1. Introduction

Throughout the paper, every time we speak about matrices we mean binary matrices (also called 01-matrices) and we omit the word binary. If we speak about a *pattern*, we again mean a binary matrix and we use the word in order to distinguish among more matrices as well as to indicate relationship.

When dealing with matrices, we always index rows and column starting with one and when we speak about a row r, we simply mean a row with index r. A *line* is a common word for both a row and a column. When we order a set of lines, we first put all rows and then all columns. For $M \in \{0,1\}^{m \times n}$, $[m]$ is a set of all rows and $[m+n]$ is a set of all lines, where m-th element is the last row. This goes with the usual notation.

Notation 1. For $n \in \mathbb{N}$ let $[n] := \{1, 2, \ldots, n\}$ and for $m \in \mathbb{N}$, where $n \leq m$ let $[n, m] := \{n, n+1, \ldots, m\}$.

Notation 2. For a matrix $M \in \{0,1\}^{m \times n}$ and $L \subseteq [m+n]$ let $M[L]$ denote a submatrix of M induced by lines in L.

Notation 3. For a matrix $M \in \{0,1\}^{m \times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M[R, C]$ denote a submatrix of M induced by rows in R and columns in C. Furthermore, for $r \in [m]$ and $c \in [n]$ let $M[r, c] := M[\{r\}, \{c\}] = M[\{r, c+m\}]$.

Definition 1. We say a matrix $M \in \{0,1\}^{m \times n}$ *contains* a pattern $P \in \{0,1\}^{k \times l}$ *as a submatrix* and denote it by $P \leq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M[R, C][r, c] = 1$.

This does not necessarily mean $P = M[R, C]$ as $M[R, C]$ can have more one-entries than P does.

Notation 4. For a matrix $M \in \{0,1\}^{m \times n}$ and $L \subseteq [m+n]$ let $M_{\preceq}[L]$ denote a matrix acquired from M by applying following operation for each $l \in L$:

- If l is the first row in L then we replace the first l rows by one row that is a bitwise OR of replaced rows.
- If l is the first column in L then we replace the first $l - m$ columns by one column that is a bitwise OR of replaced columns.
- Otherwise, we take l's predecessor $l' \in L$ in the standard ordering and replace lines $[l'+1, l]$ by one line that is a bitwise OR of replaced lines.

Notation 5. For a matrix $M \in \{0,1\}^{m \times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M_{\preceq}[R, C] := M_{\preceq}[R \cup \{c + m | c \in C\}]$.

Definition 2. We say a matrix $M \in \{0,1\}^{m \times n}$ *contains* a pattern $P \in \{0,1\}^{k \times l}$ *as an interval minor* and denote it by $P \preceq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M_{\preceq}[R, C][r, c] = 1$.

Observation 1. *For all matrices M and P, $P \leq M \Rightarrow P \preceq M$.*

Observation 2. *For all matrices M and P, if P is a permutation matrix, then $P \leq M \Leftrightarrow P \preceq M$.*

Proof. If we have $P \preceq M$, then there is a partitioning of M into rectangles and for each one-entry of P there is at least one one-entry in the corresponding rectangle of M. Since P is a permutation matrix, it is sufficient to take rows and columns having at least one one-entry in the right rectangle and we can always do so.

Together with Observation 1 this gives us the statement. □

Observation 3. *Let $M \in \{0,1\}^{m \times n}$ and $P \in \{0,1\}^{k \times l}$, $P \preceq M \Leftrightarrow P^T \preceq M^T$.*

Because of this observation we will usually only show results only for rows or columns and expect both to hold and only show results for $P \in \{0,1\}^{k \times l}$ but assume the symmetrical results for P^T.

2. Characterizations

Definition 3. A *walk* in a matrix M is a sequence of some of its entries, beginning in the top left corner and ending in the bottom right one. If an entry $M[i,j]$ is in the sequence, the next one is either $M[i+1,j]$ or $M[i,j+1]$.

Definition 4. We call a binary matrix M a *walking matrix* if there is a walk in M such that all one-entries of M are contained on the walk.

Definition 5. An *extended walk of size* $k \times l$ in a matrix M is a subset of some of its entries, beginning in the top left corner and ending in the bottom right one. If an entry $M[i,j]$ is in the subset there is also either $M[i+1,j]$ or $M[i,j+1]$. The size describes that no more than k entries directly above each other are in the subset and no more than l entries directly next to each other are in the subset. We say that an extended walk of size $k \times l$ in M starts with a walk w, if the extended walk is a subset of entries of M that

- lie on w or below w and
- lie on w shifted by $k-1$ down and by $l-1$ to the left or above it.

Definition 6. For $M \in \{0,1\}^{m \times n}$ and $r \in [m], c \in [n]$ we say $M[r,c]$ is

- *top-left empty* if $M[[r-1],[c-1]]$ is an empty matrix,
- *top-right empty* if $M[[r-1],[c+1,n]]$ is empty,
- *bottom-left empty* if $M[[r-1],[c+1,n]]$ is empty,
- *bottom-right empty* if $M[[r-1],[c+1,n]]$ is empty.

2.1 Patterns of size 2×2 and their generalization

Theorem 4. *Let $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then for all M: $P \not\prec M \Leftrightarrow M$ is a walking matrix.*

Proof. Since P is a permutation matrix, $P \not\prec M \Leftrightarrow P \not\preceq M$ and it is easy to see $P \not\preceq M \Leftrightarrow M$ is a walking matrix. $\square$

Now consider a generalization of the pattern from above:

Theorem 5. *Let $P \in \{0,1\}^{k \times l}$ be a matrix having only two one-entries – $P[1,n]$ and $P[m,1]$, then for all M: $P \not\prec M \Leftrightarrow M$ has an extended walk of size $k-1 \times l-1$ containing all one-entries.*

Proof. $\Rightarrow$ Let $P \not\prec M$ and consider the left-most top-right empty elements of M. They necessarily form a walk w. For contradiction, assume there is a one-entry e below the extended walk of size $k-1 \times l-1$ starting with w. Since e is below the extended walk, there is an element e' - the right-most element of M that is neither below e nor to the right from e and at the same time still below the extended walk (it is possible $e = e'$). Let $e = M[r,c]$ and notice $M[r-k, c-l]$ is part of walk w and because of the choice of e' neither $M[r-k-1, c-l]$ nor $M[r-k, c-l-1]$ are on the walk w and $M[r-k, c-l]$ must be a one-entry; therefore, together with e it forms the forbidden pattern in M, which is a contradiction.

$\Leftarrow$ Let $M[r,c]$ be any one-entry of M, which then necessarily lie in the extended walk. Because the size of the walk is $k-1 \times l-1$, $M[r-k+1, c-l+1]$ is top-left empty and $M[r+k-1, c+l-1]$ is bottom-right empty; therefore e cannot be a part of a mapping of P.

$\square$

Theorem 6. *Let $P = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, then for all $M \in \{0,1\}^{m\times n}$: $P \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 2.1)*

- $M[[r-1],[c-1]]$ *is empty,*
- $M[[r-1],[c+1,n]]$ *is empty,*
- $M[[r+1,m],[c-1]]$ *is empty and*
- $M[[r,m],[c,n]]$ *is a walking matrix.*

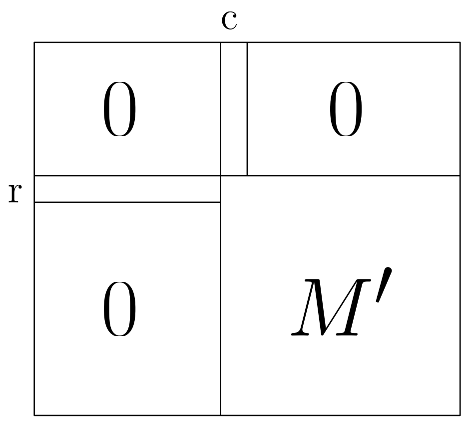

Figure 2.1: Characterization of a matrix avoiding $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ as an interval minor. Matrix M' is a walking matrix

Proof. $\Rightarrow$ If $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M$ then M is a walking matrix and we set $r = c = 1$. Otherwise, there are one-entries $M[r,c']$ and $M[r',c]$ such that $r' < r$ and $c' < c$. If there is a one-entry in regions $M[[r-1],[c-1]]$, $M[[r-1], [c+1,n]]$ or $M[[r+1,m],[c-1]]$ then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and we again get a contradiction.

$\Leftarrow$ For contradiction, assume that M described in Figure 2.1 contains P as an interval minor. It means that there is a partition of the matrix into four quadrants such that there is at least one one-entry in each quadrant besides the bottom right one. If the matrix is partitioned above the r-th row, then there is only one column containing one-entries and it is not possible for both top quadrants to have a one-entry. Similarly, if the matrix is partitioned to the left of the c-th column, there is only one row containing one-entries and there is no one-entry in either top-left or bottom-left quadrant. Therefore, the partitioning lies bellow the r-th row and to the right of the c-th column, but if the quadrants contain one-entries, there is a $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ interval minor in M', which is a contradiction with it being a walking matrix.

$\square$

Theorem 7. *Let $P \in \{0,1\}^{k \times l}$ be a matrix having only three one-entries – $P[1,1]$, $P[1,n]$ and $P[m,1]$, then for all M: $P \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 2.1 and imagine rows and columns being extended)*

- *$M[[r-1],[c-1]]$ is empty,*
- *$M[[r-1],[c+l,n]]$ is empty,*
- *$M[[r+k,m],[c-1]]$ is empty and*
- *$M[[r,m],[c,n]]$ has an extended walk of size $k-1 \times l-1$ containing all one-entries.*

Proof. Let $P' = P$ and set $P'[m,1] = 0$ (P' is a generalization of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$).

