1. Introduction

Throughout the paper, every time we speak about matrices we mean binary matrices (also called 01-matrices) and we omit the word binary. If we speak about a *pattern*, we again mean a binary matrix and we use the word in order to distinguish among more matrices as well as to indicate relationship.

When dealing with matrices, we always index rows and column starting with one and when we speak about a row r, we simply mean a row with index r. A *line* is a common word for both a row and a column. When we order a set of lines, we first put all rows and then all columns. For $M \in \{0,1\}^{m \times n}$, $[m]$ is a set of all rows and $[m+n]$ is a set of all lines, where m-th element is the last row. This goes with the usual notation.

Notation 1. For $n \in \mathbb{N}$ let $[n] := \{1, 2, \dots, n\}$ and for $m \in \mathbb{N}$, where $n \leq m$ let $[n, m] := \{n, n+1, \dots, m\}$.

Notation 2. For a matrix $M \in \{0,1\}^{m \times n}$ and $L \subseteq [m+n]$ let $M[L]$ denote a submatrix of M induced by lines in L.

Notation 3. For a matrix $M \in \{0,1\}^{m \times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M[R, C]$ denote a submatrix of M induced by rows in R and columns in C. Furthermore, for $r \in [m]$ and $c \in [n]$ let $M[r, c] := M[\{r\}, \{c\}] = M[\{r, c+m\}]$.

Definition 1. We say a matrix $M \in \{0,1\}^{m \times n}$ *contains* a pattern $P \in \{0,1\}^{k \times l}$ *as a submatrix* and denote it by $P \leq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M[R, C][r, c] = 1$.

This does not necessarily mean $P = M[R, C]$ as $M[R, C]$ can have more one-entries than P does.

Notation 4. For a matrix $M \in \{0,1\}^{m \times n}$ and $L \subseteq [m+n]$ let $M_{\preceq}[L]$ denote a matrix acquired from M by applying following operation for each $l \in L$:

- If l is the first row in L then we replace the first l rows by one row that is a bitwise OR of replaced rows.
- If l is the first column in L then we replace the first $l - m$ columns by one column that is a bitwise OR of replaced columns.
- Otherwise, we take l's predecessor $l' \in L$ in the standard ordering and replace lines $[l'+1, l]$ by one line that is a bitwise OR of replaced lines.

Notation 5. For a matrix $M \in \{0,1\}^{m \times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M_{\preceq}[R, C] := M_{\preceq}[R \cup \{c + m | c \in C\}]$.

Definition 2. We say a matrix $M \in \{0,1\}^{m \times n}$ *contains* a pattern $P \in \{0,1\}^{k \times l}$ *as an interval minor* and denote it by $P \preceq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M_{\preceq}[R, C][r, c] = 1$.

Observation 1. *For all matrices M and P, $P \leq M \Rightarrow P \preceq M$.*

Observation 2. *For all matrices M and P, if P is a permutation matrix, then $P \leq M \Leftrightarrow P \preceq M$.*

Proof. If we have $P \preceq M$, then there is a partitioning of M into rectangles and for each one-entry of P there is at least one one-entry in the corresponding rectangle of M. Since P is a permutation matrix, it is sufficient to take rows and columns having at least one one-entry in the right rectangle and we can always do so.

Together with Observation 1 this gives us the statement. □

Observation 3. *Let $M \in \{0,1\}^{m \times n}$ and $P \in \{0,1\}^{k \times l}$, $P \preceq M \Leftrightarrow P^T \preceq M^T$.*

Because of this observation we will usually only show results only for rows or columns and expect both to hold and only show results for $P \in \{0,1\}^{k \times l}$ but assume the symmetrical results for P^T.

2. Characterizations

Definition 3. A *walk* in a matrix M is a sequence of some of its entries, beginning in the top left corner and ending in the bottom right one. If an entry $M[i, j]$ is in the sequence, the next one is either $M[i + 1, j]$ or $M[i, j + 1]$.

Definition 4. We call a binary matrix M a *walking matrix* if there is a walk in M such that all one-entries of M are contained on the walk.

Definition 5. An *extended walk of size* $k \times l$ in a matrix M is a subset of some of its entries, beginning in the top left corner and ending in the bottom right one. If an entry $M[i, j]$ is in the subset there is also either $M[i+1, j]$ or $M[i, j+1]$. The size describes that no more than k entries directly above each other are in the subset and no more than l entries directly next to each other are in the subset. We say that an extended walk of size $k \times l$ in M starts with a walk w, if the extended walk is a subset of entries of M that

- lie on w or below w and
- lie on w shifted by $k - 1$ down and by $l - 1$ to the left or above it.

Definition 6. For $M \in \{0, 1\}^{m \times n}$ and $r \in [m], c \in [n]$ we say $M[r, c]$ is

- *top-left empty* if $M[[r - 1], [c - 1]]$ is an empty matrix,
- *top-right empty* if $M[[r - 1], [c + 1, n]]$ is empty,
- *bottom-left empty* if $M[[r - 1], [c + 1, n]]$ is empty,
- *bottom-right empty* if $M[[r - 1], [c + 1, n]]$ is empty.

2.1 Patterns of size 2×2 and their generalization

Theorem 4. *Let* $P = \left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$*, then for all* M*:* $P \not\prec M \Leftrightarrow M$ *is a walking matrix.*

Proof. Since P is a permutation matrix, $P \not\prec M \Leftrightarrow P \not\preceq M$ and it is easy to see $P \not\preceq M \Leftrightarrow M$ is a walking matrix. □

Now consider a generalization of the pattern from above:

Theorem 5. *Let* $P \in \{0, 1\}^{k \times l}$ *be a matrix having only two one-entries –* $P[1, n]$ *and* $P[m, 1]$*, then for all* M*:* $P \not\prec M \Leftrightarrow M$ *has an extended walk of size* $k-1 \times l-1$ *containing all one-entries.*

Proof. $\Rightarrow$ Let $P \not\prec M$ and consider the left-most top-right empty elements of M. They necessarily form a walk w. For contradiction, assume there is a one-entry e below the extended walk of size $k - 1 \times l - 1$ starting with w. Since e is below the extended walk, there is an element e' - the right-most element of M that is neither below e nor to the right from e and at the same time still below the extended walk (it is possible $e = e'$). Let $e = M[r, c]$ and notice $M[r - k, c - l]$ is part of walk w and because of the choice of e' neither $M[r - k - 1, c - l]$ nor $M[r - k, c - l - 1]$ are on the walk w and $M[r - k, c - l]$ must be a one-entry; therefore, together with e it forms the forbidden pattern in M, which is a contradiction.

$\Leftarrow$ Let $M[r,c]$ be any one-entry of M, which then necessarily lie in the extended walk. Because the size of the walk is $k-1 \times l-1$, $M[r-k+1, c-l+1]$ is top-left empty and $M[r+k-1, c+l-1]$ is bottom-right empty; therefore e cannot be a part of a mapping of P.

□

Theorem 6. *Let $P = \left(\begin{smallmatrix} \bullet & \bullet \\ \bullet & \end{smallmatrix}\right)$, then for all $M \in \{0,1\}^{m \times n}$: $P \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 2.1)*

- *$M[[r-1],[c-1]]$ is empty,*
- *$M[[r-1],[c+1,n]]$ is empty,*
- *$M[[r+1,m],[c-1]]$ is empty and*
- *$M[[r,m],[c,n]]$ is a walking matrix.*

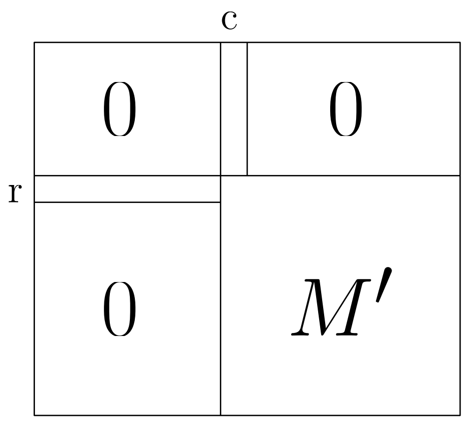


Figure 2.1: Characterization of a matrix avoiding $\left(\begin{smallmatrix} \bullet & \bullet \\ \bullet & \end{smallmatrix}\right)$ as an interval minor. Matrix M' is a walking matrix

Proof. $\Rightarrow$ If $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \not\preceq M$ then M is a walking matrix and we set $r = c = 1$. Otherwise, there are one-entries $M[r, c']$ and $M[r', c]$ such that $r' < r$ and $c' < c$. If there is a one-entry in regions $M[[r-1],[c-1]]$, $M[[r-1], [c+1, n]]$ or $M[[r+1,m],[c-1]]$ then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ and we again get a contradiction.

$\Leftarrow$ For contradiction, assume that M described in Figure 2.1 contains P as an interval minor. It means that there is a partition of the matrix into four quadrants such that there is at least one one-entry in each quadrant besides the bottom right one. If the matrix is partitioned above the r-th row, then there is only one column containing one-entries and it is not possible for both top quadrants to have a one-entry. Similarly, if the matrix is partitioned to the left of the c-th column, there is only one row containing one-entries and there is no one-entry in either top-left or bottom-left quadrant. Therefore, the partitioning lies bellow the r-th row and to the right of the c-th column, but if the quadrants contain one-entries, there is a $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ interval minor in M', which is a contradiction with it being a walking matrix.

□

Theorem 7. *Let $P \in \{0,1\}^{k\times l}$ be a matrix having only three one-entries – $P[1,1]$, $P[1,n]$ and $P[m,1]$, then for all M: $P \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 2.1 and imagine rows and columns being extended)*

- *$M[[r-1],[c-1]]$ is empty,*
- *$M[[r-1],[c+l,n]]$ is empty,*
- *$M[[r+k,m],[c-1]]$ is empty and*
- *$M[[r,m],[c,n]]$ has an extended walk of size $k-1\times l-1$ containing all one-entries.*

Proof. Let $P' = P$ and set $P'[m,1] = 0$ (P' is a generalization of $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$).

$\Rightarrow$ If $P' \not\preceq M$ then M is a matrix having an extended walk of size $k-1\times l-1$ containing all one-entries and we set $r = c = 1$. Otherwise, there are one-entries $M[r_1,c_1]$ and $M[r_2,c_2]$ such that $r_2 < r_1$ and $c_1 < c_2$. We now choose $M[r_3,c_3]$ to be the bottom-most one-entry that still forms P' with $M[r_2,c_2]$. We choose $M[r_4,c_4]$ to be the left-most one-entry that forms P' with $M[r_3,c_3]$ and set $r = r_3-k+1$ and $c = c_4-l+1$. If there is a one-entry in regions $M[[r-1],[c-1]]$, $M[[r-1],\ [c+l,n]]$ or $M[[r+k,m],[c-1]]$ then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains P' and we again get a contradiction.

$\Leftarrow$ Because of the sizes of areas with no one-entries and the condition for $M[[r,m],[c,n]]$, there cannot be P' anywhere but in $M[[r+k-1],[c+l-1]]$. Since $M[[r-1][c-1]]$ is empty, there is no one-entry to map $P[1,1]$ to; therefore, $P \not\preceq M$.

