Introduction

Throughout the paper, every time we speak about matrices we mean binary matrices (also called 01-matrices) and we omit the word binary. If we speak about a *pattern*, we again mean a binary matrix and we use the word in order to distinguish among more matrices as well as to indicate relationship.

When dealing with matrices, we always index rows and column starting with one and when we speak about a row r, we simply mean a row with index r. A *line* is a common word for both a row and a column. When we order a set of lines, we first put all rows and then all columns. For $M \in \{0,1\}^{m\times n}$, $[m]$ is a set of all rows and $[m+n]$ is a set of all lines, where m-th element is the last row. This goes with the usual notation.

Notation 1. For $n \in \mathbb{N}$ let $[n] := \{1, 2, \dots, n\}$ and for $m \in \mathbb{N}$, where $n \leq m$ let $[n, m] := \{n, n+1, \dots, m\}$.

Notation 2. For a matrix $M \in \{0,1\}^{m\times n}$ and $L \subseteq [m+n]$ let $M[L]$ denote a submatrix of M induced by lines in L.

Notation 3. For a matrix $M \in \{0,1\}^{m\times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M[R, C]$ denote a submatrix of M induced by rows in R and columns in C. Furthermore, for $r \in [m]$ and $c \in [n]$ let $M[r, c] := M[\{r\}, \{c\}] = M[\{r, c+m\}]$.

Definition 1. We say a matrix $M \in \{0,1\}^{m\times n}$ *contains* a pattern $P \in \{0,1\}^{k\times l}$ *as a submatrix* and denote it by $P \leq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M[R, C][r, c] = 1$.

This does not necessarily mean $P = M[R, C]$ as $M[R, C]$ can have more one-entries than P does.

Notation 4. For a matrix $M \in \{0,1\}^{m\times n}$ and $L \subseteq [m+n]$ let $M_{\preceq}[L]$ denote a matrix acquired from M by applying following operation for each $l \in L$:

- If l is the first row in L then we replace the first l rows by one row that is a bitwise OR of replaced rows.
- If l is the first column in L then we replace the first $l - m$ columns by one column that is a bitwise OR of replaced columns.
- Otherwise, we take l's predecessor $l' \in L$ in the standard ordering and replace lines $[l'+1, l]$ by one line that is a bitwise OR of replaced lines.

Notation 5. For a matrix $M \in \{0,1\}^{m\times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M_{\preceq}[R, C] := M_{\preceq}[R \cup \{c+m | c \in C\}]$.

Definition 2. We say a matrix $M \in \{0,1\}^{m\times n}$ *contains* a pattern $P \in \{0,1\}^{k\times l}$ *as an interval minor* and denote it by $P \preceq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M_{\preceq}[R, C][r, c] = 1$.

Observation 1. *For all matrices M and P, $P \leq M \Rightarrow P \preceq M$.*

Observation 2. *For all matrices M and P, if P is a permutation matrix, then $P \leq M \Leftrightarrow P \preceq M$.*

Proof. If we have $P \preceq M$, then there is a partitioning of M into rectangles and for each one-entry of P there is at least one one-entry in the corresponding rectangle of M. Since P is a permutation matrix, it is sufficient to take rows and columns having at least one one-entry in the right rectangle and we can always do so.

Together with Observation 1 this gives us the statement. $\square$

0.1 Characterizations

Definition 3. A *walk* in a matrix M is a sequence of some of its entries, beginning in the top left corner and ending in the bottom right one. If an entry $M[i,j]$ is in the sequence, the next one is either $M[i+1,j]$ or $M[i,j+1]$.

Definition 4. We call a binary matrix M a *walking matrix* if there is a walk in M such that all one-entries of M are contained on the walk.

Definition 5. An *extended walk of size* $k \times l$ in a matrix M is a subset of some of its entries, beginning in the top left corner and ending in the bottom right one. If an entry $M[i,j]$ is in the subset there is also either $M[i+1,j]$ or $M[i,j+1]$. The size describes that no more than k entries directly above each other are in the subset and no more than l entries directly next to each other are in the subset. We say that an extended walk of size $k \times l$ in M starts with a walk w, if the extended walk is a subset of entries of M that

- lie on w or below w and
- lie on w shifted by $k-1$ down and by $l-1$ to the left or above it.

Definition 6. For $M \in \{0,1\}^{m \times n}$ and $r \in [m], c \in [n]$ we say $M[r,c]$ is

- *top-left empty* if $M[[r-1],[c-1]]$ is an empty matrix,
- *top-right empty* if $M[[r-1],[c+1,n]]$ is empty,
- *bottom-left empty* if $M[[r-1],[c+1,n]]$ is empty,
- *bottom-right empty* if $M[[r-1],[c+1,n]]$ is empty.