⇒ If $P' \not\preceq M$ then M is a matrix having an extended walk of size $k-1 \times l-1$ containing all one-entries and we set $r = c = 1$. Otherwise, there are one-entries $M[r_1, c_1]$ and $M[r_2, c_2]$ such that $r_2 < r_1$ and $c_1 < c_2$. We now choose $M[r_3, c_3]$ to be the bottom-most one-entry that still forms P' with $M[r_2, c_2]$. We choose $M[r_4, c_4]$ to be the left-most one-entry that forms P' with $M[r_3, c_3]$ and set $r = r_3 - k + 1$ and $c = c_4 - l + 1$. If there is a one-entry in regions $M[[r-1],[c-1]]$, $M[[r-1],\ [c+l,n]]$ or $M[[r+k,m],[c-1]]$ then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains P' and we again get a contradiction.

⇐ Because of the sizes of areas with no one-entries and the condition for $M[[r,m],[c,n]]$, there cannot be P' anywhere but in $M[[r+k-1],[c+l-1]]$. Since $M[[r-1][c-1]]$ is empty, there is no one-entry to map $P[1,1]$ to; therefore, $P \not\preceq M$.

□

Lemma 8. *Let $P = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and let $M \in \{0,1\}^{m \times n}$ avoid P as an interval minor, then there exists a row r and a column c such that $M[r,c]$ is either*

1. *a one-entry and $(r,c) \in \{(1,1),(1,n),(m,1),(m,n)\}$ or*
2. *both top-left empty and bottom-right empty and $(r,c) \notin \{(1,n),(m,1)\}$ or*
3. *both top-right empty and bottom-left empty and $(r,c) \notin \{(1,1),(m,n)\}$.*

Proof. If there is a one-entry in any corner we are done. Otherwise, let A be a set of all top-left empty entries of M and B be a set of all bottom-right empty entries of M. If there is an entry $M[r,c] \in A \cap B$ different from $(1,n)$ and $(m,1)$ we are done. Assume $A \cap B = \{(1,n),(m,1)\}$. Since $(m,1) \in A$, it also holds $(m-1,1) \in A$ and because it is not in the intersection we have $(m-1,1) \notin B$. This means $M[m-1,1]$ is not bottom-right empty; therefore there is a one-entry somewhere in $M[m,[2,n]]$. Moreover, no corner contains a one-entry so the is a one-entry in $M[m,[2,n-1]]$. For simplicity, we will say that the last row in non-empty (knowing the corners are empty). Symmetrically, we also get that the first row is non-empty and both the first and the last columns are non-empty. If there is a one-entry $M[r_l,1]$ in a different row than a one-entry $M[r_r,n]$ and at the same time a one-entry $M[1, c_t]$ in a different column than a one-entry $M[m, c_b]$ then these four one-entries form a mapping of the forbidden pattern P.

This is not true!!!

Without loss of generality assume there is only one one-entry in both the first and the last column and they are both in the same row r'. Let c' be a column such that there is a one-entry $M[1, c']$. Clearly, there is no other column that contains a one-entry above r', because we would again get a contradiction. Symmetrically, let c'' be the only column containing one-entries below r'. If $c' \geq c''$ we have that both $M[r', c']$ and $M[r', c'']$ are both top-left empty and bottom-right empty, which is a contradiction with $A \cap B = \{(1, n), (m, 1)\}$. Otherwise, $c' < c''$ and both $M[r', c']$ and $M[r', c'']$ are both top-right empty and bottom-left empty where $(r', c') \notin \{(1, 1), (m, n)\}$ which concludes the proof. $\square$

Theorem 9. *Let* $P = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$*, then for all* M*:* $P \not\preceq M \Leftrightarrow M$ *looks like one of the matrices in Figure 2.2, where* $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$*,* $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) \not\preceq M_2$*,* $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_3$ *and* $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) \not\preceq M_4$*.*

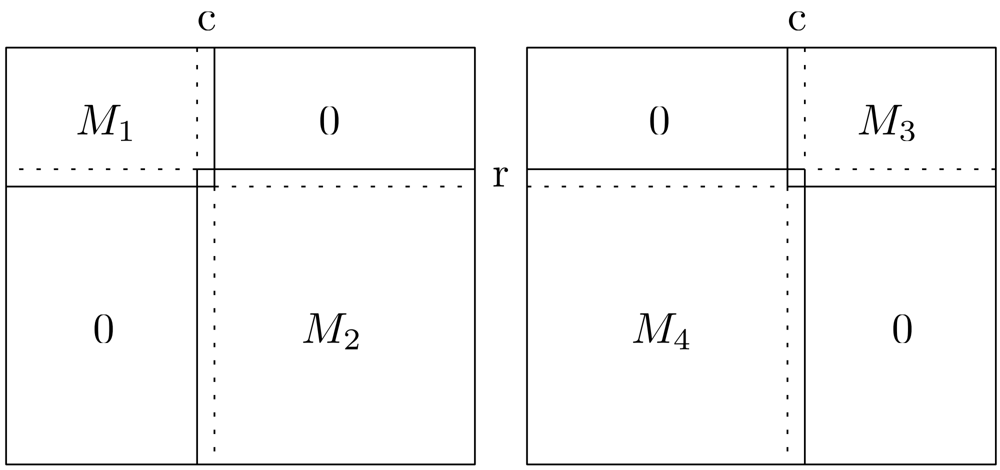

Figure 2.2: Characterization of a matrix avoiding $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ as an interval minor.

Proof.

$\Rightarrow$ We proceed by induction by the size of M.

If $M \in \{0, 1\}^{2\times 2}$ then it either avoids $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ and we are done.

For bigger M there is, from Lemma 8, $M[r, c]$ satisfying some conditions. If it is the first condition – there is a one-entry in any corner, we are done because the matrix cannot contain one of the rotations of $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$. Assume the second case – $M[r, c]$ is both top-right and bottom-left empty and $(r, c) \notin \{(1, n), (m, 1)\}$. If M_1 is non-empty, then $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) \not\preceq M_2$; otherwise, $P \preceq M$. Similarly, $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$ if M_2 is non-empty. If one of them is empty, the other is a smaller matrix avoiding P as an interval minor and by induction hypothesis, it can be partitioned. Adding empty rows and columns does not break any condition and we get a partitioning of the whole M.

$\Leftarrow$ Without loss of generality, let us assume M looks like the left matrix in Figure 2.2. For contradiction, assume $P \preceq M$. In that case, we can partition M into four quadrants such that there is at least one one-entry in each of them. It does not matter where we partition it, every time we either get $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \preceq M_1$ or $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) \preceq M_2$, which is a contradiction. $\square$

Theorem 10. *Let $P \in \{0,1\}^{k\times l}$ be a matrix having only four one-entries – $P[1,1]$, $P[1,n]$, $P[m,1]$ and $P[m,n]$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like one of the matrices in Figure 2.2, where generalized $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$, $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) \not\preceq M_2$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_3$ and $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) \not\preceq M_4$.*

2.2 Matrices of size 2×3

Theorem 11. *Let $P = \left(\begin{smallmatrix}1&0&1\\0&1&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M = M_1 \oplus_h M_2$ where $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ Let $e = [r,c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \preceq M[[m],[c-1]]$, together with e it forms P. If $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M[[m],[c,n]]$ then we are done. Let us assume it is not the case and let $e_{0,0},\ e_{1,1}$ be any two one-entries forming the forbidden pattern. Symmetrically, let $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \preceq M[[m],[c]]$ and let $e_{0,1},\ e_{1,0}$ be any two one-entries forming the forbidden pattern. Now if we take $e_{0,0},\ e_{0,1}$ and $e_{1,0}$ or $e_{1,1}$ with bigger row, we get the forbidden pattern P as an interval minor of M.

$\Leftarrow$ For contradiction, let us assume $P \preceq M$ and $M = M_1 \oplus_h M_2$. If $P \preceq M$, look at the one-entry of M where the bottom one-entry of P is mapped. If it is in M_1 then $P \not\preceq M$ because $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$. Otherwise, $P \not\preceq M$ because $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_2$.

$\square$

Lemma 12. *Let $P = \left(\begin{smallmatrix}1&1&1\\0&1&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Rightarrow M = M_1 \oplus_h M_2$ where*

1. *$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$ or*
2. *$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$.*

Proof. Let $e = [r,c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \preceq M[[m],[c-1]]$, together with e it would be the whole P. Similarly, $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M[[m],[c+1,n]]$. For contradiction with the statement, let $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \preceq M[[m],[c]]$ and $e_{0,0},\ e_{1,1}$ (none of them equal to e, since e lies in the top-right corner) be any two one-entries forming the pattern. Symmetrically, let $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \preceq M[[m],[c,n]]$ and $e_{0,1},\ e_{1,0}$ be any two one-entries forming the pattern. In that case $e_{0,0},\ e,\ e_{0,1}$ and $e_{1,0}$ or $e_{1,1}$ with bigger row give us the forbidden pattern P as an interval minor of M. $\square$

Theorem 13. *Let $P = \left(\begin{smallmatrix}1&1&1\\0&1&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like the matrix in Figure 2.3 and $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ From Lemma 12 we know $M = M_1' \oplus_h M_2'$ where $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_1'$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2'$. The second case would be dealt with symmetrically. From Theorem 6 we have that M_1' can be characterized exactly like $M[[m],[c_2-1]$ and $M[[m],[c_2,n]]$ forms a walking matrix. The only problem with our claim would be if there were two different columns having a one-entry above the r-th row. In that case, those two one-entries together with a one-entry in the r-th row between the columns c_1 and c_2 and a one-entry in the c_1-th column above the r-th row form P as an interval minor.