□

Lemma 8. *Let $P = \begin{pmatrix}\bullet & \bullet \\ \bullet & \bullet\end{pmatrix}$ and let $M \in \{0,1\}^{m\times n}$ avoid P as an interval minor, then there exists a row r and a column c such that $M[r,c]$ is either*

1. *a one-entry and $(r,c) \in \{(1,1),(1,n),(m,1),(m,n)\}$ or*
2. *both top-left empty and bottom-right empty and $(r,c) \notin \{(1,n),(m,1)\}$ or*
3. *both top-right empty and bottom-left empty and $(r,c) \notin \{(1,1),(m,n)\}$.*

Proof. If there is a one-entry in any corner we are done. Otherwise, let A be a set of all top-left empty entries of M and B be a set of all bottom-right empty entries of M. If there is an entry $M[r,c] \in A\cap B$ different from $(1,n)$ and $(m,1)$ we are done. Assume $A\cap B = \{(1,n),(m,1)\}$. Since $(m,1)\in A$, it also holds $(m-1,1)\in A$ and because it is not in the intersection we have $(m-1,1)\notin B$. This means $M[m-1,1]$ is not bottom-right empty; therefore there is a one-entry somewhere in $M[m,[2,n]]$. Moreover, no corner contains a one-entry so the is a one-entry in $M[m,[2,n-1]]$. For simplicity, we will say that the last row in non-empty (knowing the corners are empty). Symmetrically, we also get that the first row is non-empty and both the first and the last columns are non-empty. If there is a one-entry $M[r_l,1]$ in a different row than a one-entry $M[r_r,n]$ and at the same time a one-entry $M[1, c_t]$ in a different column than a one-entry $M[m, c_b]$ then these four one-entries form a mapping of the forbidden pattern P.

This is not true!!!

Without loss of generality assume there is only one one-entry in both the first and the last column and they are both in the same row r'. Let c' be a column such that there is a one-entry $M[1, c']$. Clearly, there is no other column that contains a one-entry above r', because we would again get a contradiction. Symmetrically, let c'' be the only column containing one-entries below r'. If $c' \geq c''$ we have that both $M[r', c']$ and $M[r', c'']$ are both top-left empty and bottom-right empty, which is a contradiction with $A \cap B = \{(1, n), (m, 1)\}$. Otherwise, $c' < c''$ and both $M[r', c']$ and $M[r', c'']$ are both top-right empty and bottom-left empty where $(r', c') \notin \{(1, 1), (m, n)\}$ which concludes the proof. $\square$

Theorem 9. *Let $P = \left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\bullet\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like one of the matrices in Figure 2.2, where $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right) \not\preceq M_1$, $\left(\begin{smallmatrix}&\bullet\\\bullet&\bullet\end{smallmatrix}\right) \not\preceq M_2$, $\left(\begin{smallmatrix}\bullet&\bullet\\&\bullet\end{smallmatrix}\right) \not\preceq M_3$ and $\left(\begin{smallmatrix}\bullet&\\\bullet&\bullet\end{smallmatrix}\right) \not\preceq M_4$.*

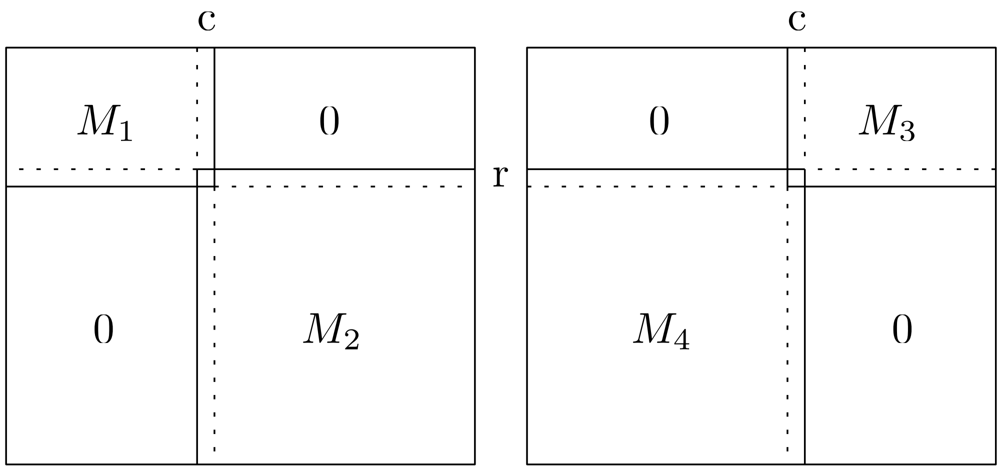


Figure 2.2: Characterization of a matrix avoiding $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\bullet\end{smallmatrix}\right)$ as an interval minor.

Proof.

$\Rightarrow$ We proceed by induction by the size of M.

If $M \in \{0, 1\}^{2\times 2}$ then it either avoids $\left(\begin{smallmatrix}&\bullet\\\bullet&\bullet\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right)$ and we are done.

For bigger M there is, from Lemma 8, $M[r, c]$ satisfying some conditions. If it is the first condition – there is a one-entry in any corner, we are done because the matrix cannot contain one of the rotations of $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right)$. Assume the second case – $M[r, c]$ is both top-right and bottom-left empty and $(r, c) \notin \{(1, n), (m, 1)\}$. If M_1 is non-empty, then $\left(\begin{smallmatrix}&\bullet\\\bullet&\bullet\end{smallmatrix}\right) \not\preceq M_2$; otherwise, $P \preceq M$. Similarly, $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right) \not\preceq M_1$ if M_2 is non-empty. If one of them is empty, the other is a smaller matrix avoiding P as an interval minor and by induction hypothesis, it can be partitioned. Adding empty rows and columns does not break any condition and we get a partitioning of the whole M.

$\Leftarrow$ Without loss of generality, let us assume M looks like the left matrix in Figure 2.2. For contradiction, assume $P \preceq M$. In that case, we can partition M into four quadrants such that there is at least one one-entry in each of them. It does not matter where we partition it, every time we either get $\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \end{smallmatrix}\right) \preceq M_1$ or $\left(\begin{smallmatrix} & \bullet\\ \bullet & \bullet\end{smallmatrix}\right) \preceq M_2$, which is a contradiction. $\square$

Theorem 10. *Let $P \in \{0,1\}^{k \times l}$ be a matrix having only four one-entries – $P[1,1]$, $P[1,n]$, $P[m,1]$ and $P[m,n]$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like one of the matrices in Figure 2.2, where generalized $\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M_1$, $\left(\begin{smallmatrix} & \bullet\\ \bullet & \bullet\end{smallmatrix}\right) \not\preceq M_2$, $\left(\begin{smallmatrix}\bullet & \bullet\\ & \bullet\end{smallmatrix}\right) \not\preceq M_3$ and $\left(\begin{smallmatrix}\bullet & \\ \bullet & \bullet\end{smallmatrix}\right) \not\preceq M_4$.*

2.2 Matrices of size 2×3

Theorem 11. *Let $P = \left(\begin{smallmatrix}\bullet & & \bullet\\ & \bullet & \end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M = M_1 \oplus_h M_2$ where $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ Let $e = [r, c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \preceq M[[m],[c-1]]$, together with e it forms P. If $\left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M[[m],[c,n]]$ then we are done. Let us assume it is not the case and let $e_{0,0}$, $e_{1,1}$ be any two one-entries forming the forbidden pattern. Symmetrically, let $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \preceq M[[m],[c]]$ and let $e_{0,1}$, $e_{1,0}$ be any two one-entries forming the forbidden pattern. Now if we take $e_{0,0}$, $e_{0,1}$ and $e_{1,0}$ or $e_{1,1}$ with bigger row, we get the forbidden pattern P as an interval minor of M.

$\Leftarrow$ For contradiction, let us assume $P \preceq M$ and $M = M_1 \oplus_h M_2$. If $P \preceq M$, look at the one-entry of M where the bottom one-entry of P is mapped. If it is in M_1 then $P \not\preceq M$ because $\left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M_1$. Otherwise, $P \not\preceq M$ because $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \not\preceq M_2$.

$\square$

Lemma 12. *Let $P = \left(\begin{smallmatrix}\bullet & \bullet & \bullet\\ & \bullet & \end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Rightarrow M = M_1 \oplus_h M_2$ where*

1. *$\left(\begin{smallmatrix}\bullet & \bullet\\ & \bullet\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M_2$ or*
2. *$\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M_2$.*

Proof. Let $e = [r, c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix}\bullet & \bullet\\ & \bullet\end{smallmatrix}\right) \preceq M[[m],[c-1]]$, together with e it would be the whole P. Similarly, $\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M[[m],[c+1,n]]$. For contradiction with the statement, let $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \preceq M[[m],[c]]$ and $e_{0,0}$, $e_{1,1}$ (none of them equal to e, since e lies in the top-right corner) be any two one-entries forming the pattern. Symmetrically, let $\left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right) \preceq M[[m],[c,n]]$ and $e_{0,1}$, $e_{1,0}$ be any two one-entries forming the pattern. In that case $e_{0,0}$, e, $e_{0,1}$ and $e_{1,0}$ or $e_{1,1}$ with bigger row give us the forbidden pattern P as an interval minor of M. $\square$

Theorem 13. *Let $P = \left(\begin{smallmatrix}\bullet & \bullet & \bullet\\ & \bullet & \end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like the matrix in Figure 2.3 and $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ From Lemma 12 we know $M = M_1' \oplus_h M_2'$ where $\left(\begin{smallmatrix}\bullet & \bullet\\ & \bullet\end{smallmatrix}\right) \not\preceq M_1'$ and $\left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right) \not\preceq M_2'$. The second case would be dealt with symmetrically. From Theorem 6 we have that M_1' can be characterized exactly like $M[[m],[c_2-1]$ and $M[[m],[c_2,n]]$ forms a walking matrix. The only problem with our claim would be if there were two different columns having a one-entry above the r-th row. In that case, those two one-entries together with a one-entry in the r-th row between the columns c_1 and c_2 and a one-entry in the c_1-th column above the r-th row form P as an interval minor.

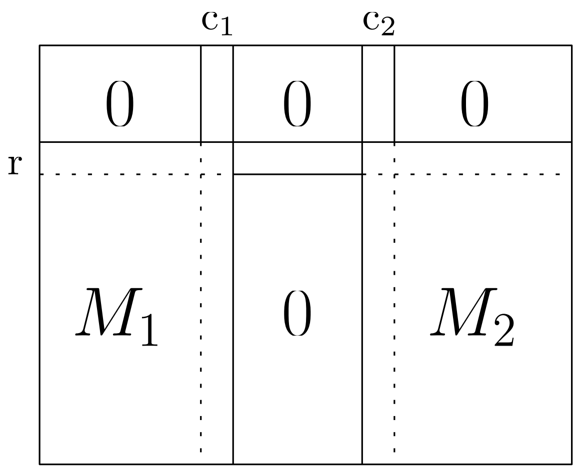


Figure 2.3: Characterization of a matrix avoiding $\begin{pmatrix} \bullet & \bullet & \bullet \\ & \bullet & \end{pmatrix}$ as an interval minor.

