Definition 1. $P \leq M$ if M *contains* P *as a submatrix.*

Definition 2. $P \preceq M$ if M *contains* P *as an interval minor.*

Observation 1. $P \leq M \Rightarrow P \preceq M$.

Observation 2. *If* P *permutation matrix, then* $P \leq M \Leftrightarrow P \preceq M$.

0.1 Characterizations

Definition 3. For $M_1 \in \{0,1\}^{n\times k}$, $M_2 \in \{0,1\}^{n\times l}$ let $M \in \{0,1\}^{n\times k+l}$ be a *horizontal join of* M_1 *and* M_2, denoted by $M = M_1 \oplus_h M_2$, if the first k columns of M give M_1 and the last l columns of M give M_2.

Definition 4. A *walk* in a matrix M is a sequence of some of its entries, beginning in the top left corner and ending in the bottom right one. If an entry at the position $[i,j]$ is in the sequence, the next one is either $[i+1,j]$ or $[i,j+1]$.

Definition 5. We call a binary matrix M a *walking matrix* if there is a walk in M such that all the one-entries of M are contained on the walk.

0.1.1 Matrices of size 2×2

Theorem 3. *Let* $P = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$*, then for all* M*:* $P \not\preceq M \Leftrightarrow M$ *is a walking matrix.*

Proof. Since P is a permutation matrix, $P \not\preceq M \Leftrightarrow P \not\leq M$ and it is easy to see $P \not\leq M \Leftrightarrow M$ is a walking matrix. $\square$

Theorem 4. *Let* $P = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$*, then for all* M*:* $P \not\preceq M \Leftrightarrow M$ *looks like the matrix in Figure 1, where* M' *is a walking matrix.*

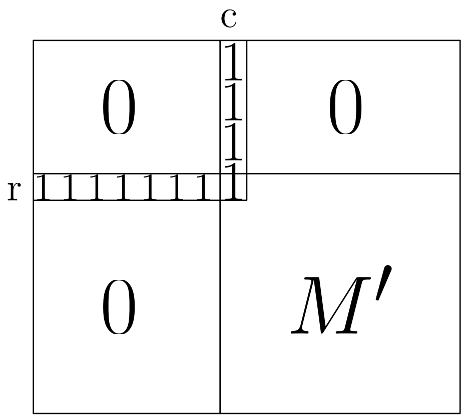

Figure 1: Characterization of a matrix avoiding $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ as an interval minor.

Proof. $\Rightarrow$ If $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M$, then $M = M'$ and we are done. Else there are one-entries at $[r,c']$ and $[r',c]$, where $r' < r$ and $c' < c$. If there was a one-entry in regions outside M', the r-th row and the c-th column, then $P \preceq M$, which would be a contradiction. If M' is not a walking matrix, then it contains $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and we again get a contradiction.

$\Leftarrow$ For contradiction, assume that M described in Figure 1 contains P as an interval minor. It means that there is a partition of the matrix into four quadrants such that there is at least one one-entry in each quadrant besides the bottom right one. If the matrix would be partitioned above the r-th row, then there would be only one column containing one-entries and it would not be possible for both top quadrants to have a one-entry. Similarly, if the matrix would be partitioned to the left of the c-th column, there would be only one row containing one-entries and there would not be one-entry in either top-left or bottom-left quadrant. Therefore, the partitioning lies bellow the r-th row and to the right of the c-th column, but if the quadrants contain one-entries, there is a $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ interval minor in M', which is a contradiction.

$\square$

Lemma 5. *TODO - lemma about existence of an element for which top-right and bottom-left submatrices are empty or symmetrically. I will need some kind of notation before I'm able to state and prove it. Was thinking about something like "an element is ... if the submatrix $M[< r, > c]$ is empty".*

Theorem 6. *Let $P = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like one of the matrices in Figure 2, where $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$, $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) \not\preceq M_2$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_3$ and $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) \not\preceq M_4$.*

