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OF MATHEMATICS
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MASTER THESIS

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Hereditary classes of binary matrices

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Study programme: Computer Science
Study branch: Discrete Models and Algorithms

Prague 2017

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Abstract: Interval minors of binary matrices were introduced by Jacob Fox in the study of Stanley-Wilf limits. We study what can be implied from their relation to the theory of pattern avoidance of submatrices, which is a very popular area of discrete mathematics. We start be characterizing matrices avoiding small interval minors. We then consider classes of matrices closed under interval minors and with some help of the operation of skew sum, we find classes of matrices that cannot be described by a finite number of forbidden interval minors. We also define and study a variant of a classical extremal Turán-type question studied in the area of combinatorics of permutations and binary matrices and in combinatorial geometry.

Keywords: binary matrix pattern–avoiding interval minor

Dedication.

Contents

1. Introduction

TODO:

- Fix or rewrite Lemma 2.10.
- Consider adding more patterns/generalizations.
- Figure out what to do with Theorem 4.31.
- Consider fixing Lemma 4.33 (currently commented).

A binary matrix (or 0–1 matrix) is a matrix with ones and zeroes as its entries. In the thesis, we only consider binary matrices and so we omit the word binary. We say that a matrix M contains a matrix P as an interval minor, if P can be created from M by a sequence of deletion of one-entries and merges of neighboring rows or columns. Otherwise, we say M avoids P. To distinguish among matrices and to indicate the relationship, we usually call the matrix P a *pattern*.

When working with matrices, we always index rows from top to bottom and columns from left to right, starting with one. When we speak about a row r, we mean a row with index r. A *line* of a matrix is either a row or a column.

1.1 The main results

While a lot is know about matrices in general, because they can intuitively represent much more complex objects, interval minors are a fairly new topic and so we have a choice of the direction from which we want to approach them.

To get familiar with definitions and pattern avoidance in general, in Chapter 2, we focus on small patterns (having up to four one-entries only) and describe the common structure of matrices avoiding them.

We then turn our focus elsewhere in Chapter 3, and instead of looking for a structure of matrices avoiding a pattern, given a class of matrices (closed under interval minors) we find the smallest set of forbidden patterns that characterizes the class. We introduce the skew sum of two matrices and show that classes of matrices closed under the skew sum can always be described by a finite number of forbidden pattern. Using the operation more, we show that there are also other classes for which this cannot be achieved.

Because it is very useful to study extremal questions like the maximum number of one-entries of a matrix from a given class of matrices, in Chapter 4, we study a variant of such complexity question, where we instead focus on the maximum number k of appearances of pairs "01" and "10" on a single line of a matrix from a given class of matrices. We show that even for classes that are described by just one forbidden pattern, k can be unbounded, and we characterize exactly for which pattern this holds. Then we generalize the approach and show what influence an intersection of classes has on the number k.

1.2 Preliminaries

Notation 1.1. For $n \in \mathbb{N}$, let $[n] := \{1, 2, \ldots, n\}$ and for $m \in \mathbb{N}$ such that $n \leq m$, let $[n, m] := \{n, n+1, \ldots, m\}$.

Notation 1.2. For a matrix $M \in \{0,1\}^{m \times n}$, $R \subseteq [m]$ and $C \subseteq [n]$, let $M[R, C]$ denote a submatrix of M induced by row indices in R and column indices in C. Furthermore, for $r \in [m]$ and $c \in [n]$, let $M[r, c] := M[\{r\}, \{c\}]$.

The pattern avoidance for matrices is a generalization of a long studied theory of pattern avoidance for permutations. There are two generally used ways to define this generalization, either we avoid a matrix pattern as a submatrix or as an interval minor. While this thesis works almost exclusively with the latter, to better introduce the whole area, we start by defining the more know of the two approaches.

Definition 1.3. We say a matrix $M \in \{0,1\}^{m \times n}$ *contains* a pattern $P \in \{0,1\}^{k \times l}$ *as a submatrix* and denote it by $P \leq M$ if there are $R \subseteq [m]$ and $C \subseteq [n]$ such that $M' = M[R, C] \in \{0,1\}^{k \times l}$ and for every $r \in R$ and $c \in C$, if $P[r, c] = 1$ then $M'[r, c] = 1$.

Every matrix $M \in \{0,1\}^{m \times n}$ can be looked at as an adjacency matrix of a bipartite graph G_M with two sets of vertices $V_1 = [m]$ and $V_2 = [n]$ such that a vertex i from V_1 is adjacent to a vertex j from V_2 if and only if $M[i, j] = 1$. The order of vertices in each set is fixed and these graphs are usually called ordered bipartite graphs. In this setting, a matrix M contains a pattern P if the ordered bipartite graph G_P is a subgraph (not necessarily induced) of the ordered bipartite graph G_M.

In graph theory, the next step is to look at graph minors. A minor is created from a graph by a repeated applying of one of three graph operations: deletion of a vertex, deletion of an edge and a contraction of an edge. The same can be represented in terms of matrices:

Definition 1.4. We say a matrix $M \in \{0,1\}^{m \times n}$ *contains* a pattern $P \in \{0,1\}^{k \times l}$ *as an interval minor* and denote it by $P \preceq M$ if there is a sequence of elementary operations that applied to M creates P. The elementary operations are:

- a deletion of a line,
- a deletion of a one-entry (a change of a one-entry to a zero-entry) and
- a merge of two neighboring rows or columns into one that is the elementwise OR of the two original lines.

For simplicity, we do not consider a deletion of a line to be a separate operation as it can be replaced by a merge of the corresponding line with a neighboring one and a series of changes of one-entries to zero-entries. Moreover, like in the realm of graphs, we can assume all merging operations are done before the deletion of one-entries. This give us an alternative way to look at the problem.

Definition 1.5. Consider matrices P and M and let $P \preceq M$. A *mapping* of P to M is a function that maps each row of P to an interval of rows of M and each column of P to an interval of columns of M in such a way that if $P[r, c] = 1$ and r is mapped to R and c is mapped to C, there is a one-entry in $M[R, C]$. An *interval of rows* (columns) is a set of consecutive rows (columns). We say that an entry $P[r, c]$ is mapped to an entry $M[r', c']$ in a fixed mapping of P to M, in which r is mapped to R and c is mapped to C, if $r' \in R$ and $c' \in C$ and if $P[r, c] = 1$ then we also require $M[r', c'] = 1$.

Each mapping of a pattern P to a matrix M corresponds to a *partitioning* of M to intervals of rows and columns that creates a block structure. On the other hand, if we find a partitioning of M to blocks such that for each one-entry $P[r, c]$ there is a one-entry in the block that can be indexed $[r, c]$ then we have a mapping of P to M and so $P \preceq M$. This means:

Observation 1.6. *For all matrices P and M, there is a mapping of P to $M \Leftrightarrow P \preceq M$.* □

While pattern avoidance in terms of submatrices and interval minors seem to be very different, they have a quite tight relationship. The next observation immediately follows from their definitions.

Observation 1.7. *For all matrices P and M, $P \leq M \Rightarrow P \preceq M$.*

As said at the beginning of the section, both approaches generalize pattern avoidance for permutations and so it makes sense that they are equal for permutation matrices – matrices having exactly one one-entry in each line.

Observation 1.8. *For all matrices P and M, if P is a permutation matrix then $P \leq M \Leftrightarrow P \preceq M$.*

Proof. If we have $P \preceq M$, then there is a mapping m of P to M. To show $P \leq M$ we need to find R, C such that $M' = M[R, C]$ has the same size as P and for every $P[r, c] = 1$ it holds $M'[r, c] = 1$. We define R and C as follows. For every row r, let R' be the interval to which r is mapped in the mapping m. There is exactly one column c such that $P[r, c] = 1$ and c is mapped to some C'. Because m is a mapping, there is a one-entry $M[r', c']$ such that $r' \in R'$ and $c' \in C'$ and we add r' to R and we add c' to C.

The other implication follows from Observation 1.7. □

Definition 1.9. A *class* of matrices $\mathcal{M}$ is a set of matrices that is closed under interval minors. It means that for every $M \in \mathcal{M}$ and every $M' \preceq M$ it holds $M' \in \mathcal{M}$.

To avoid degenerate cases, we only consider classes of matrices containing at least one matrix of size 2×1, at least one matrix of size 1×2 and at least one matrix that is non-empty.

Definition 1.10. Let $\mathcal{P}$ be a set of patterns. We denote by $Av_{\preceq}(\mathcal{P})$ the set of all matrices that avoid each $P \in \mathcal{P}$ as an interval minor.

Observation 1.11. *For all patterns P and P': $P \preceq P' \Leftrightarrow Av_{\preceq}(P) \subseteq Av_{\preceq}(P')$.*

Proof. Because $P \preceq P'$, every matrix that avoids P also avoids P'. On the other hand, if $P \not\preceq P'$ then $P' \in Av_{\preceq}(P)$. As $P' \notin Av_{\preceq}(P')$, we have $Av_{\preceq}(P) \not\subseteq Av_{\preceq}(P')$. □

The following observation goes almost without saying and we use it throughout the whole thesis to break symmetries.

Observation 1.12. *Let P and M be matrices, $P \preceq M \Leftrightarrow P^T \preceq M^T$.*

1.3 Pattern avoidance

Pattern avoidance is a general topic in combinatorics. A lot of attention is directed towards permutations, see books Bóna [2012], Kitaev [2011] for references. It is a natural generalization to regard permutations as permutation matrices and consider matrix avoidance. This is mainly studied in terms of submatrices, so we discuss some interesting results in this section.

Interval minors are, on the other hand, a fairly new topic first defined by Jacob Fox in Fox [2013] as a tool to prove results about permutations in the study of Stanley–Wilf limits. Since then, a little has been discovered about the theory of interval minors. Nevertheless, we mention some results at the end of this section.

Let us go back to submatrices for now. The question that is particularly interesting is to determine the maximum number of one-entries that a matrix avoiding a given pattern can have. This property describes complexity of a pattern and can be used for example to prove algorithmic complexity, see Efrat and Sharir [1996].

Definition 1.13. Let M be a matrix. The weight of M, denoted by $|M|$, is the number of one-entries in M.

Definition 1.14. For a pattern P and integers m, n, we define the *weight extremal function* $Ex(P, m, n) := \max\{|M|;\ M \in \{0,1\}^{m \times n} \wedge P \not\leq M\}$.

Going back to the representation of the problem in terms of ordered bipartite graphs, the question to determine $Ex(P, m, n)$ is a variant of a classical Turán extremal graph question and was studied by many authors, see for example Tardos [2005], Füredi and Hajnal [1992] or, for a wider range of variants Brass et al. [2003], Claesson et al. [2012], Klazar [2004], Pach and Tardos [2006]. Some applications associated with the weight extremal function are discussed in Fulek [2009]. There are other extremal functions that have been studied, see for instance Cibulka and Kynčl [2016], but we do not consider them in this thesis.

In the same spirit, we also define the weight extremal function for matrices avoiding patterns as interval minors.

Definition 1.15. For a pattern P and integers m, n, we define $Ex_{\preceq}(P, m, n) := \max\{|M|;\ M \in \{0,1\}^{m \times n} \wedge P \not\preceq M\}$.

Thanks to Observation 1.7 we have the following relationship between the extremal functions.

Observation 1.16. *For all patterns P and integers m, n:*
$Ex_{\preceq}(P, m, n) \leq Ex(P, m, n)$. □

From Observation 1.11 it follows:

Observation 1.17. *For all patterns P and P' and integers m, n: $P \preceq P' \Rightarrow Ex_{\preceq}(P, m, n) \leq Ex_{\preceq}(P', m, n)$.*

It was showed in Marcus and Tardos [2004] that for every permutation matrix P and every n it holds $Ex(P, n, n) \leq c_P n$. While $Ex(P, n, n)$ can become even quadratic with n, because of the previous observation and the fact that every pattern $P \in \{0,1\}^{k \times l}$ is an interval minor of some permutation pattern $P' \in \{0,1\}^{(kl)\times(kl)}$ we have the following:

Proposition 1.18. *For every pattern P and integer n: $Ex_{\preceq}(P, n, n) \leq c_P n$ for some constant c_P independent of n.* □

The following observation for $Ex(P, m, n)$ was made by several authors; see for example Cibulka [2009], Fulek [2009].

Lemma 1.19. *If $P \in \{0,1\}^{k \times l}$ has at least one one-entry, then*

$$Ex(P, m, n) \geq \begin{cases} m \cdot n & k > m \text{ or } l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Moreover, the same holds for $Ex_{\preceq}(P, m, n)$.

Proof. If $k > m \vee l > m$, we have $P \not\preceq \{1\}^{m,n}$. Otherwise, let $P[r, c] = 1$ and consider Figure 1.1. Consider a matrix M such that the first $r-1$ rows, the last $k-r$ rows, the first $c-1$ column and the last $l-c$ column contain no zero-entry and the rest is empty. Then $P \not\leq M$ and even $P \not\preceq M$. We can also see that $|M| = mn - (m-k+1)(n-l+1) = (l-1)m + (k-1)n - (k-1)(l-1)$. □

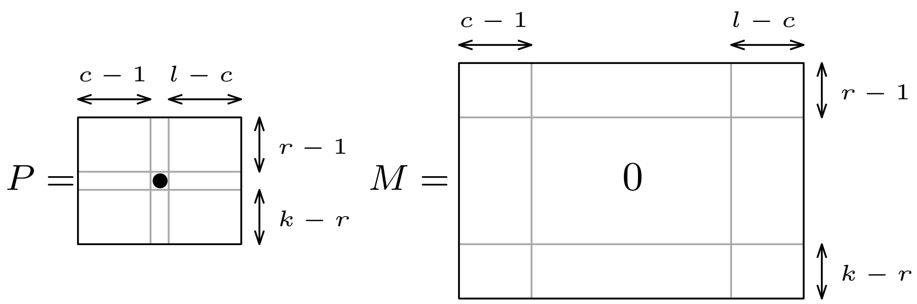


Figure 1.1: An example of a matrix M avoiding a pattern P as an interval minor.

The following definition is due to Cibulka [2013].

Definition 1.20. A pattern $P \in \{0,1\}^{k \times l}$ is *(strongly) minimalist* if

$$Ex(P, m, n) = \begin{cases} m \cdot n & k > m \text{ or } l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

We use the adjective "strongly" to further distinguish minimalist pattern from weakly minimalist patterns defined next.

Definition 1.21. A pattern $P \in \{0,1\}^{k \times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P, m, n) = \begin{cases} m \cdot n & k > m \text{ or } l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

From Observation 1.16, we immediately have:

Observation 1.22. *If a pattern P is strongly minimalist then P is weakly minimalist.*

The following result is a simplification of a lemma from Cibulka [2013].

Fact 1.23. 1. *The pattern $(\bullet)$ is strongly minimalist.*

2. *If a pattern $P \in \{0,1\}^{k\times l}$ is strongly minimalist and there is a one-entry in the last row of P in the c-th column, then $P' \in \{0,1\}^{k+1\times l}$ created from P by appending as the last row a new row having a one-entry only in the c-th column is strongly minimalist.*

3. *If a pattern P having at least two one-entries is strongly minimalist, then after changing a one-entry to a zero-entry it is still strongly minimalist.*

The following two facts come from Mohar et al. [2015]. In the article, a slightly different definition of an interval minor is used, so we show here the proofs in our setting.

Fact 1.24 (Mohar et al. [2015])**.** *Let $P = \{1\}^{2\times l}$ be a pattern, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{2\times l}$ as an interval minor and let A_i be the set of column indices j such that both $M[[i],\{j\}]$ and $M[[i+1,m],\{j\}]$ are non-empty. Clearly, $|A_i| \le l-1$; otherwise, $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M appears in at least $b_j - 1$ of sets A_i, $1 \le i \le m-1$. It follows that

$$|M| = \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} (b_j - 1) + n \le \sum_{i=1}^{m-1} |A_i| + n \le (l-1)(m-1) + n. \qquad \square$$

This result shows an example of a weakly minimalist matrix that is not strongly minimalist. Consider a matrix $\begin{pmatrix}\bullet&\bullet\\\bullet&\bullet\end{pmatrix}$. It is, thanks to Fact 1.24 weakly minimalist, but it is known due to Brown [1966] that it is not strongly minimalist.