$\Leftarrow$ The bottom-middle one-entry of P can not be mapped anywhere but to the r-th row, but in that case there are at most two columns having one-entries above it.

$\square$

Theorem 14. *Let $P = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \end{pmatrix}$, then for all M: $P \not\preceq M \Leftrightarrow M$ contains a walk w, no one-entries below the walk and for each entry $M[r, c]$ of the walk there is at most one non-empty column in $M[[r-1], [c+1, n]]$.*

Proof. $\Rightarrow$ Let w be any walk containing all the top-most and right-most entries that are bottom-left empty. From the choice of w, there are no one-entries below it and if all $M[r, c]$, $M[r-1, c]$ and $M[r, c+1]$ are on w then $M[r, c]$ is a one-entry as else $M[r, c]$ was neither top-most nor right-most bottom-left empty. As a consequence, whenever we choose $M[r, c]$ from w, it either is a one-entry or there is one-entry in the same row to the left of it. For contradiction let us now assume that there is an entry of the walk $M[r, c]$ for which there are two non-empty columns in $M[[r-1], [c+1, m]]$. Then a one-entry from each of those columns and a one-entry in $M[r, c]$ or to the left of it together give us $P \preceq M$ and consequently a contradiction.

$\Leftarrow$ For contradiction let $P \preceq M$. Without loss of generality we can assume that the bottom-left entry of P is mapped somewhere to the walk – to $M[r, c]$. But then $(\bullet\ \bullet) \preceq M[[r-1], [c+1, n]]$ which is a contradiction with it having one-entries in at most one column.

$\square$

2.3 Empty rows and columns

Observation 15. *Let $P \in \{0,1\}^{k \times l}$ and $P' \in \{0,1\}^{k \times l+1}$ such that $P' = P \oplus_h 0^{k \times 1}$, similarly let $M \in \{0,1\}^{m \times n}$ and $M' \in \{0,1\}^{m \times n+1}$ such that $M' = M \oplus_h 1^{m \times 1}$, then $P \preceq M \Leftrightarrow P' \preceq M'$.*

Proof. $\Rightarrow$ Clearly we can map the last column of P' to the last column of M' and then map (using OR) $P'[[k], [l]]$ to $M'[[m], [n]]$ the same way P is mapped to M.

$\Leftarrow$ If $P' \preceq M$ we are done. Otherwise, the last column of P' needs to be mapped to the last column of M' and by deleting both from their matrix we get $P'[[k],[l]] \preceq M'[[m],[n]]$ which is the same as $P \preceq M$.

$\square$

The same proof can be also used for adding an empty column as the first column or an empty row as the first or the last row. Using induction we can easily show that a pattern P' is avoided by a matrix M' if and only if P is avoided by M where P is derived from P' by excluding all empty beginning or ending rows and columns and M is derived from M' by excluding the same number of beginning or ending rows and columns. Therefore, when characterizing matrices avoiding a forbidden pattern, we do not need to consider patterns having empty rows or columns on their boundary.

For the following two statements, let $P \in \{0,1\}^{k\times 2}$ be a forbidden pattern and $P^+ \in \{0,1\}^{k\times 3}$ be the pattern created from P by adding a new empty column in between the two columns of P.

Lemma 16. *If $M \in Av(P^+)$ is inclusion maximal, then each row of M is either empty or it contains exactly one one-interval of length at least 2.*

Theorem 17. *For all $M \in \{0,1\}^{m\times n}$ it holds $M \in Av(P^+) \Leftrightarrow$ there exists $N \in \{0,1\}^{m\times(n-1)}$ such that $N \in Av(P)$ is inclusion maximal and M is a submatrix of $N \oplus_h 0^{m\times 1}$ placed over $0^{m\times 1} \oplus_h N$ with an operation bitwise OR.*

Proof. $\Rightarrow$ It suffices to only prove the statement for M that is inclusion maximal. To do so, we use Lemma 21. It say that each row of M contains either no one-entry or an interval of length at least two. From that we define N to be created from M by deleting the last one-entry on each row and excluding the last column. Clearly, M is equal to $N \oplus_h 0^{m\times 1}$ placed over $0^{m\times 1} \oplus_h N$ with an operation bitwise OR. If $P \preceq N$ then each mapping of P can be extended to a mapping of P^+ to M by ... How to say this?

TODO

$\Leftarrow$ It suffices to show that M that is equal to $N \oplus_h 0^{m\times 1}$ placed over $0^{m\times 1} \oplus_h N$ with an operation bitwise OR belongs to $Av(P^+)$. For contradiction, assume it does not. Then there is mapping of P^+ into elements of M and we can assume that one-entries of the first column of P^+ are mapped to those one-entries of M created from $N \oplus_h 0^{m\times 1}$. If it was not the case and there was a one-entry mapped to a one-entry of M created only from $0^{m\times 1} \oplus_h N$ we can take an element directly to its left and that is created from $N \oplus_h 0^{m\times 1}$. Symmetrically, all one-entries of the last column of P^+ are mapped to one-entries created from $0^{m\times 1} \oplus_h N$. ...

TODO

$\square$

Open questions

- insertion of multiple empty columns in between two columns of $k \times 2$ matrix
- insertion of an empty column in between all columns of P

2.4 Multiple patterns

Theorem 18. *Let $P_1 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ and $P_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$, then for all M: $P \not\preceq M \wedge P \not\preceq M \Leftrightarrow M$ contains a walk w and each one-entry e is either on the walk w or both element directly above e and directly to the right of e are on the walk w.*

Proof.

- $\Rightarrow$ Let us take a walk w containing all the left-most and bottom-most top-right empty elements of M. Clearly, every top-right "corner" entry of w ($M[r, c]$ such that both $M[r+1, c]$ and $M[r, c-1]$ are on w) is a one-entry. Now consider for contradiction there is a one-entry anywhere but on w or directly diagonally below any top-right corner of w. Then this one-entry together with at least one top-right corner of w give us either P_1 or P_2 and thus a contradiction.

- $\Leftarrow$ If we take any one-entry e, from the description of M there is no one-entry that would create either of P_1 or P_2 with e.

$\square$

3. Pattern size constrains

In the previous chapter, we characterized matrices avoiding small patterns. Their structure is very dependent on the pattern and the results are hard to generalize for arbitrary patterns. In this chapter, we look for a more general property that restricts the complexity that a class of matrices can have.

Definition 7. For matrix $M \in \{0,1\}^{m \times n}$ a *one-interval* is a sequence of consecutive one-entries in a single line of M bounded by the edge of matrix or zero-entry from both sides. In the same spirit we define *zero-interval* to be an interval of consecutive zero-entries in a single line of M bounded by one-entry or the edge of matrix from both sides.

In Section 2.1 and Section 2.2, for pattern $P \in \{0,1\}^{k \times l}$ any inclusion maximal matrix M avoiding P as an interval minor has at most $l-1$ one-intervals in each row and at most $k-1$ one-intervals in each column. A natural question is whether the size of a pattern always bounds the number of one-intervals of any inclusion maximal matrix that avoids it.

Let us present some useful notion. First of all, every time we speak about a *maximal* matrix of a class, we mean inclusion maximal – it has no zero-entry that can be changed to a one-entry so that it still belongs to the class. In terms of pattern avoidance, maximal matrices are those for which a change of a zero-entry creates a mapping of the pattern (or possibly many mappings).

Definition 8. Let P be a pattern, e a one-entry of P, M be a maximal matrix avoiding P and zi be an arbitrary zero-interval of M. We say that zi *is usable for* e if there is a zero-entry contained in zi such that if we change it to a one-entry, it creates a mapping that uses the new one-entry to map e. Note that zi can be usable for many one-entries of P at the same time.

Definition 9. Let $\mathcal{P}$ be a class of patterns. We say that $Av(\mathcal{P})$ is *bounded* if there is a function dependent only on the size of patterns in $\mathcal{P}$ and the size of $\mathcal{P}$ such that every maximal $M \in Av(\mathcal{P})$ has the number of one-intervals in each row and column bounded by the function. Otherwise, we say that $Av(\mathcal{P})$ is *unbounded*. To denote the same situation, we also say that patterns in $\mathcal{P}$ are *bounded* or *unbounded*.

Definition 10. Let $P \in \{0,1\}^{k \times l}$ be a pattern and e any of its one-entries. We say that e is *row-bounded* if there is a function $f(k,l)$ such that for each maximal matrix avoiding P and each of its rows the number of zero-intervals that are usable for e is bounded by $f(k,l)$. Symmetrically, we define *column-bounded* one-entries of e. We say that e is *bounded* if it is both row-bounded and column-bounded.

For example, let P be a pattern and assume each of its one-entries is bounded. From the definition it immediately follows that P is bounded and that is equivalent to saying that class $Av(P)$ is bounded.

The following observation follows directly the definition and we use it heavily throughout the chapter to break symmetries.

Observation 19. *For every $P \in \{0,1\}^{k \times l}$, $r \in [k]$ and $c \in [l]$, if $P[r,c]$ is row-bounded then $P^T[c,r]$ is column-bounded.*

3.1 Inserting an empty column

We start by proving Lemma 21 which we stated and used to proof Theorem 17. Even before that we show an easy lemma to get familiar with notion of one-intervals.

Lemma 20. *Let $P \in \{0,1\}^{k\times 2}$ and let $M \in \{0,1\}^{m\times n}$ be a maximal matrix avoiding P, then M contains at most one one-interval in each row.*

Proof. For contradiction, assume there are several one-intervals in a row of M. Because M is maximal, changing any zero-entry e in between two consecutive one-intervals oi_1 and oi_2 creates a mapping of the forbidden pattern. Such a mapping uses the changed one-entry to map an element $P[r',1]$ or $P[r',2]$.

In the first case, the same mapping also works if we use any one-entry of oi_1 instead of e, which gives us $P \not\preceq M$ and therefore a contradiction. In the second case, the mapping can use any one-entry of oi_2 instead of e; therefore, we again get a contradiction with $P \not\preceq M$. Since e is not usable for any one-entry of P we can change it to a one-entry and get a contradiction with M being maximal. □

Lemma 21. *Let $P \in \{0,1\}^{k\times 2}$ and let $P^+ \in \{0,1\}^{k\times 3}$ be the pattern created from P by adding a new empty column in between the two columns of P. If $M \in Av(P^+)$ is a maximal matrix, then each row of M is either empty or it contains exactly one one-interval of length at least 2.*

Proof. The same proof as in Lemma 20 can be used to show there is at most one one-interval in each row.

For contradiction assume there is exactly one one-entry $e = M[r,c]$ in row r:

- $c = 1$: we can set $M[r,2] = 1$ and the matrix stills avoid P^+, which is a contradiction with M being maximal.
- $c = n$: symmetrically with the previous case this cannot happen.
- $c \in [2, n-1]$: assume we cannot change neither $e_l = M[r, c-1]$ nor $e_r = M[r, c+1]$ to a one-entry without creating the pattern. This means e_l is usable for some $P^+[r_1, 1]$, let M_l be the mapping and e_r is usable for some $P^+[r_2, 3]$ with M_r being the mapping. We now show that the two mappings can be altered to find a mapping of P^+ to M giving a contradiction. Without loss of generality only column c of M is used to map the middle column of P^+ and we describe how to partition M into k rows. We have two cases to go through. Because the analysis is rather technical we at least provide a picture of situations in Figure 3.1:
  - $r_1 \neq r_2$: Without loss of generality we assume $r_1 > r_2$. Let r_3 be the first row used to map r_1 in M_l and let r_4 be the last row used to map r_1 in M_r. From M_l being a mapping we know that the first $r_1 - 1$ rows can be mapped to rows $[1, r_3 - 1]$ and from M_r being a mapping we know that the last $k - r_1$ rows can be mapped to rows $[r_4 + 1, m]$. We can therefore use rows $[r_3, r_4]$ to map row r_1 without using e_l and e_r.