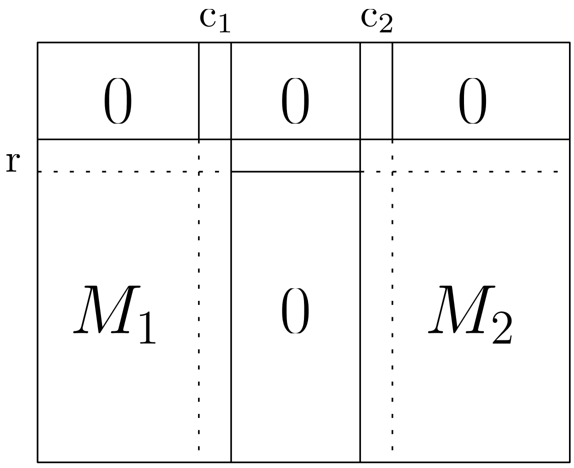

Figure 2.3: Characterization of a matrix avoiding $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ as an interval minor.

$\Leftarrow$ The bottom-middle one-entry of P can not be mapped anywhere but to the r-th row, but in that case there are at most two columns having one-entries above it.

$\square$

Theorem 14. *Let $P = \left(\begin{smallmatrix}0&1&1\\1&0&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ contains a walk w, no one-entries below the walk and for each entry $M[r, c]$ of the walk there is at most one non-empty column in $M[[r-1], [c+1, n]]$.*

Proof. $\Rightarrow$ Let w be any walk containing all the top-most and right-most entries that are bottom-left empty. From the choice of w, there are no one-entries below it and if all $M[r, c]$, $M[r-1, c]$ and $M[r, c+1]$ are on w then $M[r, c]$ is a one-entry as else $M[r, c]$ was neither top-most nor right-most bottom-left empty. As a consequence, whenever we choose $M[r, c]$ from w, it either is a one-entry or there is one-entry in the same row to the left of it. For contradiction let us now assume that there is an entry of the walk $M[r, c]$ for which there are two non-empty columns in $M[[r-1], [c+1, m]]$. Then a one-entry from each of those columns and a one-entry in $M[r, c]$ or to the left of it together give us $P \preceq M$ and consequently a contradiction.

$\Leftarrow$ For contradiction let $P \preceq M$. Without loss of generality we can assume that the bottom-left entry of P is mapped somewhere to the walk – to $M[r, c]$. But then $\left(\begin{smallmatrix}1&1\end{smallmatrix}\right) \preceq M[[r-1], [c+1, n]]$ which is a contradiction with it having one-entries in at most one column.

$\square$

2.3 Empty rows and columns

Observation 15. *Let $P \in \{0,1\}^{k \times l}$ and $P' \in \{0,1\}^{k \times l+1}$ such that $P' = P \oplus_h 0^{k \times 1}$, similarly let $M \in \{0,1\}^{m \times n}$ and $M' \in \{0,1\}^{m \times n+1}$ such that $M' = M \oplus_h 1^{m \times 1}$, then $P \preceq M \Leftrightarrow P' \preceq M'$.*

Proof. $\Rightarrow$ Clearly we can map the last column of P' to the last column of M' and then map (using OR) $P'[[k], [l]]$ to $M'[[m], [n]]$ the same way P is mapped to M.

$\Leftarrow$ If $P' \preceq M$ we are done. Otherwise, the last column of P' needs to be mapped to the last column of M' and by deleting both from their matrix we get $P'[[k],[l]] \preceq M'[[m],[n]]$ which is the same as $P \preceq M$.

□

The same proof can be also used for adding an empty column as the first column or an empty row as the first or the last row. Using induction we can easily show that a pattern P' is avoided by a matrix M' if and only if P is avoided by M where P is derived from P' by excluding all empty beginning or ending rows and columns and M is derived from M' by excluding the same number of beginning or ending rows and columns. Therefore, when characterizing matrices avoiding a forbidden pattern, we do not need to consider patterns having empty rows or columns on their boundary.

For the following two statements, let $P \in \{0,1\}^{k\times 2}$ be a forbidden pattern and $P^+ \in \{0,1\}^{k\times 3}$ be the pattern created from P by adding a new column in between the two columns of P.

Lemma 16. *Let $M \in Av(P^+)$ be an inclusion maximal matrix, then each row of M either contains no one-entries or exactly one interval of one-entries of length at least 2.*

Proof. If there are two one-entries on the same row and there is a zero-entry in between them, we can change the zero-entry into a one-entry and the new matrix will still avoid P^+.

If there is only one one-entry in a row, we can always change one of its neighbors into a one-entry. If the one-entry lies in the first column, we can insert another one-entry to the second column and the matrix will still avoid P^+, because if the new one-entry is used in a mapping of P^+ then it must as a part of the first column and we could use the one-entry to its left instead. Similarly, there is no one-entry in the last column that is the only one in it's row. For contradiction, assume there is the only one one-entry in a row, call it $e = M[r,c]$, that is not in the first nor in the last column and assume we cannot change neither $e_l = M[r,c-1]$ nor $e_r = M[r,c+1]$ to a one-entry, because then the matrix would contain the minor P^+. Clearly, all mappings of P^+ that are created by a change of e_1 contain e_1 and the element of P^+ that is mapped to e_1 comes from the first column and is on r_1-th row (there might be multiple of them so just take any), because if it came from the middle row, we could have used the original zero-entry instead as the middle columns is empty and if it came from the last row, we could have used e instead. Similarly, $P^+[r_2,3]$ is mapped to e_2 when e_2 is changed into a one-entry. We now want to show that the two partitionings can be altered to have a mapping of P^+ to M. We can always map the middle column of P^+ to the single column where e is presented and we describe how to partition M into k rows. We have two cases to go through:

$r_1 \neq r_2$ TODO

$r_1 = r_2$ TODO

□

TODO generalize the result for addition of multiple empty columns

Theorem 17. *For all $M \in \{0,1\}^{m \times n}$ it holds $M \in Av(P^+) \Leftrightarrow$ there exists $N \in \{0,1\}^{m \times (n-1)}$ such that $N \in Av(P)$ is inclusion maximal and M is a submatrix of $N \oplus_h 0^{m \times 1}$ placed over $0^{m \times 1} \oplus_h N$ with an operation bitwise OR.*

Proof. $\Rightarrow$ It suffices to only prove the statement for M that is inclusion maximal. To do so, we use Lemma 16. It say that each row of M contains either no one-entry or an interval of length at least two. From that we define N to be created from M by deleting the last one-entry on each row and excluding the last column. Clearly, M is equal to $N \oplus_h 0^{m \times 1}$ placed over $0^{m \times 1} \oplus_h N$ with an operation bitwise OR. If $P \preceq N$ then each mapping of P can be extended to a mapping of P^+ to M by ... How to say this?

TODO

$\Leftarrow$ It suffices to show that M that is equal to $N \oplus_h 0^{m \times 1}$ placed over $0^{m \times 1} \oplus_h N$ with an operation bitwise OR belongs to $Av(P^+)$. For contradiction, assume it does not. Then there is mapping of P^+ into elements of M and we can assume that one-entries of the first column of P^+ are mapped to those one-entries of M created from $N \oplus_h 0^{m \times 1}$. If it was not the case and there was a one-entry mapped to a one-entry of M created only from $0^{m \times 1} \oplus_h N$ we can take an element directly to its left and that is created from $N \oplus_h 0^{m \times 1}$. Symmetrically, all one-entries of the last column of P^+ are mapped to one-entries created from $0^{m \times 1} \oplus_h N$. ...

TODO

□

Lemma 18. *Let $P \in \{0,1\}^{k \times l}$ and let $M \in \{0,1\}^{m \times n}$ be an inclusion maximal matrix such that $P \preceq M$, then each row of M contains at most $l-1$ intervals of one-entries and each column of M contains at most $k-1$ intervals of one-entries.*

Proof. It suffices to prove the statement only for rows. Let us proceed by contradiction and let us have a row r that contains at least l intervals of one-entries. Let Z_1 denote the interval of zero-entries following the first interval of one-entries and similarly Z_i denote the interval of zero-entries following the i-th interval of one-entries for $i \in [k]$. Because M is inclusion maximal, each change of a zero-entry to a one-entry will create a copy of the forbidden minor and the new one-entry will be a part of each such copy.

We will show that changing a zero-entry from Z_i can only create a copy where the changed entry is part of the $i+1$ or higher column. We proceed by induction:

$i = 1$ For contradiction assume that changing any $e \in Z_1$ creates a copy of minor P where some element of the first column of P is mapped to e. In this case, because Z_1 follows after an interval of one-entries, any one-entry lying before e on the same row can play the same role in the mapping and we have $P \preceq M$ which is a contradiction.

$i > 1$ From the induction hypothesis, we know that changing a zero-entry of Z_{i-1} will create a copy of P where the changed element is used to map the i-th or further column of P. In particular, the first $i-1$ columns can always be mapped (even without changing any entry) to columns preceding Z_{i-1} and therefore preceding Z_i. This means that if a change of a zero-entry of

Z_i introduces a mapping of P that uses the new one-entry to map any of columns 1 to $i-1$, we can combine the mapping with the fact that the first $i-1$ columns can be mapped before Z_i and find that $P \preceq M$ in the following way:

Similarly, we are done if any change of an element e' of Z_{i-1} is used to map the $i+1$ or higher column of P; therefore, we assume each such change only allows P to map the i-th column there. The last case we need to take care of is when the change of an element $e \in Z_i$ creates a mapping of P where the i-th column uses e. Let r denote a row of P that is mapped to e when e is a one-entry and let r' denote a row of P that is mapped to e' when e' is a one-entry.

