An experimental study of the Choquet integral based on conditional aggregation operators

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Abstract

In this paper, we present an experimental study of a novel generalized Choquet integral based on conditional aggregation operators in decision making and image processing. The concept of this integral was introduced by Boczek et al. [6] recently and it is interesting from a theoretical as well as practical point of view because of his implementation in real-life situations. The standard Choquet integral is a special case of this generalization. We follow up on similar studies and their results, which were considered with respect to the standard Choquet integral and its other generalizations. We demonstrate the benefits of our approaches. For both experiments, we provide a software implementation of the generalized Choquet integral for its use in them.

Keywords: aggregation; survival function; Choquet integral; decision making; edge detection

1 Introduction

The Choquet integral is a very known concept because of its theoretical as well as practical benefits, see [11, 12, 18, 19, 25, 28, 33]. In the literature, there are known many of its generalizations, e.g. [6, 14, 21, 24, 32]. For the comprehensive study on generalizations of the Choquet integral, see [13]. In this text, we are dealing with the generalized Choquet integral based on conditional aggregation operators:

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) = \int_{[0,\infty)} \min\left\{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \leq \alpha,\, E \in \mathscr{E}\right\} \,\mathrm{d}\alpha,$$

where $[n] = \{1,\dots,n\}$, $\mathbf{x} \in [0,\infty)^{[n]}$, $\mathscr{A} = \{\mathsf{A}(\cdot|E) : E \in \mathscr{E}\}$ is a family of conditional aggregation operators with a collection $\{\emptyset,[n]\} \subseteq \mathscr{E} \subseteq 2^{[n]}$, and μ is a monotone measure on the complements of sets of $\mathscr{E}$, $n \in \mathbb{N}$. This concept was recently introduced in [6]. It generalizes previous concepts of Choquet integrals based on super level measures [15, 22, 23].

In paper [19] the author pointed out the fact that standard aggregation functions such as (weighted) arithmetic mean, maximum, minimum, or median, etc. may not be sufficient for proper data aggregation. These are mainly cases of the decision making process in which there are interactions between some criteria that cannot take into account by the mentioned aggregation functions. As a solution to this problem, he pointed to the standard Choquet integral, which, thanks to the monotone measure, is able to take these interactions into account and thus aggregate them more appropriately. In the first experimental study, or experiment, we replace the standard Choquet integral with the Choquet integral based on conditional aggregation operators. These operators, together with a suitably chosen collection, provide more information for data aggregation than the standard Choquet integral. Thus, using this generalization of the Choquet integral, it is possible to take into account the input data more sensitively, which indicates a more correct result of data aggregation. In connection with this generalization of the Choquet integral, we also deal with the experimental use of the Shapley value [36] for the correct setting of the values of the monotone measure. We observe how the outliers of the decision making process are captured.

The second area of application of the Choquet integral based on conditional aggregation operators is the image processing, namely the edge detection of an image. From a certain

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point of view, edge detection can be understood as a decision process about whether a given pixel forms an edge or not. Therefore, it is suitable to use this generalization of the Choquet integral in this case as well. We build on the ideas of the study published in [31], where using a computer vision scheme, and some other generalizations of the Choquet integral simulate the human visual system. In our approach we modify it in several ways. The main modification is to use the Choquet integral based on conditional aggregation operators with respect to four families of these operators, which, as in the previous experiment, we obtains more information about whether a given pixel forms an edge. For comparison, we also aggregate pixels using the standard Choquet integral. Further modifications of either monotone measure or pixel weighting are described and justified directly in the experimental study. We compare the results of edge detection with other methods and evaluate them qualitatively and quantitatively.

As part of an experimental study of the use of the Choquet integral based on conditional aggregation operators, we have created a software application for the decision making process and software codes (as an addition to the codes presented in [31]) for edge detection. These software supports are accessible in the online repository, which we will mention later.

This paper is organized as follows. In the following section, we present necessary notations and definitions for the mentioned experimental study. In the third section, we introduce the Choquet integral based on conditional aggregation operators. We present its basic properties that we will use and point out the approach to its calculation. In Section 4, we present our software implementations. Sections 5 and 6 contain the experiments themselves – generalized Choquet integral in decision making and image processing together with results of experiments. In the Appendix it is possible to find tables and figures supporting the experimental study.

2 Background and preliminaries

For correct definitions of terms necessary for the experimental study of the generalized Choquet integral we introduce the following notations. Since a discrete space is a base of experimental study, in the whole text we shall consider a finite set

$$[n] := \{1, 2, \dots, n\},\ n \in \mathbb{N}.$$

By $2^{[n]}$ we shall denote the power set of $[n]$. Let $\mathscr{E} \subseteq 2^{[n]}$ such that $\emptyset, [n] \in \mathscr{E}$ and $\widehat{\mathscr{E}} := \{E \in 2^{[n]} : E^c \in \mathscr{E}\}$. We shall work with nonnegative real-valued vectors, i.e. $\mathbf{x} = (x_1, \dots, x_n)$, $x_i \in [0, \infty)$, $i \in [n]$. The set of all nonnegative real-valued functions on $[n]$ (vectors) will be denoted by $[0, \infty)^{[n]}$. A nondecreasing set function $\mu\colon \widehat{\mathscr{E}} \to [0, \infty)$, i.e., $\mu(E) \leq \mu(F)$ whenever $E \subseteq F$, with $\mu(\emptyset) = 0$, will be called a *monotone* or *nonadditive measure* on $\widehat{\mathscr{E}}$. In addition, we shall suppose $\mu([n]) > 0$. A monotone measure will be called *normalized* or *capacity*, if $\mu([n]) = 1$. Monotone measure μ that is *additive*, i.e. $\mu(E \cup F) = \mu(E) + \mu(F)$ for any $E, F \in \widehat{\mathscr{E}}$, $E \cap F = \emptyset$, we shall call *measure*. Further, we put $\min \emptyset = 0$, and $\sum_{i \in \emptyset} x_i = 0$. The *survival function* of the vector $\mathbf{x}$ with respect to the monotone measure μ [6, 16] is defined by

$$\mu(\{\mathbf{x} > \alpha\}) := \mu\left(\{i \in [n] : x_i > \alpha\}\right), \quad \alpha \in [0, \infty).$$

This concept is also known as the *standard level measure* [23], the *strict level measure* [9], the *decumulative distribution* of $\mathbf{x}$ with respect to μ, see [20]. Let $\mathbf{x} = (x_1, \dots, x_n) \in [0, \infty)^{[n]}$ be an arbitrary vector. By $(\cdot)$ we denote a permutation $(\cdot)\colon [n] \to [n]$ such that $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$. Moreover, we put $x_{(0)} = 0$ and $x_{(n+1)} = \infty$ by convention. Let us realize that such a permutation $(\cdot)$ need not be unique. Further, for a fixed input vector $\mathbf{x}$ and a fixed permutation $(\cdot)$ we shall use the denotation $E_{(i)} = \{(i), \dots, (n)\}$ for any $i \in [n]$, with the convention $E_{(n+1)} = \emptyset$. We shall denote by $\mathbf{1}_E$ the indicator function of a set $E \subseteq [0, \infty)$, i.e., $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \notin E$. In particular, $\mathbf{1}_\emptyset(x) = 0$ for each $x \in E$.

In the following, we list several examples of monotone measures or (additive) measures. The other examples of monotone measures can be found in Example 3.8 or Table 2.

Example 2.1 *Let us give examples of monotone measures on $2^{[n]}$.*

(i) *The counting measure:*

$$\#(B) = |B|.$$

(ii) *The power measure:*

$$\mu(B) = \left(\frac{|B|}{n}\right)^p, \quad p \in (0,\infty).$$

(iii) *The greatest monotone measure:*

$$\mu(B) = \begin{cases} 0, & \text{if } B = \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

(iv) *The weakest monotone measure:*

$$\mu(B) = \begin{cases} 1, & \text{if } B = [n], \\ 0, & \text{otherwise.} \end{cases}$$

3 The Choquet integral based on conditional aggregation operators, definition and examples

Let us introduce the necessary terms and definitions before the definition of the generalized Choquet integral.

Definition 3.1 (cf. [6, Definition 3.1]) A *conditional aggregation operator* with respect to a set $B \in 2^{[n]} \setminus \{\emptyset\}$ is a map $\mathsf{A}(\cdot|B)\colon [0,\infty)^{[n]} \to [0,\infty)$ satisfies the following conditions:

(i) $\mathsf{A}(\mathbf{x}|B) \le \mathsf{A}(\mathbf{y}|B)$ for any $\mathbf{x}, \mathbf{y} \in [0,\infty)^{[n]}$ such that $x_i \le y_i$ for any $i \in B$,

(ii) $\mathsf{A}(\mathbf{1}_{B^c}|B) = 0$.

For a better understanding of the previous definition, we present several conditional aggregation operators in the following example.

Example 3.2 *Let $\mathbf{x} \in [0,\infty)^{[n]}$, $B \in 2^{[n]} \setminus \{\emptyset\}$ and m be a monotone measure on $2^{[n]}$.*

(i) *The sum* $\mathsf{A}^{\mathrm{sum}}(\mathbf{x}|B) = \sum_{i \in B} x_i$.

(ii) *The maximum* $\mathsf{A}^{\max}(\mathbf{x}|B) = \max_{i \in B} x_i$, *and the minimum* $\mathsf{A}^{\min}(\mathbf{x}|B) = \min_{i \in B} x_i$.

(iii) *The p-mean* $\mathsf{A}^{p-\mathrm{mean}}(\mathbf{x}|B) = \left(\frac{1}{\#(B)} \cdot \sum_{i \in B} (x_i)^p\right)^{\frac{1}{p}}$, $p \in (0,\infty)$. *For $p = 1$ we get the arithmetic mean.*

(iv) $\mathsf{A}(\mathbf{x}|B) = \mathrm{J}(\mathbf{x}\mathbf{1}_B, m)$ *(the multiplication of vectors is meant by components), where* J *is an integral defined in [7, Definition 2.2]. Namely,*

(a) $\mathsf{A}^{\mathrm{Ch}_m}(\mathbf{x}|B) = \sum_{i=1}^{n} x_{(i)} \cdot \big(m(E_{(i)} \cap B) - m(E_{(i+1)} \cap B)\big) = \sum_{i=1}^{n} (x_{(i)} - x_{(i-1)}) \cdot m(E_{(i)} \cap B)$,

(b) $\mathsf{A}^{\mathrm{Su}_m}(\mathbf{x}|B) = \max_{i \in [n]} \big\{\min\{x_{(i)}, m(E_{(i)} \cap B)\}\big\}$,

(c) $\mathsf{A}^{\mathrm{Sh}_m}(\mathbf{x}|B) = \max_{i \in [n]} \big\{x_{(i)} \cdot m(E_{(i)} \cap B)\big\}$.

In the following, we shall provide the definition of the generalized survival function, see [6, Definition 4.1.]. Let us consider a collection $\mathscr{E} \subseteq 2^{[n]}$, $\emptyset, [n] \in \mathscr{E}$, and conditional aggregation operators on sets from $\mathscr{E}$ with the property $\mathsf{A}(\cdot|\emptyset) = 0$. The set of such aggregation operators we shall denote by

$$\mathscr{A} = \{\mathsf{A}(\cdot|E) : E \in \mathscr{E}\}$$

and we shall call it a *family of conditional aggregation operators*.

Definition 3.3 (cf. [6, Definition 4.1.]) Let $\mathbf{x} \in [0,\infty)^{[n]}$ and μ be a monotone measure on $\widehat{\mathscr{E}}$. The *generalized survival function* with respect to $\mathscr{A}$ is defined as

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \min\{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \leq \alpha,\, E \in \mathscr{E}\} \tag{1}$$

for any $\alpha \in [0,\infty)$.

Because of the fact, that for any $E \in \mathscr{E}$ it holds that $E^c \in \hat{\mathscr{E}}$ is a measurable set, the presented definition is correct. Moreover, the set $\{E \in \mathscr{E} : \mathsf{A}(\mathbf{x}|E) \leq \alpha\}$ is nonempty for all $\alpha \in [0,\infty)$, because $\mathsf{A}(\cdot|\emptyset) = 0$ by convention and $\emptyset \in \mathscr{E}$. The concept of generalized survival function is a generalization of the survival function concept, since

$$\begin{aligned}\mu(\{\mathbf{x} > \alpha\}) = \mu([n] \setminus \{i \in [n] : x_i \leq \alpha\}) &= \min\left\{\mu(E^c) : (\forall i \in E)\ x_i \leq \alpha,\, E \in 2^{[n]}\right\} \\ &= \min\left\{\mu(E^c) : \mathsf{A}^{\max}(\mathbf{x}|E) \leq \alpha,\, E \in 2^{[n]}\right\}, \end{aligned} \tag{2}$$

where the last expression is the generalized survival function with the conditional aggregation operator $\mathsf{A}^{\max}$ and collection $\mathscr{E} = 2^{[n]}$, see [6, Motivation problem 1]. The motivation for dealing with the generalized survival function are real life problems such as customer's decision making problems as well as insurance, see [6, Section 6], or modification of Knapsack problem, see [5], etc. In the following we present the definition of $\mathscr{A}$-Choquet integral as a special case of $\mathscr{A}$-Choquet-Stieltjes functional [6, Definition 5.4.] (it is enough to consider $\mathbf{M_f}$ being constant and $\varphi(x) = x$).