– $r_1 = r_2$: We proceed similarly as in the previous case. Let r_3 and r_4 be the first and the last rows respectively used to map r_1 in M_l and let r_5 and r_6 be the first and the last rows respectively used to map r_1 in M_r. Without loss of generality let $r_3 < r_5$ and from M_l being a mapping we know that the first $r_1 - 1$ rows can be mapped to rows $[1, r_3 - 1]$. Again, without loss of generality let $r_4 < r_6$ and from M_r being a mapping we know that the last $k - r_1$ rows can be mapped to rows $[r_6 + 1, m]$. We can therefore use rows $[r_3, r_6]$ to map row r_1 without using e_l and e_r.

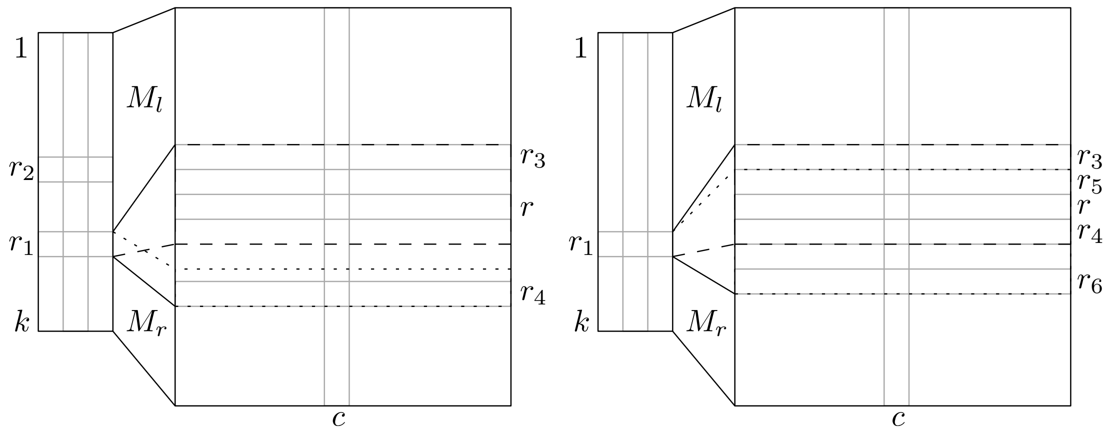


Figure 3.1: Dashed and dotted lines resembling two different mappings of a forbidden pattern, where two horizontal lines show the boundaries of the mapping of row r and the vertical lines show boundaries of the mapping of column c.

□

In Lemma 20 we proved half of the condition for a k by 2 pattern to be bounded, here comes the second half.

Observation 22. *Let $P \in \{0,1\}^{k\times l}$ and $M \in \{0,1\}^{m\times n}$ such that $P \not\preceq M$. Let $zi = M[\{r_1\}, [c_1, c_2]]$ be a zero-interval of M usable for one-entry $e = P[r, c]$. If we change a zero-entry of zi and create a mapping of P that uses the changed entry to map e, then no such mapping can map column c outside of columns $[c_1, c_2]$.*

Proof. Since the changed entry is used to map e, clearly every mapping needs to use a column from $[c_1, c_2]$ to map column c. If for contradiction after a change of a zero-entry there was a mapping using columns outside $[c_1, c_2]$ then it without loss of generality uses $c_1 - 1$ but since it bounds zero-interval zi it is a one-entry and this one-entry can be used in the mapping instead of the changed entry, which gives us a contradiction with $P \not\preceq M$. □

Lemma 23. *Let $P \in \{0,1\}^{k\times 2}$ and $M \in \{0,1\}^{m\times n}$ be a maximal matrix avoiding P, then M contains at most $2k^3 + 1$ one-intervals in each column.*

Proof. Given an arbitrary maximal matrix M avoiding P let us look at an arbitrary column c. For contradiction, assume it has at least $2k^3 + 2$ one-intervals which also means there are at least $2k^3 + 1$ zero-intervals in between. Since P

has at most $2k$ one-entries, from the Pigeonhole principle there are k^2 $(+1)$ zero-intervals $zi_1 < zi_2 < \cdots < zi_{k^2}$ usable for the same one-entry e of P. Without loss of generality let $e = P[r, 1]$.

- $\forall i < r :\ P[i, 2] = 0$: If we take the mapping created by changing a zero-entry of zi_k and to map $P[i, 1]$ we use one-intervals above zi_k we map the whole P, contradicting $P \not\preceq M$.
- $P[r, 2] = 1$: Clearly, there is a one-entry next to each zi_j and if we combine each such entry with a one-entry bounding each zi_j we find a mapping of $\{1\}^{k^2 \times 2}$, contradicting $P \not\preceq M$.
- $\exists i < r :\ P[i, 2] = 1$: If there are multiple such i, choose the biggest one, so that for all $i < j \leq r$ it holds $P[j, 2] = 0$. For all those j it holds that j-th row of P can be mapped to just one row of M. So first, if for any mapping $Map_1 \dots Map_{k^2}$, created by changing a zero-entry of $zi_1 \dots zi_{k^2}$ respectively to map e, is $i < l < r$ using two rows such that both have a one-entry in column c, we can only use the first of them, shift the mapping off all following rows up to r and find a mapping of P to M. Therefore, we assume this is not the case and every mapping uses up to $r - i \leq k$ one-entries right above zi_l it to map rows $[i, r - 1]$.

  We combine the one-entry used to map $P[i, 2]$ in mapping Map_l with a one-entry bounding zero-interval zi_l. Since we have k^2 such mappings and each of them only goes through at most k one-intervals to map $P[i, 2]$ and therefore find a one-entry in a column $c' > c$, we find at least k distinct pairs and therefore a mapping of $\{1\}^{k \times 2}$, contradicting $P \not\preceq M$.

□

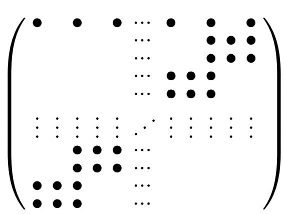
Corollary 24. *Every $P \in \{0, 1\}^{k \times 2}$ is bounded.*

3.2 Unbounded number of one-intervals

We saw that for patterns having only two rows or columns we can indeed bound the number of one-intervals of maximal matrices avoiding them. On the other hand, already for a pattern of size 3×3 we show that there are maximal matrices with arbitrarily many one-intervals.

Theorem 25. *Let $P = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$. For every $n > 1$ there is a maximal matrix M avoiding P as an interval minor having n one-intervals (P is unbounded).*

Proof. Let M be a $(2n - 1) \times (2n - 1)$ matrix described by the picture:



$P \not\preceq M$ because we always need to map $P[2, 1]$ and $P[3, 3]$ to just one "block" of one-entries of M which only leaves a zero-entry where we need to map $P[1, 2]$.

When we change any zero-entry of the first row into a one-entry we get a matrix containing a minor of $\{1\}^{3\times3}$; therefore, containing P as an interval minor. In case M is not maximal, we can add some more one-entries to make it maximal but it will still contain a row with n one-intervals. $\square$

Not only M is a maximal matrix avoiding P but it also avoids any $P' \in \{0,1\}^{3\times3}$ such that $P \preceq P'$. Its rotations avoid rotations of P and we can deduce that a big portion of patterns of size 3×3 are unbounded. Moreover, the result can be generalized also for bigger matrices. The pattern is so important that we call it P_1 for the rest of the chapter.

Theorem 26. *For every P such that $P_1 \preceq P$ and every $n > 1$ there is a maximal matrix M avoiding P as an interval minor having n one-intervals.*

Proof. First, assume there is a mapping of P_1 into $P \in \{0,1\}^{k\times l}$ that assigns a one-entry of the first row to $P_1[1,2]$, a one-entry of the first column to $P_1[2,1]$ and a one-entry of the last row and column to $P_1[3,3]$. Then, we can construct a similar matrix as we did in the proof of Theorem 25 avoiding P that after changing any zero-entry of the first row contains the whole $\{1\}^{k\times l}$.

Let P be an arbitrary pattern containing P_1. Let $P[r_1,c_1], P[r_2,c_2]$ and $P[r_3,c_3]$ be one-entries that can be used to map $P_1[1,2], P_1[2,1]$ and $P_1[3,3]$ respectively. Then we take a submatrix $P' := P[[r_1,r_3],[c_2,c_3]]$. Such a pattern fulfills assumptions of the more restricted case stated at the beginning of the proof and we can find a maximal matrix M' avoiding P' having n one-intervals. We construct M from M' by simply adding new rows and columns, all containing one-entries. We add $r_1 - 1$ rows in front of the first row and $k - r_3$ rows behind the last row. We also add $c_2 - 1$ columns in front of the first column and $l - c_3$ columns behind the last column. Constructed matrix M avoids pattern P as its submatrix P' cannot be mapped to M'. At the same time, any change of a zero-entry of the r_1-th row of M to a one-entry creates a copy of $1^{k\times l}$ so the changed matrix contains P. Constructed M can be seen in Figure 3.2.

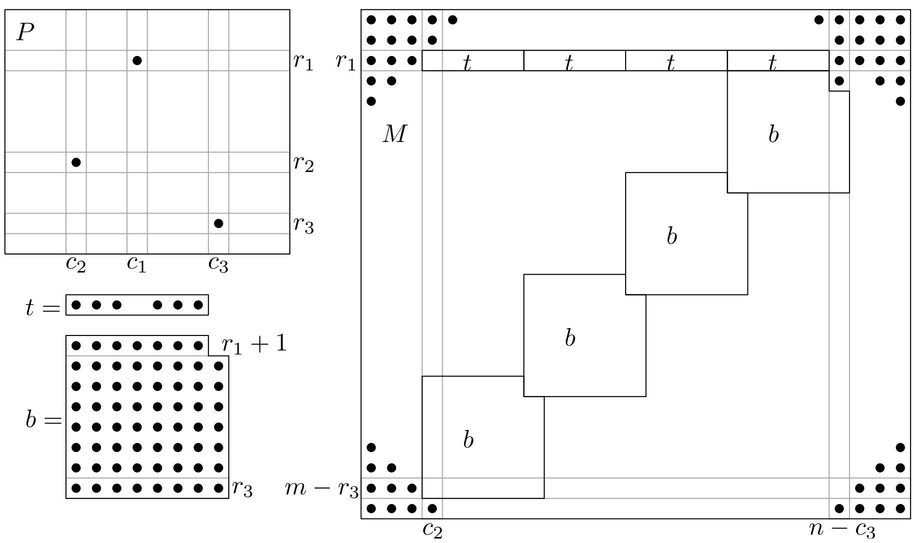


Figure 3.2: Structure of a maximal matrix avoiding P that has arbitrarily many one-intervals.

