \leq a} \mu_{(k)} > \min_{k \leq b} \mu_{(k)} = \min_{k \leq l_2} \mu_{(k)},$$
therefore $l_2 > l_1$. Thus also $\min M_2 > \min M_1$, resp. $\max M_2 > \max M_1$, that is $\psi_*(b) > \psi_*(a)$, resp. $\psi^*(b) > \psi^*(a)$. □

Remark. *In general, we cannot determine the order of the values $\mu_{(i)}$, $i \in [\kappa - 1]_0$. However, it holds*
$$\begin{aligned} &\mu_{(\psi_*(0))} \geq \mu_{(\psi_*(1))} \geq \cdots \geq \mu_{(\psi_*(\kappa-1))}, \\ \textit{and} \quad &\mu_{(\psi^*(0))} \geq \mu_{(\psi^*(1))} \geq \cdots \geq \mu_{(\psi^*(\kappa-1))}. \end{aligned}$$
This property follows directly from the definitions of ψ_* and ψ^*. Indeed, we have already shown that for $a \leq b$ we get $\psi_*(a) \leq \psi_*(b)$. In case that $\psi_*(a) = \psi_*(b)$, the result is clear. If $\psi_*(a) < \psi_*(b)$, then the result follows from Proposition 6 (vi). Similarly for ψ^*.

Lemma. *Let $\mathscr{A}$ be a FCA, $\mu \in \mathbf{M}$, $\mathbf{x} \in [0,+\infty)^{[n]}$, ψ_* and ψ^* be given as in (6.3).*

(i) *Let $i \in [\kappa-2]_0$. Then it holds*

(i1) *If $\min_{k\leq i} \mu_{(k)} > \min_{k\leq i+1} \mu_{(k)}$, then $\psi^*(i)+1 = \psi_*(i+1)$ and $\min_{k\leq i}\mu_{(k)}$ is achieved on $\big[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi_*(i+1)}\big)$. Moreover, this interval is the greatest possible.*

(i2) *If $\min_{k\leq i} \mu_{(k)} = \min_{k\leq i+1} \mu_{(k)}$, then $\psi_*(i) = \psi_*(i+1)$.*

(ii) *Let $i \in [\kappa-1]$. Then it holds*

(ii1) *If $\min_{k\leq i-1} \mu_{(k)} < \min_{k\leq i} \mu_{(k)}$, then $\psi_*(i) = \psi^*(i-1)+1$ and $\min_{k\leq i}\mu_{(k)}$ is achieved on $\Big[\mathsf{A}_{\psi^*(i-1)+1}, \mathsf{A}_{\psi^*(i)+1}\Big)$. Moreover, this interval is the greatest possible.*

(ii2) *If $\min_{k\leq i-1} \mu_{(k)} = \min_{k\leq i} \mu_{(k)}$, then $\psi^*(i-1) = \psi^*(i)$.*

Proof. Let us prove part (i).

(i1) If $\min_{k\leq i} \mu_{(k)} > \min_{k\leq i+1} \mu_{(k)}$, then directly from definitions of ψ_* and ψ^* we have $\psi^*(i) = i$ and $\psi_*(i+1) = i+1$, thus $\psi^*(i)+1 = \psi_*(i+1)$. Further, according to Proposition 6.7, we have

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) = \min_{k\leq i} \mu_{(k)} = \mu_{(\psi_*(i))}$$

for any $\alpha \in \big[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi^*(i)+1}\big)$ and this interval is the greatest possible. Using the above equality we have

$$\big[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi^*(i)+1}\big) = \big[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi_*(i+1)}\big).$$

(i2) It holds trivially.

Part (ii) can be proved analogously. □

Proof of Proposition 6.7 Because of Proposition 6 (iii) and Theorem 3.2 we know that the value $\min_{k\leq i} \mu_{(k)} = \mu_{(\psi_*(i))}$ is achieved on intervals $[\mathsf{A}_m, \mathsf{A}_{m+1})$, $m \in \{\psi_*(i), \dots, \psi^*(i)\}$. From definitions of ψ^* and ψ_* it is clear that there are no other intervals with this property. Therefore the value $\min_{k\leq i} \mu_{(k)} = \mu_{(\psi_*(i))}$ is achieved on $\bigcup_{m\in M} [\mathsf{A}_m, \mathsf{A}_{m+1}) = \big[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi^*(i)+1}\big)$. □

Proof of Proposition 6.8 Let us consider an arbitrary (fixed) $i \in [\kappa-2]_0$. If $\min_{k\leq i} \mu_{(k)} > \min_{k\leq i+1} \mu_{(k)}$, then according to previous Lemma (i1) $\min_{k\leq i} \mu_{(k)} = \mu_{(\psi_*(i))}$ is achieved on the interval $\big[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi_*(i+1)}\big)$. Moreover, this interval is the greatest possible. If $\min_{k\leq i} \mu_{(k)} = \min_{k\leq i+1} \mu_{(k)}$, then $\psi_*(i) = \psi_*(i+1)$, see previous Lemma (i2), thus $\big[\mathsf{A}_{\psi_*(i)}, \mathsf{A}_{\psi_*(i+1)}\big) = \emptyset$. This demonstrates that each value of generalized survival function is included just once in the sum and the first formula is right. The third formula follows from Lemma 4.1. The second and the fourth formula can be proved analogously. □