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Hereditary classes of binary matrices

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Abstract: Interval minors of binary matrices were introduced by Jacob Fox in the study of Stanley-Wilf limits. We study what can be implied from their relation to the theory of pattern avoidance of submatrices, which is a very popular area of discrete mathematics. We start be characterizing matrices avoiding small interval minors. We then consider classes of matrices closed under interval minors and with some help of the operation of skew sum, we find classes of matrices that cannot be described by a finite number of forbidden interval minors. We also define and study a variant of a classical extremal Turán-type question studied in the area of combinatorics of permutations and binary matrices and in combinatorial geometry.

Keywords: binary matrix pattern–avoiding interval minor

Dedication.

Contents

1. Introduction

A binary matrix (or 0–1 matrix) is a matrix with ones and zeroes as its entries. In the thesis, we only consider binary matrices and so we omit the word binary. We say that a matrix M contains a matrix P as an interval minor, if P can be created from M by a sequence of deletion of one-entries and merges of neighboring rows or columns. Otherwise, we say M avoids P. To distinguish among matrices and to indicate the relationship, we usually call the matrix P a *pattern*.

When working with matrices, we always index rows from top to bottom and columns from left to right, starting with one. When we speak about a row r, we mean a row with index r. A *line* of a matrix is either a row or a column.

1.1 The main results

While a lot is know about matrices in general, because they can intuitively represent much more complex objects, interval minors are a fairly new topic and so we have a choice of the direction from which we want to approach them.

To get familiar with definitions and pattern avoidance in general, in Chapter 2, we focus on small patterns (having up to four one-entries only) and describe the common structure of matrices avoiding them.

We then turn our focus elsewhere in Chapter 3, and instead of looking for a structure of matrices avoiding a pattern, given a class of matrices (closed under interval minors) we find the smallest set of forbidden patterns that characterizes the class. We introduce the skew sum of two matrices and show that classes of matrices closed under the skew sum can always be described by a finite number of forbidden pattern. Using the operation more, we show that there are also other classes for which this cannot be achieved.

Because it is very useful to study extremal questions like the maximum number of one-entries of a matrix from a given class of matrices, in Chapter 4, we study a variant of such complexity question, where we instead focus on the maximum number k of appearances of pairs "01" and "10" on a single line of a matrix from a given class of matrices. We show that even for classes that are described by just one forbidden pattern, k can be unbounded, and we characterize exactly for which pattern this holds. Then we generalize the approach and show what influence an intersection of classes has on the number k.

1.2 Preliminaries

Notation 1.1. For $n \in \mathbb{N}$, let $[n] := \{1, 2, \ldots, n\}$ and for $m \in \mathbb{N}$ such that $n \leq m$, let $[n, m] := \{n, n+1, \ldots, m\}$.

Notation 1.2. For a matrix $M \in \{0, 1\}^{m \times n}$, $R \subseteq [m]$ and $C \subseteq [n]$, let $M[R, C]$ denote a submatrix of M induced by row indices in R and column indices in C. Furthermore, for $r \in [m]$ and $c \in [n]$, let $M[r, c] := M[\{r\}, \{c\}]$.

The pattern avoidance for matrices is a generalization of a long studied theory of pattern avoidance for permutations. There are two generally used ways to define this generalization, either we avoid a matrix pattern as a submatrix or as an interval minor. While this thesis works almost exclusively with the latter, to better introduce the whole area, we start by defining the more know of the two approaches.

Definition 1.3. We say a matrix $M \in \{0,1\}^{m\times n}$ *contains* a pattern $P \in \{0,1\}^{k\times l}$ *as a submatrix* and denote it by $P \leq M$ if there are $R \subseteq [m]$ and $C \subseteq [n]$ such that $M' = M[R,C] \in \{0,1\}^{k\times l}$ and for every $r \in R$ and $c \in C$, if $P[r,c] = 1$ then $M'[r,c] = 1$.

Every matrix $M \in \{0,1\}^{m\times n}$ can be looked at as an adjacency matrix of a bipartite graph G_M with two sets of vertices $V_1 = [m]$ and $V_2 = [n]$ such that a vertex i from V_1 is adjacent to a vertex j from V_2 if and only if $M[i,j] = 1$. The order of vertices in each set is fixed and these graphs are usually called ordered bipartite graphs. In this setting, a matrix M contains a pattern P if the ordered bipartite graph G_P is a subgraph (not necessarily induced) of the ordered bipartite graph G_M.

In graph theory, the next step is to look at graph minors. A minor is created from a graph by a repeated applying of one of three graph operations: deletion of a vertex, deletion of an edge and a contraction of an edge. The same can be represented in terms of matrices:

Definition 1.4. We say a matrix $M \in \{0,1\}^{m\times n}$ *contains* a pattern $P \in \{0,1\}^{k\times l}$ *as an interval minor* and denote it by $P \preceq M$ if there is a sequence of elementary operations that applied to M creates P. The elementary operations are:

- a deletion of a line,
- a deletion of a one-entry (a change of a one-entry to a zero-entry) and
- a merge of two neighboring rows or columns into one that is the elementwise OR of the two original lines.

For simplicity, we do not consider a deletion of a line to be a separate operation as it can be replaced by a merge of the corresponding line with a neighboring one and a series of changes of one-entries to zero-entries. Moreover, like in the realm of graphs, we can assume all merging operations are done before the deletion of one-entries. This give us an alternative way to look at the problem.

Definition 1.5. Consider matrices P and M and let $P \preceq M$. A *mapping* of P to M is a function that maps each row of P to an interval of rows of M and each column of P to an interval of columns of M in such a way that if $P[r,c] = 1$ and r is mapped to R and c is mapped to C, there is a one-entry in $M[R,C]$. An *interval of rows* (columns) is a set of consecutive rows (columns). We say that an entry $P[r,c]$ is mapped to an entry $M[r',c']$ in a fixed mapping of P to M, in which r is mapped to R and c is mapped to C, if $r' \in R$ and $c' \in C$ and if $P[r,c] = 1$ then we also require $M[r',c'] = 1$.

Each mapping of a pattern P to a matrix M corresponds to a *partitioning* of M to intervals of rows and columns that creates a block structure. On the other hand, if we find a partitioning of M to blocks such that for each one-entry $P[r,c]$ there is a one-entry in the block that can be indexed $[r,c]$ then we have a mapping of P to M and so $P \preceq M$. This means:

Observation 1.6. *For all matrices P and M, there is a mapping of P to $M \Leftrightarrow P \preceq M$.* $\square$

While pattern avoidance in terms of submatrices and interval minors seem to be very different, they have a quite tight relationship. The next observation immediately follows from their definitions.

Observation 1.7. *For all matrices P and M, $P \leq M \Rightarrow P \preceq M$.*

As said at the beginning of the section, both approaches generalize pattern avoidance for permutations and so it makes sense that they are equal for permutation matrices – matrices having exactly one one-entry in each line.

Observation 1.8. *For all matrices P and M, if P is a permutation matrix then $P \leq M \Leftrightarrow P \preceq M$.*

Proof. If we have $P \preceq M$, then there is a mapping m of P to M. To show $P \leq M$ we need to find R, C such that $M' = M[R, C]$ has the same size as P and for every $P[r, c] = 1$ it holds $M'[r, c] = 1$. We define R and C as follows. For every row r, let R' be the interval to which r is mapped in the mapping m. There is exactly one column c such that $P[r, c] = 1$ and c is mapped to some C'. Because m is a mapping, there is a one-entry $M[r', c']$ such that $r' \in R'$ and $c' \in C'$ and we add r' to R and we add c' to C.

The other implication follows from Observation 1.7. $\square$

Definition 1.9. A *class* of matrices $\mathcal{M}$ is a set of matrices that is closed under interval minors. It means that for every $M \in \mathcal{M}$ and every $M' \preceq M$ it holds $M' \in \mathcal{M}$.

To avoid degenerate cases, we only consider classes of matrices containing at least one matrix of size 2×1, at least one matrix of size 1×2 and at least one matrix that is non-empty.

Definition 1.10. Let $\mathcal{P}$ be a set of patterns. We denote by $Av_{\preceq}(\mathcal{P})$ the set of all matrices that avoid each $P \in \mathcal{P}$ as an interval minor.

Observation 1.11. *For all patterns P and P': $P \preceq P' \Leftrightarrow Av_{\preceq}(P) \subseteq Av_{\preceq}(P')$.*

Proof. Because $P \preceq P'$, every matrix that avoids P also avoids P'. On the other hand, if $P \not\preceq P'$ then $P' \in Av_{\preceq}(P)$. As $P' \notin Av_{\preceq}(P')$, we have $Av_{\preceq}(P) \not\subseteq Av_{\preceq}(P')$. $\square$

The following observation goes almost without saying and we use it throughout the whole thesis to break symmetries.

Observation 1.12. *Let P and M be matrices, $P \preceq M \Leftrightarrow P^T \preceq M^T$.*

1.3 Pattern avoidance

Pattern avoidance is a general topic in combinatorics. A lot of attention is directed towards permutations, see books Bóna [2012], Kitaev [2011] for references. It is a natural generalization to regard permutations as permutation matrices and consider matrix avoidance. This is mainly studied in terms of submatrices, so we discuss some interesting results in this section.

Interval minors are, on the other hand, a fairly new topic first defined by Jacob Fox in Fox [2013] as a tool to prove results about permutations in the study of Stanley–Wilf limits. Since then, a little has been discovered about the theory of interval minors. Nevertheless, we mention some results at the end of this section.

Let us go back to submatrices for now. The question that is particularly interesting is to determine the maximum number of one-entries that a matrix avoiding a given pattern can have. This property describes complexity of a pattern and can be used for example to prove algorithmic complexity, see Efrat and Sharir [1996].

Definition 1.13. Let M be a matrix. The weight of M, denoted by $|M|$, is the number of one-entries in M.

Definition 1.14. For a pattern P and integers m, n, we define the *weight extremal function* $Ex(P, m, n) := \max\{|M|;\ M \in \{0,1\}^{m\times n} \wedge P \not\leq M\}$.

Going back to the representation of the problem in terms of ordered bipartite graphs, the question to determine $Ex(P, m, n)$ is a variant of a classical Turán extremal graph question and was studied by many authors, see for example Tardos [2005], Füredi and Hajnal [1992] or, for a wider range of variants Brass et al. [2003], Claesson et al. [2012], Klazar [2004], Pach and Tardos [2006]. Some applications associated with the weight extremal function are discussed in Fulek [2009]. There are other extremal functions that have been studied, see for instance Cibulka and Kynčl [2016], but we do not consider them in this thesis.

In the same spirit, we also define the weight extremal function for matrices avoiding patterns as interval minors.

Definition 1.15. For a pattern P and integers m, n, we define $Ex_{\preceq}(P, m, n) := \max\{|M|;\ M \in \{0,1\}^{m\times n} \wedge P \not\preceq M\}$.

Thanks to Observation 1.7 we have the following relationship between the extremal functions.

Observation 1.16. *For all patterns P and integers m, n:*
$Ex_{\preceq}(P, m, n) \leq Ex(P, m, n)$. □

From Observation 1.11 it follows:

Observation 1.17. *For all patterns P and P' and integers m, n: $P \preceq P' \Rightarrow Ex_{\preceq}(P, m, n) \leq Ex_{\preceq}(P', m, n)$.*

It was showed in Marcus and Tardos [2004] that for every permutation matrix P and every n it holds $Ex(P, n, n) \leq c_P n$. While $Ex(P, n, n)$ can become even quadratic with n, because of the previous observation and the fact that every pattern $P \in \{0,1\}^{k\times l}$ is an interval minor of some permutation pattern $P' \in \{0,1\}^{(kl)\times(kl)}$ we have the following:

Proposition 1.18. *For every pattern P and integer n: $Ex_{\preceq}(P,n,n) \leq c_P n$ for some constant c_P independent of n.* $\square$

The following observation for $Ex(P,m,n)$ was made by several authors; see for example Cibulka [2009], Fulek [2009].

Lemma 1.19. *If $P \in \{0,1\}^{k\times l}$ has at least one one-entry, then*

$$Ex(P,m,n) \geq \begin{cases} m \cdot n & k > m \text{ or } l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Moreover, the same holds for $Ex_{\preceq}(P,m,n)$.

Proof. If $k > m \vee l > m$, we have $P \npreceq \{1\}^{m,n}$. Otherwise, let $P[r,c] = 1$ and consider Figure 1.1. Consider a matrix M such that the first $r-1$ rows, the last $k-r$ rows, the first $c-1$ column and the last $l-c$ column contain no zero-entry and the rest is empty. Then $P \nleq M$ and even $P \npreceq M$. We can also see that $|M| = mn - (m-k+1)(n-l+1) = (l-1)m + (k-1)n - (k-1)(l-1)$. $\square$

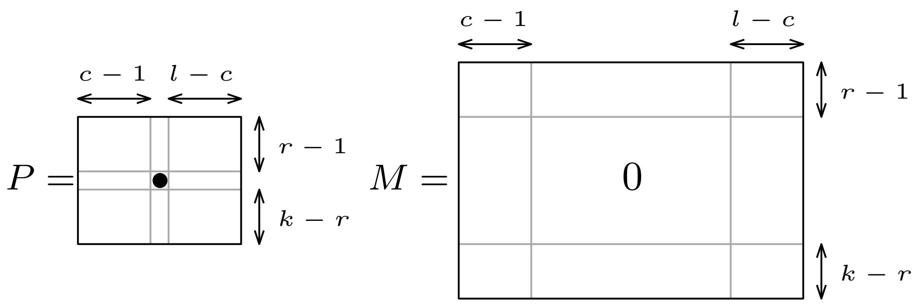


Figure 1.1: An example of a matrix M avoiding a pattern P as an interval minor.

The following definition is due to Cibulka [2013].

Definition 1.20. A pattern $P \in \{0,1\}^{k\times l}$ is *(strongly) minimalist* if

$$Ex(P,m,n) = \begin{cases} m \cdot n & k > m \text{ or } l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

We use the adjective "strongly" to further distinguish minimalist pattern from weakly minimalist patterns defined next.

Definition 1.21. A pattern $P \in \{0,1\}^{k\times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P,m,n) = \begin{cases} m \cdot n & k > m \text{ or } l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

From Observation 1.16, we immediately have:

Observation 1.22. *If a pattern P is strongly minimalist then P is weakly minimalist.*

The following result is a simplification of a lemma from Cibulka [2013].

Fact 1.23. *1. The pattern $(\bullet)$ is strongly minimalist.*

2. *If a pattern $P \in \{0,1\}^{k \times l}$ is strongly minimalist and there is a one-entry in the last row of P in the c-th column, then $P' \in \{0,1\}^{k+1 \times l}$ created from P by appending as the last row a new row having a one-entry only in the c-th column is strongly minimalist.*

3. *If a pattern P having at least two one-entries is strongly minimalist, then after changing a one-entry to a zero-entry it is still strongly minimalist.*

The following two facts come from Mohar et al. [2015]. In the article, a slightly different definition of an interval minor is used, so we show here the proofs in our setting.

Fact 1.24 (Mohar et al. [2015])**.** *Let $P = \{1\}^{2 \times l}$ be a pattern, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m \times n}$ be a matrix avoiding $P = \{1\}^{2 \times l}$ as an interval minor and let A_i be the set of column indices j such that both $M[[i], \{j\}]$ and $M[[i+1, m], \{j\}]$ are non-empty. Clearly, $|A_i| \le l - 1$; otherwise, $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M appears in at least $b_j - 1$ of sets A_i, $1 \le i \le m-1$. It follows that

$$|M| = \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} (b_j - 1) + n \le \sum_{i=1}^{m-1} |A_i| + n \le (l-1)(m-1) + n. \qquad \square$$

This result shows an example of a weakly minimalist matrix that is not strongly minimalist. Consider a matrix $\left(\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix}\right)$. It is, thanks to Fact 1.24 weakly minimalist, but it is known due to Brown [1966] that it is not strongly minimalist.

Fact 1.25 (Mohar et al. [2015])**.** *Let $P = \{1\}^{3 \times l}$ be a pattern, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m \times n}$ be a matrix avoiding $P = \{1\}^{3 \times l}$ as an interval minor and let A_i be a set of column indices j such that both $M[[i-1], \{j\}]$ and $M[[i+1, m], \{j\}]$ are non-empty and $M[i,j] = 1$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M (for which $b_j \ge 2$) appears in exactly $b_j - 2$ of sets A_i, $2 \le i \le m-1$. It follows that

$$|M| = \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} (b_j - 2) + 2n \le \sum_{i=2}^{m-1} |A_i| + 2n \le (l-1)(m-2) + 2n. \qquad \square$$

We now show that the third part of Fact 1.23 is also safe for weakly minimalist patterns.

