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Hereditary classes of binary matrices

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Title: Hereditary classes of binary matrices

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Abstract: Interval minors of binary matrices were introduced by Jacob Fox in the study of Stanley-Wilf limits. We study what can be implied from their relation to the theory of pattern avoidance of submatrices, which is a very popular area of discrete mathematics. We start by characterizing matrices avoiding small interval minors. We then consider classes of matrices closed under interval minors and with some help of the operation of skew sum, we find classes of matrices that cannot be described by a finite number of forbidden interval minors. We also define and study a variant of a classical extremal Turán-type question studied in the area of combinatorics of permutations and binary matrices and in combinatorial geometry.

Keywords: binary matrix pattern-avoiding interval minor

Contents

15		
16	1 Introduction	2
17	1.1 The main results	2
18	1.2 Preliminaries	2
19	1.3 Pattern avoidance	5
20	2 Small interval minors	9
21	2.1 Empty rows and columns	10
22	2.2 Patterns having two one-entries	13
23	2.3 Patterns having three one-entries	14
24	2.4 Patterns having four one-entries	15
25	2.5 Multiple patterns	18
26	3 The basis of a class of matrices	19
27	3.1 The skew and direct sums	19
28	3.2 Articulations	21
29	3.3 Bases	23
30	4 Zero-intervals	26
31	4.1 Pattern complexity	26
32	4.1.1 Adding empty lines	27
33	4.1.2 Non-bounding patterns	28
34	4.1.3 Bounding patterns	30
35	4.2 Chain rules	37
36	Conclusion	42
37	Bibliography	44
38	List of Figures	46

1. Introduction

A binary matrix (or 0–1 matrix) is a matrix with ones and zeroes as its entries. In the thesis, we only consider binary matrices and so we omit the word binary. We say that a matrix M contains a matrix P as an interval minor, if P can be created from M by a sequence of deletion of one-entries and merges of neighboring rows or columns. Otherwise, we say M avoids P . To distinguish among matrices and to indicate the relationship, we usually call the matrix P a *pattern*.

When working with matrices, we always index rows from top to bottom and columns from left to right, starting with one. When we speak about a row r , we mean a row with index r . A *line* of a matrix is either a row or a column.

1.1 The main results

While a lot is known about matrices in general, because they can intuitively represent much more complex objects, interval minors are a fairly new topic and so we have a choice of the direction from which we want to approach them.

To get familiar with definitions and pattern avoidance in general, in Chapter 2, we focus on small patterns (having up to four one-entries only) and describe the common structure of matrices avoiding them.

We then turn our focus elsewhere in Chapter 3, and instead of looking for a structure of matrices avoiding a pattern, given a class of matrices (closed under interval minors) we find the smallest set of forbidden patterns that characterizes the class. We introduce the skew sum of two matrices and show that classes of matrices closed under the skew sum can always be described by a finite number of forbidden patterns. Using the operation more, we show that there are also other classes for which this cannot be achieved.

Because it is very useful to study extremal questions like the maximum number of one-entries of a matrix from a given class of matrices, in Chapter 4, we study a variant of such complexity question, where we instead focus on the maximum number k of appearances of pairs “01” and “10” on a single line of a matrix from a given class of matrices. We show that even for classes that are described by just one forbidden pattern, k can be unbounded, and we characterize exactly for which pattern this holds. Then we generalize the approach and show what influence an intersection of classes has on the number k .

1.2 Preliminaries

Notation 1.1. For $n \in \mathbb{N}$, let $[n] := \{1, 2, \dots, n\}$ and for $m \in \mathbb{N}$ such that $n \leq m$, let $[n, m] := \{n, n+1, \dots, m\}$.

Notation 1.2. For a matrix $M \in \{0, 1\}^{m \times n}$, $R \subseteq [m]$ and $C \subseteq [n]$, let $M[R, C]$ denote a submatrix of M induced by row indices in R and column indices in C . Furthermore, for $r \in [m]$ and $c \in [n]$, let $M[r, c] := M[\{r\}, \{c\}]$.

The pattern avoidance for matrices is a generalization of a long studied theory of pattern avoidance for permutations. There are two generally used ways to

define this generalization, either we avoid a matrix pattern as a submatrix or as an interval minor. While this thesis works almost exclusively with the latter, to better introduce the whole area, we start by defining the more know of the two approaches.

Definition 1.3. We say a matrix $M \in \{0, 1\}^{m \times n}$ *contains* a pattern $P \in \{0, 1\}^{k \times l}$ as a *submatrix* and denote it by $P \leq M$ if there are $R \subseteq [m]$ and $C \subseteq [n]$ such that $M' = M[R, C] \in \{0, 1\}^{k \times l}$ and for every $r \in R$ and $c \in C$, if $P[r, c] = 1$ then $M'[r, c] = 1$.

Every matrix $M \in \{0, 1\}^{m \times n}$ can be looked at as an adjacency matrix of a bipartite graph G_M with two sets of vertices $V_1 = [m]$ and $V_2 = [n]$ such that a vertex i from V_1 is adjacent to a vertex j from V_2 if and only if $M[i, j] = 1$. The order of vertices in each set is fixed and these graphs are usually called ordered bipartite graphs. In this setting, a matrix M contains a pattern P if the ordered bipartite graph G_P is a subgraph (not necessarily induced) of the ordered bipartite graph G_M .

In graph theory, the next step is to look at graph minors. A minor is created from a graph by a repeated applying of one of three graph operations: deletion of a vertex, deletion of an edge and a contraction of an edge. The same can be represented in terms of matrices:

Definition 1.4. We say a matrix $M \in \{0, 1\}^{m \times n}$ *contains* a pattern $P \in \{0, 1\}^{k \times l}$ as an *interval minor* and denote it by $P \preceq M$ if there is a sequence of elementary operations that applied to M creates P . The elementary operations are:

- a deletion of a line,
- a deletion of a one-entry (a change of a one-entry to a zero-entry) and
- a merge of two neighboring rows or columns into one that is the elementwise OR of the two original lines.

For simplicity, we do not consider a deletion of a line to be a separate operation as it can be replaced by a merge of the corresponding line with a neighboring one and a series of changes of one-entries to zero-entries. Moreover, like in the realm of graphs, we can assume all merging operations are done before the deletion of one-entries. This give us an alternative way to look at the problem.

Definition 1.5. Consider matrices P and M and let $P \preceq M$. A *mapping* of P to M is a function that maps each row of P to an interval of rows of M and each column of P to an interval of columns of M in such a way that if $P[r, c] = 1$ and r is mapped to R and c is mapped to C , there is a one-entry in $M[R, C]$. An *interval of rows* (columns) is a set of consecutive rows (columns). We say that an entry $P[r, c]$ is mapped to an entry $M[r', c']$ in a fixed mapping of P to M , in which r is mapped to R and c is mapped to C , if $r' \in R$ and $c' \in C$ and if $P[r, c] = 1$ then we also require $M[r', c'] = 1$.

Each mapping of a pattern P to a matrix M corresponds to a *partitioning* of M to intervals of rows and columns that creates a block structure. On the other hand, if we find a partitioning of M to blocks such that for each one-entry $P[r, c]$ there is a one-entry in the block that can be indexed $[r, c]$ then we have a mapping of P to M and so $P \preceq M$. This means:

123 **Observation 1.6.** *For all matrices P and M , there is a mapping of P to $M \Leftrightarrow$*
 124 *$P \preceq M$.* \square

125 While pattern avoidance in terms of submatrices and interval minors seem
 126 to be very different, they have a quite tight relationship. The next observation
 127 immediately follows from their definitions.

128 **Observation 1.7.** *For all matrices P and M , $P \leq M \Rightarrow P \preceq M$.*

129 As said at the beginning of the section, both approaches generalize pattern
 130 avoidance for permutations and so it makes sense that they are equal for permu-
 131 tation matrices – matrices having exactly one one-entry in each line.

132 **Observation 1.8.** *For all matrices P and M , if P is a permutation matrix then*
 133 *$P \leq M \Leftrightarrow P \preceq M$.*

134 *Proof.* If we have $P \preceq M$, then there is a mapping m of P to M . To show $P \leq M$
 135 we need to find R, C such that $M' = M[R, C]$ has the same size as P and for
 136 every $P[r, c] = 1$ it holds $M'[r, c] = 1$. We define R and C as follows. For every
 137 row r , let R' be the interval to which r is mapped in the mapping m . There is
 138 exactly one column c such that $P[r, c] = 1$ and c is mapped to some C' . Because
 139 m is a mapping, there is a one-entry $M[r', c']$ such that $r' \in R'$ and $c' \in C'$ and
 140 we add r' to R and we add c' to C .

141 The other implication follows from Observation 1.7. \square

142 **Definition 1.9.** A *class* of matrices \mathcal{M} is a set of matrices that is closed under
 143 interval minors. It means that for every $M \in \mathcal{M}$ and every $M' \preceq M$ it holds
 144 $M' \in \mathcal{M}$.

145 To avoid degenerate cases, we only consider classes of matrices containing at
 146 least one matrix of size 2×1 , at least one matrix of size 1×2 and at least one
 147 matrix that is non-empty.

148 **Definition 1.10.** Let \mathcal{P} be a set of patterns. We denote by $Av_{\preceq}(\mathcal{P})$ the set of
 149 all matrices that avoid each $P \in \mathcal{P}$ as an interval minor.

150 **Observation 1.11.** *For all patterns P and P' : $P \preceq P' \Leftrightarrow Av_{\preceq}(P) \subseteq Av_{\preceq}(P')$.*

151 *Proof.* Because $P \preceq P'$, every matrix that avoids P also avoids P' . On the other
 152 hand, if $P \not\preceq P'$ then $P' \in Av_{\preceq}(P)$. As $P' \notin Av_{\preceq}(P')$, we have $Av_{\preceq}(P) \not\subseteq$
 153 $Av_{\preceq}(P')$. \square

154 The following observation goes almost without saying and we use it throughout
 155 the whole thesis to break symmetries.

156 **Observation 1.12.** *Let P and M be matrices, $P \preceq M \Leftrightarrow P^T \preceq M^T$.*

1.3 Pattern avoidance

Pattern avoidance is a general topic in combinatorics. A lot of attention is directed towards permutations, see books Bóna [2012], Kitaev [2011] for references. It is a natural generalization to regard permutations as permutation matrices and consider matrix avoidance. This is mainly studied in terms of submatrices, so we discuss some interesting results in this section.

Interval minors are, on the other hand, a fairly new topic first defined by Jacob Fox in Fox [2013] as a tool to prove results about permutations in the study of Stanley–Wilf limits. Since then, a little has been discovered about the theory of interval minors. Nevertheless, we mention some results at the end of this section.

Let us go back to submatrices for now. The question that is particularly interesting is to determine the maximum number of one-entries that a matrix avoiding a given pattern can have. This property describes complexity of a pattern and can be used for example to prove algorithmic complexity, see Efrat and Sharir [1996].

Definition 1.13. Let M be a matrix. The weight of M , denoted by $|M|$, is the number of one-entries in M .

Definition 1.14. For a pattern P and integers m, n , we define the *weight extremal function* $Ex(P, m, n) := \max\{|M|; M \in \{0, 1\}^{m \times n} \wedge P \not\preceq M\}$.

Going back to the representation of the problem in terms of ordered bipartite graphs, the question to determine $Ex(P, m, n)$ is a variant of a classical Turán extremal graph question and was studied by many authors, see for example Tardos [2005], Füredi and Hajnal [1992] or, for a wider range of variants Brass et al. [2003], Claesson et al. [2012], Klazar [2004], Pach and Tardos [2006]. Some applications associated with the weight extremal function are discussed in Fulek [2009]. There are other extremal functions that have been studied, see for instance Cibulka and Kynčl [2016], but we do not consider them in this thesis.

In the same spirit, we also define the weight extremal function for matrices avoiding patterns as interval minors.

Definition 1.15. For a pattern P and integers m, n , we define $Ex_{\preceq}(P, m, n) := \max\{|M|; M \in \{0, 1\}^{m \times n} \wedge P \preceq M\}$.

Thanks to Observation 1.7 we have the following relationship between the extremal functions.

Observation 1.16. For all patterns P and integers m, n :

$$Ex_{\preceq}(P, m, n) \leq Ex(P, m, n). \quad \square$$

From Observation 1.11 it follows:

Observation 1.17. For all patterns P and P' and integers m, n : $P \preceq P' \Rightarrow Ex_{\preceq}(P, m, n) \leq Ex_{\preceq}(P', m, n)$.

It was showed in Marcus and Tardos [2004] that for every permutation matrix P and every n it holds $Ex(P, n, n) \leq c_P n$. While $Ex(P, n, n)$ can become even quadratic with n , because of the previous observation and the fact that every pattern $P \in \{0, 1\}^{k \times l}$ is an interval minor of some permutation pattern $P' \in \{0, 1\}^{(kl) \times (kl)}$ we have the following:

200 **Proposition 1.18.** For every pattern P and integer n : $Ex_{\preceq}(P, n, n) \leq c_P n$ for
 201 some constant c_P independent of n . \square

202 The following observation for $Ex(P, m, n)$ was made by several authors; see
 203 for example Cibulka [2009], Fulek [2009].

204 **Lemma 1.19.** If $P \in \{0, 1\}^{k \times l}$ has at least one one-entry, then

$$205 \quad Ex(P, m, n) \geq \begin{cases} m \cdot n & k > m \text{ or } l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

206 Moreover, the same holds for $Ex_{\preceq}(P, m, n)$.

207 *Proof.* If $k > m \vee l > m$, we have $P \not\preceq \{1\}^{m,n}$. Otherwise, let $P[r, c] = 1$ and
 208 consider Figure 1.1. Consider a matrix M such that the first $r-1$ rows, the last
 209 $k-r$ rows, the first $c-1$ column and the last $l-c$ column contain no zero-entry
 210 and the rest is empty. Then $P \not\preceq M$ and even $P \not\preceq M$. We can also see that
 211 $|M| = mn - (m-k+1)(n-l+1) = (l-1)m + (k-1)n - (k-1)(l-1)$. \square

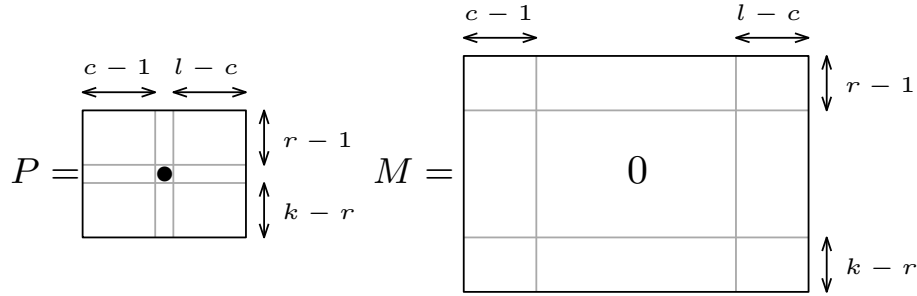


Figure 1.1: An example of a matrix M avoiding a pattern P as an interval minor.

212 The following definition is due to Cibulka [2013].

Definition 1.20. A pattern $P \in \{0, 1\}^{k \times l}$ is (strongly) *minimalist* if

$$Ex(P, m, n) = \begin{cases} m \cdot n & k > m \text{ or } l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

213 We use the adjective “strongly” to further distinguish minimalist pattern from
 214 weakly minimalist patterns defined next.

Definition 1.21. A pattern $P \in \{0, 1\}^{k \times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P, m, n) = \begin{cases} m \cdot n & k > m \text{ or } l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

215 From Observation 1.16, we immediately have:

216 **Observation 1.22.** If a pattern P is strongly minimalist then P is weakly min-
 217 imalist.

218 The following result is a simplification of a lemma from Cibulka [2013].

219 **Fact 1.23.** 1. The pattern (\bullet) is strongly minimalist.

220 2. If a pattern $P \in \{0, 1\}^{k \times l}$ is strongly minimalist and there is a one-entry in
 221 the last row of P in the c -th column, then $P' \in \{0, 1\}^{k+1 \times l}$ created from P
 222 by appending as the last row a new row having a one-entry only in the c -th
 223 column is strongly minimalist.

224 3. If a pattern P having at least two one-entries is strongly minimalist, then
 225 after changing a one-entry to a zero-entry it is still strongly minimalist.

226 The following two facts come from Mohar et al. [2015]. In the article, a slightly
 227 different definition of an interval minor is used, so we show here the proofs in our
 228 setting.

229 **Fact 1.24** (Mohar et al. [2015]). Let $P = \{1\}^{2 \times l}$ be a pattern, then P is weakly
 230 minimalist.

231 *Proof.* Let $M \in \{0, 1\}^{m \times n}$ be a matrix avoiding $P = \{1\}^{2 \times l}$ as an interval minor
 232 and let A_i be the set of column indices j such that both $M[[i], \{j\}]$ and $M[[i +$
 233 $1, m], \{j\}]$ are non-empty. Clearly, $|A_i| \leq l - 1$; otherwise, $P \preceq M$. Let b_j denote
 234 the number of one-entries in the j -th column. Each column j of M appears in at
 235 least $b_j - 1$ of sets A_i , $1 \leq i \leq m - 1$. It follows that

$$236 \quad |M| = \sum_{j=1}^n b_j = \sum_{j=1}^n (b_j - 1) + n \leq \sum_{i=1}^{m-1} |A_i| + n \leq (l - 1)(m - 1) + n. \quad \square$$

237 This result shows an example of a weakly minimalist matrix that is not
 238 strongly minimalist. Consider a matrix $\begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$. It is, thanks to Fact 1.24 weakly
 239 minimalist, but it is known due to Brown [1966] that it is not strongly minimalist.

240 **Fact 1.25** (Mohar et al. [2015]). Let $P = \{1\}^{3 \times l}$ be a pattern, then P is weakly
 241 minimalist.