$\square$

3.3 Bounded number of one-intervals

What makes it even more interesting is that any pattern avoiding all rotations of P_1 is already bounded. To prove that we need a few partial results.

Theorem 27. *Let P be a pattern avoiding all rotations of P_1, then P:*

1. *contains one non-empty line or*
2. *contains two non-empty lines or*
3. *contains three non-empty lines or*
4. *avoids $\left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$.*

Proof. Assume P has four one-entries that do not share any row or column. Then those one-entries induce a 4×4 permutation inside P and because P does not contain any rotation of P_1, the induced permutation is either 1234 or 4321. Without loss of generality, assume it is the first case and denote the one-entries by e_1, e_2, e_3 and e_4.

For contradiction with the statement, assume P also contains $P' = \left(\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right)$. Clearly, no one-entry from e_1, e_2, e_3 and e_4 can be part of any mapping of P' because it would induce a mapping of a rotation of P_1.

Let $e_2 = P[r_2, c_2]$ and $e_3 = P[r_3, c_3]$. Submatrix $P[[r_2], [c_2, l]]$ avoids P'; otherwise, together with e_1 it would give us a rotated copy of P_1. Symmetrically, $P[[r_3, k], [c_3]]$ does not contain P'. Also, $P[[r_3 - 1], [c_3 - 1]]$ and $P[[r_2 + 1, k], [c_2 + 1, l]]$ are empty; otherwise, they would together with e_2 and e_3 give us a rotation of P_1. Up to rotation, the only possible way to have $P' \preceq P$ is that $P'[1, 1]$ lies in $P[[r_3 - 1], [c_2, c_3 - 1]]$ but then this entry together with e_1 and e_3 give us a rotation of P_1 which is a contradiction. □

Now comes the hard part. For each group of patterns, we need to prove they are bounded.

Lemma 28. *Let $P \in \{0, 1\}^{k \times l}$ be a pattern having only one non-empty line. Then for every maximal matrix $M \in \{0, 1\}^{m \times n}$ avoiding P the number of one-intervals in each row and column is bounded by $k + l$.*

Proof. Without loss of generality let the non-empty line of P be a row r. Since M is maximal, $M[[r - 1], [n]]$ and $M[[m - r + 1, m], [n]]$ contain no zero-entry and each of their rows contains just one interval of one-entries. If we look at any other row, it cannot contain k one-entries, so the maximum number of one-intervals is $k - 1$.

Let us look on an arbitrary column c of M. If there is at least one one-entry in $M[[r, m - r], c]$ then because M is maximal, the whole column is made of one-entries. Otherwise, there are two intervals of one-entries – $M[[r - 1], c]$ and $M[[m - r, m], c]$. □

Lemma 29. *Let $P \in \{0, 1\}^{k \times l}$ be a pattern having two non-empty lines. Then for every maximal matrix $M \in \{0, 1\}^{m \times n}$ avoiding P the number of one-intervals in each row and column is bounded by $2k^3 + 2l^3 + 1$.*

Proof. First we assume the two non-empty lines of P are rows $r_1 < r_2$ (or symmetrically columns). From Observation 15 and maximality of M we have that $M[[r_1-1],[n]]$ and $M[[m-r_2+1,m],[n]]$ contain no zero-entry. Therefore, we may restrict ourselves to the case when $r_1 = 1$ and $r_2 = k$. From Lemma 23 we know that every maximal N avoiding $P[\{r_1, r_2\},[n]]$ has at most $2k^3 + 1$ one-intervals in each row and at most 1 one-interval in each column. From Theorem 17 we also know that for given M there is a maximal N avoiding $P[\{r_1, r_2\},[n]]$ such that M is a submatrix of shifted and OR-ed copies of N. Since M is maximal, it is equal to those shifted and OR-ed copies of N and because the number of one-intervals of N is bounded, so is the number of one-intervals of M.

Let the two non-empty lines of P be row r and column c. Because of symmetry, we only show the bound for rows. Let us take an arbitrary row of M an look at its zero-intervals. For every one-entry e of the pattern except those in the r-th row, there is at most one zero-interval usable for e. For contradiction, assume there are two such zero-intervals zi_1 and zi_2. Let Figure 3.3 illustrate the situation where dashed and dotted lines form mappings of the minor P to M when a zero-entry of zi_1 and zi_2 respectively is changed to a one-entry. When we take the outer two vertical and horizontal lines, we get a mapping of P that can use an existing one-entry in between zi_1 and zi_2 to map e. This gives us a contradiction with $P \not\preceq M$.

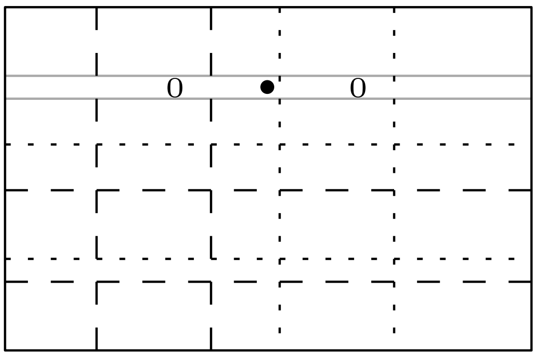


Figure 3.3: Dashed and dotted lines resembling two different mappings of a forbidden pattern, where two horizontal lines show the boundaries of the mapping of row r and the vertical lines show boundaries of the mapping of column c.

For a one-entry $e = P[r, c']$, if $c' \le c$ then there must be less than c' one-entries before any zero-intervals usable for e; otherwise, we could map $P[r,[1,c']]$ just to the single row of M. It follows that e is row-bounded. Symmetrically, the same holds in case $c' > c$. □

To make the analysis of the last two groups of patterns easier, we introduce three helpful lemmata.

Lemma 30. *Let $P \in \{0,1\}^{k \times l}$ be a pattern looking like one of the matrices in Figure 3.4. Then every one-entry from row r_2 in columns c_1 to c_2 is row-bounded.*

Proof. Let P be any of the first three described patterns and let $k' = c_2 - c_1$. We even show that for each one-entry e from row r_2 and every M maximal matrix avoiding P there is at most k' zero-intervals for which it is usable. For contradiction assume there is a row r with $k' + 1$ zero-intervals usable for e. It

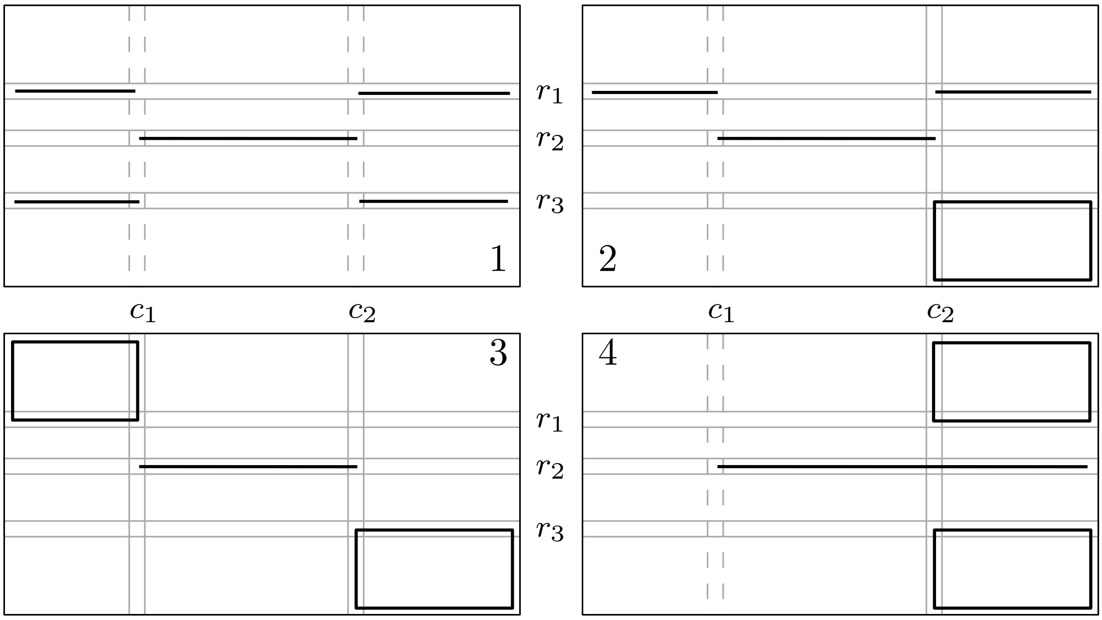


Figure 3.4: Patterns for which one-entries in row r_2 are row-bounded. One-entries may only be in the areas enclosed by bold lines.

follows that there are at least k' one-entries in between two most distant zero-intervals z_1 and z_2. Therefore, the whole row r_2 can be mapped just to r. Since changing a zero-entry of z_1 to a one-entry to which e can be mapped, there is a partitioning of M where all one-entries from columns 1 to c_1 are mapped to columns before z_1 and similarly all one-entries from columns c_2 to l are mapped to columns past z_2. To partition rows, we can simply map rows from $r_1 + 1$ to $r_3 - 1$ around row r one to one and use the rest to find enough one-entries for the one-entries of P. The partitioning using those one-entries and one-entries from r to map one-entries of r_2 together give us $P \preceq M$ and a contradiction.

Make a picture? The explanation is not clear at all.

Let us look on the fourth case. For i-th one-entry in row r_2 (ordered from left to right and only considering those in columns c_1 to c_2) no zero-interval of a maximal matrix avoiding the pattern cannot have i one-entries to the left of it and so each such one-entry is bounded by $i \geq l$. □

Lemma 31. *Let P be a pattern and c be its first non-empty column. Then every one-entry from c is row-bounded.*

Proof. The results follows immediately from the fourth case of Lemma 30 when there are no one-entries in columns before c_2. □

Lemma 32. *Let $P \in \{0,1\}^{k \times l}$ be a pattern looking like one of the matrices in Figure 3.5. Then every one-entry from column c in rows r_1 to r_2 is row-bounded.*

Proof. Let P be any of the described patterns. We even show that for each one-entry e from row r_2 and every M maximal matrix avoiding P there is at most one zero-interval for which it is usable. For contradiction assume there is a row r with two zero-intervals usable for e.

Again a picture needed – take the closer partitioning and it is done □

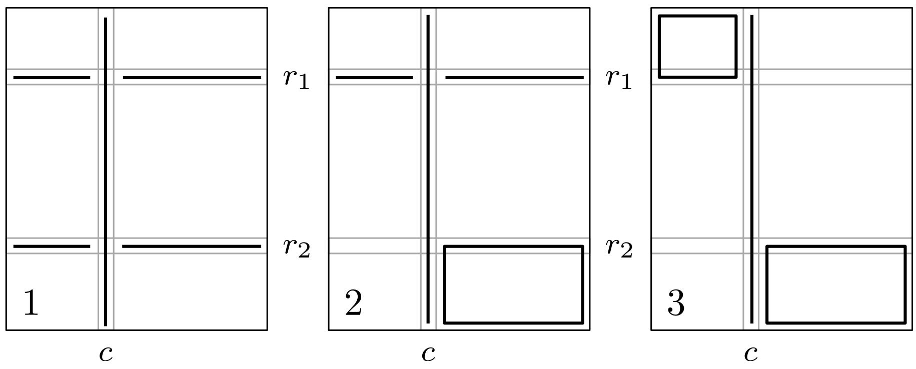


Figure 3.5: Patterns for which one-entries in column c are row-bounded. One-entries may only be in the areas enclosed by bold lines.

