Introduction

Throughout the paper, every time we speak about matrices we mean binary matrices (also called 01-matrices) and we omit the word binary. If we speak about a *pattern*, we again mean a binary matrix and we use the word in order to distinguish among more matrices as well as to indicate relationship.

When dealing with matrices, we always index rows and column starting with one and when we speak about a row r, we simply mean a row with index r. A *line* is a common word for both a row and a column. When we order a set of lines, we first put all rows and then all columns. For $M \in \{0,1\}^{m\times n}$, $[m]$ is a set of all rows and $[m+n]$ is a set of all lines, where m-th element is the last row. This goes with the usual notation.

Notation 1. For $n \in \mathbb{N}$ let $[n] := \{1, 2, \dots, n\}$ and for $m \in \mathbb{N}$, where $n \leq m$ let $[n, m] := \{n, n+1, \dots, m\}$.

Notation 2. For a matrix $M \in \{0,1\}^{m\times n}$ and $L \subseteq [m+n]$ let $M[L]$ denote a submatrix of M induced by lines in L.

Notation 3. For a matrix $M \in \{0,1\}^{m\times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M[R, C]$ denote a submatrix of M induced by rows in R and columns in C. Furthermore, for $r \in [m]$ and $c \in [n]$ let $M[r, c] := M[\{r\}, \{c\}] = M[\{r, c+m\}]$.

Definition 1. We say a matrix $M \in \{0,1\}^{m\times n}$ *contains* a pattern $P \in \{0,1\}^{k\times l}$ *as a submatrix* and denote it by $P \leq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M[R, C][r, c] = 1$.

This does not necessarily mean $P = M[R, C]$ as $M[R, C]$ can have more one-entries than P does.

Notation 4. For a matrix $M \in \{0,1\}^{m\times n}$ and $L \subseteq [m+n]$ let $M_{\preceq}[L]$ denote a matrix acquired from M by applying following operation for each $l \in L$:

- If l is the first row in L then we replace the first l rows by one row that is a bitwise OR of replaced rows.
- If l is the first column in L then we replace the first $l - m$ columns by one column that is a bitwise OR of replaced columns.
- Otherwise, we take l's predecessor $l' \in L$ in the standard ordering and replace lines $[l'+1, l]$ by one line that is a bitwise OR of replaced lines.

Notation 5. For a matrix $M \in \{0,1\}^{m\times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M_{\preceq}[R, C] := M_{\preceq}[R \cup \{c + m | c \in C\}]$.

Definition 2. We say a matrix $M \in \{0,1\}^{m\times n}$ *contains* a pattern $P \in \{0,1\}^{k\times l}$ *as an interval minor* and denote it by $P \preceq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M_{\preceq}[R, C][r, c] = 1$.

Observation 1. *For all matrices M and P, $P \leq M \Rightarrow P \preceq M$.*

Observation 2. *For all matrices M and P, if P is a permutation matrix, then $P \leq M \Leftrightarrow P \preceq M$.*

Proof. If we have $P \preceq M$, then there is a partitioning of M into rectangles and for each one-entry of P there is at least one one-entry in the corresponding rectangle of M. Since P is a permutation matrix, it is sufficient to take rows and columns having at least one one-entry in the right rectangle and we can always do so.

Together with Observation 1 this gives us the statement. □

0.1 Characterizations

Definition 3. A *walk* in a matrix M is a sequence of some of its entries, beginning in the top left corner and ending in the bottom right one. If an entry $M[i,j]$ is in the sequence, the next one is either $M[i+1,j]$ or $M[i,j+1]$.

Definition 4. We call a binary matrix M a *walking matrix* if there is a walk in M such that all one-entries of M are contained on the walk.