Definition 3.4 Let $\mathbf{x} \in [0,\infty)^{[n]}$, μ be a monotone measure on $\widehat{\mathscr{E}}$ and $\mathscr{A}$ be a family of conditional aggregation operators. The *Choquet integral of* $\mathbf{x}$ *with respect to* $\mathscr{A}$ *and* μ (*$\mathscr{A}$-Choquet integral* or *generalized Choquet integral*, for short) is defined as follows

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x}, \mu) = \int_{[0,\infty)} \mu_{\mathscr{A}}(\mathbf{x}, \alpha)\, \mathrm{d}\alpha.$$

Remark 3.5 *Let us remark that, in the introduction of this paper, we have assumed $[n] \in \mathscr{E}$ which is not necessary in general, compared with the original definition of generalized survival function [6, Definition 4.1]. Now, it turns out that this condition makes sense because it ensures the finite value of the corresponding Choquet integral, which is desirable for real applications and the software support for experiments. Indeed, if $[n] \notin \mathscr{E}$, then the generalized survival function does not need to acquire zero value resulting in the infinite value of the generalized Choquet integral. For more details see [6, Proposition 5.6].*

In the following, we summarize some properties of $\mathscr{A}$-Choquet integral. Because of them the functional given in Definition 3.4 may be called integral with respect to definition in [6, Definition 5.1](see [6, Proposition 5.6]). As we will mention later, the property (vi) of the following theorem is important for the experimental use of the generalized Choquet integral in image processing. The family of conditional aggregation operators $\mathscr{A} = \{\mathsf{A}(\cdot|E) : E \in \mathscr{E}\}$ is called *homogeneous of degree θ*, $\theta \in (0,\infty)$, if $\mathsf{A}(\cdot|E)$ is homogeneous of degree θ for any $E \in \mathscr{E}$, i.e. $\mathsf{A}(\lambda\mathbf{x}|E) = \lambda^{\theta}\mathsf{A}(\mathbf{x}|E)$ for any $\lambda \in (0,\infty)$ and $\mathbf{x} \in [0,\infty)^{[n]}$.

$\mu(E^c)$	0	0	0	0.5	0.2	0.6	0.7	1
E^c	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
E	$\{1,2,3\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{2\}$	$\{1\}$	$\emptyset$
$\mathsf{A}^{\mathrm{sum}}(\mathbf{x}\vert E)$	9	7	6	5	4	3	2	0
$\mathsf{A}^{\max}(\mathbf{x}\vert E)$	4	4	4	3	4	3	2	0
$\mathsf{A}^{2-\mathrm{mean}}(\mathbf{x}\vert E)$	3.11	3.54	3.16	2.55	4	3	2	0
$\mathsf{A}^{\mathrm{Ch}_{\#}}(\mathbf{x}\vert E)$	9	7	6	5	4	3	2	0
$\mathsf{A}^{\mathrm{Su}_{\#}}(\mathbf{x}\vert E)$	2	2	2	2	1	1	1	0
$\mathsf{A}^{\mathrm{Sh}_{\#}}(\mathbf{x}\vert E)$	6	6	4	4	4	3	2	0

Table 1: The table of crucial values for Example 3.8 with $\mathbf{x} = (2, 3, 4)$.

Theorem 3.6 *Let $\mathbf{x}, \mathbf{y} \in [0, \infty)^{[n]}$, μ, ν be a monotone measures on $\widehat{\mathscr{E}}$ and $\mathscr{A}$ be a family of conditional aggregation operators. The $\mathscr{A}$-Choquet integral has the following features:*

(i) *if $\mathbf{x} \le \mathbf{y}$, then $\mathrm{C}_{\mathscr{A}}(\mathbf{x}, \mu) \le \mathrm{C}_{\mathscr{A}}(\mathbf{y}, \mu)$,*

(ii) *if $\mu \le \nu$, then $\mathrm{C}_{\mathscr{A}}(\mathbf{x}, \mu) \le \mathrm{C}_{\mathscr{A}}(\mathbf{x}, \nu)$,*

(iii) *if $\mathscr{A}$ is homogeneous of degree θ, then for all $\lambda \in [0, \infty)$ it holds $\mathrm{C}_{\mathscr{A}}(\lambda\mathbf{x}, \mu) = \lambda^\theta \mathrm{C}_{\mathscr{A}}(\mathbf{x}, \mu)$,*

(iv) $\mathrm{C}_{\mathscr{A}}(\mathbf{0}, \mu) = 0$,

(v) $\mathrm{C}_{\mathscr{A}}(\mathbf{x}\mathbf{1}_B, \mu) = 0$ *whenever $\mu(B) = 0$,*

(vi) *if $\mathsf{A}^{\min}(\mathbf{x}|E) \le \mathsf{A}(\mathbf{x}|E) \le \mathsf{A}^{\max}(\mathbf{x}|E)$ for any $E \in \mathscr{E}$ and μ is the normalized monotone measure, then*

$$\min_{i\in[n]} x_i \le \mathrm{C}_{\mathscr{A}}(\mathbf{x}, \mu) \le \max_{i\in[n]} x_i$$

with $\mathscr{A} = \{\mathsf{A}(\mathbf{x}|E) : E \in \mathscr{E}\}$.

Proof. Properties (i), (ii), and (iv) follow from Definition 3.1, Definition 3.3, and Definition 3.4. The property (iii) follows from [6, Proposition 5.12] with φ being the identity. The part (v) follows from the fact that $\mathsf{A}(\mathbf{x}\mathbf{1}_B|B^c) = 0$, see [6, Proposition 3.3 (b)]. Then $\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = 0$ for all $\alpha \in [0, \infty)$, since $B^c \in \{E : \mathsf{A}(\mathbf{x}\mathbf{1}_B|E) \le \alpha,\ E \in \mathscr{E}\}$ for all $\alpha \in [0, \infty)$. Finally, (vi) follows from the fact that for any $E \in 2^{[n]} \setminus \{\emptyset\}$

$$\mathsf{A}^{\min}(\mathbf{x}|[n]) \le \mathsf{A}^{\min}(\mathbf{x}|E) \le \mathsf{A}(\mathbf{x}|E) \le \mathsf{A}^{\max}(\mathbf{x}|E) \le \mathsf{A}^{\max}(\mathbf{x}|[n]).$$

Then $\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \mu([n])$ for any $\alpha \in [0, \mathsf{A}^{\min}(\mathbf{x}|[n]))$ and $\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = 0$ for any $\alpha \ge \mathsf{A}^{\max}(\mathbf{x}|[n])$. Since the monotone measure is normalized and because of Definition 3.4 we get the result. □

From (2) it is immediately seen that for $\mathsf{A} = \mathsf{A}^{\max}$ and $\mathscr{E} = 2^{[n]}$, the generalized survival function shrinks to the standard one. Moreover, in [4] we have shown that also with a smaller collection of sets, we get the same result. The consequence of such a result is the fact that also the corresponding Choquet integrals are equal. Then, when we consider the conditional aggregation operator $\mathsf{A}^{\max}$ and the collection as the following proposition says (for a given fixed input vector), we get the standard Choquet integral.

Proposition 3.7 *(cf. [4, Lemma 3.2]) Let $\mathbf{x} \in [0, \infty)^{[n]}$, $\mathscr{E} \supseteq \{E^c_{(k+1)} : k \in \Psi_{\mathbf{x}}\}$ and μ be a monotone measure on $\widehat{\mathscr{E}}$. Then*

$$\mathrm{C}_{\mathscr{A}^{\max}}(\mathbf{x}, \mu) = \int_{[0,\infty)} \mu(\{\mathbf{x} > \alpha\})\, \mathrm{d}\alpha.$$

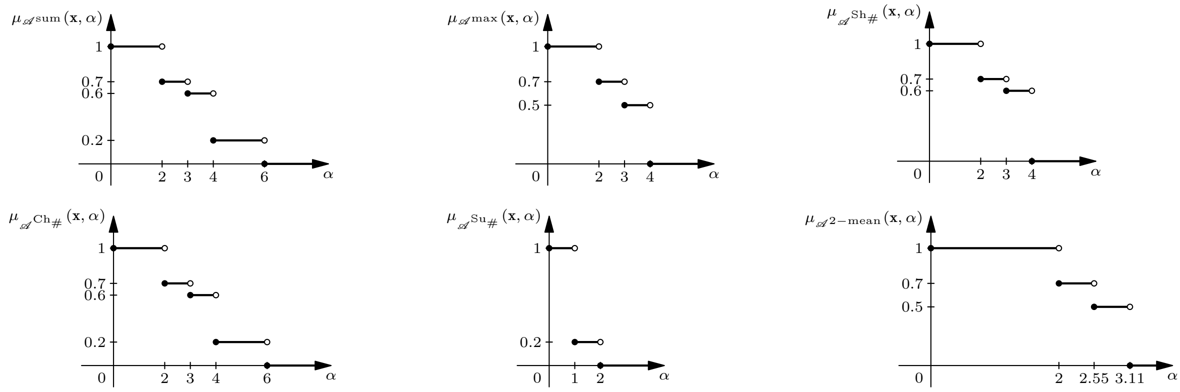


Figure 1: Graphs of generalized survival functions from Example 3.8

The other nontrivial sufficient and necessary conditions under which survival function and generalized survival function coincide can be found in paper [4], see Corollary 3.7, Corollary 3.11 or Theorem 3.17.

Example 3.8 *Let us consider $n = 3$, the input vector $\mathbf{x} = (2, 3, 4)$, $\mathscr{E} = 2^{[3]}$, the monotone measure μ with values given in Table 1 and the conditional aggregation operators $\mathsf{A}^{\mathrm{sum}}$, $\mathsf{A}^{\max}$, $\mathsf{A}^{2-\mathrm{mean}}$, $\mathsf{A}^{\mathrm{Ch}_\#}$, $\mathsf{A}^{\mathrm{Su}_\#}$, $\mathsf{A}^{\mathrm{Sh}_\#}$. The values of previously mentioned conditional aggregation operators are summarized in Table 1. Discussed survival functions take the form*

$$\begin{aligned}
\mu_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\alpha) &= \mathbf{1}_{[0,2)}(\alpha) + 0.7\cdot\mathbf{1}_{[2,3)}(\alpha) + 0.6\cdot\mathbf{1}_{[3,4)}(\alpha) + 0.2\cdot\mathbf{1}_{[4,6)}(\alpha),\\
\mu_{\mathscr{A}^{\max}}(\mathbf{x},\alpha) &= \mathbf{1}_{[0,2)}(\alpha) + 0.7\cdot\mathbf{1}_{[2,3)}(\alpha) + 0.5\cdot\mathbf{1}_{[3,4)}(\alpha),\\
\mu_{\mathscr{A}^{2-\mathrm{mean}}}(\mathbf{x},\alpha) &= \mathbf{1}_{[0,2)}(\alpha) + 0.7\cdot\mathbf{1}_{[2,2.55)}(\alpha) + 0.5\cdot\mathbf{1}_{[2.55,3.11)}(\alpha),\\
\mu_{\mathscr{A}^{\mathrm{Ch}_\#}}(\mathbf{x},\alpha) &= \mathbf{1}_{[0,2)}(\alpha) + 0.7\cdot\mathbf{1}_{[2,3)}(\alpha) + 0.6\cdot\mathbf{1}_{[3,4)}(\alpha) + 0.2\cdot\mathbf{1}_{[4,6)}(\alpha),\\
\mu_{\mathscr{A}^{\mathrm{Su}_\#}}(\mathbf{x},\alpha) &= \mathbf{1}_{[0,1)}(\alpha) + 0.2\cdot\mathbf{1}_{[1,2)}(\alpha),\\
\mu_{\mathscr{A}^{\mathrm{Sh}_\#}}(\mathbf{x},\alpha) &= \mathbf{1}_{[0,2)}(\alpha) + 0.7\cdot\mathbf{1}_{[2,3)}(\alpha) + 0.6\cdot\mathbf{1}_{[3,4)}(\alpha)
\end{aligned}$$

for $\alpha \in [0,\infty)$. Graphs of these functions can be seen in Figure 1. Generalized Choquet integrals with respect to above mentioned families of aggregation operators are

$$\begin{aligned}
&\mathrm{C}_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\mu) = 3.7, && \mathrm{C}_{\mathscr{A}^{\max}}(\mathbf{x},\mu) = 3.2, && \mathrm{C}_{\mathscr{A}^{2-\mathrm{mean}}}(\mathbf{x},\mu) \doteq 2.66,\\
&\mathrm{C}_{\mathscr{A}^{\mathrm{Ch}_\#}}(\mathbf{x},\mu) = 3.7, && \mathrm{C}_{\mathscr{A}^{\mathrm{Su}_\#}}(\mathbf{x},\mu) = 1.2, && \mathrm{C}_{\mathscr{A}^{\mathrm{Sh}_\#}}(\mathbf{x},\mu) = 3.3.
\end{aligned}$$

In the previous example we can see that $\mathrm{C}_{\mathscr{A}^{\mathrm{sum}}}(\mathbf{x},\mu) = \mathrm{C}_{\mathscr{A}^{\mathrm{Ch}_\#}}(\mathbf{x},\mu)$, which is not surprising. It is known that $\mathsf{A}^{\mathrm{Ch}_\#} = \mathsf{A}^{\mathrm{sum}}$. Indeed, the Choquet integral with respect to additive monotone measure, i.e. measure, is the Lebesgue integral, [39]. Moreover, the Lebesgue integral with respect to the counting measure is the sum.

4 Software implementation

It is clear that the calculation of the generalized survival function (Definition 3.3) based on conditional aggregation operators is significantly more complicated than the calculation of the survival function. In the most complicated case, assuming that the components of the input vector differ and the conditional aggregation operator takes different values on sets from $\mathscr{E} = 2^{[n]}$, we would have to perform 2^n calculation steps (compared to n steps in the survival function calculation), which is an exponential computational complexity. In addition, there is no small probability of making a mistake in the calculation.

For these reasons, it is advisable to automate the calculation of the generalized survival function or generalized Choquet integral, respectively, especially if a larger amount of calculations is involved. From the user's point of view and considering the optimality of the calculations, it is advisable to distinguish the number of components of the input vector.