0.1.1 Patterns of size 2×2 and their generalization

Theorem 3. *Let* $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$*, then for all* M*:* $P \not\preceq M \Leftrightarrow M$ *is a walking matrix.*

Proof. Since P is a permutation matrix, $P \not\preceq M \Leftrightarrow P \not\leq M$ and it is easy to see $P \not\leq M \Leftrightarrow M$ is a walking matrix. $\square$

Now consider a generalization of the pattern from above:

Theorem 4. *Let* $P \in \{0,1\}^{k \times l}$ *be a matrix having only two one-entries –* $P[1,n]$ *and* $P[m,1]$*, then for all* M*:* $P \not\preceq M \Leftrightarrow M$ *has an extended walk of size* $k-1 \times l-1$ *containing all one-entries.*

Proof. $\Rightarrow$ Let $P \not\preceq M$ and consider the left-most top-right empty elements of M. They necessarily form a walk w. For contradiction, assume there is a one-entry e below the extended walk of size $k-1 \times l-1$ starting with w. Since e is below the extended walk, there is an element e' - the right-most element of M that is neither below e nor to the right from e and at the same time still below the extended walk (it is possible $e = e'$). Let $e = M[r,c]$ and notice $M[r-k, c-l]$ is part of walk w and because of the choice of e' neither $M[r-k-1, c-l]$ nor $M[r-k, c-l-1]$ are on the walk w and $M[r-k, c-l]$ must be a one-entry; therefore, together with e it forms the forbidden pattern in M, which is a contradiction.

$\Leftarrow$ Let $M[r,c]$ be any one-entry of M, which then necessarily lie in the extended walk. Because the size of the walk is $k-1\times l-1$, $M[r-k+1, c-l+1]$ is top-left empty and $M[r+k-1, c+l-1]$ is bottom-right empty; therefore e cannot be a part of a mapping of P.

$\square$

Theorem 5. *Let $P = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, then for all $M \in \{0,1\}^{m\times n}$: $P \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 1)*

- $M[[r-1],[c-1]]$ *is empty,*
- $M[[r-1],[c+1,n]]$ *is empty,*
- $M[[r+1,m],[c-1]]$ *is empty and*
- $M[[r,m],[c,n]]$ *is a walking matrix.*

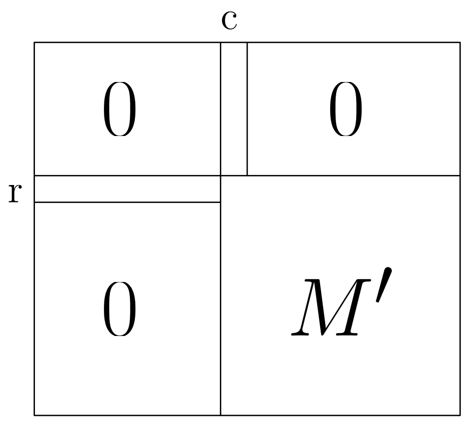

Figure 1: Characterization of a matrix avoiding $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ as an interval minor. Matrix M' is a walking matrix

Proof. $\Rightarrow$ If $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M$ then M is a walking matrix and we set $r = c = 1$. Otherwise, there are one-entries $M[r,c']$ and $M[r',c]$ such that $r' < r$ and $c' < c$. If there is a one-entry in regions $M[[r-1],[c-1]]$, $M[[r-1],[c+1,n]]$ or $M[[r+1,m],[c-1]]$ then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and we again get a contradiction.

$\Leftarrow$ For contradiction, assume that M described in Figure 1 contains P as an interval minor. It means that there is a partition of the matrix into four quadrants such that there is at least one one-entry in each quadrant besides the bottom right one. If the matrix is partitioned above the r-th row, then there is only one column containing one-entries and it is not possible for both top quadrants to have a one-entry. Similarly, if the matrix is partitioned to the left of the c-th column, there is only one row containing one-entries and there is no one-entry in either top-left or bottom-left quadrant. Therefore, the partitioning lies bellow the r-th row and to the right of the c-th column, but if the quadrants contain one-entries, there is a $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ interval minor in M', which is a contradiction with it being a walking matrix.

$\square$

Theorem 6. *Let $P \in \{0,1\}^{k \times l}$ be a matrix having only three one-entries – $P[1,1]$, $P[1,n]$ and $P[m,1]$, then for all M: $P \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 1 and imagine rows and columns being extended)*

- *$M[[r-1],[c-1]]$ is empty,*
- *$M[[r-1],[c+l,n]]$ is empty,*
- *$M[[r+k,m],[c-1]]$ is empty and*
- *$M[[r,m],[c,n]]$ has an extended walk of size $k-1 \times l-1$ containing all one-entries.*

Proof. Let $P' = P$ and set $P'[m,1] = 0$ (P' is a generalization of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$).

$\Rightarrow$ If $P' \not\preceq M$ then M is a matrix having an extended walk of size $k-1 \times l-1$ containing all one-entries and we set $r = c = 1$. Otherwise, there are one-entries $M[r_1,c_1]$ and $M[r_2,c_2]$ such that $r_2 < r_1$ and $c_1 < c_2$. We now choose $M[r_3,c_3]$ to be the bottom-most one-entry that still forms P' with $M[r_2,c_2]$. We choose $M[r_4,c_4]$ to be the left-most one-entry that forms P' with $M[r_3,c_3]$ and set $r = r_3-k+1$ and $c = c_4-l+1$. If there is a one-entry in regions $M[[r-1],[c-1]]$, $M[[r-1],\ [c+l,n]]$ or $M[[r+k,m],[c-1]]$ then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains P' and we again get a contradiction.

$\Leftarrow$ Because of the sizes of areas with no one-entries and the condition for $M[[r,m],[c,n]]$, there cannot be P' anywhere but in $M[[r+k-1],[c+l-1]]$. Since $M[[r-1][c-1]]$ is empty, there is no one-entry to map $P[1,1]$ to; therefore, $P \not\preceq M$.