0.1.1 Patterns of size 2×2

Theorem 3. *Let $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ is a walking matrix.*

Proof. Since P is a permutation matrix, $P \not\preceq M \Leftrightarrow P \not\leq M$ and it is easy to see $P \not\leq M \Leftrightarrow M$ is a walking matrix. $\square$

Theorem 4. *Let $P = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then for all $M \in \{0,1\}^{m \times n}$: $P \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 1)*

- *$M[[r-1],[c-1]]$ is empty,*
- *$M[[r-1],[c+1,n]]$ is empty,*
- *$M[[r+1,m],[c-1]]$ is empty and*
- *$M[[r,m],[c,n]]$ is a walking matrix.*

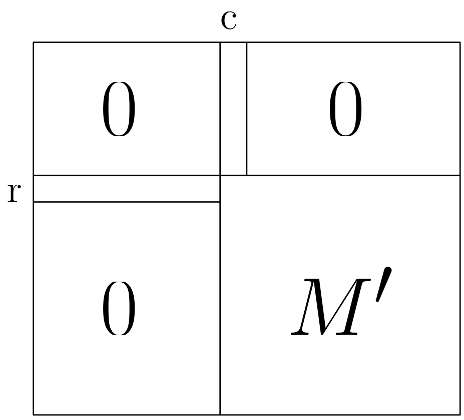

Figure 1: Characterization of a matrix avoiding $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as an interval minor.

Proof. $\Rightarrow$ If $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M$, then M is a walking matrix and we set $r = c = 1$. Otherwise, there are one-entries $M[r,c']$ and $M[r',c]$ such that $r' < r$ and $c' < c$. If there is a one-entry in regions $M[[r-1],[c-1]]$, $M[[r-1],\ [c+1,n]]$ or $M[[r+1,m],[c-1]]$ then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and we again get a contradiction.

$\Leftarrow$ For contradiction, assume that M described in Figure 1 contains P as an interval minor. It means that there is a partition of the matrix into four quadrants such that there is at least one one-entry in each quadrant besides the bottom right one. If the matrix is partitioned above the r-th row, then there is only one column containing one-entries and it is not possible for both top quadrants to have a one-entry. Similarly, if the matrix is partitioned to the left of the c-th column, there is only one row containing one-entries and there is no one-entry in either top-left or bottom-left quadrant. Therefore, the partitioning lies bellow the r-th row and to the right of the c-th column, but if the quadrants contain one-entries, there is a $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ interval minor in M', which is a contradiction with it being a walking matrix.

□

To characterize matrices avoiding $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ as an interval minor, we first need to define a few useful terms.

Definition 5. For $M \in \{0,1\}^{m\times n}$ and $r \in [m], c \in [n]$ we say $M[r,c]$ is

- *top-left empty* if $M[[r-1],[c-1]]$ is an empty matrix,
- *top-right empty* if $M[[r-1],[c+1,n]]$ is empty,
- *bottom-left empty* if $M[[r-1],[c+1,n]]$ is empty,
- *bottom-right empty* if $M[[r-1],[c+1,n]]$ is empty.

Lemma 5. *Let $P = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ and let $M \in \{0,1\}^{m\times n}$ avoid P as an interval minor, then there exists a row r and a column c such that $M[r,c]$ is either*

1. *both top-left empty and bottom-right empty and $[r,c] \notin \{[1,n],[m,1]\}$ or*
2. *both top-right empty and bottom-left empty and $[r,c] \notin \{[1,1],[m,n]\}$.*

Proof. □

Theorem 6. *Let $P = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like one of the matrices in Figure 2, where $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$, $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) \not\preceq M_2$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_3$ and $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) \not\preceq M_4$.*

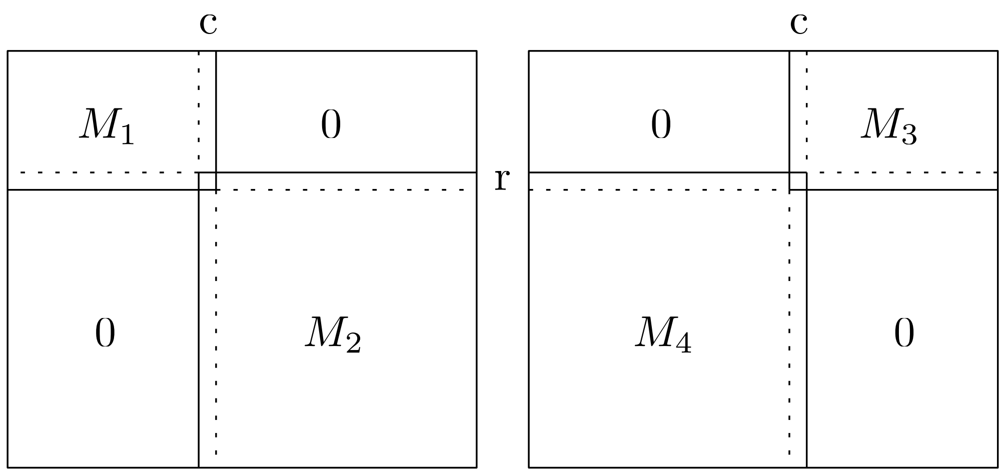

Figure 2: Characterization of a matrix avoiding $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ as an interval minor.