Software implementation for small inputs. If the input vector is of small order, it is enough to use the following intuitive and user-friendly algorithms to automate the calculations: Let $u_1, \ldots, u_p$, $p \in \mathbb{N}$, be a decreasing sequence of values that the generalized survival function achieves and $\alpha_1, \ldots, \alpha_p$ be an increasing sequence of limiting points of intervals on which the value of generalized survival function is achieved. Then the generalized survival function may be expressed in the form

$$\mu_{\mathscr{A}}(\mathbf{x}, \alpha) = \sum_{i=1}^{p} u_i \mathbf{1}_{[\alpha_{i-1}, \alpha_i)}(\alpha) \tag{3}$$

for any $\alpha \in [0, \infty)$, with $\alpha_0 = 0$. Following Algorithm 1 and Algorithm 2 provide u_i and α_i, $i = 1, 2, \ldots, p$, $p \in \mathbb{N}$, computation. Note that Algorithm 1 was created by modifying the algorithm given in [8].

Data: $\mathscr{E}$, μ, $\mathbf{x}$, $\mathsf{A}(\mathbf{x}|F)$
Result: p, $\alpha_1, \ldots, \alpha_p$, $u_1, \ldots, u_p$
$u_1 := \mu([n])$, $\alpha_1 := \mathsf{A}(\mathbf{x}|\emptyset)$, $i := 1$;
do
  $\beta := \min\{\mathsf{A}(\mathbf{x}|E) > \alpha_i,\ E \in \mathscr{E}\}$;
  $v := \min\{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) < \beta,\ E \in \mathscr{E}\}$;
  if $(v = 0)$ **then**
    break;
  end
  if $(v < u_i)$ **then**
    $i := i + 1$;
    $u_i := v$;
  end
  $\alpha_i := \beta$;
while $(\beta < \mathsf{A}(\mathbf{x}|[n]))$;
$p := i$;

Algorithm 1: u_i and α_i computation via aggregation operator

Data: $\mathscr{E}$, μ, $\mathbf{x}$, $\mathsf{A}(\mathbf{x}|F)$
Result: p, $\alpha_1, \ldots, \alpha_p$, $u_1, \ldots, u_p$
$u_1 := \mu([n])$, $\alpha_1 := 0$, $i := 1$;
do
  $v := \max\{\mu(E^c) < u_i,\ E \in \mathscr{E}\}$;
  $\beta := \min\{\mathsf{A}(\mathbf{x}|E) : \mu(E^c) = v\}$;
  if $(\beta > \alpha_i)$ **then**
    $\alpha_i := \beta$;
    $i := i + 1$;
  end
  $u_i := v$;
  $\alpha_i := \beta$;
while $(u_i > 0)$;
$p := i - 1$;

Algorithm 2: u_i and α_i computation via monotone measure

The principle of Algorithm 1 is based on searching the values of the conditional aggregation operator (interval creation process) and finding the corresponding monotone measure that is obtained. However, the opposite approach is also possible, i.e. search the values of the monotone measure and determine the corresponding interval (values of the conditional aggregation operator) on which these values are acquired – see Algorithm 2.

In the following example we describe calculation using Algorithm 1. Algorithm 2 can be used for the calculation of the generalized survival function by an analogous procedure. One can verify, that using the second algorithm we get the same result.

Example 4.1 *Let us consider Example 3.8. We describe the calculation of the generalized survival function by the Algorithm 1 for conditional aggregation operator* $\mathsf{A}^{\mathrm{Sh}_\#}$.

(i) *According to the algorithm,* $u_1 := \mu(\{1, 2, 3\}) = 1$*,* $\alpha_1 := \mathsf{A}(\mathbf{x}|\emptyset) = 0$*,* $i := 1$*. Next,* $\beta := \min\{2, 3, 4, 6\} = 2$*,* $v = \min\{1\} = 1$*. Since* $v \not< u_1$*,* $\alpha_1 := \beta = 2$*.*

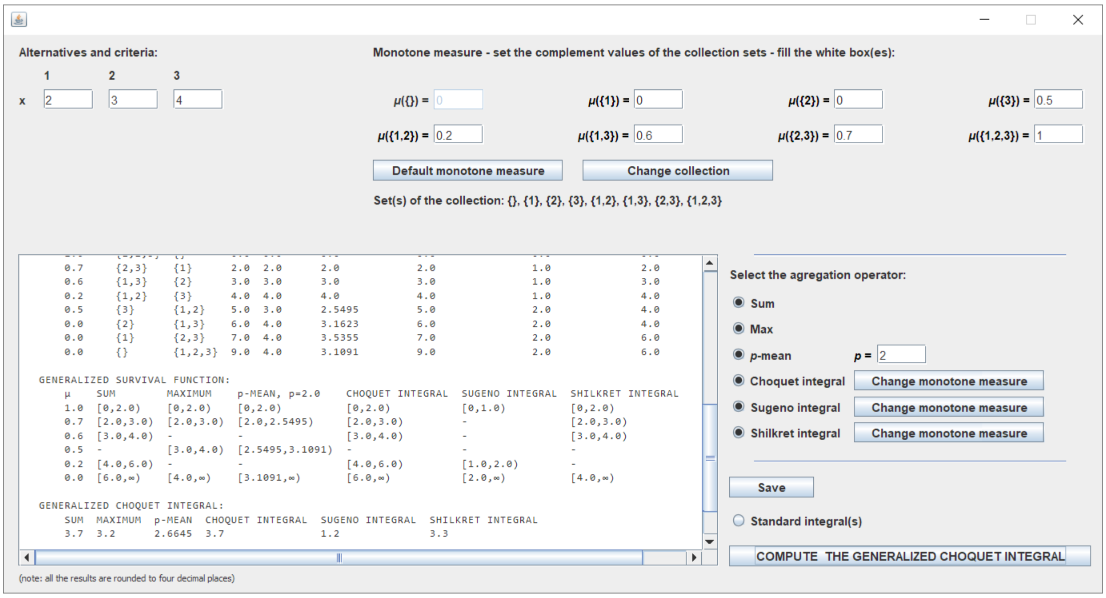


Figure 2: The main window of application – computation of the generalized Choquet integral

(ii) *In next step,* $\beta := \min\{3, 4, 6\} = 3$*,* $v := \min\{0.7, 1\} = 0.7$*. Now* $v < u_1$*, thus* $i := i+1 = 2$*,* $u_2 := v = 0.7$ *and* $\alpha_2 := \beta = 3$*. Then*

$$\mu_{\mathscr{A}^{\mathrm{Sh}_\#}}(\mathbf{x}, \alpha) = u_1 = 1 \textit{ for any } \alpha \in [\alpha_0, \alpha_1) = [0, 2).$$

(iii) *Again,* $\beta := \min\{4, 6\} = 4$*,* $v := \min\{0.6, 0.7, 1\} = 0.6$*. Since* $v < u_2$*, we put* $i := i+1 = 3$*,* $u_3 := v = 0.6$*,* $\alpha_3 := \beta = 4$*. Then*

$$\mu_{\mathscr{A}^{\mathrm{Sh}_\#}}(\mathbf{x}, \alpha) = u_2 = 0.7 \textit{ for any } \alpha \in [\alpha_1, \alpha_2) = [2, 3).$$

(iv) *Next,* $\beta := \min\{6\} = 6$*,* $v := \min\{0, 0.2, 0.5, 0.6, 0.7, 1\} = 0$*. Whereas* $v = 0$*, cycle ends and*

$$\mu_{\mathscr{A}^{\mathrm{Sh}_\#}}(\mathbf{x}, \alpha) = u_3 = 0.6 \textit{ for any } \alpha \in [\alpha_2, \alpha_3) = [3, 4).$$

By algorithm, $p := i = 3$*.*

Because of the first experiment that will be presented in the next section and verifying calculations, we create a software application that automates the calculation of the generalized survival function and the generalized Choquet integral. This software we called AGROP (abbr. of aggregation operators) and it can be downloaded with its manual from the repository

`https://github.com/Stanislav-B/CAO_Choquet_experimental_study.git`

from the *Decision making* folder. Working with this software is simple and is described in detail together with graphic elements in the mentioned manual. In AGROP software is implemented (among others) $\mathscr{A}$-Choquet integral with respect to some conditional aggregation operators from Example 3.2, and measures used in Example 2.1. The calculation of the generalized survival function and the generalized Choquet integral is running according to Algorithm 1.

In general, it can be entered up to 9 input vectors (a separate calculation will be performed for each vector) of order up to four (four-element basic set with up to 2^4 subsets). The number of parameters is limited for the user because of practical reasons. If more than 1 vector is selected at the input, the software give to output, in addition to basic information (input values) and calculations (intermediate calculation steps, generalized survival function, and generalized

Choquet integral for each input vector), the resulting descending order of vectors (relation of preference) with respect to the values of the generalized Choquet integral. This is especially beneficial for the use of this software in decision making processes, as we will show in the experiment in the next section. Figure 2 shows the main window of the application. For more information on using the software, see the manual of software application.

We note, given Proposition 3.7 and the paragraph preceding this proposition, it must hold that when choosing in the software application the conditional aggregation operator $\mathsf{A}^{\max}$ and the collection as the proposition says, we get the standard Choquet integral.[1]

Software implementation for big inputs. As we mentioned, from the Definition 3.3 of generalized survival function it is clear that for $\mathscr{E} = 2^{[n]}$ the input n-component vector is transformed into 2^n values, which means a large computational complexity for large $n \in \mathbb{N}$.

In paper [5], we dealt with calculation formulas for the generalized survival function, which differ in construction from the approach described in the previous case. The main difference is the initial arrangement of the values of the conditional aggregation operator as well as the values of the monotone measure, and adapting the calculation according to the number of their different values. This ensures computational efficiency. Using various, perhaps less intuitive, but necessary (from the point of view of optimality) maps we constructed formulas and algorithms for calculation of the generalized Choquet integral. Since we devoted the entire paper to these formulas, we will not mention them in depth in this text. In above mentioned repository in *Image processing* folder it is possible to find a software implementation for the second experiment – generalized Choquet integral in image processing. In this experiment we consider 8-component vectors and therefore a monotone measure defined on $2^8 = 256$ sets, thus a big inputs. The repository contains source Java codes (due to the possibility of entering big inputs), in which the mentioned algorithms from [5] are implemented.

5 Experiment 1 – the generalized Choquet integral in decision making process

In this subsection we shall discuss the problem of the evaluation of students in high school with respect to three subjects: mathematics, physics, and literature (criteria) proposed in [19]. Usually, the evaluation is done by a simple weighted sum, whose weights are the coefficients of the importance of the different subjects. However, this aggregation does not work very well mainly because the weighted mean does not reflect interactions among criteria. Therefore the fuzzy integrals are better. In [19] the author showed that the evaluation problem may be solved by the Choquet integral. In this part, we compare the results obtained by the Choquet integral with our approach to the generalized Choquet integral.

Let us suppose the same exemplary situation as in [19]. The school is more scientifically than literary oriented.

- The level of importance of mathematics, physics, and literature is in the ratio 3 : 3 : 2, respectively.
- Let the score vector $\mathbf{x} = (x_1, x_2, x_3)$ represents the score of a given student from mathematics, physics, and literature, respectively. The total score is 20 points for each subject.
- Let us consider two monotone measures, i.e. μ and ν, see Table 2.
- Let us consider the generalized $\mathscr{A}^{\text{sum}}$-Choquet integral with the collection $\mathscr{E} = 2^{[3]}$.

[1] In the software application also the standard integrals: the Choquet integral, the Sugeno integral, the Shilkret integral are adopted, since they are conditional aggregation operators, too. For computing only standard integrals one should choose the radio button “Standard integral(s)” and insert the values of monotone measure m.

E	$\emptyset$	$\{\mathrm{M}\}$	$\{\mathrm{P}\}$	$\{\mathrm{L}\}$	$\{\mathrm{M},\mathrm{P}\}$	$\{\mathrm{M},\mathrm{L}\}$	$\{\mathrm{P},\mathrm{L}\}$	$\{\mathrm{M},\mathrm{P},\mathrm{L}\}$
$\mu(E)$	0	0.45	0.45	0.3	0.5	0.9	0.9	1
$\nu(E)$	0	0.45	0.45	0.3	0.88	0.78	0.78	1

Table 2: The monotone measures used in the first experiment

We choose the family of conditional aggregation operators $\mathscr{A}^{\mathrm{sum}}$ because it is natural to add up (with respect to conditional sets) the individual scores of students. Let us mention that the monotone measure μ is the same as in the paper [19]. However, the question arises: Is this monotone measure well set? Respective, how to find out if the values of the monotone measure are not chosen in such a way that we achieve the result we want? In the papers, concepts such as iteration index, Banzhaff power [3], or Shapley value [36] (which we also chose due to its properties) are often used to solve this question. The formula for calculating the Shapley value

$$\varphi_i(m) = \sum_{S\subseteq[n]\setminus\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(m(S\cup\{i\}) - m(S)\right),$$

where $i \in [n]$ in our case denotes the subject (mathematics, physics, literature) and m is a monotone measure, expresses the fact that the monotone measure of the given subject does not depend only on the value of the singleton, but also on the values of the monotone measure of those sets that contain the given subject as a subset. We found that the monotone measure μ does not acquire the Shapley value, see Table 3, because the ratio of Shapley values for criteria M, P, L, respectively is not in the form $3:3:2$, so we decided to modify μ.

m	$\varphi_{\mathrm{M}}(m)$	$\varphi_{\mathrm{P}}(m)$	$\varphi_{\mathrm{L}}(m)$
μ	0.2917	0.2917	0.4167
ν	0.375	0.375	0.25

Table 3: The Shapley values of criteria

The monotone measure ν is a little bit modified by using the Shapley value, but still satisfies all interactions rules, e.g. $\nu(\{\mathrm{M},\mathrm{P}\}) < \nu(\{\mathrm{M}\})+\nu(\{\mathrm{P}\})$ (mathematics and physics are redundant, since, usually, students good at mathematics are also good at physics) and $\nu(\{\mathrm{M},\mathrm{L}\}) > \nu(\{\mathrm{M}\})+\nu(\{\mathrm{L}\})$, $\nu(\{\mathrm{P},\mathrm{L}\}) > \nu(\{\mathrm{P}\})+\nu(\{\mathrm{L}\})$ (since we have to favour students equally good at scientific subjects and literature what is rather uncommon). Moreover, the values of ν on singletons are in the ratio $3:3:2$. However, in the following table, see Table 4, we can see that the Choquet integral with respect to ν is not able to reorder the student as desired while the generalized Choquet integral is. Let us remark that the desirable result is (as the author states in [19]) to prefer student C before A because C has no weak point, he is equally enough good in each subject while A has a severe weakness in literature.

student	score vector	arithmetic mean	weighted arithmetic mean	Choquet integral	$\mathscr{A}^{\mathrm{sum}}$-Choquet integral
A	$(18, 16, 10)$	14.67	15.25	16.18	29.68
B	$(10, 12, 18)$	13.33	12.75	13.36	24.76
C	$(14, 15, 15)$	14.67	14.63	14.78	30.2
		$\mathrm{A}\sim\mathrm{C}\succ\mathrm{B}$	$\mathrm{A}\succ\mathrm{C}\succ\mathrm{B}$	$\mathrm{A}\succ\mathrm{C}\succ\mathrm{B}$	$\mathrm{C}\succ\mathrm{A}\succ\mathrm{B}$

Table 4: The evaluation of students scores with respect to different methods

As we can see small changes of μ with preserving all interaction rules lead to breaking the desired ordering. In the following, we try to do a deeper experimental study. We shall compare the Choquet integral and the generalized Choquet integrals with respect to $\mathscr{A}^{\mathrm{sum}}$ and $\mathscr{A}^{\mathrm{Ch}_{(\cdot)}}$ (generalized Choquet integral with standard Choquet integral as conditional aggregation operator with the same monotone measure) on couples of students. We shall choose the couples of students where we can judge almost unambiguously what should be the expected preference order and we shall check how many times both methods meet this preference. Let us do this experiment with respect to both measures μ and ν.