Lemma 1.26. *Let $P \in \{0,1\}^{k \times l}$ be a weakly minimalist pattern having at least two one-entries. Then a pattern P' created from P be deletion of a one-entry is also weakly minimalist.*

Proof. For contradiction, consider a matrix $M \in \{0,1\}^{m \times n}$ avoiding P' as an interval minor such that $|M| > (k-1)n + (l-1)m - (k-1)(l-1)$. The matrix M also avoids P; as otherwise, we have $P' \preceq P \preceq M$. That is a contradiction with P being weakly minimalist. $\square$

As a result, we have the following corollary:

Corollary 1.27. *Every non-empty pattern P that has at most three rows (or columns) is weakly minimalist.*

In Cibulka [2009], the author shows that for every $k \geq 1$ there is a $2k \times 2k$ permutation pattern for which $Ex[P, n] \geq k^2 n$. Because of Observation 1.8, the same construction shows that for $k \geq 2$ the patterns are not weakly minimalist. It means that the previous results cannot be easily extended. On the other hand, in Mao et al. [2015] the authors show some form of generalization and also other bounds regarding interval minors and their weight extremal function.

2. Small interval minors

Our goal in this chapter is to describe, for a given small pattern, the structure of matrices avoiding it as an interval minor.

Algorithmically speaking, deciding whether a pattern is contained in a matrix is NP-hard, even if both matrices are permutation matrices, see Bose et al. [1998]. We do not consider complexity questions here, but for small patterns, we show that matrices avoiding them have a quite simple structure. However, the structure gets significantly more complex as soon as we allow the pattern to contain at least four one-entries.

To go through cases efficiently, we first show that to some extent, we can assume, without loss of generality, there are no empty lines in studied patterns.

Before we dive into characterizations, let us introduce some useful notion.

Definition 2.1. A *walk* in a matrix M is a contiguous sequence of its entries, beginning in the top-left corner and ending in the bottom-right one. If $M[i,j]$ occurs in the sequence, its successor is either $M[i+1,j]$ or $M[i,j+1]$. Symmetrically, a *reverse walk* in M is a contiguous sequence of its entries, beginning in the top-right corner and ending in the bottom-left one.

Definition 2.2. We say a matrix M is a *walking matrix* if there is a walk in M containing all one-entries.

Definition 2.3. For a matrix $M \in \{0,1\}^{m\times n}$ and integers r, c, we say $M[r,c]$ is

- *top-left empty*, if $M[[r-1],[c-1]]$ is an empty matrix,
- *top-right empty*, if $M[[r-1],[c+1,n]]$ is empty,
- *bottom-left empty*, if $M[[r+1,m],[c-1]]$ is empty,
- *bottom-right empty*, if $M[[r+1,m],[c+1,n]]$ is empty.

Definition 2.4. For a matrix $M \in \{0,1\}^{m\times n}$ and integers r, c, we say that an entry $M[r,c]$ is *bottom-left extreme*, if it is bottom-left empty and the submatrix $M[[r,m],[c]]$ is not empty. Similarly, $M[r,c]$ is *bottom-right extreme* if it is bottom-right empty and the submatrix $M[[r,m],[c,n]]$ is not empty. A walk in M is *bottom-left extreme* if it contains all bottom-left extreme elements of M. A reverse walk in M is *bottom-right extreme* if it contains all bottom-right extreme elements of M.

It is easy to see that there is exactly one bottom-left extreme walk and exactly one bottom-right extreme walk in every non-empty matrix.

Definition 2.5. For matrices $M \in \{0,1\}^{m\times n}$ and $N \in \{0,1\}^{m\times l}$, we define $M \to N \in \{0,1\}^{m\times(n+l)}$ to be the matrix created from M by appending the columns of N at the end of M.

2.1 Empty rows and columns

From the definition of matrix containment, zero-entries of the pattern pose no restrictions on the tested matrix, so, intuitively, adding new empty lines to a pattern should not influence the structure of matrices avoiding the pattern by much.

We first show that adding empty lines as first or last lines of the pattern indeed does next to no difference. On the other hand, inserting empty lines in between non-empty lines becomes a bit more tricky and we only describe what happens when we extend a pattern of size $k \times 2$ (or symmetrically $2 \times k$) by a single empty column (row).

Observation 2.6. *For matrices $P \in \{0,1\}^{k \times l}$ and $M \in \{0,1\}^{m \times n}$, let $P' = P \to \{0\}^{k \times 1}$ and let $M' = M \to \{1\}^{m \times 1}$. Then $P \preceq M \Leftrightarrow P' \preceq M'$.*

Proof. $\Rightarrow$ The last column of P' can always be mapped just to the last column of M' and $P'[[k],[l]]$ can be mapped to $M'[[m],[n]]$ the same way P is mapped to M.

$\Leftarrow$ Taking the restriction of the mapping of P' to M', we get a mapping of P to M. $\square$

The analogous proof can be also used to characterize matrices avoiding patterns after adding an empty column as the first column or an empty row as the first or the last row. Using induction, we can easily show that a pattern P' is avoided by a matrix M' if and only if P is avoided by M, where P is derived from P' by excluding all empty leading or ending rows and columns and M is derived from M' by excluding the same number of leading or ending rows and columns. Therefore, when characterizing matrices avoiding a forbidden pattern, we do not to consider patterns having empty rows or columns on their boundary.

We now show what happens when we add an empty column in between two columns of a pattern that only has two columns. It is going to be achieved by employing a notion of intervals of one-entries. More about these intervals and their counterpart – zero-intervals can be found in the last chapter of the thesis.

Definition 2.7. A *one-interval* of a matrix M is a sequence of consecutive one-entries of a single line of M bounded from each side by a zero-entry or the edge of the matrix.

Definition 2.8. For a class of matrices $\mathcal{M}$, a matrix $M \in \mathcal{M}$ is *critical* in $\mathcal{M}$ if after a change of any zero-entry to one-entry the matrix no longer belongs to $\mathcal{M}$. For a pattern P, we denote by $Av_{crit}(P)$ the set of all matrices critical in $Av_{\preceq}(P)$.

Lemma 2.9. *Let $P \in \{0,1\}^{k \times 2}$ be a pattern and let $M \in Av_{crit}(P)$ be a matrix, then M contains at most one one-interval in each row.*

Proof. For contradiction, assume there are at least two one-intervals in a row of M. Because M is critical in $Av_{\preceq}(P)$, changing any zero-entry e in between one-intervals o_1 and o_2 creates a mapping of the forbidden pattern. Such a mapping uses the changed one-entry to map some element $P[r',1]$ or $P[r',2]$.

In the first case, the same mapping also maps P to M if we use a one-entry from o_1 instead of e; thus, $P \not\preceq M$ and we reach a contradiction. In the second case, the mapping can use a one-entry from o_2 instead of e; therefore, we again get a contradiction with $P \not\preceq M$. Since e is not usable for any one-entry of P, we can change it to a one-entry and get a contradiction with M being critical. $\square$

Lemma 2.10. *Let $P \in \{0,1\}^{k\times 3}$ be a pattern such that its middle column is empty. Every row of any matrix $M \in Av_{crit}(P)$ is either empty or it contains a single one-interval of length at least 2 (or length m if $m < 2$).*

Proof. Let a matrix $M \in Av_{crit}(P)$. The same proof as in Lemma 2.9 shows that there is at most one one-interval in each row of M. For contradiction, let there be only one one-entry $M[r,c]$ in a row r:

- $c = 1$: we can set $M[r,c+1] = 1$ and the matrix still avoids Pl, which is a contradiction with M being critical in $Av_{\preceq}(P)$.
- $c = n$: we can set $M[r,c-1] = 1$ and the matrix still avoids P, which is a contradiction with M being critical in $Av_{\preceq}(P)$.
- otherwise: consider zero-entries $e_l = M[r,c-1]$ and $e_r = M[r,c+1]$. For contradiction, assume we can change neither e_l nor e_r to a one-entry without creating a mapping of the pattern. It means that if we set $e_l = 1$ then some $P[r_1,1]$ can be mapped to it. Let m_l be the corresponding mapping. At the same time, if we set $e_r = 1$ then some $P[r_2,3]$ can be mapped to it and m_r is the corresponding mapping. We show that the two mappings can be combined to a mapping of P to M, giving a contradiction.

  Without loss of generality, in both mappings, the empty column of P is mapped exactly to the column c of M. We need to describe how to partition M into k rows. Consider Figure 2.1:

  - $r_1 \neq r_2$: Without loss of generality, we assume $r_1 > r_2$. Let r_3 be the first row of the interval where the row r_1 is mapped in m_l and let r_4 be the last row of the interval where the row r_1 is mapped in m_r. From the mapping m_l, we know that the first $r_1 - 1$ rows of P can be mapped to rows $[1, r_3 - 1]$ and from the mapping m_r, we know that the last $k - r_1$ rows of P can be mapped to rows $[r_4 + 1, m]$. From the mapping m_r, we know that the row r_1 can be mapped to rows $[r_3, r_4]$; thus, we have a mapping of P to M.
  - $r_1 = r_2$: Let $[r_3, r_4]$ be the interval where the row r_1 is mapped in m_l and let $[r_5, r_6]$ be the interval where the row r_1 is mapped in m_r. Without loss of generality, let $r_3 < r_5$. From the mapping m_l, we know that the first $r_1 - 1$ rows of P can be mapped to rows $[1, r_3 - 1]$. Without loss of generality, let $r_4 < r_6$. From the mapping m_r, we know that the last $k - r_1$ rows of P can be mapped to rows $[r_6 + 1, m]$. Therefore, we can map the row r_1 of P to the row interval $[r_3, r_6]$ without using one-entries e_l and e_r.

We showed that either e_l or e_r can be changed to a one-entry, which is a contradiction with M being critical in $Av_{\preceq}(P)$. $\square$

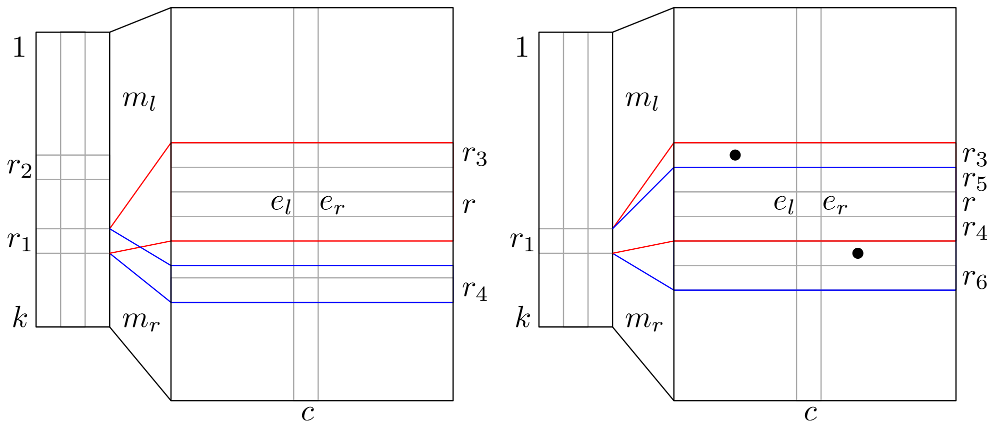


Figure 2.1: Red and blue lines representing mappings m_l and m_r of the forbidden pattern. The two horizontal lines show the boundaries of the mapping of the row r and the vertical lines show the boundaries of the mapping of the column c.

Similarly, we can prove that for every pattern $P \in \{0,1\}^{k \times l}$ such that all $(l-2)$ middle columns are empty, every matrix from $Av_{crit}(P)$ that contains at least l one-entries in each row, contains at least $l+1$ one-entries in each row. On the other hand, it cannot be generalized further, as we show in the following proposition.

Proposition 2.11. *For every integer $l > 3$, there exists a pattern $P \in \{0,1\}^{k \times l}$ such that all $(l-2)$ middle columns are empty and there exists a matrix $M \in Av_{crit}(P)$ containing a row with a single one-entry.*

Proof. We only show the construction for $l = 4$ and $l = 5$ because the first construction can easily be extended for every even l and the latter for every odd l. For $l \in \{3,4\}$, let P_l be the forbidden pattern and $M_l \in Av_{crit}(P)$ be the critical matrix that has a single one-entry in some row:

$$P_4 = \begin{pmatrix} \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \end{pmatrix} \quad M_4 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \bullet \end{pmatrix} \quad P_5 = \begin{pmatrix} \circ & \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ & \circ \end{pmatrix} \quad M_5 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \bullet & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

It is easy to check that $M_l \in Av_{\preceq}(P_l)$ and that changing a zero-entry to a one-entry creates a mapping of the forbidden pattern. □

Theorem 2.12. *Let $P \in \{0,1\}^{k \times 2}$ be a pattern and let $P' \in \{0,1\}^{k \times 3}$ be the pattern created from P by appending a new empty column in between the two columns of P. For all matrices $M \in \{0,1\}^{m \times n}$ it holds $M \in Av_{\preceq}(P') \Leftrightarrow$ there exists a matrix $N \in \{0,1\}^{m \times (n-1)}$ such that $N \in Av_{crit}(P)$ and M is a submatrix of the elementwise OR of $N \to \{0\}^{m \times 1}$ and $\{0\}^{m \times 1} \to N$.*

Proof. $\Rightarrow$ Without loss of generality, let the matrix M be critical in $Av_{\preceq}(P')$. We know from Lemma 2.10 that each row of M contains either no one-entry or a single one-interval of length at least 2. Let a matrix N be created from M by deletion of the last one-entry from each row and deletion of the last column. Clearly, M is equal to the elementwise OR of $N \to \{0\}^{m \times 1}$ and $\{0\}^{m \times 1} \to N$. If $P \preceq N$ then each mapping of P to N can be extended to a mapping of P' to M by mapping each $P'[r_1, 1]$ to the same one-entry where $P[r_1, 1]$ is mapped in $N \to \{0\}^{m \times 1}$ and mapping each $P'[r_2, 3]$ to the same one-entry where $P[r_2, 2]$ is mapped in $\{0\}^{m \times 1} \to N$.

$\Leftarrow$ Let a matrix M be equal to the elementwise OR of $N \to \{0\}^{m\times 1}$ and $\{0\}^{m\times 1} \to N$. For contradiction, assume $P' \preceq M$ and consider any mapping of P' to M. Without loss of generality, one-entries of the first column of P' are mapped to those one-entries of M created from $N \to \{0\}^{m\times 1}$. If there is a one-entry $P'[r, 1]$ mapped to a one-entry of M not created from $N \to \{0\}^{m\times 1}$, we just take the first one-entry in the row instead. Symmetrically, all one-entries of the last column of P' are mapped to one-entries created from $\{0\}^{m\times 1} \to N$. The same one-entries of N can be used to map P to N, which is a contradiction. $\square$

The symmetric characterization also holds when adding an empty row to a pattern that only has two rows. We can see in the following proposition that the straightforward generalization of the statement for bigger patterns does not hold.

Proposition 2.13. *There exists a matrix $P \in \{0,1\}^{k\times l}$ such that for each pattern $P' \in \{0,1\}^{k\times(l+1)}$ created from P by appending a new empty column in between the two existing columns, there exists a matrix $N \in Av_{\preceq}(P)$ such that the elementwise OR of $N \to \{0\}^{m\times 1}$ and $\{0\}^{m\times 1} \to N$ contains P' as an interval minor.*

Proof. Later in this chapter, we characterize the class of matrices avoiding pattern $\begin{pmatrix} \bullet & \bullet & \bullet \\ & \bullet & \end{pmatrix}$. See Proposition 2.23. Let $N \in Av_{\preceq}\left(\begin{pmatrix} \bullet & \bullet & \bullet \\ & \bullet & \end{pmatrix}\right)$ be any matrix containing $\begin{pmatrix} \bullet & & \bullet \\ & \bullet & \end{pmatrix}$ as an interval minor. Let a matrix M be equal to the elementwise OR of $N \to \{0\}^{m\times 1}$ and $\{0\}^{m\times 1} \to N$. Then $\begin{pmatrix} \bullet & \circ & \bullet & \bullet \\ & \circ & \bullet & \end{pmatrix} \preceq M$ and $\begin{pmatrix} \bullet & \bullet & \circ & \bullet \\ & \bullet & \circ & \end{pmatrix} \preceq M$. $\square$

Next, we describe the structure of matrices avoiding certain small patterns. We restrict ourselves to patterns with no empty lines. If $P \not\preceq M$ then also $P^\top \not\preceq M^\top$ and this holds for all rotations and mirrors of P and M and so we only mention these symmetries.

2.2 Patterns having two one-entries

These are, up to rotation, the only patterns having two one-entries and no empty lines:

$$P_1 = \begin{pmatrix} \bullet & \bullet \end{pmatrix} \quad P_2 = \begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$$

They can be generalized to:

$$P_1' = \begin{pmatrix} \bullet & \bullet & \cdots & \bullet & \bullet \end{pmatrix} \quad P_2' = \begin{pmatrix} & & & & \bullet \\ & & & \bullet & \\ & & \iddots & & \\ & \bullet & & & \\ \bullet & & & & \end{pmatrix}$$

Proposition 2.14. *Let $P_1' = \{1\}^{1\times k}$. For all matrices M: $P_1' \not\preceq M \Leftrightarrow M$ has at most $k-1$ non-empty columns.*

Proof. $\Rightarrow$ When a matrix M contains one-entries in k columns, then these give us a mapping of P_1'.

