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1 **MASTER THESIS**

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3 # Hereditary classes of binary matrices

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Title: Hereditary classes of binary matrices

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Abstract: Abstract.

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Dedication.

Contents

Introduction

TODO:

- Fix or rewrite Lemma 1.8.
- Characterize or exclude P_9.
- Consider adding more patterns/generalizations.
- Add an opening paragraph at the beginning of the 2nd chapter.
- Maybe rewrite Definition 2.6.
- Consider proving Proposition 2.9 (currently commented).
- Consider rewriting Observation 2.17.
- Check Theorem 2.21 and everything after it once more.
- Fix or remove Lemma 3.29.

Throughout the paper, every time we speak about matrices we mean binary matrices (also called 01-matrices) and we omit the word binary. If we speak about a *pattern*, we again mean a binary matrix and we use the word in order to distinguish among more matrices as well as to indicate relationship.

When dealing with matrices, we always index rows and column starting with one and when we speak about a row r, we simply mean a row with index r. A *line* is a common word for both a row and a column. When we order a set of lines, we first put all rows and then all columns. For $M \in \{0,1\}^{m \times n}$, $[m]$ is a set of all rows and $[m+n]$ is a set of all lines, where m-th element is the last row. This goes with the usual notation.

Notation 0.1. For $n \in \mathbb{N}$ let $[n] := \{1, 2, \dots, n\}$ and for $m \in \mathbb{N}$, where $n \leq m$ let $[n, m] := \{n, n+1, \dots, m\}$.

Notation 0.2. For a matrix $M \in \{0,1\}^{m \times n}$ and $L \subseteq [m+n]$ let $M[L]$ denote a submatrix of M induced by lines in L.

Notation 0.3. For a matrix $M \in \{0,1\}^{m \times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M[R, C]$ denote a submatrix of M induced by rows in R and columns in C. Furthermore, for $r \in [m]$ and $c \in [n]$ let $M[r, c] := M[\{r\}, \{c\}] = M[\{r, c+m\}]$.

Definition 0.4. We say a matrix $M \in \{0,1\}^{m \times n}$ *contains* a pattern $P \in \{0,1\}^{k \times l}$ *as a submatrix* and denote it by $P \leq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M[R, C][r, c] = 1$.

This does not necessarily mean $P = M[R, C]$ as $M[R, C]$ can have more one-entries than P does.

Notation 0.5. For a matrix $M \in \{0,1\}^{m \times n}$ and $L \subseteq [m+n]$ let $M_{\preceq}[L]$ denote a matrix acquired from M by applying following operation for each $l \in L$:

- If l is the first row in L then we replace the first l rows by one row that is a bitwise OR of replaced rows.
- If l is the first column in L then we replace the first $l - m$ columns by one column that is a bitwise OR of replaced columns.
- Otherwise, we take l's predecessor $l' \in L$ in the standard ordering and replace lines $[l'+1, l]$ by one line that is a bitwise OR of replaced lines.

Notation 0.6. For a matrix $M \in \{0,1\}^{m \times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M_{\preceq}[R, C] := M_{\preceq}[R \cup \{c + m | c \in C\}]$.

Definition 0.7. We say a matrix $M \in \{0,1\}^{m \times n}$ *contains* a pattern $P \in \{0,1\}^{k \times l}$ *as an interval minor* and denote it by $P \preceq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M_{\preceq}[R, C][r, c] = 1$.

Observation 0.8. *For all matrices M and P, $P \leq M \Rightarrow P \preceq M$.*

Observation 0.9. *For all matrices M and P, if P is a permutation matrix, then $P \leq M \Leftrightarrow P \preceq M$.*

Proof. If we have $P \preceq M$, then there is a partitioning of M into rectangles and for each one-entry of P there is at least one one-entry in the corresponding rectangle of M. Since P is a permutation matrix, it is sufficient to take rows and columns having at least one one-entry in the right rectangle and we can always do so.

Together with Observation 0.8 this gives us the statement. □

Observation 0.10. *Let $M \in \{0,1\}^{m \times n}$ and $P \in \{0,1\}^{k \times l}$, $P \preceq M \Leftrightarrow P^T \preceq M^T$.*

Because of this observation we will usually only show results only for rows or columns and expect both to hold and only show results for $P \in \{0,1\}^{k \times l}$ but assume the symmetrical results for P^T.

Definition 0.11. Let $\mathcal{F}$ be any class of forbidden matrices. We denote by $Av(\mathcal{F})$ the set of all matrices that avoid every $F \in \mathcal{F}$ as an interval minor.

Observation 0.12. *For all patterns P, P': $P \preceq P' \Leftrightarrow Av_{\preceq}(P) \subseteq Av_{\preceq}(P')$.*

Proof. Every $M \in Av_{\preceq}(P)$ avoids P and because $P \preceq P'$, it also avoids P'; therefore, it belongs to $Av_{\preceq}(P')$.

If $P \not\preceq P'$ then $P' \in Av_{\preceq}(P)$. As $P' \notin Av_{\preceq}(P')$ we have $Av_{\preceq}(P) \not\subseteq Av_{\preceq}(P')$. □

0.1 Extremal function

Notation 0.13. Let M be a matrix. We denote $|M|$ the weight of M, the number of one-entries in M.

Usually $|M|$ stands for a determinant of matrix M. However, in this paper we do not work with determinants at all so the notation should not lead to misunderstanding.

Definition 0.14. For a matrix P we define $Ex(P, m, n) := \max\{|M| | M \in \{0,1\}^{m\times n},\ P \not\leq M\}$. We denote $Ex(P, n) := Ex(P, n, n)$.

Definition 0.15. For a matrix P we define $Ex_{\preceq}(P, m, n) := max\{|M| | M \in \{0,1\}^{m\times n},\ P \not\preceq M\}$. We denote $Ex_{\preceq}(P, n) := Ex_{\preceq}(P, n, n)$.

Observation 0.16. *For all P, m, n; $Ex_{\preceq}(P, m, n) \leq Ex(P, m, n)$.*

Observation 0.17. *If $P \in \{0,1\}^{k\times l}$ has a one-entry at position $[a, b]$, then*

$$Ex(P, m, n) \geq \begin{cases} m \cdot n & k > m \lor l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \textit{otherwise.} \end{cases}$$

Observation 0.18. *The same holds for $Ex_{\preceq}(P, m, n)$.*

Definition 0.19. $P \in \{0,1\}^{k\times l}$ is *(strongly) minimalist* if

$$Ex(P, m, n) = \begin{cases} m \cdot n & k > m \lor l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Definition 0.20. $P \in \{0,1\}^{k\times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P, m, n) = \begin{cases} m \cdot n & k > m \lor l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Observation 0.21. *If P is strongly minimalist, then P is weakly minimalist.*

0.1.1 Known results

Fact 0.22. *1. $(\bullet)$ is strongly minimalist.*

2. If $P \in \{0,1\}^{k\times l}$ is strongly minimalist and there is a one-entry in the last row in the c-th column, then $P' \in \{0,1\}^{k+1\times l}$, which is created from P by adding a new row having a one-entry only in the c-th column, is strongly minimalist.

3. If P is strongly minimalist, then after changing a one-entry into a zero-entry it is still strongly minimalist.

Fact 0.23 (Mohar et al. [2015]). *Let $P = \{1\}^{2\times l}$, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{2\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{weight of } M[[i], \{j\}] > 0 \land \text{weight of } M[[i+1, m], \{j\} > 0]\}$. Clearly $|A_i| \leq l - 1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M appears in at least $b_j - 1$ of sets A_i, $0 \leq i \leq m - 2$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 1) + n \leq \sum_{i=0}^{m-2} |A_i| + n \leq (l-1)(m-1) + n$$

□

This result is indeed very important because it shows that there are matrices like $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, which are weakly minimalist, although it is known they are not strongly minimalist.

Fact 0.24 (Mohar et al. [2015])**.** *Let $P = \{1\}^{3\times l}$, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{3\times l}$ as an interval minor and $A_i = \{j \in [n]|$ weight of $M[[i-1],\{j\}] > 0 \wedge$ weight of $M[[i+1,m],\{j\} > 0 \wedge M[i,j]$ one-entry$]\}$. Clearly $|A_i| \leq l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M (for which $b_j \geq 2$) appears in exactly $b_j - 2$ of sets A_i, $1 \leq i \leq m-1$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 2) + 2n \leq \sum_{i=1}^{m-2} |A_i| + 2n \leq (l-1)(m-2) + 2n$$

$\square$

1. Characterizations

Our goal in this chapter is to describe what matrices avoiding small patterns as interval minors look like.

Algorithmically speaking, deciding whether a pattern is contained in a matrix is hard, even if both matrices are permutation matrices. We do not consider complexity questions here, but given a small pattern, we show that matrices avoiding the pattern have a quite simple structure. However, the structure gets significantly richer as soon as the pattern contains at least four one-entries.

To allow ourselves to go through cases efficiently, we first show that to some extend, we can assume there are no empty lines in the pattern without loss of generality.

Before we dive into the characterizations, let us introduce some useful notions.

Definition 1.1. A *walk* in a matrix M is a contiguous sequence of its entries, beginning in the top-left corner and ending in the bottom-right one. If $M[i, j]$ occurs in the sequence, its successor is either $M[i + 1, j]$ or $M[i, j + 1]$. Symmetrically, a *reverse walk* in M is a contiguous sequence of its entries, beginning in the top-right corner and ending in the bottom-left one.

Definition 1.2. We say a matrix M is a *walking matrix* if there is a walk in M containing all one-entries.

Definition 1.3. For a matrix $M \in \{0, 1\}^{m\times n}$ and integers r, c, we say $M[r, c]$ is

- *top-left empty*, if $M[[r - 1], [c - 1]]$ is an empty matrix,
- *top-right empty*, if $M[[r - 1], [c + 1, n]]$ is empty,
- *bottom-left empty*, if $M[[r - 1], [c + 1, n]]$ is empty,
- *bottom-right empty*, if $M[[r - 1], [c + 1, n]]$ is empty.

Definition 1.4. For matrices $M \in \{0, 1\}^{m\times n}$ and $N \in \{0, 1\}^{m\times l}$, we define $M \rightarrow N \in \{0, 1\}^{m\times(n+l)}$ to be the matrix created from M by adding columns of N at the end.

1.1 Empty rows and columns

From the definition of matrix containment, zero-entries of the pattern pose no restrictions on the tested matrix, so, intuitively, adding new empty lines to a pattern should not influence the structure of matrices avoiding the pattern by much.

We first show that when adding empty lines as first or last lines of the pattern, it indeed does next to no difference. On the other hand, inserting empty lines in between non-empty lines becomes a bit more tricky and we only describe what happens when we extend a pattern of size $k \times 2$ (or symmetrically $2 \times k$).

Observation 1.5. *For matrices $P \in \{0, 1\}^{k\times l}$ and $M \in \{0, 1\}^{m\times n}$, let $P' = P \rightarrow 0^{k\times 1}$ and let $M' = M \rightarrow 1^{m\times 1}$. Then $P \preceq M \Leftrightarrow P' \preceq M'$.*

Proof. $\Rightarrow$ The last column of P' can always be mapped just to the last column of M' and $P'[[k],[l]]$ can be mapped to $M'[[m],[n]]$ the same way P is mapped to M.

$\Leftarrow$ Taking the restriction of the mapping of P' to M' we get a mapping of P to M.

$\square$

The analogous proof can be also used to characterize matrices avoiding patterns after we add an empty column as the first column or an empty row as the first or the last row. Using induction, we can easily show that a pattern P' is avoided by a matrix M' if and only if P is avoided by M, where P is derived from P' by excluding all empty leading or ending rows and columns and M is derived from M' by excluding the same number of leading or ending rows and columns. Therefore, when characterizing matrices avoiding a forbidden pattern, we do not need to consider patterns having empty rows or columns on their boundary.

The following machinery shows what happens after we add empty columns in between two columns of a pattern that only has two columns. The size of the patterns is significant, because it allows us to prove that matrices avoiding them have a very simple structure. That is going to be achieved by employing a notion of intervals of one-entries. More about these intervals and their counterpart – zero-intervals can be find in the last chapter of the thesis.

Definition 1.6. A *one-interval* of a matrix M is a sequence of consecutive one-entries in a single line of M bounded from both sides by zero-entries or the edges of matrix.

Lemma 1.7. *Let $P \in \{0,1\}^{k\times 2}$ and let $M \in \{0,1\}^{m\times n}$ be an inclusion maximal matrix avoiding P, then M contains at most one one-interval in each row.*

Proof. For contradiction, assume there are at least two one-intervals in a row of M. Because M is inclusion maximal, changing any zero-entry e in between one-intervals o_1 and o_2 creates a mapping of the forbidden pattern. Such a mapping uses the changed one-entry to map some element $P[r',1]$ or $P[r',2]$.

In the first case, the same mapping also maps P to M if we use a one-entry from o_1 instead of e; thus, $P \not\preceq M$ and we reach a contradiction. In the second case, the mapping can use a one-entry from o_2 instead of e; therefore, we again get a contradiction with $P \not\preceq M$. Since e is not usable for any one-entry of P, we can change it to a one-entry and get a contradiction with M being inclusion maximal. $\square$

Lemma 1.8. *Let $P \in \{0,1\}^{k\times 2}$ and for any $l \geq 1$ let $P^l \in \{0,1\}^{k\times(l+2)}$ be a pattern created from P by adding l new empty columns in between the two columns of P. If an $m \times n$ matrix $M \in Av_{\preceq}\left(P^l\right)$ is inclusion maximal, then each row of M is either empty or it contains a single one-interval of length at least $l+1$.*

Proof. The same proof as in Lemma 1.7 shows that there is at most one one-interval in each row.

For contradiction, let there be at most l one-entries $M[\{r\},[c_1,c_2]]$ in a row r:

- $c_1 = 1$: we can set $M[r, c_2 + 1] = 1$ and the matrix still avoids P^l, which is a contradiction with M being inclusion maximal.

- $c_2 = n$: we can set $M[r, c_1 - 1] = 1$ and the matrix still avoids P^l, which is a contradiction with M being inclusion maximal.

- otherwise: let us choose zero-entries e_l and e_r in the row r such that there are exactly l columns between them and all one-entries from the row r lie in between them. For contradiction, assume we cannot change neither $e_l = M[r, c_l]$ nor $e_r = M[r, c_r]$ to a one-entry without creating the pattern. This means that if $e_l = 1$ then some $P^l[r_1, 1]$ can be mapped to it. Let m_l be the corresponding mapping. At the same time, if $e_r = 1$ then some $P^l[r_2, l+2]$ can be mapped to it and m_r is the corresponding mapping. We show that the two mappings can be combined to a mapping of P^l to M giving a contradiction. Without loss of generality, in both mappings, empty columns of P are mapped exactly to l columns of M. We need to describe how to partition M into k rows. Consider Figure 1.1:

  - $r_1 \neq r_2$: Without loss of generality, we assume $r_1 > r_2$. Let r_3 be the first row used to map r_1 in m_l and let r_4 be the last row used to map r_1 in m_r. From the mapping m_l, we know that the first $r_1 - 1$ rows of P can be mapped to rows $[1, r_3 - 1]$ of M and from the mapping m_r, we know that the last $k - r_1$ rows of P can be mapped to rows $[r_4 + 1, m]$ of M. Therefore, we can use rows $[r_3, r_4]$ of M to map row r_1 of P without using one-entries e_l and e_r.