242 *Proof.* Let $M \in \{0, 1\}^{m \times n}$ be a matrix avoiding $P = \{1\}^{3 \times l}$ as an interval minor
 243 and let A_i be a set of column indices j such that both $M[[i - 1], \{j\}]$ and $M[[i +$
 244 $1, m], \{j\}]$ are non-empty and $M[i, j] = 1$. Clearly $|A_i| \leq l - 1$, otherwise $P \preceq M$.
 245 Let b_j denote the number of one-entries in the j -th column. Each column j of M
 246 (for which $b_j \geq 2$) appears in exactly $b_j - 2$ of sets A_i , $2 \leq i \leq m - 1$. It follows
 247 that

$$248 \quad |M| = \sum_{j=1}^n b_j = \sum_{j=1}^n (b_j - 2) + 2n \leq \sum_{i=2}^{m-1} |A_i| + 2n \leq (l - 1)(m - 2) + 2n. \quad \square$$

249 We now show that the third part of Fact 1.23 is also safe for weakly minimalist
 250 patterns.

251 **Lemma 1.26.** Let $P \in \{0, 1\}^{k \times l}$ be a weakly minimalist pattern having at least
 252 two one-entries. Then a pattern P' created from P by deletion of a one-entry is
 253 also weakly minimalist.

254 *Proof.* For contradiction, consider a matrix $M \in \{0, 1\}^{m \times n}$ avoiding P' as an
 255 interval minor such that $|M| > (k - 1)n + (l - 1)m - (k - 1)(l - 1)$. The matrix M
 256 also avoids P ; as otherwise, we have $P' \preceq P \preceq M$. That is a contradiction with
 257 P being weakly minimalist. \square

258 As a result, we have the following corollary:

259 **Corollary 1.27.** *Every non-empty pattern P that has at most three rows (or*
260 *columns) is weakly minimalist.*

261 In Cibulka [2009], the author shows that for every $k \geq 1$ there is a $2k \times 2k$
262 permutation pattern for which $Ex[P, n] \geq k^2 n$. Because of Observation 1.8, the
263 same construction shows that for $k \geq 2$ the patterns are not weakly minimalist.
264 It means that the previous results cannot be easily extended. On the other hand,
265 in Mao et al. [2015] the authors show some form of generalization and also other
266 bounds regarding interval minors and their weight extremal function.

2. Small interval minors

Our goal in this chapter is to describe, for a given small pattern, the structure of matrices avoiding it as an interval minor.

Algorithmically speaking, deciding whether a pattern is contained in a matrix is NP-hard, even if both matrices are permutation matrices, see Bose et al. [1998]. We do not consider complexity questions here, but for small patterns, we show that matrices avoiding them have a quite simple structure. However, the structure gets significantly more complex as soon as we allow the pattern to contain at least four one-entries.

To go through cases efficiently, we first show that to some extent, we can assume, without loss of generality, there are no empty lines in studied patterns.

Before we dive into characterizations, let us introduce some useful notion.

Definition 2.1. A *walk* in a matrix M is a contiguous sequence of its entries, beginning in the top-left corner and ending in the bottom-right one. If $M[i, j]$ occurs in the sequence, its successor is either $M[i + 1, j]$ or $M[i, j + 1]$. Symmetrically, a *reverse walk* in M is a contiguous sequence of its entries, beginning in the top-right corner and ending in the bottom-left one.

Definition 2.2. We say a matrix M is a *walking matrix* if there is a walk in M containing all one-entries.

Definition 2.3. For a matrix $M \in \{0, 1\}^{m \times n}$ and integers r, c , we say $M[r, c]$ is

- *top-left empty*, if $M[[r - 1], [c - 1]]$ is an empty matrix,
- *top-right empty*, if $M[[r - 1], [c + 1, n]]$ is empty,
- *bottom-left empty*, if $M[[r + 1, m], [c - 1]]$ is empty,
- *bottom-right empty*, if $M[[r + 1, m], [c + 1, n]]$ is empty.

Definition 2.4. For a matrix $M \in \{0, 1\}^{m \times n}$ and integers r, c , we say that an entry $M[r, c]$ is *bottom-left extreme*, if it is bottom-left empty and the submatrix $M[[r, m], [c]]$ is not empty. Similarly, $M[r, c]$ is *bottom-right extreme* if it is bottom-right empty and the submatrix $M[[r, m], [c, n]]$ is not empty. A walk in M is *bottom-left extreme* if it contains all bottom-left extreme elements of M . A reverse walk in M is *bottom-right extreme* if it contains all bottom-right extreme elements of M .

It is easy to see that there is exactly one bottom-left extreme walk and exactly one bottom-right extreme walk in every non-empty matrix.

Definition 2.5. For matrices $M \in \{0, 1\}^{m \times n}$ and $N \in \{0, 1\}^{m \times l}$, we define $M \rightarrow N \in \{0, 1\}^{m \times (n+l)}$ to be the matrix created from M by appending the columns of N at the end of M .

303 2.1 Empty rows and columns

304 From the definition of matrix containment, zero-entries of the pattern pose no
 305 restrictions on the tested matrix, so, intuitively, adding new empty lines to a
 306 pattern should not influence the structure of matrices avoiding the pattern by
 307 much.

308 We first show that adding empty lines as first or last lines of the pattern
 309 indeed does next to no difference. On the other hand, inserting empty lines in
 310 between non-empty lines becomes a bit more tricky and we only describe what
 311 happens when we extend a pattern of size $k \times 2$ (or symmetrically $2 \times k$) by a
 312 single empty column (row).

313 **Observation 2.6.** *For matrices $P \in \{0, 1\}^{k \times l}$ and $M \in \{0, 1\}^{m \times n}$, let $P' = P \rightarrow$
 314 $\{0\}^{k \times 1}$ and let $M' = M \rightarrow \{1\}^{m \times 1}$. Then $P \preceq M \Leftrightarrow P' \preceq M'$.*

315 *Proof.* \Rightarrow The last column of P' can always be mapped just to the last column
 316 of M' and $P'[[k], [l]]$ can be mapped to $M'[[m], [n]]$ the same way P is
 317 mapped to M .

318 \Leftarrow Taking the restriction of the mapping of P' to M' , we get a mapping of P
 319 to M . \square

320 The analogous proof can be also used to characterize matrices avoiding pat-
 321 terns after adding an empty column as the first column or an empty row as the
 322 first or the last row. Using induction, we can easily show that a pattern P' is
 323 avoided by a matrix M' if and only if P is avoided by M , where P is derived from
 324 P' by excluding all empty leading or ending rows and columns and M is derived
 325 from M' by excluding the same number of leading or ending rows and columns.
 326 Therefore, when characterizing matrices avoiding a forbidden pattern, we do not
 327 to consider patterns having empty rows or columns on their boundary.

328 We now show what happens when we add an empty column in between two
 329 columns of a pattern that only has two columns. It is going to be achieved by
 330 employing a notion of intervals of one-entries. More about these intervals and
 331 their counterpart – zero-intervals can be found in the last chapter of the thesis.

332 **Definition 2.7.** A *one-interval* of a matrix M is a sequence of consecutive one-
 333 entries of a single line of M bounded from each side by a zero-entry or the edge
 334 of the matrix.

335 **Definition 2.8.** For a class of matrices \mathcal{M} , a matrix $M \in \mathcal{M}$ is *critical* in \mathcal{M}
 336 if after a change of any zero-entry to one-entry the matrix no longer belongs to
 337 \mathcal{M} . For a pattern P , we denote by $Av_{crit}(P)$ the set of all matrices critical in
 338 $Av_{\preceq}(P)$.

339 **Lemma 2.9.** *Let $P \in \{0, 1\}^{k \times 2}$ be a pattern and let $M \in Av_{crit}(P)$ be a matrix,
 340 then M contains at most one one-interval in each row.*

341 *Proof.* For contradiction, assume there are at least two one-intervals in a row of
 342 M . Because M is critical in $Av_{\preceq}(P)$, changing any zero-entry e in between one-
 343 intervals o_1 and o_2 creates a mapping of the forbidden pattern. Such a mapping
 344 uses the changed one-entry to map some element $P[r', 1]$ or $P[r', 2]$.

345 In the first case, the same mapping also maps P to M if we use a one-entry
 346 from o_1 instead of e ; thus, $P \not\leq M$ and we reach a contradiction. In the second
 347 case, the mapping can use a one-entry from o_2 instead of e ; therefore, we again
 348 get a contradiction with $P \not\leq M$. Since e is not usable for any one-entry of P , we
 349 can change it to a one-entry and get a contradiction with M being critical. \square

350 **Lemma 2.10.** *Let $P \in \{0, 1\}^{k \times 3}$ be a pattern such that its middle column is*
 351 *empty. Every row of any matrix $M \in Av_{crit}(P)$ is either empty or it contains a*
 352 *single one-interval of length at least 2 (or length m if $m < 2$).*

353 *Proof.* Let a matrix $M \in Av_{crit}(P)$. The same proof as in Lemma 2.9 shows that
 354 there is at most one one-interval in each row of M . For contradiction, let there
 355 be only one one-entry $M[r, c]$ in a row r :

- 356 • $c = 1$: we can set $M[r, c + 1] = 1$ and the matrix still avoids Pl , which is a
 357 contradiction with M being critical in $Av_{\leq}(P)$.
- 358 • $c = n$: we can set $M[r, c - 1] = 1$ and the matrix still avoids P , which is a
 359 contradiction with M being critical in $Av_{\leq}(P)$.
- 360 • otherwise: consider zero-entries $e_l = M[r, c - 1]$ and $e_r = M[r, c + 1]$. For
 361 contradiction, assume we can change neither e_l nor e_r to a one-entry without
 362 creating a mapping of the pattern. It means that if we set $e_l = 1$ then some
 363 $P[r_1, 1]$ can be mapped to it. Let m_l be the corresponding mapping. At
 364 the same time, if we set $e_r = 1$ then some $P[r_2, 3]$ can be mapped to it and
 365 m_r is the corresponding mapping. We show that the two mappings can be
 366 combined to a mapping of P to M , giving a contradiction.

367 Without loss of generality, in both mappings, the empty column of P is
 368 mapped exactly to the column c of M . We need to describe how to partition
 369 M into k rows. Consider Figure 2.1:

- 370 – $r_1 \neq r_2$: Without loss of generality, we assume $r_1 > r_2$. Let r_3 be
 371 the first row of the interval where the row r_1 is mapped in m_l and let
 372 r_4 be the last row of the interval where the row r_1 is mapped in m_r .
 373 From the mapping m_l , we know that the first $r_1 - 1$ rows of P can be
 374 mapped to rows $[1, r_3 - 1]$ and from the mapping m_r , we know that
 375 the last $k - r_1$ rows of P can be mapped to rows $[r_4 + 1, m]$. From the
 376 mapping m_r , we know that the row r_1 can be mapped to rows $[r_3, r_4]$;
 377 thus, we have a mapping of P to M .
- 378 – $r_1 = r_2$: Let $[r_3, r_4]$ be the interval where the row r_1 is mapped in
 379 m_l and let $[r_5, r_6]$ be the interval where the row r_1 is mapped in m_r .
 380 Without loss of generality, let $r_3 < r_5$. From the mapping m_l , we
 381 know that the first $r_1 - 1$ rows of P can be mapped to rows $[1, r_3 - 1]$.
 382 Without loss of generality, let $r_4 < r_6$. From the mapping m_r , we
 383 know that the last $k - r_1$ rows of P can be mapped to rows $[r_6 + 1, m]$.
 384 Therefore, we can map the row r_1 of P to the row interval $[r_3, r_6]$
 385 without using one-entries e_l and e_r .

386 We showed that either e_l or e_r can be changed to a one-entry, which is a contra-
 387 diction with M being critical in $Av_{\leq}(P)$. \square

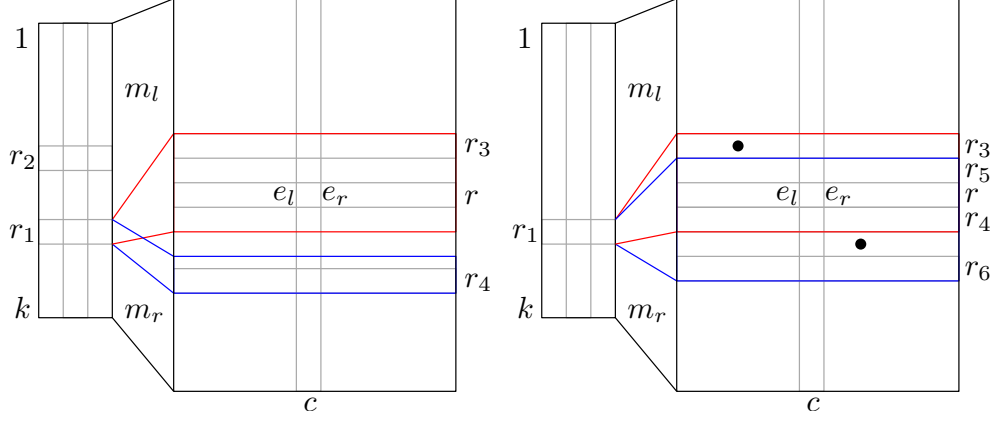


Figure 2.1: Red and blue lines representing mappings m_l and m_r of the forbidden pattern. The two horizontal lines show the boundaries of the mapping of the row r and the vertical lines show the boundaries of the mapping of the column c .

388 Similarly, we can prove that for every pattern $P \in \{0,1\}^{k \times l}$ such that all
 389 $(l - 2)$ middle columns are empty, every matrix from $Av_{crit}(P)$ that contains at
 390 least $l + 1$ one-entries in each row, contains at least $l + 1$ one-entries in each row.
 391 On the other hand, it cannot be generalized further, as we show in the following
 392 proposition.

393 **Proposition 2.11.** *For every integer $l > 3$, there exists a pattern $P \in \{0,1\}^{k \times l}$
 394 such that all $(l - 2)$ middle columns are empty and there exists a matrix $M \in$
 395 $Av_{crit}(P)$ containing a row with a single one-entry.*

396 *Proof.* We only show the construction for $l = 4$ and $l = 5$ because the first
 397 construction can easily be extended for every even l and the latter for every odd
 398 l . For $l \in \{3, 4\}$, let P_l be the forbidden pattern and $M_l \in Av_{crit}(P)$ be the
 399 critical matrix that has a single one-entry in some row:

$$400 \quad P_4 = \begin{pmatrix} \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ \end{pmatrix} \quad M_4 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} \quad P_5 = \begin{pmatrix} \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ \end{pmatrix} \quad M_5 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

401 It is easy to check that $M_l \in Av_{\preceq}(P_l)$ and that changing a zero-entry to a
 402 one-entry creates a mapping of the forbidden pattern. \square

403 **Theorem 2.12.** *Let $P \in \{0,1\}^{k \times 2}$ be a pattern and let $P' \in \{0,1\}^{k \times 3}$ be the
 404 pattern created from P by appending a new empty column in between the two
 405 columns of P . For all matrices $M \in \{0,1\}^{m \times n}$ it holds $M \in Av_{\preceq}(P') \Leftrightarrow$ there
 406 exists a matrix $N \in \{0,1\}^{m \times (n-1)}$ such that $N \in Av_{crit}(P)$ and M is a submatrix
 407 of the elementwise OR of $N \rightarrow \{0\}^{m \times 1}$ and $\{0\}^{m \times 1} \rightarrow N$.*

408 *Proof.* \Rightarrow Without loss of generality, let the matrix M be critical in $Av_{\preceq}(P')$.
 409 We know from Lemma 2.10 that each row of M contains either no one-entry
 410 or a single one-interval of length at least 2. Let a matrix N be created from
 411 M by deletion of the last one-entry from each row and deletion of the last
 412 column. Clearly, M is equal to the elementwise OR of $N \rightarrow \{0\}^{m \times 1}$ and
 413 $\{0\}^{m \times 1} \rightarrow N$. If $P \preceq N$ then each mapping of P to N can be extended
 414 to a mapping of P' to M by mapping each $P'[r_1, 1]$ to the same one-entry
 415 where $P[r_1, 1]$ is mapped in $N \rightarrow \{0\}^{m \times 1}$ and mapping each $P'[r_2, 3]$ to the
 416 same one-entry where $P[r_2, 2]$ is mapped in $\{0\}^{m \times 1} \rightarrow N$.

417 \Leftarrow Let a matrix M be equal to the elementwise OR of $N \rightarrow \{0\}^{m \times 1}$ and
 418 $\{0\}^{m \times 1} \rightarrow N$. For contradiction, assume $P' \preceq M$ and consider any mapping
 419 of P' to M . Without loss of generality, one-entries of the first column
 420 of P' are mapped to those one-entries of M created from $N \rightarrow \{0\}^{m \times 1}$.
 421 If there is a one-entry $P'[r, 1]$ mapped to a one-entry of M not created
 422 from $N \rightarrow \{0\}^{m \times 1}$, we just take the first one-entry in the row instead.
 423 Symmetrically, all one-entries of the last column of P' are mapped to one-
 424 entries created from $\{0\}^{m \times 1} \rightarrow N$. The same one-entries of N can be used
 425 to map P to N , which is a contradiction. \square

426 The symmetric characterization also holds when adding an empty row to a
 427 pattern that only has two rows. We can see in the following proposition that the
 428 straightforward generalization of the statement for bigger patterns does not hold.

429 **Proposition 2.13.** *There exists a matrix $P \in \{0, 1\}^{k \times l}$ such that for each pat-*
 430 *tern $P' \in \{0, 1\}^{k \times (l+1)}$ created from P by appending a new empty column in*
 431 *between the two existing columns, there exists a matrix $N \in Av_{\preceq}(P)$ such that*
 432 *the elementwise OR of $N \rightarrow \{0\}^{m \times 1}$ and $\{0\}^{m \times 1} \rightarrow N$ contains P' as an interval*
 433 *minor.*

434 *Proof.* Later in this chapter, we characterize the class of matrices avoiding pat-
 435 tern $(\bullet \bullet \bullet)$. See Proposition 2.23. Let $N \in Av_{\preceq}((\bullet \bullet \bullet))$ be any matrix contain-
 436 ing $(\bullet \bullet \bullet)$ as an interval minor. Let a matrix M be equal to the elementwise OR
 437 of $N \rightarrow \{0\}^{m \times 1}$ and $\{0\}^{m \times 1} \rightarrow N$. Then $(\bullet \circ \bullet \bullet) \preceq M$ and $(\bullet \bullet \circ \bullet) \preceq M$. \square

438 Next, we describe the structure of matrices avoiding certain small patterns.
 439 We restrict ourselves to patterns with no empty lines. If $P \not\preceq M$ then also
 440 $P^\top \not\preceq M^\top$ and this holds for all rotations and mirrors of P and M and so we
 441 only mention these symmetries.