$r = r'$ ... This is really hard to describe, probably will use a picture ... In this case, we take the partitioning created by both mappings and extend it to a partitioning of mapping P into M without e or e' being one-entries. We simply

□

Open questions

- insertion of an empty column in between all columns of P

2.4 Multiple patterns

Theorem 19. *Let $P_1 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ and $P_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$, then for all M: $P \not\preceq M \wedge P \not\preceq M \Leftrightarrow M$ contains a walk w and each one-entry e is either on the walk w or both element directly above e and directly to the right of e are on the walk w.*

Proof. $\Rightarrow$ Let us take a walk w containing all the left-most and bottom-most top-right empty elements of M. Clearly, every top-right "corner" entry of w ($M[r, c]$ such that both $M[r+1, c]$ and $M[r, c-1]$ are on w) is a one-entry. Now consider for contradiction there is a one-entry anywhere but on w or directly diagonally below any top-right corner of w. Then this one-entry together with at least one top-right corner of w give us either P_1 or P_2 and thus a contradiction.

$\Leftarrow$ If we take any one-entry e, from the description of M there is no one-entry that would create either of P_1 or P_2 with e.

□

3. Pattern size constrains

In the previous chapter, we characterized what matrices avoiding small patterns look like. Their structure is very dependent of the pattern and the results are hard to generalize for arbitrary patterns. In this chapter, we will look for a more general property that restricts the complexity a class of matrices avoiding a given pattern can have.

Definition 7. In a matrix $M \in \{0,1\}^{m \times n}$ a *one-interval* in a row r is a sequence of one-entries in $M[\{r\},[c_1,c_2]]$ such that either $c_1 = 1$ or $M[r, c_1 - 1]$ is a zero-entry and at the same time either $c_2 = n$ or $M[r, c_2 + 1]$ is a zero-entry. In other words, one-interval is an interval of one-entries bounded by the edge of matrix or zero-entries. We use the same name also for interval of one-entries in a column. In the same spirit we also define *zero-interval* to be an interval of zero-entries bounded by one-entries or a edge of matrix.

In Sections ref, it always holds that for a pattern P of size $k \times l$ any inclusion maximal matrix M that avoids P as an interval minor has at most $l-1$ one-intervals in each row and at most $k-1$ one-intervals in each column. A natural question is whether the size of a pattern indeed bounds the number of one-intervals of any inclusion maximal matrix that avoids it.

Before we start, let us present some useful notion. First of all, every time we speak about *maximal* matrices from a class, we mean inclusion maximal – it has no zero-entry that can be changed to a one-entry and still belong to the class. In terms of pattern avoidance maximal matrices are those for which a change of a zero-entry creates a mapping of the pattern (or possibly many mappings).

Definition 8. Let P be a pattern, e a one-entry of P, M be a maximal matrix such that $P \not\preceq M$ and z arbitrary zero-interval of M. We say that z *is usable for* e if there is a zero-entry contained in z such that if we change it to a one-entry, it creates a mapping that uses the new one-entry to map e. Note that z can be usable for many one-entries of P at the same time.

Definition 9. Let $\mathcal{P}$ be a class of patterns. We say that $Av(\mathcal{P})$ is *bounded* if there is a function dependent only on the size of patterns in $\mathcal{P}$ such that every maximal $M \in Av(\mathcal{P})$ has the number of one-intervals in each row and column bounded by the function. Otherwise, we say that $Av(\mathcal{P})$ is *unbounded.* To denote the same situation, we also say that patterns in $\mathcal{P}$ are *bounded* or *unbounded.*

Definition 10. Let $P \in \{0,1\}^{k \times l}$ be a pattern and e any of its one-entries. We say that e is *row-bounded* if there is a function of k and l such that for each maximal matrix avoiding P and each of its rows the number of zero-intervals that are usable for e is bounded by the function. In symmetric way, we also define *column-bounded* one-entries of e. We say that e is *bounded* if it is both row-bounded and column-bounded.

Let us look on how definitions relate to each other. Let us have a pattern P and assume each of its one-entries is bounded. From the definition it immediately follows that P is bounded and that is equivalent to saying that class $Av(P)$ is bounded. It also holds that if a one-entry of P is row-bounded then the same one-entry of P^T is column-bounded.

3.1 Motivation

Theorem 20. *Let $P \in \{0,1\}^{k\times 2}$ and $M \in \{0,1\}^{m\times n}$ be a maximal matrix such that $P \preceq M$, then M contains at most one one-interval in each row.*

Proof. For contradiction, assume there are several one-intervals in the same row of M. Because M is maximal avoiding P as an interval minor, changing any one-entry e in between two consecutive one-intervals creates a mapping of the forbidden pattern. Such a mapping uses the changed one-entry to map some element $P[r', 1]$ or $P[r', 0]$. In the first case, the same mapping also works if we use any one-entry that is to the left from e instead of e, which leads to $P \preceq M$ and therefore a contradiction. In the second case, the mapping can use any one-entry to the right from e instead of e and therefore, there is no mapping of P to M even if e is a one-entry. That is a contradiction with M being maximal. □

TO DO ref to what we proved by this

Theorem 21. *Let $P \in \{0,1\}^{k\times 2}$ and $M \in \{0,1\}^{m\times n}$ be a maximal matrix such that $P \preceq M$, then M contains at most $2k+2$ one-intervals in each column.*

Proof. Given an arbitrary maximal matrix M avoiding P let us look at an arbitrary column c. For contradiction, assume it has at least $2k+3$ one-intervals which also means there are at least $2k+1$ zero-intervals in between. Since P has at most $2k$ one-entries, from the Pigeonhole principle there are two zero-intervals usable for the same one-entry.

TO DO describe that mapping of the middle row is bounded by the edges of zero-interval and define the common mapping. □

Corollary 22. *Every $P \in \{0,1\}^{k\times 2}$ is bounded.*

We see that for patterns having only two rows or columns we can indeed bound the number of one-intervals of maximal matrices avoiding them. On the other hand, already for a pattern of size 3×3 we show that there are matrices with arbitrarily many intervals of one-entries and we cannot get their number smaller.

3.2 Unbounded number of one-intervals

Theorem 23. *Let $P = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$. For every $n > 1$ there is a maximal matrix M avoiding P as an interval minor having n one-intervals (P is unbounded).*

Proof. Let M be a $(2n-1) \times (2n-1)$ matrix described by the picture:

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 & 1 \\
 & & & & & \cdots & & & 1 & 1 & 1 \\
 & & & & & \cdots & & & 1 & 1 & 1 \\
 & & & & & \cdots & 1 & 1 & 1 & & \\
 & & & & & \cdots & 1 & 1 & 1 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & 1 & 1 & 1 & \cdots & & & & & \\
 & & 1 & 1 & 1 & \cdots & & & & & \\
1 & 1 & 1 & & & \cdots & & & & & \\
1 & 1 & 1 & & & \cdots & & & & &
\end{pmatrix}$$

$P \not\preceq M$ because we always need to map $P[2,1]$ and $P[3,3]$ to just one "block" of one-entries of M which only leaves zero-entries where we need to map $P[1,2]$.

When we change any zero-entry of the first row into a one-entry (getting M') we get a matrix containing a minor of $\{1\}^{3\times 3}$; therefore, containing P as an interval minor. In case M is not maximal, we can add some more one-entries to make it maximal but it will still contain a row with n intervals of one-entries. $\square$

Not only M is a maximal matrix avoiding P but also avoiding any $P' \in \{0,1\}^{3\times 3}$ such that $P \preceq P'$. Its rotations avoid rotations of P and we can deduce that a big portion of patterns of size 3×3 are unbounded. Moreover, the result can be generalized also for bigger matrices. The pattern is so important that we call it P_1 for the rest of the chapter.

Theorem 24. *For every P' such that $P_1 \preceq P'$ and every $n > 1$ there is a maximal matrix M avoiding P' as an interval minor having n one-intervals.*

Proof. First, assume there is a mapping of P_1 into $P' \in \{0,1\}^{k\times l}$ that assigns a one-entry of the first row to $P_1[1,2]$, a one-entry of the first column to $P_1[2,1]$ and a one-entry of the last row and column to $P_1[3,3]$. Then, we can construct a similar matrix as we did in the proof of Theorem 23 avoiding P' but after changing any zero-entry of the first row it contains the whole $\{1\}^{k\times l}$.

Let P' be any pattern containing P without additional restrictions. Let $P'[r_1, c_1], P'[r_2, c_2]$ and $P'[r_3, c_3]$ be one-entries of P' that can be used to map $P_1[1,2], P_1[2,1]$ and $P_1[3,3]$ to it respectively. Then we take a submatrix $P'' := P'[[r_1, r_3], [c_2, c_3]]$. Such a matrix fulfills assumptions of the more restricted case stated at the beginning of the proof and we can find a maximal matrix M' avoiding P'' having n one-intervals. We construct M from M' by simply adding new rows and columns, all containing one-entries. We add $r_1 - 1$ rows in front of the first row and $k - r_3$ rows behind the last row. We also add $c_2 - 1$ columns in front of the first column and $l - c_3$ columns behind the last column. Constructed matrix M avoids pattern P' as its submatrix P'' cannot be mapped to M without added rows and columns. At the same time, any change of a zero-entry of the r_1-th row of M to one-entry creates a copy of $1^{k\times l}$ so the changed matrix contains P'. $\square$

3.3 Bounded number of one-intervals

What makes it even more interesting is that any pattern avoiding all rotations of P_1 is already bounded. To prove that we need a few partial results.

Theorem 25. *Let P be a pattern avoiding all rotations of P_1, then P:*

1. *contains one non-empty line or*
2. *contains two non-empty lines or*
3. *contains three non-empty lines or*
4. *avoids $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$.*

Proof. Assume P has four one-entries that do not share any row or column. Then those one-entries induce a 4×4 permutation inside P and because P does not contain any rotation of P_1, the induced permutation is either 1234 or 4321.