$\Leftarrow$ A matrix M having at most $k-1$ non-empty columns avoids P_1'. $\square$

Proposition 2.15. *Let $P_2' \in \{0,1\}^{k \times k}$. For all matrices M: $P_2' \not\preceq M \Leftrightarrow$ there are $k-1$ walks in M such that each one-entry of M belongs to at least one walk.*

Proof. $\Rightarrow$ When all one-entries of a matrix M cannot fit into $k-1$ walks, then there are k one-entries such that no pair can fit to a single walk and those give us a mapping of P_2'.

$\Leftarrow$ A matrix M containing one-entries in at most $k-1$ walks avoids P_2'. $\square$

2.3 Patterns having three one-entries

These are, up to rotation and mirroring, the only patterns having three one-entries and no empty lines that we did not characterize so far:

$$P_3 = \begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix} \quad P_4 = \begin{pmatrix} \bullet & & \bullet \\ & \bullet & \end{pmatrix} \quad P_5 = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \end{pmatrix} \quad P_6 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$$

Proposition 2.16. *For all matrices $M \in \{0,1\}^{m \times n}$: $P_3 \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 2.2):*

- *$M[r,c]$ is top-left, top-right and bottom-left empty, and*
- *$M[[r,m],[c,n]]$ is a walking matrix.*

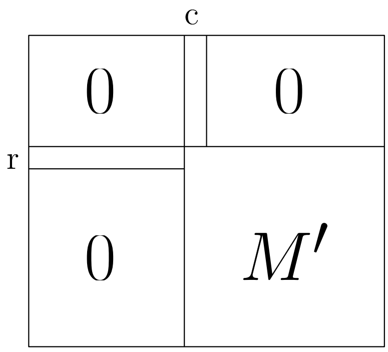


Figure 2.2: The characterization of matrices avoiding $\begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix}$ as an interval minor. The matrix M' is a walking matrix.

Proof. $\Rightarrow$ If M is a walking matrix then we set $r = c = 1$. Otherwise, there are one-entries $M[r,c']$ and $M[r',c]$ such that $r' < r$ and $c' < c$. If an entry $M[r,c]$ is not top-left, top-right or bottom-left empty then $P_3 \preceq M$. If the submatrix $M[[r,m],[c,n]]$ is not a walking matrix then it contains $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ and together with the one-entry $M[r,c']$ it gives us a mapping of P_3.

$\Leftarrow$ For contradiction, assume that a matrix M described in Figure 2.2 contains P_3 as an interval minor. Without loss of generality, let $P_3[1,1]$ be mapped to a one-entry in the r-th row. Then both $P_3[1,2]$ and $P_3[2,1]$ need to be mapped to M', which is a contradiction with it being a walking matrix. $\square$

Proposition 2.17. *For all matrices M: $P_4 \not\preceq M \Leftrightarrow$ there are matrices M_1, M_2 such that $M = M_1 \to M_2$, $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix} \not\preceq M_1$ and $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \not\preceq M_2$.*

Proof. $\Rightarrow$ Let $e = M[r, c]$ be an arbitrary top-most one-entry in M. It holds $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix} \not\preceq M[[m], [c-1]]$; otherwise, we have a mapping of P_4 to M. If we also have $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \not\preceq M[[m], [c, n]]$ then we are done. For contradiction, let e_1, e_2 be any two one-entries forming $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ in $M[[m], [c, n]]$. Symmetrically, let e'_1, e'_2 be any two one-entries forming $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix}$ in $M[[m], [c]]$. Without loss of generality, let e_2 be no higher than e'_2 and then, together with e'_1 and e_1 it gives us a mapping of P_4 to M, giving a contradiction.

$\Leftarrow$ For contradiction, let $P_4 \preceq M$ and consider an arbitrary mapping. Consider the one-entry of M, where $P_4[2, 2]$ is mapped. If it is in M_1 then $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix} \preceq M_1$ and we get a contradiction. Otherwise, we have $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \preceq M_2$, which is again a contradiction. $\square$

Proposition 2.18. *For all matrices $M \in \{0, 1\}^{m \times n}$: $P_5 \not\preceq M \Leftrightarrow$ for every one-entry $M[r, c]$ on the bottom-left extreme walk w, there is at most one non-empty column in $M[[r-1], [c+1, n]]$.*

Proof. $\Rightarrow$ For contradiction, assume there is a one-entry $M[r, c]$ on w such that there are two non-empty columns in $M[[r-1], [c+1, n]]$. Then a one-entry from each of those columns and $M[r, c]$ together give us a mapping of P_5 to M, and a contradiction.

$\Leftarrow$ For contradiction, let $P_5 \preceq M$ and consider any such mapping. Without loss of generality, $P_5[2, 1]$ is mapped to a one-entry $M[r, c]$ from w. Then $\begin{pmatrix}\bullet & \bullet\end{pmatrix} \preceq M[[r-1], [c+1, n]]$, which is a contradiction with it having one-entries in at most one column. $\square$

Proposition 2.19. *For all matrices M: $P_6 \not\preceq M \Leftrightarrow$ for every one-entry $M[r, c]$ on the bottom-right extreme reverse walk w, $M[[r-1], [c-1]]$ is a walking matrix.*

Proof. $\Rightarrow$ For contradiction, assume there are integers r, c such that $M[r, c]$ is a one-entry on w and $M[[r-1], [c-1]]$ is not a walking matrix. It means that $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix} \preceq M[[r-1], [c-1]]$ and together with $M[r, c]$ it gives us a mapping of the forbidden pattern, and a contradiction.

$\Leftarrow$ For contradiction, let $P_6 \preceq M$ and consider an arbitrary mapping of P_6. Without loss of generality, let $P_6[3, 3]$ be mapped to some one-entry $M[r, c]$ on w. Then, $M[[r], [c]]$ is not a walking matrix and we have a contradiction. $\square$

2.4 Patterns having four one-entries

These are some of the patterns having four one-entries and no empty lines that we did not characterize so far:

$$P_7 = \begin{pmatrix}\bullet & \bullet \\ \bullet & \bullet\end{pmatrix} \quad P_8 = \begin{pmatrix}\bullet & \bullet & \bullet \\ & \bullet & \end{pmatrix} \quad P_9 = \begin{pmatrix}\bullet & & & \\ & & \bullet & \\ & \bullet & & \\ & & & \bullet\end{pmatrix}$$

Lemma 2.20. *For any matrix M: $P_7 \not\preceq M \Rightarrow$ there exist integers r, c such that $M[r, c]$ is either*

1. *a one-entry and $(r, c) \in \{(1, 1), (1, n), (m, 1), (m, n)\}$, or*
2. *top-right and bottom-left empty and $(r, c) \notin \{(1, 1), (m, n)\}$, or*
3. *top-left and bottom-right empty and $(r, c) \notin \{(1, n), (m, 1)\}$.*

Proof. If there is a one-entry in any corner then the first condition is satisfied. Otherwise, consider the entry $M[2, 1]$. It is trivially bottom-left empty and if there is no one-entry in the first row of M then the second condition is satisfied. Therefore, let $M[1, c_t]$ be a one-entry in the first row. Symmetrically, let $M[m, c_b]$ be a one-entry in the last row, let $M[r_l, 1]$ be a one-entry in the first column and let $M[r_r, n]$ be a one-entry in the last column.

It cannot at the same time happen that $c_t < c_b$ and $r_r > r_l$ (or symmetrically $c_t > c_b$ and $r_r < r_l$), because then $P_7 \preceq M$. Without loss of generality, let $c_t \geq c_b$ and $r_r \geq r_l$. The submatrix $M[[r_r - 1], [c_t + 1, n]]$ is empty; otherwise, any one-entry there, together with $M[1, c_t], M[m, c_b]$ and $M[r_r, 1]$ forms the forbidden pattern. Similarly, the matrix $M[[r_r + 1, m], [c_t - 1]]$ is also empty. Thus $M[r_t, c_t]$ is top-right and bottom-left empty and it is not a corner, satisfying the second condition. $\square$

Proposition 2.21. *For all matrices $M \in \{0, 1\}^{m \times n}$: $P_7 \not\preceq M \Leftrightarrow$ there are integers r, c such that either (see Figure 2.3):*

1. *$M[r, c]$ is top-right empty and bottom-left empty, $\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M[[r], [c]]$ and $\left(\begin{smallmatrix} & \bullet\\ \bullet & \bullet\end{smallmatrix}\right) \not\preceq M[[r, m], [c, n]]$, or*
2. *$M[r, c]$ is top-left empty and bottom-right empty, $\left(\begin{smallmatrix}\bullet & \bullet\\ & \bullet\end{smallmatrix}\right) \not\preceq M[[r], [c, n]]$ and $\left(\begin{smallmatrix}\bullet & \\ \bullet & \bullet\end{smallmatrix}\right) \not\preceq M[[r, m], [c]]$.*

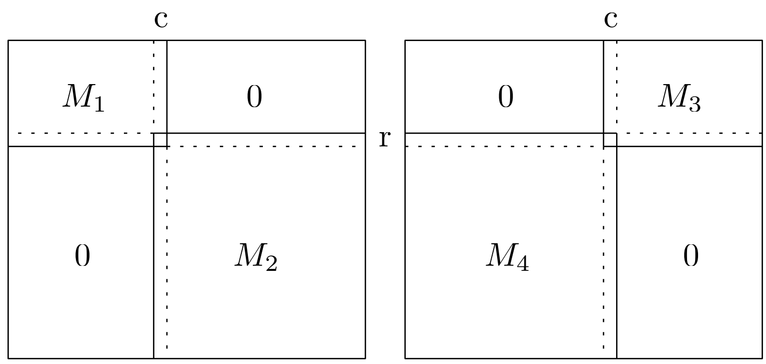


Figure 2.3: The characterization of matrices avoiding $\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right)$ as an interval minor.

Proof. We let $M_1 = M[[r], [c]]$, $M_2 = M[[r, m], [c, n]]$, $M_3 = M[[r], [c, n]]$ and $M_4 = M[[r, m], [c]]$.

$\Rightarrow$ We proceed by induction on the size of M.

If $M \in \{0, 1\}^{2 \times 2}$ then it either avoids $\left(\begin{smallmatrix} & \bullet\\ \bullet & \bullet\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \end{smallmatrix}\right)$ and we are done.

For a bigger matrix M, from Lemma 2.20, there is an entry $M[r,c]$ satisfying some conditions. If there is a one-entry in any corner, we are done because the matrix cannot contain one of the rotations of $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right)$. Otherwise, assume $M[r,c]$ is both top-right and bottom-left empty and $(r,c) \notin \{(1,1),(m,n)\}$. If M_1 is non-empty, then $\left(\begin{smallmatrix}&\bullet\\\bullet&\bullet\end{smallmatrix}\right) \not\preceq M_2$. Symmetrically, $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right) \not\preceq M_1$ if M_2 is non-empty. If one of them is empty, the other is a smaller matrix avoiding P_7 as an interval minor and the statement follows from the induction hypothesis.

$\Leftarrow$ Let $P_7 \preceq M$. Every mapping of P_7 partitions M into four non-empty quadrants; thus, there are no integers r, c satisfying the conditions. $\square$

Lemma 2.22. *For all matrices M: $P_8 \not\preceq M \Rightarrow$ there are matrices M_1, M_2 such that $M = M_1 \to M_2$ and*

1. $\left(\begin{smallmatrix}\bullet&\bullet\\&\bullet\end{smallmatrix}\right) \not\preceq M_1$ *and* $\left(\begin{smallmatrix}&\bullet\\\bullet&\end{smallmatrix}\right) \not\preceq M_2$*, or*
2. $\left(\begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}\right) \not\preceq M_1$ *and* $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right) \not\preceq M_2$.

Proof. Let $e = M[r,c]$ be an arbitrary top-most one-entry of the matrix M. It holds $\left(\begin{smallmatrix}\bullet&\bullet\\&\bullet\end{smallmatrix}\right) \not\preceq M[[m],[c-1]]$; otherwise, we have a mapping of P_8 to M. Symmetrically, $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right) \not\preceq M[[m],[c+1,n]]$. For contradiction with the statement, let $e_1,\ e_2$ (none of them equal to e) be any two one-entries forming $\left(\begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}\right)$ in $M[[m],[c]]$ and let $e_1',\ e_2'$ be any two one-entries forming $\left(\begin{smallmatrix}&\bullet\\\bullet&\end{smallmatrix}\right)$ in $M[[m],[c,n]]$. Without loss of generality, e_2' is no higher than e_2 and together with $e_1,\ e$ and e_1' it gives us a mapping of P_8 to M, which is a contradiction. $\square$

Proposition 2.23. *For all matrices $M \in \{0,1\}^{m\times n}$: $P_8 \not\preceq M \Leftrightarrow$ there are integers r, c_1 and c_2 such that all one-entries of M above the row r are in columns c_1 and c_2, $M[[r+1,m],[c_1+1,c_2-1]]$ is empty, $\left(\begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}\right) \not\preceq M[[r,m],[c_1]]$ and $\left(\begin{smallmatrix}&\bullet\\\bullet&\end{smallmatrix}\right) \not\preceq M[[r,m],[c_2,n]]$. See Figure 2.4.*

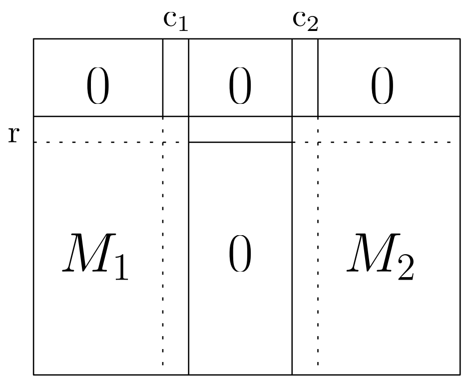


Figure 2.4: The characterization of matrices avoiding $\left(\begin{smallmatrix}\bullet&\bullet&\bullet\\&\bullet&\end{smallmatrix}\right)$ as an interval minor.

Proof. $\Rightarrow$ From Lemma 2.22, we know there are matrices M_1', M_2' such that $M = M_1' \to M_2'$, $\left(\begin{smallmatrix}\bullet&\bullet\\&\bullet\end{smallmatrix}\right) \not\preceq M_1'$ and $\left(\begin{smallmatrix}&\bullet\\\bullet&\end{smallmatrix}\right) \not\preceq M_2'$ (or symmetrically the second case). Let c_2 be the first column appended from M_2'. From Proposition 2.16, we have integers r', c' such that $M_1'[r',c']$ is top-left, top-right and bottom-right empty and $\left(\begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}\right) \preceq M_1'[[r',m],[c']] = M_1$. Let us set $r = r'$ and

$c_1 = c'$. We also have that $M[[m],[c_2,n]]$ is a walking matrix. Without loss of generality, $M[[r-1],\{c_1\}]$ and $M[\{r\},[c_1+1,c_2-1]]$ are non-empty; otherwise, we extend M_1 to cover the whole $M[[m],[c_2-1]$. There are no two different columns in M_2' having a one-entry above the r-th row; otherwise, together with one-entries in $M[[r-1],\{c_1\}]$ and $M[\{r\},[c_1+1,c_2-1]]$ they would give us a mapping of P_8 to M.

$\Leftarrow$ A one-entry $P_8[2,2]$ can not be mapped anywhere but to the r-th row, but in that case, there are at most two columns having one-entries above it. $\square$

Proposition 2.24. *For all matrices $M \in \{0,1\}^{m\times n}$: $P_9 \not\preceq M \Leftrightarrow$ for every one-entry $M[r,c]$ on the bottom-right extreme reverse walk w and every one-entry $M[r',c']$ on the top-left extreme reverse walk w', if $r > r'+3$ and $c > c'+3$ then $M[[r'+1,r-1],[c'+1,c-1]]$ is a walking matrix.*

Proof. $\Rightarrow$ If there are one-entries $M[r,c]$ on w and $M[r',c']$ on w' such that $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \preceq M[[r'+1,r-1],[c'+1,c-1]]$, we have a a mapping of P_9 to M.

$\Leftarrow$ For contradiction, let $P_9 \preceq M$ and consider any mapping. Without loss of generality, the one-entry $P_9[4,4]$ is mapped to some one-entry $M[r,c]$ on w and the one-entry $P_9[1,1]$ is mapped to some one-entry $M[r',c']$ on w'. This means that $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \preceq M[[r'+1,r-1],[c'+1,c-1]]$, which is a contradiction with it being a walking matrix. $\square$

2.5 Multiple patterns

Instead of considering matrices avoiding a single pattern, we can work with matrices avoiding a set of forbidden patterns.

We only describe the structure of matrices avoiding one particular set of patterns, because we use the simple result later.

Proposition 2.25. *Let $P_{10} = \left(\begin{smallmatrix} \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{smallmatrix}\right)$ and $P_{11} = \left(\begin{smallmatrix} \circ & \bullet \\ \circ & \circ \\ \bullet & \circ \end{smallmatrix}\right)$, then for a matrix M: $\{P_{10}, P_{11}\} \not\preceq M \Leftrightarrow$ for every $r < m$ and $c > 0$, if $M[r,c]$ is a one-entry then it either is on the bottom-left extreme walk w or both $M[r+1,c]$ and $M[r,c-1]$ are on w.*

Proof. $\Rightarrow$ Assume there are $r < m$ and $c > 0$ and a one-entry $M[r,c]$ outside of w such that $M[r+1,c]$ (or $M[r,c-1]$) is outside of w. The one-entry $M[r,c]$ is not bottom-left extreme and so there is a one-entry in $M' = M[[r+1,m],[c-1]]$. The entry $M[r+1,c]$ is not bottom-left extreme and because M' is non-empty, $M[r+1,c]$ is not bottom-left empty. Any one-entry in $M[[r+2,m],[c-1]]$ together with $M[r,c]$ give us a mapping of P_{11} (P_{10}).