442 2.2 Patterns having two one-entries

These are, up to rotation, the only patterns having two one-entries and no empty lines:

$$P_1 = (\bullet \bullet) \quad P_2 = (\bullet \bullet)$$

They can be generalized to:

$$P'_1 = (\bullet \bullet \dots \bullet \bullet) \quad P'_2 = \begin{pmatrix} & & & \bullet \\ & & \bullet & \\ & \bullet & \ddots & \\ \bullet & & & \end{pmatrix}$$

443 **Proposition 2.14.** *Let $P'_1 = \{1\}^{1 \times k}$. For all matrices M : $P'_1 \not\preceq M \Leftrightarrow M$ has at*
 444 *most $k - 1$ non-empty columns.*

445 *Proof.* \Rightarrow When a matrix M contains one-entries in k columns, then these give
 446 us a mapping of P'_1 .

447 \Leftarrow A matrix M having at most $k - 1$ non-empty columns avoids P'_1 . \square

448 **Proposition 2.15.** *Let $P'_2 \in \{0, 1\}^{k \times k}$. For all matrices M : $P'_2 \not\preceq M \Leftrightarrow$ there*
 449 *are $k - 1$ walks in M such that each one-entry of M belongs to at least one walk.*

450 *Proof.* \Rightarrow When all one-entries of a matrix M cannot fit into $k - 1$ walks, then
 451 there are k one-entries such that no pair can fit to a single walk and those
 452 give us a mapping of P'_2 .

453 \Leftarrow A matrix M containing one-entries in at most $k - 1$ walks avoids P'_2 . \square

454 2.3 Patterns having three one-entries

These are, up to rotation and mirroring, the only patterns having three one-entries and no empty lines that we did not characterize so far:

$$P_3 = (\begin{smallmatrix} \bullet & \bullet \end{smallmatrix}) \quad P_4 = (\begin{smallmatrix} \bullet & \bullet & \bullet \end{smallmatrix}) \quad P_5 = (\begin{smallmatrix} \bullet & \bullet & \bullet \end{smallmatrix}) \quad P_6 = (\begin{smallmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{smallmatrix})$$

455 **Proposition 2.16.** *For all matrices $M \in \{0, 1\}^{m \times n}$: $P_3 \not\preceq M \Leftrightarrow$ there exist a*
 456 *row r and a column c such that (see Figure 2.2):*

- 457 • $M[r, c]$ is top-left, top-right and bottom-left empty, and
- 458 • $M[[r, m], [c, n]]$ is a walking matrix.

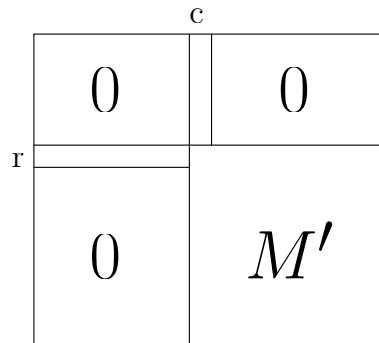


Figure 2.2: The characterization of matrices avoiding $(\begin{smallmatrix} \bullet & \bullet \end{smallmatrix})$ as an interval minor. The matrix M' is a walking matrix.

459 *Proof.* \Rightarrow If M is a walking matrix then we set $r = c = 1$. Otherwise, there
 460 are one-entries $M[r, c']$ and $M[r', c]$ such that $r' < r$ and $c' < c$. If an
 461 entry $M[r, c]$ is not top-left, top-right or bottom-left empty then $P_3 \preceq M$.
 462 If the submatrix $M[[r, m], [c, n]]$ is not a walking matrix then it contains
 463 $(\begin{smallmatrix} \bullet & \bullet \end{smallmatrix})$ and together with the one-entry $M[r, c']$ it gives us a mapping of P_3 .

464 \Leftarrow For contradiction, assume that a matrix M described in Figure 2.2 contains
 465 P_3 as an interval minor. Without loss of generality, let $P_3[1, 1]$ be mapped
 466 to a one-entry in the r -th row. Then both $P_3[1, 2]$ and $P_3[2, 1]$ need to be
 467 mapped to M' , which is a contradiction with it being a walking matrix. \square

468 **Proposition 2.17.** *For all matrices M : $P_4 \not\preceq M \Leftrightarrow$ there are matrices M_1, M_2
 469 such that $M = M_1 \rightarrow M_2$, $(\bullet \bullet) \not\preceq M_1$ and $(\bullet \bullet) \not\preceq M_2$.*

470 *Proof.* \Rightarrow Let $e = M[r, c]$ be an arbitrary top-most one-entry in M . It holds
 471 $(\bullet \bullet) \not\preceq M[[m], [c-1]]$; otherwise, we have a mapping of P_4 to M . If we also
 472 have $(\bullet \bullet) \not\preceq M[[m], [c, n]]$ then we are done. For contradiction, let e_1, e_2
 473 be any two one-entries forming $(\bullet \bullet)$ in $M[[m], [c, n]]$. Symmetrically, let
 474 e'_1, e'_2 be any two one-entries forming $(\bullet \bullet)$ in $M[[m], [c]]$. Without loss of
 475 generality, let e_2 be no higher than e'_2 and then, together with e'_1 and e_1 it
 476 gives us a mapping of P_4 to M , giving a contradiction.

477 \Leftarrow For contradiction, let $P_4 \preceq M$ and consider an arbitrary mapping. Consider
 478 the one-entry of M , where $P_4[2, 2]$ is mapped. If it is in M_1 then $(\bullet \bullet) \preceq M_1$
 479 and we get a contradiction. Otherwise, we have $(\bullet \bullet) \preceq M_2$, which is again
 480 a contradiction. \square

481 **Proposition 2.18.** *For all matrices $M \in \{0, 1\}^{m \times n}$: $P_5 \not\preceq M \Leftrightarrow$ for every one-
 482 entry $M[r, c]$ on the bottom-left extreme walk w , there is at most one non-empty
 483 column in $M[[r-1], [c+1, n]]$.*

484 *Proof.* \Rightarrow For contradiction, assume there is a one-entry $M[r, c]$ on w such that
 485 there are two non-empty columns in $M[[r-1], [c+1, n]]$. Then a one-entry
 486 from each of those columns and $M[r, c]$ together give us a mapping of P_5 to
 487 M , and a contradiction.

488 \Leftarrow For contradiction, let $P_5 \preceq M$ and consider any such mapping. Without
 489 loss of generality, $P_5[2, 1]$ is mapped to a one-entry $M[r, c]$ from w . Then
 490 $(\bullet \bullet) \preceq M[[r-1], [c+1, n]]$, which is a contradiction with it having one-
 491 entries in at most one column. \square

492 **Proposition 2.19.** *For all matrices M : $P_6 \not\preceq M \Leftrightarrow$ for every one-entry $M[r, c]$
 493 on the bottom-right extreme reverse walk w , $M[[r-1], [c-1]]$ is a walking matrix.*

494 *Proof.* \Rightarrow For contradiction, assume there are integers r, c such that $M[r, c]$ is a
 495 one-entry on w and $M[[r-1], [c-1]]$ is not a walking matrix. It means that
 496 $(\bullet \bullet) \preceq M[[r-1], [c-1]]$ and together with $M[r, c]$ it gives us a mapping
 497 of the forbidden pattern, and a contradiction.

498 \Leftarrow For contradiction, let $P_6 \preceq M$ and consider an arbitrary mapping of P_6 .
 499 Without loss of generality, let $P_6[3, 3]$ be mapped to some one-entry $M[r, c]$
 500 on w . Then, $M[[r], [c]]$ is not a walking matrix and we have a contradiction.
 501 \square

502 2.4 Patterns having four one-entries

These are some of the patterns having four one-entries and no empty lines that we did not characterize so far:

$$P_7 = (\bullet \bullet \bullet) \quad P_8 = (\bullet \bullet \bullet \bullet) \quad P_9 = \begin{pmatrix} \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \end{pmatrix}$$

503 **Lemma 2.20.** *For any matrix M : $P_7 \not\preceq M \Rightarrow$ there exist integers r, c such that*
 504 *$M[r, c]$ is either*

- 505 1. *a one-entry and $(r, c) \in \{(1, 1), (1, n), (m, 1), (m, n)\}$, or*
- 506 2. *top-right and bottom-left empty and $(r, c) \notin \{(1, 1), (m, n)\}$, or*
- 507 3. *top-left and bottom-right empty and $(r, c) \notin \{(1, n), (m, 1)\}$.*

508 *Proof.* If there is a one-entry in any corner then the first condition is satisfied.
 509 Otherwise, consider the entry $M[2, 1]$. It is trivially bottom-left empty and if
 510 there is no one-entry in the first row of M then the second condition is satisfied.
 511 Therefore, let $M[1, c_t]$ be a one-entry in the first row. Symmetrically, let $M[m, c_b]$
 512 be a one-entry in the last row, let $M[r_l, 1]$ be a one-entry in the first column and
 513 let $M[r_r, n]$ be a one-entry in the last column.

514 It cannot at the same time happen that $c_t < c_b$ and $r_r > r_l$ (or symmetrically
 515 $c_t > c_b$ and $r_r < r_l$), because then $P_7 \preceq M$. Without loss of generality, let
 516 $c_t \geq c_b$ and $r_r \geq r_l$. The submatrix $M[[r_r - 1], [c_t + 1, n]]$ is empty; otherwise, any
 517 one-entry there, together with $M[1, c_t]$, $M[m, c_b]$ and $M[r_r, 1]$ forms the forbidden
 518 pattern. Similarly, the matrix $M[[r_r + 1, m], [c_t - 1]]$ is also empty. Thus $M[r_t, c_t]$
 519 is top-right and bottom-left empty and it is not a corner, satisfying the second
 520 condition. \square

521 **Proposition 2.21.** *For all matrices $M \in \{0, 1\}^{m \times n}$: $P_7 \not\preceq M \Leftrightarrow$ there are*
 522 *integers r, c such that either (see Figure 2.3):*

- 523 1. *$M[r, c]$ is top-right empty and bottom-left empty, $(\bullet \bullet) \not\preceq M[[r], [c]]$ and*
 524 *$(\bullet \bullet) \not\preceq M[[r, m], [c, n]]$, or*
- 525 2. *$M[r, c]$ is top-left empty and bottom-right empty, $(\bullet \bullet) \not\preceq M[[r], [c, n]]$ and*
 526 *$(\bullet \bullet) \not\preceq M[[r, m], [c]]$.*

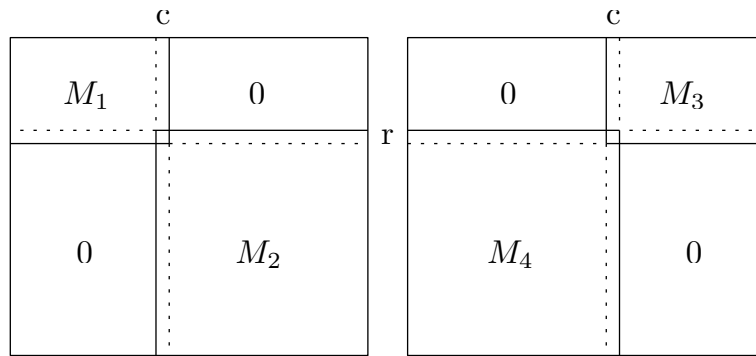


Figure 2.3: The characterization of matrices avoiding $(\bullet \bullet)$ as an interval minor.

527 *Proof.* We let $M_1 = M[[r], [c]]$, $M_2 = M[[r, m], [c, n]]$, $M_3 = M[[r], [c, n]]$ and
 528 $M_4 = M[[r, m], [c]]$.

529 \Rightarrow We proceed by induction on the size of M .

530 If $M \in \{0, 1\}^{2 \times 2}$ then it either avoids $(\bullet \bullet)$ or $(\bullet \bullet)$ and we are done.

531 For a bigger matrix M , from Lemma 2.20, there is an entry $M[r, c]$ satisfying
 532 some conditions. If there is a one-entry in any corner, we are done because
 533 the matrix cannot contain one of the rotations of $(\bullet \bullet \bullet)$. Otherwise, assume
 534 $M[r, c]$ is both top-right and bottom-left empty and $(r, c) \notin \{(1, 1), (m, n)\}$.
 535 If M_1 is non-empty, then $(\bullet \bullet \bullet) \not\leq M_2$. Symmetrically, $(\bullet \bullet \bullet) \not\leq M_1$ if M_2 is
 536 non-empty. If one of them is empty, the other is a smaller matrix avoid-
 537 ing P_7 as an interval minor and the statement follows from the induction
 538 hypothesis.

539 \Leftarrow Let $P_7 \preceq M$. Every mapping of P_7 partitions M into four non-empty
 540 quadrants; thus, there are no integers r, c satisfying the conditions. \square

541 **Lemma 2.22.** *For all matrices M : $P_8 \not\leq M \Rightarrow$ there are matrices M_1, M_2 such*
 542 *that $M = M_1 \rightarrow M_2$ and*

543 1. $(\bullet \bullet \bullet) \not\leq M_1$ and $(\bullet \bullet) \not\leq M_2$, or

544 2. $(\bullet \bullet) \not\leq M_1$ and $(\bullet \bullet \bullet) \not\leq M_2$.

545 *Proof.* Let $e = M[r, c]$ be an arbitrary top-most one-entry of the matrix M .
 546 It holds $(\bullet \bullet \bullet) \not\leq M[[m], [c - 1]]$; otherwise, we have a mapping of P_8 to M .
 547 Symmetrically, $(\bullet \bullet \bullet) \not\leq M[[m], [c + 1, n]]$. For contradiction with the statement,
 548 let e_1, e_2 (none of them equal to e) be any two one-entries forming $(\bullet \bullet)$ in
 549 $M[[m], [c]]$ and let e'_1, e'_2 be any two one-entries forming $(\bullet \bullet)$ in $M[[m], [c, n]]$.
 550 Without loss of generality, e'_2 is no higher than e_2 and together with e_1, e and e'_1
 551 it gives us a mapping of P_8 to M , which is a contradiction. \square

552 **Proposition 2.23.** *For all matrices $M \in \{0, 1\}^{m \times n}$: $P_8 \not\leq M \Leftrightarrow$ there are*
 553 *integers r, c_1 and c_2 such that all one-entries of M above the row r are in columns*
 554 *c_1 and c_2 , $M[[r + 1, m], [c_1 + 1, c_2 - 1]]$ is empty, $(\bullet \bullet) \not\leq M[[r, m], [c_1]]$ and*
 555 *$(\bullet \bullet) \not\leq M[[r, m], [c_2, n]]$. See Figure 2.4.*

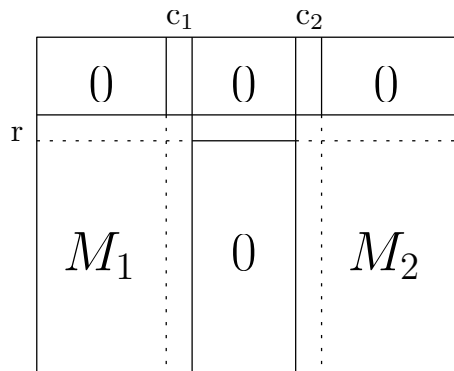


Figure 2.4: The characterization of matrices avoiding $(\bullet \bullet \bullet)$ as an interval minor.

556 *Proof.* \Rightarrow From Lemma 2.22, we know there are matrices M'_1, M'_2 such that
 557 $M = M'_1 \rightarrow M'_2$, $(\bullet \bullet \bullet) \not\leq M'_1$ and $(\bullet \bullet) \not\leq M'_2$ (or symmetrically the second
 558 case). Let c_2 be the first column appended from M'_2 . From Proposition 2.16,
 559 we have integers r', c' such that $M'_1[r', c']$ is top-left, top-right and bottom-
 560 right empty and $(\bullet \bullet) \preceq M'_1[[r', m], [c']] = M_1$. Let us set $r = r'$ and

561 $c_1 = c'$. We also have that $M[[m], [c_2, n]]$ is a walking matrix. Without
 562 loss of generality, $M[[r-1], \{c_1\}]$ and $M[\{r\}, [c_1+1, c_2-1]]$ are non-empty;
 563 otherwise, we extend M_1 to cover the whole $M[[m], [c_2-1]]$. There are no two
 564 different columns in M'_2 having a one-entry above the r -th row; otherwise,
 565 together with one-entries in $M[[r-1], \{c_1\}]$ and $M[\{r\}, [c_1+1, c_2-1]]$ they
 566 would give us a mapping of P_8 to M .

567 \Leftarrow A one-entry $P_8[2, 2]$ can not be mapped anywhere but to the r -th row, but
 568 in that case, there are at most two columns having one-entries above it. \square

569 **Proposition 2.24.** *For all matrices $M \in \{0, 1\}^{m \times n}$: $P_9 \not\preceq M \Leftrightarrow$ for every
 570 one-entry $M[r, c]$ on the bottom-right extreme reverse walk w and every one-
 571 entry $M[r', c']$ on the top-left extreme reverse walk w' , if $r > r' + 3$ and $c > c' + 3$
 572 then $M[[r' + 1, r - 1], [c' + 1, c - 1]]$ is a walking matrix.*

573 *Proof.* \Rightarrow If there are one-entries $M[r, c]$ on w and $M[r', c']$ on w' such that
 574 $(\bullet \bullet) \preceq M[[r' + 1, r - 1], [c' + 1, c - 1]]$, we have a mapping of P_9 to M .

575 \Leftarrow For contradiction, let $P_9 \preceq M$ and consider any mapping. Without loss of
 576 generality, the one-entry $P_9[4, 4]$ is mapped to some one-entry $M[r, c]$ on w
 577 and the one-entry $P_9[1, 1]$ is mapped to some one-entry $M[r', c']$ on w' . This
 578 means that $(\bullet \bullet) \preceq M[[r' + 1, r - 1], [c' + 1, c - 1]]$, which is a contradiction
 579 with it being a walking matrix. \square

580 2.5 Multiple patterns

581 Instead of considering matrices avoiding a single pattern, we can work with ma-
 582 trices avoiding a set of forbidden patterns.