  - $r_1 = r_2$: Let r_3 and r_4 be the first and the last rows respectively used to map r_1 in m_l and let r_5 and r_6 be the first and the last rows respectively used to map r_1 in m_r. Without loss of generality let $r_3 < r_5$. From m_l being a mapping, we know that the first $r_1 - 1$ rows of P can be mapped to rows $[1, r_3 - 1]$ of M. Without loss of generality let $r_4 < r_6$. From m_r being a mapping, we know that the last $k - r_1$ rows of P can be mapped to rows $[r_6 + 1, m]$ of M. Therefore, we can use rows $[r_3, r_6]$ of M to map row r_1 of P without using one-entries e_l and e_r.

We showed that either e_l or e_r can be changed to a one-entry, which is a contradiction with M being inclusion maximal.

□

Theorem 1.9. *Let $P \in \{0,1\}^{k \times 2}$ and for any integer $l \geq 1$ let $P^l \in \{0,1\}^{k \times (l+2)}$ be a pattern created from P by adding l new empty columns in between the two columns of P. For all matrices $M \in \{0,1\}^{m \times n}$ it holds $M \in Av_{\preceq}\left(P^l\right) \Leftrightarrow$ there exists a matrix $N \in \{0,1\}^{m \times (n-l)}$ such that $N \in Av_{\preceq}(P)$ is inclusion maximal and M is a submatrix of an elementwise OR of $l+1$ shifted copies of N ($N \to 0^{m \times l}, 0^{m \times 1} \to N \to 0^{m \times (l-1)}, \ldots, 0^{m \times (l-1)} \to N \to 0^{m \times 1}, 0^{m \times l} \to N$).*

Proof. $\Rightarrow$ Without loss of generality, let M be inclusion maximal. We know from Lemma 1.8 that each row of M contains either no one-entry or a single one-interval of length at least $l+1$. Let a matrix N be created from M by deleting the last l one-entries from each row and excluding the last l columns. Clearly, M is equal to an elementwise OR of $l+1$ copies of N. If

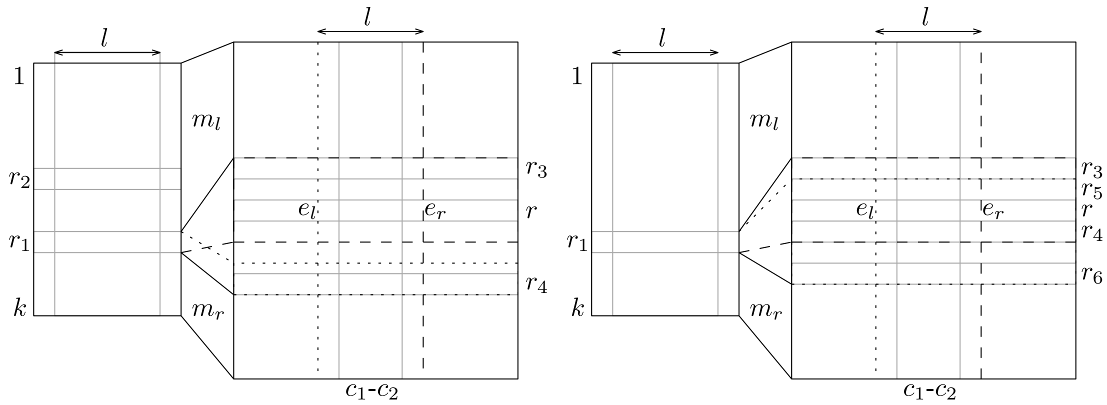


Figure 1.1: Dotted and dashed lines resembling mappings m_l and m_r of the forbidden pattern. Two horizontal lines show the boundaries of the mapping of row r and the vertical lines show boundaries of the mapping of column c.

$P \preceq N$ then each mapping of P can be extended to a mapping of P^l to M by mapping each $P^l[r_1, 1]$ to the same one-entry where $P[r_1, 1]$ is mapped in $N \to 0^{m\times l}$ and mapping each $P^l[r_2, l+2]$ to the same one-entry where $P[r_2, 2]$ is mapped in $0^{m\times l} \to N$.

$\Leftarrow$ Let M be equal to an elementwise OR of $l+1$ copies of N. For contradiction, assume $P^l \preceq M$ and consider any mapping of P^l to M. Without loss of generality, one-entries of the first column of P^l are mapped to those one-entries of M created from $N \to 0^{m\times l}$. If there is one-entry $P^l[r, 1]$ mapped to a one-entry of M not created from $N \to 0^{m\times l}$, we just take the first one-entry in the row instead. Symmetrically, all one-entries of the last column of P^l are mapped to one-entries created from $0^{m\times 1} \to N$. The same one-entries of N can be used to map P to N, which is a contradiction.

□

The symmetric characterization also holds when adding empty rows to a pattern that only has two rows. We can see in the following proposition that the straightforward generalization of the statement for bigger patterns does not hold.

Proposition 1.10. *There exists a matrix $P \in \{0,1\}^{k\times l}$ such that for each $P' \in \{0,1\}^{k\times(l+1)}$ created from P by adding a single empty column in between two existing columns, there exists a matrix $M \in \{0,1\}^{m\times n}$ such that $P' \preceq M$ and there exists $N \in \{0,1\}^{m\times(n-1)}$ such that $N \in Av_{\preceq}(P)$ is inclusion maximal and M is a submatrix of an elementwise OR of $N \to 0^{m\times 1}$ and $0^{m\times 1} \to N$.*

Proof. Later in this chapter, we characterize the class of matrices avoiding pattern P_8. For the result, look at Proposition 1.20. Let $N \in Av_{\preceq}(P_8)$ be any matrix containing P_5 as an interval minor. Let M be equal to $N \to 0^{m\times 1}$ placed over $0^{m\times 1} \to N$ with elementwise OR. Then $\left(\begin{smallmatrix}\bullet & \circ & \bullet & \bullet\\ & \circ & \bullet & \end{smallmatrix}\right), \left(\begin{smallmatrix}\bullet & \bullet & \circ & \bullet\\ & \bullet & \circ & \end{smallmatrix}\right) \preceq M$. □

Next, we describe the structure of matrices avoiding some small patterns. Because of the above results, we also characterize some of their generalizations and we completely omit empty lines in them. If $P \not\preceq M$ then also $P^\top \not\preceq M^\top$ and this holds for all rotations and mirrors of P and M and so we only mention these symmetries.

1.2 Patterns having two one-entries and their generalization

These are, up to rotation and mirroring, the only patterns having two one-entries and no empty lines:

$$P_1 = \begin{pmatrix} \bullet & \bullet \end{pmatrix} \quad P_2 = \begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$$

They can be generalized to:

$$P_1' = \begin{pmatrix} \bullet & \cdots & \bullet \end{pmatrix} \quad P_2' = \begin{pmatrix} & & \bullet \\ & \cdots & \\ \bullet & & \end{pmatrix}$$

Proposition 1.11. *Let $P_1' = 1^{1\times k}$. For all matrices M: $P_1' \not\preceq M \Leftrightarrow M$ has at most $k-1$ non-empty columns.*

Proof. $\Rightarrow$ When a matrix M contains one-entries in k columns, then these give us a mapping of P_1'.

$\Leftarrow$ A matrix M having at most $k-1$ non-empty columns avoids P_1'.

□

Proposition 1.12. *Let $P_2' \in \{0,1\}^{k\times k}$. For all matrices M: $P_2' \not\preceq M \Leftrightarrow M$ contains one-entries in at most $k-1$ walks.*

Proof. $\Rightarrow$ When one-entries of a matrix M cannot fit into $k-1$ walks, then there are k one-entries such that no pair can fit to a single walk and those give us a mapping of P_2'.

$\Leftarrow$ A matrix M containing one-entries in at most $k-1$ walks avoids P_2'.

□

1.3 Patterns having three one-entries

These are up to rotation and mirroring the only patterns having three one-entries and no empty lines that we did not characterize so far:

$$P_3 = \begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix} \quad P_4 = \begin{pmatrix} \bullet & & \bullet \\ & \bullet & \end{pmatrix} \quad P_5 = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \end{pmatrix} \quad P_6 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$$

Proposition 1.13. *For all matrices $M \in \{0,1\}^{m\times n}$: $P_3 \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 1.2):*

- *$M[r,c]$ is top-left, top-right and bottom-left empty, and*
- *$M[[r,m],[c,n]]$ is a walking matrix.*

Proof. $\Rightarrow$ If M is a walking matrix then we set $r = c = 1$. Otherwise, there are one-entries $M[r,c']$ and $M[r',c]$ such that $r' < r$ and $c' < c$. If $M[r,c]$ is not top-left, top-right or bottom-left empty then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ and together with $M[r,c']$ it gives us the forbidden pattern.

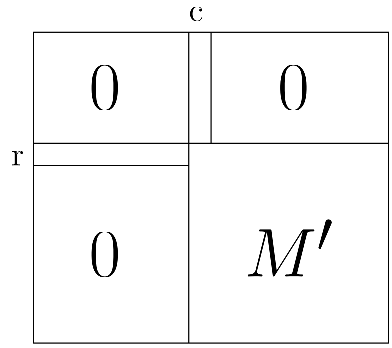


Figure 1.2: The characterization of matrices avoiding $\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \end{smallmatrix}\right)$ as an interval minor. A matrix M' is a walking matrix.

$\Leftarrow$ For contradiction, assume that a matrix M described in Figure 1.2 contains P_3 as an interval minor. Without loss of generality, let $P_3[1,1]$ be mapped to a one-entry in the r-th row. Then both $P_3[1,2]$ and $P_3[2,1]$ need to be mapped to M', which is a contradiction because it is not a walking matrix. □

Proposition 1.14. *For all matrices M: $P_4 \not\preceq M \Leftrightarrow M = M_1 \to M_2$, where $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ Let $e = M[r,c]$ be an arbitrary top-most one-entry in M. It holds $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \not\preceq M[[m],[c-1]]$, as otherwise, together with e it forms P_4. If we also have $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \not\preceq M[[m],[c,n]]$ then we are done. For contradiction, let $e_{1,2}$, $e_{2,1}$ be any two one-entries forming $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ in $M[[m],[c,n]]$. Symmetrically, let $e_{1,1}$, $e_{2,2}$ be any two one-entries forming $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right)$ in $M[[m],[c]]$. Without loss of generality, let $e_{2,1}$ be lower than $e_{2,2}$ and then, together with $e_{1,1}$ and $e_{1,2}$ it forms P_4 as an interval minor of M, giving us a contradiction.

$\Leftarrow$ For contradiction, let $P_4 \preceq M$ and consider an arbitrary mapping. Consider the one-entry of M, where $P_4[2,2]$ is mapped. If it is in M_1 then $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \preceq M_1$ and we get a contradiction. Otherwise, we have $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \preceq M_2$, which is again a contradiction. □

Proposition 1.15. *For all matrices M: $P_5 \not\preceq M \Leftrightarrow$ for the top-right most walk w in M such that there are no one-entries underneath it and for every one-entry $M[r,c]$ on w, there is at most one non-empty column in $M[[r-1],[c+1,n]]$.*

Proof. $\Rightarrow$ For contradiction, assume there is a one-entry $M[r,c]$ on w such that there are two non-empty columns in $M[[r-1],[c+1,m]]$. Then a one-entry from each of those columns and $M[r,c]$ together give us $P_5 \preceq M$ and a contradiction.

$\Leftarrow$ For contradiction, let $P_5 \preceq M$. Without loss of generality, $P_5[2,1]$ is mapped to a one-entry $M[r,c]$ from w. Then $(\bullet\,\bullet) \preceq M[[r-1],[c+1,n]]$, which is a contradiction with it having one-entries in at most one column. □

Proposition 1.16. *For all matrices M: $P_6 \not\preceq M \Leftrightarrow$ for the top-left most reverse walk w in M such that there are no one-entries underneath it and for every one-entry $M[r, c]$ on w, $M[[r-1], [c-1]]$ is a walking matrix.*

Proof. $\Rightarrow$ For contradiction, assume there are r, c such that $M[r, c]$ is a one-entry on w and $M[[r-1], [c-1]]$ is not a walking matrix. It means that $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \preceq M[[r-1], [c-1]]$ and together with $M[r, c]$ it gives us the forbidden pattern and a contradiction.

$\Leftarrow$ For contradiction, let $P_6 \preceq M$ and consider an arbitrary mapping of P_6. Without loss of generality, let $P_6[3, 3]$ be mapped to $M[r, c]$ such that there is no other one-entry in $M[[r, m], [c, n]]$. Clearly, $M[r, c]$ cannot lie on w, because then $M[[r], [c]]$ would be a walking matrix and so $M[r, c]$ could not be used to map $P_6[3, 3]$. So $M[r, c]$ lies above w but that is a contradiction with w being the top-left most reverse walk in M without one-entries underneath it.

$\square$

1.4 Patterns having four one-entries

These are some of the patterns having four one-entries and no empty lines that we did not characterize so far:

$$P_7 = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} \quad P_8 = \begin{pmatrix} \bullet & \bullet & \bullet \\ & \bullet & \end{pmatrix} \quad P_9 = \begin{pmatrix} & \bullet & & \\ \bullet & & & \\ & & & \bullet \\ & & \bullet & \end{pmatrix}$$

Lemma 1.17. *For any matrix M: $P_7 \not\preceq M \Rightarrow$ there exist integers r, c such that $M[r, c]$ is either*

1. *a one-entry and $(r, c) \in \{(1, 1), (1, n), (m, 1), (m, n)\}$ or*
2. *top-right and bottom-left empty and $(r, c) \notin \{(1, 1), (m, n)\}$ or*
3. *top-left and bottom-right empty and $(r, c) \notin \{(1, n), (m, 1)\}$.*

Proof. If there is a one-entry in any corner then the first condition is satisfied. Otherwise, consider $M[2, 1]$. It is trivially bottom-left empty and if there is no one-entry in the first row of M then the second condition is satisfied. Therefore, let $M[1, c_t]$ be a one-entry in the first row. Symmetrically, let $M[m, c_b]$ be a one-entry in the last row, let $M[r_l, 1]$ be a one-entry in the first column and let $M[r_r, n]$ be a one-entry in the last column.