Proof.

$\Rightarrow$ We proceed by induction by the size of M.

If $M \in \{0,1\}^{2\times 2}$ then it either avoids $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and we are done.

For bigger M there is, from Lemma 5, "the element". Assume the first case (top-right and bottom-left empty (will change this when I have some notation)). If M_1 is non-empty, then $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \not\preceq M_2$; otherwise, $P \preceq M$. Similarly, $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M_1$ if M_2 is non-empty. If one of them is empty, the other is a smaller matrix avoiding P as an interval minor and by induction hypothesis, it can be partitioned. Adding empty rows and columns does not break any condition and we get a partitioning of the whole M.

$\Leftarrow$ Without loss of generality, let us assume M looks like the left matrix in Figure 2. For contradiction, assume $P \preceq M$. In that case, we can partition M into four quadrants such that there is at least one one-entry in each of them. It does not matter where we partition it, every time we either get $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \preceq M_1$ or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \preceq M_2$, which is a contradiction. $\square$

0.1.2 Matrices of size 2×3

Theorem 7. *Let $P = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M = M_1 \oplus_h M_2$ where $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ Let $e = [r, c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \preceq M[[m], [c-1]]$, together with e it forms P. If $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M[[m], [c, n]]$ then we are done. Let us assume it is not the case and let $e_{0,0},\ e_{1,1}$ be any two one-entries forming the forbidden pattern. Symmetrically, let $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \preceq M[[m], [c]]$ and let $e_{0,1},\ e_{1,0}$ be any two one-entries forming the forbidden pattern. Now if we take $e_{0,0},\ e_{0,1}$ and $e_{1,0}$ or $e_{1,1}$ with bigger row, we get the forbidden pattern P as an interval minor of M.

$\Leftarrow$ For contradiction, let us assume $P \preceq M$ and $M = M_1 \oplus_h M_2$. If $P \preceq M$, look at the one-entry of M where the bottom one-entry of P is mapped. If it is in M_1 then $P \not\preceq M$ because $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M_1$. Otherwise, $P \not\preceq M$ because $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \not\preceq M_2$.

$\square$

Lemma 8. *Let $P = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Rightarrow M = M_1 \oplus_h M_2$ where*

1. *$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M_2$ or*
2. *$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M_2$.*

Proof. Let $e = [r, c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \preceq M[[m], [c-1]]$, together with e it would be the whole P. Similarly, $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M[[m], [c+1, n]]$. For contradiction with the statement, let $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \preceq M[[m], [c]]$ and $e_{0,0},\ e_{1,1}$ (none of them equal to e, since e lies in the top-right corner) be any two one-entries forming the pattern. Symmetrically, let $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \preceq M[[m], [c, n]]$ and $e_{0,1},\ e_{1,0}$ be any two one-entries forming the pattern. In that case $e_{0,0},\ e,\ e_{0,1}$ and $e_{1,0}$ or $e_{1,1}$ with bigger row give us the forbidden pattern P as an interval minor of M. $\square$

Theorem 9. *Let $P = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like the matrix in Figure 3 and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M_2$.*

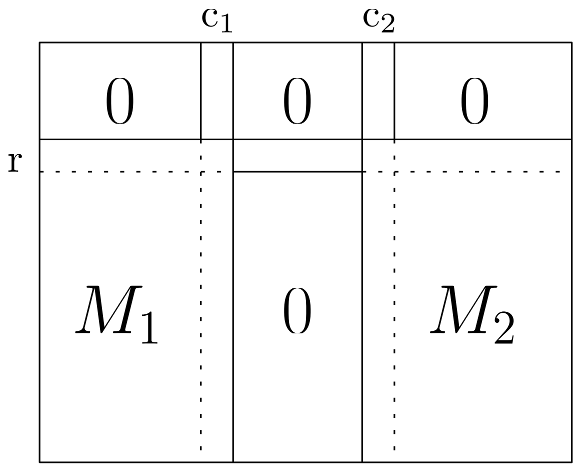

Figure 3: Characterization of a matrix avoiding $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ as an interval minor.