Let us present here several couples of students and a short explanation of which one should be considered to be better, see Table 5.

students and their score vectors		preference order
S_1: $(0,10,20)$	S_2: $(10,10,11)$	$\mathrm{S}_1 \prec \mathrm{S}_2$
S_3: $(0,0,20)$	S_4: $(5,5,0)$	$\mathrm{S}_3 \prec \mathrm{S}_4$
S_5: $(6,7,20)$	S_6: $(10,10,11)$	$\mathrm{S}_5 \prec \mathrm{S}_6$
S_7: $(3,6,15)$	S_8: $(3,9,9)$	$\mathrm{S}_7 \prec \mathrm{S}_8$

Table 5: The preference order of some students scores

- Students S_1 and S_2 have almost the same sum score. Although the student S_1 is much better in literature (full score) than S_2 (11 points), because of weakness of S_1 in mathematics (0 points) and equilibrated points of student S_2, $\mathrm{S}_2 \succ \mathrm{S}_1$.
- Although S_3 obtained two times more points than S_4, student S_3 has strong weaknesses in profile subjects, $\mathrm{S}_3 \prec \mathrm{S}_4$.
- Again, although S_5 has a greater sum score than S_6 (by two points), S_5 achieved this record because of high score from literature and from profile subjects he has fewer points that S_6. So, for scientific oriented school $\mathrm{S}_5 \prec \mathrm{S}_6$. Similar argumentation holds for S_7, S_8.

Having several similar couples of students as mentioned above, let us calculate the corresponding Choquet and the generalized Choquet integrals with respect to the conditional aggregation operators $\mathsf{A}^{\mathrm{sum}}$, $\mathsf{A}^{\mathrm{Ch}_{(\cdot)}}$ (where under $(\cdot)$ we mean the corresponding monotone measure μ or ν) and $\mathscr{E} = 2^{[3]}$ with respect to two monotone measures μ, ν, see Table 2. In Table 6 by pictogram ✓, resp. ✗ we denote whether the preference matches the expected preference.

We can see that all presented couples except the last one were ordered as it was desired with the $\mathscr{A}^{\mathrm{sum}}$-Choquet integral with respect to monotone measure ν. In the last case, we have stated that student with score vector $(18,0,18)$ should be worse than $(9,9,6)$, because he has strong weaknesses in the profile subject. This very specific and extreme case no method could take into account. However, in real life evaluation, a student with an almost full score from mathematics and 0 score from physics is less probable. In general, the results, regardless of which monotone measure we are considering, are better for the generalized Choquet integral ($\mathscr{A}^{\mathrm{sum}}$ an also $\mathscr{A}^{\mathrm{Ch}_{(\cdot)}}$). Also, regardless of which integral we are considering the results are better for monotone measure ν.

In general, from the table, we can see that regardless of which monotone measure we consider (μ or ν) in both cases the generalized Choquet integral based on the sum conditional aggregation operator $\mathsf{A}^{\mathrm{sum}}$, with the collection $\mathscr{E} = 2^{[3]}$ gives the best results. So, it seems that the generalized Choquet integral takes into account outliers better, e.g. the student with score vector $(0,10,20)$ is according to the Choquet integral w.r.t. ν on the 4th place in the ranking, according to generalized Choquet integral it is ranking lower (15th place), see Table A.3 or the reordering of students for each method separately, see Table A.1, A.2 in Appendix. Results dealing with the

Choquet integral as the conditional aggregation operator instead of the sum as the conditional aggregation operator are very similar.

input		expected result	real result			
			monotone measure	Choquet integral	$\mathscr{A}^{\mathrm{sum}}$-Choquet integral	$\mathscr{A}^{\mathrm{Ch}(\cdot)}$-Choquet integral
$(0,0,20)$	$(8,7,4)$	$<$	μ	$6 > 5.95$ ✗	$6 < 10.5$ ✓	$1.8 < 3.18$ ✓
20	19		ν	$6 < 7.09$ ✓	$6 < 12.76$ ✓	$1.8 < 5.05$ ✓
$(0,0,20)$	$(5,5,0)$	$<$	μ	$6 > 2.5$ ✗	$6 > 4.75$ ✗	$1.8 > 1.24$ ✗
20	10		ν	$6 > 4.4$ ✗	$6 < 6.65$ ✓	$1.8 < 2.95$ ✓
$(0,0,20)$	$(10,10,0)$	$<$	μ	$6 > 5$ ✗	$6 < 9.5$ ✓	$1.8 < 2.48$ ✓
20	20		ν	$6 < 8.8$ ✓	$6 < 13.3$ ✓	$1.8 < 5.9$ ✓
$(0,10,20)$	$(10,10,11)$	$<$	μ	$12 > 10.3$ ✗	$15 < 18.7$ ✓	$6.3 > 5.74$ ✗
30	31		ν	$10.8 > 10.3$ ✗	$13.8 < 21.1$ ✓	$5.76 < 7.93$ ✓
$(7,7,18)$	$(10,10,11)$	$<$	μ	$10.3 = 10.3$ ✗	$18.7 = 18.7$ ✗	$5.51 < 5.74$ ✓
32	31		ν	$10.3 = 10.3$ ✗	$17.86 < 21.1$ ✓	$5.46 < 7.93$ ✓
$(8,8,17)$	$(10,10,11)$	$<$	μ	$10.7 > 10.3$ ✗	$20.3 > 18.7$ ✗	$5.97 > 5.74$ ✗
33	31		ν	$10.7 > 10.3$ ✗	$19.34 < 21.1$ ✓	$5.92 < 7.93$ ✓
$(6,7,20)$	$(10,10,11)$	$<$	μ	$10.8 > 10.3$ ✗	$18.3 < 18.7$ ✓	$5.58 < 5.74$ ✓
33	31		ν	$10.68 > 10.3$ ✗	$17.46 < 21.1$ ✓	$5.49 < 7.93$ ✓
$(9,10,9)$	$(9,9,10)$	$>$	μ	$9.45 > 9.3$ ✓	$16.65 < 16.9$ ✗	$5.18 < 5.19$ ✗
28	28		ν	$9.45 > 9.3$ ✓	$19.17 > 19.02$ ✓	$7.14 < 7.16$ ✗
$(9,10,10)$	$(9,9,12)$	$>$	μ	$9.9 = 9.9$ ✗	$17.4 < 18.3$ ✗	$5.46 < 5.67$ ✗
29	30		ν	$9.78 < 9.9$ ✗	$19.8 > 19.62$ ✓	$7.37 < 7.61$ ✗
$(10,0,18)$	$(9,9,6)$	$<$	μ	$11.4 > 7.5$ ✗	$14.4 > 13.65$ ✗	$6.12 > 4.05$ ✗
28	24		ν	$10.2 > 8.64$ ✗	$13.2 < 16.47$ ✓	$5.22 < 6.39$ ✓
$(9,0,18)$	$(9,9,6)$	$<$	μ	$10.8 > 7.5$ ✗	$13.5 < 13.65$ ✓	$5.67 > 4.05$ ✗
27	24		ν	$9.72 > 8.64$ ✗	$12.42 < 16.47$ ✓	$4.86 < 6.39$ ✓
$(20,13,17)$	$(18,18,12)$	$<$	μ	$17.95 > 15$ ✗	$29.55 > 27.3$ ✗	$9.87 > 8.1$ ✗
50	48		ν	$17.47 > 17.28$ ✗	$32.71 < 32.94$ ✓	$12.37 < 12.78$ ✓
$(11,9,5)$	$(8,12,7)$	$<$	μ	$7.9 < 9.3$ ✓	$13.7 < 15.35$ ✓	$4.2 < 5$ ✓
25	27		ν	$9.42 < 9.68$ ✓	$16.62 < 17.69$ ✓	$6.62 < 6.79$ ✓
$(3,6,15)$	$(3,9,9)$	$<$	μ	$8.4 = 8.4$ ✗	$12.9 > 12.6$ ✗	$4.37 < 4.58$ ✓
24	21		ν	$8.04 > 7.68$ ✗	$12.18 < 12.72$ ✓	$4.62 < 4.77$ ✓
$(18,0,18)$	$(9,9,6)$	$<$	μ	$16.2 > 7.5$ ✗	$21.6 > 13.65$ ✗	$8.51 > 4.05$ ✗
36	24		ν	$14.04 > 8.64$ ✗	$19.44 > 16.47$ ✗	$7.21 > 6.39$ ✗

Table 6: Couples of students with different score vectors

6 Experiment 2 – the generalized Choquet integral in image processing

The subject of the second experiment is the image edge detection. The term *edge* is not exactly defined in the literature, but under it we mean features or contours of the objects forming the image. In the paper [1], the authors analyzed and divided the known approaches to edge detection into four categories: gradient-based methods, region-based segmentation methods, methods based on machine learning, and fuzzy-logic-based methods. Of these, gradient-based methods are perhaps the most well-known, such as Prewit [34] and Sobel [38] edge detectors, Marr-Hildreth Laplacian-based edge detection method [37], Canny edge detectors [10] and others. Prewitt and Sobel edge detectors find the edges by approximating the gradient magnitude of the image. The Canny method computes the gradient of the input image using the derivative of the Gaussian filter. These approaches have certain advantages and disadvantages. For example, the Sobel operator finds more edges or makes edges more visible as compared to the Prewitt Operator. The Sobel edge detection method cannot produce smooth and thin edges compared

(a) Input image (b) (SM1) (c) (SM2) (d) (SM3) (e) (SM4)

Figure 3: Image 118035 from BSDS500 database, together with its (SM1) – (SM4) smoothing

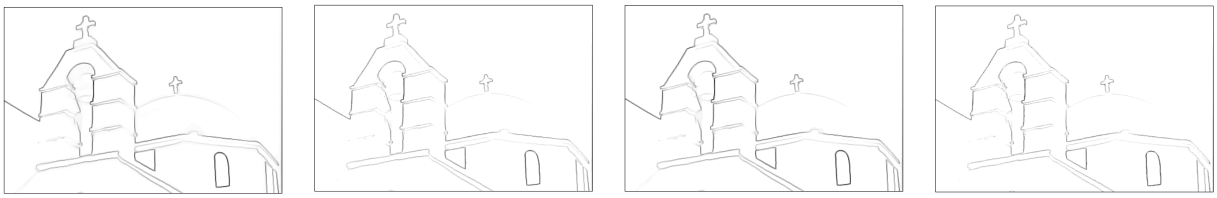
to the Canny method. But like other methods, Sobel and Canny methods are very sensitive to noise pixels. With the advent of convolution neural networks, edge detection has also developed quickly in this direction [26] in an attempt to simulate neural processes in the human visual system.

Just the idea of connecting human visual system with fuzzy-logic-based methods of edge detection became the basis of paper [31]. For edge detection, they used various modifications of the Choquet integral incorporated into the simulation of biological processes in the human visual system. It is a set of organs that ensure the reception, transmission, and processing of information brought by a light stimulus into a complex of nerve irritations, the result of which is visual perception. Ganglion cells in the retina itself are known to use organized cellular structures to search for primary image elements. Specifically, humans have receptive fields on the retina of the eye, whose information is further combined in the visual cortex to create features, contours, etc. As stated in [31], this nontrivial biological discovery had a huge impact on computer vision in the early 1980s. Computer vision is a scheme that consists of four phases: conditioning, feature extraction, blending, and scaling of the image.

In the following, we build on the ideas, procedures, and part of the software codes presented in paper [31] using the computer vision scheme, however with four modifications:

- In the feature extraction phase we we do not arrange the values of the vector in nondecreasing order. This preserves the information about the positions of pixels.
- Inspired by the Sobel operator, we assign a weight to each pixel (to components of the vector in the feature extraction phase).
- In the blending phase we blend feature using $\mathscr{A}$-Choquet integral with three families of the conditional aggregation operators, and standard Choquet integral.
- We modify the monotone measure, and create necessary software codes.