Lemma 33. *Let $P \in \{0,1\}^{k \times l}$ be a pattern avoiding $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ (or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$). Then for every maximal matrix $M \in \{0,1\}^{m \times n}$ avoiding P the number of one-intervals in each row and column is bounded.*

Proof. From Theorem 4 we know that P is a walking pattern. Every one-entry of P satisfies either conditions of the third case of Lemma 30 or it satisfies conditions of the third case of Lemma 32 and therefore is row-bounded. To prove it is also column-bounded, we can look at P^T and show that its one-entries are row-bounded. Since it is again a walking pattern, we can use the same arguments. $\square$

Lemma 34. *Let $P \in \{0,1\}^{k \times l}$ be a pattern having three non-empty lines and avoiding all rotations of P_1. Then for every maximal matrix $M \in \{0,1\}^{m \times n}$ avoiding P the number of one-intervals in each row and column is bounded.*

Proof. First of all, if P avoids $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$ we can use Lemma 33. Without loss of generality we assume it contains both but avoids every rotation of P_1.

Let us prove that each pattern having one-entries in three rows is bounded. If all one-entries are in up to two columns then we are again done. Therefore, P has one-entries in at least three columns and so it contains a three by three permutation matrix as a submatrix (or an interval minor). Since rotations of P_1 are avoided, that permutation is either 123 or 321 and without loss of generality we assume the first case. In Figure 3.6 we see the structure of each such pattern. Capital letters stand for one-entries of the permutation, letters $a - f$ stand each for a potential one-entry and greek letters stand each for a potential sequence of one-entries and zero-entries. Everything else is zero. Not all one-entries can be present at the same time, because that would create a mapping of P_1 or its rotation and we also need to find $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$. The following analysis will only use hereditary arguments. This mean that if we prove P is bounded, we also prove that each submatrix of P is bounded. With this in mind, we restrict ourselves to maximal patterns.

- γ contains a one-entry $\Rightarrow f = 0 \Rightarrow$ because $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$ needs to be there it holds $a = 1 \Rightarrow \alpha = 0$
  - $d = 1 \Rightarrow b = 0,\ \beta = 0,\ e = 0,\ c = ?$:
    Lemma 30 (case 4): one-entries in c, C, γ are row-bounded.
    Lemma 31: a and A are row-bounded.

  Lemma 32 (case 1): d and B are row-bounded.

  Lemma 31: all one-entries except for B are column-bounded.
  Lemma 30 (case 1): B is column-bounded.

- $d = 0$
  - $c = 1 \Rightarrow \beta = 0,\ e = 0,\ b =?$:
    Lemma 30 (case 4): one-entries in c, C, γ are row-bounded.
    Lemma 31: a, b, A are row-bounded.
    Lemma 30 (case 1): B is row-bounded.

    Lemma 31: one-entries in the first and the third non-empty rows are column-bounded.
    Lemma 30 (case 2): b, B are column-bounded.
  - $c = 0 \Rightarrow$ in the maximal case $b = 1,\ e = 1,\ \gamma$ contains a one-entry:
    Lemma 30 (case 4): one-entries in c, C, γ are row-bounded.
    Lemma 31: one-entries in the first non-empty column are row-bounded.
    Lemma 30 (case 1): one-entries in the middle non-empty row are row-bounded.

    Lemma 31: one-entries in the first and the third non-empty rows are column-bounded.
    Lemma 32 (case 2): one-entries in the middle non-empty row are column-bounded.

- $\gamma = 0$
  - α contains a one-entry $\Rightarrow a = 0,\ b = 0$:
    Every such pattern has already been dealt with as we can rotate it by 180 degrees, map A and α to γ, map d to C and so on.
  - $\alpha = 0$:
    Without loss of generality, we can assume that $a = 1$, because there needs to be $\left({}_{\bullet}{}^{\bullet}\right)$ and if we set $a = 0$, it must hold $f = 1$ and then we can just rotate the pattern by 180 degrees and get the case $a = 1$.
    - $d = 1 \Rightarrow b = 0,\ e = 0,\ \beta = 0,\ c =?,\ f =?$:
      Lemma 31: a, f, A and C are row-bounded.
      Lemma 32 (case 1): c, d and B are row-bounded.

      Lemma 31: one-entries in the first and third non-empty rows are column-bounded.
      Lemma 30 (case 1): B is column-bounded.
    - $d = 0$
      - $e = 1 \Rightarrow c = 0,\ b =?,\ f =?$:
        Since $\alpha = 0$ it follows that if there is a one-entry in β only if it can be in e.
        Lemma 31: a, f, A and C are row-bounded.
        Lemma 30 (case 1): one-entries in b, e, B and β are row-

bounded.

Lemma 31: a, f, A and C are column-bounded.
Lemma 32 (case 1): one-entries in b, e, B and β are column-bounded.

· $e = 0$:
We can assume $c = 1$ as else e can be 1 and we have already dealt with that case. We can also assume $b = 1$ since otherwise, we would have a submatrix of the case dealt with when $d = 1$:
Lemma 31: a, b and A are row-bounded.
Lemma 30 (case 4): c and C are row-bounded.
Lemma 30 (case 1): B is row-bounded.

Because the pattern is symmetric, it is also column-bounded.

The same analysis also proves that if the pattern with the same restrictions only has three non-empty column it is bounded.

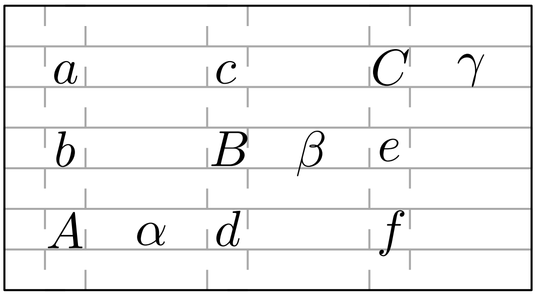


Figure 3.6: Structure of a pattern only having three non-empty rows and avoiding all rotations of P_1.

Let us now look at the case where the pattern that only have one-entries in either one of two rows r_1, r_2 or one column c_1. Without loss of generality we again assume permutation 123 is present and we distinguish three cases. Consider Figure 3.7:

- C lies in column c_1
  - $a = 1 \Rightarrow b = 0,\ \alpha = 0$ and everything else can be one:
    Lemma 30 (case 2): one-entries in a, c, B and β are row-bounded.
    Lemma 31: all other one-entries are row-bounded.

    Lemma 30 (case 4): one-entries in c, C and γ are column-bounded.
    Lemma 32 (case 1): one-entries in a, c, B and β are column-bounded.
    Lemma 31: d and A are column-bounded.
  - $a = 0$ and everything else can be one:
    Lemma 30 (case 4): one-entries in b, A and α are row-bounded.
    Lemma 30 (case 2): one-entries in c, B and β are row-bounded.
    Lemma 31: one-entries in c, d, C and γ are row-bounded.

Lemma 30 (case 4): one-entries in c, C and γ are column-bounded.
Lemma 32 (case 2): one-entries in c, B and β are column-bounded.
Lemma 31: one-entries in b, d, A and α are column-bounded.

- B lies in column c_1
  - $a = 1 \Rightarrow \alpha = 0$
    - $d = 1 \Rightarrow \gamma = 0$:
      Lemma 32 (case 1): all one-entries in column c_1 are row-bounded.
      Lemma 31: all other one-entries are row-bounded.

      Lemma 30 (case 1): all one-entries in column c_1 are column-bounded.
      Lemma 31: all other one-entries are column-bounded.
    - $d = 0$:
      Lemma 32 (case 1): all one-entries in column c_1 are row-bounded.
      Lemma 31: a and A are row-bounded.
      Lemma 30 (case 4): one-entries in C and γ are row-bounded.

      Lemma 30 (case 1): all one-entries in column c_1 are column-bounded.
      Lemma 31: all other one-entries are column-bounded.
  - $a = 0$
    - $d = 1 \Rightarrow \gamma = 0$:
      Lemma 32 (case 1): all one-entries in column c_1 are row-bounded.
      Lemma 31: d and C are row-bounded.
      Lemma 30 (case 4): one-entries in A and α are row-bounded.

      Lemma 30 (case 1): all one-entries in column c_1 are column-bounded.
      Lemma 31: all other one-entries are column-bounded.
    - $d = 0$:
      Lemma 32 (case 1): all one-entries in column c_1 are row-bounded.
      Lemma 30 (case 4): one-entries in A, C, α and γ are row-bounded.

      Lemma 30 (case 1): all one-entries in column c_1 are column-bounded.
      Lemma 31: all other one-entries are column-bounded.
- A lies in column c_1:
  This is the first situation rotated by 180 degrees.

The same analysis also proves that if one-entries of a pattern with the same restrictions are in one row or two columns then the pattern it is bounded. □

Combining all the lemmata we finally get the following result.

Theorem 35. *Let P be a pattern avoiding all rotations of $P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$, then P is bounded.*

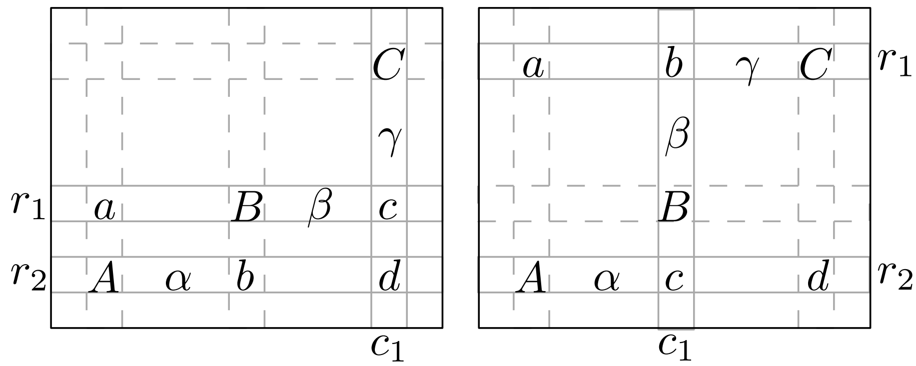


Figure 3.7: Structure of a pattern only having one-entries in two rows and one column that avoids all rotations of P_1.

3.4 Chain rules

Theorem 36. *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite classes of patterns. If both $\mathcal{P}$ and $\mathcal{Q}$ are bounded then $Av(\mathcal{P} \cup \mathcal{Q})$ is bounded.*

Proof. For a bounded pattern P, let $f(P)$ be a bound of the number of one-intervals of any maximal matrix avoiding P. For $\mathcal{P}$ let $f(\mathcal{P}) = \sum_{P \in \mathcal{P}} f(P)$. Let M be a maximal matrix avoiding $\mathcal{P} \cup \mathcal{Q}$ and take the line with the highest number of one-intervals. For every $P \in \mathcal{P} \cup \mathcal{Q}$ there is at most $f(P) + 1$ zero-intervals usable for P's one-entries. For contradiction assume there are more and change all zero-entries that are not usable for P's one-entries with one-entries. This way we end up with a maximal matrix avoiding P and a contradiction to $f(P)$ being the bound on the number of one-intervals of such matrix.