Fact 1.25 (Mohar et al. [2015])**.** *Let $P = \{1\}^{3\times l}$ be a pattern, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{3\times l}$ as an interval minor and let A_i be a set of column indices j such that both $M[[i-1],\{j\}]$ and $M[[i+1,m],\{j\}]$ are non-empty and $M[i,j] = 1$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M (for which $b_j \ge 2$) appears in exactly $b_j - 2$ of sets A_i, $2 \le i \le m-1$. It follows that

$$|M| = \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} (b_j - 2) + 2n \le \sum_{i=2}^{m-1} |A_i| + 2n \le (l-1)(m-2) + 2n. \qquad \square$$

We now show that the third part of Fact 1.23 is also safe for weakly minimalist patterns.

Lemma 1.26. *Let $P \in \{0,1\}^{k\times l}$ be a weakly minimalist pattern having at least two one-entries. Then a pattern P' created from P be deletion of a one-entry is also weakly minimalist.*

Proof. For contradiction, consider a matrix $M \in \{0,1\}^{m \times n}$ avoiding P' as an interval minor such that $|M| > (k-1)n + (l-1)m - (k-1)(l-1)$. The matrix M also avoids P; as otherwise, we have $P' \preceq P \preceq M$. That is a contradiction with P being weakly minimalist. $\square$

As a result, we have the following corollary:

Corollary 1.27. *Every non-empty pattern P that has at most three rows (or columns) is weakly minimalist.*

In Cibulka [2009], the author shows that for every $k \geq 1$ there is a $2k \times 2k$ permutation pattern for which $Ex[P, n] \geq k^2 n$. Because of Observation 1.8, the same construction shows that for $k \geq 2$ the patterns are not weakly minimalist. It means that the previous results cannot be easily extended. On the other hand, in Mao et al. [2015] the authors show some form of generalization and also other bounds regarding interval minors and their weight extremal function.

2. Characterizations

Our goal in this chapter is to describe what matrices avoiding small patterns as interval minors look like.

Algorithmically speaking, deciding whether a pattern is contained in a matrix is hard, even if both matrices are permutation matrices, see Bose et al. [1998]. We do not consider complexity questions here, but given a small pattern, we show that matrices avoiding the pattern have a quite simple structure. However, the structure gets significantly richer as soon as the pattern contains at least four one-entries.

To allow ourselves to go through cases efficiently, we first show that to some extent, we can assume there are no empty lines in the pattern without loss of generality.

Before we dive into the characterizations, let us introduce some useful notions.

Definition 2.1. A *walk* in a matrix M is a contiguous sequence of its entries, beginning in the top-left corner and ending in the bottom-right one. If $M[i, j]$ occurs in the sequence, its successor is either $M[i + 1, j]$ or $M[i, j + 1]$. Symmetrically, a *reverse walk* in M is a contiguous sequence of its entries, beginning in the top-right corner and ending in the bottom-left one.

Definition 2.2. We say a matrix M is a *walking matrix* if there is a walk in M containing all one-entries.

Definition 2.3. For a matrix $M \in \{0, 1\}^{m\times n}$ and integers r, c, we say $M[r, c]$ is

- *top-left empty*, if $M[[r - 1], [c - 1]]$ is an empty matrix,
- *top-right empty*, if $M[[r - 1], [c + 1, n]]$ is empty,
- *bottom-left empty*, if $M[[r + 1, m], [c - 1]]$ is empty,
- *bottom-right empty*, if $M[[r + 1, m], [c + 1, n]]$ is empty.

Definition 2.4. For a matrix $M \in \{0, 1\}^{m\times n}$ and integers r, c, we say $M[r, c]$ is *bottom-left extreme* if it is bottom-left empty and the submatrix $M[[r, m], [c]]$ is not empty. Similarly, $M[r, c]$ is *bottom-right extreme* if it is bottom-right empty and the submatrix $M[[r, m], [c, n]]$ is not empty. A walk in M is *bottom-left extreme* if it contains all bottom-left extreme elements of M. A reverse walk in M is *bottom-right extreme* if it contains all bottom-right extreme elements of M.

It is easy to see that there is exactly one bottom-left extreme walk and exactly one bottom-right extreme walk in every matrix.

Definition 2.5. For matrices $M \in \{0, 1\}^{m\times n}$ and $N \in \{0, 1\}^{m\times l}$, we define $M \to N \in \{0, 1\}^{m\times(n+l)}$ to be the matrix created from M by appending the columns of N at the end of M.

2.1 Empty rows and columns

From the definition of matrix containment, zero-entries of the pattern pose no restrictions on the tested matrix, so, intuitively, adding new empty lines to a pattern should not influence the structure of matrices avoiding the pattern by much.

We first show that adding empty lines as first or last lines of the pattern indeed does next to no difference. On the other hand, inserting empty lines in between non-empty lines becomes a bit more tricky and we only describe what happens when we extend a pattern of size $k \times 2$ (or symmetrically $2 \times k$).

Observation 2.6. *For matrices $P \in \{0,1\}^{k \times l}$ and $M \in \{0,1\}^{m \times n}$, let $P' = P \to \{0\}^{k \times 1}$ and let $M' = M \to \{1\}^{m \times 1}$. Then $P \preceq M \Leftrightarrow P' \preceq M'$.*

Proof. $\Rightarrow$ The last column of P' can always be mapped just to the last column of M' and $P'[[k],[l]]$ can be mapped to $M'[[m],[n]]$ the same way P is mapped to M.

$\Leftarrow$ Taking the restriction of the mapping of P' to M' we get a mapping of P to M. □

The analogous proof can be also used to characterize matrices avoiding patterns after we add an empty column as the first column or an empty row as the first or the last row. Using induction, we can easily show that a pattern P' is avoided by a matrix M' if and only if P is avoided by M, where P is derived from P' by excluding all empty leading or ending rows and columns and M is derived from M' by excluding the same number of leading or ending rows and columns. Therefore, when characterizing matrices avoiding a forbidden pattern, we do not need to consider patterns having empty rows or columns on their boundary.

The following machinery shows what happens after we add empty columns in between two columns of a pattern that only has two columns. The size of the patterns is significant, because it allows us to prove that matrices avoiding them have a very simple structure. That is going to be achieved by employing a notion of intervals of one-entries. More about these intervals and their counterpart – zero-intervals can be find in the last chapter of the thesis.

Definition 2.7. A *one-interval* of a matrix M is a sequence of consecutive one-entries in a single line of M bounded from both sides by zero-entries or the edges of matrix.

Definition 2.8. A matrix M avoiding a pattern P is *critical* if after a change of any zero-entry to one-entry M no longer avoids P.

Lemma 2.9. *Let $P \in \{0,1\}^{k \times 2}$ and let $M \in \{0,1\}^{m \times n}$ be a critical matrix avoiding P, then M contains at most one one-interval in each row.*

Proof. For contradiction, assume there are at least two one-intervals in a row of M. Because M is critical, changing any zero-entry e in between one-intervals o_1 and o_2 creates a mapping of the forbidden pattern. Such a mapping uses the changed one-entry to map some element $P[r',1]$ or $P[r',2]$.

In the first case, the same mapping also maps P to M if we use a one-entry from o_1 instead of e; thus, $P \not\preceq M$ and we reach a contradiction. In the second case, the mapping can use a one-entry from o_2 instead of e; therefore, we again get a contradiction with $P \not\preceq M$. Since e is not usable for any one-entry of P, we can change it to a one-entry and get a contradiction with M being critical. $\square$

Lemma 2.10. *Let $P \in \{0,1\}^{k\times 2}$ and for any $l \geq 1$ let $P^l \in \{0,1\}^{k\times(l+2)}$ be a pattern created from P by adding l new empty columns in between the two columns of P. If an $m \times n$ matrix $M \in Av_{\preceq}\left(P^l\right)$ is critical, then each row of M is either empty or it contains a single one-interval of length at least $l+1$ (or length m if $m < l+1$).*

Proof. The same proof as in Lemma 2.9 shows that there is at most one one-interval in each row.

For contradiction, let there be at most l one-entries $M[\{r\},[c_1,c_2]]$ in a row r:

- $c_1 = 1$: we can set $M[r, c_2+1] = 1$ and the matrix still avoids P^l, which is a contradiction with M being critical.
- $c_2 = n$: we can set $M[r, c_1-1] = 1$ and the matrix still avoids P^l, which is a contradiction with M being critical.
- otherwise: let us choose zero-entries e_l and e_r in the row r such that there are exactly l columns between them and all one-entries from the row r lie in between them. For contradiction, assume we can change neither $e_l = M[r, c_l]$ nor $e_r = M[r, c_r]$ to a one-entry without creating the pattern. This means that if $e_l = 1$ then some $P^l[r_1, 1]$ can be mapped to it. Let m_l be the corresponding mapping. At the same time, if $e_r = 1$ then some $P^l[r_2, l+2]$ can be mapped to it and m_r is the corresponding mapping. We show that the two mappings can be combined to a mapping of P^l to M giving a contradiction. Without loss of generality, in both mappings, empty columns of P are mapped exactly to l columns of M. We need to describe how to partition M into k rows. Consider Figure 2.1:
  - $r_1 \neq r_2$: Without loss of generality, we assume $r_1 > r_2$. Let r_3 be the first row used to map r_1 in m_l and let r_4 be the last row used to map r_1 in m_r. From the mapping m_l, we know that the first $r_1 - 1$ rows of P can be mapped to rows $[1, r_3 - 1]$ of M and from the mapping m_r, we know that the last $k - r_1$ rows of P can be mapped to rows $[r_4+1, m]$ of M. Therefore, we can use rows $[r_3, r_4]$ of M to map row r_1 of P without using one-entries e_l and e_r.
  - $r_1 = r_2$: Let r_3 and r_4 be the first and the last rows respectively used to map r_1 in m_l and let r_5 and r_6 be the first and the last rows respectively used to map r_1 in m_r. Without loss of generality let $r_3 < r_5$. From m_l being a mapping, we know that the first $r_1 - 1$ rows of P can be mapped to rows $[1, r_3 - 1]$ of M. Without loss of generality let $r_4 < r_6$. From m_r being a mapping, we know that the last $k - r_1$ rows of P can be mapped to rows $[r_6+1, m]$ of M. Therefore, we can use rows $[r_3, r_6]$ of M to map row r_1 of P without using one-entries e_l and e_r.

We showed that either e_l or e_r can be changed to a one-entry, which is a contradiction with M being critical. $\square$

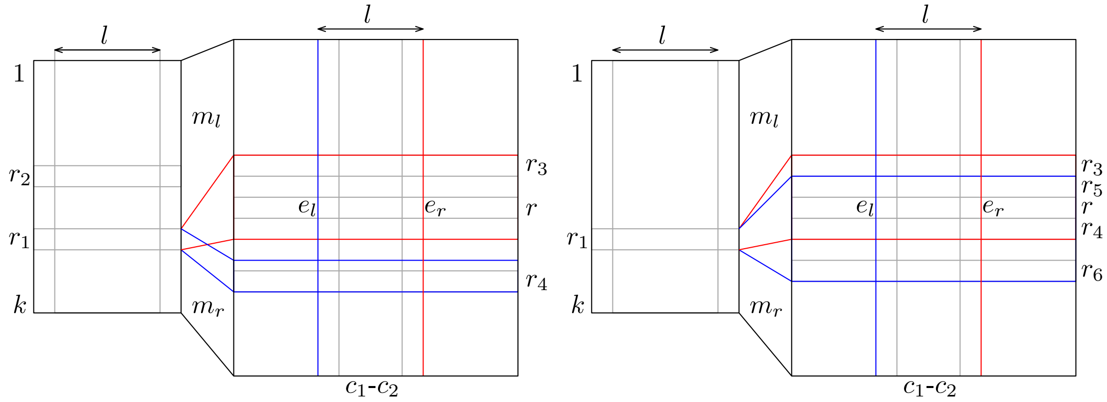


Figure 2.1: Red and blue lines representing mappings m_l and m_r of the forbidden pattern. The two horizontal lines show the boundaries of the mapping of row r and the vertical lines show the boundaries of the mapping of column c.

Theorem 2.11. *Let $P \in \{0,1\}^{k\times 2}$ and for any integer $l \geq 1$ let $P^l \in \{0,1\}^{k\times(l+2)}$ be a pattern created from P by adding l new empty columns in between the two columns of P. For all matrices $M \in \{0,1\}^{m\times n}$ it holds $M \in Av_{\preceq}\left(P^l\right) \Leftrightarrow$ there exists a matrix $N \in \{0,1\}^{m\times(n-l)}$ such that $N \in Av_{\preceq}(P)$ is critical and M is a submatrix of an elementwise OR of $l+1$ shifted copies of N ($N \to \{0\}^{m\times l}, \{0\}^{m\times 1} \to N \to \{0\}^{m\times(l-1)}, \ldots, \{0\}^{m\times(l-1)} \to N \to \{0\}^{m\times 1}, \{0\}^{m\times l} \to N$).*

Proof. $\Rightarrow$ Without loss of generality, let M be critical. We know from Lemma 2.10 that each row of M contains either no one-entry or a single one-interval of length at least $l+1$. Let a matrix N be created from M by deleting the last l one-entries from each row and excluding the last l columns. Clearly, M is equal to an elementwise OR of $l+1$ copies of N. If $P \preceq N$ then each mapping of P can be extended to a mapping of P^l to M by mapping each $P^l[r_1, 1]$ to the same one-entry where $P[r_1, 1]$ is mapped in $N \to \{0\}^{m\times l}$ and mapping each $P^l[r_2, l+2]$ to the same one-entry where $P[r_2, 2]$ is mapped in $\{0\}^{m\times l} \to N$.

$\Leftarrow$ Let M be equal to an elementwise OR of $l+1$ copies of N. For contradiction, assume $P^l \preceq M$ and consider any mapping of P^l to M. Without loss of generality, one-entries of the first column of P^l are mapped to those one-entries of M created from $N \to \{0\}^{m\times l}$. If there is one-entry $P^l[r, 1]$ mapped to a one-entry of M not created from $N \to \{0\}^{m\times l}$, we just take the first one-entry in the row instead. Symmetrically, all one-entries of the last column of P^l are mapped to one-entries created from $\{0\}^{m\times 1} \to N$. The same one-entries of N can be used to map P to N, which is a contradiction. $\square$

The symmetric characterization also holds when adding empty rows to a pattern that only has two rows. We can see in the following proposition that the straightforward generalization of the statement for bigger patterns does not hold.

Proposition 2.12. *There exists a matrix $P \in \{0,1\}^{k\times l}$ such that for each $P' \in \{0,1\}^{k\times(l+1)}$ created from P by adding a single empty column in between two existing columns, there exists a matrix N avoiding P such that the elementwise OR of $N \to \{0\}^{m\times 1}$ and $\{0\}^{m\times 1} \to N$ contains P' as an interval minor.*

Proof. Later in this chapter, we characterize the class of matrices avoiding pattern P_8. For the result, look at Proposition 2.22. Let $N \in Av_{\preceq}(P_8)$ be any matrix containing P_5 as an interval minor. Let a matrix M be equal the elementwise OR of $N \to \{0\}^{m\times 1}$ and $\{0\}^{m\times 1} \to N$. Then $\begin{pmatrix} \bullet & \circ & \bullet & \bullet \\ & \circ & \bullet & \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & \circ & \bullet \\ & \bullet & \circ & \end{pmatrix} \preceq M$. $\square$

Next, we describe the structure of matrices avoiding certain small patterns. We restrict ourselves to patterns with no empty lines. If $P \npreceq M$ then also $P^\top \npreceq M^\top$ and this holds for all rotations and mirrors of P and M and so we only mention these symmetries.

2.2 Patterns having two one-entries and their generalization

These are, up to rotation and mirroring, the only patterns having two one-entries and no empty lines:

$$P_1 = \begin{pmatrix} \bullet & \bullet \end{pmatrix} \quad P_2 = \begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$$

They can be generalized to:

$$P_1' = \begin{pmatrix} \bullet & \bullet & \cdots & \bullet & \bullet \end{pmatrix} \quad P_2' = \begin{pmatrix} & & & & \bullet \\ & & & \bullet & \\ & & \iddots & & \\ & \bullet & & & \\ \bullet & & & & \end{pmatrix}$$

Proposition 2.13. *Let $P_1' = 1^{1\times k}$. For all matrices M: $P_1' \npreceq M \Leftrightarrow M$ has at most $k-1$ non-empty columns.*

Proof. $\Rightarrow$ When a matrix M contains one-entries in k columns, then these give us a mapping of P_1'.

$\Leftarrow$ A matrix M having at most $k-1$ non-empty columns avoids P_1'. $\square$

Proposition 2.14. *Let $P_2' \in \{0,1\}^{k\times k}$. For all matrices M: $P_2' \npreceq M \Leftrightarrow$ there are $k-1$ walks in M such that each one-entry of M belongs to at least one walk.*

Proof. $\Rightarrow$ When one-entries of a matrix M cannot fit into $k-1$ walks, then there are k one-entries such that no pair can fit to a single walk and those give us a mapping of P_2'.