$\square$

Lemma 7. *Let $P = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ and let $M \in \{0,1\}^{m \times n}$ avoid P as an interval minor, then there exists a row r and a column c such that $M[r,c]$ is either*

1. *a one-entry and $(r,c) \in \{(1,1),(1,n),(m,1),(m,n)\}$ or*
2. *both top-left empty and bottom-right empty and $(r,c) \notin \{(1,n),(m,1)\}$ or*
3. *both top-right empty and bottom-left empty and $(r,c) \notin \{(1,1),(m,n)\}$.*

Proof. If there is a one-entry in any corner we are done. Otherwise, let A be a set of all top-left empty entries of M and B be a set of all bottom-right empty entries of M. If there is an entry $M[r,c] \in A \cap B$ different from $(1,n)$ and $(m,1)$ we are done. Assume $A \cap B = \{(1,n),(m,1)\}$. Since $(m,1) \in A$, it also holds $(m-1,1) \in A$ and because it is not in the intersection we have $(m-1,1) \notin B$. This means $M[m-1,1]$ is not bottom-right empty; therefore there is a one-entry somewhere in $M[m,[2,n]]$. Moreover, no corner contains a one-entry so the is a one-entry in $M[m,[2,n-1]]$. For simplicity, we will say that the last row in non-empty (knowing the corners are empty). Symmetrically, we also get that the first row is non-empty and both the first and the last columns are non-empty. If there is a one-entry $M[r_l,1]$ in a different row than a one-entry $M[r_r,n]$ and at the same time a one-entry $M[1, c_t]$ in a different column than a one-entry $M[m, c_b]$ then these four one-entries form a mapping of the forbidden pattern P.

This is not true!!!

Without loss of generality assume there is only one one-entry in both the first and the last column and they are both in the same row r'. Let c' be a column such that there is a one-entry $M[1, c']$. Clearly, there is no other column that contains a one-entry above r', because we would again get a contradiction. Symmetrically, let c'' be the only column containing one-entries below r'. If $c' \geq c''$ we have that both $M[r', c']$ and $M[r', c'']$ are both top-left empty and bottom-right empty, which is a contradiction with $A \cap B = \{(1, n), (m, 1)\}$. Otherwise, $c' < c''$ and both $M[r', c']$ and $M[r', c'']$ are both top-right empty and bottom-left empty where $(r', c') \notin \{(1, 1), (m, n)\}$ which concludes the proof. $\square$

Theorem 8. *Let $P = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like one of the matrices in Figure 2, where $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M_1$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \not\preceq M_2$, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \not\preceq M_3$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \not\preceq M_4$.*

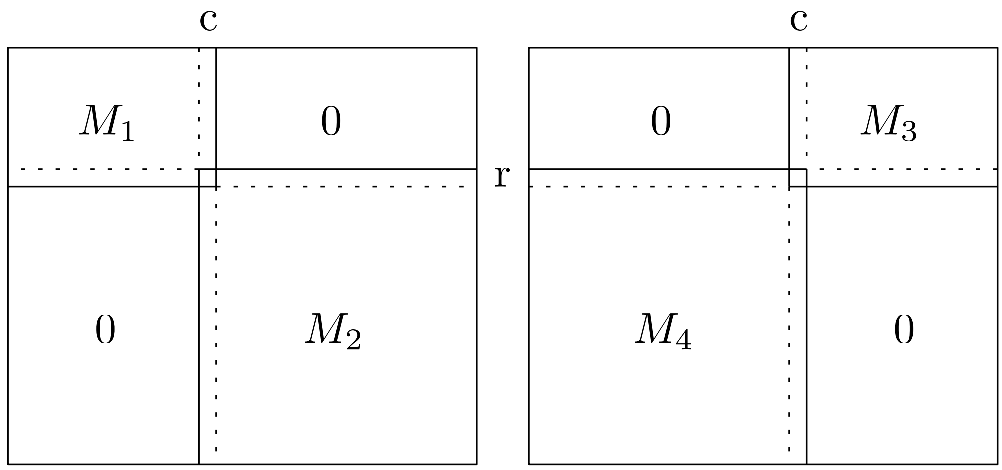

Figure 2: Characterization of a matrix avoiding $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ as an interval minor.

Proof.

$\Rightarrow$ We proceed by induction by the size of M.

If $M \in \{0, 1\}^{2\times 2}$ then it either avoids $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and we are done.

For bigger M there is, from Lemma 7, $M[r, c]$ satisfying some conditions. If it is the first condition – there is a one-entry in any corner, we are done because the matrix cannot contain one of the rotations of $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Assume the second case – $M[r, c]$ is both top-right and bottom-left empty and $(r, c) \notin \{(1, n), (m, 1)\}$. If M_1 is non-empty, then $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \not\preceq M_2$; otherwise, $P \preceq M$. Similarly, $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \not\preceq M_1$ if M_2 is non-empty. If one of them is empty, the other is a smaller matrix avoiding P as an interval minor and by induction hypothesis, it can be partitioned. Adding empty rows and columns does not break any condition and we get a partitioning of the whole M.