It cannot happen that $c_t < c_b$ and $r_r > r_l$ (or symmetrically $c_t > c_b$ and $r_r < r_l$), because then $P_7 \preceq M$. Without loss of generality, let $c_t \geq c_b$ and $r_r \geq r_l$. A matrix $M[[r_r - 1], [c_t + 1, n]]$ is empty; otherwise, any one-entry there, together with $M[1, c_t], M[m, c_b]$ and $M[r_r, 1]$ forms the forbidden pattern. Similarly, a matrix $M[[r_r + 1, m], [c_t - 1]]$ is also empty. Thus $M[r_t, c_t]$ is top-right and bottom-left empty and it is not a corner, because those are empty. $\square$

Proposition 1.18. *For all matrices M: $P_7 \not\preceq M \Leftrightarrow M$ looks like one of the matrices in Figure 1.3, where $\begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix} \not\preceq M_1$, $\begin{pmatrix} & \bullet \\ \bullet & \bullet \end{pmatrix} \not\preceq M_2$, $\begin{pmatrix} \bullet & \bullet \\ & \bullet \end{pmatrix} \not\preceq M_3$ and $\begin{pmatrix} \bullet & \\ \bullet & \bullet \end{pmatrix} \not\preceq M_4$.*

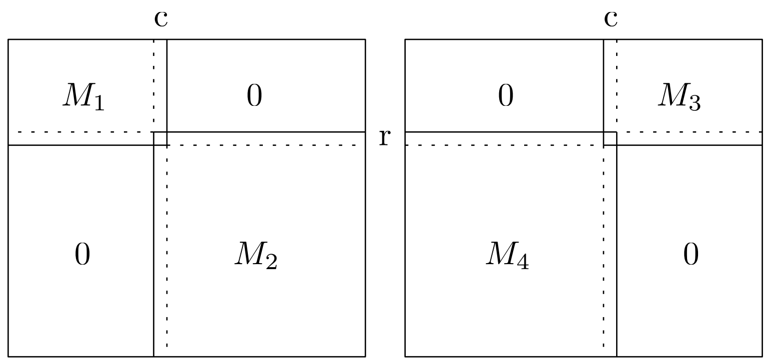


Figure 1.3: The characterization of matrices avoiding $\begin{pmatrix}\bullet&\bullet\\\bullet&\bullet\end{pmatrix}$ as an interval minor.

Proof. $\Rightarrow$ We proceed by induction on the size of M.

If $M \in \{0,1\}^{2\times 2}$ then it either avoids $\begin{pmatrix}&\bullet\\\bullet&\bullet\end{pmatrix}$ or $\begin{pmatrix}\bullet&\bullet\\\bullet&\end{pmatrix}$ and we are done.

For a bigger matrix M, from Lemma 1.17, there is an element $M[r,c]$ satisfying some conditions. If there is a one-entry in any corner, we are done because the matrix cannot contain one of the rotations of $\begin{pmatrix}\bullet&\bullet\\\bullet&\end{pmatrix}$. Otherwise, assume $M[r,c]$ is both top-right and bottom-left empty and $(r,c) \notin \{(1,1),(1,1)\}$. Let $M_1 = M[[r],[c]]$ and $M_2 = M[[r,m],[c,n]]$. If M_1 is non-empty, then $\begin{pmatrix}&\bullet\\\bullet&\bullet\end{pmatrix} \not\preceq M_2$. Symmetrically, $\begin{pmatrix}\bullet&\bullet\\\bullet&\end{pmatrix} \not\preceq M_1$ if M_2 is non-empty. If one of them is empty, the other is a smaller matrix avoiding P as an interval minor and the statement follows from the induction.

$\Leftarrow$ Without loss of generality, assume a matrix M looks like the left matrix in Figure 1.3. For contradiction, let $P \preceq M$. We can partition M into four quadrants such that there is at least one one-entry in each of them. It does not matter where we partition it, every time we either get $\begin{pmatrix}\bullet&\bullet\\\bullet&\end{pmatrix} \preceq M_1$ or $\begin{pmatrix}&\bullet\\\bullet&\bullet\end{pmatrix} \preceq M_2$, which is a contradiction.

$\square$

Lemma 1.19. *For all matrices M: $P_8 \not\preceq M \Rightarrow M = M_1 \to M_2$ where*

1. $\begin{pmatrix}\bullet&\bullet\\&\bullet\end{pmatrix} \not\preceq M_1$ *and* $\begin{pmatrix}&\bullet\\\bullet&\end{pmatrix} \not\preceq M_2$ *or*
2. $\begin{pmatrix}\bullet&\\&\bullet\end{pmatrix} \not\preceq M_1$ *and* $\begin{pmatrix}\bullet&\bullet\\\bullet&\end{pmatrix} \not\preceq M_2$.

Proof. Let $e = M[r,c]$ be an arbitrary top-most one-entry of M. It holds $\begin{pmatrix}\bullet&\bullet\\&\bullet\end{pmatrix} \not\preceq M[[m],[c-1]]$; otherwise, together with e it would form the whole P_8. Symmetrically, $\begin{pmatrix}\bullet&\bullet\\\bullet&\end{pmatrix} \not\preceq M[[m],[c+1,n]]$. For contradiction with statement, let $e_{1,1}$, $e_{2,2}$ (none of them equal to e) be any two one-entries forming $\begin{pmatrix}\bullet&\\&\bullet\end{pmatrix}$ in $M[[m],[c]]$ and let $e_{1,2}$, $e_{2,1}$ be any two one-entries forming $\begin{pmatrix}&\bullet\\\bullet&\end{pmatrix}$ in $M[[m],[c,n]]$. Without loss of generality, $e_{2,1}$ is lower than $e_{2,2}$ and together with $e_{1,1}$, e and $e_{1,2}$ it gives us a mapping of P_8 to M, which is a contradiction. $\square$

Proposition 1.20. *For all matrices M: $P_8 \not\preceq M \Leftrightarrow M$ looks like the matrix in Figure 1.4, where $\begin{pmatrix}\bullet&\\&\bullet\end{pmatrix} \not\preceq M_1$ and $\begin{pmatrix}&\bullet\\\bullet&\end{pmatrix} \not\preceq M_2$.*

Proof. $\Rightarrow$ From Lemma 1.19, we know $M = M_1' \to M_2'$, where $\begin{pmatrix}\bullet&\bullet\\&\bullet\end{pmatrix} \not\preceq M_1'$ and $\begin{pmatrix}&\bullet\\\bullet&\end{pmatrix} \not\preceq M_2'$ (or symmetrically the second case). From Proposition 1.13,

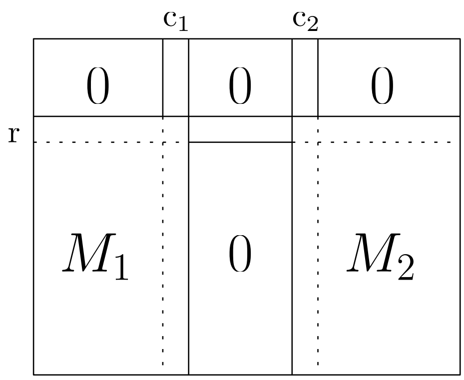


Figure 1.4: The characterization of matrices avoiding $\begin{pmatrix} \bullet & \bullet & \bullet \\ & \bullet & \end{pmatrix}$ as an interval minor.

we have that M_1' looks like $M[[m],[c_2-1]$ in Figure 1.4 and $M[[m],[c_2,n]]$ forms a walking matrix. Without loss of generality, $M[[r-1],\{c_1\}]$ and $M[\{r\},[c_1+1,c_2-1]]$ are non-empty; otherwise, we extend M_1 to cover the whole $M[[m],[c_2-1]$. If there are two different columns in M_2' having a one-entry above the r-th row, together with one-entries in $M[[r-1],\{c_1\}]$ and $M[\{r\},[c_1+1,c_2-1]]$ they form a mapping of P_8.

$\Leftarrow$ A one-entry $P_8[2,2]$ can not be mapped anywhere but to the r-th row, but in that case, there are at most two columns having one-entries above it.

$\square$

1.5 Multiple patterns

Instead of considering matrices avoiding a single pattern, we can work with matrices avoiding a set of forbidden patterns.

We only describe the structure of matrices avoiding one particular set of patterns, because we use the simple result later.

Proposition 1.21. *Let* $P_{10} = \begin{pmatrix} \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{pmatrix}$ *and* $P_{11} = \begin{pmatrix} \circ & \bullet \\ \circ & \circ \\ \bullet & \circ \end{pmatrix}$*, then for all matrices* M*:* $\{P_{10}, P_{11}\} \not\preceq M \Leftrightarrow$ *for the top-right most walk* w *in* M *such that there are no one-entries underneath it, each one-entry* $M[r,c]$ *is either on* w *or both* $M[r+1,c]$ *and* $M[r,c-1]$ *are on* w*.*

Proof. $\Rightarrow$ For contradiction, assume there is a one-entry anywhere but on w or directly diagonally next to any bottom-left corner of w. Then this one-entry together with at least one bottom-left corner of w give us a mapping of P_{10} or P_{11} and a contradiction.

$\Leftarrow$ For any one-entry e, from the description of M, there is no one-entry that creates P_{10} or P_{11} with e.

$\square$

2. Operations with matrices

When speaking about class of matrices, unless stated otherwise, they are closed under interval minors, which means that whenever a matrix belong to a class, all its minors belong there too. All classes discussed are also non-trivial. This means, there is at least one matrix of size 2×1, at least one matrix of size 1×2 and at least one matrix is non-empty in each class.

Definition 2.1. Let $\mathcal{M}$ be a class of matrices. The *basis* of $\mathcal{M}$ is a set of all minimal (with respect to minors) matrices $\mathcal{P}$ such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$.

Let us start with a few simple observations, regarding classes of matrices and their bases.

Observation 2.2. *Let $\mathcal{M} = Av_{\preceq}(\mathcal{P})$ for some set of matrices $\mathcal{P}$. Then $\mathcal{M}$ is closed under interval minors.*

Observation 2.3. *Every finite class of matrices has a finite basis.*

2.1 The skew and direct sums

In the realm of permutations, the skew and direct sums are very useful operations. What follows is a direct generalization to our settings and a few simple results. More interesting statements and the relation with interval minors follow in the next section.

Definition 2.4. For matrices $A \in \{0,1\}^{m \times n}$ and $B \in \{0,1\}^{k \times l}$ we define their *skew sum* as a matrix $C := A \nearrow B \in \{0,1\}^{(m+k)\times(n+l)}$ such that $C[[k+1, m+k],[n]] = A$, $C[[k],[n+1,n+l]] = B$ and the rest is empty. Symmetrically, we define their *direct sum* $D := A \searrow B \in \{0,1\}^{(m+k)\times(n+l)}$ such that $D[[m],[n]] = A$, $D[[m+1,m+k],[n+1,n+l]] = B$ and the rest is empty.

Using this notation, we can very easily rewrite the results from the previous chapter. Here is an example of Proposition 1.13 and Proposition 1.18:

Proposition 2.5. $Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \circ\end{smallmatrix}\right)\right) = Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ\end{smallmatrix}\right)\right) \searrow Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\right)\right)$

Proposition 2.6.
$$\begin{aligned} Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \bullet\end{smallmatrix}\right)\right) = {} & \left(Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ\end{smallmatrix}\right)\right) \searrow Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\right)\right) \searrow Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \circ \\ \circ & \bullet\end{smallmatrix}\right)\right)\right) \cup \\ & \left(Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \circ \\ \bullet & \circ\end{smallmatrix}\right)\right) \nearrow Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)\right) \nearrow Av_{\preceq}\left(\left(\begin{smallmatrix}\circ & \bullet \\ \circ & \circ\end{smallmatrix}\right)\right)\right). \end{aligned}$$

Something, we get a great use of later is a closure under the skew sum.

Definition 2.7. For a set of matrices $\mathcal{M}$, let $Cl(\mathcal{M})$ denote a class of matrices containing each $M \in \mathcal{M}$ and closed under the skew sum and interval minors.

When speaking about graph minors, we can always imagine that the contractions of edges are done after all deletions. Similarly, an element derived from a matrix M by reapplying the skew sum and taking its interval minor can be also derived by taking an interval minor of the skew sum of an appropriate number of copies of M.

Observation 2.8. *For every set of matrices $\mathcal{P}$, each $M \in Cl\left(\mathcal{P}\right)$ is an interval minor of the skew sum of multiple copies of P.*

What follows are two simple results of the relation of closures under the skew sum and the description using interval minors that we greatly generalize in the next section.

Proposition 2.9. $Cl\left(\begin{pmatrix}\bullet&\circ\\\circ&\bullet\end{pmatrix}\right) = Av_{\preceq}\left(\begin{pmatrix}\bullet&\circ&\circ\\\circ&\circ&\bullet\end{pmatrix}, \begin{pmatrix}\bullet&\circ\\\circ&\circ\\\circ&\bullet\end{pmatrix}\right)$.

Proof. The skew sum of an arbitrary number of copies of $\begin{pmatrix}\bullet&\circ\\\circ&\bullet\end{pmatrix}$ avoids both forbidden patterns and because the relation of being an interval minor is transitive, we have $Cl\left(\begin{pmatrix}\bullet&\circ\\\circ&\bullet\end{pmatrix}\right) \subseteq Av_{\preceq}\left(\begin{pmatrix}\bullet&\circ&\circ\\\circ&\circ&\bullet\end{pmatrix}, \begin{pmatrix}\bullet&\circ\\\circ&\circ\\\circ&\bullet\end{pmatrix}\right)$.

From Proposition 1.21, for every matrix $M \in Av_{\preceq}\left(\begin{pmatrix}\bullet&\circ&\circ\\\circ&\circ&\bullet\end{pmatrix}, \begin{pmatrix}\bullet&\circ\\\circ&\circ\\\circ&\bullet\end{pmatrix}\right)$, it holds that for the top-right most walk w in M such that there are no one-entries underneath it, each one-entry $M[r,c]$ is either on w or both $M[r+1,c]$ and $M[r,c-1]$ are on w. Clearly, $\begin{pmatrix}\bullet&\bullet\\\bullet&\bullet\end{pmatrix}$ is an interval minor of the skew sum of three copies of $\begin{pmatrix}\bullet&\circ\\\circ&\bullet\end{pmatrix}$ and by the skew sum of multiple copies of $\begin{pmatrix}\bullet&\bullet\\\bullet&\bullet\end{pmatrix}$ we can then create the whole w and all one-entries outside of it. Thus, we have the other inclusion. $\square$

While it does not make sense for permutations, we can generalize the skew sum to also allow some overlap between the summed matrices.

Definition 2.10. For matrices $A \in \{0,1\}^{m\times n}$, $B \in \{0,1\}^{k\times l}$ and integers a, b, let a matrix $C := A \nearrow_{a\times b} B \in \{0,1\}^{(m+k-a)\times(n+l-b)}$ such that $C[[k+1, m+k], [n]] = A$, $C[[k], [n+1, n+l]] = B$, the part that overlaps is an elementwise OR of both submatrices and the rest of C is empty. We say C is the *skew sum with $a \times b$ overlap* of A and B.

Theorem 2.11. *For integers a, b, m, n such that $a \le m \le 2a$ and $b \le n \le 2b$, let $\mathcal{M}$ be an arbitrary set of matrices, not necessarily closed under interval minors, such that:*

- *$\mathcal{M}$ is closed under deletion of one-entries,*
- *$\mathcal{M}$ is closed under the skew sum with $a \times b$ overlap and*
- *there is a $m \times n$ matrix $M \in \mathcal{M}$,*

then $\mathcal{M}$ is also closed under the skew sum with $(2a-m) \times (2b-n)$ overlap.

Proof. Given any $A, B \in \mathcal{M}$ and a matrix $M \in \mathcal{M}$ such that $M \in \{0,1\}^{m\times n}$, let $C = A \nearrow_{a\times b} M \nearrow_{a\times b} B$. It has the same size as $D = A \nearrow_{(2a-m)\times(2b-n)} B$, whose set of one-entries is a subset of one-entries of $C \in \mathcal{M}$; therefore, $D \in \mathcal{M}$. $\square$

We see that already with pretty reasonable assumptions, whenever a set of matrices is closed under the skew sum with some overlap, it is also closed under the skew sum with smaller overlap. On the other hand, in general the opposite does not hold even if we work with classes of matrices.