This approach is suitable because the (generalized) Choquet integral can, analogously to e.g. the Laplacian kernel used in classical methods, consider relations or the interactions between elements (pixels), but in a much more sophisticated way, which is the significance of this approach. Before describing the phases of the computer vision scheme in more detail, let us introduce the basic terms. By *image* we mean the function $D\colon R \times C \to L$, where $R = \{1, \ldots, r\}$ represent rows, and $C = \{1, \ldots, c\}$ columns of pixels forming the image, $r, c \in \mathbb{N}$. Let L represent the set of pixel colours. For a colour image $L = \{0, \ldots, 255\}^3$ (RGB scale, where 0 represents the black and 255 the white colour), for a grayscale image $L = \{0, \ldots, 255\}$ (components of the RGB scale have the same value), and for a binary image $L = \{0, 1\}$. Let each pixel has coordinates $[x, y]$, $x \in R$, $y \in C$, and colour $D([x, y]) \in L$. In the experiment, we use the Berkeley Segmentation Dataset and Benchmark (BSDS500), see [2]. This database contains 200 images together with their over 1000 ground truth labellings. We use truth labelling for the final statistical evaluation of the agreement of edge detection with detections described by humans. Following [31], let us consider image 118035 from BSDS500, see Figure 6 (left), as the main input image.



(a) (standard) Choq. int. (b) $\mathscr{A}^{\mathrm{Ch}_m}$-Choq. int. (c) $\mathscr{A}^{\mathrm{maxMin}}$-Choq. int. (d) $\mathscr{A}^{1-\mathrm{mean}}$-Choq. int.

Figure 4: Blending phase of input image smoothed by (SM2)

Conditioning This phase aims at removing additional information or noise not suitable for the edge detection process. It helps to remove spurious artefacts in image acquisition, as well as to reduce the effect of textures in edge detection. In the first, the input image is converted to grayscale. This preserves the edges of the image, and instead of three components of the RGB scale, it is enough to consider one representative value. Following [31], for reduce of noise we use two types of image smoothing – Gaussian smoothing and Gravitational smoothing [30], now with following parameters:

(SM1) Gaussian smoothing with $\sigma = 2$, i.e. 5×5 convolution kernel,

(SM2) Gravitational smoothing with $G = 0.05$, $cF = 20$, $t = 30$,

(SM3) Gravitational smoothing with $G = 0.05$, $cF = 50$, $t = 40$,

(SM4) Gravitational smoothing with $G = 0.05$, $cF = 70$, $t = 30$.

The main advantage of Gravitation smoothing, unlike Gaussian smoothing, which works with pixels regardless of their colour in the same way, is the fact that it takes these colours into account, and therefore a more correct output can be expected in the end. The input image in grayscale and its (SM1) – (SM4) smoothing can be seen in the Figure 3.

Feature extraction The aim is to gather simplistic edge cues at each pixel. In other words, to collect information about the occurrence of an edge. For each pixel $[x, y] \in R \times C$ of the image D, we create 8-component vector $\mathbf{x} = (x_1, \ldots, x_8) \in [0, \infty)^{[8]}$ by considering the absolute values of the differences this pixel and its neighbours, i.e.

$$x_k = |D(x, y) - D(x + i, y + j)|$$

for $((i, j))_{k=1}^{8} = ((-1, -1), (0, -1), (1, -1), (-1, 0), (1, 0), (-1, 1), (0, 1), (1, 1))$. In this way the information about the edges is abstracted. If two pixels are approximately the same colour (no edge present), the difference is close to 0, which represents black. However, if two pixels have a different colour (edge is present), the difference value is close to 255 (white colour). The image is transformed into 8 layers – for a given pixel position, each component of the vector $\mathbf{x}$ represents one layer. It is an adaptation of the layers of the visual cortex in the human visual system.

Blending The n layers from the previous phase we aggregate into one resulting layer. This is the phase in which we experimentally use the generalized Choquet integral. For aggregating the vector $\mathbf{x}$ for each position in the image from the previous phase we use

(i) the (standard) Choquet integral.

(ii) the $\mathscr{A}$-Choquet integrals with the following families of conditional aggregation operators:

- $\mathscr{A}^{1-\mathrm{mean}}$, see Example 3.2,
- $\mathscr{A}^{\mathrm{Ch}_m}$, where for $\mathsf{A}^{\mathrm{Ch}_m}(\cdot|E)$, $E \in \mathscr{E}$, we use the capacity m on $2^{[n]}$ such that $m(C) = 1$ for each $C \supseteq E$, and $m(C) = \left(\frac{|C|}{|E|}\right)^q$ otherwise. We consider $q = 0.9$.

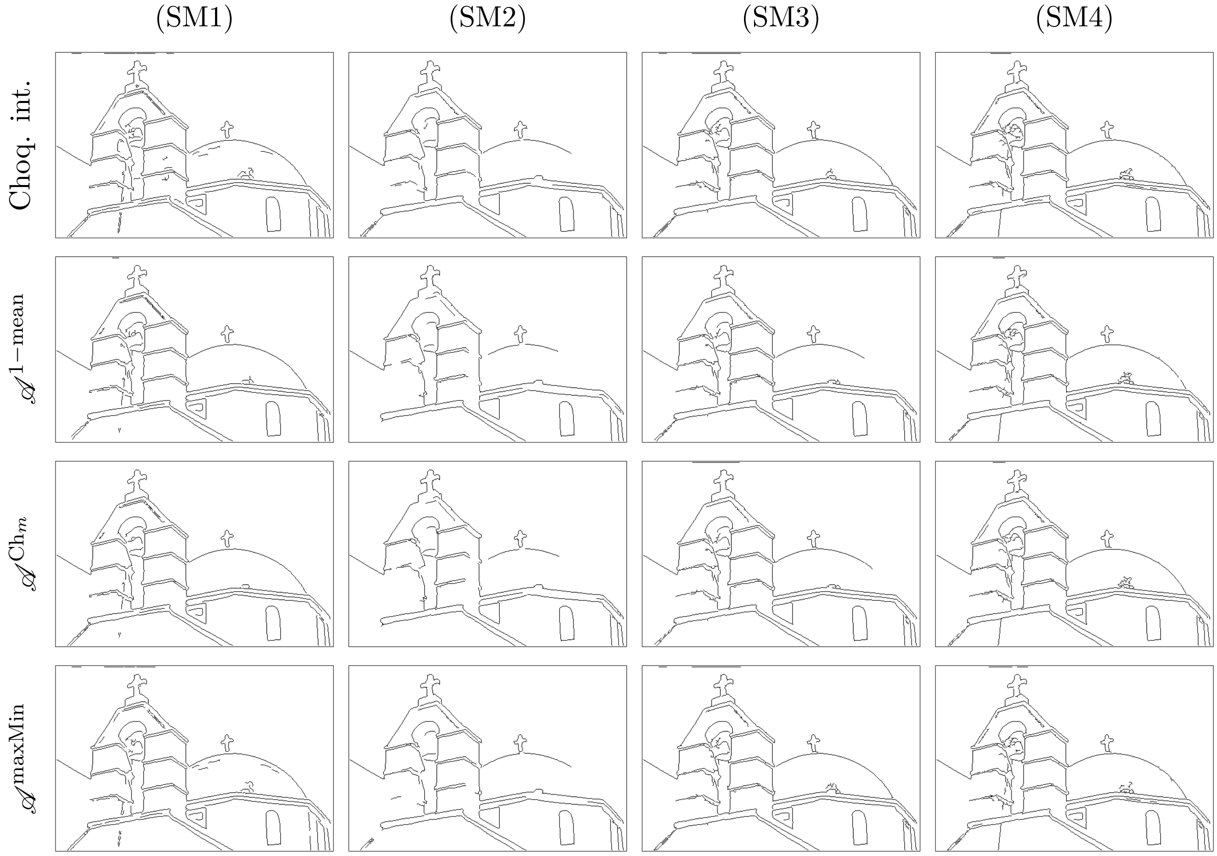


Figure 5: Scaling phase – detected edges

- $\mathscr{A}^{\text{maxMin}} = \left\{ \frac{\mathsf{A}^{\max}(\cdot|E)+\mathsf{A}^{\min}(\cdot|E)}{2} : E \in \mathscr{E} \right\}$

  all with $\mathscr{E} = 2^{[n]}$. This collection ensures that all possible combinations of differences x_k, $k \in [n]$, are aggregated. This gives us more information about the occurrence of an edge. The choice of these families of conditional aggregation operators is natural. $\mathscr{A}^{1-\text{mean}}$ returns the average of the aggregated colour values, $\mathscr{A}^{\text{maxMin}}$ returns the average of the maximum and minimum values, and $\mathscr{A}^{\text{Ch}_m}$ detects edges using a standard Choquet integral like the conditional aggregation operator.

We calculate the standard and the $\mathscr{A}$-Choquet integrals with respect to following (modified) monotone measure

$$\mu(B) = \left(\frac{|B|}{n}\right)^p \cdot \left(\frac{1}{|B|}\sum_{i \in B} w_i\right) \cdot k,$$

where $B \in \widehat{\mathscr{E}}$, $p \in (0, \infty)$, vector $(w_1, \dots, w_n)$ express weights assigned to components of the aggregated vector $\mathbf{x}$, and the coefficient $k = \frac{n}{\sum_{i=1}^{n} w_i}$ ensures that μ is the capacity (normalized monotone measure). Whereas $\mathbf{x} \in [0, \infty)^{[8]}$, we have $n = 8$. Let us use parameter $p = 0.9$, and the following vector of weights

$$(w_1, \dots, w_8) = \left(\tfrac{1}{\sqrt{2}}, 1, \tfrac{1}{\sqrt{2}}, 1, 1, \tfrac{1}{\sqrt{2}}, 1, \tfrac{1}{\sqrt{2}}\right),$$

which represents the reciprocal of the Euclidean distance values of the neighboring pixels from the central pixel.

The resulting value after aggregation must be from the range of values for the RGB scale. Let us note that this property is equivalent to the idempotency of the aggregation operator. Standard

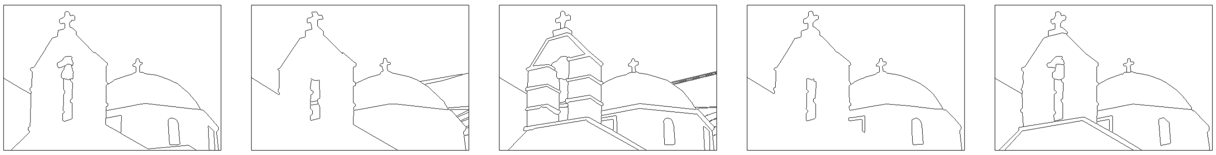


Figure 6: Several hand-labelled edges by human of image 118035 (see Figure 3(a)) from BSDS500 dataset

Choquet integral is idempotent if we consider the capacity as a monotone measure, see [20, Theorem 4.24]. The generalized Choquet integral is to be an averaging type of aggregation, if the assumption of Theorem 3.6(vi) is fulfilled. It is easy to verify that above mentioned $\mathscr{A}$-Choquet integrals acquired the stated condition.

In Figure 4 we can see the blending phase, i.e. aggregation of layers obtained in the feature extraction phase for input image (see Figure 3(a)) smoothed by (SM2). The software implementation of phases feature extraction and blending in Java codes can be found in GitHub repository[2].

Scaling In the last phase, the image from the previous stage is converted to a binary image with thin lines. First, non-maxima suppression [35] is applied to thin the edges, and then hysteresis for binarization. For these methods, we use online available codes presented in [31]. The scaling phase for standard and $\mathscr{A}$-Choquet integrals with respect to (SM1) – (SM4) smoothing can be seen in Figure 5.

Finally, it is important to evaluate the edge detection method. It can be evaluated either with a subjective opinion (qualitatively) or, if possible, quantitatively. Whereas the BSDS500 dataset [2] contains several hand-labelled edges by humans, we can evaluate methods in both ways. Thus in both cases, the evaluation depends on the individual human perception of the edge. In other words, with regard to the individual, the expected results may differ, see Figure 6. The result of edge detection is a binary image, so quantitative edge evaluation can be taken as a binary classification problem. To compute the displacement-tolerant correspondence of edges, the standard procedure Estrada and Jepson [17] is selected. The spatial tolerance is set to 2.5 % of the length of the diagonal of the image. Quantitative comparison of hand-labelled edges with a edge detection method is based on $\mathrm{F}_{0.5}$ measure, see [27],

$$\mathrm{F}_\alpha = \frac{\mathrm{Prec}\cdot\mathrm{Rec}}{\alpha\cdot\mathrm{Prec}+(1-\alpha)\cdot\mathrm{Rec}},$$

where

$$\mathrm{Prec} = \frac{\mathrm{TP}}{\mathrm{TP}+\mathrm{FP}} \quad \text{and} \quad \mathrm{Rec} = \frac{\mathrm{TP}}{\mathrm{TP}+\mathrm{FN}}.$$

In Table 7 we can see the quantitative comparison of results $(\mathrm{Prec}, \mathrm{Rec}, \mathrm{F}_{0.5})$ of edge detection using the Choquet and $\mathscr{A}$-Choquet integrals with respect to smoothings (SM1) – (SM4).

method	(SM1)			(SM2)			(SM3)			(SM4)		
	Prec	Rec	$\mathrm{F}_{0.5}$	Prec	Rec	$\mathrm{F}_{0.5}$	Prec	Rec	$\mathrm{F}_{0.5}$	Prec	Rec	$\mathrm{F}_{0.5}$
Choq. int	0.637	0.921	0.740	0.714	0.896	0.783	0.684	0.920	0.772	0.670	0.922	0.763
$\mathscr{A}^{\text{1-mean}}$	0.679	0.923	0.769	0.745	0.865	0.790	0.694	0.897	0.771	0.678	0.924	0.770
$\mathscr{A}^{\mathrm{Ch}_m}$	0.685	0.923	0.772	0.745	0.864	0.789	0.688	0.907	0.770	0.681	0.924	0.772
$\mathscr{A}^{\mathrm{maxMin}}$	0.660	0.920	0.755	0.734	0.889	0.794	0.686	0.921	0.774	0.673	0.922	0.766

Table 7: Quantitative comparison method of edge detection

The edge detection of others images from BSDS500 database can be found in Figures A.2 – A.3 in Appendix. Their quantitative analysis are available online at the mentioned repository.