$\Leftarrow$ Without loss of generality, let us assume M looks like the left matrix in Figure 2. For contradiction, assume $P \preceq M$. In that case, we can partition M into four quadrants such that there is at least one one-entry in each of them. It does not matter where we partition it, every time we either get $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \preceq M_1$ or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \preceq M_2$, which is a contradiction. $\square$

Theorem 9. *Let $P \in \{0,1\}^{k \times l}$ be a matrix having only four one-entries – $P[1,1]$, $P[1,n]$, $P[m,1]$ and $P[m,n]$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like one of the matrices in Figure 2, where generalized $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$, $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) \not\preceq M_2$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_3$ and $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) \not\preceq M_4$.*

0.1.2 Matrices of size 2×3

Theorem 10. *Let $P = \left(\begin{smallmatrix}1&0&1\\0&1&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M = M_1 \oplus_h M_2$ where $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ Let $e = [r, c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \preceq M[[m], [c-1]]$, together with e it forms P. If $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M[[m], [c, n]]$ then we are done. Let us assume it is not the case and let $e_{0,0}$, $e_{1,1}$ be any two one-entries forming the forbidden pattern. Symmetrically, let $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \preceq M[[m], [c]]$ and let $e_{0,1}$, $e_{1,0}$ be any two one-entries forming the forbidden pattern. Now if we take $e_{0,0}$, $e_{0,1}$ and $e_{1,0}$ or $e_{1,1}$ with bigger row, we get the forbidden pattern P as an interval minor of M.

$\Leftarrow$ For contradiction, let us assume $P \preceq M$ and $M = M_1 \oplus_h M_2$. If $P \preceq M$, look at the one-entry of M where the bottom one-entry of P is mapped. If it is in M_1 then $P \not\preceq M$ because $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_1$. Otherwise, $P \not\preceq M$ because $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_2$.

□

Lemma 11. *Let $P = \left(\begin{smallmatrix}1&1&1\\0&1&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Rightarrow M = M_1 \oplus_h M_2$ where*

1. *$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$ or*
2. *$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$.*

Proof. Let $e = [r, c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \preceq M[[m], [c-1]]$, together with e it would be the whole P. Similarly, $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right) \not\preceq M[[m], [c+1, n]]$. For contradiction with the statement, let $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \preceq M[[m], [c]]$ and $e_{0,0}$, $e_{1,1}$ (none of them equal to e, since e lies in the top-right corner) be any two one-entries forming the pattern. Symmetrically, let $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \preceq M[[m], [c, n]]$ and $e_{0,1}$, $e_{1,0}$ be any two one-entries forming the pattern. In that case $e_{0,0}$, e, $e_{0,1}$ and $e_{1,0}$ or $e_{1,1}$ with bigger row give us the forbidden pattern P as an interval minor of M. □

Theorem 12. *Let $P = \left(\begin{smallmatrix}1&1&1\\0&1&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ looks like the matrix in Figure 3 and $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ From Lemma 11 we know $M = M_1' \oplus_h M_2'$ where $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \not\preceq M_1'$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \not\preceq M_2'$. The second case would be dealt with symmetrically. From Theorem 5 we have that M_1' can be characterized exactly like $M[[m], [c_2-1]$ and $M[[m], [c_2, n]]$ forms a walking matrix. The only problem with our claim would be if there were two different columns having a one-entry above the r-th row. In that case, those two one-entries together with a one-entry in the r-th row between the columns c_1 and c_2 and a one-entry in the c_1-th column above the r-th row form P as an interval minor.

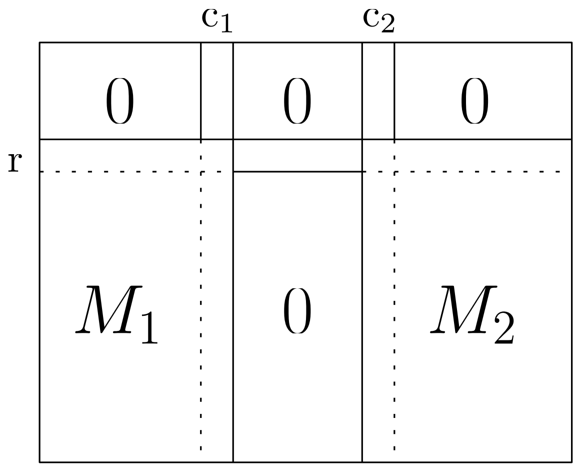

Figure 3: Characterization of a matrix avoiding $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ as an interval minor.

$\Leftarrow$ The bottom-middle one-entry of P can not be mapped anywhere but to the r-th row, but in that case there are at most two columns having one-entries above it.

□

Theorem 13. *Let $P = \left(\begin{smallmatrix}0&1&1\\1&0&0\end{smallmatrix}\right)$, then for all M: $P \not\preceq M \Leftrightarrow M$ contains a walk w, no one-entries below the walk and for each entry $M[r,c]$ of the walk there is at most one non-empty column in $M[[r-1],[c+1,n]]$.*

Proof. $\Rightarrow$ Let w be any walk containing all the top-most and right-most entries that are bottom-left empty. From the choice of w, there are no one-entries below it and if all $M[r,c]$, $M[r-1,c]$ and $M[r,c+1]$ are on w then $M[r,c]$ is a one-entry as else $M[r,c]$ was neither top-most nor right-most bottom-left empty. As a consequence, whenever we choose $M[r,c]$ from w, it either is a one-entry or there is one-entry in the same row to the left of it. For contradiction let us now assume that there is an entry of the walk $M[r,c]$ for which there are two non-empty columns in $M[[r-1],[c+1,m]]$. Then a one-entry from each of those columns and a one-entry in $M[r,c]$ or to the left of it together give us $P \preceq M$ and consequently a contradiction.