Together we then have $f(\mathcal{P} \cup \mathcal{Q}) \geq f(\mathcal{P}) + |P| + f(\mathcal{Q}) + |Q| + 1$ and since all numbers are finite, we have that $\mathcal{P} \cup \mathcal{Q}$ is bounded. $\square$

Using induction, we can show that also a union of finite number of bounded classes of finite size are bounded. Interestingly enough, unbounded classes are not closed the same way.

Fact 37 (Higman's lemma)**.** *Let A be a finite alphabet and A^* be a set of finite sequences over A. Then A^* is well quasi ordered with respect to the subsequence relation.*

Theorem 38. $\sigma = Av\left(\begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}, \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix}, \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}\right)$ *is bounded. (Moreover, every subclass is bounded – dont see how this follows.)*

Proof. From Theorem 27 we know that elements of σ fall into finitely many classes. For each we need to prove that it is bounded and also that it does not contain an infinite anti-chain. Knowing that we use Theorem 36 to obtain the result. Let us consider m by n matrix $M \in \sigma$:

- M only contains up to three non-empty rows (columns):
  Clearly, if M is maximal then it contains three rows made of one-entries and everything else is zero, so the number of one-intervals is bounded by three.

We use alphabet $A = \{a, b, c, d, e, f, g, h, i, j\}$ to describe each M as follows. Let r_1, r_2 and r_3 be the non-empty rows (if less then three are non-empty, we choose $r_3 = m$ and $r_2 = m-1$ and $r_1 = m-2$ if needed). We define w_M as follows. First we use letter g $r_1 - 1$ times, letter h $r_2 - r_1$ times, letter i $r_3 - r_2$ times and letter j $m - r_3$ times to describe the number of rows of M. Then we describe columns from the first one to the last one as follows. For each 0 in r_1 we use letter a, for 1, we use ab. For each 0 in r_2 we use letter c, for 1, we use cd. For each 0 in r_3 we use letter e, for 1, we use ef.

If we have $w_M, w_{M'} \in A^*$ such that w_M is a subsequence of $w_{M'}$ then we want to show that M is an interval minor of M'. Let r_1, r_2, r_3 and r'_1, r'_2, r'_3 be the non-empty rows of M and M' respectively. Since the number of leading letters g is not bigger in w_M, M does not have more empty rows before r_1 than M' does before r'_1 and similarly it has at most as many empty rows in between r_1, r_2 and r_2, r_3 and after r_3.

Now consider there is ab in w_M and it corresponds to some $a \dots b$ in $w_{M'}$. We can always assume that in $w_{M'}$ the "a" is the one exactly before b. Because of the order $abcdefab\dots$ it can only happen that $abcdeface$ is a subsequence of $\mathbf{ab}cea\mathbf{cd}eac\mathbf{eface}$ if the bold letters are used and since they correspond to one-entries lying in the following columns, this indeed corresponds to an interval minor.

From Fact 37 we have that A^* is well ordered which mean that matrices having at most three non-empty rows (columns) are well ordered (this holds for every finite number of non-empty rows) and so they does not have an infitely long anti-chain.

- M has at most three non-empty lines – at most two rows and one column (or vice versa):
  The number of one-intervals of any such maximal M is bounded by two.

  We use alphabet $A = \{a, b, c, d, e, f, g\}$ and for non-empty rows r_1, r_2 and column c_1 we define w_M as follows. We first encode each column in such a way that for each 0 in r_1 we use letter a, for 1, we use ab. For each 0 in r_2 we use letter c, for 1, we use cd. Right before and after column c_1 we put letter g. Next we encode each row in such a way that for each 0 in c_1 we use letter e and for each 1 letters ef. Right before and after rows r_1 and r_2 we again place letter g.

  Because of the distinct letters for encoding rows and columns we can apply the same analysis as we did in the previous case and since entries at $M[r_1, c_1]$ and $M[r_2, c_1]$ are separated from the rest by a special letter there is no way to find a one-entry if it is not there.

- M avoids $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ (or $\begin{pmatrix} \bullet & \\ & \bullet \end{pmatrix}$):
  From Theorem 4 we know M is a walking matrix and any such maximal matrix only contains at most one one-intervals in each row and column.

  We use alphabet $A = \{a, b, c, d\}$ and encode M as follows. We choose a walk of M containing all one-entries and index its entries as $w_1 \dots w_{m+n-1}$.

Starting from w_1 we encode w_i is such a way that a stands for 0 and ab for 1 if w_{i+1} lies in the same row as w_i and we use c for 0 and cd for 1 if w_{i+1} lies in the same column as w_i.

$\square$

Observation 39. *There exists a non-trivial bounded pattern P having an unbounded subset of $Av(P)$.*

Proof. Let $P = I_n$ (identity matrix) for $n > 3$. From Lemma 33 we have that P is bounded. On the other hand, $Av(I_n, P_1)$ is unbounded, because the construction used in the proof of Theorem 25 also works for this class. $\square$

Open questions:

- $\mathcal{C}$ row-bounded $\Rightarrow$ $\mathcal{C}$ column-bounded
- $Av\left(\begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}, \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix}\right)$ bounded (hereditary)
- $Av\left(\begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}\right)$ bounded (hereditary)

4. Extremal function

Notation 6. Let M be a matrix. We denote $|M|$ the weight of M, the number of one-entries in M.

Usually $|M|$ stands for a determinant of matrix M. However, in this paper we do not work with determinants at all so the notation should not lead to misunderstanding.

Definition 11. For a matrix P we define $Ex(P,m,n) := \max\{|M| | M \in \{0,1\}^{m\times n},\ P \not\leq M\}$. We denote $Ex(P,n) := Ex(P,n,n)$.

Definition 12. For a matrix P we define $Ex_{\preceq}(P,m,n) := max\{|M| | M \in \{0,1\}^{m\times n},\ P \npreceq M\}$. We denote $Ex_{\preceq}(P,n) := Ex_{\preceq}(P,n,n)$.

Observation 40. *For all P,m,n; $Ex_{\preceq}(P,m,n) \leq Ex(P,m,n)$.*

Observation 41. *If $P \in \{0,1\}^{k\times l}$ has a one-entry at position $[a,b]$, then*

$$Ex(P,m,n) \geq \begin{cases} m\cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \textit{otherwise.} \end{cases}$$

Observation 42. *The same holds for $Ex_{\preceq}(P,m,n)$.*

Definition 13. $P \in \{0,1\}^{k\times l}$ is *(strongly) minimalist* if

$$Ex(P,m,n) = \begin{cases} m\cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Definition 14. $P \in \{0,1\}^{k\times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P,m,n) = \begin{cases} m\cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Observation 43. *If P is strongly minimalist, then P is weakly minimalist.*

4.1 Known results

Fact 44.

1. *$(\bullet)$ is strongly minimalist.*
2. *If $P \in \{0,1\}^{k\times l}$ is strongly minimalist and there is a one-entry in the last row in the c-th column, then $P' \in \{0,1\}^{k+1\times l}$, which is created from P by adding a new row having a one-entry only in the c-th column, is strongly minimalist.*
3. *If P is strongly minimalist, then after changing a one-entry into a zero-entry it is still strongly minimalist.*

Fact 45. *Let $P = \begin{pmatrix} \bullet & \cdots & \bullet \\ \bullet & \cdots & \bullet \end{pmatrix}$ have l columns, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{2\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{weight of } M[[i], \{j\}] > 0 \wedge \text{weight of } M[[i+1,m], \{j\} > 0]\}$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M appears in at least $b_j - 1$ of sets A_i, $0 \le i \le m-2$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 1) + n \le \sum_{i=0}^{m-2} |A_i| + n \le (l-1)(m-1) + n$$

□

This result is indeed very important because it shows that there are matrices like $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, which are weakly minimalist, although it is known they are not strongly minimalist.

Fact 46. *Let* $P = \begin{pmatrix} \bullet & \cdots & \bullet \\ \bullet & \cdots & \bullet \\ \bullet & \cdots & \bullet \end{pmatrix}$ *have l columns, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{3\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{ weight of } M[[i-1], \{j\}] > 0 \wedge \text{ weight of } M[[i+1,m], \{j\} > 0 \wedge M[i,j] \text{ one-entry}]\}$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M (for which $b_j \ge 2$) appears in exactly $b_j - 2$ of sets A_i, $1 \le i \le m-1$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 2) + 2n \le \sum_{i=1}^{m-2} |A_i| + 2n \le (l-1)(m-2) + 2n$$

□

5. Operations with matrices

Notation 7. When speaking about a class of matrices, unless stated otherwise, we always expect the class to be closed under minors. Also, all classes discussed are non-trivial. That means that there is at least one matrix of size 2 by 1 and at least one matrix of size 1 by 2 in each class. Moreover, at least one matrix is non-empty.

Definition 15. Let $\mathcal{F}$ be any class of forbidden matrices. We denote by $Av(\mathcal{F})$ the set of all matrices that avoid every $F \in \mathcal{F}$ as an interval minor.

Observation 47. *Let $\mathcal{T} = Av(\mathcal{F})$ for some $\mathcal{F}$. Then $\mathcal{T}$ is closed under minors.*

Observation 48. *Let $\mathcal{M}$ be a finite class of matrices. There exists a finite set $\mathcal{F}$ such that $\mathcal{M} = Av(\mathcal{F})$.*

Definition 16. For matrices $A \in \{0,1\}^{m \times n}$ and $B \in \{0,1\}^{k \times l}$ we define their *direct sum* as a matrix $C := A \nearrow B \in \{0,1\}^{m+k \times n+l}$ such that $D[[k+1, m+k], [n]] = A$, $D[[k], [n+1, n+l]] = B$ and the rest is empty. Symmetrically, we define $D := A \searrow B \in \{0,1\}^{m+k \times n+l}$ such that $C[[m], [n]] = A$, $C[[m+1, m+k], [n+1, n+l]] = B$ and the rest is empty.

Theorem 49. $Av((\begin{smallmatrix}\bullet&\bullet\\ \bullet&\bullet\end{smallmatrix})) = (Av((\begin{smallmatrix}\bullet&\\ &\end{smallmatrix})) \searrow Av((\begin{smallmatrix}&\bullet\\ \bullet&\end{smallmatrix})) \searrow Av((\begin{smallmatrix}&\\ &\bullet\end{smallmatrix}))) \cup$
$\cup\, (Av((\begin{smallmatrix}&\\ \bullet&\end{smallmatrix})) \nearrow Av((\begin{smallmatrix}\bullet&\\ &\bullet\end{smallmatrix})) \nearrow Av((\begin{smallmatrix}&\bullet\\ &\end{smallmatrix})))$.