$\Leftarrow$ A matrix M containing one-entries in at most $k-1$ walks avoids P_2'. $\square$

2.3 Patterns having three one-entries

These are up to rotation and mirroring the only patterns having three one-entries and no empty lines that we did not characterize so far:

$$P_3 = \begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix} \quad P_4 = \begin{pmatrix} \bullet & & \bullet \\ & \bullet & \end{pmatrix} \quad P_5 = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \end{pmatrix} \quad P_6 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$$

Proposition 2.15. *For all matrices $M \in \{0,1\}^{m\times n}$: $P_3 \npreceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 2.2):*

- *$M[r,c]$ is top-left, top-right and bottom-left empty, and*

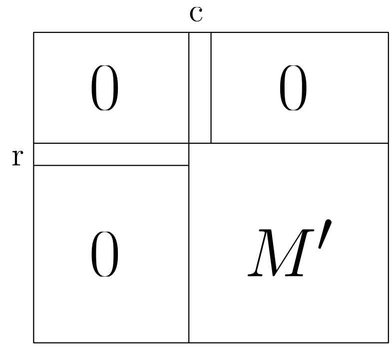


Figure 2.2: The characterization of matrices avoiding $\begin{pmatrix}\bullet & \bullet \\ \bullet & \end{pmatrix}$ as an interval minor. The matrix M' is a walking matrix.

- $M[[r,m],[c,n]]$ *is a walking matrix.*

Proof. $\Rightarrow$ If M is a walking matrix then we set $r = c = 1$. Otherwise, there are one-entries $M[r,c']$ and $M[r',c]$ such that $r' < r$ and $c' < c$. If $M[r,c]$ is not top-left, top-right or bottom-left empty then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ and together with $M[r,c']$ it gives us the forbidden pattern.

$\Leftarrow$ For contradiction, assume that a matrix M described in Figure 2.2 contains P_3 as an interval minor. Without loss of generality, let $P_3[1,1]$ be mapped to a one-entry in the r-th row. Then both $P_3[1,2]$ and $P_3[2,1]$ need to be mapped to M', which is a contradiction because it is not a walking matrix. $\square$

Proposition 2.16. *For all matrices M: $P_4 \not\preceq M \Leftrightarrow M = M_1 \to M_2$, where $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix} \not\preceq M_1$ and $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \not\preceq M_2$.*

Proof. $\Rightarrow$ Let $e = M[r,c]$ be an arbitrary top-most one-entry in M. It holds $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix} \not\preceq M[[m],[c-1]]$, as otherwise, together with e it forms P_4. If we also have $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \not\preceq M[[m],[c,n]]$ then we are done. For contradiction, let e_1, e_2 be any two one-entries forming $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ in $M[[m],[c,n]]$. Symmetrically, let e_1', e_2' be any two one-entries forming $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix}$ in $M[[m],[c]]$. Without loss of generality, let e_2 be lower than e_2' and then, together with e_1' and e_1 it forms P_4 as an interval minor of M, giving us a contradiction.

$\Leftarrow$ For contradiction, let $P_4 \preceq M$ and consider an arbitrary mapping. Consider the one-entry of M, where $P_4[2,2]$ is mapped. If it is in M_1 then $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix} \preceq M_1$ and we get a contradiction. Otherwise, we have $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \preceq M_2$, which is again a contradiction. $\square$

Proposition 2.17. *For all matrices $M \in \{0,1\}^{m \times n}$: $P_5 \not\preceq M \Leftrightarrow$ for every one-entry $M[r,c]$ on the bottom-left extreme walk w, there is at most one non-empty column in $M[[r-1],[c+1,n]]$.*

Proof. $\Rightarrow$ For contradiction, assume there is a one-entry $M[r,c]$ on w such that there are two non-empty columns in $M[[r-1],[c+1,n]]$. Then a one-entry from each of those columns and $M[r,c]$ together give us $P_5 \preceq M$ and a contradiction.

$\Leftarrow$ For contradiction, let $P_5 \preceq M$. Without loss of generality, $P_5[2,1]$ is mapped to a one-entry $M[r,c]$ from w. Then $\begin{pmatrix} \bullet & \bullet \end{pmatrix} \preceq M[[r-1],[c+1,n]]$, which is a contradiction with it having one-entries in at most one column. $\square$

Proposition 2.18. *For all matrices M: $P_6 \not\preceq M \Leftrightarrow$ for every one-entry $M[r,c]$ on the bottom-right extreme reverse walk w, $M[[r-1],[c-1]]$ is a walking matrix.*

Proof. $\Rightarrow$ For contradiction, assume there are r,c such that $M[r,c]$ is a one-entry on w and $M[[r-1],[c-1]]$ is not a walking matrix. It means that $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \preceq M[[r-1],[c-1]]$ and together with $M[r,c]$ it gives us the forbidden pattern and a contradiction.

$\Leftarrow$ For contradiction, let $P_6 \preceq M$ and consider an arbitrary mapping of P_6. Without loss of generality, let $P_6[3,3]$ be mapped to $M[r,c]$ such that there is no other one-entry in $M[[r,m],[c,n]]$. Then, $M[r,c]$ lies on w and $M[[r],[c]]$ is a walking matrix and so $M[r,c]$ cannot be used to map $P_6[3,3]$, which is a contradiction. $\square$

2.4 Patterns having four one-entries

These are some of the patterns having four one-entries and no empty lines that we did not characterize so far:

$$P_7 = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} \quad P_8 = \begin{pmatrix} \bullet & \bullet & \bullet \\ & \bullet & \end{pmatrix} \quad P_9 = \begin{pmatrix} \bullet & & & \\ & & \bullet & \\ & \bullet & & \\ & & & \bullet \end{pmatrix}$$

Lemma 2.19. *For any matrix M: $P_7 \not\preceq M \Rightarrow$ there exist integers r,c such that $M[r,c]$ is either*

1. *a one-entry and $(r,c) \in \{(1,1),(1,n),(m,1),(m,n)\}$ or*
2. *top-right and bottom-left empty and $(r,c) \notin \{(1,1),(m,n)\}$ or*
3. *top-left and bottom-right empty and $(r,c) \notin \{(1,n),(m,1)\}$.*

Proof. If there is a one-entry in any corner then the first condition is satisfied. Otherwise, consider $M[2,1]$. It is trivially bottom-left empty and if there is no one-entry in the first row of M then the second condition is satisfied. Therefore, let $M[1,c_t]$ be a one-entry in the first row. Symmetrically, let $M[m,c_b]$ be a one-entry in the last row, let $M[r_l,1]$ be a one-entry in the first column and let $M[r_r,n]$ be a one-entry in the last column.

It cannot happen that $c_t < c_b$ and $r_r > r_l$ (or symmetrically $c_t > c_b$ and $r_r < r_l$), because then $P_7 \preceq M$. Without loss of generality, let $c_t \geq c_b$ and $r_r \geq r_l$. The matrix $M[[r_r-1],[c_t+1,n]]$ is empty; otherwise, any one-entry there, together with $M[1,c_t], M[m,c_b]$ and $M[r_r,1]$ forms the forbidden pattern. Similarly, the matrix $M[[r_r+1,m],[c_t-1]]$ is also empty. Thus $M[r_t,c_t]$ is top-right and bottom-left empty and it is not a corner, because those are empty. $\square$

Proposition 2.20. *For all matrices $M \in \{0,1\}^{m\times n}$: $P_7 \not\preceq M \Leftrightarrow$ there are integers r,c such that either (see Figure 2.3)*

1. *$M[r,c]$ is top-right empty and bottom-left empty, $\begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix} \not\preceq M[[r],[c]]$ and $\begin{pmatrix} & \bullet \\ \bullet & \bullet \end{pmatrix} \not\preceq M[[r,m],[c,n]]$, or*

2. *$M[r,c]$ is top-left empty and bottom-right empty, $\begin{pmatrix}\bullet & \bullet\\ & \bullet\end{pmatrix} \not\preceq M[[r],[c,n]]$ and $\begin{pmatrix}\bullet & \\ \bullet & \bullet\end{pmatrix} \not\preceq M[[r,m],[c]]$.*

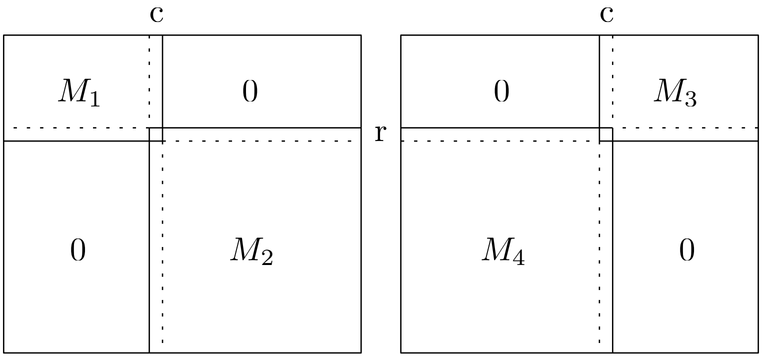


Figure 2.3: The characterization of matrices avoiding $\begin{pmatrix}\bullet & \bullet\\ \bullet & \bullet\end{pmatrix}$ as an interval minor.

Proof. We let $M_1 = M[[r],[c]]$, $M_2 = M[[r,m],[c,n]]$, $M_3 = M[[r],[c,n]]$ and $M_4 = M[[r,m],[c]]$.

$\Rightarrow$ We proceed by induction on the size of M.

If $M \in \{0,1\}^{2\times 2}$ then it either avoids $\begin{pmatrix} & \bullet\\ \bullet & \bullet\end{pmatrix}$ or $\begin{pmatrix}\bullet & \bullet\\ \bullet & \end{pmatrix}$ and we are done.

For a bigger matrix M, from Lemma 2.19, there is an element $M[r,c]$ satisfying some conditions. If there is a one-entry in any corner, we are done because the matrix cannot contain one of the rotations of $\begin{pmatrix}\bullet & \bullet\\ \bullet & \end{pmatrix}$. Otherwise, assume $M[r,c]$ is both top-right and bottom-left empty and $(r,c) \notin \{(1,1),(1,1)\}$. Let $M_1 = M[[r],[c]]$ and $M_2 = M[[r,m],[c,n]]$. If M_1 is non-empty, then $\begin{pmatrix} & \bullet\\ \bullet & \bullet\end{pmatrix} \not\preceq M_2$. Symmetrically, $\begin{pmatrix}\bullet & \bullet\\ \bullet & \end{pmatrix} \not\preceq M_1$ if M_2 is non-empty. If one of them is empty, the other is a smaller matrix avoiding P as an interval minor and the statement follows from the induction.

$\Leftarrow$ Without loss of generality, assume a matrix M looks like the left matrix in Figure 2.3. For contradiction, let $P \preceq M$. We can partition M into four quadrants such that there is at least one one-entry in each of them. It does not matter where we partition it, every time we either get $\begin{pmatrix}\bullet & \bullet\\ \bullet & \end{pmatrix} \preceq M_1$ or $\begin{pmatrix} & \bullet\\ \bullet & \bullet\end{pmatrix} \preceq M_2$, which is a contradiction. $\square$

Lemma 2.21. *For all matrices M: $P_8 \not\preceq M \Rightarrow M = M_1 \to M_2$ where*

1. *$\begin{pmatrix}\bullet & \bullet\\ & \bullet\end{pmatrix} \not\preceq M_1$ and $\begin{pmatrix} & \bullet\\ \bullet & \end{pmatrix} \not\preceq M_2$ or*

2. *$\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix} \not\preceq M_1$ and $\begin{pmatrix}\bullet & \bullet\\ \bullet & \end{pmatrix} \not\preceq M_2$.*

Proof. Let $e = M[r,c]$ be an arbitrary top-most one-entry of M. It holds $\begin{pmatrix}\bullet & \bullet\\ & \bullet\end{pmatrix} \not\preceq M[[m],[c-1]]$; otherwise, together with e it would form the whole P_8. Symmetrically, $\begin{pmatrix}\bullet & \bullet\\ \bullet & \end{pmatrix} \not\preceq M[[m],[c+1,n]]$. For contradiction with statement, let e_1, e_2 (none of them equal to e) be any two one-entries forming $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix}$ in $M[[m],[c]]$ and let e_1', e_2' be any two one-entries forming $\begin{pmatrix} & \bullet\\ \bullet & \end{pmatrix}$ in $M[[m],[c,n]]$. Without loss of generality, e_2' is lower than e_2 and together with e_1, e and e_1' it gives us a mapping of P_8 to M, which is a contradiction. $\square$

Proposition 2.22. *For all matrices $M \in \{0,1\}^{m\times n}$: $P_8 \not\preceq M \Leftrightarrow$ there are integers r, c_1 and c_2 such that all one-entries of M above the row r are in columns c_1 and c_2, $M[[r+1,m],[c_1+1,c_2-1]]$ is empty, $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix} \not\preceq M[[r,m],[c_1]]$ and $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \not\preceq M[[r,m],[c_2,n]]$. See Figure 2.4.*

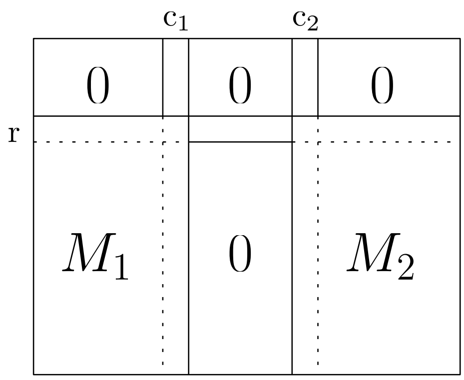


Figure 2.4: The characterization of matrices avoiding $\begin{pmatrix}\bullet & & \bullet\\ & \bullet & \\ & \bullet & \end{pmatrix}$ as an interval minor.

Proof. $\Rightarrow$ From Lemma 2.21, we know $M = M_1' \to M_2'$, where $\begin{pmatrix}\bullet & \\ & \bullet\\ & \bullet\end{pmatrix} \not\preceq M_1'$ and $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \not\preceq M_2'$ (or symmetrically the second case). From Proposition 2.15, we have that M_1' looks like $M[[m],[c_2-1]$ in Figure 2.4 and $M[[m],[c_2,n]]$ forms a walking matrix. Without loss of generality, $M[[r-1],\{c_1\}]$ and $M[\{r\},[c_1+1,c_2-1]]$ are non-empty; otherwise, we extend M_1 to cover the whole $M[[m],[c_2-1]$. If there are two different columns in M_2' having a one-entry above the r-th row, together with one-entries in $M[[r-1],\{c_1\}]$ and $M[\{r\},[c_1+1,c_2-1]]$ they form a mapping of P_8.

$\Leftarrow$ A one-entry $P_8[2,2]$ can not be mapped anywhere but to the r-th row, but in that case, there are at most two columns having one-entries above it. $\square$

Proposition 2.23. *For all matrices $M \in \{0,1\}^{m\times n}$: $P_9 \not\preceq M \Leftrightarrow$ for every one-entry $M[r,c]$ on the bottom-right extreme reverse walk w and every one-entry $M[r',c']$ on the top-left extreme reverse walk w', if $r > r'+3$ and $c > c'+3$ then $M[[r'+1,r-1],[c'+1,c-1]]$ is a walking matrix.*

Proof. $\Rightarrow$ Let w be the bottom-right extreme reverse walk and let w' be the top-left extreme reverse walk in M. If there are one-entries $M[r,c]$ on w and $M[r',c']$ such that $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \preceq M[[r'+1,r-1],[c'+1,c-1]]$, we have a contradiction with $P_9 \not\preceq M$.

$\Leftarrow$ For contradiction, let $P_9 \preceq M$. Without loss of generality, in any mapping of P_9, the element $P_9[4,4]$ is mapped to some one-entry $M[r,c]$ on w and the element $P_9[1,1]$ is mapped to some one-entry $M[r',c']$ on w'. This means that $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \preceq M[[r'+1,r-1],[c'+1,c-1]]$, which is a contradiction with it being a walking matrix. $\square$

2.5 Multiple patterns

Instead of considering matrices avoiding a single pattern, we can work with matrices avoiding a set of forbidden patterns.

We only describe the structure of matrices avoiding one particular set of patterns, because we use the simple result later.