$\Leftarrow$ For any one-entry $M[r,c]$, there are one-entries in neither $M[[r+2,m],[c-1]]$ nor $M[[r+1,m],[c-2]]$. $\square$

3. The basis of a class of matrices

In this chapter, we look at classes of matrices from a different perspective. Unlike in the previous chapter, where we studied the structure of matrices avoiding a given set of forbidden patterns (usually just one), we are given a class of matrices and a question whether the class can be described by forbidden patterns.

Recall that a class of matrices is set of matrices closed under interval minors. While it is obvious that any class of matrices can be described by a set of forbidden patterns, as it suffices to forbid all matrices not contained in the class, it is no longer clear how complex the set can be.

Definition 3.1. Let $\mathcal{M}$ be a class of matrices. The *basis* of $\mathcal{M}$ is a set of all minimal (with respect to minors) matrices $\mathcal{P}$ such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$.

We show that there are many classes of matrices having a finite basis but there are also classes that have an infinite basis. Let us start with a few simple observations, regarding classes of matrices and their bases.

Observation 3.2. *Let $\mathcal{M} = Av_{\preceq}(\mathcal{P})$ for some set of matrices $\mathcal{P}$. Then $\mathcal{M}$ is closed under interval minors.*

Observation 3.3. *Every finite class of matrices has a finite basis.*

Proof. Let $\mathcal{M}$ be a finite class of matrices, let m be the maximum number of rows a matrix from $\mathcal{M}$ has and let n be the maximum number of columns a matrix from $\mathcal{M}$ has. We define a set of matrices $\mathcal{P}$ to contain all matrices of size smaller or equal to $m \times n$ that do not belong to $\mathcal{M}$ and we add $\{0\}^{m\times(n+1)}$ and $\{0\}^{(m+1)\times n}$. Clearly, $\mathcal{P}$ is finite and $\mathcal{M} = Av_{\preceq}(\mathcal{P})$. $\square$

3.1 The skew and direct sums

In the realm of permutations, the skew and direct sums are very useful operations. What follows is a direct generalization to our settings and a few simple results. More interesting statements and the relation with interval minors follow in the next section.

Definition 3.4. For matrices $A \in \{0,1\}^{m\times n}$ and $B \in \{0,1\}^{k\times l}$ we define their *skew sum* as a matrix $C := A \nearrow B \in \{0,1\}^{(m+k)\times(n+l)}$ such that the submatrix $C[[k+1, m+k], [n]] = A$, $C[[k], [n+1, n+l]] = B$ and the rest is empty. Symmetrically, we define their *direct sum* $D := A \searrow B \in \{0,1\}^{(m+k)\times(n+l)}$ such that $D[[m], [n]] = A$, $D[[m+1, m+k], [n+1, n+l]] = B$ and the rest is empty.

Using this notation, we can very easily rewrite the results from the previous chapter. Here is an example of Proposition 2.16 and Proposition 2.21:

Proposition 3.5. $Av_{\preceq}\left(\left(\begin{smallmatrix} \bullet & \bullet \\ \bullet & \circ \end{smallmatrix}\right)\right) = Av_{\preceq}\left(\left(\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix}\right)\right) \searrow Av_{\preceq}\left(\left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix}\right)\right)$

Proposition 3.6.

$$\begin{aligned} Av_{\preceq}\left(\left(\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix}\right)\right) = {} & \left(Av_{\preceq}\left(\left(\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix}\right)\right) \searrow Av_{\preceq}\left(\left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix}\right)\right) \searrow Av_{\preceq}\left(\left(\begin{smallmatrix} \circ & \circ \\ \circ & \bullet \end{smallmatrix}\right)\right)\right) \cup \\ & \left(Av_{\preceq}\left(\left(\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix}\right)\right) \nearrow Av_{\preceq}\left(\left(\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}\right)\right) \nearrow Av_{\preceq}\left(\left(\begin{smallmatrix} \circ & \bullet \\ \circ & \circ \end{smallmatrix}\right)\right)\right). \end{aligned}$$

Something, we get a great use of later is the closure under the skew sum.

Definition 3.7. For a set of matrices $\mathcal{M}$, let $Cl(\mathcal{M})$ denote the smallest class of matrices containing each $M \in \mathcal{M}$ that is closed under the skew sum and interval minors.

When speaking about interval minors, we suppose, without loss of generality, that the merges of neighboring lines are done after all deletions of one-entries. Similarly, a matrix created from a matrix M by reapplying the skew sum and taking its interval minor can be also created by taking an interval minor of the skew sum of an appropriate number of copies of M.

Observation 3.8. *For every set of matrices $\mathcal{P}$, each $M \in Cl(\mathcal{P})$ is an interval minor of the skew sum of multiple copies of P.*

What follows is a simple example of the relation between the closure under the skew sum and the description using interval minors. We greatly generalize this result in the next section.

Proposition 3.9. $Cl\left(\left(\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)\right)\right) = Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \circ & \circ \\ \circ & \circ & \bullet\end{smallmatrix}\right), \left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ \\ \circ & \bullet\end{smallmatrix}\right)\right)$.

Proof. The skew sum of an arbitrary number of copies of $\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$ avoids both forbidden patterns and because the relation of being an interval minor is transitive, we have $Cl\left(\left(\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)\right)\right) \subseteq Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \circ & \circ \\ \circ & \circ & \bullet\end{smallmatrix}\right), \left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ \\ \circ & \bullet\end{smallmatrix}\right)\right)$.

From Proposition 2.25, for every matrix $M \in Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \circ & \circ \\ \circ & \circ & \bullet\end{smallmatrix}\right), \left(\begin{smallmatrix}\bullet & \circ \\ \circ & \circ \\ \circ & \bullet\end{smallmatrix}\right)\right)$, it holds that for every $r < m$ and $c > 0$, if $M[r, c]$ is a one-entry then it either is on the bottom-left extreme walk w or both $M[r+1, c]$ and $M[r, c-1]$ are on w. Clearly, $\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \bullet\end{smallmatrix}\right)$ is an interval minor of the skew sum of three copies of $\left(\begin{smallmatrix}\bullet & \circ \\ \circ & \bullet\end{smallmatrix}\right)$ and from the skew sum of multiple copies of $\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \bullet\end{smallmatrix}\right)$, we can create the walk w and all one-entries outside of it by taking an interval minor. $\square$

We generalize the skew sum to also allow an overlap between the summed matrices.

Definition 3.10. For matrices $A \in \{0,1\}^{m \times n}$, $B \in \{0,1\}^{k \times l}$ and integers a, b, let $C := A \nearrow_{a \times b} B \in \{0,1\}^{(m+k-a) \times (n+l-b)}$ be a matrix such that the submatrix $C[[k+1, m+k], [n]] = A$, $C[[k], [n+1, n+l]] = B$, the part that overlaps is the elementwise OR of the overlapping submatrices and the rest of C is empty. We say C is the *skew sum with $a \times b$ overlap* of A and B.

Proposition 3.11. *For integers a, b, m, n such that $a \leq m \leq 2a$ and $b \leq n \leq 2b$, let $\mathcal{M}$ be an arbitrary set of matrices, not necessarily closed under interval minors, such that:*

- *$\mathcal{M}$ is closed under deletion of one-entries,*
- *$\mathcal{M}$ is closed under the skew sum with $a \times b$ overlap and*
- *there is a $m \times n$ matrix $M \in \mathcal{M}$,*

then $\mathcal{M}$ is also closed under the skew sum with $(2a - m) \times (2b - n)$ overlap.

Proof. Given any $A, B \in \mathcal{M}$ and a matrix $M \in \mathcal{M}$ such that $M \in \{0,1\}^{m \times n}$, let $C = A \nearrow_{a \times b} M \nearrow_{a \times b} B$. It has the same size as $D = A \nearrow_{(2a-m) \times (2b-n)} B$, whose set of one-entries is a subset of one-entries of $C \in \mathcal{M}$; therefore, $D \in \mathcal{M}$. $\square$

We see that with some reasonable assumptions, whenever a set of matrices is closed under the skew sum with an overlap, it is also closed under the skew sum with a smaller overlap. On the other hand, in general, the opposite does not hold even if we work with classes of matrices.

Observation 3.12. *There is a class of matrices closed under the skew sum with* 1×1 *overlap that is not closed under the skew sum with* 2×2 *overlap.*

Proof. Let $\mathcal{M} = Av_{\preceq}\left(\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right)\right)$. Clearly, $\mathcal{M}$ is hereditary and closed under the skew sum with 1×1 overlap. On the other hand, $\mathcal{M}$ is not closed under the skew sum with 2×2 overlap, because, for matrices $\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \end{smallmatrix}\right), \left(\begin{smallmatrix} & \bullet \\ \bullet & \bullet\end{smallmatrix}\right) \in \mathcal{M}$ it holds $\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \end{smallmatrix}\right) \nearrow_{2\times 2} \left(\begin{smallmatrix} & \bullet \\ \bullet & \bullet\end{smallmatrix}\right) = \left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \bullet\end{smallmatrix}\right) \notin \mathcal{M}$. □

A similar proof shows that for all $a \geq 1, b > 1$ there is a class of matrices closed under the skew sum with $a \times b$ overlap that is not closed under the skew sum with $(a+1) \times b$ (or $a \times (b+1)$) overlap. Luckily for us, this does not hold for $a = 0$ or $b = 0$:

Proposition 3.13. *Every class of matrices is closed under the skew sum* $\Leftrightarrow$ *it is closed under the skew sum with* 1×1 *overlap.*

Proof.
- $\Rightarrow$ If a class is closed under the skew sum then, because it is also closed under interval minors, it is closed under the skew sum with 1×1 overlap.
- $\Leftarrow$ Let $\mathcal{M}$ be a class closed under the skew sum with 1×1 overlap. Using the assumption that $\mathcal{M}$ is non-trivial, it contains matrices $M_1 \in \{0,1\}^{2\times 1}$ and $M_2 \in \{0,1\}^{1\times 2}$. For $M = M_1 \nearrow_{1\times 1} M_2$, we have $M \in \mathcal{M}$ and we can use Proposition 3.11 to show $\mathcal{M}$ is closed under the skew sum. □

3.2 Articulations

Our next goal is to show that the closure under the skew sum of a single matrix creates a class with finite basis. In order to prove it, we define and get familiar with articulations.

Definition 3.14. Let $M \in \{0,1\}^{m\times n}$ be a matrix. An element $M[r,c]$ is an *articulation* if it is top-left empty ($M[[r-1],[c-1]]$ is empty) and bottom-right empty ($M[[r+1,m],[c+1,n]]$ is empty). We say that an articulation $M[r,c]$ is *trivial* if $(r,c) \in \{(m,1),(1,n)\}$.

Observation 3.15. *Let M be a matrix. If there are integers r, c such that the entry $M[r,c]$ is an articulation, then for every matrix P such that $P \preceq M$, if $P[r',c']$ can be mapped to a block containing $M[r,c]$ then $P[r',c']$ is an articulation.*

Observation 3.16. *Let $P \in \{0,1\}^{k\times l}$ be a matrix. There are P_1, P_2 non-empty interval minors of P such that $P = P_1 \nearrow_{1\times 1} P_2$ $\Leftrightarrow$ there exist integers r, c such that $P[r,c]$ is an articulation and $P[[r,k],[c]], P[[r],[c,l]]$ are non-empty.*

Lemma 3.17. *Let $\mathcal{P}$ be a set of matrices. There is a minimal (with respect to interval minors) matrix $P \in \mathcal{P}$ and there are P_1, P_2 non-empty interval minors of P such that $P = P_1 \nearrow_{1\times 1} P_2$ $\Leftrightarrow$ $Av_{\preceq}(\mathcal{P})$ is not closed under the skew sum with 1×1 overlap.*

Proof. $\Rightarrow$ Let $P_1 \in \{0,1\}^{k_1\times l_1}$ and $P_2 \in \{0,1\}^{k_2\times l_2}$. While P is an interval minor of neither $P_1 \nearrow_{1\times 1} \{0\}^{k_2\times l_2}$ nor $\{0\}^{k_1\times l_1} \nearrow_{1\times 1} P_2$, we have that $P \preceq P_1 \nearrow_{1\times 1} \{0\}^{k_2\times l_2} \nearrow \{0\}^{k_1\times l_1} \nearrow_{1\times 1} P_2$.

$\Leftarrow$ Let there be non-empty matrices $M_1, M_2 \in Av_{\preceq}(\mathcal{P})$ such that there is a minimal $P \in \mathcal{P}$ of size $k \times l$ for which $P \preceq M = M_1 \nearrow_{1\times 1} M_2$. From Observation 3.15 there exists an articulation $P[r,c]$ and both submatrices $P[[r,k],[c]], P[[r],[c,l]]$ are non-empty.

It remains to show that assuming M_1 and M_2 are non-empty is without loss of generality. If $Av_{\preceq}(\mathcal{P})$ is non-trivial, it contains a non-empty matrix M. We can keep applying the skew sum with 1×1 to M until we either find M_1, M_2 non-empty such that $M_1 \nearrow_{1\times 1} M_2 \notin Av_{\preceq}(\mathcal{P})$ or create a big enough non-empty matrix such that for any empty $M' \in Av_{\preceq}(P)$ it holds $M' \preceq M$.

This does not work only if $M = \{1\}^{1\times 1}$ and it is the only non-empty matrix in $Av_{\preceq}(\mathcal{P})$. But then consider the matrix M' that is equal to the skew sum with 1×1 overlap of M and any non-empty matrix $M'' \in Av_{\preceq}(\mathcal{P})$ of size bigger than 1×1 (the class is non-trivial so there is one). There is a pattern $P \in \mathcal{P}$ such that $P \preceq M'$ and because $P \not\preceq M''$, $P \in \{0,1\}^{k\times l}$ contains exactly one one-entry $P[r,c]$. It means that $P[r,c]$ is an articulation and P is the skew sum with 1×1 overlap of its non-empty interval minors, as $P = P[[r,k],[c]] \nearrow_{1\times 1} P[[r],[c,l]]$. $\square$

In what follows, we always assume that all articulations are on a reverse walk (no two articulations form $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right)$) and a matrix between two articulations $M[r_1,c_1]$ and $M[r_2,c_2]$ is the matrix $M[[r_2,r_1],[c_1,c_2]]$.

Lemma 3.18. *Let $\mathcal{P}$ be a set of matrices, then for all matrices $M \in \{0,1\}^{m\times n}$ it holds that $M \in Cl(\mathcal{P}) \Leftrightarrow$ there exists a sequence of articulations of M on a reverse walk such that for each matrix M' in between two consecutive articulations there exists a pattern $P \in \mathcal{P}$ such that $M' \preceq (\,1\,) \nearrow P \nearrow (\,1\,)$.*

Proof. $\Rightarrow$ Having Proposition 3.13 in mind, consider the skew sum with 1×1 overlap of multiple copies of elements of $\mathcal{P}$ and let the sequence contain an articulation between each pair of consecutive copies of matrices from P, together with the trivial articulations $M[m,1]$ and $M[1,n]$.

Between each pair of consecutive articulations, we have a matrix from $\mathcal{P}$ and so the statement holds. When we take an arbitrary interval minor and keep original articulations, each matrix between two consecutive articulations only contains at most one original copy of some matrix $P \in \mathcal{P}$, but it may happen that the bottom-left and top-right corners become one-entries even though they were zero-entries before. The matrix does not have to be an interval minor of P anymore, but it is an interval minor of $(\,1\,) \nearrow P \nearrow (\,1\,)$ for the corresponding $P \in \mathcal{P}$.

$\Leftarrow$ We can simply blow up each matrix M' between two consecutive articulation to the skew sum of three copies of the corresponding matrix P and because $M' \preceq (\,1\,) \nearrow P \nearrow (\,1\,) \preceq P \nearrow P \nearrow P$ it holds $M \in Cl(\mathcal{P})$. $\square$

Finally, we show that a closure under the skew sum can always be described by a finite number of forbidden patterns.

Theorem 3.19. *For all matrices $M \in \{0,1\}^{m \times n}$, $Cl\,(M)$ has a finite basis.*

Proof. Let $\mathcal{F}$ be the set of all minimal (with respect to interval minors) matrices such that $Cl\,(M) = Av_{\preceq}\,(\mathcal{F})$. We need to prove that $\mathcal{F}$ is finite. Thanks to Proposition 3.13, $Av_{\preceq}\,(\mathcal{F})$ is closed under the direct sum with 1×1 overlap and from Lemma 3.17 follows that for no $F \in \mathcal{F}$ there are its non-empty interval minors F_1, F_2 such that $F = F_1 \nearrow 1 \times 1 F_2$. We denote by $\mathcal{P}$ a set of matrices $F \in \mathcal{F}$ such that F has at most $2m + 4$ rows and $2n + 4$ columns. We want to show $Cl\,(M) = Av_{\preceq}\,(\mathcal{P})$.