Observation 2.12. *There is a class of matrices closed under the skew sum with 1×1 overlap that is not closed under the skew sum with 2×2 overlap.*

Proof. Let $\mathcal{M} = Av_{\preceq}\left(\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix}\right)$. Clearly, $\mathcal{M}$ is hereditary and closed under the skew sum with 1×1 overlap. On the other hand, $\mathcal{M}$ is not closed under the skew sum with 2×2 overlap, because for matrices $\begin{pmatrix}\bullet & \bullet \\ \bullet & \end{pmatrix}, \begin{pmatrix} & \bullet \\ \bullet & \bullet\end{pmatrix} \in \mathcal{M}$, it holds $\begin{pmatrix}\bullet & \bullet \\ \bullet & \end{pmatrix} \nearrow_{2\times 2} \begin{pmatrix} & \bullet \\ \bullet & \bullet\end{pmatrix} = \begin{pmatrix}\bullet & \bullet \\ \bullet & \bullet\end{pmatrix} \notin \mathcal{M}$. $\square$

A similar proof shows that for all $a \geq 1, b > 1$ there is a class of matrices closed under the skew sum with $a \times b$ overlap that is not closed under the skew sum with $(a+1) \times b$ (or $a \times (b+1)$) overlap. Luckily for us, this does not hold for $a = 0$ or $b = 0$:

Observation 2.13. *Every class of matrices closed under the skew sum is also closed under the skew sum with* 1×1 *overlap.*

2.2 Articulations

Our next goal is to show that whenever we have a matrix closed under the skew sum and interval minors, the obtained class has a finite basis. In order to prove it, we define and get familiar with articulations.

Definition 2.14. Let $M \in \{0,1\}^{m\times n}$ be a matrix. An element $M[r,c]$ is an *articulation* if it is top-left empty ($M[[r-1],[c-1]]$ is empty) and bottom-right empty ($M[[r+1,m],[c+1,n]]$ is empty). We say that an articulation $M[r,c]$ is *trivial* if $(r,c) \in \{(m,1),(1,n)\}$.

Whenever $P \preceq M$, for every $M[r,c]$ there is some $P[r',c']$ that can be mapped to $M[r,c]$; therefore, the following observation shows that once there is an articulation in M, it also exists in P and it is not necessarily trivial.

Observation 2.15. *Let M be a matrix. If there are integers r,c such that $M[r,c]$ is an articulation, then for every matrix P such that $P \preceq M$, if $P[r',c']$ can be mapped to $M[r,c]$ then it is an articulation.*

Observation 2.16. *Let $P \in \{0,1\}^{k\times l}$ be a matrix. There are P_1, P_2 non-empty interval minors of P such that $P = P_1 \nearrow_{1\times 1} P_2 \Leftrightarrow$ there exist integers r,c such that $P[r,c]$ is an articulation and $P[[r,k],[c]], P[[r],[c,l]]$ are non-empty.*

Observation 2.17. *Let $\mathcal{P}$ be a set of matrices. There is a minimal (with respect to interval minors) matrix $P \in \mathcal{P}$ and there are P_1, P_2 non-empty interval minors of P such that $P = P_1 \nearrow_{1\times 1} P_2 \Leftrightarrow Av_{\preceq}(\mathcal{P})$ is not closed under the skew sum with 1×1 overlap.*

Proof. $\Rightarrow$ Let $P_1 \in \{0,1\}^{k_1\times l_1}$ and $P_2 \in \{0,1\}^{k_2\times l_2}$. While $P \not\preceq P_1 \nearrow_{1\times 1} 0^{k_2\times l_2}$ and $P \not\preceq 0^{k_1\times l_1} \nearrow_{1\times 1} P_2$, we have $P \preceq P_1 \nearrow_{1\times 1} 0^{k_2\times l_2} \nearrow 0^{k_1\times l_1} \nearrow_{1\times 1} P_2$.

$\Leftarrow$ If there is no minimal matrix $P \in \mathcal{P}$ that is the skew sum of its non-empty interval minors, we want to show it makes $Av_{\preceq}(\mathcal{P})$ closed under the skew sum with 1×1 overlap. From Observation 2.16, for every $P\mathcal{P}$ there are no r,c that $P[r,c]$ is an articulation and $P[[r,k],[c]], P[[r],[c,l]]$ are non-empty. Let $M_1, M_2 \in Av_{\preceq}(P)$ be arbitrary matrices and let $M = M_1 \nearrow_{1\times 1} M_2$. The matrix M contains a non-trivial articulation and from Observation 2.15 it follows $M \in Av_{\preceq}(P)$ for each minimal $P \in \mathcal{P}$; thus, $M \in Av_{\preceq}(\mathcal{P})$.

$\square$

In the following, we always expect articulations to be on a reverse walk (no two articulations forming $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix}$) and by a matrix between two articulations $M[r_1, c_1]$ and $M[r_2, c_2]$ we mean the matrix $M[[r_2, r_1], [c_1, c_2]]$.

Lemma 2.18. *Let $\mathcal{P}$ be a set of matrices, then for all matrices $M \in \{0,1\}^{m\times n}$ it holds that $M \in Cl\,(\mathcal{P}) \Leftrightarrow$ there exists a sequence of articulations of M on a reverse walk such that for each matrix M' in between two consecutive articulations of M there exists $P \in \mathcal{P}$ such that $M' \preceq \begin{pmatrix}1\end{pmatrix} \nearrow P \nearrow \begin{pmatrix}1\end{pmatrix}$.*

Proof. $\Rightarrow$ With Observation 2.13 in mind, consider the skew sum with 1×1 overlap of multiple copies of elements of $\mathcal{P}$ and let the sequence contain an articulation between each pair of consecutive copies of matrices from P, together with the trivial articulations $M[m, 1]$ and $M[1, n]$.

Between each pair of consecutive articulations, we have a matrix from $\mathcal{P}$ and so the statement holds. When we take an arbitrary interval minor and keep original articulations, each matrix between two consecutive articulations only contains at most one original copy of some matrix $P \in \mathcal{P}$, but it may happen that the bottom-left and top-right corners become one-entries even though they were zero-entries before. The matrix does not have to be an interval minor of P anymore, but it is an interval minor of $\begin{pmatrix}1\end{pmatrix} \nearrow P \nearrow \begin{pmatrix}1\end{pmatrix}$ for the corresponding $P \in \mathcal{P}$.

$\Leftarrow$ We can simply blow up each matrix M' between two consecutive articulation to the skew sum of three copies of the corresponding matrix P and because $M' \preceq \begin{pmatrix}1\end{pmatrix} \nearrow P \nearrow \begin{pmatrix}1\end{pmatrix} \preceq P \nearrow P \nearrow P$ it holds $M \in Cl\,(\mathcal{P})$.

$\square$

Finally, we show that a closure under the skew sum can always be described by a finite number of forbidden patterns.

Theorem 2.19. *For all matrices $M \in \{0,1\}^{m\times n}$, $Cl\,(M)$ has a finite basis.*

Proof. Let $\mathcal{F}$ be the set of all minimal (with respect to interval minors) matrices such that $Cl\,(M) = Av_{\preceq}(\mathcal{F})$. We need to prove that $\mathcal{F}$ is finite. Thanks to Observation 2.13, $Av_{\preceq}(\mathcal{F})$ is closed under the direct sum with 1×1 overlap and from Observation 2.17 follows that for no $F \in \mathcal{F}$ there are its non-empty interval minors F_1, F_2 such that $F = F_1 \nearrow 1 \times 1 F_2$. We denote by $\mathcal{P}$ a set of matrices $F \in \mathcal{F}$ such that F has at most $2m + 4$ rows and $2n + 4$ columns. We want to show $Cl\,(M) = Av_{\preceq}(\mathcal{P})$.

$\subseteq$ Clearly, $\mathcal{P}$ is finite and we immediately see that $Cl\,(M) \subseteq Av_{\preceq}(\mathcal{P})$.

$\supseteq$ For contradiction, consider a minimal matrix $X \in Av_{\preceq}(\mathcal{P}) - Cl\,(M)$. There are no X_1, X_2 non-empty interval minors of X such that $X = X_1 \nearrow 1 \times 1 X_2$; otherwise, as $X_1, X_2 \in Av_{\preceq}(\mathcal{P})$ and X is the minimum matrix such that $X \notin Cl\,(M)$, we would have $X_1, X_2 \in Cl\,(M)$; therefore, $X \in Cl\,(M)$ and a contradiction.

Without loss of generality, we assume $X \in \{0,1\}^{k\times l}$ has at least $2m + 5$ rows. Let X' denote a matrix created from X by deletion of the first row. We have $X' \in Av_{\preceq}(\mathcal{P})$ and from minimality of X also $X' \in Cl\,(M)$. From

Lemma 2.18, there is a sequence of articulations of X' on a reverse walk such that each matrix between two consecutive articulations is an interval minor of $(1) \nearrow M \nearrow (1)$. Let $X'[r, c]$ be the first articulation from the sequence (sorted by the second coordinate in ascending order) for which $c > 1$. The matrix between $X'[r, c]$ and the previous articulation in the sequence is an interval minor of $(1) \nearrow M \nearrow (1)$, which also means that $c \leq n+2$. Since $X[r, c]$ is not an articulation, it must hold that $X[1, c_1] = 1$ for some $c_1 < c \leq n + 2$. Symmetrically, let X'' denote a matrix created from X by deletion of the last row. Following the same steps we did before, we get the last articulation $X''[r, c]$ such that $c < l$ and the observation that $c \geq l - n - 1$. Since $X[r, c]$ is not an articulation, it must hold that $X[k, c_2] = 1$ for some $c_2 > c \geq l - n - 1$.

We showed that a matrix $Y \in \{0, 1\}^{(m+1)\times 2}$ such that the only one-entries are $Y[1, 1]$ and $Y[m + 1, 2]$ is an interval minor of X. To reach a contradiction, it suffices to show that there is a matrix $P \in \mathcal{P}$ such that $P \preceq Y$. For contradiction, let $Y \in Av_{\preceq}(\mathcal{P})$ and since $Y \preceq X$ and X is minimum such that $X \notin Cl(M)$ it holds $Y \in Cl(M)$. But this cannot be, because Y contains no non-trivial articulation and from Observation 2.15, we know that every matrix $Z \in Cl(M)$ bigger than $m \times n$ contains at least one.

□

2.3 Basis

We recall that the basis of a class of matrices $\mathcal{M}$ is a set of all minimal (with respect to interval minors) matrices $\mathcal{P}$ such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$. It goes without saying that it does not make sense to consider a basis of a set of matrices that is not closed under interval minors.

So far, we showed that whenever $\mathcal{M}$ is finite, its basis is also finite. The same hold when $\mathcal{M} = Cl(M)$ for some M. We show next that, unlike in graph theory, there are classes that does not have a finite basis. Moreover, we show that even for a class $\mathcal{M}$ with finite basis, its closure $Cl(\mathcal{M})$ can have an infinite basis.

Definition 2.20. Let P be a matrix. We denote by $\mathcal{R}(P)$ a set of all minimal (with respect to minors) matrices P' such that $P \preceq P'$ and P' is not the skew sum with 1×1 overlap of non-empty interval minors of P'. For a set of matrices $\mathcal{P}$, let $\mathcal{R}(\mathcal{P})$ denote a set of all minimal (with respect to minors) matrices from the set $\bigcup_{P \in \mathcal{P}} \mathcal{R}(P)$.

Theorem 2.21. *Let $\mathcal{M}$ and $\mathcal{P}$ be sets of matrices such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$, then $Cl(\mathcal{M}) = Av_{\preceq}(\mathcal{R}(\mathcal{P}))$.*

Proof. $\subseteq$ Assume $M \notin Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ and without loss of generality, because $Cl(\mathcal{M})$ is hereditary, let M be minimal (with respect to interval minors). It follows that $M \in \mathcal{R}(\mathcal{P})$. As such, the matrix M is not a skew sum with 1×1 overlap of non-empty interval minors of M; therefore, according to Observation 2.16, there is no articulations $M[r, c]$ such that $M[[r, k], [c]], M[[r], [c, l]]$ are non-empty. For contradiction, assume $M \in Cl(\mathcal{M})$. According to Lemma 2.18 and the fact M contains no non-trivial articulation, M is a minor of $(1) \nearrow M' \nearrow (1)$ for some $M' \in \mathcal{M}$. Because the trivial articulations (top-right and bottom-left corners) contain zero-entries, it even holds $M \preceq M'$. We also have $M \preceq P$ for some $P \in \mathcal{P}$, which together give us a contradiction with $\mathcal{M} = Av_{\preceq}(\mathcal{P})$.

$\supseteq$ First of all, $Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ is closed under the skew sum with 1×1 overlap. For contradiction, assume there are matrices $M_1, M_2 \in Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ but $M = M_1 \nearrow_{1\times 1} M_2 \notin Av_{\preceq}(\mathcal{R}(\mathcal{P}))$. Then there exists a matrix $P \in \mathcal{R}(\mathcal{P})$ such that $P \preceq M$. Because P is not a skew sum with 1×1 overlap of non-empty interval minors of P, it follows that either $P \preceq M_1$ or $P \preceq M_2$ and we have a contradiction.

It suffices to show that the inclusion holds for any $M \in Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ that is not a skew sum with 1×1 overlap of non-empty interval minors of M. From Observation 2.16, we know that M does not contain any non-trivial articulation and those trivial ones are empty. Thus, $M \in Av_{\preceq}(\mathcal{P}) = \mathcal{M}$ and so $M \in Cl(\mathcal{M})$.

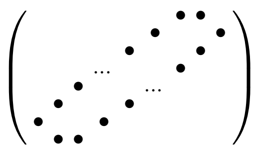
$\square$

Corollary 2.22. *Let $\mathcal{M}$ and $\mathcal{P}$ be sets of matrices such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$, then $\mathcal{R}(\mathcal{P})$ is the basis of $Cl(\mathcal{M})$.*

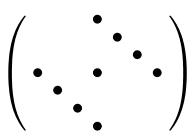
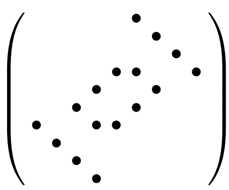
What follows is a construction of parameterized matrices that become the main tool of finding a class of matrices with an infinite basis.

Definition 2.23. Let $Nucleus_1 = (\bullet)$ and for $n > 1$ let $Nucleus_n \in \{0,1\}^{n \times n+1}$ be a matrix described by the examples:

$$Nucleus_2 = \begin{pmatrix} \bullet & \bullet & \\ & \bullet & \bullet \end{pmatrix} \quad Nucleus_3 = \begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & \bullet & \end{pmatrix} \quad Nucleus_n =$$



Definition 2.24. Let $Candy_{k,n,l}$ be a matrix given by $I_k \nearrow_{1\times 2} Nucleus_n \nearrow_{1\times 2} I_l$, where I_k, I_l are unit matrices of sizes $k \times k$ and $l \times l$ respectively.