Proposition 2.24. *Let* $P_{10} = \begin{pmatrix} \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{pmatrix}$ *and* $P_{11} = \begin{pmatrix} \circ & \bullet \\ \circ & \circ \\ \bullet & \circ \end{pmatrix}$*, then for all matrices* M*:* $\{P_{10}, P_{11}\} \not\preceq M \Leftrightarrow$ *for the bottom-left extreme walk* w *in* M*, each one-entry* $M[r, c]$ *is either on* w *or both* $M[r+1, c]$ *and* $M[r, c-1]$ *are on* w*.*

Proof.

$\Rightarrow$ For contradiction, assume there is a one-entry anywhere but on w or directly diagonally next to any bottom-left corner of w. Then this one-entry together with at least one bottom-left corner of w give us a mapping of P_{10} or P_{11} and a contradiction.

$\Leftarrow$ For any one-entry e, from the description of M, there is no one-entry that creates P_{10} or P_{11} with e. $\square$

3. Operations with matrices

In this chapter, we look at classes of matrices from a different perspective. Unlike in the previous chapter, where we studied the structure of matrices avoiding a given set of forbidden pattern (usually just one), now we are given a class of matrices and we ask whether it can be described by forbidden patterns.

Recall that a class of matrices is set of matrices closed under interval minors. While it is obvious that any class of matrices can be described by a set of forbidden patterns, as it is enough to forbid all matrices not contained in the class, it is no longer clear how complex can the forbidden set be.

Definition 3.1. Let $\mathcal{M}$ be a class of matrices. The *basis* of $\mathcal{M}$ is a set of all minimal (with respect to minors) matrices $\mathcal{P}$ such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$.

We show that there are many classes of matrices having a finite basis but there are also classes that have an infinite basis. Let us start with a few simple observations, regarding classes of matrices and their bases.

Observation 3.2. *Let $\mathcal{M} = Av_{\preceq}(\mathcal{P})$ for some set of matrices $\mathcal{P}$. Then $\mathcal{M}$ is closed under interval minors.*

Observation 3.3. *Every finite class of matrices has a finite basis.*

3.1 The skew and direct sums

In the realm of permutations, the skew and direct sums are very useful operations. What follows is a direct generalization to our settings and a few simple results. More interesting statements and the relation with interval minors follow in the next section.

Definition 3.4. For matrices $A \in \{0,1\}^{m\times n}$ and $B \in \{0,1\}^{k\times l}$ we define their *skew sum* as a matrix $C := A \nearrow B \in \{0,1\}^{(m+k)\times(n+l)}$ such that $C[[k+1, m+k], [n]] = A$, $C[[k], [n+1, n+l]] = B$ and the rest is empty. Symmetrically, we define their *direct sum* $D := A \searrow B \in \{0,1\}^{(m+k)\times(n+l)}$ such that $D[[m], [n]] = A$, $D[[m+1, m+k], [n+1, n+l]] = B$ and the rest is empty.

Using this notation, we can very easily rewrite the results from the previous chapter. Here is an example of Proposition 2.15 and Proposition 2.20:

Proposition 3.5. $Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \circ\end{smallmatrix}\right)\right) = Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \circ\\ \circ & \circ\end{smallmatrix}\right)\right) \searrow Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\right)\right)$

Proposition 3.6.
$$Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right)\right) = \left(Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \circ\\ \circ & \circ\end{smallmatrix}\right)\right) \searrow Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \bullet\\ \bullet & \circ\end{smallmatrix}\right)\right) \searrow Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \circ\\ \circ & \bullet\end{smallmatrix}\right)\right)\right) \cup$$
$$\left(Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \circ\\ \bullet & \circ\end{smallmatrix}\right)\right) \nearrow Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \circ\\ \circ & \bullet\end{smallmatrix}\right)\right) \nearrow Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \bullet\\ \circ & \circ\end{smallmatrix}\right)\right)\right).$$

Something, we get a great use of later is a closure under the skew sum.

Definition 3.7. For a set of matrices $\mathcal{M}$, let $Cl(\mathcal{M})$ denote the smallest class of matrices containing each $M \in \mathcal{M}$ that is closed under the skew sum and interval minors.

When speaking about graph minors, we can always imagine that the contractions of edges are done after all deletions. Similarly, an element derived from a matrix M by reapplying the skew sum and taking its interval minor can be also derived by taking an interval minor of the skew sum of an appropriate number of copies of M.

Observation 3.8. *For every set of matrices $\mathcal{P}$, each $M \in Cl(\mathcal{P})$ is an interval minor of the skew sum of multiple copies of P.*

What follows is a simple result of the relation of a closure under the skew sum and the description using interval minors. We greatly generalize this result in the next section.

Proposition 3.9. $Cl\left(\begin{pmatrix} \bullet & \circ \\ \circ & \bullet \end{pmatrix}\right) = Av_{\preceq}\left(\begin{pmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \end{pmatrix}, \begin{pmatrix} \bullet & \circ \\ \circ & \circ \\ \circ & \bullet \end{pmatrix}\right)$.

Proof. The skew sum of an arbitrary number of copies of $\begin{pmatrix} \bullet & \circ \\ \circ & \bullet \end{pmatrix}$ avoids both forbidden patterns and because the relation of being an interval minor is transitive, we have $Cl\left(\begin{pmatrix} \bullet & \circ \\ \circ & \bullet \end{pmatrix}\right) \subseteq Av_{\preceq}\left(\begin{pmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \end{pmatrix}, \begin{pmatrix} \bullet & \circ \\ \circ & \circ \\ \circ & \bullet \end{pmatrix}\right)$.

From Proposition 2.24, for every matrix $M \in Av_{\preceq}\left(\begin{pmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \end{pmatrix}, \begin{pmatrix} \bullet & \circ \\ \circ & \circ \\ \circ & \bullet \end{pmatrix}\right)$, it holds that for the bottom-left extreme walk w in M, each one-entry $M[r, c]$ is either on w or both $M[r+1, c]$ and $M[r, c-1]$ are on w. Clearly, $\begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$ is an interval minor of the skew sum of three copies of $\begin{pmatrix} \bullet & \circ \\ \circ & \bullet \end{pmatrix}$ and by the skew sum of multiple copies of $\begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$ we can then create the whole w and all one-entries outside of it. Thus, we have the other inclusion. $\square$

While it does not make sense for permutations, we can generalize the skew sum to also allow some overlap between the summed matrices.

Definition 3.10. For matrices $A \in \{0,1\}^{m \times n}$, $B \in \{0,1\}^{k \times l}$ and integers a, b, let a matrix $C := A \nearrow_{a \times b} B \in \{0,1\}^{(m+k-a) \times (n+l-b)}$ such that $C[[k+1, m+k], [n]] = A$, $C[[k], [n+1, n+l]] = B$, the part that overlaps is an elementwise OR of both submatrices and the rest of C is empty. We say C is the *skew sum with $a \times b$ overlap* of A and B.

Theorem 3.11. *For integers a, b, m, n such that $a \leq m \leq 2a$ and $b \leq n \leq 2b$, let $\mathcal{M}$ be an arbitrary set of matrices, not necessarily closed under interval minors, such that:*

- *$\mathcal{M}$ is closed under deletion of one-entries,*
- *$\mathcal{M}$ is closed under the skew sum with $a \times b$ overlap and*
- *there is a $m \times n$ matrix $M \in \mathcal{M}$,*

then $\mathcal{M}$ is also closed under the skew sum with $(2a - m) \times (2b - n)$ overlap.

Proof. Given any $A, B \in \mathcal{M}$ and a matrix $M \in \mathcal{M}$ such that $M \in \{0,1\}^{m \times n}$, let $C = A \nearrow_{a \times b} M \nearrow_{a \times b} B$. It has the same size as $D = A \nearrow_{(2a-m) \times (2b-n)} B$, whose set of one-entries is a subset of one-entries of $C \in \mathcal{M}$; therefore, $D \in \mathcal{M}$. $\square$

We see that already with pretty reasonable assumptions, whenever a set of matrices is closed under the skew sum with some overlap, it is also closed under the skew sum with smaller overlap. On the other hand, in general the opposite does not hold even if we work with classes of matrices.

Observation 3.12. *There is a class of matrices closed under the skew sum with* 1×1 *overlap that is not closed under the skew sum with* 2×2 *overlap.*

Proof. Let $\mathcal{M} = Av_{\preceq}\left(\begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}\right)$. Clearly, $\mathcal{M}$ is hereditary and closed under the skew sum with 1×1 overlap. On the other hand, $\mathcal{M}$ is not closed under the skew sum with 2×2 overlap, because for matrices $\begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix}, \begin{pmatrix} & \bullet \\ \bullet & \bullet \end{pmatrix} \in \mathcal{M}$, it holds $\begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix} \nearrow_{2\times 2} \begin{pmatrix} & \bullet \\ \bullet & \bullet \end{pmatrix} = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} \notin \mathcal{M}$. □

A similar proof shows that for all $a \geq 1, b > 1$ there is a class of matrices closed under the skew sum with $a \times b$ overlap that is not closed under the skew sum with $(a+1) \times b$ (or $a \times (b+1)$) overlap. Luckily for us, this does not hold for $a = 0$ or $b = 0$:

Observation 3.13. *Every class of matrices closed under the skew sum is also closed under the skew sum with* 1×1 *overlap.*

3.2 Articulations

Our next goal is to show that whenever we have a matrix closed under the skew sum and interval minors, the obtained class has a finite basis. In order to prove it, we define and get familiar with articulations.

Definition 3.14. Let $M \in \{0,1\}^{m \times n}$ be a matrix. An element $M[r, c]$ is an *articulation* if it is top-left empty ($M[[r-1],[c-1]]$ is empty) and bottom-right empty ($M[[r+1, m],[c+1, n]]$ is empty). We say that an articulation $M[r, c]$ is *trivial* if $(r, c) \in \{(m, 1), (1, n)\}$.

Whenever $P \preceq M$, for every $M[r, c]$ there is some $P[r', c']$ that can be mapped to $M[r, c]$; therefore, the following observation shows that once there is an articulation in M, it also exists in P and it is not necessarily trivial.

Observation 3.15. *Let* M *be a matrix. If there are integers* r, c *such that* $M[r, c]$ *is an articulation, then for every matrix* P *such that* $P \preceq M$*, if* $P[r', c']$ *can be mapped to* $M[r, c]$ *then it is an articulation.*

Observation 3.16. *Let* $P \in \{0,1\}^{k \times l}$ *be a matrix. There are* P_1, P_2 *non-empty interval minors of* P *such that* $P = P_1 \nearrow_{1\times 1} P_2 \Leftrightarrow$ *there exist integers* r, c *such that* $P[r, c]$ *is an articulation and* $P[[r, k],[c]], P[[r],[c, l]]$ *are non-empty.*

Observation 3.17. *Let* $\mathcal{P}$ *be a set of matrices. There is a minimal (with respect to interval minors) matrix* $P \in \mathcal{P}$ *and there are* P_1, P_2 *non-empty interval minors of* P *such that* $P = P_1 \nearrow_{1\times 1} P_2 \Leftrightarrow Av_{\preceq}(\mathcal{P})$ *is not closed under the skew sum with* 1×1 *overlap.*

Proof. $\Rightarrow$ Let $P_1 \in \{0,1\}^{k_1 \times l_1}$ and $P_2 \in \{0,1\}^{k_2 \times l_2}$. While $P \npreceq P_1 \nearrow_{1\times 1} 0^{k_2 \times l_2}$ and $P \npreceq 0^{k_1 \times l_1} \nearrow_{1\times 1} P_2$, we have $P \preceq P_1 \nearrow_{1\times 1} 0^{k_2 \times l_2} \nearrow 0^{k_1 \times l_1} \nearrow_{1\times 1} P_2$.

$\Leftarrow$ If there is no minimal matrix $P \in \mathcal{P}$ that is the skew sum of its non-empty interval minors, we want to show it makes $Av_{\preceq}(\mathcal{P})$ closed under the skew sum with 1×1 overlap. From Observation 3.16, for every $P \in \mathcal{P}$ there are no r, c that $P[r, c]$ is an articulation and $P[[r, k], [c]], P[[r], [c, l]]$ are non-empty. Let $M_1, M_2 \in Av_{\preceq}(P)$ be arbitrary matrices and let $M = M_1 \nearrow_{1\times 1} M_2$. The matrix M contains a non-trivial articulation and from Observation 3.15 it follows $M \in Av_{\preceq}(P)$ for each minimal $P \in \mathcal{P}$; thus, $M \in Av_{\preceq}(\mathcal{P})$. $\square$

In the following, we always expect articulations to be on a reverse walk (no two articulations forming $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right)$) and by a matrix between two articulations $M[r_1, c_1]$ and $M[r_2, c_2]$ we mean the matrix $M[[r_2, r_1], [c_1, c_2]]$.

Lemma 3.18. *Let $\mathcal{P}$ be a set of matrices, then for all matrices $M \in \{0, 1\}^{m\times n}$ it holds that $M \in Cl(\mathcal{P}) \Leftrightarrow$ there exists a sequence of articulations of M on a reverse walk such that for each matrix M' in between two consecutive articulations of M there exists $P \in \mathcal{P}$ such that $M' \preceq \left(\begin{smallmatrix}1\end{smallmatrix}\right) \nearrow P \nearrow \left(\begin{smallmatrix}1\end{smallmatrix}\right)$.*

Proof. $\Rightarrow$ With Observation 3.13 in mind, consider the skew sum with 1×1 overlap of multiple copies of elements of $\mathcal{P}$ and let the sequence contain an articulation between each pair of consecutive copies of matrices from P, together with the trivial articulations $M[m, 1]$ and $M[1, n]$.

Between each pair of consecutive articulations, we have a matrix from $\mathcal{P}$ and so the statement holds. When we take an arbitrary interval minor and keep original articulations, each matrix between two consecutive articulations only contains at most one original copy of some matrix $P \in \mathcal{P}$, but it may happen that the bottom-left and top-right corners become one-entries even though they were zero-entries before. The matrix does not have to be an interval minor of P anymore, but it is an interval minor of $\left(\begin{smallmatrix}1\end{smallmatrix}\right) \nearrow P \nearrow \left(\begin{smallmatrix}1\end{smallmatrix}\right)$ for the corresponding $P \in \mathcal{P}$.

$\Leftarrow$ We can simply blow up each matrix M' between two consecutive articulation to the skew sum of three copies of the corresponding matrix P and because $M' \preceq \left(\begin{smallmatrix}1\end{smallmatrix}\right) \nearrow P \nearrow \left(\begin{smallmatrix}1\end{smallmatrix}\right) \preceq P \nearrow P \nearrow P$ it holds $M \in Cl(\mathcal{P})$. $\square$

Finally, we show that a closure under the skew sum can always be described by a finite number of forbidden patterns.

Theorem 3.19. *For all matrices $M \in \{0, 1\}^{m\times n}$, $Cl(M)$ has a finite basis.*

Proof. Let $\mathcal{F}$ be the set of all minimal (with respect to interval minors) matrices such that $Cl(M) = Av_{\preceq}(\mathcal{F})$. We need to prove that $\mathcal{F}$ is finite. Thanks to Observation 3.13, $Av_{\preceq}(\mathcal{F})$ is closed under the direct sum with 1×1 overlap and from Observation 3.17 follows that for no $F \in \mathcal{F}$ there are its non-empty interval minors F_1, F_2 such that $F = F_1 \nearrow 1 \times 1 F_2$. We denote by $\mathcal{P}$ a set of matrices $F \in \mathcal{F}$ such that F has at most $2m + 4$ rows and $2n + 4$ columns. We want to show $Cl(M) = Av_{\preceq}(\mathcal{P})$.

$\subseteq$ Clearly, $\mathcal{P}$ is finite and we immediately see that $Cl(M) \subseteq Av_{\preceq}(\mathcal{P})$.

$\supseteq$ For contradiction, consider a minimal matrix $X \in Av_{\preceq}(\mathcal{P}) - Cl(M)$. There are no X_1, X_2 non-empty interval minors of X such that $X = X_1 \nearrow$

$1 \times 1 X_2$; otherwise, as $X_1, X_2 \in Av_{\preceq}(\mathcal{P})$ and X is the minimum matrix such that $X \notin Cl(M)$, we would have $X_1, X_2 \in Cl(M)$; therefore, $X \in Cl(M)$ and a contradiction.