$\subseteq$ Clearly, $\mathcal{P}$ is finite and we immediately see that $Cl\,(M) \subseteq Av_{\preceq}\,(\mathcal{P})$.

$\supseteq$ For contradiction, consider a minimal matrix $X \in Av_{\preceq}\,(\mathcal{P}) - Cl\,(M)$. There are no X_1, X_2 non-empty interval minors of X such that $X = X_1 \nearrow 1 \times 1 X_2$; otherwise, as $X_1, X_2 \in Av_{\preceq}\,(\mathcal{P})$ and X is the minimum matrix such that $X \notin Cl\,(M)$, we would have $X_1, X_2 \in Cl\,(M)$; therefore, $X \in Cl\,(M)$ and a contradiction.

Without loss of generality, we assume $X \in \{0,1\}^{k \times l}$ has at least $2m + 5$ rows. Let X' denote a matrix created from X by deletion of the first row. We have $X' \in Av_{\preceq}\,(\mathcal{P})$ and from minimality of X also $X' \in Cl\,(M)$. From Lemma 3.18, there is a sequence of articulations of X' on a reverse walk such that each matrix between two consecutive articulations is an interval minor of $(1) \nearrow M \nearrow (1)$. Let $X'[r,c]$ be the first articulation from the sequence (sorted by the second coordinate in ascending order) for which $c > 1$. The matrix between $X'[r,c]$ and the previous articulation in the sequence is an interval minor of $(1) \nearrow M \nearrow (1)$, which also means that $c \le n+2$. Since $X[r,c]$ is not an articulation, it must hold that $X[1,c_1] = 1$ for some $c_1 < c \le n + 2$. Symmetrically, let X'' denote a matrix created from X by deletion of the last row. Following the same steps we did before, we get the last articulation $X''[r,c]$ such that $c < l$ and the observation that $c \ge l - n - 1$. Since $X[r,c]$ is not an articulation, it must hold that $X[k,c_2] = 1$ for some $c_2 > c \ge l - n - 1$.

We showed that a matrix $Y \in \{0,1\}^{(m+1)\times 2}$ such that the only one-entries are $Y[1,1]$ and $Y[m+1,2]$ is an interval minor of X. To reach a contradiction, it suffices to show that there is a matrix $P \in \mathcal{P}$ such that $P \preceq Y$. For contradiction, let $Y \in Av_{\preceq}\,(\mathcal{P})$ and since $Y \preceq X$ and X is minimum such that $X \notin Cl\,(M)$ it holds $Y \in Cl\,(M)$. But this cannot be, because Y contains no non-trivial articulation and from Observation 3.15, we know that every matrix $Z \in Cl\,(M)$ bigger than $m \times n$ contains at least one. $\square$

3.3 Bases

We recall that the basis of a class of matrices $\mathcal{M}$ is a set of all minimal (with respect to interval minors) matrices $\mathcal{P}$ such that $\mathcal{M} = Av_{\preceq}\,(\mathcal{P})$. It goes without saying that it does not make sense to consider a basis of a set of matrices that is not closed under interval minors.

So far, we showed that whenever $\mathcal{M}$ is finite, its basis is also finite. The same hold when $\mathcal{M} = Cl\,(M)$ for some M. We show next that, unlike in graph theory, there are classes that does not have a finite basis. Moreover, we show that even for a class $\mathcal{M}$ with finite basis, its closure $Cl\,(\mathcal{M})$ can have an infinite basis.

Definition 3.20. Let P be a matrix. We denote by $\mathcal{R}\,(P)$ a set of all minimal (with respect to minors) matrices P' such that $P \preceq P'$ and P' is not the skew sum with 1×1 overlap of non-empty interval minors of P'. For a set of matrices $\mathcal{P}$, let $\mathcal{R}\,(\mathcal{P})$ denote a set of all minimal (with respect to minors) matrices from the set $\bigcup_{P \in \mathcal{P}} \mathcal{R}\,(P)$.

Theorem 3.21. *Let $\mathcal{M}$ and $\mathcal{P}$ be sets of matrices such that $\mathcal{M} = Av_{\preceq}\,(\mathcal{P})$, then $Cl\,(\mathcal{M}) = Av_{\preceq}\,(\mathcal{R}\,(\mathcal{P}))$.*

Proof. $\subseteq$ Consider a matrix $M \notin Av_{\preceq}\,(\mathcal{R}\,(\mathcal{P}))$ and without loss of generality, because $Cl\,(\mathcal{M})$ is closed under interval minors, let M be minimal (with respect to interval minors). It follows that $M \in \mathcal{R}\,(\mathcal{P})$. Then, the matrix M is not a skew sum with 1×1 overlap of non-empty interval minors of M; therefore, according to Observation 3.16, there is no articulations $M[r, c]$ such that $M[[r, k], [c]], M[[r], [c, l]]$ are non-empty.

For contradiction with the statement, assume $M \in Cl\,(\mathcal{M})$. According to Lemma 3.18 and the fact M contains no non-trivial articulation, it holds $M \preceq (\,1\,) \nearrow M' \nearrow (\,1\,)$ for some $M' \in \mathcal{M}$. Because the trivial articulations contain zero-entries, it even holds $M \preceq M'$. We also know $P \preceq M$ for some $P \in \mathcal{P}$, which together give us a contradiction with $\mathcal{M} = Av_{\preceq}\,(\mathcal{P})$.

$\supseteq$ First of all, $Av_{\preceq}\,(\mathcal{R}\,(\mathcal{P}))$ is closed under the skew sum with 1×1 overlap. For contradiction, assume there are matrices $M_1, M_2 \in Av_{\preceq}\,(\mathcal{R}\,(\mathcal{P}))$ but $M = M_1 \nearrow_{1\times 1} M_2 \notin Av_{\preceq}\,(\mathcal{R}\,(\mathcal{P}))$. Then there exists a matrix $P \in \mathcal{R}\,(\mathcal{P})$ such that $P \preceq M$. Because P is not a skew sum with 1×1 overlap of non-empty interval minors of P, it follows that either $P \preceq M_1$ or $P \preceq M_2$ and we have a contradiction.

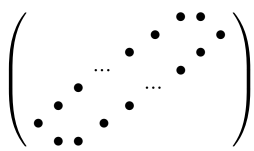
It suffices to show that the inclusion holds for any matrix $M \in Av_{\preceq}\,(\mathcal{R}\,(\mathcal{P}))$ that is not a skew sum with 1×1 overlap of non-empty interval minors of M. From Observation 3.16, we know that M does not contain any non-trivial articulation and those trivial ones are empty. Thus, $M \in Av_{\preceq}\,(\mathcal{P}) = \mathcal{M}$ and so $M \in Cl\,(\mathcal{M})$. $\square$

Corollary 3.22. *Let $\mathcal{M}$ and $\mathcal{P}$ be sets of matrices such that $\mathcal{M} = Av_{\preceq}\,(\mathcal{P})$, then $\mathcal{R}\,(\mathcal{P})$ is the basis of $Cl\,(\mathcal{M})$.*

What follows is a construction of parameterized matrices that become the main tool of finding a class of matrices with an infinite basis.

Definition 3.23. Let $Nucleus_1 = (\,\bullet\,)$ and for $n > 1$ let $Nucleus_n \in \{0, 1\}^{n \times n+1}$ be a matrix described by the examples:

$$Nucleus_2 = \begin{pmatrix} \bullet & \bullet & \\ & \bullet & \bullet \end{pmatrix} \quad Nucleus_3 = \begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & \bullet & \end{pmatrix} \quad Nucleus_n =$$



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Definition 3.24. Let $Candy_{k,n,l}$ be a matrix given by $I_k \nearrow_{1\times 2} Nucleus_n \nearrow_{1\times 2} I_l$, where I_k, I_l are unit matrices of sizes $k \times k$ and $l \times l$ respectively.

$$Candy_{4,1,4} = \begin{pmatrix} &&&\bullet&&& \\ &&&&\bullet&& \\ &&&&&\bullet& \\ \bullet&&&\bullet&&&\bullet \\ &\bullet&&&&& \\ &&\bullet&&&& \\ &&&\bullet&&& \end{pmatrix} \quad Candy_{4,4,4} = \begin{pmatrix} &&&&&\bullet&&& \\ &&&&&&\bullet&& \\ &&&&&&&\bullet& \\ &&&&\bullet&\bullet&&&\bullet \\ &&&\bullet&&&\bullet&& \\ &&\bullet&&&\bullet&&& \\ \bullet&&&\bullet&\bullet&&&& \\ &\bullet&&&&&&& \\ &&\bullet&&&&&& \\ &&&\bullet&&&&& \end{pmatrix}$$

Theorem 3.25. *There exists a matrix P for which $\mathcal{R}(P)$ is infinite.*

Proof. Let $P = Candy_{4,1,4}$. For all $n > 3$ it holds $P \preceq Candy_{4,n,4}$ and it suffices to show that each $Candy_{4,n,4}$ is a minimal matrix (with respect to minors) and it is not the skew sum of two of its non-empty interval minors. According to Observation 3.16, the second condition holds as $Candy_{4,n,4}$ contains no non-trivial articulation and the trivial ones are empty. To show it is minimal, we need to consider any matrix $M \preceq Candy_{4,n,4}$ and argue that either $P \npreceq M$ or M contains an articulation.

Thanks to Observation 3.15, as soon as we find a non-trivial articulation $M[r,c]$ such that $M[[r,k],[c]], M[[r],[c,l]]$ are non-empty, it will stay there in any interval minor, because we cannot delete one-entries $M[1,n-3], M[2,n-2], M[3,n-1]$ and $M[4,n]$ (and symmetrically $M[m-3,1], M[m-2,2], M[m-1,3], M[m,4]$) without loosing the condition $P \preceq M$. Therefore, we can only consider one minoring operation at a time.

It is easy to see that when a one-entry is changed to a zero-entry, then the matrix does not belong to $\mathcal{R}(P)$ anymore. Consider that rows $r_1, r_2, \ldots, r_k$ are chosen to be merged into one with an elementwise OR. If $r_1 < 4$ or $r_k > n+3$ then P is no longer an interval minor of such matrix. Otherwise, the original $Candy_{4,n,4}[r_1, n-r_1+2]$ becomes an articulation. Symmetrically, the same holds for columns which concludes the proof. $\square$

Corollary 3.26. *There exists a class of matrices $\mathcal{M}$ having a finite basis such that $Cl(\mathcal{M})$ has an infinite basis.*

Proof. From Theorem 3.25, we have a matrix P for which $\mathcal{R}(P)$ is infinite. Class $\mathcal{M} = Av_{\preceq}(P)$ has a finite basis. On the other hand, from Theorem 3.21, we have $Cl(\mathcal{M}) = Av_{\preceq}(\mathcal{R}(P))$. $\square$

4. Zero-intervals

In Chapter 2, we characterized matrices avoiding small patterns. Their structure is very dependent on the pattern they avoid and the results are hard to generalize for arbitrary patterns. In this chapter, we look for a more general property that restricts the complexity of a class of matrices.

Definition 4.1. For a matrix $M \in \{0,1\}^{m \times n}$, a row interval $M[\{r\}, [c_1, c_2]]$ is a *zero-interval* if all entries are zero-entries, $c_1 = 0$ or $M[r, c_1 - 1] = 1$ and $c_2 = n$ or $M[r, c_2 + 1] = 1$. In other words, it is an interval of zero-entries bounded by one-entries. Symmetrically, we also call a column interval $M[[r_1, r_2, \{c\}]]$ a *zero-interval* if all entries are zero-entries, $r_1 = 0$ or $M[r_1 - 1, c] = 1$ and $r_2 = m$ or $M[r_2 + 1, c] = 1$. In the same spirit, we define a *one-interval* to be an interval of one-entries in a single line of M bounded by zero-entries (or edges of the matrix).

Definition 4.2. For a class of matrices $\mathcal{M}$, we say that a matrix $M \in \mathcal{M}$ is *critical* in $\mathcal{M}$ if the change of any zero-entry to a one-entry creates a matrix that does not belong to $\mathcal{M}$. For any set of matrices $\mathcal{P}$, let $Av_{crit}(\mathcal{P})$ be a set of all critical matrices avoiding $\mathcal{P}$ as an interval minor.

In Chapter 2, for a pattern $P \in \{0,1\}^{k \times l}$ it very often holds that any matrix from $Av_{crit}(P)$ has at most k zero-intervals in each row and at most l zero-intervals in each column. The main goal of this chapter is to describe patterns P for which there can be arbitrarily many zero-intervals in matrices from $Av_{crit}(P)$.

4.1 Pattern complexity

We define the complexity of a class of matrices as the maximum number of zero-intervals (or one intervals as they go in pair) a critical matrix from the class can have.

Definition 4.3. For a class of matrices $\mathcal{M}$, we define its *row-complexity* $r(\mathcal{M})$ to be the supremum of the number of zero-intervals in a single row of any critical matrix $M \in \mathcal{M}$. We say that $\mathcal{M}$ is *row-bounded*, if its row-complexity is finite, and *row-unbounded* otherwise. Symmetrically, we define its *column-complexity* $c(\mathcal{M})$ and the property of being *column-bounded* and *column-unbounded*. The class $\mathcal{M}$ is *bounded* if it is both row-bounded and column-bounded; otherwise, it is *unbounded*.

Definition 4.4. We say that a set of patterns $\mathcal{P}$ is *bounding*, if the class $Av_{\preceq}(\mathcal{P})$ is bounded; otherwise, it is *non-bounding*.

Now that we introduced the most essential definitions in this chapter, it is time to state the main theorem:

Theorem 4.5. *A pattern P is bounding $\Leftrightarrow P_i \npreceq P$ for all $1 \le i \le 4$.*

$$P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix} \quad P_2 = \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix} \quad P_3 = \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix} \quad P_4 = \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}$$

We prove the statement in several steps. We show the first implication in Subsection 4.1.2, then we proof multiple lemmata so that we finally show the other implication at the end of Subsection 4.1.3. Before we start proving the main result, we introduce some useful notation and get more familiar with zero-intervals.

Definition 4.6. Let P be a pattern, let e be a one-entry of P, consider a matrix $M \in Av_{\preceq}(P)$ and let z be an arbitrary zero-interval of M. We say that z is *usable for* e if there is a zero-entry contained in z such that if we change it to a one-entry, it creates a mapping of P to M that uses the new one-entry to map e. This way, z can be usable for many one-entries of P at once.

Observation 4.7. *Let $P \in \{0,1\}^{k\times l}$ and $M \in \{0,1\}^{m\times n}$ be matrices such that $P \not\preceq M$. Let $z = M[\{r_1\},[c_1,c_2]]$ be a zero-interval of M usable for a one-entry $e = P[r,c]$. If we change a zero-entry of z and create a mapping of P that uses the changed entry to map e, then the mapping can only map column c of P to columns $[c_1,c_2]$ of M.*

Proof. Since the changed entry is used to map e, clearly the mapping needs to use a column from $[c_1,c_2]$ to map column c. If, for contradiction, the mapping uses columns outside $[c_1,c_2]$ then, without loss of generality, it uses the column c_1-1. Since that column bounds the zero-interval z, $M[r_1,c_1-1]=1$ and this one-entry can be used in the mapping instead of the changed entry, which gives us a contradiction with $P \not\preceq M$. □

Definition 4.8. Let $\mathcal{P}$ be a set of patterns and let e be a one-entry of any matrix $P \in \mathcal{P}$. We define the *row-complexity* of e, $r\left(Av_{\preceq}(\mathcal{P}),e\right)$ to be the supremum of the number of zero-intervals of a single row of any $M \in Av_{crit}(\mathcal{P})$ that are usable for e. We say that e is *row-unbounded* in $Av_{\preceq}(\mathcal{P})$ if $r\left(Av_{\preceq}(\mathcal{P}),e\right)=\infty$ and *row-bounded* otherwise. Symmetrically, we define the *column-complexity* of e, $c\left(Av_{\preceq}(\mathcal{P}),e\right)$ to be the maximum number of zero-intervals of a single column of any matrix from $Av_{crit}(\mathcal{P})$ that are usable for e, and we say e is *column-unbounded* if it is infinite and *column-bounded* otherwise.

The following observation follows directly from the definition and we use it heavily throughout the chapter to break symmetries.

Observation 4.9. *For every set $\mathcal{M}$, $\mathcal{M}$ is row-bounded $\Leftrightarrow$ $\mathcal{M}^\top$ is column-bounded.*

4.1.1 Adding empty lines

As in Chapter 2, we show that we do not need to consider patterns with leading and ending empty rows and columns.

Observation 4.10. *For a matrix $P \in \{0,1\}^{k\times l}$ and an integer n, let $P' = P \to 0^{k\times n}$. The matrix P is bounding $\Leftrightarrow$ P' is bounding. Moreover, if P is bounding, then $r\left(Av_{\preceq}(P')\right) \le r\left(Av_{\preceq}(P)\right)+1$.*

Lemma 4.11. *Let $P \in \{0,1\}^{2\times k}$ be a matrix and for any $l \ge 1$, let $P^l \in \{0,1\}^{(l+2)\times k}$ be a pattern created from P by adding l new empty rows in between the two row of P. For every one-entry e of P^l it holds $r\left(Av_{\preceq}\left(P^l\right),e\right) \le k^2$.*

Proof. Given a matrix $M \in Av_{crit}(P)$, consider an arbitrary row r of M. Without loss of generality, assume $e = P[1, c]$. For contradiction, assume there are $k^2 + 1$ zero-intervals $z_1, \ldots, z_{k^2+1}$ in r usable for e. In particular, the first k^2 of them are bounded by a one-entry from the right side.