$\Leftarrow$ For contradiction let $P \preceq M$. Without loss of generality we can assume that the bottom-left entry of P is mapped somewhere to the walk – to $M[r,c]$. But then $\left(\begin{smallmatrix}1&1\end{smallmatrix}\right) \preceq M[[r-1],[c+1,n]]$ which is a contradiction with it having one-entries in at most one column.

□

0.1.3 Empty rows and columns

Observation 14. *Let $P \in \{0,1\}^{k\times l}$ and $P' \in \{0,1\}^{k\times l+1}$ such that $P' = P \oplus_h 0^{k\times 1}$, similarly let $M \in \{0,1\}^{m\times n}$ and $M' \in \{0,1\}^{m\times n+1}$ such that $M' = M \oplus_h 0^{m\times 1}$, then $P \preceq M \Leftrightarrow P' \preceq M'$.*

Proof. $\Rightarrow$ Clearly we can map the last column of P' to the last column of M' and then map (using OR) $P'[[k],[l]]$ to $M'[[m],[n]]$ the same way P is mapped to M.

$\Leftarrow$ If $P' \preceq M$ we are done. Otherwise, the last column of P' needs to be mapped to the last column of M' and by deleting both from their matrix we get $P'[[k],[l]] \preceq M'[[m],[n]]$ which is the same as $P \preceq M$.

$\square$

The same proof can be also used for adding an empty column as the first column or an empty row as the first or the last row. Using induction we can easily show that a pattern P' is avoided by a matrix M' if and only if P is avoided by M where P is derived from P' by excluding all empty beginning or ending rows and columns and M is derived from M' by excluding the same number of beginning or ending rows and columns. Therefore, when characterizing matrices avoiding a forbidden pattern, we do not need to consider patterns having empty rows or columns on their boundary.

For the following two statements, let $P \in \{0,1\}^{k\times 2}$ be a forbidden pattern and $P^+ \in \{0,1\}^{k\times 3}$ be the pattern created from P by adding a new column in between the two columns of P.

Lemma 15. *Let $M \in Av(P^+)$ be an inclusion maximal matrix in $Av(P^+)$, then each row of M contains one interval of one-entries of length different from one (but can be zero).*

Proof. If there are two one-entries on the same row and there is a zero-entry in between them, we can change the zero-entry into a one-entry and the new matrix will still avoid P^+.

If there is only one one-entry in a row, we can always change one of its neighbors into a one-entry. If the one-entry lies in the first column, we can insert another one-entry to the second column and the matrix will still avoid P^+, because if the new one-entry is used in a mapping of P^+ then it must as a part of the first column and we could use the one-entry to its left instead. Similarly, there is no one-entry in the last column that is the only one in it's row. For contradiction, assume there is the only one one-entry in a row, call it $e = M[r,c]$, that is not in the first nor in the last column and assume we cannot change neither $e_l = M[r,c-1]$ nor $e_r = M[r,c+1]$ to a one-entry, because then the matrix would contain the minor P^+.

TODO $\square$

Theorem 16. *For all $M \in \{0,1\}^{m\times n}$ it holds $M \in Av(P^+) \Leftrightarrow$ there exists $N \in \{0,1\}^{m\times(n-1)}$ such that $N \in Av(P)$ and M is a submatrix of $N \oplus_h 0^{m\times 1}$ placed over $0^{m\times 1} \oplus_h N$ with an operation bitwise OR.*

Proof. $\Rightarrow$ It suffices to only prove the statement for M that is inclusion maximal. To do so, we use Lemma 15. It say that each row of M contains either no one-entry or an interval of length at least two. From that we define N to be created from M by deleting the last one-entry on each row and excluding the last column. Clearly, M is equal to $N \oplus_h 0^{m\times 1}$ placed over $0^{m\times 1} \oplus_h N$ with an operation bitwise OR. If $P \preceq N$ then each mapping of P can be extended to a mapping of P^+ to M by ... How to say this?

TODO

$\Leftarrow$ It suffices to show that M that is equal to $N \oplus_h 0^{m\times 1}$ placed over $0^{m\times 1} \oplus_h N$ with an operation bitwise OR belongs to $Av(P^+)$. For contradiction, assume it does not. Then there is mapping of P^+ into elements of M and we can assume that one-entries of the first column of P^+ are mapped to those one-entries of M created from $N \oplus_h 0^{m\times 1}$. If it was not the case and there was a one-entry mapped to a one-entry of M created only from $0^{m\times 1} \oplus_h N$ we can take an element directly to its left and that is created from $N \oplus_h 0^{m\times 1}$. Symmetrically, all one-entries of the last column of P^+ are mapped to one-entries created from $0^{m\times 1} \oplus_h N$. ...

TODO

□

Open question:

- 2 rows, 1 interval of one-entries, 3 rows, 2 intervals... and generalization
- using the previous point to characterize insertion of an empty column to a bigger matrix
- insertion of an empty column in between all columns of P

0.1.4 Multiple patterns

Theorem 17. *Let* $P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ *and* $P_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$*, then for all* M*:* $P \not\preceq M \wedge P \not\preceq M \Leftrightarrow M$ *contains a walk* w *and each one-entry* e *is either on the walk* w *or both element directly above* e *and directly to the right of* e *are on the walk* w*.*

Proof. $\Rightarrow$ Let us take a walk w containing all the left-most and bottom-most top-right empty elements of M. Clearly, every top-right "corner" entry of w ($M[r,c]$ such that both $M[r+1,c]$ and $M[r,c-1]$ are on w) is a one-entry. Now consider for contradiction there is a one-entry anywhere but on w or directly diagonally below any top-right corner of w. Then this one-entry together with at least one top-right corner of w give us either P_1 or P_2 and thus a contradiction.