[2] `https://github.com/Stanislav-B/CAO_Choquet_experimental_study.git`

Results Let us make results with respect to

- *images from database BSDS500, see Figure 3(a), and inputs in Figure A.2 (resp. Figures A.3 – A.5):*

  The qualitative (visual) results agree with the results reported in [31]. As seen in Figure 4 and Figure A.1 (see Appendix), already at the level of the blending phase we can see the presence of false elements that could be detected as edges in the final phase. This is due to the fact that the Gaussian smoothing does not adjust the smoothing according to the pixel values but smooths them all in the same way. Gravitational smoothing removed false elements better, and let us notice that it creates thinner edges than Gaussian smoothing. These observations are fully manifested in the scaling phase. As can be seen in Figure A.2, the use of Gaussian smoothing causes the detection of many spurious edges in each of the given images. Similarly, but to a lesser extent, the detection of false edges also smoothing (SM4). The reason is the configuration of Gravitational smoothing parameters. Setting cF to higher values causes more emphasis to be placed on pixel colour value rather than pixel distance, see [29]. Smaller values of cF cause pixels to be more likely to change their colour values in regions where colour is alike.

  Based on the edge detection shown in Figure 5 and Figures A.2 – A.5, (SM2) appears to be the best choice for smoothing, followed by (SM3). Note that (SM2) is the configuration with which the best results were also obtained in [31]. These qualitative assessments are also confirmed by quantitative results, see Table 7. Indeed, the presented edge detection methods give the best results with respect to (SM2) smoothing and, moreover, achieve higher scores than in [31]. The most optimal combination is (SM2) smoothing with $\mathscr{A}^{\max}$-Choquet integral. In addition, regardless of the type of image smoothing, each above presented method achieves higher scores than classical methods, see Table 3 in [31]. Regarding these results, it can be concluded that the four presented changes in the computer vision scheme together with the construction of the generalized Choquet integral are the suitable apparatus for fuzzy detection of image edges.

- *Figure A.6 and its edge detections on Figure A.7:*

  In this case, smoothing the image with configuration (SM2) gives the worst edge detection, which may be surprising given the previous results. It is here that the absence of an exact definition of the edge becomes apparent – as can be seen in Figure 6, different people detect edges differently, which affects the statistical results based on these detections. For Figure A.6, smoothing (SM4) with $\mathscr{A}^{\mathrm{maxMin}}$-Choquet integral appears as the optimal configuration. Therefore, it is important to choose the correct smoothing of the image based on the characteristics, features, etc. of the image. It can be considered as the subject of further research.

Considering the above conclusions, it can be concluded that with a proper choice of image smoothing, the $\mathscr{A}$-Choquet integral provides a suitable tool for edge detection, achieving better results than standard methods in many cases. For the sake of interest, in Figure A.8 we add for comparison edge detection of Figure A.6 by classical methods (Prewitt, Sobel, Robert, Marr-Hildreth, and Canny edge detector). Edge detection of other presented images by the standard method can be found in [31].

Conclusion

In this paper, we have presented the possible applications and advantages of the Choquet integral based on conditional aggregation operators. This new concept introduced by Boczek et al. [6] we have studied experimentally in two areas – decision making processes and image processing. We have supported both experimental studies either by creating the AGROP software application (based on algorithms presented in Section 4) and Java codes presented in [5], for the generalized Choquet integral in decision making processes, but also with Java codes for image processing with computer visual system modifications also using the generalized Choquet integral. In the first experimental study, we have also demonstrated the usefulness of such a concept in the problem of student evaluation. It turns out that $\mathscr{A}$-Choquet integral in comparison with the Choquet integral ordered the outliers better, i.e. students with low scores in profile subjects were ranked lower. In the second experimental study, we have shown that with the correct configuration of smoothing the input image, $\mathscr{A}$-Choquet integral achieves better results than standard edge detection methods, or in several cases better score than the standard Choquet integral, which we have also verified statistically.

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Appendix

In the following, we present additional tables and figures supporting the presented experimental study of the use of $\mathscr{A}$-Choquet integral in decision making and image processing.

Choquet integral			$\mathscr{A}^{\mathrm{sum}}$-Choquet integral			$\mathscr{A}^{\mathrm{Ch}(\cdot)}$-Choquet integral		
μ			μ			μ		
01.	$(20,13,17)$	[17.95]	01.	$(20,13,17)$	[29.55]	01.	$(20,13,17)$	[9.87]
02.	$(18,0,18)$	[16.2]	02.	$(18,18,12)$	[27.3]	02.	$(18,0,18)$	[8.51]
03.	$(18,18,12)$	[15]	03.	$(18,0,18)$	[21.6]	03.	$(18,18,12)$	[8.1]
04.	$(0,10,20)$	[12]	04.	$(8,8,17)$	[20.3]	04.	$(0,10,20)$	[6.3]
05.	$(10,0,18)$	[11.4]	05.	$(10,10,11)$	[18.7]	05.	$(10,0,18)$	[6.12]
06.	$(6,7,20)$	[10.8]	05.	$(7,7,18)$	[18.7]	06.	$(8,8,17)$	[5.97]
06.	$(9,0,18)$	[10.8]	06.	$(6,7,20)$	[18.3]	07.	$(10,10,11)$	[5.74]
07.	$(8,8,17)$	[10.7]	06.	$(9,9,12)$	[18.3]	08.	$(9,0,18)$	[5.67]
08.	$(7,7,18)$	[10.3]	07.	$(9,10,10)$	[17.4]	08.	$(9,9,12)$	[5.67]
08.	$(10,10,11)$	[10.3]	08.	$(9,9,10)$	[16.9]	09.	$(6,7,20)$	[5.58]
09.	$(9,10,10)$	[9.9]	09.	$(9,10,9)$	[16.65]	10.	$(7,7,18)$	[5.51]
09.	$(9,9,12)$	[9.9]	10.	$(8,12,7)$	[15.35]	11.	$(9,10,10)$	[5.46]
10.	$(9,10,9)$	[9.45]	11.	$(0,10,20)$	[15]	12.	$(9,9,10)$	[5.19]
11.	$(9,9,10)$	[9.3]	12.	$(10,0,18)$	[14.4]	13.	$(9,10,9)$	[5.18]
11.	$(8,12,7)$	[9.3]	13.	$(11,9,5)$	[13.7]	14.	$(8,12,7)$	[5]
12.	$(3,6,15)$	[8.4]	14.	$(9,9,6)$	[13.65]	15.	$(3,9,9)$	[4.58]
12.	$(3,9,9)$	[8.4]	15.	$(9,0,18)$	[13.5]	16.	$(3,6,15)$	[4.37]
13.	$(11,9,5)$	[7.9]	16.	$(3,6,15)$	[12.9]	17.	$(11,9,5)$	[4.2]
14.	$(9,9,6)$	[7.5]	17.	$(3,9,9)$	[12.6]	18.	$(9,9,6)$	[4.05]
15.	$(0,0,20)$	[6]	18.	$(8,7,4)$	[10.5]	19.	$(8,7,4)$	[3.18]
16.	$(8,7,4)$	[5.95]	19.	$(10,10,0)$	[9.5]	20.	$(10,10,0)$	[2.48]
17.	$(10,10,0)$	[5]	20.	$(0,0,20)$	[6]	21.	$(0,0,20)$	[1.8]
18.	$(5,5,0)$	[2.5]	21.	$(5,5,0)$	[4.75]	22.	$(5,5,0)$	[1.24]

Table A.1: The evaluation of students scores with respect to different methods – monotone measure μ

Choquet integral			$\mathscr{A}^{\mathrm{sum}}$-Choquet integral			$\mathscr{A}^{\mathrm{Ch}(\cdot)}$-Choquet integral		
ν			ν			ν		
01.	$(20,13,17)$	[17.47]	01.	$(18,18,12)$	[32.94]	01.	$(18,18,12)$	[12.78]
02.	$(18,18,12)$	[17.28]	02.	$(20,13,17)$	[32.71]	02.	$(20,13,17)$	[12.37]
03.	$(18,0,18)$	[14.04]	03.	$(10,10,11)$	[21.1]	03.	$(10,10,11)$	[7.93]
04.	$(0,10,20)$	[10.8]	04.	$(9,10,10)$	[19.8]	04.	$(9,9,12)$	[7.61]
05.	$(8,8,17)$	[10.7]	05.	$(9,9,12)$	[19.62]	05.	$(8,8,17)$	[7.38]
06.	$(6,7,20)$	[10.68]	06.	$(18,0,18)$	[19.44]	06.	$(9,10,10)$	[7.37]
07.	$(10,10,11)$	[10.3]	07.	$(8,8,17)$	[19.34]	07.	$(18,0,18)$	[7.21]
07.	$(7,7,18)$	[10.3]	08.	$(9,10,9)$	[19.17]	08.	$(9,9,10)$	[7.16]
08.	$(10,0,18)$	[10.2]	09.	$(9,9,10)$	[19.02]	09.	$(9,10,9)$	[7.14]
09.	$(9,9,12)$	[9.9]	10.	$(7,7,18)$	[17.86]	10.	$(8,12,7)$	[6.79]
10.	$(9,10,10)$	[9.78]	11.	$(8,12,7)$	[17.69]	11.	$(7,7,18)$	[6.74]
11.	$(9,0,18)$	[9.72]	12.	$(6,7,20)$	[17.46]	12.	$(11,9,5)$	[6.62]
12.	$(8,12,7)$	[9.68]	13.	$(11,9,5)$	[16.62]	13.	$(6,7,20)$	[6.55]
13.	$(9,10,9)$	[9.45]	14.	$(9,9,6)$	[16.47]	14.	$(9,9,6)$	[6.39]
14.	$(11,9,5)$	[9.42]	15.	$(0,10,20)$	[13.8]	15.	$(10,10,0)$	[5.9]
15.	$(9,9,10)$	[9.3]	16.	$(10,10,0)$	[13.3]	16.	$(0,10,20)$	[5.4]
16.	$(10,10,0)$	[8.8]	17.	$(10,0,18)$	[13.2]	17.	$(10,0,18)$	[5.22]
17.	$(9,9,6)$	[8.64]	18.	$(8,7,4)$	[12.76]	18.	$(8,7,4)$	[5.05]
18.	$(3,6,15)$	[8.04]	19.	$(3,9,9)$	[12.72]	19.	$(9,0,18)$	[4.86]
19.	$(3,9,9)$	[7.68]	20.	$(9,0,18)$	[12.42]	20.	$(3,9,9)$	[4.77]
20.	$(8,7,4)$	[7.09]	21.	$(3,6,15)$	[12.18]	21.	$(3,6,15)$	[4.62]
21.	$(0,0,20)$	[6]	22.	$(5,5,0)$	[6.65]	22.	$(5,5,0)$	[2.95]
22.	$(5,5,0)$	[4.4]	23.	$(0,0,20)$	[6]	23.	$(0,0,20)$	[1.8]

Table A.2: The evaluation of students scores with respect to different methods – monotone measure ν

vector	Choquet integral				$\mathscr{A}^{\text{sum}}$-Choquet integral				$\mathscr{A}^{\text{Ch}(\cdot)}$-Choquet integral			
	μ		ν		μ		ν		μ		ν	
$(20, 13, 17)$	01.	[17.95]	01.	[17.47]	01.	[29.55]	02.	[32.71]	01.	[9.87]	02.	[12.37]
$(18, 0, 18)$	02.	[16.2]	03.	[14.04]	03.	[21.6]	06.	[19.44]	02.	[8.51]	07.	[7.21]
$(18, 18, 12)$	03.	[15]	02.	[17.28]	02.	[27.3]	01.	[32.94]	03.	[8.1]	01.	[12.78]
$(0, 10, 20)$	04.	[12]	04.	[10.8]	11.	[15]	15.	[13.8]	04.	[6.3]	16.	[5.4]
$(10, 0, 18)$	05.	[11.4]	08.	[10.2]	12.	[14.4]	17.	[13.2]	05.	[6.12]	17.	[5.22]
$(6, 7, 20)$	06.	[10.8]	06.	[10.68]	06.	[18.3]	12.	[17.46]	09.	[5.58]	13.	[6.55]
$(9, 0, 18)$	06.	[10.8]	11.	[9.72]	15.	[13.5]	20.	[12.42]	08.	[5.67]	19.	[4.86]
$(8, 8, 17)$	07.	[10.7]	05.	[10.7]	04.	[20.3]	07.	[19.34]	06.	[5.97]	05.	[7.38]
$(7, 7, 18)$	08.	[10.3]	07.	[10.3]	05.	[18.7]	10.	[17.86]	10.	[5.51]	11.	[6.74]
$(10, 10, 11)$	08.	[10.3]	07.	[10.3]	05.	[18.7]	03.	[21.1]	07.	[5.74]	03.	[7.93]
$(9, 10, 10)$	09.	[9.9]	10.	[9.78]	07.	[17.4]	04.	[19.8]	11.	[5.46]	06.	[7.37]
$(9, 9, 12)$	09.	[9.9]	09.	[9.9]	06.	[18.3]	05.	[19.62]	08.	[5.67]	04.	[7.61]
$(9, 10, 9)$	10.	[9.45]	13.	[9.45]	09.	[16.65]	08.	[19.17]	13.	[5.18]	09.	[7.14]
$(9, 9, 10)$	11.	[9.3]	15.	[9.3]	08.	[16.9]	09.	[19.02]	12.	[5.19]	08.	[7.16]
$(8, 12, 7)$	11.	[9.3]	12.	[9.68]	10.	[15.35]	11.	[17.69]	14.	[5]	10.	[6.79]
$(3, 6, 15)$	12.	[8.4]	18.	[8.04]	16.	[12.9]	21.	[12.18]	16.	[4.37]	21.	[4.62]
$(3, 9, 9)$	12.	[8.4]	19.	[7.68]	17.	[12.6]	19.	[12.72]	15.	[4.58]	20.	[4.77]
$(11, 9, 5)$	13.	[7.9]	14.	[9.42]	13.	[13.7]	13.	[16.62]	17.	[4.2]	12.	[6.62]
$(9, 9, 6)$	14.	[7.5]	17.	[8.64]	14.	[13.65]	14.	[16.47]	18.	[4.05]	14.	[6.39]
$(0, 0, 20)$	15.	[6]	21.	[6]	20.	[6]	23.	[6]	21.	[1.8]	23.	[1.8]
$(8, 7, 4)$	16.	[5.95]	20.	[7.09]	18.	[10.5]	18.	[12.76]	19.	[3.18]	18.	[5.05]
$(10, 10, 0)$	17.	[5]	16.	[8.8]	19.	[9.5]	16.	[13.3]	20.	[2.48]	15.	[5.9]
$(5, 5, 0)$	18.	[2.5]	22.	[4.4]	21.	[4.75]	22.	[6.65]	22.	[1.24]	22.	[2.95]