$Candy_{4,1,4} =$  $Candy_{4,4,4} =$ 

Theorem 2.25. *There exists a matrix P for which $\mathcal{R}(P)$ is infinite.*

Proof. Let $P = Candy_{4,1,4}$. For all $n > 3$ it holds $P \preceq Candy_{4,n,4}$ and it suffices to show that each $Candy_{4,n,4}$ is a minimal matrix (with respect to minors) that is not the skew sum of two of its non-empty interval minors. According to Observation 2.16, the second condition holds as $Candy_{4,n,4}$ contains no non-trivial articulation. To show it is minimal, we need to consider any matrix $M \preceq Candy_{4,n,4}$ and argue that either $P \not\preceq M$ or M contains an articulation.

Thanks to Observation 2.15, we can only consider one minoring operation at a time. It is easy to see that when a one-entry is changed to a zero-entry, then the matrix does not belong to $\mathcal{R}(P)$ anymore. Consider that rows $r_1, r_2, \ldots, r_k$

are chosen to be merged into one with an elementwise OR. If $r_1 < 4$ or $r_k > n+3$ then P is no longer an interval minor of such matrix. Otherwise, the original $Candy_{4,n,4}[r_1, n-r_1+2]$ becomes an articulation. Symmetrically, the same holds for columns which concludes the proof. □

Corollary 2.26. *There exists a class of matrices $\mathcal{M}$ having a finite basis such that $Cl(\mathcal{M})$ has an infinite basis.*

Proof. From Theorem 2.25, we have a matrix P for which $\mathcal{R}(P)$ is infinite. Class $\mathcal{M} = Av_{\preceq}(P)$ has a finite basis. On the other hand, from Theorem 2.21, we have $Cl(\mathcal{M}) = Av_{\preceq}(\mathcal{R}(P))$. □

3. Zero-intervals

In Chapter 1, we characterized matrices avoiding small patterns. Their structure is very dependent on the pattern and the results are hard to generalize for arbitrary patterns. In this chapter, we look for a more general property that restricts the complexity that a class of matrices can have.

Definition 3.1. For a matrix $M \in \{0,1\}^{m \times n}$, a row interval $M[\{r\},[c_1,c_2]]$ is a *zero-interval* if all entries are zero-entries, $c_1 = 0$ or $M[r,c_1-1] = 1$ and $c_2 = n$ or $M[r,c_2+1] = 1$. In other words, it is an interval of zero-entries bounded by one-entries. Symmetrically, we also call a single column sequence $M[[r_1,r_2],\{c\}]$ a *zero-interval* if all entries are zero-entries, $r_1 = 0$ or $M[r_1-1,c] = 1$ and $r_2 = m$ or $M[r_2+1,c] = 1$. In the same spirit, we define a *one-interval* to be an interval of one-entries in a single line of M bounded by zero-entries (or edges of the matrix).

In the previous chapter, for pattern $P \in \{0,1\}^{k \times l}$ it very often holds that any inclusion maximal matrix M avoiding P as an interval minor has at most l zero-intervals in each row and at most k zero-intervals in each column. The main goal of this chapter is to describe patterns for which the size of a pattern bounds the number of zero-intervals of any inclusion maximal matrix that avoids it.

$$P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix} \; P_2 = \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix} \; P_3 = \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix} \; P_4 = \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}$$

Ultimately, we show that for every matrix P, there is an inclusion maximal matrix $M \in Av_{\preceq}(P)$ with arbitrarily many zero-intervals if and only if P contains an interval minor P_1, P_2, P_3 or P_4.

3.1 Pattern complexity

Let us present some useful notion. First of all, every time we speak about a *maximal* matrix of a class, we mean inclusion maximal – it has no zero-entry that can be changed to a one-entry so that it still belongs to the class. In terms of pattern avoidance, maximal matrices are those for which a change of a zero-entry creates a mapping of the pattern (or possibly many mappings).

Definition 3.2. For any matrix P, let $Av_{max}(P)$ be a set of all maximal matrices avoiding P as an interval minor.

Definition 3.3. Let P be a pattern, let e a one-entry of P, $M \in Av_{\preceq}(P)$ and let z be an arbitrary zero-interval of M. We say that z is *usable for* e if there is a zero-entry contained in z such that if we change it to a one-entry, it creates a mapping that uses the new one-entry to map e. This way, z can be usable for many one-entries of P at the same time.

Observation 3.4. *Let $P \in \{0,1\}^{k \times l}$ and $M \in \{0,1\}^{m \times n}$ be matrices such that $P \not\preceq M$. Let $z = M[\{r_1\},[c_1,c_2]]$ be a zero-interval of M usable for a one-entry $e = P[r,c]$. If we change a zero-entry of z and create a mapping of P that uses the changed entry to map e, then no such mapping can map column c outside of columns $[c_1,c_2]$ of M.*

Proof. Since the changed entry is used to map e, clearly every mapping needs to use a column from $[c_1, c_2]$ to map column c. If, for contradiction, after a change of a zero-entry there is a mapping using columns outside $[c_1, c_2]$ then it, without loss of generality. uses $c_1 - 1$ but since it bounds zero-interval z, it is a one-entry and this one-entry can be used in the mapping instead of the changed entry, which gives us a contradiction with $P \not\preceq M$. $\square$

Definition 3.5. For a class of matrices $\mathcal{M}$, we define its *row-complexity*, $r(\mathcal{M})$ to be the supremum of the number of zero-intervals in a single row of any maximal $M \in \mathcal{M}$. We say that $\mathcal{M}$ is *row-bounded*, if its row-complexity is finite, and *row-unbounded* otherwise. Symmetrically, we define *column-complexity* $c(\mathcal{M})$ and the property of being *column-bounded* and *column-unbounded*. Class $\mathcal{M}$ is *bounded* if it is both row-bounded and column-bounded and it is *unbounded* otherwise.

Definition 3.6. We say that a set of pattern $\mathcal{P}$ is *bounding*, if the class $Av_{\preceq}(\mathcal{P})$ is bounded and is *non-bounding* otherwise.

Definition 3.7. Let $\mathcal{P}$ be a set of patterns and let e be a one-entry of any $P \in \mathcal{P}$. We define the *row-complexity* of e, $r(Av_{\preceq}(\mathcal{P}), e)$ to be the supremum of the number of zero-intervals of a single row of any $M \in Av_{max}(\mathcal{P})$ that are usable for e. We say that e is *row-unbounded* in $Av_{\preceq}(\mathcal{P})$ if $r(Av_{\preceq}(\mathcal{P}), e) = \infty$ and *row-bounded* otherwise. Symmetrically, we define the *column-complexity* e, $c(Av_{\preceq}(\mathcal{P}), e)$ to be the maximum number of zero-intervals of a single column of any matrix from $Av_{max}(\mathcal{P})$ that are usable for e and say e is *column-unbounded* if it is infinite and *column-bounded* otherwise.

The following observation follows directly from the definition and we use it heavily throughout the chapter to break symmetries.

Observation 3.8. *For every set $\mathcal{M}$, $\mathcal{M}$ is row-bounded if and only if $\mathcal{M}^\top$ is column-bounded.*

3.1.1 Adding empty lines

Similarly, as we did in Chapter 1, we show that we do not need to consider patterns with leading (and ending) empty rows (and columns).

Observation 3.9. *For a matrix $P \in \{0,1\}^{k \times l}$ and integer n, let $P' = P \to 0^{k \times n}$. Matrix P is bounding if and only if P' is bounding. Moreover, if P is bounding, then $r(Av_{\preceq}(P')) = r(Av_{\preceq}(P)) + 1$.*

Lemma 3.10. *Let $P \in \{0,1\}^{2 \times k}$ and for any $l \geq 1$ let $P^l \in \{0,1\}^{(l+2) \times k}$ be a pattern created from P by adding l new empty rows in between the two row of P. For every one-entry e of P^l $r\left(Av_{\preceq}\left(P^l\right), e\right) \leq k^2$.*

Proof. Given $M \in Av_{max}(P)$, let us look at an arbitrary row r of M. Without loss of generality assume $e = P[1, c]$. For contradiction, assume there are $k^2 + 1$ zero-intervals $z_1, \ldots, z_{k^2+1}$ in r usable for e.

- $P[2, c] = 1$: Clearly, there is a one-entry in rows $[r + l + 1, m]$ underneath each z_j and if we combine each such one-entry with a one-entry bounding corresponding z_j, we find a mapping of $\left(\{1\}^{2 \times k^2}\right)^l$, contradicting $P \not\preceq M$.

- $P[2, c] = 0$: For each $i \in [t]$, we define an extended interval z_i^* to be the interval containing z_i and also all elements of r between z_i and z_{i+1}. Because of the Pigeonhole principle, we can find either k consecutive extended intervals such that there are no one-entries in rows $[r + l + 1, m]$ underneath them, or k extended intervals such that there is a one-entry in rows $[r+l+1, m]$ underneath each of them. Because each extended interval contains a one-entry, in the second case we find $\left(\{1\}^{k\times 2}\right)^l$ as an intervals minor. In the first case, without loss of generality, assume $P[2, c_1] = 1$ and it is the minimum such $c_1 > c$. Also let $z_{first}, \ldots, z_{last}$ be the consecutive zero-intervals. Now consider the mapping of P^l created when a zero-entry of z_{first} gets changed to a one-entry used to map e. Since $P[2, c_1] = 1$ and there are no one-entries in rows $[r+l+1, m]$ underneath extended intervals z_{first}-z_{last}, $P^l[l+2, c_1]$ has to be mapped to the columns of M after the end of z_{last}. This leaves k one-entries to be used to map potential one-entries in $P^l[\{l+2\}, [c, c_2 - 1]]$ and so $P^l \preceq M$.

$\square$

Corollary 3.11. *Let $P \in \{0,1\}^{k\times 2}$ and for any $l \geq 1$ let $P^l \in \{0,1\}^{k\times(l+2)}$ be a pattern created from P by adding l new empty columns in between the two columns of P. Then $Av_{\preceq}\left(P^l\right)$ is bounded for any $l \geq 1$.*

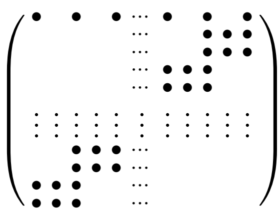
Proof. We know $Av_{\preceq}\left(P^l\right)$ is row-bounded from Lemma 1.7. From Lemma 3.10 and Observation 3.8 we have that the class is also column-bounded. $\square$

3.1.2 Non-bounding patterns

We see that for patterns having only two rows or columns we can indeed bound the number of zero-intervals of maximal matrices avoiding them. On the other hand, already for a pattern of size 3×3 we show that there are maximal matrices with arbitrarily many zero-intervals.

Lemma 3.12. *Class $Av_{\preceq}(P_1)$ is unbounded.*

Proof. For given n, let M be a $(2n+1)\times(2n+1)$ matrix described by the picture:



We see that $P_1 \npreceq M$ because we always need to map $P_1[2, 1]$ and $P_1[3, 3]$ to just one "block" of one-entries, which only leaves a zero-entry for $P_1[1, 2]$.

If we change any zero-entry of the first row into a one-entry we get a matrix containing an interval minor of $\{1\}^{3\times 3}$; therefore, containing P_1 as an interval minor. In case M is not maximal, we can add some more one-entries to make it maximal but it will still contain a row with n zero-intervals. $\square$

Not only $M \in Av_{max}(P_1)$ but it also avoids any $P \in \{0,1\}^{3\times 3}$ such that $P_1 \preceq P$. Its rotations avoid rotations of P_1 and we can deduce that a big portion of patterns of size 3×3 are non-bounding. Moreover, the result can be generalized also for bigger matrices.

Theorem 3.13. *For every P such that $P_1 \preceq P$, $Av_{\preceq}(P)$ is unbounded.*

Proof. First, assume there is a mapping of P_1 into $P \in \{0,1\}^{k\times l}$ that assigns a one-entry of the first row to $P_1[1,2]$, a one-entry of the first column to $P_1[2,1]$ and a one-entry of the last row and column to $P_1[3,3]$. Then, we use a similar construction to what we did in the proof of Lemma 3.12 to find a matrix $M \in Av_{max}(P)$ with n zero-intervals for any n.

Let P be an arbitrary pattern containing P_1 as an interval minor. Let $P[r_1,c_1], P[r_2,c_2]$ and $P[r_3,c_3]$ be one-entries that can be used to map $P_1[1,2], P_1[2,1]$ and $P_1[3,3]$ respectively. We take a submatrix $P' := P[[r_1,r_3],[c_2,c_3]]$. Such a pattern fulfills assumptions of the more restricted case above and we can find a matrix $M' \in Av_{max}(P')$ having n zero-intervals. We construct M from M' by simply adding new rows and columns containing only one-entries. We add $r_1 - 1$ rows in front of the first row and $k - r_3$ rows behind the last row. We also add $c_2 - 1$ columns in front of the first column and $l - c_3$ columns behind the last column. Constructed matrix M avoids P an an interval minor because its submatrix P' cannot be mapped to M'. At the same time, any change of a zero-entry of the r_1-th row of M to a one-entry creates a copy of $1^{k\times l}$. Constructed M can be seen in Figure 3.1.

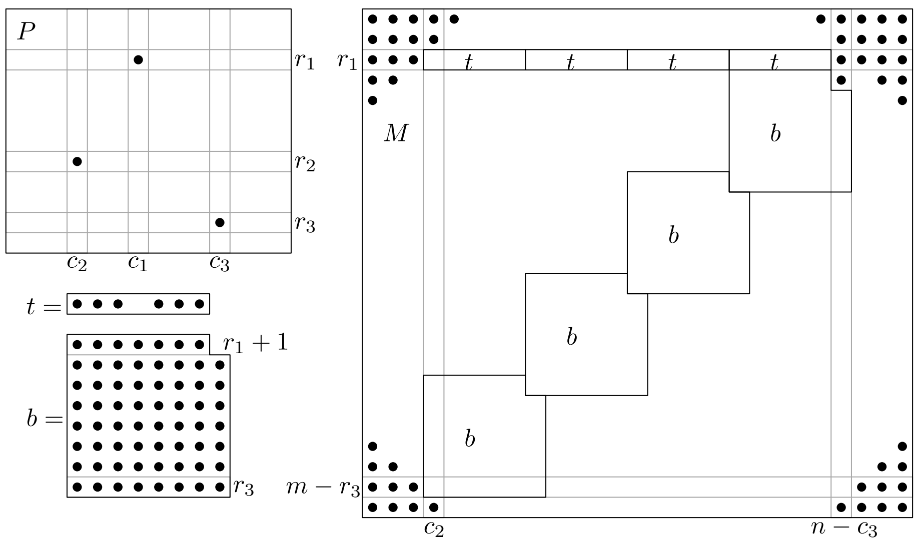


Figure 3.1: Structure of a maximal matrix avoiding P that has arbitrarily many one-intervals.

□

3.1.3 Bounding patterns

What makes it even more interesting is that any pattern avoiding all rotations of P_1 is already bounding.

Theorem 3.14. *Let P be a pattern avoiding all rotations of P_1, then P:*

1. *contains at most three non-empty lines or*
2. *avoids $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix}$ or $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$.*

Proof. Assume P has four one-entries that do not share any row or column. Then those one-entries induce a 4×4 permutation inside P and because P does not contain any rotation of P_1, the induced permutation is either 1234 or 4321. Without loss of generality, assume it is the first one and denote its one-entries by e_1, e_2, e_3 and e_4.