$\Leftarrow$ If we take any one-entry e, from the description of M there is no one-entry that would create either of P_1 or P_2 with e.

□

0.2 Extremal function

Notation 6. Let M be a matrix. We denote $|M|$ the weight of M, the number of one-entries in M.

Usually $|M|$ stands for a determinant of matrix M. However, in this paper we do not work with determinants at all so the notation should not lead to misunderstanding.

Definition 7. For a matrix P we define $Ex(P, m, n) := \max\{|M||M \in \{0,1\}^{m \times n}, P \not\prec M\}$. We denote $Ex(P, n) := Ex(P, n, n)$.

Definition 8. For a matrix P we define $Ex_{\preceq}(P, m, n) := max\{|M||M \in \{0,1\}^{m \times n}, P \not\preceq M\}$. We denote $Ex_{\preceq}(P, n) := Ex_{\preceq}(P, n, n)$.

Observation 18. *For all P, m, n; $Ex_{\preceq}(P, m, n) \leq Ex(P, m, n)$.*

Observation 19. *If $P \in \{0,1\}^{k \times l}$ has a one-entry at position $[a, b]$, then*

$$Ex(P, m, n) \geq \begin{cases} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \textit{otherwise.} \end{cases}$$

Observation 20. *The same holds for $Ex_{\preceq}(P, m, n)$.*

Definition 9. $P \in \{0,1\}^{k \times l}$ is *(strongly) minimalist* if

$$Ex(P, m, n) = \begin{cases} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Definition 10. $P \in \{0,1\}^{k \times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P, m, n) = \begin{cases} m \cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Observation 21. *If P is strongly minimalist, then P is weakly minimalist.*

0.2.1 Known results

Fact 22. 1. *$\left(1 \right)$ is strongly minimalist.*

2. *If $P \in \{0,1\}^{k \times l}$ is strongly minimalist and there is a one-entry in the last row in the c-th column, then $P' \in \{0,1\}^{k+1 \times l}$, which is created from P by adding a new row having a one-entry only in the c-th column, is strongly minimalist.*

3. *If P is strongly minimalist, then after changing a one-entry into a zero-entry it is still strongly minimalist.*

Fact 23. *Let $P = \left(\begin{smallmatrix} 1 \cdots 1 \\ 1 \ldots 1 \end{smallmatrix} \right)$ have l columns, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{2\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{weight of } M[[i],\{j\}] > 0 \wedge \text{weight of } M[[i+1,m],\{j\} > 0]\}$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M appears in at least $b_j - 1$ of sets A_i, $0 \le i \le m-2$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n}(b_j - 1) + n \le \sum_{i=0}^{m-2} |A_i| + n \le (l-1)(m-1) + n$$

□

This result is indeed very important because it shows that there are matrices like $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, which are weakly minimalist, although it is known they are not strongly minimalist.

Fact 24. *Let* $P = \left(\begin{smallmatrix}1\dots1\\1\dots1\\1\dots1\end{smallmatrix}\right)$ *have* l *columns, then* P *is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{3\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{ weight of } M[[i-1],\{j\}] > 0 \wedge \text{ weight of } M[[i+1,m],\{j\} > 0 \wedge M[i,j] \text{ one-entry}]\}$. Clearly $|A_i| \le l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M (for which $b_j \ge 2$) appears in exactly $b_j - 2$ of sets A_i, $1 \le i \le m-1$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n}(b_j - 2) + 2n \le \sum_{i=1}^{m-2} |A_i| + 2n \le (l-1)(m-2) + 2n$$

□

0.3 Operations with matrices

Notation 7. When speaking about a class of matrices, unless stated otherwise, we always expect the class to be closed under minors. Also, all classes discussed are non-trivial. That means that there is at least one matrix of size 2 by 1 and at least one matrix of size 1 by 2 in each class. Moreover, at least one matrix is non-empty.

Definition 11. Let $\mathcal{F}$ be any class of forbidden matrices. We denote by $Av(\mathcal{F})$ the set of all matrices that avoid every $F \in \mathcal{F}$ as an interval minor.

Observation 25. *Let $\mathcal{T} = Av(\mathcal{F})$ for some $\mathcal{F}$. Then $\mathcal{T}$ is closed under minors.*

Observation 26. *Let $\mathcal{M}$ be a finite class of matrices. There exists a finite set $\mathcal{F}$ such that $\mathcal{M} = Av(\mathcal{F})$.*

Definition 12. For matrices $A \in \{0,1\}^{m\times n}$ and $B \in \{0,1\}^{k\times l}$ we define their *direct sum* as a matrix $C := A \nearrow B \in \{0,1\}^{m+k\times n+l}$ such that $D[[k+1, m+k], [n]] = A$, $D[[k], [n+1, n+l]] = B$ and the rest is empty. Symmetrically, we define $D := A \searrow B \in \{0,1\}^{m+k\times n+l}$ such that $C[[m], [n]] = A$, $C[[m+1, m+k], [n+1, n+l]] = B$ and the rest is empty.