583 We only describe the structure of matrices avoiding one particular set of pat-
 584 terns, because we use the simple result later.

585 **Proposition 2.25.** *Let $P_{10} = (\begin{smallmatrix} \circ & \circ & \circ \\ \bullet & \circ & \circ \end{smallmatrix})$ and $P_{11} = (\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \\ \bullet & \circ \end{smallmatrix})$, then for a matrix M :
 586 $\{P_{10}, P_{11}\} \not\preceq M \Leftrightarrow$ for every $r < m$ and $c > 0$, if $M[r, c]$ is a one-entry then it
 587 either is on the bottom-left extreme walk w or both $M[r+1, c]$ and $M[r, c-1]$ are
 588 on w .*

589 *Proof.* \Rightarrow Assume there are $r < m$ and $c > 0$ and a one-entry $M[r, c]$ outside of
 590 w such that $M[r+1, c]$ (or $M[r, c-1]$) is outside of w . The one-entry $M[r, c]$
 591 is not bottom-left extreme and so there is a one-entry in $M' = M[[r+1, m], [c-1]]$.
 592 The entry $M[r+1, c]$ is not bottom-left extreme and because
 593 M' is non-empty, $M[r+1, c]$ is not bottom-left empty. Any one-entry in
 594 $M[[r+2, m], [c-1]]$ together with $M[r, c]$ give us a mapping of P_{11} (P_{10}).

595 \Leftarrow For any one-entry $M[r, c]$, there are one-entries in neither $M[[r+2, m], [c-1]]$
 596 nor $M[[r+1, m], [c-2]]$. \square

3. The basis of a class of matrices

In this chapter, we look at classes of matrices from a different perspective. Unlike in the previous chapter, where we studied the structure of matrices avoiding a given set of forbidden patterns (usually just one), we are given a class of matrices and a question whether the class can be described by forbidden patterns.

Recall that a class of matrices is set of matrices closed under interval minors. While it is obvious that any class of matrices can be described by a set of forbidden patterns, as it suffices to forbid all matrices not contained in the class, it is no longer clear how complex the set can be.

Definition 3.1. Let \mathcal{M} be a class of matrices. The *basis* of \mathcal{M} is a set of all minimal (with respect to minors) matrices \mathcal{P} such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$.

We show that there are many classes of matrices having a finite basis but there are also classes that have an infinite basis. Let us start with a few simple observations, regarding classes of matrices and their bases.

Observation 3.2. Let $\mathcal{M} = Av_{\preceq}(\mathcal{P})$ for some set of matrices \mathcal{P} . Then \mathcal{M} is closed under interval minors.

Observation 3.3. Every finite class of matrices has a finite basis.

Proof. Let \mathcal{M} be a finite class of matrices, let m be the maximum number of rows a matrix from \mathcal{M} has and let n be the maximum number of columns a matrix from \mathcal{M} has. We define a set of matrices \mathcal{P} to contain all matrices of size smaller or equal to $m \times n$ that do not belong to \mathcal{M} and we add $\{0\}^{m \times (n+1)}$ and $\{0\}^{(m+1) \times n}$. Clearly, \mathcal{P} is finite and $\mathcal{M} = Av_{\preceq}(\mathcal{P})$. \square

3.1 The skew and direct sums

In the realm of permutations, the skew and direct sums are very useful operations. What follows is a direct generalization to our settings and a few simple results. More interesting statements and the relation with interval minors follow in the next section.

Definition 3.4. For matrices $A \in \{0, 1\}^{m \times n}$ and $B \in \{0, 1\}^{k \times l}$ we define their *skew sum* as a matrix $C := A \nearrow B \in \{0, 1\}^{(m+k) \times (n+l)}$ such that $C[[k+1, m+k], [n]] = A$, $C[[k], [n+1, n+l]] = B$ and the rest is empty. Symmetrically, we define their *direct sum* $D := A \searrow B \in \{0, 1\}^{(m+k) \times (n+l)}$ such that $D[[m], [n]] = A$, $D[[m+1, m+k], [n+1, n+l]] = B$ and the rest is empty.

Using this notation, we can very easily rewrite the results from the previous chapter. Here is an example of Proposition 2.16 and Proposition 2.21:

Proposition 3.5. $Av_{\preceq}((\bullet \bullet \bullet)) = Av_{\preceq}((\bullet \circ \circ)) \searrow Av_{\preceq}((\circ \bullet \bullet))$

Proposition 3.6. $Av_{\preceq}((\bullet \bullet \bullet)) = (Av_{\preceq}((\bullet \circ \circ)) \searrow Av_{\preceq}((\circ \bullet \bullet)) \searrow Av_{\preceq}((\circ \circ \bullet))) \cup (Av_{\preceq}((\bullet \circ \circ)) \nearrow Av_{\preceq}((\circ \bullet \bullet)) \nearrow Av_{\preceq}((\circ \circ \bullet)))$.

Something, we get a great use of later is a closure under the skew sum.

Definition 3.7. For a set of matrices \mathcal{M} , let $Cl(\mathcal{M})$ denote the smallest class of matrices containing each $M \in \mathcal{M}$ that is closed under the skew sum and interval minors.

When speaking about graph minors, we can always imagine that the contractions of edges are done after all deletions. Similarly, an element derived from a matrix M by reapplying the skew sum and taking its interval minor can be also derived by taking an interval minor of the skew sum of an appropriate number of copies of M .

Observation 3.8. For every set of matrices \mathcal{P} , each $M \in Cl(\mathcal{P})$ is an interval minor of the skew sum of multiple copies of P .

What follows is a simple result of the relation of a closure under the skew sum and the description using interval minors. We greatly generalize this result in the next section.

Proposition 3.9. $Cl((\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix})) = Av_{\preceq}((\begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \bullet \end{smallmatrix}), (\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}))$.

Proof. The skew sum of an arbitrary number of copies of $(\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix})$ avoids both forbidden patterns and because the relation of being an interval minor is transitive, we have $Cl((\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix})) \subseteq Av_{\preceq}((\begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \bullet \end{smallmatrix}), (\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}))$.

From Proposition 2.25, for every matrix $M \in Av_{\preceq}((\begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \bullet \end{smallmatrix}), (\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}))$, it holds that for the bottom-left extreme walk w in M , each one-entry $M[r, c]$ is either on w or both $M[r+1, c]$ and $M[r, c-1]$ are on w . Clearly, $(\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix})$ is an interval minor of the skew sum of three copies of $(\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix})$ and by the skew sum of multiple copies of $(\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix})$ we can then create the whole w and all one-entries outside of it. Thus, we have the other inclusion. \square

While it does not make sense for permutations, we can generalize the skew sum to also allow some overlap between the summed matrices.

Definition 3.10. For matrices $A \in \{0, 1\}^{m \times n}$, $B \in \{0, 1\}^{k \times l}$ and integers a, b , let a matrix $C := A \nearrow_{a \times b} B \in \{0, 1\}^{(m+k-a) \times (n+l-b)}$ such that $C[[k+1, m+k], [n]] = A$, $C[[k], [n+1, n+l]] = B$, the part that overlaps is an elementwise OR of both submatrices and the rest of C is empty. We say C is the *skew sum with $a \times b$ overlap* of A and B .

Theorem 3.11. For integers a, b, m, n such that $a \leq m \leq 2a$ and $b \leq n \leq 2b$, let \mathcal{M} be an arbitrary set of matrices, not necessarily closed under interval minors, such that:

- \mathcal{M} is closed under deletion of one-entries,
- \mathcal{M} is closed under the skew sum with $a \times b$ overlap and
- there is a $m \times n$ matrix $M \in \mathcal{M}$,

then \mathcal{M} is also closed under the skew sum with $(2a - m) \times (2b - n)$ overlap.

Proof. Given any $A, B \in \mathcal{M}$ and a matrix $M \in \mathcal{M}$ such that $M \in \{0, 1\}^{m \times n}$, let $C = A \nearrow_{a \times b} M \nearrow_{a \times b} B$. It has the same size as $D = A \nearrow_{(2a-m) \times (2b-n)} B$, whose set of one-entries is a subset of one-entries of $C \in \mathcal{M}$; therefore, $D \in \mathcal{M}$. \square

675 We see that already with pretty reasonable assumptions, whenever a set of
676 matrices is closed under the skew sum with some overlap, it is also closed under
677 the skew sum with smaller overlap. On the other hand, in general the opposite
678 does not hold even if we work with classes of matrices.

679 **Observation 3.12.** *There is a class of matrices closed under the skew sum with*
680 *1×1 overlap that is not closed under the skew sum with 2×2 overlap.*

681 *Proof.* Let $\mathcal{M} = Av_{\preceq}((\bullet \bullet))$. Clearly, \mathcal{M} is hereditary and closed under the
682 skew sum with 1×1 overlap. On the other hand, \mathcal{M} is not closed under the
683 skew sum with 2×2 overlap, because for matrices $(\bullet \bullet), (\bullet \bullet) \in \mathcal{M}$, it holds
684 $(\bullet \bullet) \nearrow_{2 \times 2} (\bullet \bullet) = (\bullet \bullet) \notin \mathcal{M}$. \square

685 A similar proof shows that for all $a \geq 1, b > 1$ there is a class of matrices
686 closed under the skew sum with $a \times b$ overlap that is not closed under the skew
687 sum with $(a + 1) \times b$ (or $a \times (b + 1)$) overlap. Luckily for us, this does not hold
688 for $a = 0$ or $b = 0$:

689 **Observation 3.13.** *Every class of matrices closed under the skew sum is also*
690 *closed under the skew sum with 1×1 overlap.*

691 3.2 Articulations

692 Our next goal is to show that whenever we have a matrix closed under the skew
693 sum and interval minors, the obtained class has a finite basis. In order to prove
694 it, we define and get familiar with articulations.

695 **Definition 3.14.** Let $M \in \{0, 1\}^{m \times n}$ be a matrix. An element $M[r, c]$ is an
696 *articulation* if it is top-left empty ($M[[r - 1], [c - 1]]$ is empty) and bottom-right
697 empty ($M[[r + 1, m], [c + 1, n]]$ is empty). We say that an articulation $M[r, c]$ is
698 *trivial* if $(r, c) \in \{(m, 1), (1, n)\}$.

699 Whenever $P \preceq M$, for every $M[r, c]$ there is some $P[r', c']$ that can be mapped
700 to $M[r, c]$; therefore, the following observation shows that once there is an articulation in M , it also exists in P and it is not necessarily trivial.

702 **Observation 3.15.** *Let M be a matrix. If there are integers r, c such that $M[r, c]$*
703 *is an articulation, then for every matrix P such that $P \preceq M$, if $P[r', c']$ can be*
704 *mapped to $M[r, c]$ then it is an articulation.*

705 **Observation 3.16.** *Let $P \in \{0, 1\}^{k \times l}$ be a matrix. There are P_1, P_2 non-empty*
706 *interval minors of P such that $P = P_1 \nearrow_{1 \times 1} P_2 \Leftrightarrow$ there exist integers r, c such*
707 *that $P[r, c]$ is an articulation and $P[[r, k], [c]], P[[r], [c, l]]$ are non-empty.*

708 **Observation 3.17.** *Let \mathcal{P} be a set of matrices. There is a minimal (with respect*
709 *to interval minors) matrix $P \in \mathcal{P}$ and there are P_1, P_2 non-empty interval minors*
710 *of P such that $P = P_1 \nearrow_{1 \times 1} P_2 \Leftrightarrow Av_{\preceq}(\mathcal{P})$ is not closed under the skew sum*
711 *with 1×1 overlap.*

712 *Proof.* \Rightarrow Let $P_1 \in \{0, 1\}^{k_1 \times l_1}$ and $P_2 \in \{0, 1\}^{k_2 \times l_2}$. While $P \not\preceq P_1 \nearrow_{1 \times 1} 0^{k_2 \times l_2}$
713 and $P \not\preceq 0^{k_1 \times l_1} \nearrow_{1 \times 1} P_2$, we have $P \preceq P_1 \nearrow_{1 \times 1} 0^{k_2 \times l_2} \nearrow 0^{k_1 \times l_1} \nearrow_{1 \times 1} P_2$.

714 \Leftarrow If there is no minimal matrix $P \in \mathcal{P}$ that is the skew sum of its non-empty
715 interval minors, we want to show it makes $Av_{\preceq}(\mathcal{P})$ closed under the skew
716 sum with 1×1 overlap. From Observation 3.16, for every $P \in \mathcal{P}$ there are no
717 r, c that $P[r, c]$ is an articulation and $P[[r, k], [c]], P[[r], [c, l]]$ are non-empty.
718 Let $M_1, M_2 \in Av_{\preceq}(P)$ be arbitrary matrices and let $M = M_1 \nearrow_{1 \times 1} M_2$.
719 The matrix M contains a non-trivial articulation and from Observation 3.15
720 it follows $M \in Av_{\preceq}(P)$ for each minimal $P \in \mathcal{P}$; thus, $M \in Av_{\preceq}(\mathcal{P})$. \square

721 In the following, we always expect articulations to be on a reverse walk (no two
722 articulations forming $(\bullet \bullet)$) and by a matrix between two articulations $M[r_1, c_1]$
723 and $M[r_2, c_2]$ we mean the matrix $M[[r_2, r_1], [c_1, c_2]]$.

724 **Lemma 3.18.** *Let \mathcal{P} be a set of matrices, then for all matrices $M \in \{0, 1\}^{m \times n}$
725 it holds that $M \in Cl(\mathcal{P}) \Leftrightarrow$ there exists a sequence of articulations of M on a
726 reverse walk such that for each matrix M' in between two consecutive articulations
727 of M there exists $P \in \mathcal{P}$ such that $M' \preceq \begin{pmatrix} 1 \end{pmatrix} \nearrow P \nearrow \begin{pmatrix} 1 \end{pmatrix}$.*

728 *Proof.* \Rightarrow With Observation 3.13 in mind, consider the skew sum with 1×1
729 overlap of multiple copies of elements of \mathcal{P} and let the sequence contain
730 an articulation between each pair of consecutive copies of matrices from \mathcal{P} ,
731 together with the trivial articulations $M[m, 1]$ and $M[1, n]$.

732 Between each pair of consecutive articulations, we have a matrix from \mathcal{P} and
733 so the statement holds. When we take an arbitrary interval minor and keep
734 original articulations, each matrix between two consecutive articulations
735 only contains at most one original copy of some matrix $P \in \mathcal{P}$, but it may
736 happen that the bottom-left and top-right corners become one-entries even
737 though they were zero-entries before. The matrix does not have to be an
738 interval minor of P anymore, but it is an interval minor of $\begin{pmatrix} 1 \end{pmatrix} \nearrow P \nearrow \begin{pmatrix} 1 \end{pmatrix}$
739 for the corresponding $P \in \mathcal{P}$.

740 \Leftarrow We can simply blow up each matrix M' between two consecutive articulation
741 to the skew sum of three copies of the corresponding matrix P and because
742 $M' \preceq \begin{pmatrix} 1 \end{pmatrix} \nearrow P \nearrow \begin{pmatrix} 1 \end{pmatrix} \preceq P \nearrow P \nearrow P$ it holds $M \in Cl(\mathcal{P})$. \square

743 Finally, we show that a closure under the skew sum can always be described
744 by a finite number of forbidden patterns.

745 **Theorem 3.19.** *For all matrices $M \in \{0, 1\}^{m \times n}$, $Cl(M)$ has a finite basis.*

746 *Proof.* Let \mathcal{F} be the set of all minimal (with respect to interval minors) matrices
747 such that $Cl(M) = Av_{\preceq}(\mathcal{F})$. We need to prove that \mathcal{F} is finite. Thanks to
748 Observation 3.13, $Av_{\preceq}(\mathcal{F})$ is closed under the direct sum with 1×1 overlap and
749 from Observation 3.17 follows that for no $F \in \mathcal{F}$ there are its non-empty interval
750 minors F_1, F_2 such that $F = F_1 \nearrow 1 \times 1 F_2$. We denote by \mathcal{P} a set of matrices
751 $F \in \mathcal{F}$ such that F has at most $2m + 4$ rows and $2n + 4$ columns. We want to
752 show $Cl(M) = Av_{\preceq}(\mathcal{P})$.

753 \subseteq Clearly, \mathcal{P} is finite and we immediately see that $Cl(M) \subseteq Av_{\preceq}(\mathcal{P})$.

754 \supseteq For contradiction, consider a minimal matrix $X \in Av_{\preceq}(\mathcal{P}) - Cl(M)$. There
755 are no X_1, X_2 non-empty interval minors of X such that $X = X_1 \nearrow$

756 $1 \times 1 X_2$; otherwise, as $X_1, X_2 \in Av_{\preceq}(\mathcal{P})$ and X is the minimum ma-
 757 trix such that $X \notin Cl(M)$, we would have $X_1, X_2 \in Cl(M)$; therefore,
 758 $X \in Cl(M)$ and a contradiction.

759 Without loss of generality, we assume $X \in \{0, 1\}^{k \times l}$ has at least $2m + 5$
 760 rows. Let X' denote a matrix created from X by deletion of the first row.
 761 We have $X' \in Av_{\preceq}(\mathcal{P})$ and from minimality of X also $X' \in Cl(M)$. From
 762 Lemma 3.18, there is a sequence of articulations of X' on a reverse walk
 763 such that each matrix between two consecutive articulations is an interval
 764 minor of $(1) \nearrow M \nearrow (1)$. Let $X'[r, c]$ be the first articulation from the
 765 sequence (sorted by the second coordinate in ascending order) for which
 766 $c > 1$. The matrix between $X'[r, c]$ and the previous articulation in the
 767 sequence is an interval minor of $(1) \nearrow M \nearrow (1)$, which also means that
 768 $c \leq n + 2$. Since $X[r, c]$ is not an articulation, it must hold that $X[1, c_1] = 1$
 769 for some $c_1 < c \leq n + 2$. Symmetrically, let X'' denote a matrix created
 770 from X by deletion of the last row. Following the same steps we did before,
 771 we get the last articulation $X''[r, c]$ such that $c < l$ and the observation
 772 that $c \geq l - n - 1$. Since $X[r, c]$ is not an articulation, it must hold that
 773 $X[k, c_2] = 1$ for some $c_2 > c \geq l - n - 1$.