Without loss of generality, assume is the the first case and denote the one-entries by e_1, e_2, e_3 and e_4.

For contradiction with the statement, assume P also contains $P' = \left(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right)$. Clearly, any one-entry from e_1, e_2, e_3 and e_4 cannot be a part of any mapping of P' because it would introduce a mapping of a rotation of P_1. Let $e_2 = P[r_2, c_2]$ and $e_3 = P[r_3, c_3]$. First, the submatrix $P[[r_2], [c_2, l]]$ does not contain P' because then this copy of P' together with e_4 would give us a rotated copy of P_1. Symmetrically, $P[[r_3, k], [c_3]]$ does not contain P'. Also, $P[[r_3 - 1], [c_3 - 1]]$ and $P[[r_2 + 1, k], [c_2 + 1, l]]$ are empty (else they would together with e_2 and e_3 give us a mapping of a rotation of P_1). Up to rotation, the only possible way to have $P' \preceq P$ is that the top one-entry of P' is in the submatrix $P[[r_3 - 1], [c_2, c_3 - 1]]$ but then this entry together with e_1 and e_3 give us a mapping of a rotation of P_1 which is a contradiction. ☐

Now comes the hard part. For each group of patterns, we need to prove all of them are bounded.

Lemma 26. *Let $P \in \{0,1\}^{k \times l}$ be a pattern having only one non-empty line. Then for every maximal matrix $M \in \{0,1\}^{m \times n}$ avoiding P the number of one-intervals in each row and column is bounded by $O(k+1)$.*

Proof. Let the non-empty line of P be a row r. Since M is maximal, submatrices $M[[r-1], [n]]$ and $M[[m-r+1, m], [n]]$ contain no zero-entry and each of their rows contains just one interval of one-entries. If we look at any other row, it cannot contain k one-entries, so the maximum number of one-intervals is $k-1$.

Let us look on an arbitrary column c of M. If there is at least one one-entry in $M[[r, m-r], c]$ then because M is maximal, the whole column is made of one-entries. Otherwise, there are two intervals of one-entries – $M[[r-1], c]$ and $M[[m-r, m], c]$. ☐

Lemma 27. *Let $P \in \{0,1\}^{k \times l}$ be a pattern having two non-empty lines. Then for every maximal matrix $M \in \{0,1\}^{m \times n}$ avoiding P the number of one-intervals in each row and column is bounded by $2k + 2l + 3$.*

Proof. First we assume the two non-empty lines of P are rows $r_1 < r_2$ (or symmetrically columns). From Observation 15 and maximality of M we have that $M[[r_1 - 1], [n]]$ and $M[[m - r_2 + 1, m], [n]]$ contain no zero-entry. Therefore, we may restrict ourselves to case $r_1 = 1$ and $r_2 = k$. From Theorem 21 we know that every maximal N avoiding $P[\{r_1, r_2\}, [n]]$ has at most $2k+3$ one-intervals of one-entries in each row and at most 1 one-interval in each column. From Theorem 17 we also know that for given M there is a maximal N avoiding $P[\{r_1, r_2\}, [n]]$ such that M is submatrix of shifted and OR-ed copies of N. Since M is maximal it is equal to those shifted and OR-ed copies of N and since the number of one-intervals of N is bounded, so is the number of one-intervals of M.

Assume the two non-empty lines of P are row r and column c. Because the proof is symmetric, we only show the bound for rows. Let us take an arbitrary row of M an look at its zero-intervals. For every one-entry e of the pattern except those in the r-th row, there is at most one zero-interval usable for e. For contradiction, assume there are two such zero-intervals. Let Figure 3.1 illustrate the situation where dashed and dotted lines form a partitioning of the minor

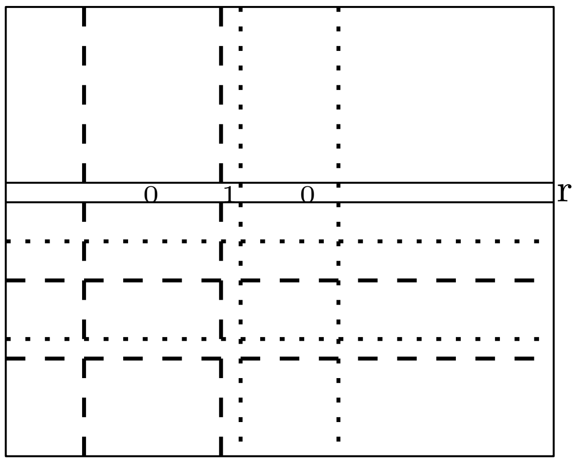

Figure 3.1: Dashed and dotted lines resembling two different mappings of a forbidden pattern, where two horizontal lines show the boundaries of the mapping of row r and the vertical lines show boundaries of the mapping of column c.

P in M when a respective zero-entry of our two zero-intervals is changed to a one-entry. When we take the outer two vertical and horizontal lines, we get a mapping of P that can use an existing one-entry in between the two zero-intervals to also map e. This gives us a contradiction with $P \not\prec M$. Therefore, for every one-entry e of P from the r-th row there is at most one zero-interval usable for it. This gives us the bound. □

To argue about the last two cases more easily, we introduce two helpful lemmata.

Lemma 28. *Let $P \in \{0,1\}^{k \times l}$ be a pattern looking like one of the matrices in Figure 3.2. Then every one-entry from row r_2 in columns c_1 to c_2 is row-bounded.*

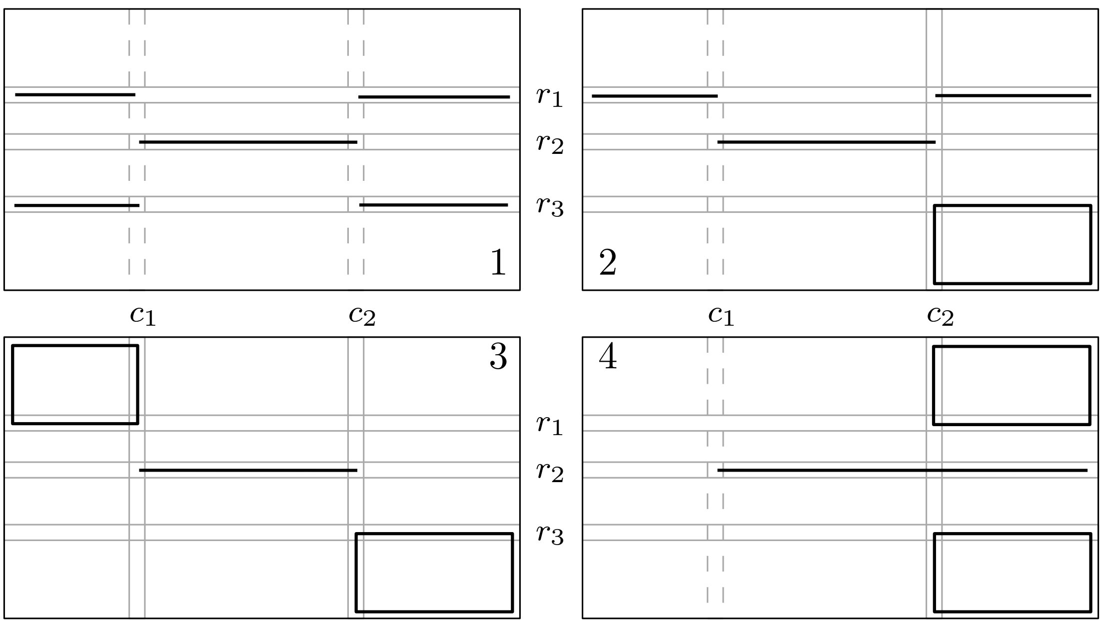

Figure 3.2: Patterns for which one-entries in row r_2 are row-bounded. One-entries may only be in the areas enclosed by bold lines.

Proof. Let P be any of the first three described patterns and let $k' = c_2 - c_1$. We even show that for each one-entry e from row r_2 and every M maximal matrix avoiding P there is at most k' zero-intervals for which it is usable. For contradiction assume there is a row r with $k'+1$ zero-intervals usable for e. It follows that there are at least k' one-entries in between two most distant zero-intervals z_1 and z_2. Therefore, the whole row r_2 can be mapped just to r. Since changing a zero-entry of z_1 to a one-entry to which e can be mapped, there is a partitioning of M where all one-entries from columns 1 to c_1 are mapped to columns before z_1 and similarly all one-entries from columns c_2 to l are mapped to columns past z_2. To partition rows, we can simply map rows from $r_1 + 1$ to $r_3 - 1$ around row r one to one and use the rest to find enough one-entries for the one-entries of P. The partitioning using those one-entries and one-entries from r to map one-entries of r_2 together give us $P \preceq M$ and a contradiction.

Make a picture? The explanation is not clear at all.

Let us look on the fourth case. For i-th one-entry in row r_2 (ordered from left to right and only considering those in columns c_1 to c_2) no zero-interval of a maximal matrix avoiding the pattern cannot have i one-entries to the left of it and so each such one-entry is bounded by $i \geq l$. □

Lemma 29. *Let P be a pattern and c be its first non-empty column. Then every one-entry from c is row-bounded.*

Proof. The results follows immediately from the fourth case of Lemma 28 when there are no one-entries in columns before c_2. □

Lemma 30. *Let $P \in \{0,1\}^{k \times l}$ be a pattern looking like one of the matrices in Figure 3.3. Then every one-entry from column c in rows r_1 to r_2 is row-bounded.*

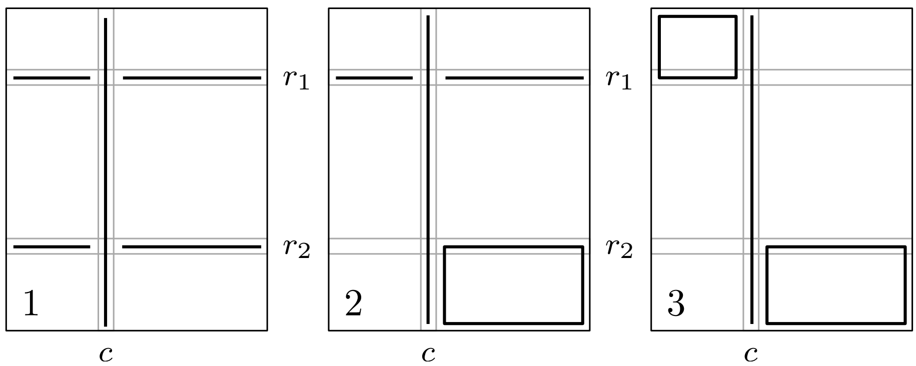

Figure 3.3: Patterns for which one-entries in column c are row-bounded. One-entries may only be in the areas enclosed by bold lines.