Proof. If follows from Theorem 9 and $Av((\begin{smallmatrix}\bullet&\bullet\\ \bullet&\end{smallmatrix})) = Av((\begin{smallmatrix}\bullet&\\ &\end{smallmatrix})) \searrow Av((\begin{smallmatrix}&\bullet\\ \bullet&\end{smallmatrix}))$. □

Notation 8. Let $\mathcal{M}$ be a class of matrices. Denote by $Cl(\mathcal{M})$ a set containing each $M \in \mathcal{M}$ closed under direct sum and minors.

Definition 17. Let $M \in \{0,1\}^{m \times n}$ be a matrix. We call an element $M[r,c]$ an *articulation* of M if both $M[[r-1],[c-1]]$ and $M[[r+1,m],[c+1,n]]$ are empty.

Lemma 50. *Let $M \in \{0,1\}^{k \times l}$, then for all $X \in \{0,1\}^{m \times n}$ it holds $X \in Cl(M) \Leftrightarrow$ there exists a sequence of articulations of X such that each matrix in between two consecutive articulations of X is a minor of $(\,1\,) \nearrow M \nearrow (\,1\,)$.*

Proof. $\Rightarrow$

$\Leftarrow$

□

Theorem 51. *For all $M \in \{0,1\}^{k \times l}$ there exists $\mathcal{F}$ finite such that $Cl(M) = Av(\mathcal{F})$.*

Proof. Using Lemma 50 □

Theorem 52. $Cl((\begin{smallmatrix}1&0\\0&1\end{smallmatrix})) = Av\left((\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}), \left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)\right)$.

Proof. $\subseteq$

$\supseteq$

□

Theorem 53. $Cl\left(\left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)\right) = Av\left(\left(\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&0\\0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\\0&1\end{smallmatrix}\right)\right).$

We can generalize direct sum to allow the matrices to overlap.

Definition 18. TODO $A \oplus_{k\times l} B$

Theorem 54. *Let $\mathcal{C}$ be any class of matrices such that*

- *$\mathcal{C}$ is closed under deleting of one-entries and*
- *$\mathcal{C}$ is closed under the direct sum with $k \times l$ overlap and*
- *there is any $M \in \{0,1\}^{m\times n}$ in $\mathcal{C}$*

then $\mathcal{C}$ is also closed under direct sum with $m - 2k \times n - 2l$ overlap.

Proof. Choose any two $A, B \in \mathcal{C}$ and $C\mathcal{C}$ such that $C \in \{0,1\}^{m\times n}$. Let $D \in \mathcal{C}$ denote the direct sum with $k \times l$ overlap of A and C. Finally, let E be the direct sum with $k \times l$ overlap of D and B. It has the same size as F, the direct sum with $m - 2k \times n - 2l$ overlap of A and B, which set of one-entries is also a subset of one-entries of $E \in \mathcal{C}$; therefore $F \in \mathcal{C}$. □

Theorem 55. *Let $\mathcal{C}$ be any class of matrices that is hereditary according to interval minors then for all m, n, k, l if $\mathcal{C}$ is closed under the direct sum with $m \times n$ overlap then is is also closed under the direct sum with $m + k \times n + l$ overlap.*

Proof. For contradiction, assume there are $A, B \in \mathcal{C}$ such that $A \oplus_{m+k\times n+l} B \notin \mathcal{C}$. □

Observation 56. *There is a $\mathcal{C}$ hereditary according to submatrices such that it is closed under the direct sum but it is not closed under the direct sum with 1×1 overlap.*

Proof. Let $\mathcal{C}$ be a class of all matrices obtained by applying the direct sum on $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$. Clearly, it is closed under the direct sum. On the other hand, $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \oplus_{1\times 1} \left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) = \left(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \end{smallmatrix}\right) \notin \mathcal{C}$. □

Notation 9. We define $\mathrm{Av}(M)$ to be a class of all matrices avoiding M as

We state following characterization only for the direct sum with 1×1 overlap but, because of Theorem 55, it also holds for any other size of overlap.

Theorem 57. *Let M be a matrix. There are M_1, M_2 proper submatrices of M such that $M = M_1 \oplus_{1\times 1} M_2 \Leftrightarrow Av(M)$ is not closed under the direct sum with 1×1 overlap.*

Proof. $\Rightarrow$

$\Leftarrow$

□

Observation 58. *Let M be a matrix. There are M_1, M_2 proper submatrices of M such that $M = M_1 \oplus_{1\times 1} M_2 \Leftrightarrow$ exists r, c such that either*

1. *$M[r,c]$ is a one-entry and $(r,c) \in \{(1,1),(m,n)\}$ or*

2. *$M[r,c]$ is both top-right and bottom-left empty and $(r,c) \notin \{(1,1),(m,n)\}$*

Definition 19. Let F be a matrix. We denote $\mathcal{R}(F)$ to be a set of all minimal (relating to minors) matrices F' such that $F \preceq F'$ and F' is not a direct sum with 1×1 overlap of proper submatrices of F'. For a class of matrices $\mathcal{F}$ let $\mathcal{R}(\mathcal{F})$ denote a set of all minimal (relating to minors) matrices from the set $\bigcup_{F\in\mathcal{F}} \mathcal{R}(F)$.

Theorem 59. *Let $\mathcal{T}$ and $\mathcal{F}$ be classes of matrices such that $\mathcal{T} = Av(\mathcal{F})$, then $Cl(\mathcal{T}) = Av(\mathcal{R}(\mathcal{F}))$.*

Proof. Need to change the proof a bit probably after changing the statement

- $\subseteq$ Instead of proving $M \in Cl(\mathcal{T}) \Rightarrow M \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ we show $M \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F)) \Rightarrow M \notin Cl(\mathcal{T})$. Assume $M \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$. It follow from the definition that $M \in \bigcup_{F\in\mathcal{F}} \mathcal{R}(F)$, in particular, $M \in \mathcal{R}(F)$ for some $f \in \mathcal{F}$. Because of the definition of $\mathcal{R}(F)$, M is not a direct sum with 1×1 overlap of proper submatrices of M which means, according to Observation 58, there are no non-trivial articulations and both top-right and bottom-left corners are empty. For contradiction, assume $M \in Cl(\mathcal{T})$, then, according to a generalization of Lemma 50, there exists a sequence of articulations of M such that each matrix in between two consecutive articulations of M is a minor of $(\,1\,) \nearrow T \nearrow (\,1\,)$ for some $T \in \mathcal{T}$. Since M has only trivial articulations and they are both empty, it holds $M \preceq T$ and because of the choice of M is also holds $M \preceq F$ for some $F \in \mathcal{F}$ which together give us a contradiction with $\mathcal{T} = Av(\mathcal{F})$.

- $\supseteq$ First of all, $Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ is closed under the direct sum with 1×1 overlap. For contradiction, assume there are $M_1, M_2 \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ but $M = M_1 \nearrow_1 M_2 \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$. Then there exists $F' \in \mathcal{R}(F)$ for some $F \in \mathcal{F}$ such that $F' \preceq M$. Because F' is not a direct sum with 1×1 overlap of proper submatrices of F', it follows that either $F' \preceq M_1$ or $F' \preceq M_2$ and since $F \preceq F'$ we have a contradiction.

  Now that we know that both sides are closed under the direct sum with 1×1 overlap it sufficient to show that the inclusion holds for any $M \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ that is not a direct sum with 1×1 overlap of proper submatrices of M. Such M does not contain (again from Observation 58) any non-trivial articulation and those trivial ones are empty. Because of that it holds $F \not\preceq M$ for every $F \in \mathcal{F}$; otherwise either $M \in \mathcal{R}(F)$ or its minor would be there. Therefore $M \in \mathcal{T}$ and also $M \in Cl(\mathcal{T})$.

□

Definition 20. Let T be a class of matrices. The *basis* of T is a set of all minimal (relating to minors) matrices that do not belong to T.

Corollary 60. *Let $\mathcal{T}$ and $\mathcal{F}$ be classes of matrices such that $\mathcal{T} = Av(\mathcal{F})$, then $\mathcal{R}(\mathcal{F})$ is a basis of $Cl(\mathcal{T})$.*

Proof. The proof follow directly from Theorem 59. □

A natural question follows, whether the closure under direct sum of a class with finite basis has final basis. We prove that this is not the case.

Definition 21. Let $Nucleus_1 = (\bullet)$ and for $n > 1$ let $Nucleus_n \in \{0,1\}^{n\times n+1}$ be a matrix described by the examples:
$Nucleus_2 = \begin{pmatrix} \bullet & \bullet & \\ & \bullet & \bullet \end{pmatrix}$, $Nucleus_3 = \begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & \bullet & \end{pmatrix}$, $Nucleus_4 = \begin{pmatrix} & & \bullet & \bullet & \\ & \bullet & & & \bullet \\ \bullet & & & \bullet & \\ & \bullet & \bullet & & \end{pmatrix}$,

$Nucleus_5 = \begin{pmatrix} & & & \bullet & \bullet & \\ & & \bullet & & & \bullet \\ & \bullet & & & \bullet & \\ \bullet & & & \bullet & & \\ & \bullet & \bullet & & & \end{pmatrix}$, $Nucleus_n = \begin{pmatrix} & & & & & & & \bullet & \bullet & \\ & & & & & & \bullet & & & \bullet \\ & & & & \bullet & & & & \bullet & \\ & & & \cdots & & & & \bullet & & \\ & & \bullet & & & \cdots & & & & \\ & \bullet & & & \bullet & & & & & \\ \bullet & & & \bullet & & & & & & \\ & \bullet & \bullet & & & & & & & \end{pmatrix}$.

Definition 22. Let $Candy_{k,n,l}$ be a matrix given by $I_k \nearrow_{1\times 2} Nucleus_n \nearrow_{1\times 2} I_l$, where I_k, I_l are unit matrices of sizes $k \times k$ and $l \times l$ respectively.

$$Candy_{4,1,4} \begin{pmatrix} & & & \bullet & & & \\ & & & & \bullet & & \\ & & & & & \bullet & \\ \bullet & & & \bullet & & & \bullet \\ & \bullet & & & & & \\ & & \bullet & & & & \\ & & & \bullet & & & \end{pmatrix} \quad Candy_{4,4,4} \begin{pmatrix} & & & & & \bullet & & & \\ & & & & & & \bullet & & \\ & & & & & & & \bullet & \\ & & & & \bullet & \bullet & & & \bullet \\ & & & \bullet & & & \bullet & & \\ & & \bullet & & & \bullet & & & \\ \bullet & & & \bullet & \bullet & & & & \\ & \bullet & & & & & & & \\ & & \bullet & & & & & & \\ & & & \bullet & & & & & \end{pmatrix}$$

Theorem 61. *There exists a matrix F such that $\mathcal{R}(F)$ is infinite.*

Proof. □

Corollary 62. *There exists a class of matrices $\mathcal{C}$ having a finite basis such that $Cl(\mathcal{C})$ has an infinite basis.*

Proof. From Theorem 61, we have a matrix F for which $\mathcal{R}(F)$ is infinite. Let $\mathcal{C} = Av(F)$. Clearly, $\mathcal{C}$ has a finite basis. On the other hand, from Theorem 59 we have $Cl(\mathcal{C}) = Av(\mathcal{R}(F))$ and $\mathcal{R}(F)$ is infinite from the choice of F. □