774 We showed that a matrix $Y \in \{0, 1\}^{(m+1) \times 2}$ such that the only one-entries
 775 are $Y[1, 1]$ and $Y[m + 1, 2]$ is an interval minor of X . To reach a contra-
 776 diction, it suffices to show that there is a matrix $P \in \mathcal{P}$ such that $P \preceq Y$.
 777 For contradiction, let $Y \in Av_{\preceq}(\mathcal{P})$ and since $Y \preceq X$ and X is minimum
 778 such that $X \notin Cl(M)$ it holds $Y \in Cl(M)$. But this cannot be, because
 779 Y contains no non-trivial articulation and from Observation 3.15, we know
 780 that every matrix $Z \in Cl(M)$ bigger than $m \times n$ contains at least one. \square

781 3.3 Bases

782 We recall that the basis of a class of matrices \mathcal{M} is a set of all minimal (with
 783 respect to interval minors) matrices \mathcal{P} such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$. It goes without
 784 saying that it does not make sense to consider a basis of a set of matrices that is
 785 not closed under interval minors.

786 So far, we showed that whenever \mathcal{M} is finite, its basis is also finite. The same
 787 hold when $\mathcal{M} = Cl(M)$ for some M . We show next that, unlike in graph theory,
 788 there are classes that does not have a finite basis. Moreover, we show that even
 789 for a class \mathcal{M} with finite basis, its closure $Cl(\mathcal{M})$ can have an infinite basis.

790 **Definition 3.20.** Let P be a matrix. We denote by $\mathcal{R}(P)$ a set of all minimal
 791 (with respect to minors) matrices P' such that $P \preceq P'$ and P' is not the skew
 792 sum with 1×1 overlap of non-empty interval minors of P' . For a set of matrices
 793 \mathcal{P} , let $\mathcal{R}(\mathcal{P})$ denote a set of all minimal (with respect to minors) matrices from
 794 the set $\bigcup_{P \in \mathcal{P}} \mathcal{R}(P)$.

795 **Theorem 3.21.** Let \mathcal{M} and \mathcal{P} be sets of matrices such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$, then
 796 $Cl(\mathcal{M}) = Av_{\preceq}(\mathcal{R}(\mathcal{P}))$.

797 *Proof.* \subseteq Consider a matrix $M \notin Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ and without loss of generality,
 798 because $Cl(\mathcal{M})$ is closed under interval minors, let M be minimal (with

respect to interval minors). It follows that $M \in \mathcal{R}(\mathcal{P})$. Then, the matrix M is not a skew sum with 1×1 overlap of non-empty interval minors of M ; therefore, according to Observation 3.16, there is no articulations $M[r, c]$ such that $M[[r, k], [c]], M[[r], [c, l]]$ are non-empty.

For contradiction with the statement, assume $M \in Cl(\mathcal{M})$. According to Lemma 3.18 and the fact M contains no non-trivial articulation, it holds $M \preceq (1) \nearrow M' \nearrow (1)$ for some $M' \in \mathcal{M}$. Because the trivial articulations contain zero-entries, it even holds $M \preceq M'$. We also know $P \preceq M$ for some $P \in \mathcal{P}$, which together give us a contradiction with $\mathcal{M} = Av_{\preceq}(\mathcal{P})$.

\supseteq First of all, $Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ is closed under the skew sum with 1×1 overlap. For contradiction, assume there are matrices $M_1, M_2 \in Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ but $M = M_1 \nearrow_{1 \times 1} M_2 \notin Av_{\preceq}(\mathcal{R}(\mathcal{P}))$. Then there exists a matrix $P \in \mathcal{R}(\mathcal{P})$ such that $P \preceq M$. Because P is not a skew sum with 1×1 overlap of non-empty interval minors of P , it follows that either $P \preceq M_1$ or $P \preceq M_2$ and we have a contradiction.

It suffices to show that the inclusion holds for any matrix $M \in Av_{\preceq}(\mathcal{R}(\mathcal{P}))$ that is not a skew sum with 1×1 overlap of non-empty interval minors of M . From Observation 3.16, we know that M does not contain any non-trivial articulation and those trivial ones are empty. Thus, $M \in Av_{\preceq}(\mathcal{P}) = \mathcal{M}$ and so $M \in Cl(\mathcal{M})$. \square

Corollary 3.22. *Let \mathcal{M} and \mathcal{P} be sets of matrices such that $\mathcal{M} = Av_{\preceq}(\mathcal{P})$, then $\mathcal{R}(\mathcal{P})$ is the basis of $Cl(\mathcal{M})$.*

What follows is a construction of parameterized matrices that become the main tool of finding a class of matrices with an infinite basis.

Definition 3.23. Let $Nucleus_1 = (\bullet)$ and for $n > 1$ let $Nucleus_n \in \{0, 1\}^{n \times n+1}$ be a matrix described by the examples:

$$Nucleus_2 = \begin{pmatrix} \bullet & \bullet & \bullet \end{pmatrix} \quad Nucleus_3 = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \quad Nucleus_n = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}.$$

Definition 3.24. Let $Candy_{k,n,l}$ be a matrix given by $I_k \nearrow_{1 \times 2} Nucleus_n \nearrow_{1 \times 2} I_l$, where I_k, I_l are unit matrices of sizes $k \times k$ and $l \times l$ respectively.

$$Candy_{4,1,4} = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} \quad Candy_{4,4,4} = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Theorem 3.25. *There exists a matrix P for which $\mathcal{R}(P)$ is infinite.*

Proof. Let $P = Candy_{4,1,4}$. For all $n > 3$ it holds $P \preceq Candy_{4,n,4}$ and it suffices to show that each $Candy_{4,n,4}$ is a minimal matrix (with respect to minors) and it is not the skew sum of two of its non-empty interval minors. According to Observation 3.16, the second condition holds as $Candy_{4,n,4}$ contains no non-trivial articulation and the trivial ones are empty. To show it is minimal, we need to

835 consider any matrix $M \preceq Candy_{4,n,4}$ and argue that either $P \not\preceq M$ or M contains
 836 an articulation.

837 Thanks to Observation 3.15, as soon as we find a non-trivial articulation
 838 $M[r, c]$ such that $M[[r, k], [c]], M[[r], [c, l]]$ are non-empty, it will stay there in
 839 any interval minor, because we cannot delete one-entries $M[1, n-3], M[2, n-2],$
 840 $M[3, n-1]$ and $M[4, n]$ (and symmetrically $M[m-3, 1], M[m-2, 2], M[m-1, 3],$
 841 $M[m, 4]$) without losing the condition $P \preceq M$. Therefore, we can only
 842 consider one minoring operation at a time.

843 It is easy to see that when a one-entry is changed to a zero-entry, then the
 844 matrix does not belong to $\mathcal{R}(P)$ anymore. Consider that rows r_1, r_2, \dots, r_k are
 845 chosen to be merged into one with an elementwise OR. If $r_1 < 4$ or $r_k > n+3$
 846 then P is no longer an interval minor of such matrix. Otherwise, the original
 847 $Candy_{4,n,4}[r_1, n-r_1+2]$ becomes an articulation. Symmetrically, the same holds
 848 for columns which concludes the proof. \square

849 **Corollary 3.26.** *There exists a class of matrices \mathcal{M} having a finite basis such*
 850 *that $Cl(\mathcal{M})$ has an infinite basis.*

851 *Proof.* From Theorem 3.25, we have a matrix P for which $\mathcal{R}(P)$ is infinite. Class
 852 $\mathcal{M} = Av_{\preceq}(P)$ has a finite basis. On the other hand, from Theorem 3.21, we have
 853 $Cl(\mathcal{M}) = Av_{\preceq}(\mathcal{R}(P))$. \square

4. Zero-intervals

In Chapter 2, we characterized matrices avoiding small patterns. Their structure is very dependent on the pattern they avoid and the results are hard to generalize for arbitrary patterns. In this chapter, we look for a more general property that restricts the complexity of a class of matrices.

Definition 4.1. For a matrix $M \in \{0, 1\}^{m \times n}$, a row interval $M[\{r\}, [c_1, c_2]]$ is a *zero-interval* if all entries are zero-entries, $c_1 = 0$ or $M[r, c_1 - 1] = 1$ and $c_2 = n$ or $M[r, c_2 + 1] = 1$. In other words, it is an interval of zero-entries bounded by one-entries. Symmetrically, we also call a column interval $M[[r_1, r_2], \{c\}]$ a *zero-interval* if all entries are zero-entries, $r_1 = 0$ or $M[r_1 - 1, c] = 1$ and $r_2 = m$ or $M[r_2 + 1, c] = 1$. In the same spirit, we define a *one-interval* to be an interval of one-entries in a single line of M bounded by zero-entries (or edges of the matrix).

Definition 4.2. For a class of matrices \mathcal{M} , we say that a matrix $M \in \mathcal{M}$ is *critical* in \mathcal{M} if the change of any zero-entry to a one-entry creates a matrix that does not belong to \mathcal{M} . For any set of matrices \mathcal{P} , let $Av_{crit}(\mathcal{P})$ be a set of all critical matrices avoiding \mathcal{P} as an interval minor.

In Chapter 2, for a pattern $P \in \{0, 1\}^{k \times l}$ it very often holds that any matrix from $Av_{crit}(P)$ has at most k zero-intervals in each row and at most l zero-intervals in each column. The main goal of this chapter is to describe patterns P for which there can be arbitrarily many zero-intervals in matrices from $Av_{crit}(P)$.

4.1 Pattern complexity

We define the complexity of a class of matrices as the maximum number of zero-intervals (or one intervals as they go in pair) a critical matrix from the class can have.

Definition 4.3. For a class of matrices \mathcal{M} , we define its *row-complexity* $r(\mathcal{M})$ to be the supremum of the number of zero-intervals in a single row of any critical matrix $M \in \mathcal{M}$. We say that \mathcal{M} is *row-bounded*, if its row-complexity is finite, and *row-unbounded* otherwise. Symmetrically, we define its *column-complexity* $c(\mathcal{M})$ and the property of being *column-bounded* and *column-unbounded*. The class \mathcal{M} is *bounded* if it is both row-bounded and column-bounded; otherwise, it is *unbounded*.

Definition 4.4. We say that a set of patterns \mathcal{P} is *bounding*, if the class $Av_{\leq}(\mathcal{P})$ is bounded; otherwise, it is *non-bounding*.

Now that we introduced the most essential definitions in this chapter, it is time to state the main theorem:

Theorem 4.5. A pattern P is bounding $\Leftrightarrow P_i \not\leq P$ for all $1 \leq i \leq 4$.

$$P_1 = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \quad P_2 = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \quad P_3 = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \quad P_4 = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

889 We prove the statement in several steps. We show the first implication in
 890 Subsection 4.1.2, then we proof multiple lemmata so that we finally show the
 891 other implication at the end of Subsection 4.1.3. Before we start proving the
 892 main result, we introduce some useful notation and get more familiar with zero-
 893 intervals.

894 **Definition 4.6.** Let P be a pattern, let e be a one-entry of P , consider a matrix
 895 $M \in Av_{\preceq}(P)$ and let z be an arbitrary zero-interval of M . We say that z is
 896 *usable for e* if there is a zero-entry contained in z such that if we change it to a
 897 one-entry, it creates a mapping of P to M that uses the new one-entry to map e .
 898 This way, z can be usable for many one-entries of P at once.

899 **Observation 4.7.** Let $P \in \{0, 1\}^{k \times l}$ and $M \in \{0, 1\}^{m \times n}$ be matrices such that
 900 $P \not\preceq M$. Let $z = M[\{r_1\}, [c_1, c_2]]$ be a zero-interval of M usable for a one-
 901 entry $e = P[r, c]$. If we change a zero-entry of z and create a mapping of P that
 902 uses the changed entry to map e , then the mapping can only map column c of P
 903 to columns $[c_1, c_2]$ of M .

904 *Proof.* Since the changed entry is used to map e , clearly the mapping needs to
 905 use a column from $[c_1, c_2]$ to map column c . If, for contradiction, the mapping
 906 uses columns outside $[c_1, c_2]$ then, without loss of generality, it uses the column
 907 $c_1 - 1$. Since that column bounds the zero-interval z , $M[r_1, c_1 - 1] = 1$ and this
 908 one-entry can be used in the mapping instead of the changed entry, which gives
 909 us a contradiction with $P \not\preceq M$. \square

910 **Definition 4.8.** Let \mathcal{P} be a set of patterns and let e be a one-entry of any ma-
 911 trix $P \in \mathcal{P}$. We define the *row-complexity* of e , $r(Av_{\preceq}(\mathcal{P}), e)$ to be the supremum
 912 of the number of zero-intervals of a single row of any $M \in Av_{crit}(\mathcal{P})$ that are
 913 usable for e . We say that e is *row-unbounded* in $Av_{\preceq}(\mathcal{P})$ if $r(Av_{\preceq}(\mathcal{P}), e) = \infty$
 914 and *row-bounded* otherwise. Symmetrically, we define the *column-complexity* of e ,
 915 $c(Av_{\preceq}(\mathcal{P}), e)$ to be the maximum number of zero-intervals of a single column of
 916 any matrix from $Av_{crit}(\mathcal{P})$ that are usable for e , and we say e is *column-unbounded*
 917 if it is infinite and *column-bounded* otherwise.

918 The following observation follows directly from the definition and we use it
 919 heavily throughout the chapter to break symmetries.

920 **Observation 4.9.** For every set \mathcal{M} , \mathcal{M} is row-bounded $\Leftrightarrow \mathcal{M}^\top$ is column-
 921 bounded.

922 4.1.1 Adding empty lines

923 As in Chapter 2, we show that we do not need to consider patterns with leading
 924 and ending empty rows and columns.

925 **Observation 4.10.** For a matrix $P \in \{0, 1\}^{k \times l}$ and an integer n , let $P' = P \rightarrow$
 926 $0^{k \times n}$. The matrix P is bounding $\Leftrightarrow P'$ is bounding. Moreover, if P is bounding,
 927 then $r(Av_{\preceq}(P')) \leq r(Av_{\preceq}(P)) + 1$.

928 **Lemma 4.11.** Let $P \in \{0, 1\}^{2 \times k}$ be a matrix and for any $l \geq 1$, let $P^l \in$
 929 $\{0, 1\}^{(l+2) \times k}$ be a pattern created from P by adding l new empty rows in between
 930 the two row of P . For every one-entry e of P^l it holds $r(Av_{\preceq}(P^l), e) \leq k^2$.

931 *Proof.* Given a matrix $M \in Av_{crit}(P)$, consider an arbitrary row r of M . Without
 932 loss of generality, assume $e = P[1, c]$. For contradiction, assume there are $k^2 + 1$
 933 zero-intervals z_1, \dots, z_{k^2+1} in r usable for e . In particular, the first k^2 of them
 934 are bounded by a one-entry from the right side.

935 • $P[2, c] = 1$: Clearly, there is a one-entry in rows $[r + l + 1, m]$ underneath
 936 each z_j and if we combine each such one-entry with a one-entry bounding
 937 corresponding z_j , we find a mapping of $(\{1\}^{2 \times k^2})^l$, contradicting $P \not\leq M$.

938 • $P[2, c] = 0$: For each $i \in [k^2]$, we define an extended interval z_i^* to be
 939 the interval containing z_i and also all entries on the row r between z_i and
 940 z_{i+1} . Because of the Pigeonhole principle, we can find either k consecutive
 941 extended intervals such that there are no one-entries in rows $[r + l + 1, m]$
 942 underneath them, or k (not necessarily consecutive) extended intervals such
 943 that there is a one-entry in rows $[r + l + 1, m]$ underneath each of them.
 944 Because each extended interval contains a one-entry, in the second case we
 945 find $(\{1\}^{k \times 2})^l$ as an intervals minor.

946 In the first case, without loss of generality, assume $P[2, c_1] = 1$ and it is
 947 the minimum such $c_1 > c$. Let z'_1, \dots, z'_k be the consecutive zero-intervals.
 948 Consider the mapping of P^l created when a zero-entry of z'_1 is changed to
 949 a one-entry used to map e . Since $P[2, c_1] = 1$ and there are no one-entries
 950 in rows $[r + l + 1, m]$ underneath extended intervals z'_1, \dots, z'_k , $P^l[l + 2, c_1]$
 951 has to be mapped to the columns of M after the end of z'_k . This leaves k
 952 one-entries to be used to map potential one-entries in $P^l[\{l + 2\}, [c, c_1 - 1]]$
 953 and so $P^l \leq M$, which is again a contradiction. \square

954 **Corollary 4.12.** Let $P \in \{0, 1\}^{k \times 2}$ be a matrix and for any $l \geq 1$, let $P^l \in$
 955 $\{0, 1\}^{k \times (l+2)}$ be a matrix created from P by adding l new empty columns in between
 956 the two columns of P . Then $Av_{\leq}(P^l)$ is bounded for any $l \geq 1$.

957 *Proof.* We know $Av_{\leq}(P^l)$ is row-bounded from Lemma 2.9. From Lemma 4.11
 958 and Observation 4.9 we have that the class is also column-bounded. \square

959 4.1.2 Non-bounding patterns

960 We see that for patterns having only two non-empty rows or columns we can
 961 indeed bound the number of zero-intervals of critical matrices avoiding them. On
 962 the other hand, already for a pattern of size 3×3 we show that there are maximal
 963 matrices with arbitrarily many zero-intervals.

964 **Lemma 4.13.** A class $Av_{\leq}(P_1)$ is unbounded.

Proof. For a given integer n , let M be a $(2n + 1) \times (2n + 1)$ matrix described by
 the picture:

$$\begin{pmatrix} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet \\ & & & \cdots & & & & \\ & & & \cdots & & & & \\ & & & \cdots & & & & \\ & & & \cdots & & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

965 We see that $P_1 \not\preceq M$ because we always need to map $P_1[2, 1]$ and $P_1[3, 3]$ to just
 966 one “block” of one-entries, which only leaves a zero-entry for $P_1[1, 2]$.