Table A.3: The evaluation of students scores with respect to different methods

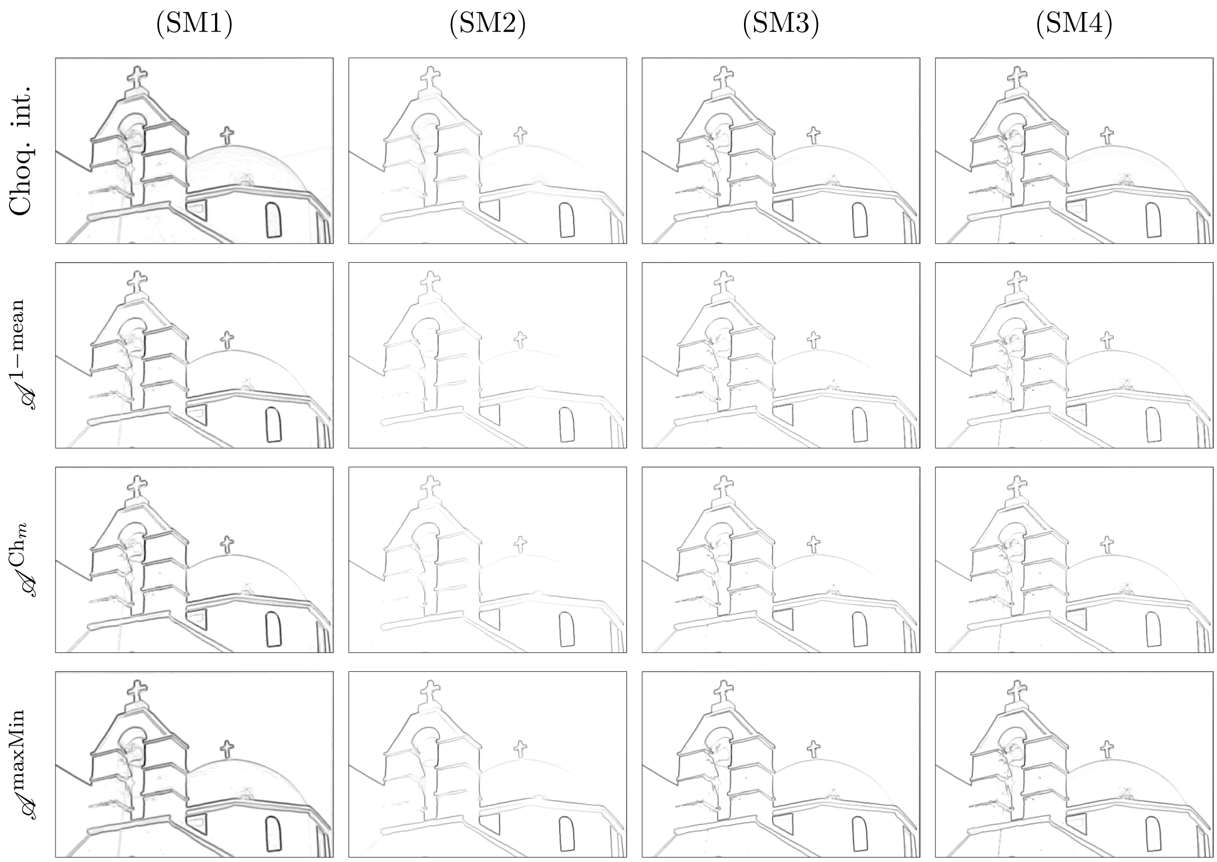

Figure A.1: Blending phase – detected edges before scaling

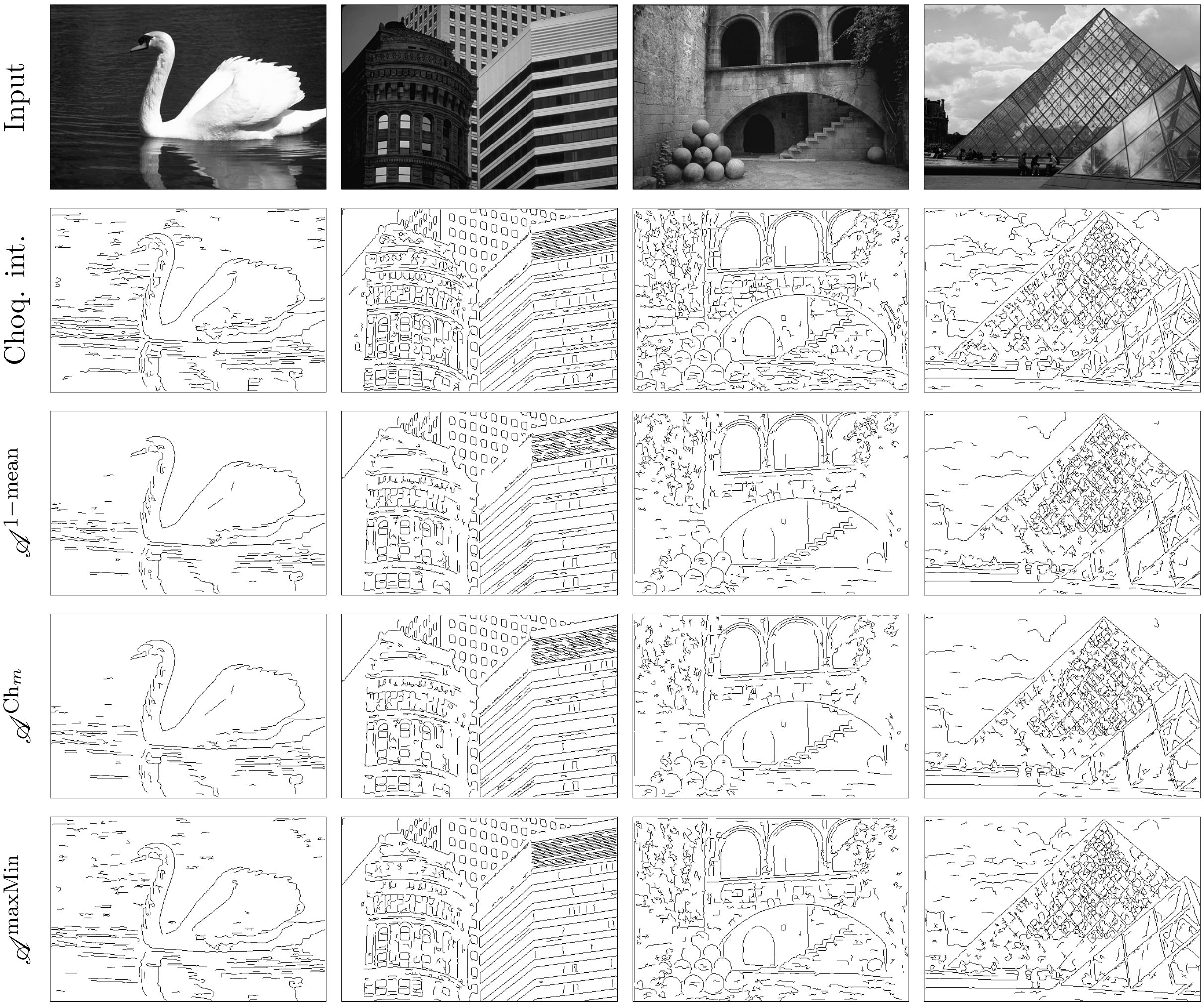

Figure A.2: Comparison of edge detection of images 8068, 48017, 118072, and 223060 from BSDS500 database smoothed with (SM1)

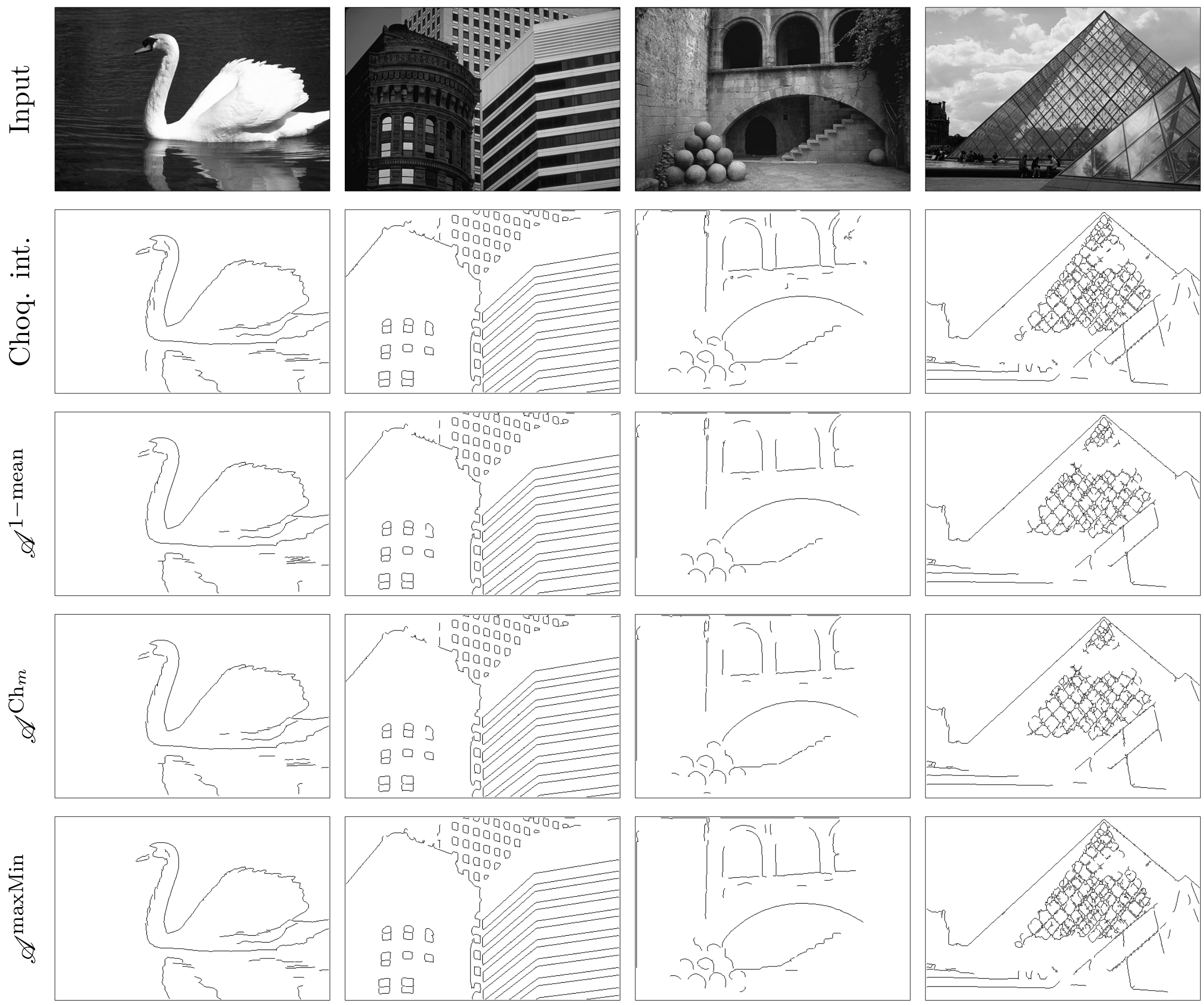

Figure A.3: Comparison of edge detection of images 8068, 48017, 118072, and 223060 from BSDS500 database smoothed with (SM2)