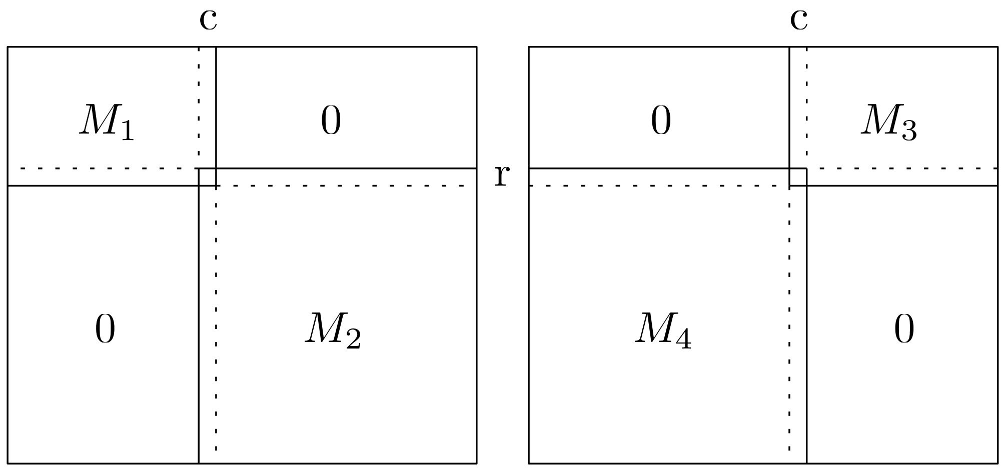

Figure 2: Characterization of a matrix avoiding $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ as an interval minor.

Proof.

$\Rightarrow$ We proceed by induction by the size of M.

If $M \in \{0,1\}^{2\times 2}$, then it either avoids $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ and we are done.

For bigger M, there is, from Lemma 5, "the element". Assume the first case (top-right and bottom-left empty (will change this when I have some notation)). If M_1 is non-empty, then $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) \not\preceq M_2$; otherwise, $P \preceq M$. Similarly, $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$ if M_2 is non-empty. If one of them is empty, the other is a smaller matrix avoiding P as an interval minor and by induction hypothesis, it can be partitioned. Adding empty rows and columns does not break any condition and we get a partitioning of the whole M.

$\Leftarrow$ Let us assume M looks like the left matrix in Figure 2, for the other one we would argue symmetrically. For contradiction, assume $P \preceq M$. In that case, we can partition M into four quadrants such that there is at least one one-entry in each of them. It does not matter where we partition it, every time we either get $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \preceq M_1$ or $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) \preceq M_2$, which is a contradiction. $\square$

0.1.2 Matrices of size 2×3

Theorem 7. *Let $P = \left(\begin{smallmatrix}1&0&1\\0&1&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M = M_1 \oplus_h M_2$, where $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ Let $e = [r, c]$ be the top-most one-entry of M. If there was $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ as an interval minor in the first $c-1$ columns of M, together with e it would be the whole P; therefore, it is not. If the rest of the columns besides the first $c-1$ avoid $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ as an interval minor, we are done. Let us assume it is not the case and $e_{0,0}, e_{1,1}$ be any two one-entries forming the forbidden pattern. Similarly, let the first c columns contain $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ as an interval minor (else $P \preceq M$) and $e_{0,1}, e_{1,0}$ be any two one-entries forming the forbidden pattern. Now if we take $e_{0,0}, e_{0,1}$ and a lower of $e_{1,0}$ and $e_{1,1}$ we get forbidden pattern P as an interval minor, which is a contradiction.

$\Leftarrow$ For contradiction let us assume $P \preceq M$ and $M = M_1 \oplus_h M_2$. If P is an interval minor of M, let us look where is the one-entry of M, where the bottom one of P can be mapped. If it is in M_1, then $P \not\preceq M$ because $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$. Similarly, if it is in M_2, then $P \not\preceq M$ because $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_2$.