Proof. Let P be any of the described patterns. We even show that for each one-entry e from row r_2 and every M maximal matrix avoiding P there is at most one zero-interval for which it is usable. For contradiction assume there is a row r with two zero-intervals usable for e.

Again a picture needed – take the closer partitioning and it is done □

Lemma 31. *Let $P \in \{0,1\}^{k \times l}$ be a pattern avoiding $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ (or $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$). Then for every maximal matrix $M \in \{0,1\}^{m \times n}$ avoiding P the number of one-intervals in each row and column is bounded.*

Proof. From Theorem 4 we know that P is a walking pattern. Every one-entry of P satisfies either conditions of the third case of Lemma 28 or it satisfies conditions of the third case of Lemma 30 and therefore is row-bounded. To prove it is also column-bounded, we can look at P^T and show that its one-entries are row-bounded. Since it is again a walking pattern, we can use the same arguments. $\square$

Lemma 32. *Let $P \in \{0,1\}^{k\times l}$ be a pattern having three non-empty lines and avoiding all rotations of P_1. Then for every maximal matrix $M \in \{0,1\}^{m\times n}$ avoiding P the number of one-intervals in each row and column is bounded.*

Proof. First of all, if P avoids $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ we can use Lemma 31. Without loss of generality we assume it contains both but avoids every rotation of P_1.

Let us prove that each pattern having one-entries in three rows is bounded. If all one-entries are in up to two columns then we are again done. Therefore, P has one-entries in at least three columns and so it contains a three by three permutation matrix as a submatrix (or an interval minor). Since rotations of P_1 are avoided, that permutation is either 123 or 321 and without loss of generality we assume the first case. In Figure 3.4 we see the structure of each such pattern. Capital letters stand for one-entries of the permutation, letters $a-f$ stand each for a potential one-entry and greek letters stand each for a potential sequence of one-entries and zero-entries. Everything else is zero. Not all one-entries can be present at the same time, because that would create a mapping of P_1 or its rotation and we also need to find $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. The following analysis will only use hereditary arguments. This mean that if we prove P is bounded, we also prove that each submatrix of P is bounded. With this in mind, we restrict ourselves to maximal patterns.

- γ contains a one-entry $\Rightarrow f=0 \Rightarrow$ because $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ needs to be there it holds $a=1 \Rightarrow \alpha=0$
  - $d=1 \Rightarrow b=0,\ \beta=0,\ e=0,\ c=?$:
    Lemma 28 (case 4): one-entries in c, C, γ are row-bounded.
    Lemma 29: a and A are row-bounded.
    Lemma 30 (case 1): d and B are row-bounded.

    Lemma 29: all one-entries except for B are column-bounded.
    Lemma 28 (case 1): B is column-bounded.
  - $d=0$
    * $c=1 \Rightarrow \beta=0,\ e=0,\ b=?$:
      Lemma 28 (case 4): one-entries in c, C, γ are row-bounded.
      Lemma 29: a, b, A are row-bounded.
      Lemma 28 (case 1): B is row-bounded.

      Lemma 29: one-entries in the first and the third non-empty rows are column-bounded.
      Lemma 28 (case 2): b, B are column-bounded.
    * $c=0 \Rightarrow$ in the maximal case $b=1,\ e=1,\ \gamma$ contains a one-entry:
      Lemma 28 (case 4): one-entries in c, C, γ are row-bounded.

Lemma 29: one-entries in the first non-empty column are row-bounded.
Lemma 28 (case 1): one-entries in the middle non-empty row are row-bounded.

Lemma 29: one-entries in the first and the third non-empty rows are column-bounded.
Lemma 30 (case 2): one-entries in the middle non-empty row are column-bounded.

- $\gamma = 0$
  - α contains a one-entry $\Rightarrow a = 0,\ b = 0$:
    Every such pattern has already been dealt with as we can rotate it by 180 degrees, map A and α to γ, map d to C and so on.
  - $\alpha = 0$:
    Without loss of generality, we can assume that $a = 1$, because there needs to be $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and if we set $a = 0$, it must hold $f = 1$ and then we can just rotate the pattern by 180 degrees and get the case $a = 1$.
    * $d = 1 \Rightarrow b = 0,\ e = 0,\ \beta = 0,\ c = ?,\ f = ?$:
      Lemma 29: a, f, A and C are row-bounded.
      Lemma 30 (case 1): c, d and B are row-bounded.

      Lemma 29: one-entries in the first and third non-empty rows are column-bounded.
      Lemma 28 (case 1): B is column-bounded.
    * $d = 0$
      · $e = 1 \Rightarrow c = 0,\ b = ?,\ f = ?$:
        Since $\alpha = 0$ it follows that if there is a one-entry in β only if it can be in e.
        Lemma 29: a, f, A and C are row-bounded.
        Lemma 28 (case 1): one-entries in b, e, B and β are row-bounded.

        Lemma 29: a, f, A and C are column-bounded.
        Lemma 30 (case 1): one-entries in b, e, B and β are column-bounded.
      · $e = 0$:
        We can assume $c = 1$ as else e can be 1 and we have already dealt with that case. We can also assume $b = 1$ since otherwise, we would have a submatrix of the case dealt with when $d = 1$:
        Lemma 29: a, b and A are row-bounded.
        Lemma 28 (case 4): c and C are row-bounded.
        Lemma 28 (case 1): B is row-bounded.

        Because the pattern is symmetric, it is also column-bounded.

The same analysis also proves that if the pattern with the same restrictions only has three non-empty column it is bounded.

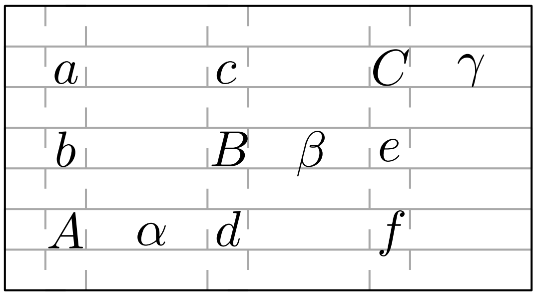

Figure 3.4: Structure of a pattern only having three non-empty rows and avoiding all rotations of P_1.

Let us now look at the case where the pattern that only have one-entries in either one of two rows r_1, r_2 or one column c_1. Without loss of generality we again assume permutation 123 is present and we distinguish three cases. Consider Figure 3.5:

- C lies in column c_1
  - $a = 1 \Rightarrow b = 0,\ \alpha = 0$ and everything else can be one:
    Lemma 28 (case 2): one-entries in a, c, B and β are row-bounded.
    Lemma 29: all other one-entries are row-bounded.

    Lemma 28 (case 4): one-entries in c, C and γ are column-bounded.
    Lemma 30 (case 1): one-entries in a, c, B and β are column-bounded.
    Lemma 29: d and A are column-bounded.
  - $a = 0$ and everything else can be one:
    Lemma 28 (case 4): one-entries in b, A and α are row-bounded.
    Lemma 28 (case 2): one-entries in c, B and β are row-bounded.
    Lemma 29: one-entries in c, d, C and γ are row-bounded.

    Lemma 28 (case 4): one-entries in c, C and γ are column-bounded.
    Lemma 30 (case 2): one-entries in c, B and β are column-bounded.
    Lemma 29: one-entries in b, d, A and α are column-bounded.
- B lies in column c_1
  - $a = 1 \Rightarrow \alpha = 0$
    * $d = 1 \Rightarrow \gamma = 0$:
      Lemma 30 (case 1): all one-entries in column c_1 are row-bounded.
      Lemma 29: all other one-entries are row-bounded.

      Lemma 28 (case 1): all one-entries in column c_1 are column-bounded.
      Lemma 29: all other one-entries are column-bounded.

* $d = 0$:
  Lemma 30 (case 1): all one-entries in column c_1 are row-bounded.
  Lemma 29: a and A are row-bounded.
  Lemma 28 (case 4): one-entries in C and γ are row-bounded.

  Lemma 28 (case 1): all one-entries in column c_1 are column-bounded.
  Lemma 29: all other one-entries are column-bounded.

- $a = 0$
  * $d = 1 \Rightarrow \gamma = 0$:
    Lemma 30 (case 1): all one-entries in column c_1 are row-bounded.
    Lemma 29: d and C are row-bounded.
    Lemma 28 (case 4): one-entries in A and α are row-bounded.

    Lemma 28 (case 1): all one-entries in column c_1 are column-bounded.
    Lemma 29: all other one-entries are column-bounded.
  * $d = 0$:
    Lemma 30 (case 1): all one-entries in column c_1 are row-bounded.
    Lemma 28 (case 4): one-entries in A, C, α and γ are row-bounded.

    Lemma 28 (case 1): all one-entries in column c_1 are column-bounded.
    Lemma 29: all other one-entries are column-bounded.