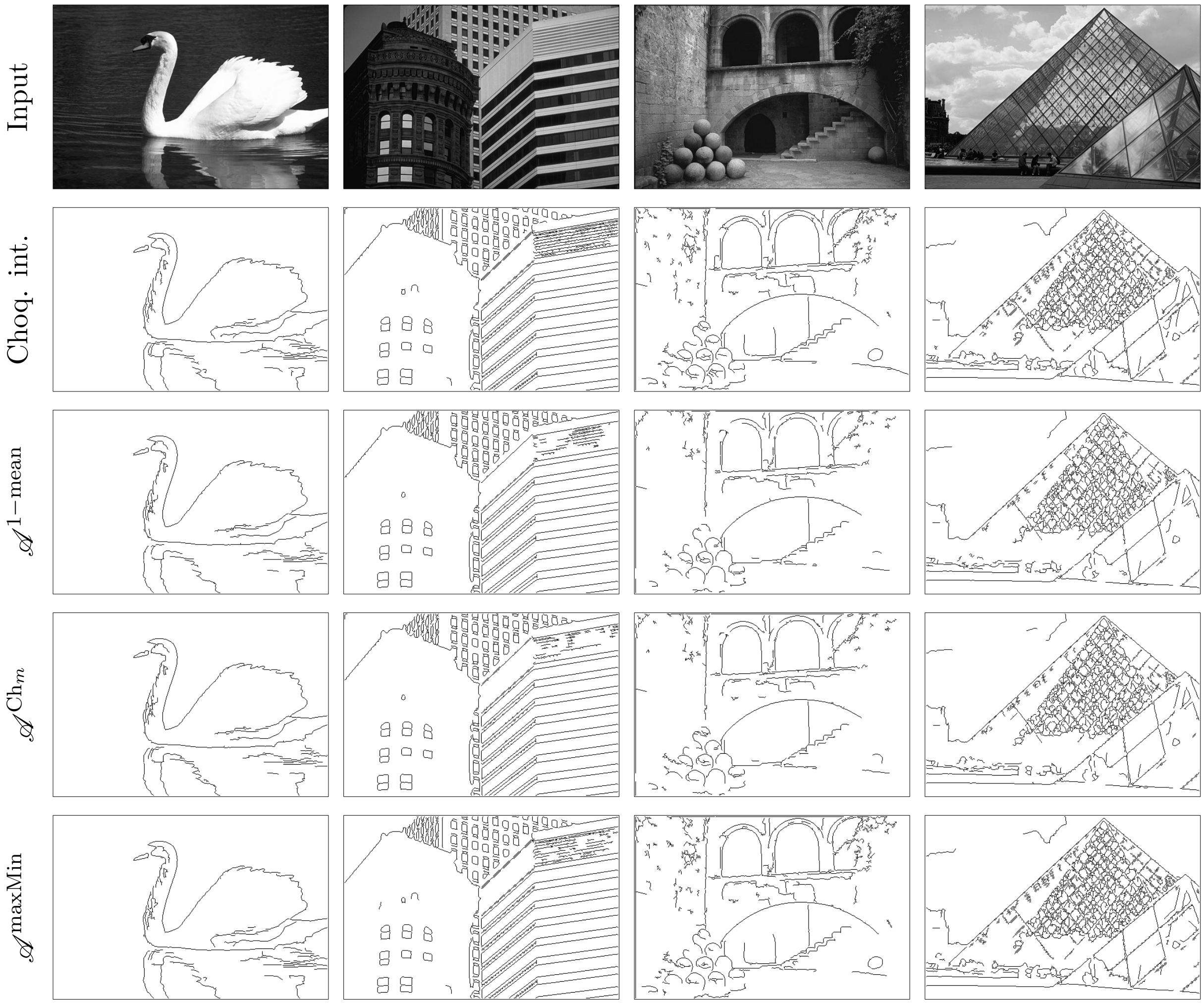

Figure A.4: Comparison of edge detection of images 8068, 48017, 118072, and 223060 from BSDS500 database smoothed with (SM3)

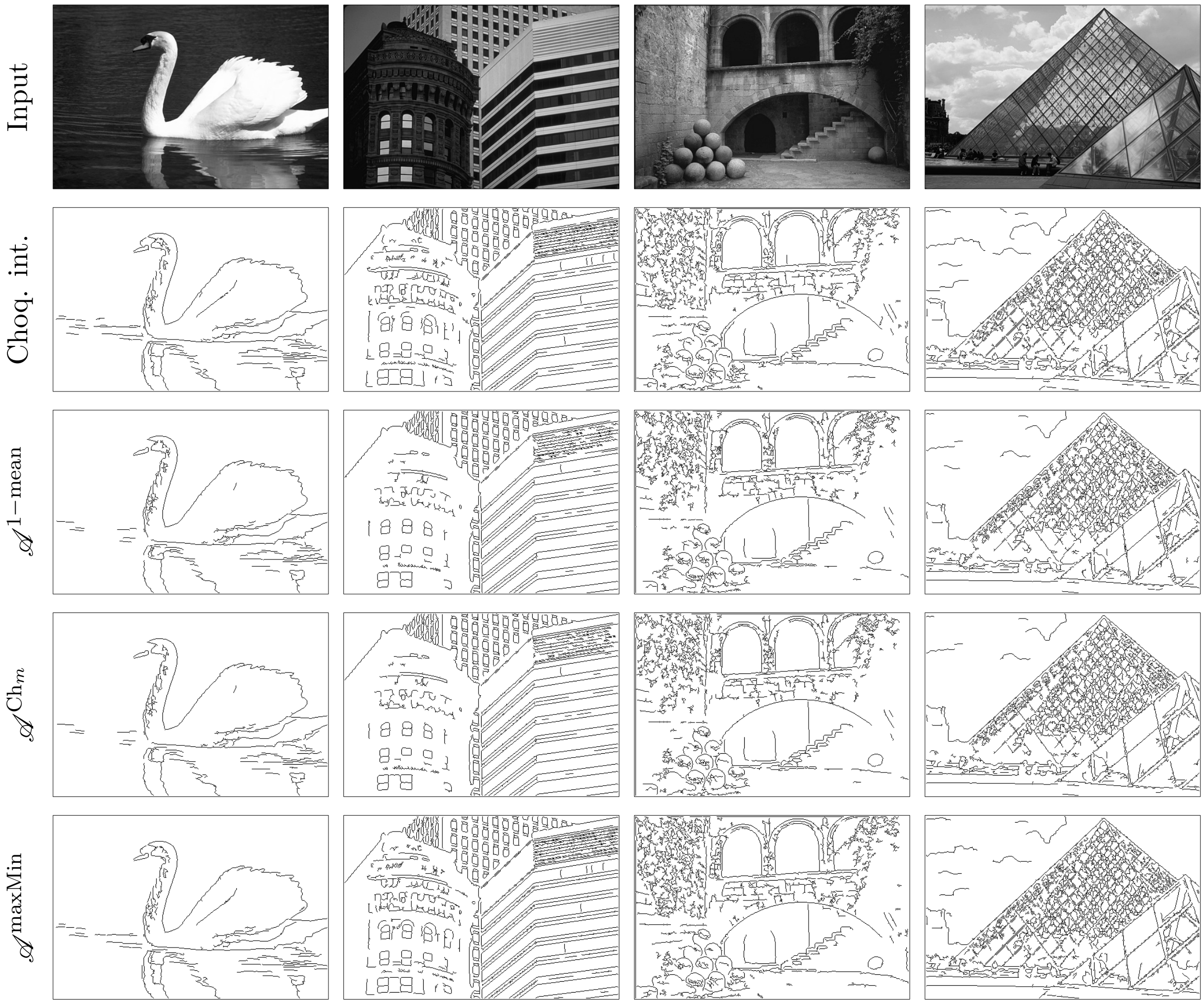

Figure A.5: Comparison of edge detection of images 8068, 48017, 118072, and 223060 from BSDS500 database smoothed with (SM4)

Figure A.6: The Lenna image

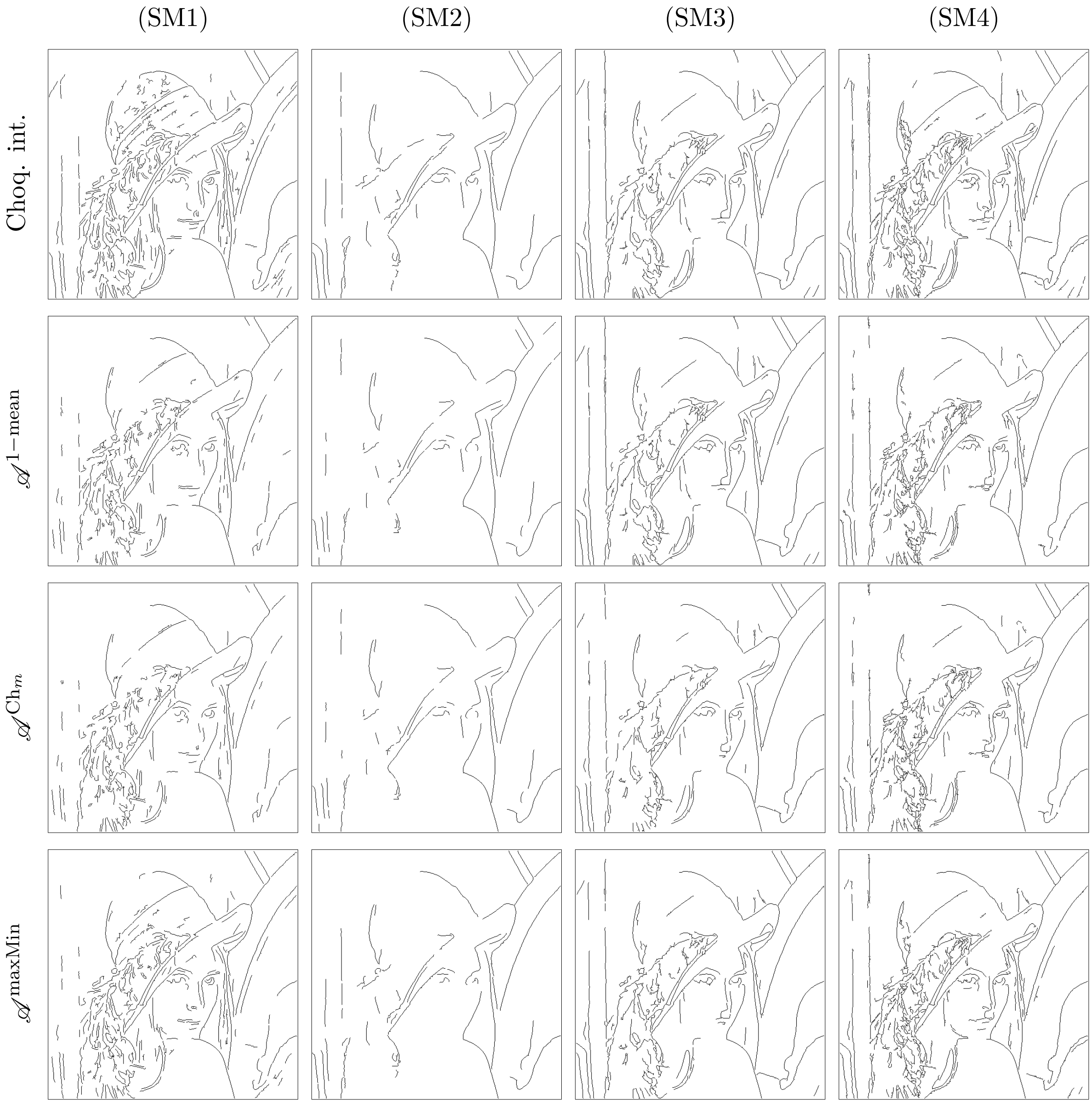

Figure A.7: Detected edges of Lenna image

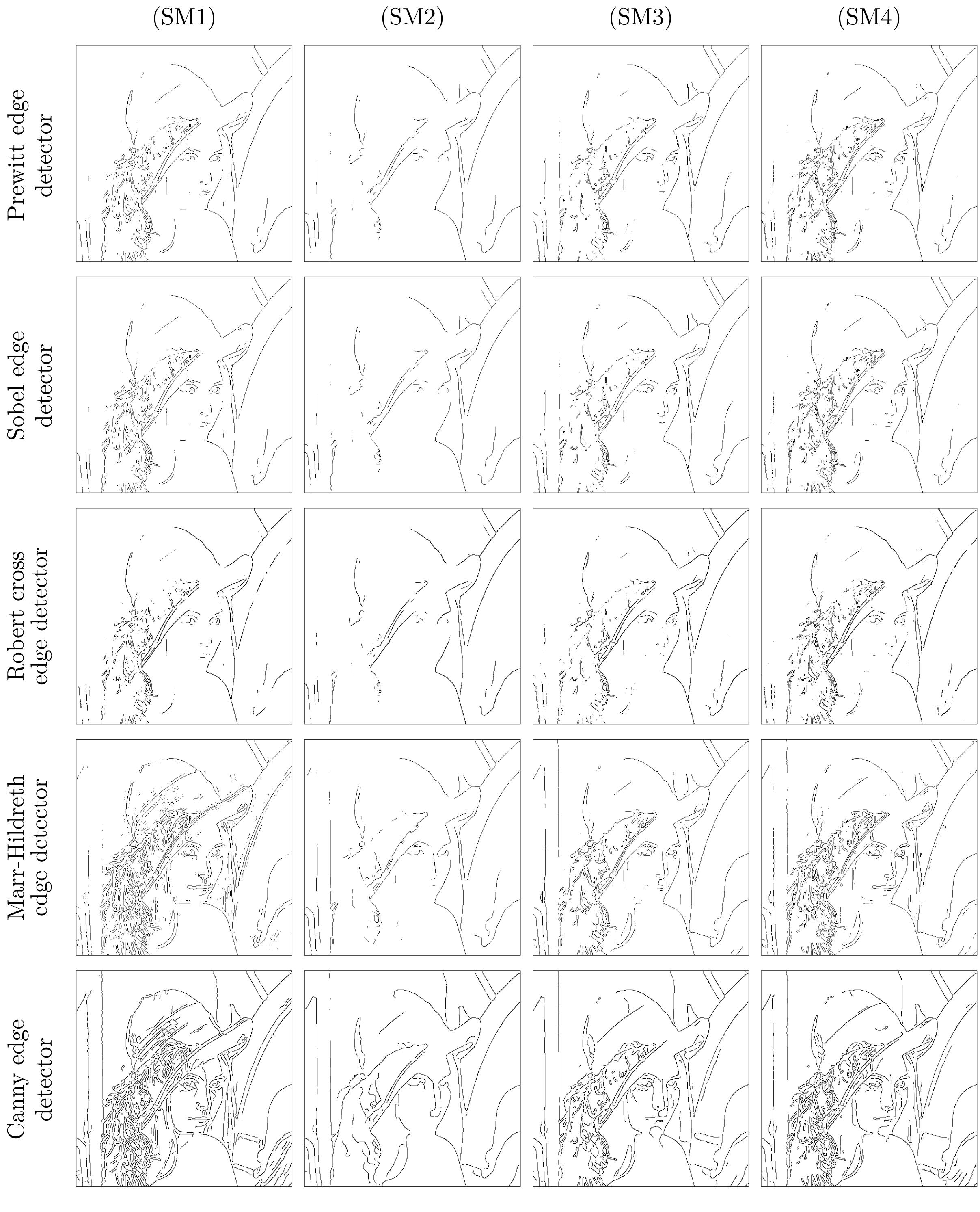

Figure A.8: Detected edges of Lenna image by standard methods