For contradiction, assume P also contains $P' = \left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$. Clearly, no one-entry from e_1, e_2, e_3 and e_4 can be part of any mapping of P' because it would induce a mapping of a rotation of P_1.

Let $e_2 = P[r_2, c_2]$ and $e_3 = P[r_3, c_3]$. The submatrix $P[[r_2], [c_2, l]]$ avoids P'; otherwise, together with e_1 it would give us a rotated copy of P_1. Symmetrically, $P[[r_3, k], [c_3]]$ does not contain P'. Also, $P[[r_3 - 1], [c_3 - 1]]$ and $P[[r_2 + 1, k], [c_2 + 1, l]]$ are empty; otherwise, they would together with e_2 and e_3 give us a rotation of P_1. Up to rotation, the only possible way to have $P' \preceq P$ is that $P'[1, 1]$ is mapped to a one-entry from $P[[r_3 - 1], [c_2, c_3 - 1]]$ but then this entry together with e_1 and e_3 give us a rotation of P_1, which is a contradiction. □

Lemma 3.15. *Let $P \in \{0, 1\}^{k \times l}$ be a pattern having one non-empty line. Then $r\,(Av_{\preceq}\,(P)) \leq k$ and $c\,(Av_{\preceq}\,(P)) \leq l$.*

Proof. Without loss of generality, let the non-empty line be a row r. Consider any $MAv_{max}\,(P)$. Matrices $M[[r - 1], [n]]$ and $M[[m - r + 1, m], [n]]$ contain no zero-entry. If we look at any other row, it cannot contain k one-entries, so the maximum number of zero-intervals is k.

Consider a column c of M. If there is at least one one-entry in $M[[r, m-r], c]$ then because M is maximal, the whole column is made of one-entries. Otherwise, there are two one-intervals $M[[r - 1], c]$ and $M[[m - r, m], c]$. □

Lemma 3.16. *Let $P \in \{0, 1\}^{k \times l}$ be a pattern having two non-empty lines. Then $r\,(Av_{\preceq}\,(P)) \leq k^2 + l$ and $c\,(Av_{\preceq}\,(P)) \leq l^2 + k$.*

Proof. First, we assume the two non-empty lines of P are rows $r_1 < r_2$ (or symmetrically columns). From Observation 1.5 and maximality of M we have that $M[[r_1 - 1], [n]]$ and $M[[m - r_2 + 1, m], [n]]$ contain no zero-entry. Therefore, we may restrict ourselves to the case when $r_1 = 1$ and $r_2 = k$. From Corollary 3.11, we have that there are at most k^2 zero-intervals in each $M \in Av_{max}\,(P)$.

Let the two non-empty lines of P be a row r and a column c. Because of symmetry, we only show the bound for rows. Let us take an arbitrary row of M an look at its zero-intervals. For every one-entry e of the pattern except those in the r-th row, there is at most one zero-interval usable for e. For contradiction, assume there are two such zero-intervals z_1 and z_2. Let Figure 3.2 illustrate the situation where dashed and dotted lines form mappings of an interval minor P to M when a zero-entry of z_1 and z_2 respectively is changed to a one-entry. When we take the outer two vertical and horizontal lines, we get a mapping of P that can use an existing one-entry in between z_1 and z_2 to map e. This gives us a contradiction with $P \not\preceq M$.

For a one-entry $e = P[r, c']$, if $c' \leq c$ then there must be less than c' one-entries before any zero-intervals usable for e; otherwise, we could map $P[r, [1, c']]$ just to the single row of M. It follows that e is row-bounded. Symmetrically, the same holds in case $c' > c$ and together we have at most $k + l$ zero-intervals in each $M \in Av_{max}\,(P)$. □

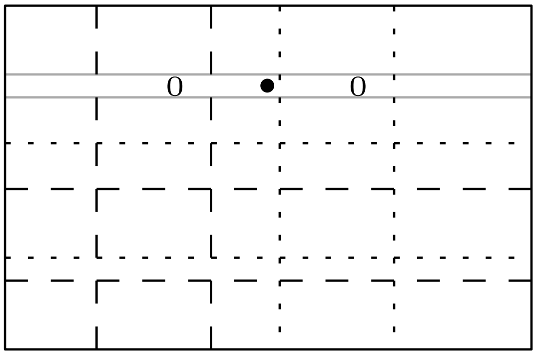


Figure 3.2: Dashed and dotted lines resembling two different mappings of a forbidden pattern, where two horizontal lines show the boundaries of the mapping of row r and the vertical lines show boundaries of the mapping of column c.

Lemma 3.17. *Let $P \in \{0,1\}^{k \times l}$ be a pattern structured like one of the matrices in Figure 3.3. Then every one-entry in $P[\{r_2\}, [c_1, c_2]]$ is row-bounded.*

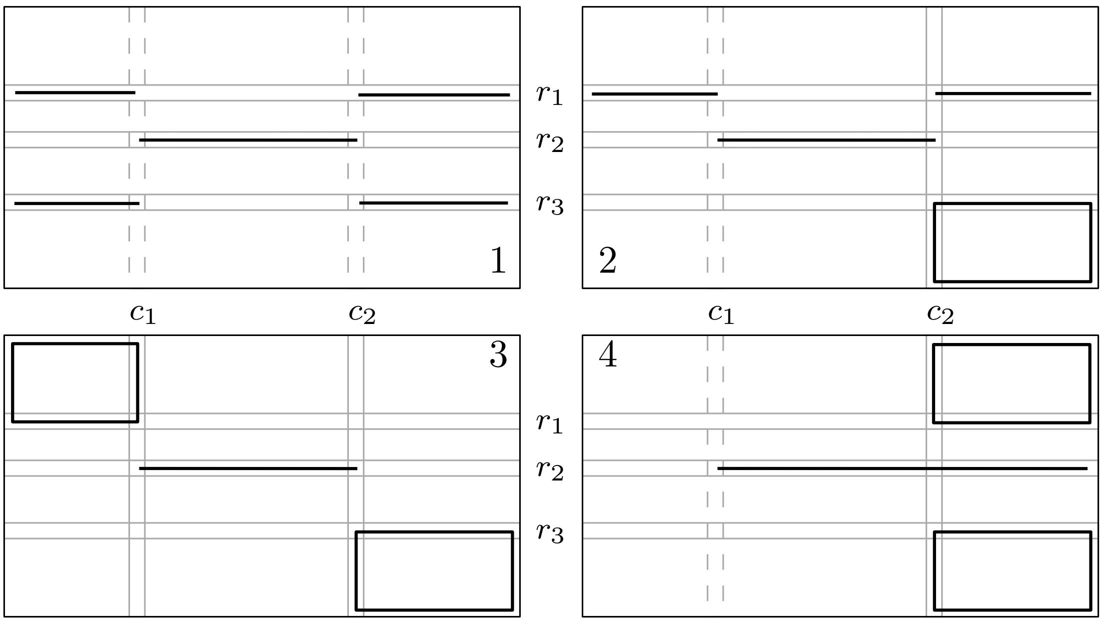


Figure 3.3: Patterns for which one-entries in row r_2 and columns c_1 to c_2 are row-bounded. One-entries may only be in the areas enclosed by bold lines.

Proof. Let P be the first described pattern and let $k' = c_2 - c_1$. We show that for each one-entry e from row r_2 and every $M \in Av_{max}(P)$ there is at most k' zero-intervals for which it is usable. For contradiction assume there is a row r with $k' + 1$ zero-intervals usable for e. It follows that there are at least k' one-entries in between two most distant zero-intervals z_1 and z_2. Therefore, the whole row r_2 can be mapped just to r. Since changing a zero-entry of z_1 to a one-entry to which e can be mapped creates a partitioning of M where all one-entries from columns 1 to c_1 are mapped to columns up to z_1 and similarly all one-entries from columns c_2 to l can be mapped to columns from and past z_2, we can simply map empty rows from $r_1 + 1$ to $r_3 - 1$ around row r and use the rest to map rows r_1 and r_2. Described partitioning gives us $P \preceq M$ and a contradiction. We can see the partitioning in Figure 3.4.

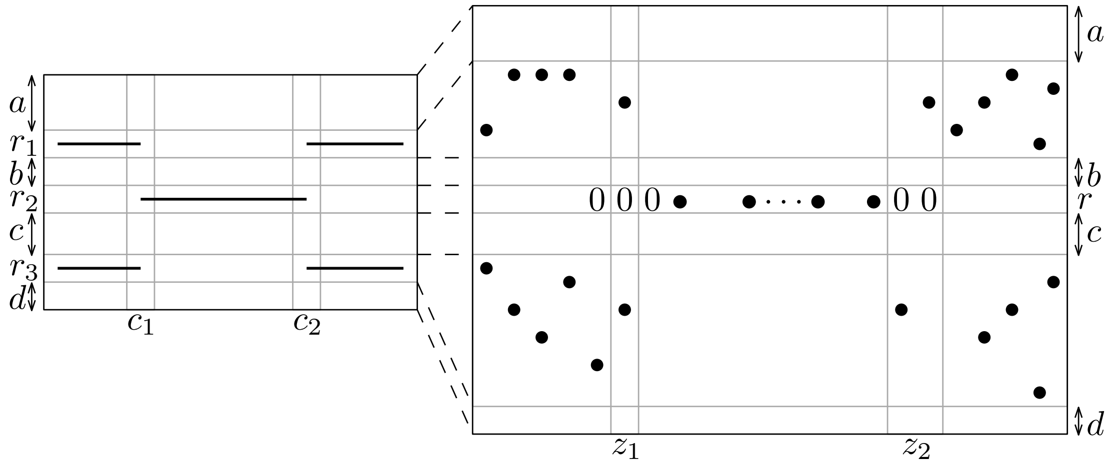


Figure 3.4: Mapping of a pattern into a matrix only using one line to map an empty line of the pattern and only using one line to map row r_2.

Proofs of cases two and three are similar to the first one and we skip them.

Let us look on the fourth case. For i-th one-entry in row r_2 (ordered from left to right and only considering those in columns c_1 to c_2) no zero-interval of a maximal matrix avoiding the pattern cannot have i one-entries to the left of it and so each such one-entry is bounded by $i \geq l$.

It is important to realize we could not have used the same proof we used for the first three cases also for the fourth case, because we can never rely on the fact a mapping of P only uses one row of M to map row r_2. This is because in the fourth case, unlike the first three, there are also potential one-entries in $P[\{r_2\}, [c_2, l]]$. $\square$

Lemma 3.18. *Let $P \in \{0,1\}^{k \times l}$ be a pattern structured like one of the matrices in Figure 3.5. Then every one-entry in $P[[r_1+1, r_2-1], \{c\}]$ is row-bounded. Moreover, in the first two cases, if $c = l-1$ and there are no one-entries in $P[[r_1-1], \{c\}]$ and $P[[r_2+1, k], \{c\}]$, then also one-entries $P[r_1, c]$ and $P[r_2, c]$ are row-bounded.*

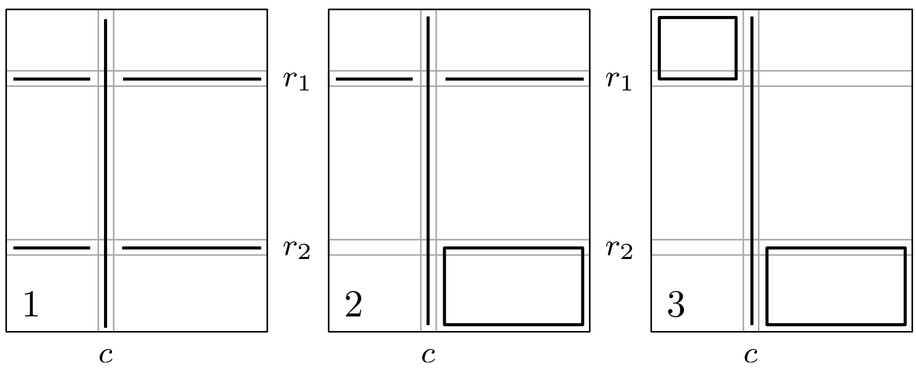


Figure 3.5: Patterns for which one-entries in column c and rows $r_1 + 1$ to $r_2 - 1$ are row-bounded. One-entries may only be in the areas enclosed by bold lines.

Proof. Let P be the first described pattern. We show that for each one-entry from $P[[r_1+1, r_2-1], \{c\}]$ and every M maximal matrix avoiding P there is at most one zero-interval for which it is usable. For contradiction assume there is a row r with two zero-intervals z_1 and z_2 usable for e. Look at Figure 3.6 and let the dashed partitioning be a mapping of P to M when a zero-entry of z_1 is changed to a one-entry used to map e and let the dotted partitioning be a mapping of P to M when a zero-entry of z_2 is changed to a one-entry used to map e. If we map column c to where it is mapped in both mappings together and map rows r_1 and r_2 as suggested in the picture, we get a partitioning of P inside M and so a contradiction with $P \not\prec M$.

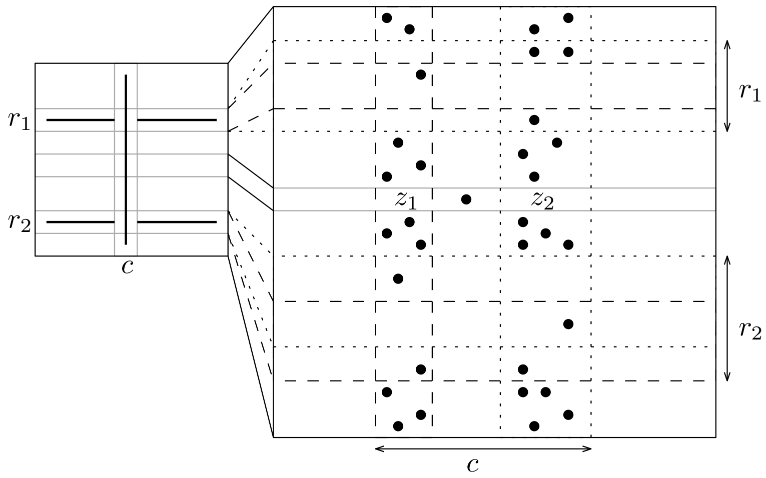


Figure 3.6:

Proofs of cases two and three are similar to the first one and we skip them.