- A lies in column c_1:
  This is the first situation rotated by 180 degrees.

The same analysis also proves that if one-entries of a pattern with the same restrictions are in one row or two columns then the pattern it is bounded. □

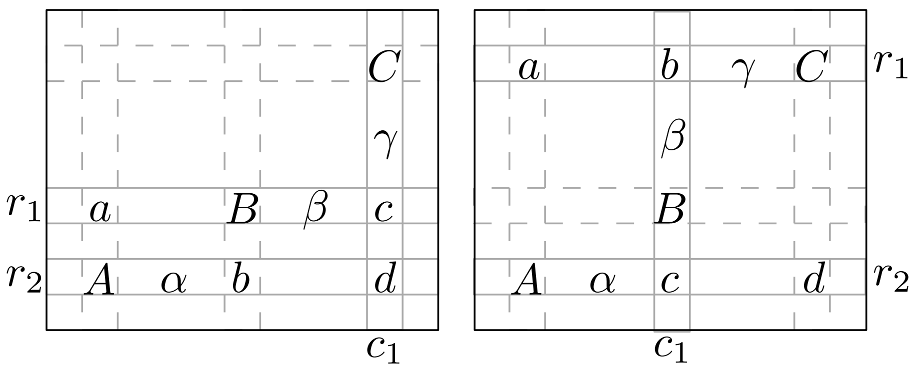

Figure 3.5: Structure of a pattern only having one-entries in two rows and one column that avoids all rotations of P_1.

Combining all the lemmata we finally get the following result.

Theorem 33. *Let P be a pattern avoiding all rotations of $P_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$, then P is bounded.*

3.4 Chain rules

Theorem 34. *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite classes of patterns. If both $\mathcal{P}$ and $\mathcal{Q}$ are bounded then $Av(\mathcal{P} \cup \mathcal{Q})$ is bounded.*

Proof. For a bounded pattern P, let $f(P)$ be a bound of the number of one-intervals of any maximal matrix avoiding P. For $\mathcal{P}$ let $f(\mathcal{P}) = \sum_{P \in \mathcal{P}} f(P)$. Let M be a maximal matrix avoiding $\mathcal{P} \cup \mathcal{Q}$ and take the line with the highest number of one-intervals. For every $P \in \mathcal{P} \cup \mathcal{Q}$ there is at most $f(P) + 1$ zero-intervals usable for P's one-entries. For contradiction assume there are more and change all zero-entries that are not usable for P's one-entries with one-entries. This way we end up with a maximal matrix avoiding P and a contradiction to $f(P)$ being the bound on the number of one-intervals of such matrix.

Together we then have $f(\mathcal{P} \cup \mathcal{Q}) \geq f(\mathcal{P}) + |P| + f(\mathcal{Q}) + |Q| + 1$ and since all numbers are finite, we have that $\mathcal{P} \cup \mathcal{Q}$ is bounded. $\square$

Using induction, we can show that also a union of finite number of bounded classes of finite size are bounded. Interestingly enough, unbounded classes are not closed the same way.

Fact 35 (Higman's lemma)**.** *Let A be a finite alphabet and A^* be a set of finite sequences over A. Then A^* is well quasi ordered with respect to the subsequence relation.*

Theorem 36. $Av\left(\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}\right)$ *is bounded. Moreover, every subclass is bounded.*

Observation 37. *There exists a non-trivial bounded pattern P having an unbounded subset of $Av(P)$.*

Proof. Let $P = I_n$ (identity matrix) for $n > 3$. From Lemma 31 we have that P is bounded. On the other hand, $Av(I_n, P_1)$ is unbounded, because the construction used in the proof of Theorem 23 also works for this class. $\square$

Open questions:

- $\mathcal{C}$ row-bounded $\Rightarrow$ $\mathcal{C}$ column-bounded
- $Av\left(\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}\right)$ bounded (hereditary)
- $Av\left(\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\right)$ bounded (hereditary)

4. Extremal function

Notation 6. Let M be a matrix. We denote $|M|$ the weight of M, the number of one-entries in M.

Usually $|M|$ stands for a determinant of matrix M. However, in this paper we do not work with determinants at all so the notation should not lead to misunderstanding.

Definition 11. For a matrix P we define $Ex(P, m, n) := \max\{|M| | M \in \{0,1\}^{m \times n},\ P \not\leq M\}$. We denote $Ex(P, n) := Ex(P, n, n)$.

Definition 12. For a matrix P we define $Ex_{\preceq}(P, m, n) := max\{|M| | M \in \{0,1\}^{m \times n},\ P \not\preceq M\}$. We denote $Ex_{\preceq}(P, n) := Ex_{\preceq}(P, n, n)$.

Observation 38. *For all P, m, n; $Ex_{\preceq}(P, m, n) \leq Ex(P, m, n)$.*

Observation 39. *If $P \in \{0,1\}^{k \times l}$ has a one-entry at position $[a, b]$, then*

$$Ex(P, m, n) \geq \begin{cases} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \textit{otherwise.} \end{cases}$$

Observation 40. *The same holds for $Ex_{\preceq}(P, m, n)$.*

Definition 13. $P \in \{0,1\}^{k \times l}$ is *(strongly) minimalist* if

$$Ex(P, m, n) = \begin{cases} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Definition 14. $P \in \{0,1\}^{k \times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P, m, n) = \begin{cases} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Observation 41. *If P is strongly minimalist, then P is weakly minimalist.*

4.1 Known results

Fact 42. *1. $\left(\begin{smallmatrix} 1 \end{smallmatrix}\right)$ is strongly minimalist.*

2. *If $P \in \{0,1\}^{k \times l}$ is strongly minimalist and there is a one-entry in the last row in the c-th column, then $P' \in \{0,1\}^{k+1 \times l}$, which is created from P by adding a new row having a one-entry only in the c-th column, is strongly minimalist.*

3. *If P is strongly minimalist, then after changing a one-entry into a zero-entry it is still strongly minimalist.*

Fact 43. *Let $P = \left(\begin{smallmatrix} 1 \cdots 1 \\ 1 \cdots 1 \end{smallmatrix}\right)$ have l columns, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{2\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{weight of } M[[i],\{j\}] > 0 \wedge \text{weight of } M[[i+1,m],\{j\} > 0]\}$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M appears in at least $b_j - 1$ of sets A_i, $0 \le i \le m-2$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 1) + n \le \sum_{i=0}^{m-2} |A_i| + n \le (l-1)(m-1) + n$$

□

This result is indeed very important because it shows that there are matrices like $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, which are weakly minimalist, although it is known they are not strongly minimalist.

Fact 44. *Let* $P = \left(\begin{smallmatrix}1 \dots 1\\1 \dots 1\\1 \dots 1\end{smallmatrix}\right)$ *have* l *columns, then* P *is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{3\times l}$ as an interval minor and $A_i = \{j \in [n] |$ weight of $M[[i-1],\{j\}] > 0 \wedge$ weight of $M[[i+1,m],\{j\} > 0 \wedge M[i,j]$ one-entry$]\}$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M (for which $b_j \ge 2$) appears in exactly $b_j - 2$ of sets A_i, $1 \le i \le m-1$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 2) + 2n \le \sum_{i=1}^{m-2} |A_i| + 2n \le (l-1)(m-2) + 2n$$

□

5. Operations with matrices

Notation 7. When speaking about a class of matrices, unless stated otherwise, we always expect the class to be closed under minors. Also, all classes discussed are non-trivial. That means that there is at least one matrix of size 2 by 1 and at least one matrix of size 1 by 2 in each class. Moreover, at least one matrix is non-empty.

Definition 15. Let $\mathcal{F}$ be any class of forbidden matrices. We denote by $Av(\mathcal{F})$ the set of all matrices that avoid every $F \in \mathcal{F}$ as an interval minor.

Observation 45. *Let $\mathcal{T} = Av(\mathcal{F})$ for some $\mathcal{F}$. Then $\mathcal{T}$ is closed under minors.*

Observation 46. *Let $\mathcal{M}$ be a finite class of matrices. There exists a finite set $\mathcal{F}$ such that $\mathcal{M} = Av(\mathcal{F})$.*

Definition 16. For matrices $A \in \{0,1\}^{m \times n}$ and $B \in \{0,1\}^{k \times l}$ we define their *direct sum* as a matrix $C := A \nearrow B \in \{0,1\}^{m+k \times n+l}$ such that $D[[k+1, m+k], [n]] = A$, $D[[k], [n+1, n+l]] = B$ and the rest is empty. Symmetrically, we define $D := A \searrow B \in \{0,1\}^{m+k \times n+l}$ such that $C[[m],[n]] = A$, $C[[m+1, m+k], [n+1, n+l]] = B$ and the rest is empty.

Theorem 47. $Av((\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})) = (Av((\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})) \searrow Av((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})) \searrow Av((\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}))) \cup$
$\cup\, (Av((\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})) \nearrow Av((\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})) \nearrow Av((\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})))$.

Proof. If follows from Theorem 9 and $Av((\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix})) = Av((\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})) \searrow Av((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$. □

Notation 8. Let $\mathcal{M}$ be a class of matrices. Denote by $Cl(\mathcal{M})$ a set containing each $M \in \mathcal{M}$ closed under direct sum and minors.

Definition 17. Let $M \in \{0,1\}^{m \times n}$ be a matrix. We call an element $M[r,c]$ an *articulation* of M if both $M[[r-1],[c-1]]$ and $M[[r+1,m],[c+1,n]]$ are empty.