Proof. $\Rightarrow$ From Lemma 8 we know $M = M_1' \oplus_h M_2'$ where $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_1'$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2'$. The second case would be dealt with symmetrically. From Theorem 4 we have that M_1' can be characterized exactly like $M[[m],[c_2-1]$ and $M[[m],[c_2,n]]$ forms a walking matrix. The only problem with our claim would be if there were two different columns having a one-entry above the r-th row. In that case, those two one-entries together with a one-entry in the r-th row between the columns c_1 and c_2 and a one-entry in the c_1-th column above the r-th row form P as an interval minor.

$\Leftarrow$ The bottom-middle one-entry of P can not be mapped anywhere but to the r-th row, but in that case there are at most two columns having one-entries above it.

$\square$

0.2 Extremal function

Notation 6. Let M be a matrix. We denote $|M|$ the weight of M, the number of one-entries in M.

Usually $|M|$ stands for a determinant of matrix M. However, in this paper we do not work with determinants at all so the notation should not lead to misunderstanding.

Definition 6. For a matrix P we define $Ex(P, m, n) := \max\{|M| | M \in \{0,1\}^{m \times n},\ P \not\leq M\}$. We denote $Ex(P, n) := Ex(P, n, n)$.

Definition 7. For a matrix P we define $Ex_{\preceq}(P, m, n) := max\{|M| | M \in \{0,1\}^{m \times n},\ P \not\preceq M\}$. We denote $Ex_{\preceq}(P, n) := Ex_{\preceq}(P, n, n)$.

Observation 10. *For all P, m, n; $Ex_{\preceq}(P, m, n) \leq Ex(P, m, n)$.*

Observation 11. *If $P \in \{0,1\}^{k \times l}$ has a one-entry at position $[a, b]$, then*

$$Ex(P, m, n) \geq \begin{cases} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \textit{otherwise.} \end{cases}$$

Observation 12. *The same holds for $Ex_{\preceq}(P, m, n)$.*

Definition 8. $P \in \{0,1\}^{k \times l}$ is *(strongly) minimalist* if

$$Ex(P, m, n) = \begin{cases} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Definition 9. $P \in \{0,1\}^{k \times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P, m, n) = \begin{cases} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Observation 13. *If P is strongly minimalist, then P is weakly minimalist.*

0.2.1 Known results

Fact 14.

1. *$\left(\begin{smallmatrix}1\end{smallmatrix}\right)$ is strongly minimalist.*
2. *If $P \in \{0,1\}^{k \times l}$ is strongly minimalist and there is a one-entry in the last row in the c-th column, then $P' \in \{0,1\}^{k+1 \times l}$, which is created from P by adding a new row having a one-entry only in the c-th column, is strongly minimalist.*
3. *If P is strongly minimalist, then after changing a one-entry into a zero-entry it is still strongly minimalist.*

Fact 15. *Let $P = \left(\begin{smallmatrix}1 \cdots 1\\ 1 \cdots 1\end{smallmatrix}\right)$ have l columns, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{2\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{weight of } M[[i],\{j\}] > 0 \wedge \text{weight of } M[[i+1,m],\{j\} > 0]\}$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M appears in at least $b_j - 1$ of sets A_i, $0 \le i \le m-2$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 1) + n \le \sum_{i=0}^{m-2} |A_i| + n \le (l-1)(m-1) + n$$

□

This result is indeed very important because it shows that there are matrices like $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, which are weakly minimalist, although it is known they are not strongly minimalist.

Fact 16. *Let* $P = \begin{pmatrix} 1\ldots1 \\ 1\ldots1 \\ 1\ldots1 \end{pmatrix}$ *have l columns, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{3\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{ weight of } M[[i-1],\{j\}] > 0 \wedge \text{ weight of } M[[i+1,m],\{j\} > 0 \wedge M[i,j] \text{ one-entry}]\}$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M (for which $b_j \ge 2$) appears in exactly $b_j - 2$ of sets A_i, $1 \le i \le m-1$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 2) + 2n \le \sum_{i=1}^{m-2} |A_i| + 2n \le (l-1)(m-2) + 2n$$

□