- $P[2, c] = 1$: Clearly, there is a one-entry in rows $[r + l + 1, m]$ underneath each z_j and if we combine each such one-entry with a one-entry bounding corresponding z_j, we find a mapping of $\left(\{1\}^{2\times k^2}\right)^l$, contradicting $P \not\preceq M$.

- $P[2, c] = 0$: For each $i \in [k^2]$, we define an extended interval z_i^* to be the interval containing z_i and also all entries on the row r between z_i and z_{i+1}. Because of the Pigeonhole principle, we can find either k consecutive extended intervals such that there are no one-entries in rows $[r + l + 1, m]$ underneath them, or k (not necessarily consecutive) extended intervals such that there is a one-entry in rows $[r + l + 1, m]$ underneath each of them. Because each extended interval contains a one-entry, in the second case we find $\left(\{1\}^{k\times 2}\right)^l$ as an intervals minor.

  In the first case, without loss of generality, assume $P[2, c_1] = 1$ and it is the minimum such $c_1 > c$. Let $z'_1, \ldots, z'_k$ be the consecutive zero-intervals. Consider the mapping of P^l created when a zero-entry of z'_1 is changed to a one-entry used to map e. Since $P[2, c_1] = 1$ and there are no one-entries in rows $[r + l + 1, m]$ underneath extended intervals $z'_1, \ldots, z'_k$, $P^l[l + 2, c_1]$ has to be mapped to the columns of M after the end of z'_k. This leaves k one-entries to be used to map potential one-entries in $P^l[\{l + 2\}, [c, c_1 - 1]]$ and so $P^l \preceq M$, which is again a contradiction. □

Corollary 4.12. *Let $P \in \{0, 1\}^{k\times 2}$ be a matrix and for any $l \geq 1$, let $P^l \in \{0, 1\}^{k\times(l+2)}$ be a matrix created from P by adding l new empty columns in between the two columns of P. Then $Av_{\preceq}\left(P^l\right)$ is bounded for any $l \geq 1$.*

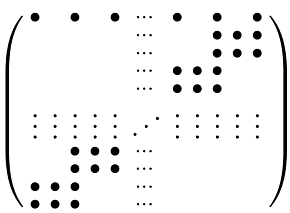
Proof. We know $Av_{\preceq}\left(P^l\right)$ is row-bounded from Lemma 2.9. From Lemma 4.11 and Observation 4.9 we have that the class is also column-bounded. □

4.1.2 Non-bounding patterns

We see that for patterns having only two non-empty rows or columns we can indeed bound the number of zero-intervals of critical matrices avoiding them. On the other hand, already for a pattern of size 3×3 we show that there are maximal matrices with arbitrarily many zero-intervals.

Lemma 4.13. *A class $Av_{\preceq}(P_1)$ is unbounded.*

Proof. For a given integer n, let M be a $(2n + 1) \times (2n + 1)$ matrix described by the picture:



We see that $P_1 \not\preceq M$ because we always need to map $P_1[2,1]$ and $P_1[3,3]$ to just one "block" of one-entries, which only leaves a zero-entry for $P_1[1,2]$.

If we change any zero-entry of the first row into a one-entry, we get a matrix containing an interval minor of $\{1\}^{3\times 3}$; therefore, containing P_1 as an interval minor. In case M is not critical, we add some more one-entries to make it critical but it will still contain a row with n zero-intervals. $\square$

Not only $M \in Av_{crit}(P_1)$ but it also avoids any $P \in \{0,1\}^{3\times 3}$ such that $P_1 \preceq P$. Its rotations avoid rotations of P_1 and we conclude that a big portion of patterns of size 3×3 are non-bounding. Moreover, the result can be generalized also for bigger matrices.

Theorem 4.14. *For every matrix P such that $P_1 \preceq P$, $Av_{\preceq}(P)$ is unbounded.*

Proof. First, assume there is a mapping of P_1 into $P \in \{0,1\}^{k\times l}$ that maps $P_1[1,2]$ to a one-entry of the first row of P, $P_1[2,1]$ to a one-entry of the first column of P and $P_1[3,3]$ to the bottom-right corner of P. Then, we use a similar construction as we did in the proof of Lemma 4.13 to find a matrix $M \in Av_{crit}(P)$ with n zero-intervals for any n.

Let P be an arbitrary pattern containing P_1 as an interval minor. Let $P[r_1, c_1], P[r_2, c_2]$ and $P[r_3, c_3]$ be one-entries that can be used to map $P_1[1,2]$, $P_1[2,1]$ and $P_1[3,3]$ respectively. We take a submatrix $P' := P[[r_1, r_3], [c_2, c_3]]$. Such a matrix fulfills assumptions of the more restricted case above and we find a matrix $M' \in Av_{crit}(P')$ having n zero-intervals. We construct M from M' by simply adding new rows and columns containing only one-entries. We add $r_1 - 1$ rows in front of the first row and $k - r_3$ rows behind the last row. We also add $c_2 - 1$ columns in front of the first column and $l - c_3$ columns behind the last column. The constructed matrix M avoids P an an interval minor because its submatrix P' cannot be mapped to M'. At the same time, any change of a zero-entry of the r_1-th row of M to a one-entry creates a copy of $1^{k\times l}$. The constructed matrix M can be seen in Figure 4.1. $\square$

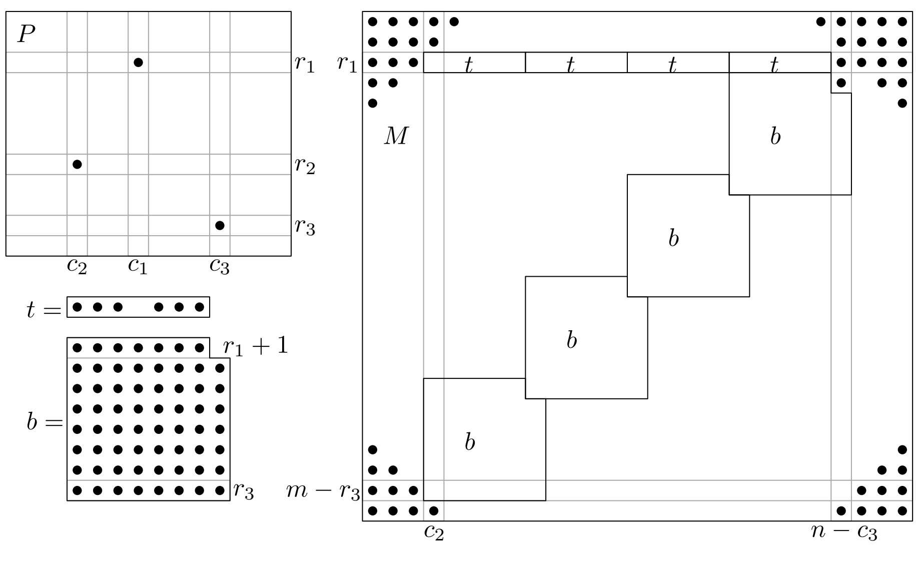


Figure 4.1: The structure of a critical matrix avoiding P that has arbitrarily many zero-intervals.

4.1.3 Bounding patterns

What makes it even more interesting is that any pattern avoiding all rotations of P_1 as interval minors is already bounding. For simplicity, whenever we say that a matrix has only k non-empty lines, we mean that every one-entry belongs to one of the k lines.

Theorem 4.15. *Let P be a pattern avoiding all rotations of P_1, then P*

1. *contains at most three non-empty lines or*

2. *avoids $\begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}$ or $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$.*

Proof. Assume P has four one-entries that do not share any row or column. Then those one-entries induce a 4×4 permutation inside P and because P does not contain any rotation of P_1, the induced permutation is either 1234 or 4321. Without loss of generality, assume it is the first one and denote its one-entries by e_1, e_2, e_3 and e_4. Clearly, no one-entry from e_1, e_2, e_3 and e_4 can be part of any mapping of $P' = \begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}$ because it would induce a mapping of a rotation of P_1.

Let $e_2 = P[r_2, c_2]$ and $e_3 = P[r_3, c_3]$. The submatrix $P[[r_2], [c_2, l]]$ avoids P'; otherwise, together with e_1 it would give P_2 as an interval minor. Symmetrically, $P' \not\preceq P[[r_3, k], [c_3]]$. The submatrix $P[[r_3 - 1], [c_3 - 1]]$ is empty; as otherwise, any one-entry would create a rotation of P_1 with e_3 and either e_1 or e_2. Symmetrically, the submatrix $P[[r_2 - 1], [c_2 - 1]]$ is also empty. This leave no one-entry in P to be used to map $P'[1, 1]$ and so $P' \not\preceq P$. $\square$

We now need to prove that whenever P avoids all rotations of P_1 (and satisfies one of the conditions we just showed) it is bounding.

Lemma 4.16. *Let $P \in \{0,1\}^{k \times l}$ be a pattern having one non-empty line. Then $r\left(Av_{\preceq}(P)\right) \leq k$ and $c\left(Av_{\preceq}(P)\right) \leq l$.*

Proof. Without loss of generality, let the non-empty line be a row r. Consider any matrix $M \in Av_{crit}(P)$. Submatrices $M[[r-1], [n]]$ and $M[[m-r+1, m], [n]]$ contain no zero-entry. If we look at any other row, it cannot contain k one-entries, so the maximum number of zero-intervals is k.

Consider a column c of M. If there is at least one one-entry in $M[[r, m - r - 1], c]$ then because M is critical, the whole column is made of one-entries. Otherwise, there are two one-intervals $M[[r-1], c]$ and $M[[m-r, m], c]$. $\square$

Lemma 4.17. *Let $P \in \{0,1\}^{k \times l}$ be a pattern having two non-empty lines. Then $r\left(Av_{\preceq}(P)\right) \leq k^2 + l$ and $c\left(Av_{\preceq}(P)\right) \leq l^2 + k$.*

Proof. First, we assume the two non-empty lines of P are rows $r_1 < r_2$ (or symmetrically columns) and consider any matrix $M \in Av_{crit}(P)$. From Observation 2.6 and maximality of M, we have that the submatrices $M[[r_1 - 1], [n]]$ and $M[[m - r_2 + 1, m], [n]]$ contain no zero-entry. Therefore, we may restrict ourselves to the case when $r_1 = 1$ and $r_2 = k$. From Corollary 4.12, we have that there are at most k^2 zero-intervals in each row of M and there are at most two zero-intervals in each column of M.

Let the two non-empty lines of P be a row r and a column c. Because of symmetry, we only show the bound for rows. For every one-entry e of P, except those in the row r, there is at most one zero-interval usable for e in each row of any $M Av_{crit}(P)$. For contradiction, assume there are two such zero-intervals z_1 and z_2 in the same row. Let Figure 4.2 illustrate the situation where red and blue lines form two mappings of P to M when a zero-entry of z_1 and z_2 respectively is changed to a one-entry used to map e. When we take the outer two vertical and horizontal lines, we get a mapping of P that uses an existing one-entry in between z_1 and z_2 to map e. This is a contradiction with $P \not\prec M$.

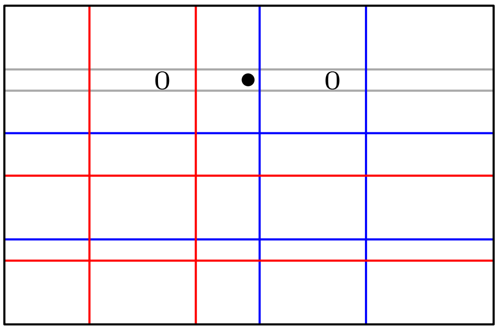


Figure 4.2: Red and blue lines representing two different mappings of a forbidden pattern. The two horizontal lines show the boundaries of the mapping of row r and the vertical lines show the boundaries of the mapping of column c.

For a one-entry $e = P[r, c']$, if $c' \leq c$ then there must be less than c' one-entries before any zero-intervals usable for e; otherwise, we could map $P[r, [1, c']]$ just to the single row of M. It follows that e is row-bounded. Symmetrically, the same holds in case $c' > c$ and together we have at most $k + l$ zero-intervals in each $M \in Av_{crit}(P)$. $\square$

Before we proof the other cases, let us introduce three useful lemmata that make the future case analysis bearable.

Lemma 4.18. *Let $P \in \{0,1\}^{k \times l}$ be one of the four matrices in Figure 4.3. Then every one-entry in $P[\{r_2\}, [c_1, c_2]]$ is row-bounded. Moreover, the same also holds if we change some one-entries to zero-entries.*

Proof. Let a pattern P be the first described matrix and let $k' = c_2 - c_1$. We show that for each one-entry $e \in P[\{r_2\}, [c_1, c_2]]$ and every matrix $M \in Av_{crit}(P)$ there are at most k' zero-intervals usable for e in each row of M. For contradiction, assume there is a row r with $k'+1$ zero-intervals usable for some e. It follows that there are at least k' one-entries in between the two most distant zero-intervals z_1 and z_2. Therefore, the whole row r_2 can be mapped just to the row r. Changing a zero-entry of z_1 to a one-entry, to which e can be mapped, creates a mapping of P to M, in which all one-entries from columns $[c_1]$ are mapped to columns before z_1 (and z_1) and similarly all one-entries from columns $[c_2, l]$ can be mapped to columns past z_2 (and z_2). It also holds that all the one-entries from the row r_1 are mapped (in both mappings) to one-entries of M in rows $[r - r_2 + r_1]$ (and symmetrically for one-entries from the row r_3). Thus, we can simply map empty rows $[r_1 + 1, r_3 - 1]$ around row r and use the rest to map rows r_1 and r_2.

Proofs of cases two and three are similar to the first one and we skip them.

Let a pattern P be the fourth described matrix and consider any matrix $M \in Av_{crit}(P)$. For the i-th one-entry e in the row r_2 (ordered from left to right

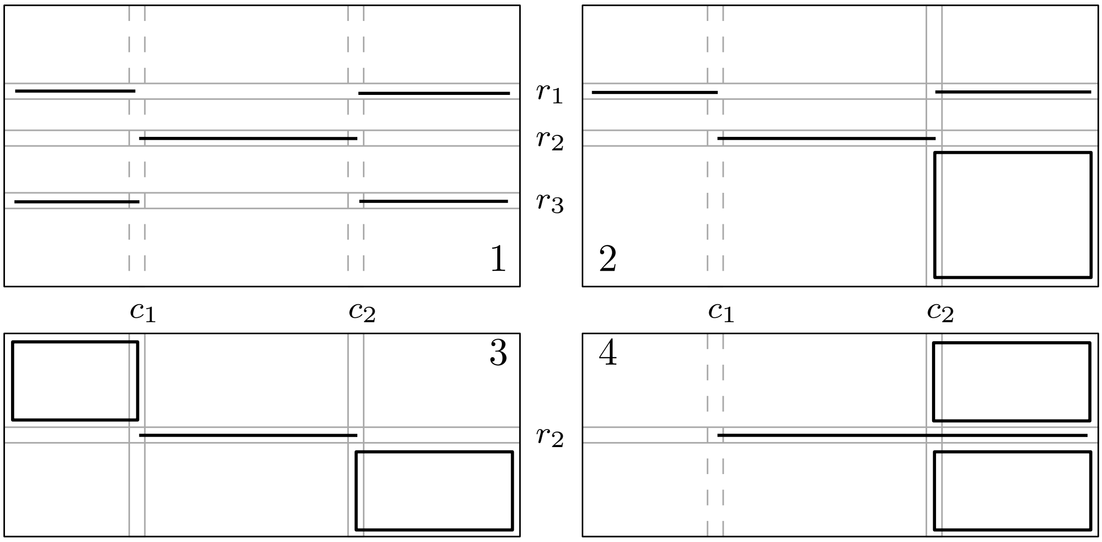


Figure 4.3: The patterns for which all one-entries in the row r_2 and the columns $[c_1, c_2]$ are row-bounded. One-entries of the patterns are inside the bold rectangles and on the bold lines.

and only considering those in columns $[c_1, c_2]$) no zero-interval of M usable for e cannot have i one-entries before it and so the row-complexity of each such one-entry is bounded by $i \geq l$.

Throughout the proof, we have never used as a fact that an entry of M is a one-entry and so the proof also holds for any pattern P created from any of the fourth described matrices by deletion of one-entries. $\square$

It is important to realize that we could not have used the same proof we used for the first three cases also for the fourth case, because we can never rely on the fact a mapping of P only uses one row of M to map the row r_2. This is because in the fourth case, there are also potential one-entries in $P[\{r_2\}, [c_2 + 1, l]]$.

What follows is a direct corollary of the fourth case of just stated Lemma 4.18. Even though it is very simple and straightforward, it is going to be used so often that it is worth stating it apart from the rest.