Theorem 27. $Av(\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)) = (Av(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)) \searrow Av(\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)) \searrow Av(\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right))) \cup$
$\cup\, (Av(\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)) \nearrow Av(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)) \nearrow Av(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)))$.

Proof. If follows from Theorem 8 and $Av(\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)) = Av(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)) \searrow Av(\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right))$. □

Notation 8. Let $\mathcal{M}$ be a class of matrices. Denote by $Cl(\mathcal{M})$ a set containing each $M \in \mathcal{M}$ closed under direct sum and minors.

Definition 13. Let $M \in \{0,1\}^{m\times n}$ be a matrix. We call an element $M[r,c]$ an *articulation* of M if both $M[[r-1],[c-1]]$ and $M[[r+1,m],[c+1,n]]$ are empty.

Lemma 28. *Let $M \in \{0,1\}^{k\times l}$, then for all $X \in \{0,1\}^{m\times n}$ it holds $X \in Cl(M) \Leftrightarrow$ there exists a sequence of articulations of X such that each matrix in between two consecutive articulations of X is a minor of $(\,1\,) \nearrow M \nearrow (\,1\,)$.*

Proof. $\Rightarrow$

$\Leftarrow$

□

Theorem 29. *For all $M \in \{0,1\}^{k\times l}$ there exists $\mathcal{F}$ finite such that $Cl(M) = Av(\mathcal{F})$.*

Proof. Using Lemma 28 □

Theorem 30. $Cl(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)) = Av\left(\left(\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)\right)$.

Proof. $\subseteq$

$\supseteq$

□

Theorem 31. *Cl* $\left(\left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)\right) = Av\left(\left(\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&0\\0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\\0&1\end{smallmatrix}\right)\right)$.

We can generalize direct sum to allow the matrices to overlap.

Definition 14. TODO $A \oplus_{k\times l} B$

Theorem 32. *Let $\mathcal{C}$ be any class of matrices such that*

- *$\mathcal{C}$ is closed under deleting of one-entries and*
- *$\mathcal{C}$ is closed under the direct sum with $k \times l$ overlap and*
- *there is any $M \in \{0,1\}^{m\times n}$ in $\mathcal{C}$*

then $\mathcal{C}$ is also closed under direct sum with $m-2k \times n-2l$ overlap.

Proof. Choose any two $A, B \in \mathcal{C}$ and $C\mathcal{C}$ such that $C \in \{0,1\}^{m\times n}$. Let $D \in \mathcal{C}$ denote the direct sum with $k \times l$ overlap of A and C. Finally, let E be the direct sum with $k \times l$ overlap of D and B. It has the same size as F, the direct sum with $m-2k \times n-2l$ overlap of A and B, which set of one-entries is also a subset of one-entries of $E \in \mathcal{C}$; therefore $F \in \mathcal{C}$. ☐

Theorem 33. *Let $\mathcal{C}$ be any class of matrices that is hereditary according to interval minors then for all m, n, k, l if $\mathcal{C}$ is closed under the direct sum with $m \times n$ overlap then is is also closed under the direct sum with $m+k \times n+l$ overlap.*

Proof. For contradiction, assume there are $A, B \in \mathcal{C}$ such that $A \oplus_{m+k\times n+l} B \notin \mathcal{C}$. ☐

Observation 34. *There is a $\mathcal{C}$ hereditary according to submatrices such that it is closed under the direct sum but it is not closed under the direct sum with 1×1 overlap.*

Proof. Let $\mathcal{C}$ be a class of all matrices obtained by applying the direct sum on $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Clearly, it is closed under the direct sum. On the other hand, $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \oplus_{1\times 1} \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1&0\\1&0&1\\0&1&0\end{smallmatrix}\right) \notin \mathcal{C}$. ☐

Notation 9. We define $\mathrm{Av}(M)$ to be a class of all matrices avoiding M as

We state following characterization only for the direct sum with 1×1 overlap but, because of Theorem 33, it also holds for any other size of overlap.

Theorem 35. *Let M be a matrix. There are M_1, M_2 proper submatrices of M such that $M = M_1 \oplus_{1\times 1} M_2 \Leftrightarrow$ Av(M) is not closed under the direct sum with 1×1 overlap.*

Proof. $\Rightarrow$

$\Leftarrow$

☐

Observation 36. *Let M be a matrix. There are M_1, M_2 proper submatrices of M such that $M = M_1 \oplus_{1\times 1} M_2 \Leftrightarrow$ exists r, c such that either*

1. *$M[r,c]$ is a one-entry and $(r,c) \in \{(1,1),(m,n)\}$ or*

2. *$M[r,c]$ is both top-right and bottom-left empty and $(r,c) \notin \{(1,1),(m,n)\}$*

Definition 15. Let F be a matrix. We denote $\mathcal{R}(F)$ to be a set of all minimal (relating to minors) matrices F' such that $F \preceq F'$ and F' is not a direct sum with 1×1 overlap of proper submatrices of F'. For a class of matrices $\mathcal{F}$ let $\mathcal{R}(\mathcal{F})$ denote a set of all minimal (relating to minors) matrices from the set $\bigcup_{F\in\mathcal{F}} \mathcal{R}(F)$.