Without loss of generality, we assume $X \in \{0,1\}^{k \times l}$ has at least $2m+5$ rows. Let X' denote a matrix created from X by deletion of the first row. We have $X' \in Av_{\preceq}(\mathcal{P})$ and from minimality of X also $X' \in Cl(M)$. From Lemma 3.18, there is a sequence of articulations of X' on a reverse walk such that each matrix between two consecutive articulations is an interval minor of $(1) \nearrow M \nearrow (1)$. Let $X'[r,c]$ be the first articulation from the sequence (sorted by the second coordinate in ascending order) for which $c > 1$. The matrix between $X'[r,c]$ and the previous articulation in the sequence is an interval minor of $(1) \nearrow M \nearrow (1)$, which also means that $c \leq n+2$. Since $X[r,c]$ is not an articulation, it must hold that $X[1,c_1] = 1$ for some $c_1 < c \leq n+2$. Symmetrically, let X'' denote a matrix created from X by deletion of the last row. Following the same steps we did before, we get the last articulation $X''[r,c]$ such that $c < l$ and the observation that $c \geq l-n-1$. Since $X[r,c]$ is not an articulation, it must hold that $X[k,c_2] = 1$ for some $c_2 > c \geq l-n-1$.

We showed that a matrix $Y \in \{0,1\}^{(m+1)\times 2}$ such that the only one-entries are $Y[1,1]$ and $Y[m+1,2]$ is an interval minor of X. To reach a contradiction, it suffices to show that there is a matrix $P \in \mathcal{P}$ such that $P \preceq Y$. For contradiction, let $Y \in Av_{\preceq}(\mathcal{P})$ and since $Y \preceq X$ and X is minimum such that $X \notin Cl(M)$ it holds $Y \in Cl(M)$. But this cannot be, because Y contains no non-trivial articulation and from Observation 3.15, we know that every matrix $Z \in Cl(M)$ bigger than $m \times n$ contains at least one. $\square$

3.3 Basis

We recall that the basis of a class of matrices $\mathcal{M}$ is a set of all minimal (with respect to interval minors) matrices $\mathcal{P}$ such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$. It goes without saying that it does not make sense to consider a basis of a set of matrices that is not closed under interval minors.

So far, we showed that whenever $\mathcal{M}$ is finite, its basis is also finite. The same hold when $\mathcal{M} = Cl(M)$ for some M. We show next that, unlike in graph theory, there are classes that does not have a finite basis. Moreover, we show that even for a class $\mathcal{M}$ with finite basis, its closure $Cl(\mathcal{M})$ can have an infinite basis.

Definition 3.20. Let P be a matrix. We denote by $\mathcal{R}(P)$ a set of all minimal (with respect to minors) matrices P' such that $P \preceq P'$ and P' is not the skew sum with 1×1 overlap of non-empty interval minors of P'. For a set of matrices $\mathcal{P}$, let $\mathcal{R}(\mathcal{P})$ denote a set of all minimal (with respect to minors) matrices from the set $\bigcup_{P \in \mathcal{P}} \mathcal{R}(P)$.

Theorem 3.21. *Let $\mathcal{M}$ and $\mathcal{P}$ be sets of matrices such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$, then $Cl(\mathcal{M}) = Av_{\preceq}(\mathcal{R}(\mathcal{P}))$.*

Proof. $\subseteq$ Consider a matrix $M \notin Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ and without loss of generality, because $Cl(\mathcal{M})$ is closed under interval minors, let M be minimal (with respect to interval minors). It follows that $M \in \mathcal{R}(\mathcal{P})$. Then, the matrix M is not a skew sum with 1×1 overlap of non-empty interval minors of M; therefore, according to Observation 3.16, there is no articulations $M[r, c]$ such that $M[[r, k], [c]], M[[r], [c, l]]$ are non-empty.

For contradiction with the statement, assume $M \in Cl(\mathcal{M})$. According to Lemma 3.18 and the fact M contains no non-trivial articulation, it holds $M \preceq (\,1\,) \nearrow M' \nearrow (\,1\,)$ for some $M' \in \mathcal{M}$. Because the trivial articulations contain zero-entries, it even holds $M \preceq M'$. We also know $P \preceq M$ for some $P \in \mathcal{P}$, which together give us a contradiction with $\mathcal{M} = Av_{\preceq}(\mathcal{P})$.

$\supseteq$ First of all, $Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ is closed under the skew sum with 1×1 overlap. For contradiction, assume there are matrices $M_1, M_2 \in Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ but $M = M_1 \nearrow_{1\times 1} M_2 \notin Av_{\preceq}(\mathcal{R}(\mathcal{P}))$. Then there exists a matrix $P \in \mathcal{R}(\mathcal{P})$ such that $P \preceq M$. Because P is not a skew sum with 1×1 overlap of non-empty interval minors of P, it follows that either $P \preceq M_1$ or $P \preceq M_2$ and we have a contradiction.

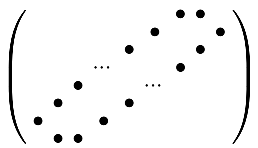
It suffices to show that the inclusion holds for any matrix $M \in Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ that is not a skew sum with 1×1 overlap of non-empty interval minors of M. From Observation 3.16, we know that M does not contain any non-trivial articulation and those trivial ones are empty. Thus, $M \in Av_{\preceq}(\mathcal{P}) = \mathcal{M}$ and so $M \in Cl(\mathcal{M})$. $\square$

Corollary 3.22. *Let $\mathcal{M}$ and $\mathcal{P}$ be sets of matrices such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$, then $\mathcal{R}(\mathcal{P})$ is the basis of $Cl(\mathcal{M})$.*

What follows is a construction of parameterized matrices that become the main tool of finding a class of matrices with an infinite basis.

Definition 3.23. Let $Nucleus_1 = (\,\bullet\,)$ and for $n > 1$ let $Nucleus_n \in \{0,1\}^{n \times n+1}$ be a matrix described by the examples:

$$Nucleus_2 = \begin{pmatrix} \bullet & \bullet & \\ & \bullet & \bullet \end{pmatrix} \quad Nucleus_3 = \begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & \bullet & \end{pmatrix} \quad Nucleus_n =$$



Definition 3.24. Let $Candy_{k,n,l}$ be a matrix given by $I_k \nearrow_{1\times 2} Nucleus_n \nearrow_{1\times 2} I_l$, where I_k, I_l are unit matrices of sizes $k \times k$ and $l \times l$ respectively.

$$Candy_{4,1,4} = \begin{pmatrix} & & & \bullet & & & \\ & & & & \bullet & & \\ & & & & & \bullet & \\ \bullet & & & \bullet & & & \bullet \\ & \bullet & & & & & \\ & & \bullet & & & & \\ & & & \bullet & & & \end{pmatrix} \quad Candy_{4,4,4} = \begin{pmatrix} & & & & & \bullet & & & \\ & & & & & & \bullet & & \\ & & & & & & & \bullet & \\ & & & & \bullet & \bullet & & & \bullet \\ & & & \bullet & & & \bullet & & \\ & & \bullet & & & \bullet & & & \\ \bullet & & & \bullet & \bullet & & & & \\ & \bullet & & & & & & & \\ & & \bullet & & & & & & \\ & & & \bullet & & & & & \end{pmatrix}$$

Theorem 3.25. *There exists a matrix P for which $\mathcal{R}(P)$ is infinite.*

Proof. Let $P = Candy_{4,1,4}$. For all $n > 3$ it holds $P \preceq Candy_{4,n,4}$ and it suffices to show that each $Candy_{4,n,4}$ is a minimal matrix (with respect to minors) and it is not the skew sum of two of its non-empty interval minors. According to Observation 3.16, the second condition holds as $Candy_{4,n,4}$ contains no non-trivial articulation and the trivial ones are empty. To show it is minimal, we need to consider any matrix $M \preceq Candy_{4,n,4}$ and argue that either $P \not\preceq M$ or M contains an articulation.

Thanks to Observation 3.15, as soon as we find a non-trivial articulation $M[r,c]$ such that $M[[r,k],[c]], M[[r],[c,l]]$ are non-empty, it will stay there in any interval minor, because we cannot delete one-entries $M[1,n-3], M[2,n-2], M[3,n-1]$ and $M[4,n]$ (and symmetrically $M[m-3,1], M[m-2,2], M[m-1,3], M[m,4]$) without loosing the condition $P \preceq M$. Therefore, we can only consider one minoring operation at a time.

It is easy to see that when a one-entry is changed to a zero-entry, then the matrix does not belong to $\mathcal{R}(P)$ anymore. Consider that rows $r_1, r_2, \ldots, r_k$ are chosen to be merged into one with an elementwise OR. If $r_1 < 4$ or $r_k > n+3$ then P is no longer an interval minor of such matrix. Otherwise, the original $Candy_{4,n,4}[r_1, n-r_1+2]$ becomes an articulation. Symmetrically, the same holds for columns which concludes the proof. $\square$

Corollary 3.26. *There exists a class of matrices $\mathcal{M}$ having a finite basis such that $Cl(\mathcal{M})$ has an infinite basis.*

Proof. From Theorem 3.25, we have a matrix P for which $\mathcal{R}(P)$ is infinite. Class $\mathcal{M} = Av_{\preceq}(P)$ has a finite basis. On the other hand, from Theorem 3.21, we have $Cl(\mathcal{M}) = Av_{\preceq}(\mathcal{R}(P))$. $\square$

4. Zero-intervals

In Chapter 2, we characterized matrices avoiding small patterns. Their structure is very dependent on the pattern they avoid and the results are hard to generalize for arbitrary patterns. In this chapter, we look for a more general property that restricts the complexity of a class of matrices.

Definition 4.1. For a matrix $M \in \{0,1\}^{m \times n}$, a row interval $M[\{r\},[c_1,c_2]]$ is a *zero-interval* if all entries are zero-entries, $c_1 = 0$ or $M[r, c_1 - 1] = 1$ and $c_2 = n$ or $M[r, c_2 + 1] = 1$. In other words, it is an interval of zero-entries bounded by one-entries. Symmetrically, we also call a column interval $M[[r_1, r_2, \{c\}]]$ a *zero-interval* if all entries are zero-entries, $r_1 = 0$ or $M[r_1 - 1, c] = 1$ and $r_2 = m$ or $M[r_2 + 1, c] = 1$. In the same spirit, we define a *one-interval* to be an interval of one-entries in a single line of M bounded by zero-entries (or edges of the matrix).

Definition 4.2. For a class of matrices $\mathcal{M}$, we say that a matrix $M \in \mathcal{M}$ is *critical* in $\mathcal{M}$ if the change of any zero-entry to a one-entry creates a matrix that does not belong to $\mathcal{M}$. For any set of matrices $\mathcal{P}$, let $Av_{crit}(\mathcal{P})$ be a set of all critical matrices avoiding $\mathcal{P}$ as an interval minor.

In Chapter 2, for a pattern $P \in \{0,1\}^{k \times l}$ it very often holds that any matrix from $Av_{crit}(P)$ has at most k zero-intervals in each row and at most l zero-intervals in each column. The main goal of this chapter is to describe patterns P for which there can be arbitrarily many zero-intervals in matrices from $Av_{crit}(P)$.

4.1 Pattern complexity

We define the complexity of a class of matrices as the maximum number of zero-intervals (or one intervals as they go in pair) a critical matrix from the class can have.

Definition 4.3. For a class of matrices $\mathcal{M}$, we define its *row-complexity* $r(\mathcal{M})$ to be the supremum of the number of zero-intervals in a single row of any critical matrix $M \in \mathcal{M}$. We say that $\mathcal{M}$ is *row-bounded*, if its row-complexity is finite, and *row-unbounded* otherwise. Symmetrically, we define its *column-complexity* $c(\mathcal{M})$ and the property of being *column-bounded* and *column-unbounded*. The class $\mathcal{M}$ is *bounded* if it is both row-bounded and column-bounded; otherwise, it is *unbounded*.

Definition 4.4. We say that a set of patterns $\mathcal{P}$ is *bounding*, if the class $Av_{\preceq}(\mathcal{P})$ is bounded; otherwise, it is *non-bounding*.

Now that we introduced the most essential definitions in this chapter, it is time to state the main theorem:

Theorem 4.5. *A pattern P is bounding $\Leftrightarrow P_i \not\preceq P$ for all $1 \leq i \leq 4$.*

$$P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix} \quad P_2 = \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix} \quad P_3 = \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix} \quad P_4 = \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}$$

We prove the statement in several steps. We show the first implication in Subsection 4.1.2, then we proof multiple lemmata so that we finally show the other implication at the end of Subsection 4.1.3. Before we start proving the main result, we introduce some useful notation and get more familiar with zero-intervals.

Definition 4.6. Let P be a pattern, let e be a one-entry of P, consider a matrix $M \in Av_{\preceq}(P)$ and let z be an arbitrary zero-interval of M. We say that z is *usable for* e if there is a zero-entry contained in z such that if we change it to a one-entry, it creates a mapping of P to M that uses the new one-entry to map e. This way, z can be usable for many one-entries of P at once.

Observation 4.7. *Let $P \in \{0,1\}^{k\times l}$ and $M \in \{0,1\}^{m\times n}$ be matrices such that $P \not\preceq M$. Let $z = M[\{r_1\},[c_1,c_2]]$ be a zero-interval of M usable for a one-entry $e = P[r,c]$. If we change a zero-entry of z and create a mapping of P that uses the changed entry to map e, then the mapping can only map column c of P to columns $[c_1,c_2]$ of M.*

Proof. Since the changed entry is used to map e, clearly the mapping needs to use a column from $[c_1,c_2]$ to map column c. If, for contradiction, the mapping uses columns outside $[c_1,c_2]$ then, without loss of generality, it uses the column c_1-1. Since that column bounds the zero-interval z, $M[r_1,c_1-1]=1$ and this one-entry can be used in the mapping instead of the changed entry, which gives us a contradiction with $P \not\preceq M$. □

Definition 4.8. Let $\mathcal{P}$ be a set of patterns and let e be a one-entry of any matrix $P \in \mathcal{P}$. We define the *row-complexity* of e, $r\left(Av_{\preceq}(\mathcal{P}),e\right)$ to be the supremum of the number of zero-intervals of a single row of any $M \in Av_{crit}(\mathcal{P})$ that are usable for e. We say that e is *row-unbounded* in $Av_{\preceq}(\mathcal{P})$ if $r\left(Av_{\preceq}(\mathcal{P}),e\right)=\infty$ and *row-bounded* otherwise. Symmetrically, we define the *column-complexity* of e, $c\left(Av_{\preceq}(\mathcal{P}),e\right)$ to be the maximum number of zero-intervals of a single column of any matrix from $Av_{crit}(\mathcal{P})$ that are usable for e, and we say e is *column-unbounded* if it is infinite and *column-bounded* otherwise.

The following observation follows directly from the definition and we use it heavily throughout the chapter to break symmetries.

Observation 4.9. *For every set $\mathcal{M}$, $\mathcal{M}$ is row-bounded $\Leftrightarrow$ $\mathcal{M}^\top$ is column-bounded.*

4.1.1 Adding empty lines

As in Chapter 2, we show that we do not need to consider patterns with leading and ending empty rows and columns.

Observation 4.10. *For a matrix $P \in \{0,1\}^{k\times l}$ and an integer n, let $P' = P \to 0^{k\times n}$. The matrix P is bounding $\Leftrightarrow$ P' is bounding. Moreover, if P is bounding, then $r\left(Av_{\preceq}(P')\right) \le r\left(Av_{\preceq}(P)\right)+1$.*

Lemma 4.11. *Let $P \in \{0,1\}^{2\times k}$ be a matrix and for any $l \ge 1$, let $P^l \in \{0,1\}^{(l+2)\times k}$ be a pattern created from P by adding l new empty rows in between the two row of P. For every one-entry e of P^l it holds $r\left(Av_{\preceq}\left(P^l\right),e\right) \le k^2$.*

Proof. Given a matrix $M \in Av_{crit}(P)$, consider an arbitrary row r of M. Without loss of generality, assume $e = P[1, c]$. For contradiction, assume there are $k^2 + 1$ zero-intervals $z_1, \ldots, z_{k^2+1}$ in r usable for e. In particular, the first k^2 of them are bounded by a one-entry from the right side.

- $P[2, c] = 1$: Clearly, there is a one-entry in rows $[r + l + 1, m]$ underneath each z_j and if we combine each such one-entry with a one-entry bounding corresponding z_j, we find a mapping of $\left(\{1\}^{2\times k^2}\right)^l$, contradicting $P \not\preceq M$.

- $P[2, c] = 0$: For each $i \in [k^2]$, we define an extended interval z_i^* to be the interval containing z_i and also all entries on the row r between z_i and z_{i+1}. Because of the Pigeonhole principle, we can find either k consecutive extended intervals such that there are no one-entries in rows $[r + l + 1, m]$ underneath them, or k (not necessarily consecutive) extended intervals such that there is a one-entry in rows $[r + l + 1, m]$ underneath each of them. Because each extended interval contains a one-entry, in the second case we find $\left(\{1\}^{k\times 2}\right)^l$ as an intervals minor.