Lemma 4.19. *Let P be a matrix and let c be its first non-empty column. Then every one-entry from c is row-bounded.* $\square$

Lemma 4.20. *Let $P \in \{0, 1\}^{k \times l}$ be one of the three matrices in Figure 4.4. Then every one-entry in $P[[r_1 + 1, r_2 - 1], \{c\}]$ is row-bounded. Moreover, the same also holds if we change some one-entries to zero-entries.*

Proof. Let P be a submatrix of the first described matrix. We show that for each one-entry e from $P[[r_1 + 1, r_2 - 1], \{c\}]$ and every matrix $M \in Av_{crit}(P)$ there is at most one zero-interval usable for e in M. For contradiction, assume there is a row r with two zero-intervals z_1 and z_2 usable for e. Consider Figure 4.5, where the red lines show a mapping of P to M created when a zero-entry of z_1 is changed to a one-entry used to map e and the blue lines show a mapping of P to M created when a zero-entry of z_2 is changed to a one-entry used to map e. If we map the column c to the columns of M enclosed by the two outer vertical lines and map rows r_1 and r_2 again to rows enclosed by the corresponding two

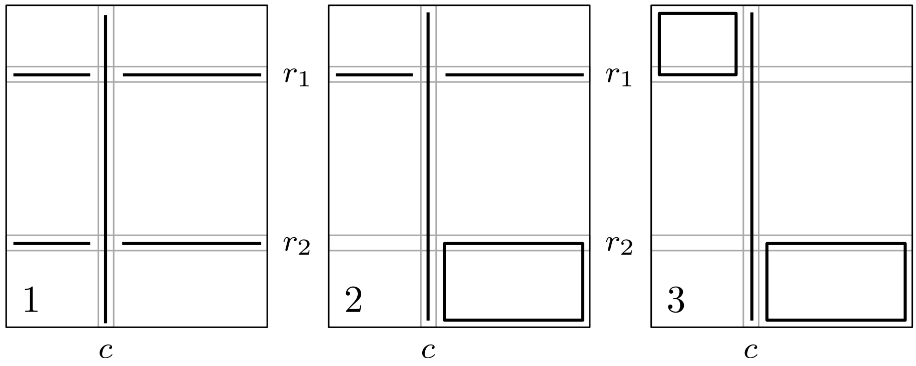


Figure 4.4: The patterns for which all one-entries in the column c and the rows $[r_1+1, r_2-1]$ are row-bounded. One-entries of the patterns are inside the bold rectangles and on the bold lines.

outer horizontal lines, we get a mapping of P to M and so a contradiction with $P \not\prec M$.

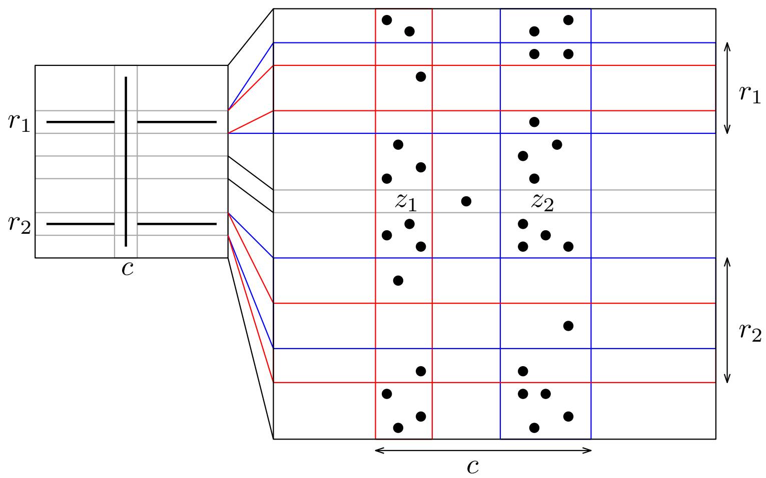


Figure 4.5: Red and blue lines representing two different mappings of a forbidden pattern. The four horizontal lines show the boundaries of the mapping of rows r_1 and r_2 and the vertical lines show the boundaries of the mapping of the column c.

Proofs of cases two and three are similar to the first one and we skip them.

Throughout the proof, we have never used as a fact that an entry of M is a one-entry and so the proof also holds for any pattern P created from any of the fourth described matrices by deletion of one-entries. $\square$

Lemma 4.21. *Let a pattern $P \in \{0,1\}^{k\times l}$ be created from one of the matrices in Figure 4.6 by deletion of one-entries and let $c = l-1$. Then every one-entry in $P[[r_1, r_2], \{c\}]$ is row-bounded.*

Proof. Let a pattern P be created from the first described matrix. From 4.20, we know that all one-entries in $P[[r_1+1, r_2-1], \{c\}]$ are row-bounded. Thank to symmetry, it suffices to show that the one-entry $e = P[r_1, c]$ is row-bounded. Without loss of generality, we have $P[r_2, l] = 1$; otherwise, we can use the fourth case of Lemma 4.3 to prove that e is row-bounded.

Consider any matrix $M \in Av_{crit}(P)$ and let $z_1 < z_2$ be any two zero-intervals from the same row usable for e. Without loss of generality, in any mapping of P

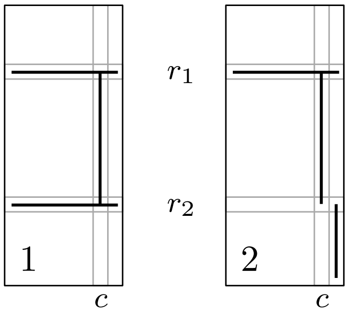


Figure 4.6: The patterns for which all one-entries in the column c and the rows $[r_1, r_2]$ are row-bounded. One-entries of the patterns are on the bold lines and the column c is the second last.

to M, created when a zero-entry of z_1 is changed to a one-entry used to map e, the one-entry $P[r_2, l]$ is mapped to a column before z_2. Otherwise, if we map e to the one-entry between z_1 and z_2 and map $P[r_1, l]$ to any one-entry behind z_2 we get a mapping showing $P \preceq M$.

We prove there are at most l zero-intervals usable for e on every row of M. For contradiction, let there be such zero-intervals $z_1, \ldots, z_l$ that there is a one-entry behind each of them. For each zero-interval z_i, let e_i be any one-entry of M that can be used to map the one-entry $P[r_2, l]$ if a zero-entry of z_i is changed to a one-entry used to map e. In the sequence $e_1, \ldots, e_l$ there either are two one-entries $M[r_1', c_1'], M[r_2', c_2']$ such that $r_1' \leq r_2'$, or the rows of one-entries form a decreasing sequence.

Let us first consider the first case and let $e_i = M[r_1', c_1']$ and $e_j = M[r_2', c_2']$. Consider a mapping of P to M created when a zero-entry of z_i is changed to a one-entry used to map e. If in this mapping, we map e to a one-entry between z_i and z_j, map $P[r_1, l]$ to a one-entry behind z_j, map $P[r_2, l-1]$ to e_i and map $P[r_2, l]$ to e_j, we get a mapping of P to M, which is a contradiction.

And so it holds that the one-entries $e_1, \ldots, e_l$ form a row decreasing sequence. We can pair every e_i with a one-entry bounding z_i from the right and so we can map the whole submatrix $P[[k], [l-2]]$ just to columns before z_{l-1} of M. Because z_l is usable for e, there are enough one-entries to map the whole column c there and there are one-entries where $P[r_1, l]$ and $P[r_2, l]$ can be mapped. The only problem is that e is mapped to a one-entry created by changing a zero-entry of z_l but we can also map it to a one-entry between zero-intervals z_{l-1} and z_l and we have $P \preceq M$ and a contradiction.

Let a pattern P be created from the second described matrix. All one-entries in $P[[r_1 + 1, r_2 - 1], \{c\}]$ are row-bounded thanks to (the second case of) Lemma 4.20. From the fourth case of Lemma 4.18, the one-entry $P[r_1, c]$ is also row-bounded. So we only need to prove that the one-entry $P[r_2, c]$ is row-bounded.

Without loss of generality, $P[r_1, l] = 1$; otherwise, $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \not\preceq P$ and in the following Lemma 4.22, we show that every such P is bounding. We once again define one-entries $e_1, \ldots, e_l$ and use the same analysis as we did in the first case.

$\square$

Now that the very technical lemmata are stated, we just use them to easily prove that the remaining patterns described in Theorem 4.15 are also bounding.

Lemma 4.22. *Let $P \in \{0,1\}^{k \times l}$ be a pattern avoiding $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$. Then P is bounding.*

Proof. From Proposition 2.15, we know that P is a walking pattern. Every one-entry of P satisfies either conditions of the third case of Lemma 4.18 or it satisfies conditions of the third case of Lemma 4.20 and therefore is row-bounded. From Observation 4.9, we know it is also column-bounded. □

What follows is the last and the most difficult case of our analysis. Its length is caused by the fact that it is harder to describe symmetries than it is to just use the previous lemmata to show that each pattern is bounding.

Lemma 4.23. *Let $P \in \{0,1\}^{k \times l}$ be a pattern having three non-empty lines and avoiding all rotations of P_1. Then P is bounding.*

Proof. First of all, if P avoids $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$, we use Lemma 4.22.

Let the three non-empty lines be three rows and let a pattern P have one-entries in at least three columns. Then it contains a 3×3 permutation matrix as a submatrix. Since the rotations of P_1 are avoided, the only feasible permutations are 123 and 321 and without loss of generality, we assume the first case. In Figure 4.7 we see the structure of P. The capital letters stand for one-entries of the permutation and are chosen to be the left-most possible, letters $a - f$ stand each for a potential one-entry and the Greek letters stand each for a potential sequence of one-entries. Everything else is empty. Not all one-entries can be there at the same time, because that would create a mapping of P_1 or its rotation. We also need to find $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right) \preceq P$. The following analysis only uses hereditary arguments, which means that if we prove that P is bounding, we also prove that each submatrix of P is bounding. With this in mind, we restrict ourselves to critical patterns.

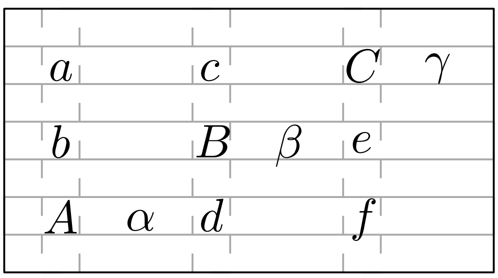


Figure 4.7: The structure of a pattern only having three non-empty rows and avoiding all rotations of P_1.

1. $\gamma = 1 \Rightarrow f = 0 \Rightarrow$ because $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right) \preceq P$, it holds $a = 1 \Rightarrow \alpha = 0$
   (a) $d = 1 \Rightarrow b = 0,\ \beta = 0,\ e = 0$
   (b) $d = 0$
      i. $c = 1 \Rightarrow \beta = 0,\ e = 0$
      ii. $c = 0$

2. $\gamma = 0$

   (a) $\alpha = 1 \Rightarrow a = 0,\ b = 0$. If $f = 0$ we have case 1. (b) ii.; otherwise, we have case 1. (a).

   (b) $\alpha = 0$

      i. $c = 1,\ d = 1 \Rightarrow b = 0,\ e = 0,\ \beta = 0$

      ii. $c = 1,\ d = 0 \Rightarrow e = 0,\ \beta = 0$ and without loss of generality, $b = 1$. Otherwise, we have the previous case. Therefore, $f = 0$

      iii. $c = 0,\ d = 1 \Rightarrow b = 0$. Without loss of generality, $e = 1,\ \beta = 1$. Otherwise, we have the case $c = 1,\ d = 1$. Therefore, $a = 0$

      iv. $c = 0,\ d = 0$

The same analysis also proves that if a pattern with the same restrictions only has three non-empty columns then it is bounding.

Let P be a pattern having two non-empty rows r_1, r_2 and one non-empty column c_1. Without loss of generality, we again assume permutation 123 is present and we distinguish three cases. Consider Figure 4.8:

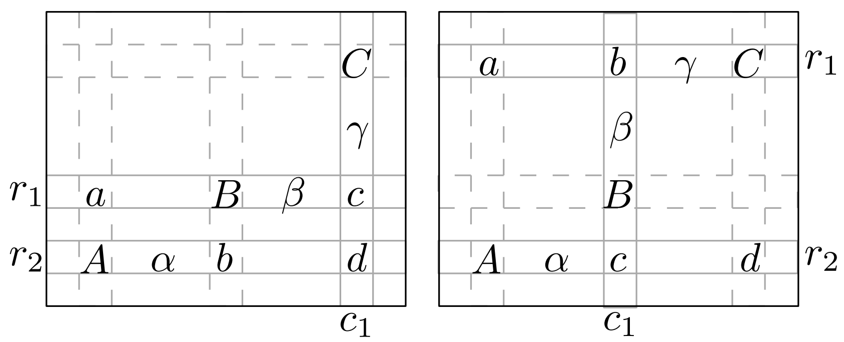


Figure 4.8: The structure of a pattern only having one-entries in two rows and one column that avoids all rotations of P_1.

1. C lies in column c_1

   (a) $a = 0$

   (b) $a = 1 \Rightarrow b = 0,\ \alpha = 0$

2. B lies in column c_1

   (a) $a = 1,\ d = 1 \Rightarrow \alpha = 0,\ \gamma = 0$

   (b) $a = 1,\ d = 0 \Rightarrow \alpha = 0$

   (c) $a = 0,\ d = 1 \Rightarrow \gamma = 0$

   (d) $a = 0,\ d = 0$. The pattern avoids $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right)$.

3. A lies in column c_1. This is symmetric to the first situation.

The same analysis also proves that if a pattern P has two non-empty columns and one non-empty row then the pattern is bounding. □

Combining the lemmata we finally get the following result.

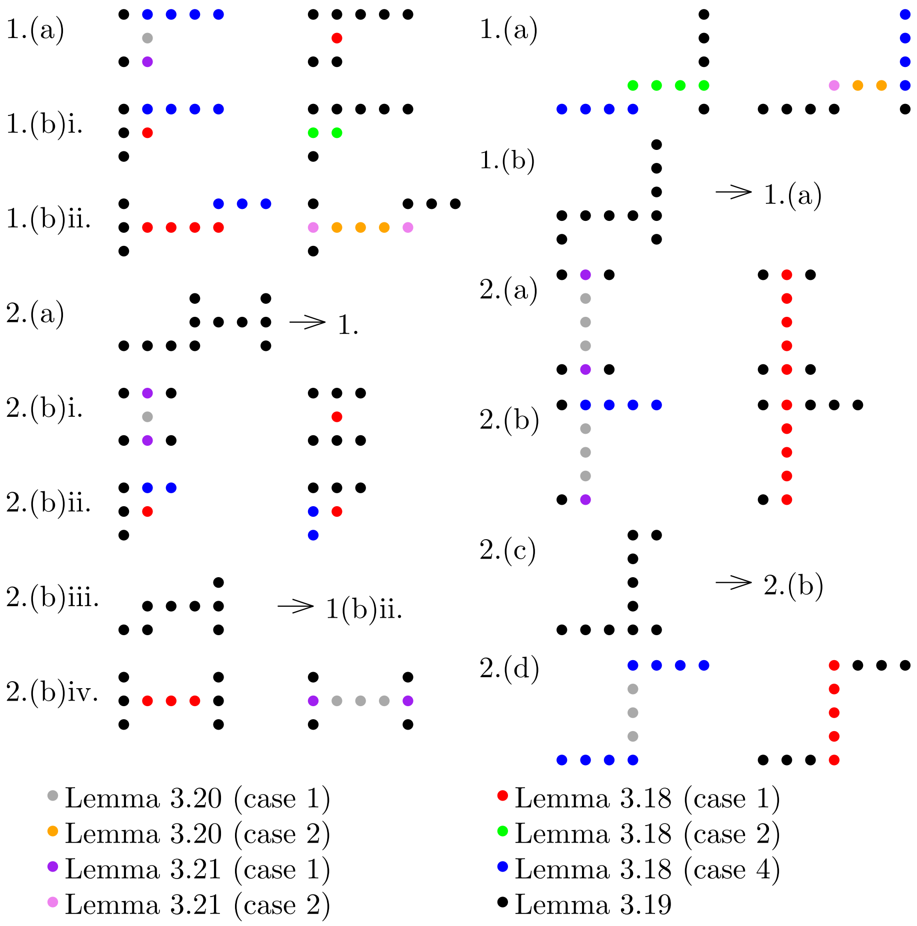


Figure 4.9: A figure showing which lemma can be used to prove that each one-entry of patterns discussed in the case analysis is bounded. The patterns from the left half of the picture only contain three non-empty rows and the patterns from the right half only contain two non-empty rows and one non-empty column. Each case either contains a picture showing that each one-entry is row-bounded and column-bounded, or an arrow describing that the case can be reduced to a different one.

Theorem 4.24. *Let P be a pattern avoiding all rotations of P_1, then P is bounding.* $\square$

A lot can be implied from this theorem. Here are two straightforward corollaries for which we do not know any other proof.