967 If we change any zero-entry of the first row into a one-entry, we get a matrix
 968 containing an interval minor of $\{1\}^{3 \times 3}$; therefore, containing P_1 as an interval
 969 minor. In case M is not critical, we add some more one-entries to make it critical
 970 but it will still contain a row with n zero-intervals. \square

971 Not only $M \in Av_{crit}(P_1)$ but it also avoids any $P \in \{0, 1\}^{3 \times 3}$ such that
 972 $P_1 \preceq P$. Its rotations avoid rotations of P_1 and we conclude that a big portion of
 973 patterns of size 3×3 are non-bounding. Moreover, the result can be generalized
 974 also for bigger matrices.

975 **Theorem 4.14.** *For every matrix P such that $P_1 \preceq P$, $Av_{\preceq}(P)$ is unbounded.*

976 *Proof.* First, assume there is a mapping of P_1 into $P \in \{0, 1\}^{k \times l}$ that maps $P_1[1, 2]$
 977 to a one-entry of the first row of P , $P_1[2, 1]$ to a one-entry of the first column of P
 978 and $P_1[3, 3]$ to the bottom-right corner of P . Then, we use a similar construction
 979 as we did in the proof of Lemma 4.13 to find a matrix $M \in Av_{crit}(P)$ with n
 980 zero-intervals for any n .

981 Let P be an arbitrary pattern containing P_1 as an interval minor. Let
 982 $P[r_1, c_1]$, $P[r_2, c_2]$ and $P[r_3, c_3]$ be one-entries that can be used to map $P_1[1, 2]$,
 983 $P_1[2, 1]$ and $P_1[3, 3]$ respectively. We take a submatrix $P' := P[[r_1, r_3], [c_2, c_3]]$.
 984 Such a matrix fulfills assumptions of the more restricted case above and we find
 985 a matrix $M' \in Av_{crit}(P')$ having n zero-intervals. We construct M from M' by
 986 simply adding new rows and columns containing only one-entries. We add $r_1 - 1$
 987 rows in front of the first row and $k - r_3$ rows behind the last row. We also add
 988 $c_2 - 1$ columns in front of the first column and $l - c_3$ columns behind the last
 989 column. The constructed matrix M avoids P as an interval minor because its
 990 submatrix P' cannot be mapped to M' . At the same time, any change of a zero-
 991 entry of the r_1 -th row of M to a one-entry creates a copy of $1^{k \times l}$. The constructed
 992 matrix M can be seen in Figure 4.1. \square

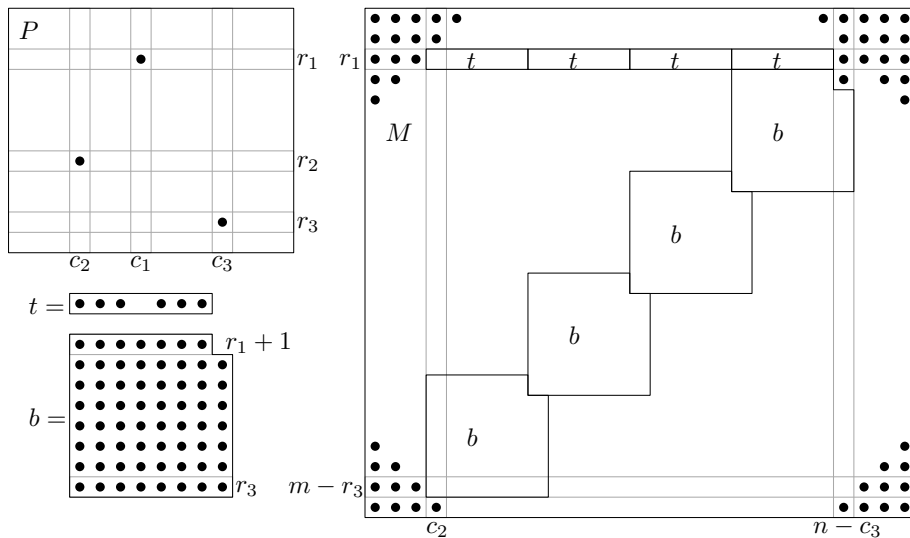


Figure 4.1: The structure of a critical matrix avoiding P that has arbitrarily many zero-intervals.

993 4.1.3 Bounding patterns

994 What makes it even more interesting is that any pattern avoiding all rotations of
 995 P_1 as interval minors is already bounding. For simplicity, whenever we say that
 996 a matrix has only k non-empty lines, we mean that every one-entry belongs to
 997 one of the k lines.

998 **Theorem 4.15.** *Let P be a pattern avoiding all rotations of P_1 , then P*

999 *1. contains at most three non-empty lines or*

1000 *2. avoids $(\bullet \bullet)$ or $(\bullet \bullet)$.*

1001 *Proof.* Assume P has four one-entries that do not share any row or column.
 1002 Then those one-entries induce a 4×4 permutation inside P and because P does
 1003 not contain any rotation of P_1 , the induced permutation is either 1234 or 4321.
 1004 Without loss of generality, assume it is the first one and denote its one-entries by
 1005 e_1, e_2, e_3 and e_4 . Clearly, no one-entry from e_1, e_2, e_3 and e_4 can be part of any
 1006 mapping of $P' = (\bullet \bullet)$ because it would induce a mapping of a rotation of P_1 .

1007 Let $e_2 = P[r_2, c_2]$ and $e_3 = P[r_3, c_3]$. The submatrix $P[[r_2], [c_2, l]]$ avoids P' ;
 1008 otherwise, together with e_1 it would give P_2 as an interval minor. Symmetrically,
 1009 $P' \not\leq P[[r_3, k], [c_3]]$. The submatrix $P[[r_3 - 1], [c_3 - 1]]$ is empty; as otherwise, any
 1010 one-entry would create a rotation of P_1 with e_3 and either e_1 or e_2 . Symmetrically,
 1011 the submatrix $P[[r_2 - 1], [c_2 - 1]]$ is also empty. This leave no one-entry in P to
 1012 be used to map $P'[1, 1]$ and so $P' \not\leq P$. \square

1013 We now need to prove that whenever P avoids all rotations of P_1 (and satisfies
 1014 one of the conditions we just showed) it is bounding.

1015 **Lemma 4.16.** *Let $P \in \{0, 1\}^{k \times l}$ be a pattern having one non-empty line. Then*
 1016 *$r(Av_{\leq}(P)) \leq k$ and $c(Av_{\leq}(P)) \leq l$.*

1017 *Proof.* Without loss of generality, let the non-empty line be a row r . Consider
 1018 any matrix $M \in Av_{crit}(P)$. Submatrices $M[[r - 1], [n]]$ and $M[[m - r + 1, m], [n]]$
 1019 contain no zero-entry. If we look at any other row, it cannot contain k one-entries,
 1020 so the maximum number of zero-intervals is k .

1021 Consider a column c of M . If there is at least one one-entry in $M[[r, m -$
 1022 $r - 1], c]$ then because M is critical, the whole column is made of one-entries.
 1023 Otherwise, there are two one-intervals $M[[r - 1], c]$ and $M[[m - r, m], c]$. \square

1024 **Lemma 4.17.** *Let $P \in \{0, 1\}^{k \times l}$ be a pattern having two non-empty lines. Then*
 1025 *$r(Av_{\leq}(P)) \leq k^2 + l$ and $c(Av_{\leq}(P)) \leq l^2 + k$.*

1026 *Proof.* First, we assume the two non-empty lines of P are rows $r_1 < r_2$ (or
 1027 symmetrically columns) and consider any matrix $M \in Av_{crit}(P)$. From Obser-
 1028 vation 2.6 and maximality of M , we have that the submatrices $M[[r_1 - 1], [n]]$
 1029 and $M[[m - r_2 + 1, m], [n]]$ contain no zero-entry. Therefore, we may restrict
 1030 ourselves to the case when $r_1 = 1$ and $r_2 = k$. From Corollary 4.12, we have that
 1031 there are at most k^2 zero-intervals in each row of M and there are at most two
 1032 zero-intervals in each column of M .

1033 Let the two non-empty lines of P be a row r and a column c . Because of
 1034 symmetry, we only show the bound for rows. For every one-entry e of P , except

1035 those in the row r , there is at most one zero-interval usable for e in each row of
 1036 any $MAv_{crit}(P)$. For contradiction, assume there are two such zero-intervals z_1
 1037 and z_2 in the same row. Let Figure 4.2 illustrate the situation where red and blue
 1038 lines form two mappings of P to M when a zero-entry of z_1 and z_2 respectively
 1039 is changed to a one-entry used to map e . When we take the outer two vertical
 1040 and horizontal lines, we get a mapping of P that uses an existing one-entry in
 1041 between z_1 and z_2 to map e . This is a contradiction with $P \not\leq M$.

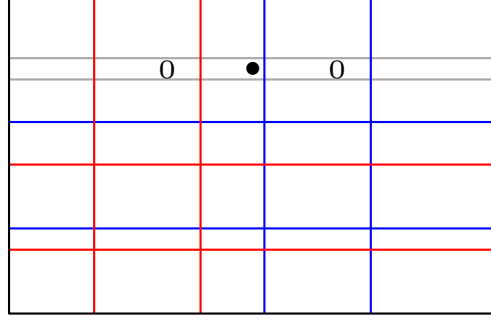


Figure 4.2: Red and blue lines representing two different mappings of a forbidden pattern. The two horizontal lines show the boundaries of the mapping of row r and the vertical lines show the boundaries of the mapping of column c .

1042 For a one-entry $e = P[r, c']$, if $c' \leq c$ then there must be less than c' one-entries
 1043 before any zero-intervals usable for e ; otherwise, we could map $P[r, [1, c']]$ just to
 1044 the single row of M . It follows that e is row-bounded. Symmetrically, the same
 1045 holds in case $c' > c$ and together we have at most $k + l$ zero-intervals in each
 1046 $M \in Av_{crit}(P)$. \square

1047 Before we proof the other cases, let us introduce three useful lemmata that
 1048 make the future case analysis bearable.

1049 **Lemma 4.18.** *Let $P \in \{0, 1\}^{k \times l}$ be one of the four matrices in Figure 4.3. Then
 1050 every one-entry in $P[\{r_2\}, [c_1, c_2]]$ is row-bounded. Moreover, the same also holds
 1051 if we change some one-entries to zero-entries.*

1052 *Proof.* Let a pattern P be the first described matrix and let $k' = c_2 - c_1$. We show
 1053 that for each one-entry $e \in P[\{r_2\}, [c_1, c_2]]$ and every matrix $M \in Av_{crit}(P)$ there
 1054 are at most k' zero-intervals usable for e in each row of M . For contradiction,
 1055 assume there is a row r with $k' + 1$ zero-intervals usable for some e . It follows that
 1056 there are at least k' one-entries in between the two most distant zero-intervals z_1
 1057 and z_2 . Therefore, the whole row r_2 can be mapped just to the row r . Changing
 1058 a zero-entry of z_1 to a one-entry, to which e can be mapped, creates a mapping of
 1059 P to M , in which all one-entries from columns $[c_1]$ are mapped to columns before
 1060 z_1 (and z_1) and similarly all one-entries from columns $[c_2, l]$ can be mapped to
 1061 columns past z_2 (and z_2). It also holds that all the one-entries from the row r_1
 1062 are mapped (in both mappings) to one-entries of M in rows $[r - r_2 + r_1]$ (and
 1063 symmetrically for one-entries from the row r_3). Thus, we can simply map empty
 1064 rows $[r_1 + 1, r_3 - 1]$ around row r and use the rest to map rows r_1 and r_2 .

1065 Proofs of cases two and three are similar to the first one and we skip them.

1066 Let a pattern P be the fourth described matrix and consider any matrix $M \in$
 1067 $Av_{crit}(P)$. For the i -th one-entry e in the row r_2 (ordered from left to right

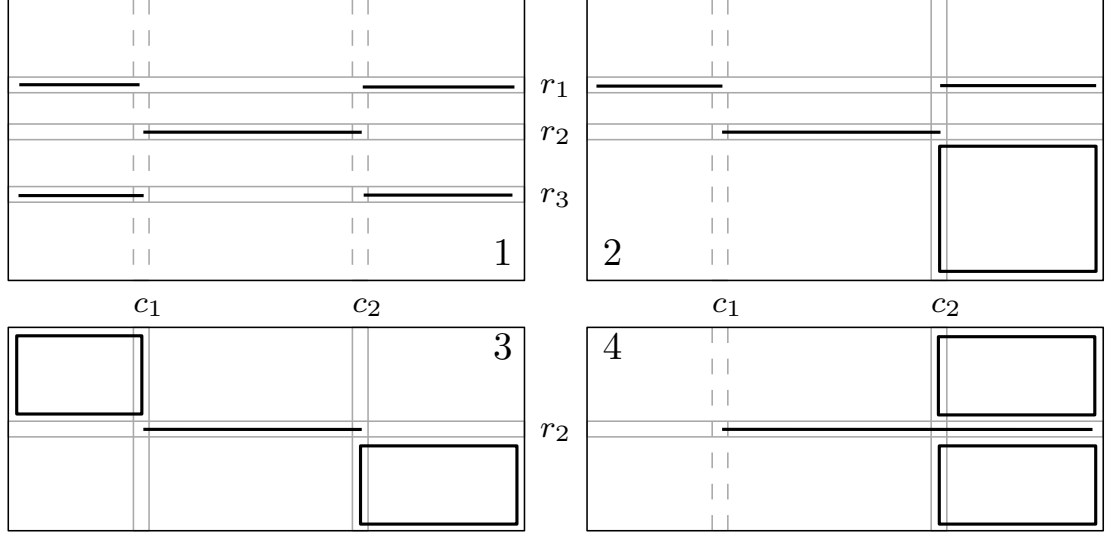


Figure 4.3: The patterns for which all one-entries in the row r_2 and the columns $[c_1, c_2]$ are row-bounded. One-entries of the patterns are inside the bold rectangles and on the bold lines.

1068 and only considering those in columns $[c_1, c_2]$) no zero-interval of M usable for
 1069 e cannot have i one-entries before it and so the row-complexity of each such
 1070 one-entry is bounded by $i \geq l$.

1071 Throughout the proof, we have never used as a fact that an entry of M is a
 1072 one-entry and so the proof also holds for any pattern P created from any of the
 1073 fourth described matrices by deletion of one-entries. \square

1074 It is important to realize that we could not have used the same proof we used
 1075 for the first three cases also for the fourth case, because we can never rely on the
 1076 fact a mapping of P only uses one row of M to map the row r_2 . This is because
 1077 in the fourth case, there are also potential one-entries in $P[\{r_2\}, [c_2 + 1, l]]$.

1078 What follows is a direct corollary of the fourth case of just stated Lemma 4.18.
 1079 Even though it is very simple and straightforward, it is going to be used so often
 1080 that it is worth stating it apart from the rest.

1081 **Lemma 4.19.** *Let P be a matrix and let c be its first non-empty column. Then*
 1082 *every one-entry from c is row-bounded.* \square

1083 **Lemma 4.20.** *Let $P \in \{0, 1\}^{k \times l}$ be one of the three matrices in Figure 4.4. Then*
 1084 *every one-entry in $P[[r_1 + 1, r_2 - 1], \{c\}]$ is row-bounded. Moreover, the same also*
 1085 *holds if we change some one-entries to zero-entries.*

1086 *Proof.* Let P be a submatrix of the first described matrix. We show that for each
 1087 one-entry e from $P[[r_1 + 1, r_2 - 1], \{c\}]$ and every matrix $M \in Av_{crit}(P)$ there
 1088 is at most one zero-interval usable for e in M . For contradiction, assume there
 1089 is a row r with two zero-intervals z_1 and z_2 usable for e . Consider Figure 4.5,
 1090 where the red lines show a mapping of P to M created when a zero-entry of z_1
 1091 is changed to a one-entry used to map e and the blue lines show a mapping of P
 1092 to M created when a zero-entry of z_2 is changed to a one-entry used to map e .
 1093 If we map the column c to the columns of M enclosed by the two outer vertical
 1094 lines and map rows r_1 and r_2 again to rows enclosed by the corresponding two

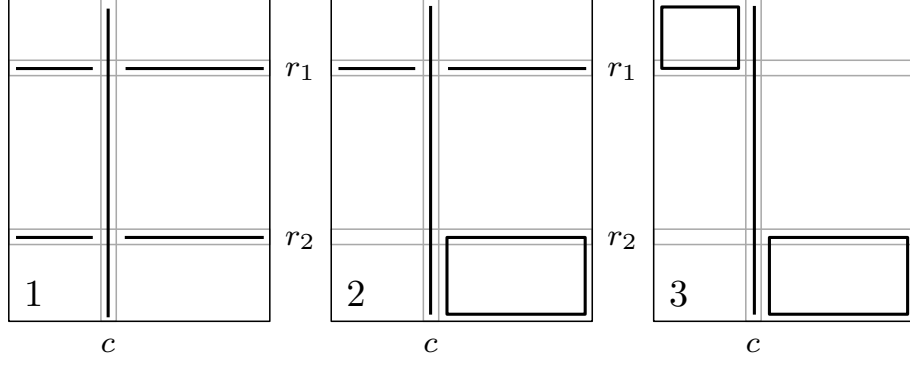


Figure 4.4: The patterns for which all one-entries in the column c and the rows $[r_1 + 1, r_2 - 1]$ are row-bounded. One-entries of the patterns are inside the bold rectangles and on the bold lines.

1095 outer horizontal lines, we get a mapping of P to M and so a contradiction with
 1096 $P \not\leq M$.

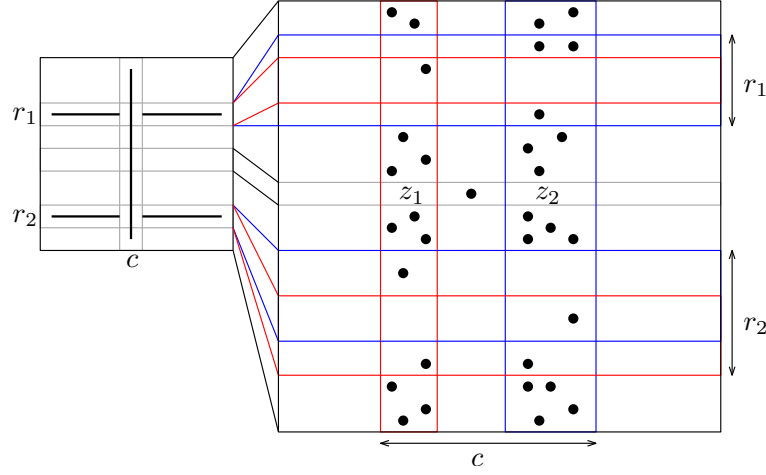


Figure 4.5: Red and blue lines representing two different mappings of a forbidden pattern. The four horizontal lines show the boundaries of the mapping of rows r_1 and r_2 and the vertical lines show the boundaries of the mapping of the column c .