  In the first case, without loss of generality, assume $P[2, c_1] = 1$ and it is the minimum such $c_1 > c$. Let $z'_1, \ldots, z'_k$ be the consecutive zero-intervals. Consider the mapping of P^l created when a zero-entry of z'_1 is changed to a one-entry used to map e. Since $P[2, c_1] = 1$ and there are no one-entries in rows $[r + l + 1, m]$ underneath extended intervals $z'_1, \ldots, z'_k$, $P^l[l + 2, c_1]$ has to be mapped to the columns of M after the end of z'_k. This leaves k one-entries to be used to map potential one-entries in $P^l[\{l + 2\}, [c, c_1 - 1]]$ and so $P^l \preceq M$, which is again a contradiction. $\square$

Corollary 4.12. *Let $P \in \{0, 1\}^{k\times 2}$ be a matrix and for any $l \geq 1$, let $P^l \in \{0, 1\}^{k\times(l+2)}$ be a matrix created from P by adding l new empty columns in between the two columns of P. Then $Av_{\preceq}\left(P^l\right)$ is bounded for any $l \geq 1$.*

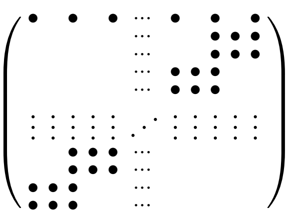
Proof. We know $Av_{\preceq}\left(P^l\right)$ is row-bounded from Lemma 2.9. From Lemma 4.11 and Observation 4.9 we have that the class is also column-bounded. $\square$

4.1.2 Non-bounding patterns

We see that for patterns having only two non-empty rows or columns we can indeed bound the number of zero-intervals of critical matrices avoiding them. On the other hand, already for a pattern of size 3×3 we show that there are maximal matrices with arbitrarily many zero-intervals.

Lemma 4.13. *A class $Av_{\preceq}(P_1)$ is unbounded.*

Proof. For a given integer n, let M be a $(2n + 1) \times (2n + 1)$ matrix described by the picture:



We see that $P_1 \not\preceq M$ because we always need to map $P_1[2,1]$ and $P_1[3,3]$ to just one "block" of one-entries, which only leaves a zero-entry for $P_1[1,2]$.

If we change any zero-entry of the first row into a one-entry, we get a matrix containing an interval minor of $\{1\}^{3\times3}$; therefore, containing P_1 as an interval minor. In case M is not critical, we add some more one-entries to make it critical but it will still contain a row with n zero-intervals. $\square$

Not only $M \in Av_{crit}(P_1)$ but it also avoids any $P \in \{0,1\}^{3\times3}$ such that $P_1 \preceq P$. Its rotations avoid rotations of P_1 and we conclude that a big portion of patterns of size 3×3 are non-bounding. Moreover, the result can be generalized also for bigger matrices.

Theorem 4.14. *For every matrix P such that $P_1 \preceq P$, $Av_{\preceq}(P)$ is unbounded.*

Proof. First, assume there is a mapping of P_1 into $P \in \{0,1\}^{k\times l}$ that maps $P_1[1,2]$ to a one-entry of the first row of P, $P_1[2,1]$ to a one-entry of the first column of P and $P_1[3,3]$ to the bottom-right corner of P. Then, we use a similar construction as we did in the proof of Lemma 4.13 to find a matrix $M \in Av_{crit}(P)$ with n zero-intervals for any n.

Let P be an arbitrary pattern containing P_1 as an interval minor. Let $P[r_1,c_1], P[r_2,c_2]$ and $P[r_3,c_3]$ be one-entries that can be used to map $P_1[1,2]$, $P_1[2,1]$ and $P_1[3,3]$ respectively. We take a submatrix $P' := P[[r_1,r_3],[c_2,c_3]]$. Such a matrix fulfills assumptions of the more restricted case above and we find a matrix $M' \in Av_{crit}(P')$ having n zero-intervals. We construct M from M' by simply adding new rows and columns containing only one-entries. We add $r_1 - 1$ rows in front of the first row and $k - r_3$ rows behind the last row. We also add $c_2 - 1$ columns in front of the first column and $l - c_3$ columns behind the last column. The constructed matrix M avoids P an an interval minor because its submatrix P' cannot be mapped to M'. At the same time, any change of a zero-entry of the r_1-th row of M to a one-entry creates a copy of $1^{k\times l}$. The constructed matrix M can be seen in Figure 4.1. $\square$

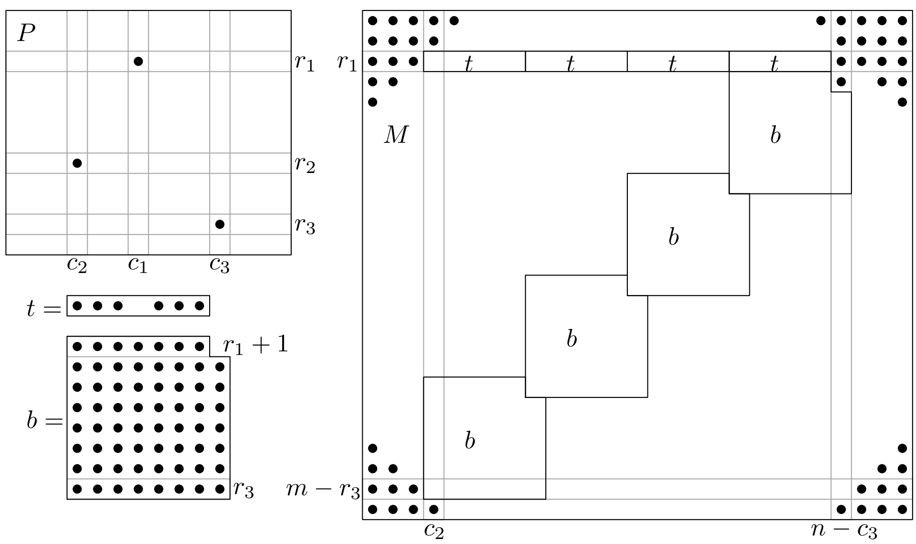


Figure 4.1: The structure of a critical matrix avoiding P that has arbitrarily many zero-intervals.

4.1.3 Bounding patterns

What makes it even more interesting is that any pattern avoiding all rotations of P_1 as interval minors is already bounding. For simplicity, whenever we say that a matrix has only k non-empty lines, we mean that every one-entry belongs to one of the k lines.

Theorem 4.15. *Let P be a pattern avoiding all rotations of P_1, then P*

1. *contains at most three non-empty lines or*
2. *avoids $\begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}$ or $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$.*

Proof. Assume P has four one-entries that do not share any row or column. Then those one-entries induce a 4×4 permutation inside P and because P does not contain any rotation of P_1, the induced permutation is either 1234 or 4321. Without loss of generality, assume it is the first one and denote its one-entries by e_1, e_2, e_3 and e_4. Clearly, no one-entry from e_1, e_2, e_3 and e_4 can be part of any mapping of $P' = \begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}$ because it would induce a mapping of a rotation of P_1.

Let $e_2 = P[r_2, c_2]$ and $e_3 = P[r_3, c_3]$. The submatrix $P[[r_2], [c_2, l]]$ avoids P'; otherwise, together with e_1 it would give P_2 as an interval minor. Symmetrically, $P' \not\preceq P[[r_3, k], [c_3]]$. The submatrix $P[[r_3 - 1], [c_3 - 1]]$ is empty; as otherwise, any one-entry would create a rotation of P_1 with e_3 and either e_1 or e_2. Symmetrically, the submatrix $P[[r_2 - 1], [c_2 - 1]]$ is also empty. This leave no one-entry in P to be used to map $P'[1, 1]$ and so $P' \not\preceq P$. $\square$

We now need to prove that whenever P avoids all rotations of P_1 (and satisfies one of the conditions we just showed) it is bounding.

Lemma 4.16. *Let $P \in \{0,1\}^{k \times l}$ be a pattern having one non-empty line. Then $r\left(Av_{\preceq}\left(P\right)\right) \leq k$ and $c\left(Av_{\preceq}\left(P\right)\right) \leq l$.*

Proof. Without loss of generality, let the non-empty line be a row r. Consider any matrix $M \in Av_{crit}\left(P\right)$. Submatrices $M[[r-1], [n]]$ and $M[[m-r+1, m], [n]]$ contain no zero-entry. If we look at any other row, it cannot contain k one-entries, so the maximum number of zero-intervals is k.

Consider a column c of M. If there is at least one one-entry in $M[[r, m - r - 1], c]$ then because M is critical, the whole column is made of one-entries. Otherwise, there are two one-intervals $M[[r-1], c]$ and $M[[m-r, m], c]$. $\square$

Lemma 4.17. *Let $P \in \{0,1\}^{k \times l}$ be a pattern having two non-empty lines. Then $r\left(Av_{\preceq}\left(P\right)\right) \leq k^2 + l$ and $c\left(Av_{\preceq}\left(P\right)\right) \leq l^2 + k$.*

Proof. First, we assume the two non-empty lines of P are rows $r_1 < r_2$ (or symmetrically columns) and consider any matrix $M \in Av_{crit}\left(P\right)$. From Observation 2.6 and maximality of M, we have that the submatrices $M[[r_1 - 1], [n]]$ and $M[[m - r_2 + 1, m], [n]]$ contain no zero-entry. Therefore, we may restrict ourselves to the case when $r_1 = 1$ and $r_2 = k$. From Corollary 4.12, we have that there are at most k^2 zero-intervals in each row of M and there are at most two zero-intervals in each column of M.

Let the two non-empty lines of P be a row r and a column c. Because of symmetry, we only show the bound for rows. For every one-entry e of P, except those in the row r, there is at most one zero-interval usable for e in each row of any $MAv_{crit}(P)$. For contradiction, assume there are two such zero-intervals z_1 and z_2 in the same row. Let Figure 4.2 illustrate the situation where red and blue lines form two mappings of P to M when a zero-entry of z_1 and z_2 respectively is changed to a one-entry used to map e. When we take the outer two vertical and horizontal lines, we get a mapping of P that uses an existing one-entry in between z_1 and z_2 to map e. This is a contradiction with $P \not\preceq M$.

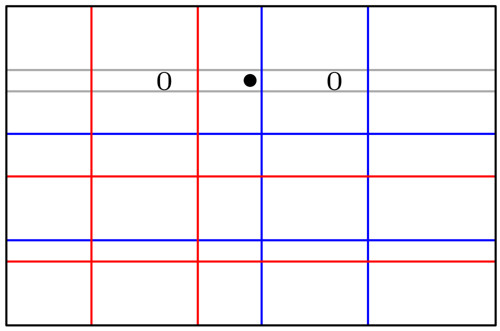


Figure 4.2: Red and blue lines representing two different mappings of a forbidden pattern. The two horizontal lines show the boundaries of the mapping of row r and the vertical lines show the boundaries of the mapping of column c.

For a one-entry $e = P[r, c']$, if $c' \leq c$ then there must be less than c' one-entries before any zero-intervals usable for e; otherwise, we could map $P[r, [1, c']]$ just to the single row of M. It follows that e is row-bounded. Symmetrically, the same holds in case $c' > c$ and together we have at most $k + l$ zero-intervals in each $M \in Av_{crit}(P)$. $\square$

Before we proof the other cases, let us introduce three useful lemmata that make the future case analysis bearable.

Lemma 4.18. *Let $P \in \{0, 1\}^{k \times l}$ be one of the four matrices in Figure 4.3. Then every one-entry in $P[\{r_2\}, [c_1, c_2]]$ is row-bounded. Moreover, the same also holds if we change some one-entries to zero-entries.*

Proof. Let a pattern P be the first described matrix and let $k' = c_2 - c_1$. We show that for each one-entry $e \in P[\{r_2\}, [c_1, c_2]]$ and every matrix $M \in Av_{crit}(P)$ there are at most k' zero-intervals usable for e in each row of M. For contradiction, assume there is a row r with $k' + 1$ zero-intervals usable for some e. It follows that there are at least k' one-entries in between the two most distant zero-intervals z_1 and z_2. Therefore, the whole row r_2 can be mapped just to the row r. Changing a zero-entry of z_1 to a one-entry, to which e can be mapped, creates a mapping of P to M, in which all one-entries from columns $[c_1]$ are mapped to columns before z_1 (and z_1) and similarly all one-entries from columns $[c_2, l]$ can be mapped to columns past z_2 (and z_2). It also holds that all the one-entries from the row r_1 are mapped (in both mappings) to one-entries of M in rows $[r - r_2 + r_1]$ (and symmetrically for one-entries from the row r_3). Thus, we can simply map empty rows $[r_1 + 1, r_3 - 1]$ around row r and use the rest to map rows r_1 and r_2.

Proofs of cases two and three are similar to the first one and we skip them.

Let a pattern P be the fourth described matrix and consider any matrix $M \in Av_{crit}(P)$. For the i-th one-entry e in the row r_2 (ordered from left to right

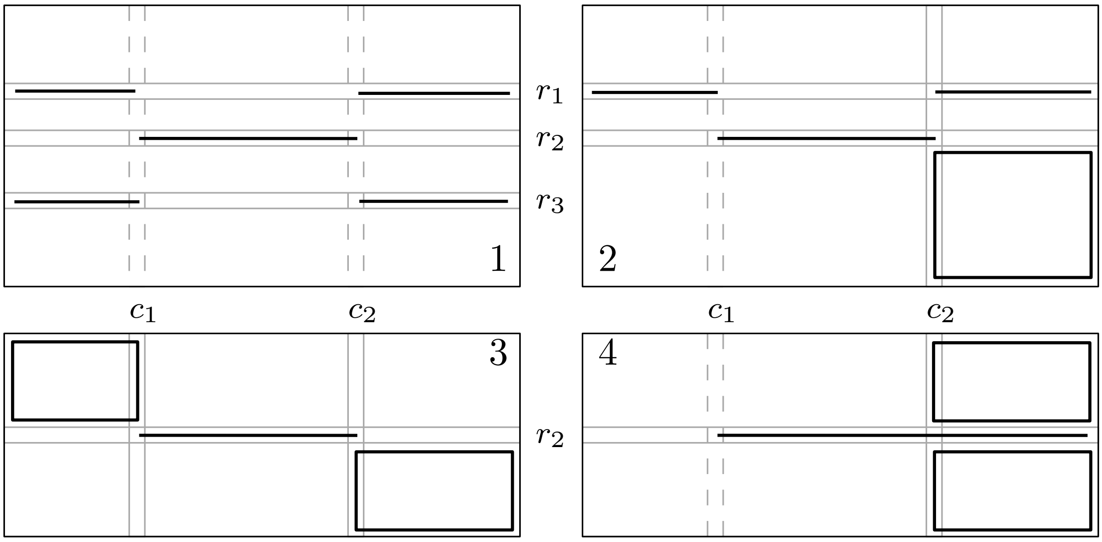


Figure 4.3: The patterns for which all one-entries in the row r_2 and the columns $[c_1, c_2]$ are row-bounded. One-entries of the patterns are inside the bold rectangles and on the bold lines.

and only considering those in columns $[c_1, c_2]$) no zero-interval of M usable for e cannot have i one-entries before it and so the row-complexity of each such one-entry is bounded by $i \geq l$.

Throughout the proof, we have never used as a fact that an entry of M is a one-entry and so the proof also holds for any pattern P created from any of the fourth described matrices by deletion of one-entries. $\square$

It is important to realize that we could not have used the same proof we used for the first three cases also for the fourth case, because we can never rely on the fact a mapping of P only uses one row of M to map the row r_2. This is because in the fourth case, there are also potential one-entries in $P[\{r_2\}, [c_2 + 1, l]]$.

What follows is a direct corollary of the fourth case of just stated Lemma 4.18. Even though it is very simple and straightforward, it is going to be used so often that it is worth stating it apart from the rest.