$\square$

Lemma 8. *Let $P = \left(\begin{smallmatrix}1&1&1\\0&1&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Rightarrow M = M_1 \oplus_h M_2$, where $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$ or $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$.*

Proof. Let $e = [r, c]$ be the top-most one-entry of M. If there was $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ as an interval minor in the first $c-1$ columns of M, together with e it would be the whole P; therefore, it is not. Similarly, there is not $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ as an interval minor in the rest of columns besides the first c. Now, in order not to decompose M, the first c columns induce $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ as an interval minor, where $e_{0,0}$ and $e_{1,1}$ (non of them equal to e, since e lies in the top-right corner) are any two one-entries forming the pattern, and the rest of columns besides the first $c-1$ induce $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ as an interval minor and $e_{0,1}, e_{1,0}$ are any two one-entries forming the pattern. In that case $e_{0,0}, e, e_{0,1}$ and a lower one of $e_{1,0}$ and $e_{1,1}$ give us the forbidden pattern P as an interval minor, which is a contradiction; therefore there exists described decomposition of M. $\square$

Theorem 9. *Let $P = \left(\begin{smallmatrix}1&1&1\\0&1&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like the matrix in Figure 3 and $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ From Lemma 8 we know $M = M_1' \oplus_h M_2'$, where $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_1'$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2'$. The other case would be dealt with similarly. As Theorem 4 says, M_1' can be characterized exactly like the first $c_2 - 1$ columns of M and the rest then form a walking matrix. The only problem with our claim would be if there were two different columns having a one-entry above the

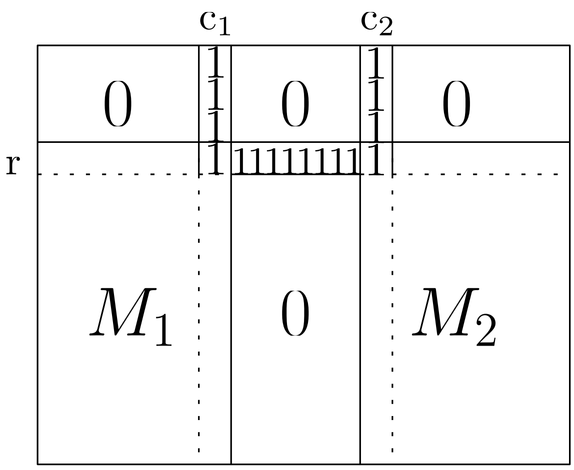

Figure 3: Characterization of a matrix avoiding $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ as an interval minor.

r-th row. In that case those two one-entries together with a one-entry in the r-th row between the columns c_1 and c_2 and a one-entry in the c_1-th column above the r-th row form P interval minor, which is a contradiction with $P \not\preceq M$.

$\Leftarrow$ The bottom-middle one-entry of P can not be mapped anywhere but to the r-th row, but in that case there are at most two columns having one-entries above it. (will do better hopefully)

□

0.2 Extremal function

Definition 6. $Ex(P, m, n) := \{oc(M) | M \in \{0,1\}^{m \times n}, P \not\leq M\}$

Definition 7. $Ex_{\preceq}(P, m, n) := \{oc(M) | M \in \{0,1\}^{m \times n}, P \not\preceq M\}$

Observation 10. *For all P: $Ex_{\preceq}(P, m, n) \leq Ex(P, m, n)$.*

Observation 11. *If $P \in \{0,1\}^{k \times l}$ has a one-entry at position $[a, b]$, then*

$$Ex(P, m, n) \geq \begin{array}{ll} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \textit{otherwise.} \end{array}$$

Observation 12. *The same holds for $Ex_{\preceq}(P, m, n)$.*

Definition 8. $P \in \{0,1\}^{k \times l}$ is *(strongly) minimalist* if

$$Ex(P, m, n) = \begin{array}{ll} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{array}$$

Definition 9. $P \in \{0,1\}^{k \times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P, m, n) = \begin{array}{ll} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{array}$$

Observation 13. *If P is strongly minimalist, then P is weakly minimalist.*

Fact 14. 1. $\begin{pmatrix} 1 \end{pmatrix}$ *is strongly minimalist.*

2. *If* $P \in \{0,1\}^{k \times l}$ *is strongly minimalist and there is a one-entry in the last row in the c-th column, then* $P' \in \{0,1\}^{k+1 \times l}$*, which is created from* P *by adding a new row having a one-entry only in the c-th row, is strongly minimalist.*

3. *If* P *is strongly minimalist, then after changing a one-entry into a zero-entry it is still strongly minimalist.*