Relativistic beaming of CMB

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1 What do we see when moving relativistically?

Suppose a stationary observer Alice sees an object in the direction γ, where γ can be thought of as a unit vector representing a point on the celestial sphere. Suppose that Bob is moving relativistically and Λ is a Lorentz transform from Alice's inertial frame to Bob's inertial frame. When we say that Alice sees the object in the direction $\vec{\gamma}$, we mean that she detects photons with four-velocity $\gamma^\mu = (1, -\gamma)$. Bob detects the same photons, but in his inertial frame they have four-velocity $\tilde{\eta}^\nu = \Lambda^\nu_\mu \gamma^\mu = R \cdot (1, -\tilde{\gamma})$, where R is the blueshift coefficient. So, Bob sees the same object in the direction $\tilde{\gamma}$ with blueshift R.

We want to understand the transformation $\gamma \mapsto \tilde{\gamma}$ better.

Theorem 1. *The celestial sphere is transformed conformally with conformal factor R^{-2}, the inverse squared blueshift (which depends on the point on the sphere, obviously).*

Proof. Suppose we have a curve $\gamma(x)$ drawn on the celestial sphere, as seen by Alice. The same curve as seen by Bob is $\tilde{\gamma}(x)$. Let's look at how the tangent vector $\frac{d}{dx}\gamma(x)$ is transformed. We have

$$\frac{d}{dx}(1, \tilde{\gamma}(x)) = \frac{d}{dx}\left(\frac{\eta^\mu(x)}{\eta^0(x)}\right) = \frac{d}{dx}\eta^\mu(x) R^{-1}(x) = R^{-1}(x)\frac{d}{dx}\eta^\mu(x) + \eta^\mu(x)\frac{d}{dx}(R^{-1}(x)),$$

$$\left|\frac{d}{dx}(1, \tilde{\gamma}(x))\right|^2 = R^{-2}(x)\left|\frac{d}{dx}\eta^\mu(x)\right|^2 + A\left\langle \frac{d}{dx}\eta^\mu(x), \eta^\mu \right\rangle + B\left\langle \eta^\mu, \eta^\mu \right\rangle.$$

Now, using the fact that Λ preserves Lorentz dot-products and that $\frac{d}{dx}\gamma^\mu$ is Lorentz orthogonal to γ^μ, we have

$$\left|\frac{d}{dx}(1, \tilde{\gamma}(x))\right|^2 = R^{-2}(x)\left|\frac{d}{dx}(1, \gamma(x))\right|^2,$$

$$\left|\frac{d}{dx}\tilde{\gamma}(x)\right|^2 = R^{-2}(x)\left|\frac{d}{dx}\gamma(x)\right|^2.$$

So, we see that the celestial sphere is transformed conformally with conformal factor R^{-2}. □

Note. We know that conformal maps of sphere to itself map circles to circles. So, circular objects will still appear circular to a moving observer. We might expect them to be Lorentz contracted and appear as ellipses, but Lorentz contraction is about the transormation of coordinates, while what we care about here is what we actually *see*.

Theorem 2. *The flux per unit solid angle is multiplied by R^4.*

Proof. Let's consider a small region S of the celestial sphere (as seen by Alice), such that redshifts for all directions in that region are approximately the same. Let's call the same region as seen by Bob $\tilde{S}$. Let's temporarily consider only light coming from the region S and turn off all other light in the Universe. The energy-momentum of electromagnetic field in Alice's frame is $T^{\mu\nu} \approx I\gamma^\mu\gamma^\nu$, where I is the flux from S as seen by Alice. In the Bob's frame we have $\tilde{T}^{\mu\nu} \approx I\eta^\mu\eta^\nu = IR^2(1, \tilde{\gamma})^\mu(1, \tilde{\gamma})^\nu$, because we know how the energy-momentum tensor is transformed. We see that the flux coming from $\tilde{S}$ as seen by Bob is IR^2. But the solid angle of $\tilde{S}$ is R^{-2} larger than the solid angle of S by the previous theorem, so the flux per unit solid angle is actually R^4 times larger. □

Corollary 3. *The flux per unit solid angle is changed in such a way, that the spectrum remains a black-body spectrum. So, we only need to calculate aberration and redshifts, intensity is going to remain that of a black-body.*

Let's think about CMB now. If we move with velocity $(1 - \varepsilon) \cdot c$, then the objects in the direction of our motion will be blueshifted with $R = (2\varepsilon)^{-1/2}$. To actually see something with naked eye we need blueshifts of the order several hundred, so $\varepsilon \approx 10^{-5}$. We will see a bright region $\tilde{S}$ on the sky in from of us, where in $\tilde{S}$ blueshifts are of the order several hundred. The solid angle of $\tilde{S}$ is $R^{-2} \approx 2\varepsilon$ times the solid angle of the original region S in Alice's frame by the first theorem. The original region can't be more than the whole sphere, so the solid angle of $\tilde{S}$ is at most $8\pi\varepsilon$. So, we will see a small bright spot in from of us, approximately Sun-like.