Corollary 4.25. *For every pattern P: $Av_{\preceq}(P)$ is row-bounded $\Leftrightarrow$ $Av_{\preceq}(P)$ is column-bounded.*

Corollary 4.26. *For every bounding pattern P and every $P' \preceq P$ it holds P' is bounding.*

4.2 Chain rules

Now that we know exactly what patterns are bounding, it is time to speak about the complexity of classes more in general. We are still going to be concerned with classes of matrices avoiding patterns, but they will avoid a set of patterns rather than just one pattern.

First, we show that Corollary 4.25 does not hold in general. Next, we show that bounded classes are closed to intersection. At the end of the chapter, we prove the same is not true for unbounded classes of matrices and even more, an intersection of a few unbounded classes can be bounded hereditarily, which means that its every subset is bounded.

It is easy to see that Lemma 4.18, Lemma 4.19, Lemma 4.20, Lemma 4.21 and Lemma 4.22 can be generalized to our settings. Their proofs without change show that for every set of patterns $\mathcal{P}$, if a pattern $P \in \mathcal{P}$ looks like a described pattern, then any one-entry of P is (row-)bounded in $Av_{\preceq}(\mathcal{P})$. Therefore, we use the lemmata without restating them.

We define classes of matrices to be bounded if they are both row-bounded and column-bounded. From what we proved so far, we see that for a pattern P, the class $Av_{\preceq}(P)$ is row-bounded if and only of it is column-bounded. Once we consider classes avoiding sets of patterns, this does not have to be true.

Lemma 4.27. *There exists a set of patters $\mathcal{P}$ such that the class $Av_{\preceq}(P)$ is row-bounded but column-unbounded.*

Proof. Let $\mathcal{P} = \left\{ P = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & \bullet & \\ & & \bullet \end{pmatrix}, I_4 = \begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix} \right\}$. We can use a similar construction to what we did in Lemma 4.13, to prove $Av_{\preceq}(\mathcal{P})$ is column-unbounded. The only difference is that the "blocks" are of size 4×2 and the whole matrix is transposed.

To prove that the class $Av_{\preceq}(\mathcal{P})$ is row-bounded, we take an arbitrary matrix $M \in Av_{crit}(\mathcal{P})$ and consider any row r of M. We need to prove that every one-entry of I_4 and P is row-bounded.

From Lemma 4.22, we know that every one-entry of I_4 is row-bounded (and column-bounded) in $Av_{\preceq}(\mathcal{P})$. From Lemma 4.19, one-entries $P[2,1]$ and $P[4,3]$ are row-bounded in $Av_{\preceq}(\mathcal{P})$. From the first case of Lemma 4.20, the one-entry $P[3,2]$ is row-bounded in $Av_{\preceq}(\mathcal{P})$.

We prove that there are at most two zero-intervals usable for $P[1,2]$ in the row r. For contradiction, let there be three zero-intervals $z_1 < z_2 < z_3$. Consider a mapping of P to M created when a zero-entry of z_3 is changed to a one-entry used to map $P[1,2]$. Without loss of generality, the one-entry used to map $P[2,1]$ lies in columns of z_3 or just under the one-entry e bounding z_3 from left; otherwise, we could use e to map $P[1,2]$ and find the pattern in M. Then, a one-entry between zero-intervals z_1 and z_2 together with the one-entries used to map $P[2,1], P[3,2]$ and $P[4,3]$ give us a mapping of I_4 and so a contradiction with $M \in Av_{\preceq}(\mathcal{P})$. $\square$

Theorem 4.28. *Let $\mathcal{P}$ and $\mathcal{Q}$ be classes of patterns. If both classes $Av_{\preceq}(\mathcal{P})$ and $Av_{\preceq}(\mathcal{Q})$ are bounded then $Av_{\preceq}(\mathcal{P} \cup \mathcal{Q})$ is bounded.*

Proof. Let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$. We show that $comp_{\mathcal{R}} \leq comp_{\mathcal{P}} + comp_{\mathcal{Q}} = C$.

For contradiction, let a matrix $M \in Av_{crit}(\mathcal{R})$ have at least $C+1$ zero-intervals in a single row (or column). Without loss of generality, it means there is more than $comp_{\mathcal{P}}$ zero-intervals usable for one-entries of the patterns from $\mathcal{P}$. Let $M' \in Av_{\preceq}(\mathcal{P})$ be a matrix created from M by changing as many zero-entries to one-entries as possible. Clearly, it still contains more than $comp_{\mathcal{P}}$ zero-intervals usable for one-entries of the patterns from $\mathcal{P}$, which is a contradiction with the value of $comp_{\mathcal{P}}$. ☐

Using induction, we can show that also a union of a finite number of bounded classes of finite sizes is bounded. Interestingly enough, unbounded classes are not closed the same way.

Theorem 4.29. *For every* $1 \leq i < j \leq 4]$ *is* $Av_{\preceq}(\{P_i, P_j\})$ *bounded.*

Proof. We only show that $Av_{\preceq}(P_1, P_2)$ is bounded. To prove $Av_{\preceq}(P_1, P_3)$ is bounded, we can use the same steps. All other pairs are then symmetric to these two.

- $Av_{\preceq}(P_1, P_2)$ is row-bounded:
  From Lemma 4.19, we have that one-entries $P_1[2,1], P_1[3,3], P_2[2,3]$ and $P_3[3,1]$ are row-bounded. For $P_1[1,2]$ and $P_2[1,2]$, we prove there are at most two zero-intervals usable for each of them in each row of any matrix $M \in Av_{crit}(P_1, P_2)$. For contradiction, let $z_1 < z_2 < z_3$ be three zero-intervals usable for $P_1[1,2]$ in a row r of M. The one-entries used to map $P_1[2,1]$ and $P_1[3,3]$ in a mapping created when a zero-entry of z_1 is changed to a one-entry used to map $P_1[1,2]$, together with a one-entry in between z_2 and z_3 give us a mapping of P_2 to M. Symmetrically, the same goes for $P_2[1,2]$.

- $Av_{\preceq}(P_1, P_2)$ is column-bounded:
  The proof that all one-entries of P_1 and P_2 are column-bounded is the same. ☐

We prove even stronger result for the class $Av_{\preceq}(P_1, P_2, P_3, P_4)$ by using a well known fact from the theory of ordered sets. It is due to Higman [1952] and states the following:

Fact 4.30 (Higman's lemma)**.** *Let A be a finite alphabet and A^* be a set of finite sequences over A partially ordered by the subsequence relation. Then A^* is well-quasi-ordered.*

In other words, whenever we have a potentially infinite $S \subseteq A^*$, there are sequences $a, b \in S$ such that a is a subsequence of b. This also means that no such S contains an infinite anti-chain.

Theorem 4.31. *The class* $\sigma = Av_{\preceq}(P_1, P_2, P_3, P_4)$ *is bounded. Moreover, every subclass of* σ *is bounded.*

Proof. We first prove that σ is bounded. Consider any critical matrix $M \in \sigma$. If it avoids $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ (or $\begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}$), in which case it is a walking matrix then it has at most two zero-intervals in each row and column. If M contains at most three non-empty rows (columns) then from the case analysis in Lemma 4.23, we see that there are at most four zero-intervals in each row and trivially, there are at most four zero-intervals in each column. Otherwise, M contains at most two non-empty rows and one non-empty column (or vice versa), and we again see from the case analysis of Lemma 4.23 that there are at most four zero-intervals in each row and column.

Now consider an arbitrary $\mathcal{M} \subseteq \sigma$. In terms of forbidden patterns, we have $\mathcal{M} = Av_{\preceq}(\{P_1, P_2, P_3, P_4\} \cup \mathcal{P})$ for some set of matrices $\mathcal{P} \subseteq \sigma$. If $\mathcal{P}$ is finite then we can use iterated Theorem 4.28 to show that $\mathcal{M}$ is bounded.

Assume that $\mathcal{P}$ is infinite. Then we want to find a finite subset $\mathcal{P}'$ such that for every $P \in \mathcal{P}$ there is $P' \in \mathcal{P}'$ with $P' \preceq P$. In other words, we need to prove that no $\mathcal{P}$ contains an infinite anti-chain. To do so, we use Fact 4.30.

As the relation of being interval minor is a partial ordering on any set of matrices, we define a finite alphabet A and define a word $w_M \in A^*$ for every matrix $M \in \sigma$ is such a way, that for every two words $w_P, w_M \in A^*$ it holds that if w_P is a subsequence of w_M then $P \preceq M$.

- For all matrices $M \in \sigma$ that have at most three non-empty rows (we proceed symmetrically if it has at most three non-empty columns), we use words over alphabet $A = \{a, b, c, d, e, f, g, h, i, j\}$. Let $r_1 < r_2 < r_3$ be the non-empty rows (if less then three are non-empty then we choose extra values arbitrarily). We define $w_M \in A^*$ as follows. First, we use the letter g r_1-times, the letter h $(r_2 - r_1)$-times, the letter i $(r_3 - r_2)$-times and the letter j $(m - r_3)$-times to describe the number of rows of M and the position of non-empty rows. Then we describe the matrix column by column as follows. For each 0 in r_1, we use the letter a and for 1, we use letters ab. For each 0 in r_2, we use the letter c and for 1, we use letters cd. For each 0 in r_3, we use the letter e and for 1, we use letters ef.

  Let $w_P, w_M \in A^*$ be two words such that w_P is a subsequence of w_M. Let r_1, r_2, r_3 and r_1', r_2', r_3' be the non-empty rows of P and M respectively. Since the number of leading letters g is not bigger in w_P, P does not have more empty rows before r_1 than M does before r_1' and similarly for the other pairs of non-empty rows.

  Now consider there is a sequence ab in w_P and it corresponds to some $a \cdots b$ in w_M. Without loss of generality, the letter a in w_P is the one exactly before the letter b. Clearly, one-entries of P can be mapped to one-entries of M and we only need to check that two one-entries of two different columns of P are not mapped to two one-entries of the same column of M. This is not hard to see and we have $P \preceq M$ (but it does not have to hold that $P \leq M$).

- For all matrices $M \in \sigma$ that have at most two non-empty rows and a non-empty column (we proceed symmetrically if it has at most two non-empty columns and a non-empty row), we use words over alphabet $A = \{a, b, c, d, e, f, g\}$ and for non-empty rows r_1, r_2 and a column c_1, we define w_M as follows. We first encode the matrix column by column in such a way that for each 0 in r_1, we use the letter a and for 1, we use letters ab. For each 0 in r_2, we use the letter c and for 1, we use letters cd. Right before and right after the description of the column c_1, we put the letter g. Next, we encode each row in such a way that for each 0 in c_1 we use the letter e and for each 1, letters ef. Right before and right after the descriptions of rows r_1 and r_2 we again place the letter g.

  Because of the distinct letters for encoding rows and columns we can apply the same analysis as we did in the previous case and since the entries

$M[r_1, c_1]$ and $M[r_2, c_1]$ are separated from the rest by the letter g there is no way to find a one-entry where it is not.

- For all matrices $M \in \sigma$ avoiding $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ (we proceed symmetrically if it avoids $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$), we use words over alphabet $A = \{a, b, c, d\}$ and encode the matrix as follows. We choose an arbitrary walk of M containing all one-entries and index its entries as $w_1 \dots w_{m+n-1}$. Starting from w_1, we encode w_i so that the letter a stands for 0 and letters ab for 1, if w_{i+1} lies in the same row as w_i, and we use the letter c for 0 and letters cd for 1, if w_{i+1} lies in the same column as w_i. We always use a or ab for the last entry.

  We again need to check that if w_P is a subsequence of w_M then $P \preceq M$. For contradiction, assume that two one-entries of two different rows of P are mapped to two one-entries e, e' in the same row of M. Then in w_P the corresponding one-entries are separated by (or equal to) the letter c and so the letter also appear in w_M, which is a contradiction with the one-entries e, e' being in the same row of M.

In the construction of words corresponding to matrices, we only make sure that if w_P is a subsequence of w_M then $P \preceq M$ and the other implication does not need to hold. A different construction may lead to equivalence, but it is not necessary for our purposes.

We use distinct alphabets to describe matrices from different categories and when given a potentially infinite class of matrices $\mathcal{P}$, we know from Fact 4.30 that inside each category there is at most finite number of minimal (with respect to interval minors) matrices. Using induction on Theorem 4.28, we have that each $\mathcal{M} \subseteq \sigma$ is bounded. $\square$

Observation 4.32. *There exists a bounding pattern P having an unbounded subclass of $Av_{\preceq}(P)$.*

Proof. Let $P = I_n$ (identity matrix) for $n > 3$. From Lemma 4.22, we have that P is bounding. On the other hand, $Av_{\preceq}(I_n, P_1)$ is unbounded, because the construction used in the proof of Lemma 4.13 also works for this class. $\square$

Conclusion

Throughout the thesis, we have been looking from multiple angles at classes of binary matrices. In particular, we studied properties of matrices containing or avoiding other, smaller matrices as interval minors.

Small interval minors We started by describing the structure of matrices avoiding some small patterns. We managed to characterize all matrices avoiding patterns having up to three one-entries and also showed how to generalize some of the characterizations for much bigger patterns. Even for very small patterns (only having four one-entries), the structure of matrices avoiding them became very rich and hard to properly describe.

So instead, we began to consider how a small change of the pattern influences the structure of matrices avoiding the pattern. This was mainly looked at when adding empty lines to the pattern. We showed for patterns of size $k \times 2$ what matrices avoiding its blown version look like, but we are unable to generalize the result (or show something similar) when the pattern we are working with is bigger. So the question, answer for which could later be used to describe complexity of some patterns or help enumerate matrices from some class, is:

Question 4.33. What can we say about matrices avoiding a pattern with added empty line with respect to the matrices avoiding the original pattern?

The basis of a class of matrices After dealing with small patterns, we defined an operation of the direct sum and a closure under the direct and began to explore how it relates with classes of matrices. Once again we started by considering some small cases, where we observed that the closure can be describe by forbidden patterns very naturally. Later, we considered the direct sum with an overlap, which allowed us to restate the characterizations from the previous chapter in a much easier way.

We also introduced a notion of articulation that allowed us to proof a strong result saying that any matrix closed under the direct sum can always be describe by a finite set of forbidden patterns.

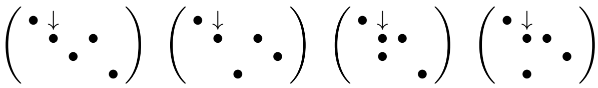
In order to generalize the result, we started looking at sets of matrices and ultimately, we showed that there are set of matrices with finite basis, whose closure cannot be described with a finite number of forbidden minors.

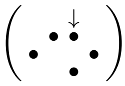
Zero-intervals In the last chapter, we studies a property of a class of matrices that in some terms describes the complexity of inclusion maximal matrices from the class.

To bring the notion back to the pattern avoiding, we defined a pattern P to be bounding if and only of the class $Av_{\preceq}(P)$ is bounded and we showed that a pattern is bounding if and only if it avoids all rotations of $P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$.

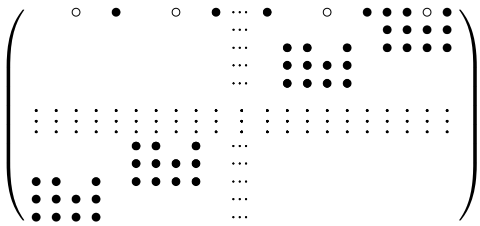
To show for a pattern that it is bounding meant to show that each one-entry it contains is bounded. It may be an interesting generalization to show for each one-entry whether it is bounded or unbounded in its pattern. On our way, we only saw one type of unbounded one-entries, and those are $P_1[2,1]$ for rows, $P_1[1,2]$ for columns and corresponding one-entries in the rotations of P_1. Let us call these one-entries *trivially unbounded*.

Considering this generalization, there are one-entries that are unbounded but not trivially unbounded. Let us mention some of them (arrows point to row-unbounded one-entries):



Proposition 4.34. *Let* $P =$ . *For every integer n there is a matrix $M \in Av_{crit}(P)$ having at least n zero-intervals.*

Proof. Let M be a matrix described by the picture:



We see that $P \not\preceq M$ because we always need to map $P[2,1]$ and $P[3,3]$ to just one "block" of one-entries of M, which only leaves a zero-entry where we need to map $P[1,3]$ or $P[2,4]$.

When we change any marked zero-entry of the first row into a one-entry, we get a matrix containing a minor of $\{1\}^{3\times 4}$; therefore, containing P as an interval minor. In case M is not maximal, we can add more one-entries to make it maximal but it will still contain a row with n one-intervals. ☐

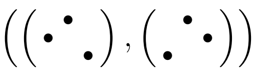
Our tools are not strong enough to let us characterize unbounded one-entries. Based on our attempts, we state the following conjecture:

Conjecture 4.35. Every row-unbounded one-entry share a row with a trivially row-unbounded one-entry.

Throughout the chapter, we work with arguments such that if something holds for a matrix, it also holds for every submatrix. While it seems completely natural, we are unable to decide the following question:

Question 4.36. Can a non-bounding pattern become bounding after a one-entry is changed to a zero-entry?

Using our machinery, we showed that while the union of bounding sets of patterns is always bounding again, the union of non-bounding sets may become bounding. For the class of matrices avoiding all rotations of P_1, we even showed that every subclass is also bounded. The same remains open for other sets of patterns:

Question 4.37. Is $Av_{\preceq}$  hereditarily bounded?

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List of Figures