1097 Proofs of cases two and three are similar to the first one and we skip them.

1098 Throughout the proof, we have never used as a fact that an entry of M is a
 1099 one-entry and so the proof also holds for any pattern P created from any of the
 1100 fourth described matrices by deletion of one-entries. \square

1101 **Lemma 4.21.** *Let a pattern $P \in \{0, 1\}^{k \times l}$ be created from one of the matrices in*
 1102 *Figure 4.6 by deletion of one-entries and let $c = l - 1$. Then every one-entry in*
 1103 *$P[[r_1, r_2], \{c\}]$ is row-bounded.*

1104 *Proof.* Let a pattern P be created from the first described matrix. From 4.20,
 1105 we know that all one-entries in $P[[r_1 + 1, r_2 - 1], \{c\}]$ are row-bounded. Thank
 1106 to symmetry, it suffices to show that the one-entry $e = P[r_1, c]$ is row-bounded.
 1107 Without loss of generality, we have $P[r_2, l] = 1$; otherwise, we can use the fourth
 1108 case of Lemma 4.3 to prove that e is row-bounded.

1109 Consider any matrix $M \in Av_{crit}(P)$ and let $z_1 < z_2$ be any two zero-intervals
 1110 from the same row usable for e . Without loss of generality, in any mapping of P

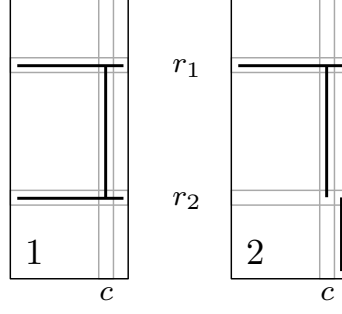


Figure 4.6: The patterns for which all one-entries in the column c and the rows $[r_1, r_2]$ are row-bounded. One-entries of the patterns are on the bold lines and the column c is the second last.

1111 to M , created when a zero-entry of z_1 is changed to a one-entry used to map e ,
 1112 the one-entry $P[r_2, l]$ is mapped to a column before z_2 . Otherwise, if we map e
 1113 to the one-entry between z_1 and z_2 and map $P[r_1, l]$ to any one-entry behind z_2
 1114 we get a mapping showing $P \preceq M$.

1115 We prove there are at most l zero-intervals usable for e on every row of M .
 1116 For contradiction, let there be such zero-intervals z_1, \dots, z_l that there is a one-
 1117 entry behind each of them. For each zero-interval z_i , let e_i be any one-entry of
 1118 M that can be used to map the one-entry $P[r_2, l]$ if a zero-entry of z_i is changed
 1119 to a one-entry used to map e . In the sequence e_1, \dots, e_l there either are two
 1120 one-entries $M[r'_1, c'_1], M[r'_2, c'_2]$ such that $r'_1 \leq r'_2$, or the rows of one-entries form
 1121 a decreasing sequence.

1122 Let us first consider the first case and let $e_i = M[r'_1, c'_1]$ and $e_j = M[r'_2, c'_2]$.
 1123 Consider a mapping of P to M created when a zero-entry of z_i is changed to a
 1124 one-entry used to map e . If in this mapping, we map e to a one-entry between
 1125 z_i and z_j , map $P[r_1, l]$ to a one-entry behind z_j , map $P[r_2, l - 1]$ to e_i and map
 1126 $P[r_2, l]$ to e_j , we get a mapping of P to M , which is a contradiction.

1127 And so it holds that the one-entries e_1, \dots, e_l form a row decreasing sequence.
 1128 We can pair every e_i with a one-entry bounding z_i from the right and so we can
 1129 map the whole submatrix $P[[k], [l - 2]]$ just to columns before z_{l-1} of M . Because
 1130 z_l is usable for e , there are enough one-entries to map the whole column c there
 1131 and there are one-entries where $P[r_1, l]$ and $P[r_2, l]$ can be mapped. The only
 1132 problem is that e is mapped to a one-entry created by changing a zero-entry of
 1133 z_l but we can also map it to a one-entry between zero-intervals z_{l-1} and z_l and
 1134 we have $P \preceq M$ and a contradiction.

1135

1136 Let a pattern P be created from the second described matrix. All one-
 1137 entries in $P[[r_1 + 1, r_2 - 1], \{c\}]$ are row-bounded thanks to (the second case
 1138 of) Lemma 4.20. From the fourth case of Lemma 4.18, the one-entry $P[r_1, c]$
 1139 is also row-bounded. So we only need to prove that the one-entry $P[r_2, c]$ is
 1140 row-bounded.

1141 Without loss of generality, $P[r_1, l] = 1$; otherwise, $(\bullet, \bullet) \not\preceq P$ and in the
 1142 following Lemma 4.22, we show that every such P is bounding. We once again
 1143 define one-entries e_1, \dots, e_l and use the same analysis as we did in the first case.

1144 \square

1145 Now that the very technical lemmata are stated, we just use them to easily

1146 prove that the remaining patterns described in Theorem 4.15 are also bounding.

1147 **Lemma 4.22.** *Let $P \in \{0, 1\}^{k \times l}$ be a pattern avoiding $(\bullet \bullet)$ or $(\bullet \bullet)$. Then P is*
 1148 *bounding.*

1149 *Proof.* From Proposition 2.15, we know that P is a walking pattern. Every one-
 1150 entry of P satisfies either conditions of the third case of Lemma 4.18 or it satisfies
 1151 conditions of the third case of Lemma 4.20 and therefore is row-bounded. From
 1152 Observation 4.9, we know it is also column-bounded. \square

1153 What follows is the last and the most difficult case of our analysis. Its length
 1154 is caused by the fact that it is harder to describe symmetries than it is to just
 1155 use the previous lemmata to show that each pattern is bounding.

1156 **Lemma 4.23.** *Let $P \in \{0, 1\}^{k \times l}$ be a pattern having three non-empty lines and*
 1157 *avoiding all rotations of P_1 . Then P is bounding.*

1158 *Proof.* First of all, if P avoids $(\bullet \bullet)$ or $(\bullet \bullet)$, we use Lemma 4.22.

1159 Let the three non-empty lines be three rows and let a pattern P have one-
 1160 entries in at least three columns. Then it contains a 3×3 permutation matrix as a
 1161 submatrix. Since the rotations of P_1 are avoided, the only feasible permutations
 1162 are 123 and 321 and without loss of generality, we assume the first case. In
 1163 Figure 4.7 we see the structure of P . The capital letters stand for one-entries of
 1164 the permutation and are chosen to be the left-most possible, letters $a - f$ stand
 1165 each for a potential one-entry and the Greek letters stand each for a potential
 1166 sequence of one-entries. Everything else is empty. Not all one-entries can be there
 1167 at the same time, because that would create a mapping of P_1 or its rotation.
 1168 We also need to find $(\bullet \bullet) \preceq P$. The following analysis only uses hereditary
 1169 arguments, which means that if we prove that P is bounding, we also prove that
 1170 each submatrix of P is bounding. With this in mind, we restrict ourselves to
 critical patterns.

	a		c		C	γ	
	b		B	β	e		
	A	α	d		f		

Figure 4.7: The structure of a pattern only having three non-empty rows and avoiding all rotations of P_1 .

1171

1172 1. $\gamma = 1 \Rightarrow f = 0 \Rightarrow$ because $(\bullet \bullet) \preceq P$, it holds $a = 1 \Rightarrow \alpha = 0$

1173 (a) $d = 1 \Rightarrow b = 0, \beta = 0, e = 0$

1174 (b) $d = 0$

1175 i. $c = 1 \Rightarrow \beta = 0, e = 0$

1176 ii. $c = 0$

1177 2. $\gamma = 0$

1178 (a) $\alpha = 1 \Rightarrow a = 0, b = 0$. If $f = 0$ we have case 1. (b) ii.; otherwise, we
1179 have case 1. (a).

1180 (b) $\alpha = 0$

1181 i. $c = 1, d = 1 \Rightarrow b = 0, e = 0, \beta = 0$

1182 ii. $c = 1, d = 0 \Rightarrow e = 0, \beta = 0$ and without loss of generality, $b = 1$.
1183 Otherwise, we have the previous case. Therefore, $f = 0$

1184 iii. $c = 0, d = 1 \Rightarrow b = 0$. Without loss of generality, $e = 1, \beta = 1$.
1185 Otherwise, we have the case $c = 1, d = 1$. Therefore, $a = 0$

1186 iv. $c = 0, d = 0$

1187 The same analysis also proves that if a pattern with the same restrictions only
1188 has three non-empty columns then it is bounding.

1189 Let P be a pattern having two non-empty rows r_1, r_2 and one non-empty
1190 column c_1 . Without loss of generality, we again assume permutation 123 is present
and we distinguish three cases. Consider Figure 4.8:

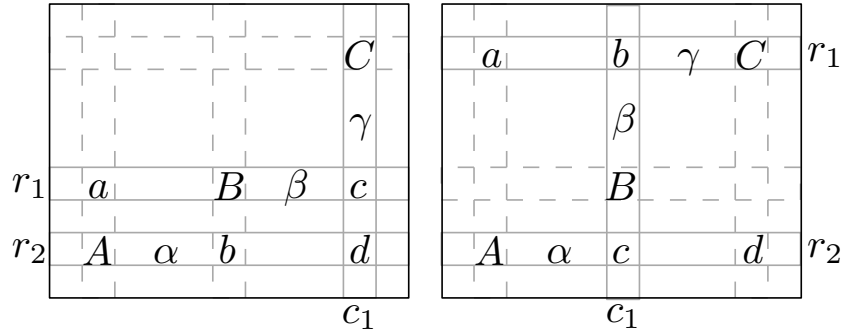


Figure 4.8: The structure of a pattern only having one-entries in two rows and one column that avoids all rotations of P_1 .

1191

1192 1. C lies in column c_1

1193 (a) $a = 0$

1194 (b) $a = 1 \Rightarrow b = 0, \alpha = 0$

1195 2. B lies in column c_1

1196 (a) $a = 1, d = 1 \Rightarrow \alpha = 0, \gamma = 0$

1197 (b) $a = 1, d = 0 \Rightarrow \alpha = 0$

1198 (c) $a = 0, d = 1 \Rightarrow \gamma = 0$

1199 (d) $a = 0, d = 0$. The pattern avoids $(\bullet \bullet)$.

1200 3. A lies in column c_1 . This is symmetric to the first situation.

1201 The same analysis also proves that if a pattern P has two non-empty columns
1202 and one non-empty row then the pattern is bounding. \square

1203 Combining the lemmata we finally get the following result.

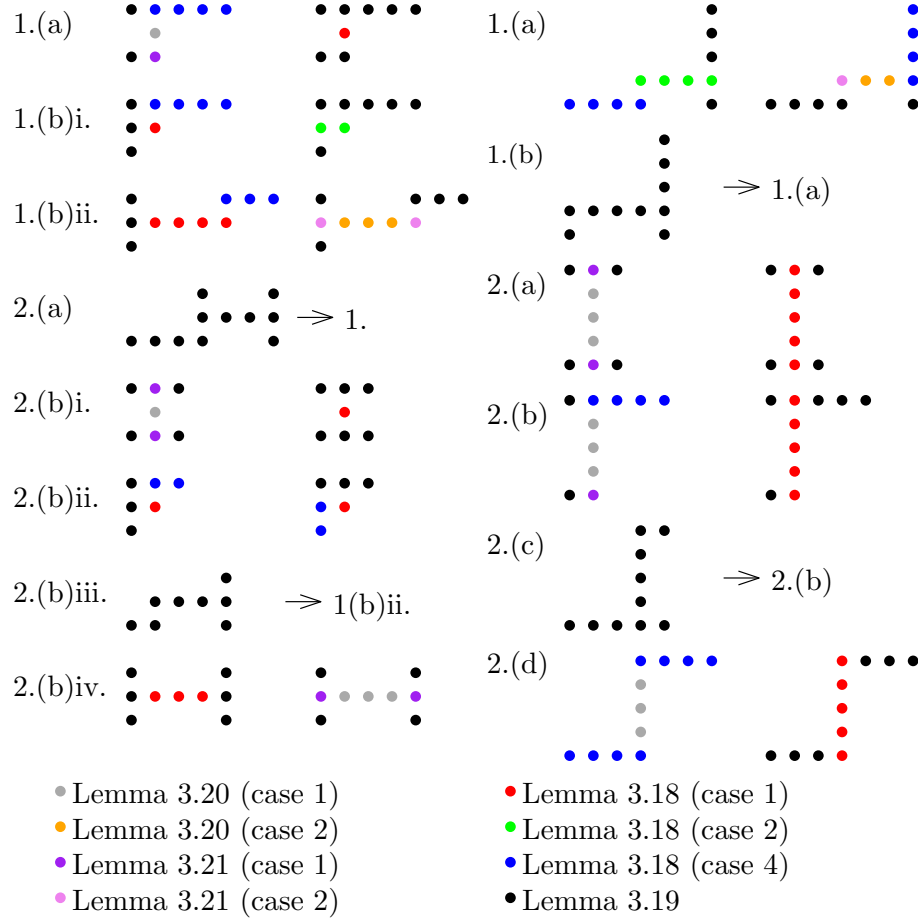


Figure 4.9: A figure showing which lemma can be used to prove that each one-entry of patterns discussed in the case analysis is bounded. The patterns from the left half of the picture only contain three non-empty rows and the patterns from the right half only contain two non-empty rows and one non-empty column. Each case either contains a picture showing that each one-entry is row-bounded and column-bounded, or an arrow describing that the case can be reduced to a different one.

1204 **Theorem 4.24.** *Let P be a pattern avoiding all rotations of P_1 , then P is bound-*
 1205 *ing.* □

1206 A lot can be implied from this theorem. Here are two straightforward corol-
 1207 laries for which we do not know any other proof.

1208 **Corollary 4.25.** *For every pattern P : $Av_{\preceq}(P)$ is row-bounded $\Leftrightarrow Av_{\preceq}(P)$ is*
 1209 *column-bounded.*

1210 **Corollary 4.26.** *For every bounding pattern P and every $P' \preceq P$ it holds P' is*
 1211 *bounding.*

1212 4.2 Chain rules

1213 Now that we know exactly what patterns are bounding, it is time to speak about
 1214 the complexity of classes more in general. We are still going to be concerned with

1215 classes of matrices avoiding patterns, but they will avoid a set of patterns rather
1216 than just one pattern.

1217 First, we show that Corollary 4.25 does not hold in general. Next, we show
1218 that bounded classes are closed to intersection. At the end of the chapter, we
1219 prove the same is not true for unbounded classes of matrices and even more, an
1220 intersection of a few unbounded classes can be bounded hereditarily, which means
1221 that its every subset is bounded.

1222 It is easy to see that Lemma 4.18, Lemma 4.19, Lemma 4.20, Lemma 4.21
1223 and Lemma 4.22 can be generalized to our settings. Their proofs without change
1224 show that for every set of patterns \mathcal{P} , if a pattern $P \in \mathcal{P}$ looks like a described
1225 pattern, then any one-entry of P is (row-)bounded in $Av_{\leq}(\mathcal{P})$. Therefore, we use
1226 the lemmata without restating them.

1227 We define classes of matrices to be bounded if they are both row-bounded
1228 and column-bounded. From what we proved so far, we see that for a pattern P ,
1229 the class $Av_{\leq}(P)$ is row-bounded if and only if it is column-bounded. Once we
1230 consider classes avoiding sets of patterns, this does not have to be true.

1231 **Lemma 4.27.** *There exists a set of patterns \mathcal{P} such that the class $Av_{\leq}(\mathcal{P})$ is*
1232 *row-bounded but column-unbounded.*

1233 *Proof.* Let $\mathcal{P} = \left\{ P = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}, I_4 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right\}$. We can use a similar construc-
1234 tion to what we did in Lemma 4.13, to prove $Av_{\leq}(\mathcal{P})$ is column-unbounded. The
1235 only difference is that the “blocks” are of size 4×2 and the whole matrix is
1236 transposed.

1237 To prove that the class $Av_{\leq}(\mathcal{P})$ is row-bounded, we take an arbitrary ma-
1238 trix $M \in Av_{crit}(\mathcal{P})$ and consider any row r of M . We need to prove that every
1239 one-entry of I_4 and P is row-bounded.

1240 From Lemma 4.22, we know that every one-entry of I_4 is row-bounded (and
1241 column-bounded) in $Av_{\leq}(\mathcal{P})$. From Lemma 4.19, one-entries $P[2, 1]$ and $P[4, 3]$
1242 are row-bounded in $Av_{\leq}(\mathcal{P})$. From the first case of Lemma 4.20, the one-
1243 entry $P[3, 2]$ is row-bounded in $Av_{\leq}(\mathcal{P})$.

1244 We prove that there are at most two zero-intervals usable for $P[1, 2]$ in the
1245 row r . For contradiction, let there be three zero-intervals $z_1 < z_2 < z_3$. Consider a
1246 mapping of P to M created when a zero-entry of z_3 is changed to a one-entry used
1247 to map $P[1, 2]$. Without loss of generality, the one-entry used to map $P[2, 1]$ lies
1248 in columns of z_3 or just under the one-entry e bounding z_3 from left; otherwise, we
1249 could use e to map $P[1, 2]$ and find the pattern in M . Then, a one-entry between
1250 zero-intervals z_1 and z_2 together with the one-entries used to map $P[2, 1], P[3, 2]$
1251 and $P[4, 3]$ give us a mapping of I_4 and so a contradiction with $M \in Av_{\leq}(\mathcal{P})$. \square

1252 **Theorem 4.28.** *Let \mathcal{P} and \mathcal{Q} be classes of patterns. If both classes $Av_{\leq}(\mathcal{P})$ and*
1253 *$Av_{\leq}(\mathcal{Q})$ are bounded then $Av_{\leq}(\mathcal{P} \cup \mathcal{Q})$ is bounded.*

1254 *Proof.* Let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$. We show that $comp_{\mathcal{R}} \leq comp_{\mathcal{P}} + comp_{\mathcal{Q}} = C$.