Lemma 4.19. *Let P be a matrix and let c be its first non-empty column. Then every one-entry from c is row-bounded.* $\square$

Lemma 4.20. *Let $P \in \{0, 1\}^{k \times l}$ be one of the three matrices in Figure 4.4. Then every one-entry in $P[[r_1 + 1, r_2 - 1], \{c\}]$ is row-bounded. Moreover, the same also holds if we change some one-entries to zero-entries.*

Proof. Let P be a submatrix of the first described matrix. We show that for each one-entry e from $P[[r_1 + 1, r_2 - 1], \{c\}]$ and every matrix $M \in Av_{crit}(P)$ there is at most one zero-interval usable for e in M. For contradiction, assume there is a row r with two zero-intervals z_1 and z_2 usable for e. Consider Figure 4.5, where the red lines show a mapping of P to M created when a zero-entry of z_1 is changed to a one-entry used to map e and the blue lines show a mapping of P to M created when a zero-entry of z_2 is changed to a one-entry used to map e. If we map the column c to the columns of M enclosed by the two outer vertical lines and map rows r_1 and r_2 again to rows enclosed by the corresponding two

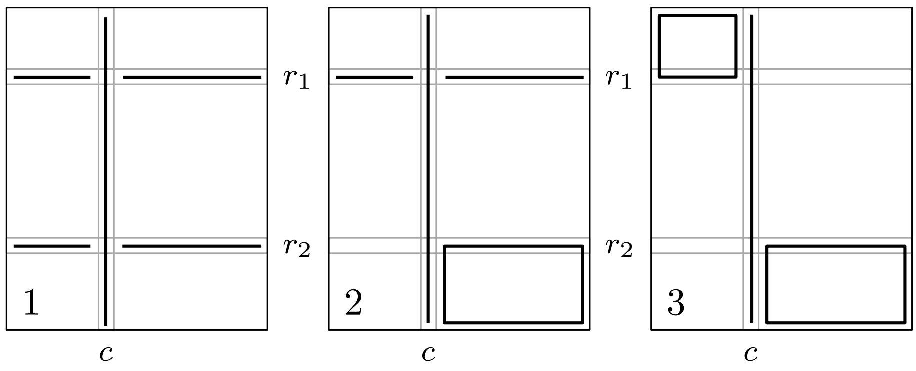


Figure 4.4: The patterns for which all one-entries in the column c and the rows $[r_1 + 1, r_2 - 1]$ are row-bounded. One-entries of the patterns are inside the bold rectangles and on the bold lines.

outer horizontal lines, we get a mapping of P to M and so a contradiction with $P \not\prec M$.

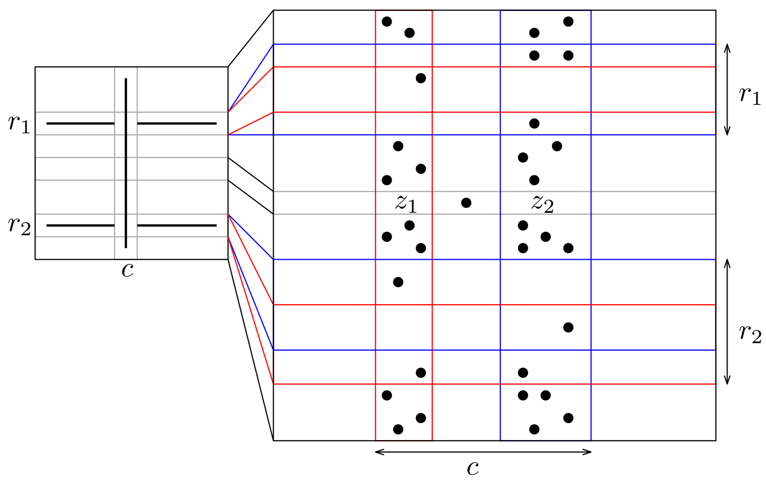


Figure 4.5: Red and blue lines representing two different mappings of a forbidden pattern. The four horizontal lines show the boundaries of the mapping of rows r_1 and r_2 and the vertical lines show the boundaries of the mapping of the column c.

Proofs of cases two and three are similar to the first one and we skip them.

Throughout the proof, we have never used as a fact that an entry of M is a one-entry and so the proof also holds for any pattern P created from any of the fourth described matrices by deletion of one-entries. $\square$

Lemma 4.21. *Let a pattern $P \in \{0,1\}^{k \times l}$ be created from one of the matrices in Figure 4.6 by deletion of one-entries and let $c = l - 1$. Then every one-entry in $P[[r_1, r_2], \{c\}]$ is row-bounded.*

Proof. Let a pattern P be created from the first described matrix. From 4.20, we know that all one-entries in $P[[r_1 + 1, r_2 - 1], \{c\}]$ are row-bounded. Thank to symmetry, it suffices to show that the one-entry $e = P[r_1, c]$ is row-bounded. Without loss of generality, we have $P[r_2, l] = 1$; otherwise, we can use the fourth case of Lemma 4.3 to prove that e is row-bounded.

Consider any matrix $M \in Av_{crit}(P)$ and let $z_1 < z_2$ be any two zero-intervals from the same row usable for e. Without loss of generality, in any mapping of P

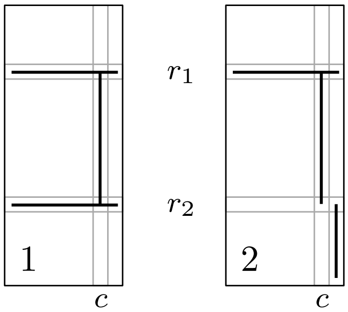


Figure 4.6: The patterns for which all one-entries in the column c and the rows $[r_1, r_2]$ are row-bounded. One-entries of the patterns are on the bold lines and the column c is the second last.

to M, created when a zero-entry of z_1 is changed to a one-entry used to map e, the one-entry $P[r_2, l]$ is mapped to a column before z_2. Otherwise, if we map e to the one-entry between z_1 and z_2 and map $P[r_1, l]$ to any one-entry behind z_2 we get a mapping showing $P \preceq M$.

We prove there are at most l zero-intervals usable for e on every row of M. For contradiction, let there be such zero-intervals $z_1, \dots, z_l$ that there is a one-entry behind each of them. For each zero-interval z_i, let e_i be any one-entry of M that can be used to map the one-entry $P[r_2, l]$ if a zero-entry of z_i is changed to a one-entry used to map e. In the sequence $e_1, \dots, e_l$ there either are two one-entries $M[r'_1, c'_1], M[r'_2, c'_2]$ such that $r'_1 \leq r'_2$, or the rows of one-entries form a decreasing sequence.

Let us first consider the first case and let $e_i = M[r'_1, c'_1]$ and $e_j = M[r'_2, c'_2]$. Consider a mapping of P to M created when a zero-entry of z_i is changed to a one-entry used to map e. If in this mapping, we map e to a one-entry between z_i and z_j, map $P[r_1, l]$ to a one-entry behind z_j, map $P[r_2, l-1]$ to e_i and map $P[r_2, l]$ to e_j, we get a mapping of P to M, which is a contradiction.

And so it holds that the one-entries $e_1, \dots, e_l$ form a row decreasing sequence. We can pair every e_i with a one-entry bounding z_i from the right and so we can map the whole submatrix $P[[k], [l-2]]$ just to columns before z_{l-1} of M. Because z_l is usable for e, there are enough one-entries to map the whole column c there and there are one-entries where $P[r_1, l]$ and $P[r_2, l]$ can be mapped. The only problem is that e is mapped to a one-entry created by changing a zero-entry of z_l but we can also map it to a one-entry between zero-intervals z_{l-1} and z_l and we have $P \preceq M$ and a contradiction.

Let a pattern P be created from the second described matrix. All one-entries in $P[[r_1+1, r_2-1], \{c\}]$ are row-bounded thanks to (the second case of) Lemma 4.20. From the fourth case of Lemma 4.18, the one-entry $P[r_1, c]$ is also row-bounded. So we only need to prove that the one-entry $P[r_2, c]$ is row-bounded.

Without loss of generality, $P[r_1, l] = 1$; otherwise, $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \not\preceq P$ and in the following Lemma 4.22, we show that every such P is bounding. We once again define one-entries $e_1, \dots, e_l$ and use the same analysis as we did in the first case.

□

Now that the very technical lemmata are stated, we just use them to easily prove that the remaining patterns described in Theorem 4.15 are also bounding.

Lemma 4.22. *Let* $P \in \{0,1\}^{k\times l}$ *be a pattern avoiding* $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$*. Then* P *is bounding.*

Proof. From Proposition 2.14, we know that P is a walking pattern. Every one-entry of P satisfies either conditions of the third case of Lemma 4.18 or it satisfies conditions of the third case of Lemma 4.20 and therefore is row-bounded. From Observation 4.9, we know it is also column-bounded. □

What follows is the last and the most difficult case of our analysis. Its length is caused by the fact that it is harder to describe symmetries than it is to just use the previous lemmata to show that each pattern is bounding.

Lemma 4.23. *Let* $P \in \{0,1\}^{k\times l}$ *be a pattern having three non-empty lines and avoiding all rotations of* P_1*. Then* P *is bounding.*

Proof. First of all, if P avoids $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$, we use Lemma 4.22.

Let the three non-empty lines be three rows and let a pattern P have one-entries in at least three columns. Then it contains a 3×3 permutation matrix as a submatrix. Since the rotations of P_1 are avoided, the only feasible permutations are 123 and 321 and without loss of generality, we assume the first case. In Figure 4.7 we see the structure of P. The capital letters stand for one-entries of the permutation and are chosen to be the left-most possible, letters $a - f$ stand each for a potential one-entry and the Greek letters stand each for a potential sequence of one-entries. Everything else is empty. Not all one-entries can be there at the same time, because that would create a mapping of P_1 or its rotation. We also need to find $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right) \preceq P$. The following analysis only uses hereditary arguments, which means that if we prove that P is bounding, we also prove that each submatrix of P is bounding. With this in mind, we restrict ourselves to critical patterns.

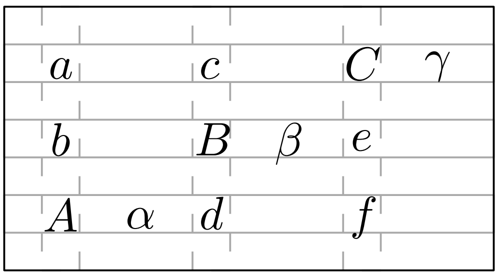


Figure 4.7: The structure of a pattern only having three non-empty rows and avoiding all rotations of P_1.

1. $\gamma = 1 \Rightarrow f = 0 \Rightarrow$ because $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right) \preceq P$, it holds $a = 1 \Rightarrow \alpha = 0$
    (a) $d = 1 \Rightarrow b = 0,\ \beta = 0,\ e = 0$
    (b) $d = 0$
        i. $c = 1 \Rightarrow \beta = 0,\ e = 0$
        ii. $c = 0$

2. $\gamma = 0$

   (a) $\alpha = 1 \Rightarrow a = 0,\ b = 0$. If $f = 0$ we have case 1. (b) ii.; otherwise, we have case 1. (a).

   (b) $\alpha = 0$

      i. $c = 1,\ d = 1 \Rightarrow b = 0,\ e = 0,\ \beta = 0$

      ii. $c = 1,\ d = 0 \Rightarrow e = 0,\ \beta = 0$ and without loss of generality, $b = 1$. Otherwise, we have the previous case. Therefore, $f = 0$

      iii. $c = 0,\ d = 1 \Rightarrow b = 0$. Without loss of generality, $e = 1,\ \beta = 1$. Otherwise, we have the case $c = 1,\ d = 1$. Therefore, $a = 0$

      iv. $c = 0,\ d = 0$

The same analysis also proves that if a pattern with the same restrictions only has three non-empty columns then it is bounding.

Let P be a pattern having two non-empty rows r_1, r_2 and one non-empty column c_1. Without loss of generality, we again assume permutation 123 is present and we distinguish three cases. Consider Figure 4.8:

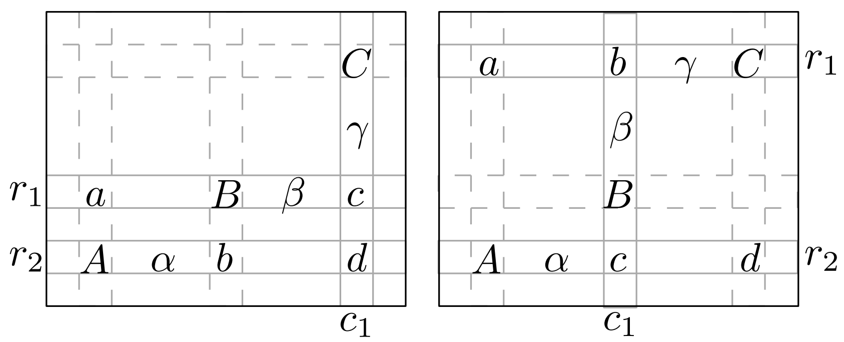


Figure 4.8: The structure of a pattern only having one-entries in two rows and one column that avoids all rotations of P_1.

1. C lies in column c_1

   (a) $a = 0$

   (b) $a = 1 \Rightarrow b = 0,\ \alpha = 0$

2. B lies in column c_1

   (a) $a = 1,\ d = 1 \Rightarrow \alpha = 0,\ \gamma = 0$

   (b) $a = 1,\ d = 0 \Rightarrow \alpha = 0$

   (c) $a = 0,\ d = 1 \Rightarrow \gamma = 0$

   (d) $a = 0,\ d = 0$. The pattern avoids $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right)$.

3. A lies in column c_1. This is symmetric to the first situation.

The same analysis also proves that if a pattern P has two non-empty columns and one non-empty row then the pattern is bounding. $\square$

Combining the lemmata we finally get the following result.

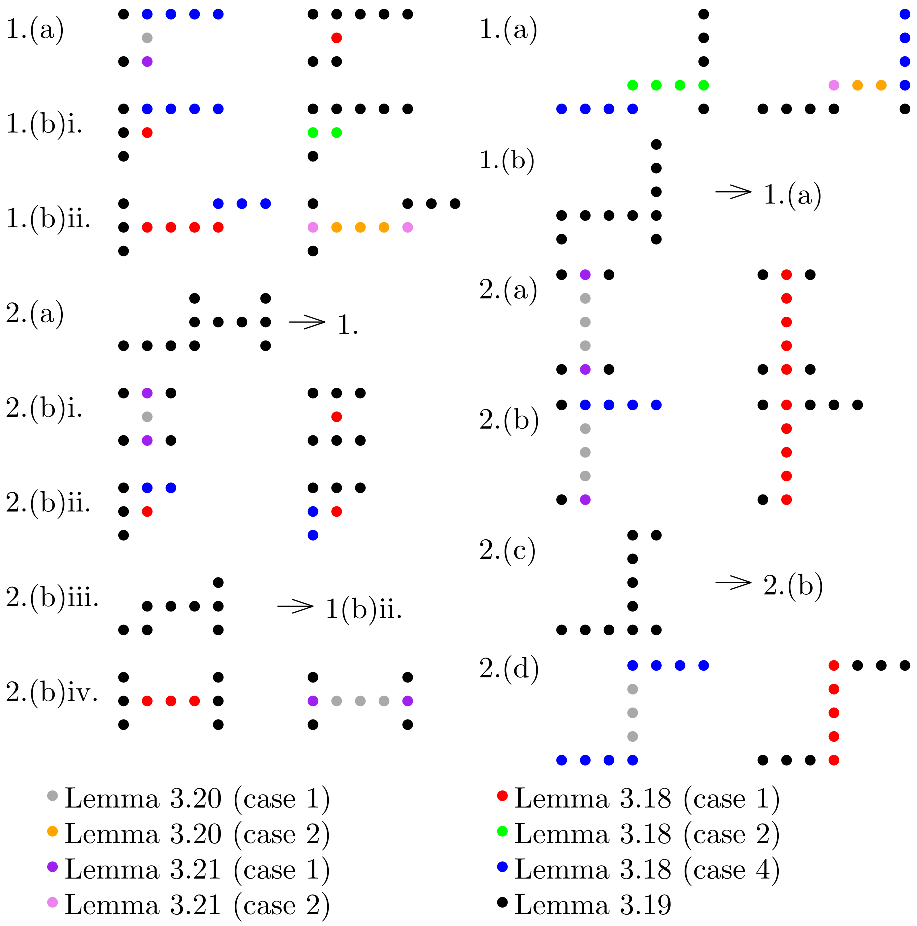


Figure 4.9: A figure showing which lemma can be used to prove that each one-entry of patterns discussed in the case analysis is bounded. The patterns from the left half of the picture only contain three non-empty rows and the patterns from the right half only contain two non-empty rows and one non-empty column. Each case either contains a picture showing that each one-entry is row-bounded and column-bounded, or an arrow describing that the case can be reduced to a different one.

Theorem 4.24. *Let P be a pattern avoiding all rotations of P_1, then P is bounding.* □

A lot can be implied from this theorem. Here are two straightforward corollaries for which we do not know any other proof.

Corollary 4.25. *For every pattern P: $Av_{\preceq}(P)$ is row-bounded $\Leftrightarrow$ $Av_{\preceq}(P)$ is column-bounded.*

Corollary 4.26. *For every bounding pattern P and every $P' \preceq P$ it holds P' is bounding.*

4.2 Chain rules

Now that we know exactly what patterns are bounding, it is time to speak about the complexity of classes more in general. We are still going to be concerned with classes of matrices avoiding patterns, but they will avoid a set of patterns rather than just one pattern.