Theorem 37. *Let $\mathcal{T}$ and $\mathcal{F}$ be classes of matrices such that $\mathcal{T} = Av(\mathcal{F})$, then $Cl(\mathcal{T}) = Av(\mathcal{R}(\mathcal{F}))$.*

Proof. Need to change the proof a bit probably after changing the statement

- $\subseteq$ Instead of proving $M \in Cl(\mathcal{T}) \Rightarrow M \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ we show $M \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F)) \Rightarrow M \notin Cl(\mathcal{T})$. Assume $M \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$. It follow from the definition that $M \in \bigcup_{F\in\mathcal{F}} \mathcal{R}(F)$, in particular, $M \in \mathcal{R}(F)$ for some $f \in \mathcal{F}$. Because of the definition of $\mathcal{R}(F)$, M is not a direct sum with 1×1 overlap of proper submatrices of M which means, according to Observation 36, there are no non-trivial articulations and both top-right and bottom-left corners are empty. For contradiction, assume $M \in Cl(\mathcal{T})$, then, according to a generalization of Lemma 28, there exists a sequence of articulations of M such that each matrix in between two consecutive articulations of M is a minor of $\begin{pmatrix} 1 \end{pmatrix} \nearrow T \nearrow \begin{pmatrix} 1 \end{pmatrix}$ for some $T \in \mathcal{T}$. Since M has only trivial articulations and they are both empty, it holds $M \preceq T$ and because of the choice of M is also holds $M \preceq F$ for some $F \in \mathcal{F}$ which together give us a contradiction with $\mathcal{T} = Av(\mathcal{F})$.

- $\supseteq$ First of all, $Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ is closed under the direct sum with 1×1 overlap. For contradiction, assume there are $M_1, M_2 \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ but $M = M_1 \nearrow_1 M_2 \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$. Then there exists $F' \in \mathcal{R}(F)$ for some $F \in \mathcal{F}$ such that $F' \preceq M$. Because F' is not a direct sum with 1×1 overlap of proper submatrices of F', it follows that either $F' \preceq M_1$ or $F' \preceq M_2$ and since $F \preceq F'$ we have a contradiction.

  Now that we know that both sides are closed under the direct sum with 1×1 overlap it sufficient to show that the inclusion holds for any $M \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ that is not a direct sum with 1×1 overlap of proper submatrices of M. Such M does not contain (again from Observation 36) any non-trivial articulation and those trivial ones are empty. Because of that it holds $F \npreceq M$ for every $F \in \mathcal{F}$; otherwise either $M \in \mathcal{R}(F)$ or its minor would be there. Therefore $M \in \mathcal{T}$ and also $M \in Cl(\mathcal{T})$.

□

Definition 16. Let T be a class of matrices. The *basis* of T is a set of all minimal (relating to minors) matrices that do not belong to T.

Corollary. Let $\mathcal{T}$ and $\mathcal{F}$ be classes of matrices such that $\mathcal{T} = Av(\mathcal{F})$, then $\mathcal{R}(\mathcal{F})$ is a basis of $Cl(\mathcal{T})$.

Proof. The proof follow directly from Theorem 37. □

A natural question follows, whether the closure under direct sum of a class with finite basis has final basis. We prove that this is not the case.

Definition 17. Let $Nucleus_n$ be a matrix described by the examples below

$$Nucleus_1 = \begin{pmatrix} 1 \end{pmatrix},\ Nucleus_2 = \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \end{pmatrix},\ Nucleus_3 = \begin{pmatrix} & 1 & 1 & \\ 1 & & & 1 \\ & 1 & 1 & \end{pmatrix},\ Nucleus_4 =$$
$$\begin{pmatrix} & & 1 & 1 & \\ & 1 & & & 1 \\ 1 & & & 1 & \\ & 1 & 1 & & \end{pmatrix},\ Nucleus_n = \begin{pmatrix} & & & & & & 1 & 1 & \\ & & & & & 1 & & & 1 \\ & & & & 1 & & & 1 & \\ & & & \cdots & & & 1 & & \\ & & 1 & & & \cdots & & & \\ & 1 & & & 1 & & & & \\ 1 & & & 1 & & & & & \\ & 1 & 1 & & & & & & \end{pmatrix}$$

Definition 18. Let $Candy_{k,n,l}$ be a matrix given by $I_k \nearrow_{1\times 2} Nucleus_n \nearrow_{1\times 2} I_l$, where I_k, I_l are unit matrices of sizes $k \times k$ and $l \times l$ respectively.

$$Candy_{4,1,4} \begin{pmatrix} & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ 1 & & & 1 & & & 1 \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \end{pmatrix} \quad Candy_{4,4,4} \begin{pmatrix} & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & 1 & 1 & & & 1 \\ & & & 1 & & & 1 & & \\ & & 1 & & & 1 & & & \\ 1 & & & 1 & 1 & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \end{pmatrix}$$

Theorem 38. *There exists a matrix F such that $\mathcal{R}(F)$ is infinite.*

Proof. □

Corollary. There exists a class of matrices $\mathcal{C}$ having a finite basis such that $Cl(\mathcal{C})$ has an infinite basis.

Proof. From Theorem 38, we have a matrix F for which $\mathcal{R}(F)$ is infinite. Let $\mathcal{C} = Av(F)$. Clearly, $\mathcal{C}$ has a finite basis. On the other hand, from Theorem 37 we have $Cl(\mathcal{C}) = Av(\mathcal{R}(F))$ and $\mathcal{R}(F)$ is infinite from the choice of F. □