Lemma 48. *Let $M \in \{0,1\}^{k \times l}$, then for all $X \in \{0,1\}^{m \times n}$ it holds $X \in Cl(M) \Leftrightarrow$ there exists a sequence of articulations of X such that each matrix in between two consecutive articulations of X is a minor of $(\,1\,) \nearrow M \nearrow (\,1\,)$.*

Proof. $\Rightarrow$

$\Leftarrow$

□

Theorem 49. *For all $M \in \{0,1\}^{k \times l}$ there exists $\mathcal{F}$ finite such that $Cl(M) = Av(\mathcal{F})$.*

Proof. Using Lemma 48 □

Theorem 50. $Cl((\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})) = Av\left((\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$.

Proof. $\subseteq$

$\supseteq$

□

Theorem 51. *$Cl\left(\left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)\right) = Av\left(\left(\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&0\\0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\\0&1\end{smallmatrix}\right)\right)$.*

We can generalize direct sum to allow the matrices to overlap.

Definition 18. TODO $A \oplus_{k\times l} B$

Theorem 52. *Let $\mathcal{C}$ be any class of matrices such that*

- *$\mathcal{C}$ is closed under deleting of one-entries and*
- *$\mathcal{C}$ is closed under the direct sum with $k \times l$ overlap and*
- *there is any $M \in \{0,1\}^{m\times n}$ in $\mathcal{C}$*

then $\mathcal{C}$ is also closed under direct sum with $m - 2k \times n - 2l$ overlap.

Proof. Choose any two $A, B \in \mathcal{C}$ and $C\mathcal{C}$ such that $C \in \{0,1\}^{m\times n}$. Let $D \in \mathcal{C}$ denote the direct sum with $k \times l$ overlap of A and C. Finally, let E be the direct sum with $k \times l$ overlap of D and B. It has the same size as F, the direct sum with $m - 2k \times n - 2l$ overlap of A and B, which set of one-entries is also a subset of one-entries of $E \in \mathcal{C}$; therefore $F \in \mathcal{C}$. □

Theorem 53. *Let $\mathcal{C}$ be any class of matrices that is hereditary according to interval minors then for all m, n, k, l if $\mathcal{C}$ is closed under the direct sum with $m \times n$ overlap then is is also closed under the direct sum with $m + k \times n + l$ overlap.*

Proof. For contradiction, assume there are $A, B \in \mathcal{C}$ such that $A \oplus_{m+k\times n+l} B \notin \mathcal{C}$. □

Observation 54. *There is a $\mathcal{C}$ hereditary according to submatrices such that it is closed under the direct sum but it is not closed under the direct sum with 1×1 overlap.*

Proof. Let $\mathcal{C}$ be a class of all matrices obtained by applying the direct sum on $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Clearly, it is closed under the direct sum. On the other hand, $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \oplus_{1\times 1} \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1&0\\1&0&1\\0&1&0\end{smallmatrix}\right) \notin \mathcal{C}$. □

Notation 9. We define $\mathrm{Av}(M)$ to be a class of all matrices avoiding M as

We state following characterization only for the direct sum with 1×1 overlap but, because of Theorem 53, it also holds for any other size of overlap.

Theorem 55. *Let M be a matrix. There are M_1, M_2 proper submatrices of M such that $M = M_1 \oplus_{1\times 1} M_2 \Leftrightarrow Av(M)$ is not closed under the direct sum with 1×1 overlap.*

Proof. $\Rightarrow$

$\Leftarrow$

□

Observation 56. *Let M be a matrix. There are M_1, M_2 proper submatrices of M such that $M = M_1 \oplus_{1\times 1} M_2 \Leftrightarrow$ exists r, c such that either*

1. $M[r,c]$ *is a one-entry and* $(r,c) \in \{(1,1),(m,n)\}$ *or*

2. $M[r,c]$ *is both top-right and bottom-left empty and* $(r,c) \notin \{(1,1),(m,n)\}$

Definition 19. Let F be a matrix. We denote $\mathcal{R}(F)$ to be a set of all minimal (relating to minors) matrices F' such that $F \preceq F'$ and F' is not a direct sum with 1×1 overlap of proper submatrices of F'. For a class of matrices $\mathcal{F}$ let $\mathcal{R}(\mathcal{F})$ denote a set of all minimal (relating to minors) matrices from the set $\bigcup_{F\in\mathcal{F}} \mathcal{R}(F)$.

Theorem 57. *Let* $\mathcal{T}$ *and* $\mathcal{F}$ *be classes of matrices such that* $\mathcal{T} = Av(\mathcal{F})$*, then* $Cl(\mathcal{T}) = Av(\mathcal{R}(\mathcal{F}))$*.*

Proof. Need to change the proof a bit probably after changing the statement

- $\subseteq$ Instead of proving $M \in Cl(\mathcal{T}) \Rightarrow M \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ we show $M \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F)) \Rightarrow M \notin Cl(\mathcal{T})$. Assume $M \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$. It follow from the definition that $M \in \bigcup_{F\in\mathcal{F}} \mathcal{R}(F)$, in particular, $M \in \mathcal{R}(F)$ for some $f \in \mathcal{F}$. Because of the definition of $\mathcal{R}(F)$, M is not a direct sum with 1×1 overlap of proper submatrices of M which means, according to Observation 56, there are no non-trivial articulations and both top-right and bottom-left corners are empty. For contradiction, assume $M \in Cl(\mathcal{T})$, then, according to a generalization of Lemma 48, there exists a sequence of articulations of M such that each matrix in between two consecutive articulations of M is a minor of $(\,1\,) \nearrow T \nearrow (\,1\,)$ for some $T \in \mathcal{T}$. Since M has only trivial articulations and they are both empty, it holds $M \preceq T$ and because of the choice of M is also holds $M \preceq F$ for some $F \in \mathcal{F}$ which together give us a contradiction with $\mathcal{T} = Av(\mathcal{F})$.

- $\supseteq$ First of all, $Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ is closed under the direct sum with 1×1 overlap. For contradiction, assume there are $M_1, M_2 \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ but $M = M_1 \nearrow_1 M_2 \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$. Then there exists $F' \in \mathcal{R}(F)$ for some $F \in \mathcal{F}$ such that $F' \preceq M$. Because F' is not a direct sum with 1×1 overlap of proper submatrices of F', it follows that either $F' \preceq M_1$ or $F' \preceq M_2$ and since $F \preceq F'$ we have a contradiction.

  Now that we know that both sides are closed under the direct sum with 1×1 overlap it sufficient to show that the inclusion holds for any $M \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ that is not a direct sum with 1×1 overlap of proper submatrices of M. Such M does not contain (again from Observation 56) any non-trivial articulation and those trivial ones are empty. Because of that it holds $F \not\preceq M$ for every $F \in \mathcal{F}$; otherwise either $M \in \mathcal{R}(F)$ or its minor would be there. Therefore $M \in \mathcal{T}$ and also $M \in Cl(\mathcal{T})$.

$\square$

Definition 20. Let T be a class of matrices. The *basis* of T is a set of all minimal (relating to minors) matrices that do not belong to T.

Corollary 58. *Let* $\mathcal{T}$ *and* $\mathcal{F}$ *be classes of matrices such that* $\mathcal{T} = Av(\mathcal{F})$*, then* $\mathcal{R}(\mathcal{F})$ *is a basis of* $Cl(\mathcal{T})$*.*

Proof. The proof follow directly from Theorem 57. $\square$

A natural question follows, whether the closure under direct sum of a class with finite basis has final basis. We prove that this is not the case.

Definition 21. Let $Nucleus_1 = \left(1 \right)$ and for $n > 1$ let $Nucleus_n \in \{0,1\}^{n \times n+1}$ be a matrix described by the examples:
$Nucleus_2 = \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \end{pmatrix}$, $Nucleus_3 = \begin{pmatrix} & 1 & 1 & \\ 1 & & & 1 \\ & 1 & 1 & \end{pmatrix}$, $Nucleus_4 = \begin{pmatrix} & & 1 & 1 & \\ & 1 & & & 1 \\ 1 & & & 1 & \\ & 1 & 1 & & \end{pmatrix}$,

$Nucleus_5 = \begin{pmatrix} & & & 1 & 1 & \\ & & 1 & & & 1 \\ & 1 & & & 1 & \\ 1 & & & 1 & & \\ & 1 & 1 & & & \end{pmatrix}$, $Nucleus_n = \begin{pmatrix} & & & & & & 1 & 1 & \\ & & & & & 1 & & & 1 \\ & & & & 1 & & & 1 & \\ & & & \cdots & & & 1 & & \\ & & 1 & & & \cdots & & & \\ & 1 & & & 1 & & & & \\ 1 & & & 1 & & & & & \\ & 1 & 1 & & & & & & \end{pmatrix}$.

Definition 22. Let $Candy_{k,n,l}$ be a matrix given by $I_k \nearrow_{1\times 2} Nucleus_n \nearrow_{1\times 2} I_l$, where I_k, I_l are unit matrices of sizes $k \times k$ and $l \times l$ respectively.

$$Candy_{4,1,4} \begin{pmatrix} & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ 1 & & & 1 & & & 1 \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \end{pmatrix} \quad Candy_{4,4,4} \begin{pmatrix} & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & 1 & 1 & & & 1 \\ & & & 1 & & & 1 & & \\ & & 1 & & & 1 & & & \\ 1 & & & 1 & 1 & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \end{pmatrix}$$

Theorem 59. *There exists a matrix F such that $\mathcal{R}(F)$ is infinite.*

Proof. □

Corollary 60. *There exists a class of matrices $\mathcal{C}$ having a finite basis such that $Cl(\mathcal{C})$ has an infinite basis.*

Proof. From Theorem 59, we have a matrix F for which $\mathcal{R}(F)$ is infinite. Let $\mathcal{C} = Av(F)$. Clearly, $\mathcal{C}$ has a finite basis. On the other hand, from Theorem 57 we have $Cl(\mathcal{C}) = Av(\mathcal{R}(F))$ and $\mathcal{R}(F)$ is infinite from the choice of F. □