From now on, consider there are no one-entries in $P[[r_1 - 1], \{c\}]$ and $P[[r_2 + 1, k], \{c\}]$. Let P be the second described pattern and let $c = l - 1$. One-entry in $P[r_1, c]$ is row-bounded thanks to the fourth case of Lemma 3.17. Without loss of generality assume $P[r_1, l] = 1$, as otherwise, the pattern avoids $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ and in Lemma 3.20 we will show that each one-entry is then row-bounded. Without loss of generality, when a zero-entry of a zero-interval is changed to a one-entry that is used to map $P[r_2, c]$, the row r_2 is mapped to just one row because we can always use the one-entry bounding the corresponding interval to map $P[r_2, l]$ (if we do not consider the only potential zero-interval that is bounded by the edge of matrix). If $z_1 < z_2$ are two zero-intervals usable for $P[r_2, c]$ then in each mapping created by changing a zero-entry of z_1 to a one-entry used to map $P[r_2, c]$, one-entry $P[r_1, l]$ is mapped to a column smaller than the first column of z_2. Otherwise, we could combine the mapping with a one-entry in between z_1 and z_2 and a mapping created when a zero-entry of z_2 is changed to a one-entry to find a mapping of P. Assume, there are l zero-intervals usable for $P[r_2, c]$ and for each consider a one-entry used to map $P[r_1, l]$ in the corresponding mapping created when a zero-entry is changed to a one-entry. If there is a non-decreasing pair of them, the corresponding mappings can be combined to find a mapping of P. Otherwise, the one-entries form a decreasing sequence of length l and if we consider the last used zero-interval and its mapping, we can use the decreasing sequence of one-entries to map all one-entries from row r_1 and we can still take a one-entry bounding the zero-interval from left and use it to map $P[r_2, c]$. This proves there are at most $l + 1$ zero-intervals usable for $P[r_2, c]$.

The proof that $P[r_1, c]$ and $P[r_2, c]$ are row-bounded in the same setting when P is described by the first picture is analogous. $\square$

Lemma 3.19. *Let P be a pattern and c be its first non-empty column. Then every one-entry from c is row-bounded.*

Proof. The result follows immediately from the fourth case of Lemma 3.17. $\square$

Lemma 3.20. *Let $P \in \{0,1\}^{k\times l}$ be a pattern avoiding $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ (or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$). Then $Av_{\preceq}(P)$ is bounded.*

Proof. From Proposition 1.12 we know that P is a walking pattern. Every one-entry of P satisfies either conditions of the third case of Lemma 3.17 or it satisfies conditions of the third case of Lemma 3.18 and therefore is row-bounded. From Observation 3.8, we know it is also column-bounded. $\square$

Lemma 3.21. *Let $P \in \{0,1\}^{k\times l}$ be a pattern having three non-empty lines and avoiding all rotations of P_1. Then $Av_{\preceq}(P)$ is bounded.*

Proof. First of all, if P avoids $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$, we use Lemma 3.20. From now on, we assume it contains both.

Let us prove that each pattern having one-entries in three rows is bounded. Pattern P has one-entries in at least three columns; therefore, it contains a three by three permutation matrix as a submatrix. Since rotations of P_1 are avoided, only feasible permutations are 123 and 321 and without loss of generality we assume the first case. In Figure 3.7 we see the structure of each such pattern. Capital letters stand for one-entries of the permutation, letters $a - f$ stand each for a potential one-entry and Greek letters stand each for a potential sequence of one-entries and zero-entries. Everything else is empty. Not all one-entries can be there at the same time, because that would create a mapping of P_1 or its rotation. We also need to find $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$. The following analysis only uses hereditary arguments, which means that if we prove P is bounded, we also prove that each submatrix of P is bounded. With this in mind, we restrict ourselves to maximal patterns.

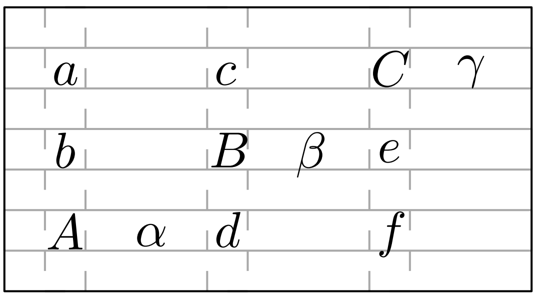


Figure 3.7: Structure of a pattern only having three non-empty rows and avoiding all rotations of P_1.

1. γ contains a one-entry $\Rightarrow f = 0 \Rightarrow$ because $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right) \preceq P$, it holds $a = 1 \Rightarrow \alpha = 0$

   (a) $d = 1 \Rightarrow b = 0,\ \beta = 0,\ e = 0$

   (b) $d = 0$
      i. $c = 1 \Rightarrow \beta = 0,\ e = 0$
      ii. $c = 0$

2. $\gamma = 0$

   (a) α contains a one-entry $\Rightarrow a = 0,\ b = 0$. If $f = 0$ we have case 1.(b) ii. otherwise, we have case 1.(a).

   (b) $\alpha = 0$
      i. $c = 1,\ d = 1 \Rightarrow b = 0,\ e = 0,\ \beta = 0$
      ii. $c = 1,\ d = 0 \Rightarrow e = 0,\ \beta = 0$ and without loss of generality, $b = 1$. Otherwise, we have the previous case. Therefore, $f = 0$
      iii. $c = 0,\ d = 1 \Rightarrow b = 0$: Without loss of generality, $e = 1$ or β contains a one-entry. Otherwise, we have the case $c = 1,\ d = 1$. Therefore, $a = 0$
      iv. $c = 0,\ d = 0$

The same analysis also proves that if a pattern with the same restrictions only has three non-empty columns then it is bounding.

Let us now look at the case when all one-entries of the pattern are in either one of two rows r_1, r_2 or in a column c_1. Without loss of generality, we again assume permutation 123 is present and we distinguish three cases. Consider Figure 3.8:

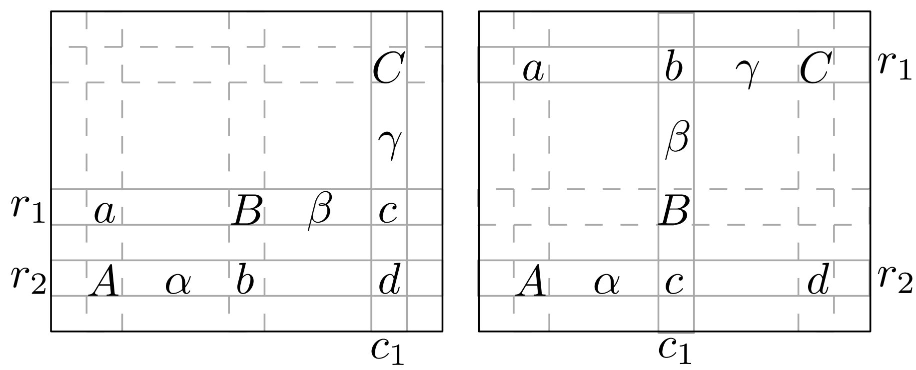


Figure 3.8: Structure of a pattern only having one-entries in two rows and one column that avoids all rotations of P_1.

1. C lies in column c_1

   (a) $a = 0$

   (b) $a = 1 \Rightarrow b = 0,\ \alpha = 0$

2. B lies in column c_1: Thanks to Lemma 3.19 are one-entries in a, d, A, C row-bounded and one-entries in a, b, c, d, A, C, α and γ column-bounded. From the first case of Lemma 3.18, we have that one-entries in B and β are row-bounded and from the first case of Lemma 3.17, one-entries in b, c, B and β are column-bounded. Thus, every one-entry is column-bounded.

   (a) $a = 1,\ d = 1 \Rightarrow \alpha = 0,\ \gamma = 0$

(b) $a = 1,\ d = 0 \Rightarrow \alpha = 0$

(c) $a = 0,\ d = 1 \Rightarrow \gamma = 0$

(d) $a = 0,\ d = 0$: The pattern avoids $(\,^{\bullet}{}_{\bullet}\,)$ so it is bounded according to Lemma 3.20.

3. A lies in column c_1: This is symmetric to the first situation.

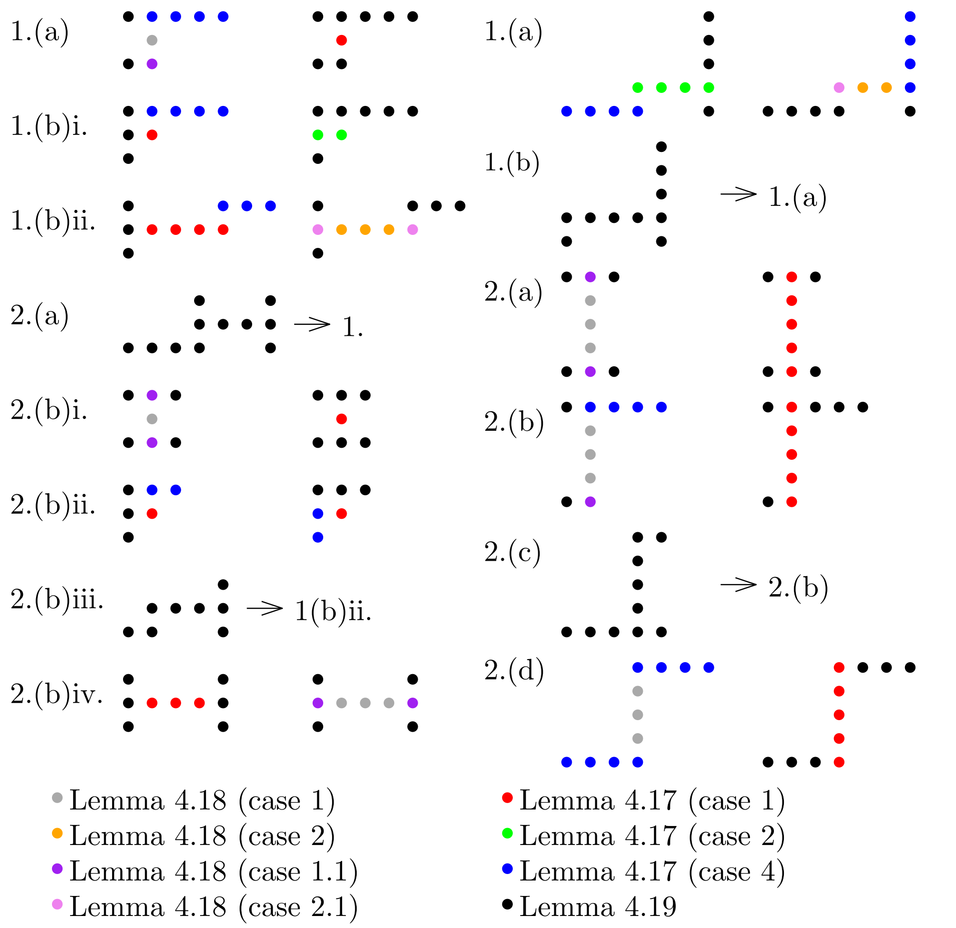


Figure 3.9: A figure showing which lemma can be used to proof row-boundness and column-boundness for each one-entry of patterns discussed in the case analysis. The left half of the picture deals with the situation where there are three non-empty rows and the right half with the situation where there are two non-empty rows and one non-empty column. Each case either contains a picture showing row-boundedness and column-boundeness or an arrow describing that the case can be easily reduced to a different one.

The same analysis also proves that if one-entries of a pattern with the same restrictions are in one row or two columns then the pattern is bounded. $\square$

Combining all the lemmata we finally get the following result.

Theorem 3.22. *Let P be a pattern avoiding all rotations of P_1, then $Av_{\preceq}(P)$ is bounded.* $\square$

3.2 Chain rules

In this section, we study what happens when we combine multiple classes that are bounded or unbounded.

Theorem 3.23. *Let $\mathcal{P}$ and $\mathcal{Q}$ be classes of patterns. If both $\mathcal{P}$ and $\mathcal{Q}$ are bounded then $Av(\mathcal{P} \cup \mathcal{Q})$ is bounded.*

Proof. We show $comp_{\mathcal{P}\cup\mathcal{Q}} \leq comp_{\mathcal{P}} + comp_{\mathcal{Q}} = C$.

For contradiction, let M be a maximal matrix avoiding $\mathcal{P} \cup \mathcal{Q}$ having at least $C+1$ zero-intervals in a single row (or column). Without loss of generality it means there is more than $comp_{\mathcal{P}}$ zero-intervals usable for one-entries of the patterns from $\mathcal{P}$. Not let us change some zero-entries of M to one-entries to get $M' \in Av(\mathcal{P})$. Clearly, it still contains more than $comp_{\mathcal{P}}$ zero-intervals usable for one-entries of the patterns from $\mathcal{P}$, which is a contradiction with the definition of $comp_{\mathcal{P}}$.

Similarly, the same inequality holds also for the column-complexity of $\mathcal{P} \cup \mathcal{Q}$ and so the union is bounded. □

Using induction, we can show that also a union of a finite number of bounded classes of finite sizes is bounded. Interestingly enough, unbounded classes are not closed the same way.

Theorem 3.24. *For every $1 \leq i < j \leq 4]$ is $\{P_i, P_j\}$ bounded.*

Proof. Due to symmetries it is enough to only consider $i = 1$ and $j = [1, 2]$.

- $\{P_1, P_2\}$ is row-bounded: from Lemma 3.19 we have that one-entries $P_1[2,1], P_1[3,3], P_2[2,$ and $P_3[3,1]$ are row-bounded. For $P_1[1,2]$ and $P_2[1,2]$ we prove there are at most two zero-intervals usable for each of them. Otherwise, if there are three zero-intervals $z_1 < z_2 < z_3$ usable for $P_1[1,2]$ then the one-entries used to map $P_1[2,1]$ and $P_1[3,3]$ in a mapping created when a zero-entry of z_1 changes to one-entry used to map $P_1[1,2]$ together with a one-entry in between z_2 and z_3 give us a mapping of P_2 to M. Symmetrically, the same goes for $P_2[1,2]$ and z_3'.

- $\{P_1, P_2\}$ is column-bounded: from Lemma 3.19 combined with Observation 3.8 we have that one-entries $P_1[1,2], P_1[3,3], P_2[1,2]$ and $P_3[3,1]$ are column-bounded. For $P_1[2,1]$ and $P_2[2,3]$ we prove there are at most two zero-intervals usable for each of them. Otherwise, if there are three zero-intervals $z_1 < z_2 < z_3$ (from top down) usable for $P_[2,1]$ then the one-entries used to map $P_1[1,2]$ and $P_1[3,3]$ in a mapping created when a zero-entry of z_1 changes to one-entry used to map $P_1[1,2]$ together with a one-entry in between z_2 and z_3 give us a mapping of P_2 to M. Symmetrically, the same goes for $P_2[2,3]$ and z_3'.

- $\{P_1, P_3\}$ is row-bounded: we can use the same proof as when showing that $\{P_1, P_2\}$ is column-bounded.

- $\{P_1, P_3\}$ is column-bounded: we can use the same proof as when showing that $\{P_1, P_2\}$ is row-bounded.

$\square$

We prove even stronger result by using a well known fact from the theory of ordered sets.

Fact 3.25 (Higman's lemma)**.** *Let A be a finite alphabet and A^* be a set of finite sequences over A. Then A^* is well quasi ordered with respect to the subsequence relation.*

Theorem 3.26. $\sigma = Av\left(\begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}, \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix}, \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}\right)$ *is bounded. Moreover, every subclass is bounded.*

Proof. From Theorem 3.14 we know that elements of σ fall into finitely many classes. For each we need to prove that it is bounded and also that it does not contain an infinite anti-chain. Knowing that we use Theorem 3.23 to obtain the result. Let us consider an m by n matrix $M \in \sigma$:

- M only contains up to three non-empty rows (columns):
  Clearly, if M is maximal then it contains three rows made of one-entries and everything else is zero, so the number of one-intervals is bounded by three.