First, we show that Corollary 4.25 does not hold in general. Next, we show that bounded classes are closed to intersection. At the end of the chapter, we prove the same is not true for unbounded classes of matrices and even more, an intersection of a few unbounded classes can be bounded hereditarily, which means that its every subset is bounded.

It is easy to see that Lemma 4.18, Lemma 4.19, Lemma 4.20, Lemma 4.21 and Lemma 4.22 can be generalized to our settings. Their proofs without change show that for every set of patterns $\mathcal{P}$, if a pattern $P \in \mathcal{P}$ looks like a described pattern, then any one-entry of P is (row-)bounded in $Av_{\preceq}(\mathcal{P})$. Therefore, we use the lemmata without restating them.

We define classes of matrices to be bounded if they are both row-bounded and column-bounded. From what we proved so far, we see that for a pattern P, the class $Av_{\preceq}(P)$ is row-bounded if and only of it is column-bounded. Once we consider classes avoiding sets of patterns, this does not have to be true.

Lemma 4.27. *There exists a set of patters $\mathcal{P}$ such that the class $Av_{\preceq}(P)$ is row-bounded but column-unbounded.*

Proof. Let $\mathcal{P} = \left\{ P = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & \bullet & \\ & & \bullet \end{pmatrix}, I_4 = \begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix} \right\}$. We can use a similar construction to what we did in Lemma 4.13, to prove $Av_{\preceq}(\mathcal{P})$ is column-unbounded. The only difference is that the "blocks" are of size 4×2 and the whole matrix is transposed.

To prove that the class $Av_{\preceq}(\mathcal{P})$ is row-bounded, we take an arbitrary matrix $M \in Av_{crit}(\mathcal{P})$ and consider any row r of M. We need to prove that every one-entry of I_4 and P is row-bounded.

From Lemma 4.22, we know that every one-entry of I_4 is row-bounded (and column-bounded) in $Av_{\preceq}(\mathcal{P})$. From Lemma 4.19, one-entries $P[2,1]$ and $P[4,3]$ are row-bounded in $Av_{\preceq}(\mathcal{P})$. From the first case of Lemma 4.20, the one-entry $P[3,2]$ is row-bounded in $Av_{\preceq}(\mathcal{P})$.

We prove that there are at most two zero-intervals usable for $P[1,2]$ in the row r. For contradiction, let there be three zero-intervals $z_1 < z_2 < z_3$. Consider a mapping of P to M created when a zero-entry of z_3 is changed to a one-entry used to map $P[1,2]$. Without loss of generality, the one-entry used to map $P[2,1]$ lies in columns of z_3 or just under the one-entry e bounding z_3 from left; otherwise, we could use e to map $P[1,2]$ and find the pattern in M. Then, a one-entry between zero-intervals z_1 and z_2 together with the one-entries used to map $P[2,1], P[3,2]$ and $P[4,3]$ give us a mapping of I_4 and so a contradiction with $M \in Av_{\preceq}(\mathcal{P})$. $\square$

Theorem 4.28. *Let $\mathcal{P}$ and $\mathcal{Q}$ be classes of patterns. If both classes $Av_{\preceq}(\mathcal{P})$ and $Av_{\preceq}(\mathcal{Q})$ are bounded then $Av_{\preceq}(\mathcal{P} \cup \mathcal{Q})$ is bounded.*

Proof. Let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$. We show that $comp_{\mathcal{R}} \leq comp_{\mathcal{P}} + comp_{\mathcal{Q}} = C$.

For contradiction, let a matrix $M \in Av_{crit}(\mathcal{R})$ have at least $C + 1$ zero-intervals in a single row (or column). Without loss of generality, it means there is more than $comp_{\mathcal{P}}$ zero-intervals usable for one-entries of the patterns from $\mathcal{P}$. Let $M' \in Av_{\preceq}(\mathcal{P})$ be a matrix created from M by changing as many zero-entries to one-entries as possible. Clearly, it still contains more than $comp_{\mathcal{P}}$ zero-intervals usable for one-entries of the patterns from $\mathcal{P}$, which is a contradiction with the value of $comp_{\mathcal{P}}$. $\square$

Using induction, we can show that also a union of a finite number of bounded classes of finite sizes is bounded. Interestingly enough, unbounded classes are not closed the same way.

Theorem 4.29. *For every $1 \leq i < j \leq 4]$ is $Av_{\preceq}(\{P_i, P_j\})$ bounded.*

Proof. We only show that $Av_{\preceq}(P_1, P_2)$ is bounded. To prove $Av_{\preceq}(P_1, P_3)$ is bounded, we can use the same steps. All other pairs are then symmetric to these two.

- $Av_{\preceq}(P_1, P_2)$ is row-bounded:
  From Lemma 4.19, we have that one-entries $P_1[2,1], P_1[3,3], P_2[2,3]$ and $P_3[3,1]$ are row-bounded. For $P_1[1,2]$ and $P_2[1,2]$, we prove there are at most two zero-intervals usable for each of them in each row of any matrix $M \in Av_{crit}(P_1, P_2)$. For contradiction, let $z_1 < z_2 < z_3$ be three zero-intervals usable for $P_1[1,2]$ in a row r of M. The one-entries used to map $P_1[2,1]$ and $P_1[3,3]$ in a mapping created when a zero-entry of z_1 is changed to a one-entry used to map $P_1[1,2]$, together with a one-entry in between z_2 and z_3 give us a mapping of P_2 to M. Symmetrically, the same goes for $P_2[1,2]$.

- $Av_{\preceq}(P_1, P_2)$ is column-bounded:
  The proof that all one-entries of P_1 and P_2 are column-bounded is the same. $\square$

We prove even stronger result for the class $Av_{\preceq}(P_1, P_2, P_3, P_4)$ by using a well known fact from the theory of ordered sets. It is due to Higman [1952] and states the following:

Fact 4.30 (Higman's lemma)**.** *Let A be a finite alphabet and A^* be a set of finite sequences over A partially ordered by the subsequence relation. Then A^* is well-quasi-ordered.*

In other words, whenever we have a potentially infinite $S \subseteq A^*$, there are sequences $a, b \in S$ such that a is a subsequence of b. This also means that no such S contains an infinite anti-chain.

Theorem 4.31. *The class $\sigma = Av_{\preceq}(P_1, P_2, P_3, P_4)$ is bounded. Moreover, every subclass of σ is bounded.*

Proof. From Theorem 4.15, we know that σ is a union of finitely many classes of matrices. For each such class $\mathcal{M}$, we prove that every subclass $\mathcal{M}' \subseteq \mathcal{M}$ is also bounded, which we do by showing that there is only a finite number of critical matrices in $\mathcal{M}'$. Then we use Theorem 4.28 to obtain the result.

- Let $\mathcal{M}_1 \subseteq \sigma$ be the class containing all matrices from σ that have their one-entries in at most three rows and let $\mathcal{M} \subseteq \mathcal{M}_1$ be an arbitrary subclass.

  To proof there is no infinite anti-chain of critical matrices from $\mathcal{M}$, we use Fact 4.30. To describe $M \in \mathcal{M}$,č we use words over alphabet $A =$

$\{a, b, c, d, e, f, g, h, i, j\}$. Let $r_1 < r_2 < r_3$ be the non-empty rows (if less then three are non-empty we choose extra values arbitrarily). We define $w_M \in A^*$ as follows. First, we use a letter g r_1 times, letter h $r_2 - r_1$ times, letter i $r_3 - r_2$ times and letter j $m - r_3$ times to describe the number of rows of M and position of non-empty rows. Then we describe columns from the first one to the last one as follows. For each 0 in r_1 we use a letter a and for 1, we use letters ab. For each 0 in r_2 we use a letter c and for 1, we use letters cd. For each 0 in r_3 we use a letter e and for 1, we use letters ef.

If we have $w_M, w_{M'} \in A^*$ such that w_M is a subsequence of $w_{M'}$, then we want to show that M is an interval minor of M'. Let r_1, r_2, r_3 and r'_1, r'_2, r'_3 be the non-empty rows of M and M' respectively. Since the number of leading letters g is not bigger in w_M, M does not have more empty rows before r_1 than M' does before r'_1 and similarly for the other pairs of non-empty rows.

Now consider there is ab in w_M and it corresponds to some $a \ldots b$ in $w_{M'}$. Without loss of generality, the letter a in $w_{M'}$ is the one exactly before b. Clearly, one-entries of M can be mapped to one-entries in M' and we only need to check that two one-entries of two different columns of M are not mapped to two one-entries of the same column of M'. But this is not hard to see and we have $M \preceq M'$ (but it does not have to hold that $M \leq M'$).

From Fact 4.30, we have that A^* is well ordered, which means that matrices having at most three non-empty rows (columns) are well ordered and so they does not have an infitely long anti-chain.

- Let $\mathcal{M}_2 \subseteq \sigma$ be the class containing all matrices from σ that have their one-entries in at most three non-empty columns:

- M only contains at most two rows and one column (or vice versa):
  The number of one-intervals of any critical matrix M is bounded by two.

  We use words over alphabet $A = \{a, b, c, d, e, f, g\}$ and for non-empty rows r_1, r_2 and column c_1, we define w_M as follows. We first encode each column in such a way that for each 0 in r_1 we use a letter a and for 1, we use letters ab. For each 0 in r_2 we use a letter c and for 1, we use letters cd. Right before and after the description of column c_1, we put a letter g. Next, we encode each row in such a way that for each 0 in c_1 we use a letter e and for each 1 letters ef. Right before and after the descriptions of rows r_1 and r_2 we again place a letter g.

  Because of the distinct letters for encoding rows and columns we can apply the same analysis as we did in the previous case and since entries at $M[r_1, c_1]$ and $M[r_2, c_1]$ are separated from the rest by a special letter g there is no way to find a one-entry if it is not there.

- M avoids $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ (or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$):
  From Proposition 2.14 we know M is a walking matrix and any such critical matrix only contains at most one one-intervals in each row and column.

> We use words over alphabet $A = \{a, b, c, d\}$ and encode M as follows. We choose an arbitrary walk of M containing all one-entries and index its entries as $w_1 \dots w_{m+n-1}$. Starting from w_1, we encode w_i so that a letter a stands for 0 and letters ab for 1, if w_{i+1} lies in the same row as w_i, and we use a letter c for 0 and letters cd for 1, if w_{i+1} lies in the same column as w_i. We always use a or ab for the last entry.

In the construction of words corresponding to matrices, we only made sure that $w_M \subseteq w_{M'} \Rightarrow M \preceq M'$ and the other implication does not need to hold. A different construction may lead to equivalence, but that is not necessary for our result.

We use distinct alphabets to describe different categories and when given a potentially infinite class of matrices from σ, we know that inside each category there is at most finite number of minimal matrices such that all of the rest contain a smaller one as an interval minor. Using induction on Theorem 4.28, we have that each category is bounded and by applying induction with Theorem 4.28 once again, we get that the union of the categories is also bounded. □

Observation 4.32. *There exists a bounding pattern P having an unbounded subclass of $Av_{\preceq}(P)$.*

Proof. Let $P = I_n$ (identity matrix) for $n > 3$. From Lemma 4.22, we have that P is bounding. On the other hand, $Av_{\preceq}(I_n, P_1)$ is unbounded, because the construction used in the proof of Lemma 4.13 also works for this class. □

Conclusion

Throughout the thesis, we have been looking from multiple angles at classes binary matrices. In particular, we studied properties of matrices containing or avoiding other, smaller matrices as interval minors.

Characterizations We started by describing the structure of matrices avoiding some small patterns. We managed to characterize all matrices avoiding patterns having up to three one-entries and also showed how to generalize some of the characterizations for much bigger patterns. Even for very small patterns (only having four one-entries), the structure of matrices avoiding them became very rich and hard to properly describe.

So instead, we began to consider how a small change of the pattern influences the structure of matrices avoiding the pattern. This was mainly looked at when adding empty lines to the pattern. We showed for patterns of size $k \times 2$ what matrices avoiding its blown version look like, but we are unable to generalize the result (or show something similar) when the pattern we are working with is bigger. So the question, answer for which could later be used to describe complexity of some patterns or help enumerate matrices from some class, is:

Question 4.33. What can we say about matrices avoiding a pattern with added empty line with respect to the matrices avoiding the original pattern?

Operations with matrices After dealing with small patterns, we defined an operation of the direct sum and a closure under the direct and began to explore how it relates with classes of matrices. Once again we started by considering some small cases, where we observed that the closure can be describe by forbidden patterns very naturally. Later, we considered the direct sum with an overlap, which allowed us to restate the characterizations from the previous chapter in a much easier way.

We also introduced a notion of articulation that allowed us to proof a strong result saying that any matrix closed under the direct sum can always be describe by a finite set of forbidden patterns.

In order to generalize the result, we started looking at sets of matrices and ultimately, we showed that there are set of matrices with finite basis, whose closure cannot be described with a finite number of forbidden minors.

Zero-intervals In the last chapter, we studies a property of a class of matrices that in some terms describes the complexity of inclusion maximal matrices from the class.

To bring the notion back to the pattern avoiding, we defined a pattern P to be bounding if and only of the class $Av_{\preceq}(P)$ is bounded and we showed that a pattern is bounding if and only if it avoids all rotations of $P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$.

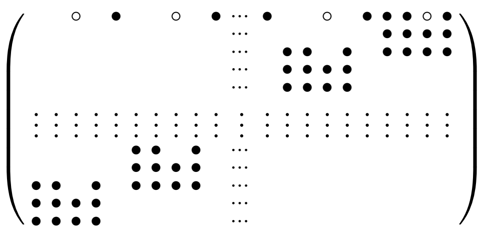
To show for a pattern that it is bounding meant to show that each one-entry it contains is bounded. It may be an interesting generalization to show for each one-entry whether it is bounded or unbounded in its pattern. On our way, we only saw one type of unbounded one-entries, and those are $P_1[2,1]$ for rows, $P_1[1,2]$ for columns and corresponding one-entries in the rotations of P_1. Let us call these one-entries *trivially unbounded*.

Considering this generalization, there are one-entries that are unbounded but not trivially unbounded. Let us mention some of them (arrows point to row-unbounded one-entries):

$$\begin{pmatrix} \bullet & & & & \\ & \overset{\downarrow}{\bullet} & & \bullet & \\ & & \bullet & & \\ & & & & \bullet \end{pmatrix} \begin{pmatrix} \bullet & & & & \\ & \overset{\downarrow}{\bullet} & & \bullet & \\ & & & & \bullet \\ & & \bullet & & \end{pmatrix} \begin{pmatrix} \bullet & & & \\ & \overset{\downarrow}{\bullet} & \bullet & \\ & \bullet & & \\ & & & \bullet \end{pmatrix} \begin{pmatrix} \bullet & & & \\ & \overset{\downarrow}{\bullet} & \bullet & \\ & & & \bullet \\ & \bullet & & \end{pmatrix}$$

Proposition 4.34. *Let* $P = \begin{pmatrix} & \bullet & \overset{\downarrow}{\bullet} & \\ \bullet & & & \bullet \\ & & \bullet & \end{pmatrix}$. *For every integer n there is a matrix $M \in Av_{crit}(P)$ having at least n zero-intervals.*

Proof. Let M be a matrix described by the picture:



We see that $P \npreceq M$ because we always need to map $P[2,1]$ and $P[3,3]$ to just one "block" of one-entries of M, which only leaves a zero-entry where we need to map $P[1,3]$ or $P[2,4]$.

When we change any marked zero-entry of the first row into a one-entry, we get a matrix containing a minor of $\{1\}^{3\times 4}$; therefore, containing P as an interval minor. In case M is not maximal, we can add more one-entries to make it maximal but it will still contain a row with n one-intervals. $\square$

Our tools are not strong enough to let us characterize unbounded one-entries. Based on our attempts, we state the following conjecture:

Conjecture 4.35. Every row-unbounded one-entry share a row with a trivially row-unbounded one-entry.

Throughout the chapter, we work with arguments such that if something holds for a matrix, it also holds for every submatrix. While it seems completely natural, we are unable to decide the following question:

Question 4.36. Can a non-bounding pattern become bounding after a one-entry is changed to a zero-entry?

Using our machinery, we showed that while the union of bounding sets of patterns is always bounding again, the union of non-bounding sets may become bounding. For the class of matrices avoiding all rotations of P_1, we even showed that every subclass is also bounded. The same remains open for other sets of patterns:

Question 4.37. Is $Av_{\preceq}\left(\begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}\right)$ hereditarily bounded?

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