1255 For contradiction, let a matrix $M \in Av_{crit}(\mathcal{R})$ have at least $C + 1$ zero-
1256 intervals in a single row (or column). Without loss of generality, it means there is
1257 more than $comp_{\mathcal{P}}$ zero-intervals usable for one-entries of the patterns from \mathcal{P} . Let
1258 $M' \in Av_{\leq}(\mathcal{P})$ be a matrix created from M by changing as many zero-entries to
1259 one-entries as possible. Clearly, it still contains more than $comp_{\mathcal{P}}$ zero-intervals

usable for one-entries of the patterns from \mathcal{P} , which is a contradiction with the value of $comp_{\mathcal{P}}$. \square

Using induction, we can show that also a union of a finite number of bounded classes of finite sizes is bounded. Interestingly enough, unbounded classes are not closed the same way.

Theorem 4.29. *For every $1 \leq i < j \leq 4$ is $Av_{\preceq}(\{P_i, P_j\})$ bounded.*

Proof. We only show that $Av_{\preceq}(P_1, P_2)$ is bounded. To prove $Av_{\preceq}(P_1, P_3)$ is bounded, we can use the same steps. All other pairs are then symmetric to these two.

- $Av_{\preceq}(P_1, P_2)$ is row-bounded:
 From Lemma 4.19, we have that one-entries $P_1[2, 1], P_1[3, 3], P_2[2, 3]$ and $P_3[3, 1]$ are row-bounded. For $P_1[1, 2]$ and $P_2[1, 2]$, we prove there are at most two zero-intervals usable for each of them in each row of any matrix $M \in Av_{crit}(P_1, P_2)$. For contradiction, let $z_1 < z_2 < z_3$ be three zero-intervals usable for $P_1[1, 2]$ in a row r of M . The one-entries used to map $P_1[2, 1]$ and $P_1[3, 3]$ in a mapping created when a zero-entry of z_1 is changed to a one-entry used to map $P_1[1, 2]$, together with a one-entry in between z_2 and z_3 give us a mapping of P_2 to M . Symmetrically, the same goes for $P_2[1, 2]$.
- $Av_{\preceq}(P_1, P_2)$ is column-bounded:
 The proof that all one-entries of P_1 and P_2 are column-bounded is the same. \square

We prove even stronger result for the class $Av_{\preceq}(P_1, P_2, P_3, P_4)$ by using a well known fact from the theory of ordered sets. It is due to Higman [1952] and states the following:

Fact 4.30 (Higman's lemma). *Let A be a finite alphabet and A^* be a set of finite sequences over A partially ordered by the subsequence relation. Then A^* is well-quasi-ordered.*

In other words, whenever we have a potentially infinite $S \subseteq A^*$, there are sequences $a, b \in S$ such that a is a subsequence of b . This also means that no such S contains an infinite anti-chain.

Theorem 4.31. *The class $\sigma = Av_{\preceq}(P_1, P_2, P_3, P_4)$ is bounded. Moreover, every subclass of σ is bounded.*

Proof. We first prove that σ is bounded. Consider any critical matrix $M \in \sigma$. If it avoids $(\bullet \bullet)$ (or $(\bullet \bullet)$), in which case it is a walking matrix then it has at most two zero-intervals in each row and column. If M contains at most three non-empty rows (columns) then from the case analysis in Lemma 4.23, we see that there are at most four zero-intervals in each row and trivially, there are at most four zero-intervals in each column. Otherwise, M contains at most two non-empty rows and one non-empty column (or vice versa), and we again see from the case analysis of Lemma 4.23 that there are at most four zero-intervals in each row and column.

1302 Now consider an arbitrary $\mathcal{M} \subseteq \sigma$. In terms of forbidden patterns, we have
 1303 $\mathcal{M} = Av_{\preceq}(\{P_1, P_2, P_3, P_4\} \cup \mathcal{P})$ for some set of matrices $\mathcal{P} \subseteq \sigma$. If \mathcal{P} is finite
 1304 then we can use iterated Theorem 4.28 to show that \mathcal{M} is bounded.

1305 Assume that \mathcal{P} is infinite. Then we want to find a finite subset \mathcal{P}' such that
 1306 for every $P \in \mathcal{P}$ there is $P' \in \mathcal{P}'$ with $P' \preceq P$. In other words, we need to prove
 1307 that no \mathcal{P} contains an infinite anti-chain. To do so, we use Fact 4.30.

1308 As the relation of being interval minor is a partial ordering on any set of
 1309 matrices, we define a finite alphabet A and define a word $w_M \in A^*$ for every
 1310 matrix $M \in \sigma$ in such a way, that for every two words $w_P, w_M \in A^*$ it holds that
 1311 if w_P is a subsequence of w_M then $P \preceq M$.

1312 • For all matrices $M \in \sigma$ that have at most three non-empty rows (we proceed
 1313 symmetrically if it has at most three non-empty columns), we use words
 1314 over alphabet $A = \{a, b, c, d, e, f, g, h, i, j\}$. Let $r_1 < r_2 < r_3$ be the non-
 1315 empty rows (if less than three are non-empty then we choose extra values
 1316 arbitrarily). We define $w_M \in A^*$ as follows. First, we use the letter g r_1 -
 1317 times, the letter h $(r_2 - r_1)$ -times, the letter i $(r_3 - r_2)$ -times and the letter j
 1318 $(m - r_3)$ -times to describe the number of rows of M and the position of non-
 1319 empty rows. Then we describe the matrix column by column as follows. For
 1320 each 0 in r_1 , we use the letter a and for 1, we use letters ab . For each 0 in
 1321 r_2 , we use the letter c and for 1, we use letters cd . For each 0 in r_3 , we use
 1322 the letter e and for 1, we use letters ef .

1323 Let $w_P, w_M \in A^*$ be two words such that w_P is a subsequence of w_M . Let
 1324 r_1, r_2, r_3 and r'_1, r'_2, r'_3 be the non-empty rows of P and M respectively. Since
 1325 the number of leading letters g is not bigger in w_P , P does not have more
 1326 empty rows before r_1 than M does before r'_1 and similarly for the other
 1327 pairs of non-empty rows.

1328 Now consider there is a sequence ab in w_P and it corresponds to some $a \cdots b$
 1329 in w_M . Without loss of generality, the letter a in w_P is the one exactly before
 1330 the letter b . Clearly, one-entries of P can be mapped to one-entries of M
 1331 and we only need to check that two one-entries of two different columns of
 1332 P are not mapped to two one-entries of the same column of M . This is not
 1333 hard to see and we have $P \preceq M$ (but it does not have to hold that $P \leq M$).

1334 • For all matrices $M \in \sigma$ that have at most two non-empty rows and a
 1335 non-empty column (we proceed symmetrically if it has at most two non-
 1336 empty columns and a non-empty row), we use words over alphabet $A =$
 1337 $\{a, b, c, d, e, f, g\}$ and for non-empty rows r_1, r_2 and a column c_1 , we define
 1338 w_M as follows. We first encode the matrix column by column in such a way
 1339 that for each 0 in r_1 , we use the letter a and for 1, we use letters ab . For
 1340 each 0 in r_2 , we use the letter c and for 1, we use letters cd . Right before
 1341 and right after the description of the column c_1 , we put the letter g . Next,
 1342 we encode each row in such a way that for each 0 in c_1 we use the letter e
 1343 and for each 1, letters ef . Right before and right after the descriptions of
 1344 rows r_1 and r_2 we again place the letter g .

1345 Because of the distinct letters for encoding rows and columns we can ap-
 1346 ply the same analysis as we did in the previous case and since the entries

1347 $M[r_1, c_1]$ and $M[r_2, c_1]$ are separated from the rest by the letter g there is
 1348 no way to find a one-entry where it is not.

1349 • For all matrices $M \in \sigma$ avoiding $(\bullet \bullet)$ (we proceed symmetrically if it avoids
 1350 $(\bullet \bullet)$), we use words over alphabet $A = \{a, b, c, d\}$ and encode the matrix
 1351 as follows. We choose an arbitrary walk of M containing all one-entries and
 1352 index its entries as $w_1 \dots w_{m+n-1}$. Starting from w_1 , we encode w_i so that
 1353 the letter a stands for 0 and letters ab for 1, if w_{i+1} lies in the same row as
 1354 w_i , and we use the letter c for 0 and letters cd for 1, if w_{i+1} lies in the same
 1355 column as w_i . We always use a or ab for the last entry.

1356 We again need to check that if w_P is a subsequence of w_M then $P \preceq M$.
 1357 For contradiction, assume that two one-entries of two different rows of P
 1358 are mapped to two one-entries e, e' in the same row of M . Then in w_P
 1359 the corresponding one-entries are separated by (or equal to) the letter c
 1360 and so the letter also appear in w_M , which is a contradiction with the
 1361 one-entries e, e' being in the same row of M .

1362 In the construction of words corresponding to matrices, we only make sure
 1363 that if w_P is a subsequence of w_M then $P \preceq M$ and the other implication does
 1364 not need to hold. A different construction may lead to equivalence, but it is not
 1365 necessary for our purposes.

1366 We use distinct alphabets to describe matrices from different categories and
 1367 when given a potentially infinite class of matrices \mathcal{P} , we know from Fact 4.30 that
 1368 inside each category there is at most finite number of minimal (with respect to
 1369 interval minors) matrices. Using induction on Theorem 4.28, we have that each
 1370 $\mathcal{M} \subseteq \sigma$ is bounded. \square

1371 **Observation 4.32.** *There exists a bounding pattern P having an unbounded sub-*
 1372 *class of $Av_{\preceq}(P)$.*

1373 *Proof.* Let $P = I_n$ (identity matrix) for $n > 3$. From Lemma 4.22, we have
 1374 that P is bounding. On the other hand, $Av_{\preceq}(I_n, P_1)$ is unbounded, because the
 1375 construction used in the proof of Lemma 4.13 also works for this class. \square

Conclusion

Throughout the thesis, we have been looking from multiple angles at classes of binary matrices. In particular, we studied properties of matrices containing or avoiding other, smaller matrices as interval minors.

Small interval minors We started by describing the structure of matrices avoiding some small patterns. We managed to characterize all matrices avoiding patterns having up to three one-entries and also showed how to generalize some of the characterizations for much bigger patterns. Even for very small patterns (only having four one-entries), the structure of matrices avoiding them became very rich and hard to properly describe.

So instead, we began to consider how a small change of the pattern influences the structure of matrices avoiding the pattern. This was mainly looked at when adding empty lines to the pattern. We showed for patterns of size $k \times 2$ what matrices avoiding its blown version look like, but we are unable to generalize the result (or show something similar) when the pattern we are working with is bigger. So the question, answer for which could later be used to describe complexity of some patterns or help enumerate matrices from some class, is:

Question 4.33. What can we say about matrices avoiding a pattern with added empty line with respect to the matrices avoiding the original pattern?

The basis of a class of matrices After dealing with small patterns, we defined an operation of the direct sum and a closure under the direct and began to explore how it relates with classes of matrices. Once again we started by considering some small cases, where we observed that the closure can be describe by forbidden patterns very naturally. Later, we considered the direct sum with an overlap, which allowed us to restate the characterizations from the previous chapter in a much easier way.

We also introduced a notion of articulation that allowed us to proof a strong result saying that any matrix closed under the direct sum can always be describe by a finite set of forbidden patterns.

In order to generalize the result, we started looking at sets of matrices and ultimately, we showed that there are set of matrices with finite basis, whose closure cannot be described with a finite number of forbidden minors.

Zero-intervals In the last chapter, we studies a property of a class of matrices that in some terms describes the complexity of inclusion maximal matrices from the class.

To bring the notion back to the pattern avoiding, we defined a pattern P to be bounding if and only of the class $Av_{\leq}(P)$ is bounded and we showed that a pattern is bounding if and only if it avoids all rotations of $P_1 = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$.

To show for a pattern that it is bounding meant to show that each one-entry it contains is bounded. It may be an interesting generalization to show for each one-entry whether it is bounded or unbounded in its pattern. On our way, we only saw one type of unbounded one-entries, and those are $P_1[2, 1]$ for rows, $P_1[1, 2]$ for

Bibliography

- Miklós Bóna. *Combinatorics of permutations*. CRC Press, 2012.
- Prosenjit Bose, Jonathan F Buss, and Anna Lubiw. Pattern matching for permutations. *Information Processing Letters*, 65(5):277–283, 1998.
- Peter Brass, Gyula Károlyi, and Pavel Valtr. A turán-type extremal theory of convex geometric graphs. In *Discrete and computational geometry*, pages 275–300. Springer, 2003.
- William G Brown. On graphs that do not contain a Thomsen graph. *Canad. Math. Bull.*, 9(2):1–2, 1966.
- Josef Cibulka. On constants in the Füredi–Hajnal and the Stanley–Wilf conjecture. *Journal of Combinatorial Theory, Series A*, 116(2):290–302, 2009.
- Josef Cibulka. *Extremal combinatorics of matrices, sequences and sets of permutations*. PhD thesis, Ph. D. Thesis, Charles University, Prague, 2013.
- Josef Cibulka and Jan Kynčl. Füredi–Hajnal limits are typically subexponential. *arXiv preprint arXiv:1607.07491*, 2016.
- Anders Claesson, Vít Jelínek, and Einar Steingrímsson. Upper bounds for the Stanley–Wilf limit of 1324 and other layered patterns. *Journal of Combinatorial Theory, Series A*, 119(8):1680–1691, 2012.
- Alon Efrat and Micha Sharir. A near-linear algorithm for the planar segment-center problem. *Discrete & Computational Geometry*, 16(3):239–257, 1996.
- Jacob Fox. Stanley–Wilf limits are typically exponential. *arXiv:1310.8378*, 2013.
- Radoslav Fulek. Linear bound on extremal functions of some forbidden patterns in 0–1 matrices. *Discrete Mathematics*, 309(6):1736–1739, 2009.
- Zoltán Füredi and Péter Hajnal. Davenport–Schinzel theory of matrices. *Discrete Mathematics*, 103(3):233–251, 1992.
- Graham Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, 3(1):326–336, 1952.
- Sergey Kitaev. *Patterns in permutations and words*. Springer Science & Business Media, 2011.
- Martin Klazar. Extremal problems for ordered (hyper) graphs: applications of Davenport–Schinzel sequences. *European Journal of Combinatorics*, 25(1):125–140, 2004.
- Yaping Mao, Hongjian Lai, Zhao Wang, and Zhiwei Guo. Interval minors of complete multipartite graphs. *arXiv preprint arXiv:1508.01263*, 2015.

- 1478 Adam Marcus and Gábor Tardos. Excluded permutation matrices and the
1479 Stanley–Wilf conjecture. *Journal of Combinatorial Theory, Series A*, 107(1):
1480 153–160, 2004.
- 1481 Bojan Mohar, Arash Rafiey, Behruz Tayfeh-Rezaie, and Hehui Wu. Interval
1482 minors of complete bipartite graphs. *Journal of Graph Theory*, 2015.
- 1483 János Pach and Gábor Tardos. Forbidden paths and cycles in ordered graphs and
1484 matrices. *Israel Journal of Mathematics*, 155(1):359–380, 2006.
- 1485 Gábor Tardos. On 0–1 matrices and small excluded submatrices. *Journal of*
1486 *Combinatorial Theory, Series A*, 111(2):266–288, 2005.

List of Figures

1487			
1488	1.1	An example of a matrix M avoiding a pattern P as an interval	
1489		minor.	6
1490	2.1	Red and blue lines representing mappings m_l and m_r of the forbid-	
1491		den pattern. The two horizontal lines show the boundaries of the	
1492		mapping of the row r and the vertical lines show the boundaries	
1493		of the mapping of the column c	12
1494	2.2	The characterization of matrices avoiding $(\begin{smallmatrix} \bullet & \bullet \end{smallmatrix})$ as an interval mi-	
1495		nor. The matrix M' is a walking matrix.	14
1496	2.3	The characterization of matrices avoiding $(\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix})$ as an interval minor.	16
1497	2.4	The characterization of matrices avoiding $(\begin{smallmatrix} \bullet & \bullet & \bullet \end{smallmatrix})$ as an interval minor.	17
1498	4.1	The structure of a critical matrix avoiding P that has arbitrarily	
1499		many zero-intervals.	29
1500	4.2	Red and blue lines representing two different mappings of a for-	
1501		bidden pattern. The two horizontal lines show the boundaries of	
1502		the mapping of row r and the vertical lines show the boundaries	
1503		of the mapping of column c	31
1504	4.3	The patterns for which all one-entries in the row r_2 and the columns	
1505		$[c_1, c_2]$ are row-bounded. One-entries of the patterns are inside the	
1506		bold rectangles and on the bold lines.	32
1507	4.4	The patterns for which all one-entries in the column c and the	
1508		rows $[r_1 + 1, r_2 - 1]$ are row-bounded. One-entries of the patterns	
1509		are inside the bold rectangles and on the bold lines.	33
1510	4.5	Red and blue lines representing two different mappings of a for-	
1511		bidden pattern. The four horizontal lines show the boundaries	
1512		of the mapping of rows r_1 and r_2 and the vertical lines show the	
1513		boundaries of the mapping of the column c	33
1514	4.6	The patterns for which all one-entries in the column c and the	
1515		rows $[r_1, r_2]$ are row-bounded. One-entries of the patterns are on	
1516		the bold lines and the column c is the second last.	34
1517	4.7	The structure of a pattern only having three non-empty rows and	
1518		avoiding all rotations of P_1	35
1519	4.8	The structure of a pattern only having one-entries in two rows and	
1520		one column that avoids all rotations of P_1	36
1521	4.9	A figure showing which lemma can be used to prove that each	
1522		one-entry of patterns discussed in the case analysis is bounded.	
1523		The patterns from the left half of the picture only contain three	
1524		non-empty rows and the patterns from the right half only contain	
1525		two non-empty rows and one non-empty column. Each case either	
1526		contains a picture showing that each one-entry is row-bounded	
1527		and column-bounded, or an arrow describing that the case can be	
1528		reduced to a different one.	37