  We use words over alphabet $A = \{a, b, c, d, e, f, g, h, i, j\}$ to describe each M as follows. Let $r_1 < r_2 < r_3$ be the non-empty rows (if less then three are non-empty we choose extra values arbitrarily). We define $w_M \in A^*$ as follows. First, we use letter g r_1 times, letter h $r_2 - r_1$ times, letter i $r_3 - r_2$ times and letter j $m - r_3$ times to describe the number of rows of M. Then we describe columns from the first one to the last one as follows. For each 0 in r_1 we use letter a and for 1, we use ab. For each 0 in r_2 we use letter c and for 1, we use cd. For each 0 in r_3 we use letter e and for 1, we use ef.

  If we have $w_M, w_{M'} \in A^*$ such that w_M is a subsequence of $w_{M'}$ then we want to show that M is an interval minor of M'. Let r_1, r_2, r_3 and r'_1, r'_2, r'_3 be the non-empty rows of M and M' respectively. Since the number of leading letters g is not bigger in w_M, M does not have more empty rows before r_1 than M' does before r'_1 and similarly it has at most as many empty rows in between r_1, r_2 and r_2, r_3 and after r_3.

  Now consider there is ab in w_M and it corresponds to some $a \dots b$ in $w_{M'}$. We can always assume that in $w_{M'}$ the "a" is the one exactly before b. It can only happen that $abcdeface$ is a subsequence of **ab**cea**cd**eac**eface** if the bold letters are used and since they correspond to one-entries lying in the following columns, this indeed corresponds to an interval minor (but it clearly does not have to mean that M is a submatrix of M').

  From Fact 3.25 we have that A^* is well ordered which means that matrices having at most three non-empty rows (columns) are well ordered (the construction can be extended to every fixed number of non-empty rows) and so they does not have an infitely long anti-chain.

- one-entries of M lie in at most two rows and one column (or vice versa):
  The number of one-intervals of any such maximal M is bounded by two.

We use words over alphabet $A = \{a, b, c, d, e, f, g\}$ and for non-empty rows r_1, r_2 and column c_1 we define w_M as follows. We first encode each column in such a way that for each 0 in r_1 we use letter a and for 1, we use ab. For each 0 in r_2 we use letter c and for 1, we use cd. Right before and after the description of column c_1 we put letter g. Next we encode each row in such a way that for each 0 in c_1 we use letter e and for each 1 letters ef. Right before and after the descriptions of rows r_1 and r_2 we again place letter g.

Because of the distinct letters for encoding rows and columns we can apply the same analysis as we did in the previous case and since entries at $M[r_1, c_1]$ and $M[r_2, c_1]$ are separated from the rest by a special letter g there is no way to find a one-entry if it is not there.

- M avoids $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ (or $\begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}$):
  From Proposition 1.12 we know M is a walking matrix and any such maximal matrix only contains at most one one-intervals in each row and column.

  We use words over alphabet $A = \{a, b, c, d\}$ and encode M as follows. We choose an arbitrary walk of M containing all one-entries and index its entries as $w_1 \dots w_{m+n-1}$. Starting from w_1 we encode w_i so that a stands for 0 and ab for 1 if w_{i+1} lies in the same row as w_i and we use c for 0 and cd for 1 if w_{i+1} lies in the same column as w_i.

In the construction of words corresponding to matrices, we only made sure that $w_M \subseteq w_{M'} \Rightarrow M \preceq M'$ and the other impication does not hold. A different construction may lead to equivalence, but that is not necessary for our result.

We now use distinct alphabets to discribe different classes and when we given a potentialy infinite class of matrices from σ, we know that inside each class there is at most finite number of minimal matrices such that all of the rest contain a smaller one inside. Using induction on Theorem 3.23, we have that each class is bounded and by applying induction with Theorem 3.23 once again we get that the union of the classes is also bounded. $\square$

Observation 3.27. *There exists a bounding pattern P having an unbounded subset of $Av(P)$.*

Proof. Let $P = I_n$ (identity matrix) for $n > 3$. From Lemma 3.20 we have that P is bounding. On the other hand, $Av(I_n, P_1)$ is unbounded, because the construction used in the proof of Lemma 3.12 also works for this class. $\square$

We define matrices to be bounded if they are both row-bounded and column-bounded. From what we proved so far, we see that a pattern P is row-bounded if and only of it is column-bounded. But once we look at collections of patterns, this does not have to be true.

Lemma 3.28. *There exists a class of patters $\mathcal{P}$, which is row-bounded but column-unbounded.*

Proof. Let $\mathcal{P} = \left\{ P = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & \bullet & \\ & & \bullet \end{pmatrix}, I_4 = \begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix} \right\}$. We can use the same construction as we did in Lemma 3.12, just transposed, to prove $Av(\mathcal{P})$ is column-unbounded.

To prove that $\mathcal{P}$ is row-bounded, we take any M maximal avoiding $\mathcal{P}$ and look at an arbitrary row. In Lemma 3.20 we proved that patterns avoiding $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ are bounded and so every one-entry of I_4 is row-bounded. We need to proof the same for P. Using Lemma 3.19, $P[2,1]$ and $P[4,3]$ are row-bounded. Using the first case of Lemma 3.18, $P[3,2]$ is row-bounded. We prove that there are at most two zero-intervals usable for $P[1,2]$. For contradiction, let there be three – $z_1 < z_2 < z_3$. It means there are at least two one-entries $e_1 < e_2$ in between them. Now consider the partitioning of P into M when a zero-entry of z_3 is changed to one-entry used to map $P[1,2]$. Clearly, the one-entry used for mapping $P[2,1]$ lies under the left one-entry e bounding z_3 or in a latter column; otherwise we could use e to map $P[1,2]$ and find the pattern in M. It may happen $e = e_2$, but still e_1 and the one-entries used for mapping $P[2,1], P[3,2]$ and $P[4,3]$ together give us a mapping of I_4 and so a contradiction with $M \in Av(\mathcal{P})$. It means that each one-entry of P is also row-bounded and $Av(\mathcal{P})$ is row-bounded. □

3.3 Complexity of one-entries

So far we have been working with the whole patterns and determining their complexity. To make the results even more general, we can analyze the complexity of each one-entry.

In spare time, I will have a look at this.

Lemma 3.29. *Let $P \in \{0,1\}^{k\times l}$ be a pattern such that all its one-entries are either in rows r_1, r_2 ($r_1 < r_2$) and $P[[r_2], \{c\}]$. Then $P[r_1, c]$ is row-bounded.*

Proof. We prove there are at most k^4 zero-intervals usable for $P[r_1, c]$ in each row of any maximal matrix M avoiding P. For contradiction, let there be more than k^4 of them $(zi_1, \ldots, zi_{k^4})$ in some row and for each of them, consider the top most row r'_j used to map r_2-th row of P in a mapping created when a zero-entry of zi_j is changed to a one-entry used to map $P[r_1, c]$. Then pairs $[zi_1, r'_1], [zi_2, r'_2], \ldots, [zi_{k^4}, r'_{k^4}]$ form a sequence of distinct pairs and thanks to the Pigeonhole principle, there is a subsequence of length at least k^2 such that the values of r'_j are either non-increasing or non-decreasing. Without loss of generality, assume they are non-decreasing and let $zi'_1, \ldots, zi'_{k^2}$ be their corresponding zero-intervals.

What if $P[r_2, c] = 0$? TODO □

Theorem 3.30. *Let P be a pattern. Any one-entry $P[r, c]$ is row-unbounded if (and only if) there is a trivially unbounded one-entry $P[r, c']$ and we cannot apply the fourth case of Lemma 3.17 nor Lemma 3.29 to $P[r, c]$.*

Proof. Without loss of generality, let $P[r, c']$ be part of mapping of P_1, where $P_1[1,2]$ is mapped to it. Let $P_1[2,1]$ be mapped to $P[r_2, c_2]$ and $P_1[3,3]$ be mapped to $P[r_3, c_3]$. We go through all potential one-entries $P[r, c]$ and show that either we can use one of the lemmata mentioned in the statement or the one-entry is row-unbounded.

- $c < c_2$: If there is no one-entry in $P[[r-1], [c-1]]$ nor $P[[r+1, k], [c-1]]$, then the fourth case of Lemma 3.17 can be used for $P[r, c]$. Otherwise, first consider there is a one-entry in $P[[r-1], [c-1]]$, then we can use the construction from Lemma **??**. In the last case, assume there is a one-entry $P[r', c']$ in $P[[r+1, k], [c-1]]$, then if $r'! = r_2$, entries $P[r, c], P[r', c']$ and $P[r_2, c_2]$ form either P_1 or P_2 and $P[r, c]$ is trivially row-unbounded. If $r' = r_2$, then we use $P[r, c], P[r', c']$ and $P[r_3, c_3]$ to again find either P_1 or P_2 and $P[r, c]$ is trivially row-unbounded once again.

- $c = c_2$: If there is no one-entry in $P[[r-1], [c-1]]$ nor $P[[r+1, k], [c-1]]$, then the fourth case of Lemma 3.17 can be used for $P[r, c]$. Otherwise, first assume there is a one-entry in $P[[r-1], [c-1]]$, then we can use the construction from Lemma **??**. In the last case, assume there is a one-entry $P[r', c']$ in $P[[r+1, k], [c-1]]$, then if $r'! = r_3$, entries $P[r, c], P[r', c']$ and $P[r_3, c_3]$ form either P_1 or P_2 and $P[r, c]$ is trivially row-unbounded. If $r' = r_3$, then what?

  Cannot just use lemma even if it was proved.

  TOOD

- $c_2 < c < c_3$: In this case $P[r, c]$ is trivially unbounded as together with $P[r_2, c_2]$ and $P[r_3, c_3]$ it forms P_1.

- $c = c_3$: If there is no one-entry in $P[[r-1], [c+1, l]]$ nor $P[[r+1, k], [c+1, l]]$, then the fourth case of Lemma 3.17 can be used for $P[r, c]$. Otherwise, first consider there is a one-entry in $P[[r-1], [c+1, l]]$, then we can use the construction from Lemma **??**. In the last case, assume there is a one-entry $P[r', c']$ in $P[[r+1, k], [c-1]]$, then if $r'! = r_2$, entries $P[r, c], P[r', c']$ and $P[r_2, c_2]$ form either P_1 or P_2 and $P[r, c]$ is trivially row-unbounded. If $r' = r_2$, then we use the construction from Lemma **??** to show $P[r, c]$ is row-unbounded once again.

- $c > c_3$: There are three cases to go through and we can handle them the same way as we did in case $c < c_2$.

$\square$

Conclusion

Throughout the thesis, we have been looking from multiple angles at classes binary matrices. In particular, we studied properties of matrices containing or avoiding other, smaller matrices as interval minors.

Characterizations We started by describing the structure of matrices avoiding some small patterns. We managed to characterize all matrices avoiding patterns having up to three one-entries and also showed how to generalize some of the characterizations for much bigger patterns. Even for very small patterns (only having four one-entries), the structure of matrices avoiding them became very rich and hard to properly describe.

So instead, we began to consider how a small change of the pattern influences the structure of matrices avoiding the pattern. This was mainly looked at when adding empty lines to the pattern. We showed for patterns of size $k \times 2$ what matrices avoiding its blown version look like, but we are unable to generalize the result (or show something similar) when the pattern we are working with is bigger. So the question, answer for which could later be used to describe complexity of some patterns or help enumerate matrices from some class, is:

Question 3.31. What can we say about matrices avoiding a pattern with added empty line with respect to the matrices avoiding the original pattern?

Operations with matrices After dealing with small patterns, we defined an operation of the direct sum and a closure under the direct and began to explore how it relates with classes of matrices. Once again we started by considering some small cases, where we observed that the closure can be describe by forbidden patterns very naturally. Later, we considered the direct sum with an overlap, which allowed us to restate the characterizations from the previous chapter in a much easier way.

We also introduced a notion of articulation that allowed us to proof a strong result saying that any matrix closed under the direct sum can always be describe by a finite set of forbidden patterns.

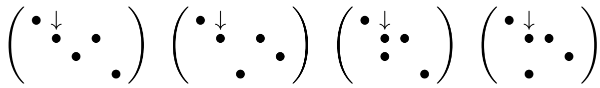
In order to generalize the result, we started looking at sets of matrices and ultimately, we showed that there are set of matrices with finite basis, whose closure cannot be described with a finite number of forbidden minors.

Zero-intervals In the last chapter, we studies a property of a class of matrices that in some terms describes the complexity of inclusion maximal matrices from the class.

To bring the notion back to the pattern avoiding, we defined a pattern P to be bounding if and only of the class $Av_{\preceq}(P)$ is bounded and we showed that a pattern is bounding if and only if it avoids all rotations of $P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$.

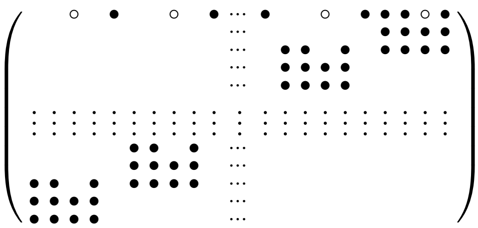
To show for a pattern that it is bounding meant to show that each one-entry it contains is bounded. It may be an interesting generalization to show for each one-entry whether it is bounded or unbounded in its pattern. On our way, we only saw one type of unbounded one-entries, and those are $P_1[2, 1]$ for rows, $P_1[1, 2]$ for columns and corresponding one-entries in the rotations of P_1. Let us call these one-entries *trivially unbounded*.

Considering this generalization, there are one-entries that are unbounded but not trivially unbounded. Let us mention some of them (arrows point to row-unbounded one-entries):



Proposition 3.32. *Let* $P = \begin{pmatrix} & \bullet & \overset{\downarrow}{\bullet} & \\ \bullet & & & \bullet \\ & & \bullet & \end{pmatrix}$. *For every integer* n *there is a matrix* $M \in Av_{max}(P)$ *having at least* n *zero-intervals.*

Proof. Let M be a matrix described by the picture:



We see that $P \not\preceq M$ because we always need to map $P[2,1]$ and $P[3,3]$ to just one "block" of one-entries of M, which only leaves a zero-entry where we need to map $P[1,3]$ or $P[2,4]$.

When we change any marked zero-entry of the first row into a one-entry, we get a matrix containing a minor of $\{1\}^{3\times 4}$; therefore, containing P as an interval minor. In case M is not maximal, we can add more one-entries to make it maximal but it will still contain a row with n one-intervals. $\square$

Our tools are not strong enough to let us characterize unbounded one-entries. Based on our attempts, we state the following conjecture:

Conjecture 3.33. Every row-unbounded one-entry share a row with a trivially row-unbounded one-entry.

Throughout the chapter, we work with arguments such that if something holds for a matrix, it also holds for every submatrix. While it seems completely natural, we are unable to decide the following question:

Question 3.34. Can a non-bounding pattern become bounding after a one-entry is changed to a zero-entry?

Using our machinery, we showed that while the union of bounding sets of patterns is always bounding again, the union of non-bounding sets may become bounding. For the class of matrices avoiding all rotations of P_1, we even showed that every subclass is also bounded. The same remains open for other sets of patterns:

Question 3.35. Is $Av_{\preceq}\left(\begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}\right)$ hereditarily bounded?

Bibliography

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List of Figures