1. Introduction

Throughout the paper, every time we speak about matrices we mean binary matrices (also called 01-matrices) and we omit the word binary. If we speak about a *pattern*, we again mean a binary matrix and we use the word in order to distinguish among more matrices as well as to indicate relationship.

When dealing with matrices, we always index rows and column starting with one and when we speak about a row r, we simply mean a row with index r. A *line* is a common word for both a row and a column. When we order a set of lines, we first put all rows and then all columns. For $M \in \{0,1\}^{m\times n}$, $[m]$ is a set of all rows and $[m+n]$ is a set of all lines, where m-th element is the last row. This goes with the usual notation.

Notation 1. For $n \in \mathbb{N}$ let $[n] := \{1, 2, \ldots, n\}$ and for $m \in \mathbb{N}$, where $n \leq m$ let $[n, m] := \{n, n+1, \ldots, m\}$.

Notation 2. For a matrix $M \in \{0,1\}^{m\times n}$ and $L \subseteq [m+n]$ let $M[L]$ denote a submatrix of M induced by lines in L.

Notation 3. For a matrix $M \in \{0,1\}^{m\times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M[R, C]$ denote a submatrix of M induced by rows in R and columns in C. Furthermore, for $r \in [m]$ and $c \in [n]$ let $M[r, c] := M[\{r\}, \{c\}] = M[\{r, c+m\}]$.

Definition 1. We say a matrix $M \in \{0,1\}^{m\times n}$ *contains* a pattern $P \in \{0,1\}^{k\times l}$ *as a submatrix* and denote it by $P \leq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M[R, C][r, c] = 1$.

This does not necessarily mean $P = M[R, C]$ as $M[R, C]$ can have more one-entries than P does.

Notation 4. For a matrix $M \in \{0,1\}^{m\times n}$ and $L \subseteq [m+n]$ let $M_{\preceq}[L]$ denote a matrix acquired from M by applying following operation for each $l \in L$:

- If l is the first row in L then we replace the first l rows by one row that is a bitwise OR of replaced rows.
- If l is the first column in L then we replace the first $l - m$ columns by one column that is a bitwise OR of replaced columns.
- Otherwise, we take l's predecessor $l' \in L$ in the standard ordering and replace lines $[l'+1, l]$ by one line that is a bitwise OR of replaced lines.

Notation 5. For a matrix $M \in \{0,1\}^{m\times n}$, $R \subseteq [m]$ and $C \subseteq [n]$ let $M_{\preceq}[R, C] := M_{\preceq}[R \cup \{c + m | c \in C\}]$.

Definition 2. We say a matrix $M \in \{0,1\}^{m\times n}$ *contains* a pattern $P \in \{0,1\}^{k\times l}$ *as an interval minor* and denote it by $P \preceq M$ if there are $R \in [m]$ and $C \in [n]$ such that $|R| = k$, $|C| = l$ and for every $r \in R$ and $c \in C$ if $P[r, c] = 1$, then $M_{\preceq}[R, C][r, c] = 1$.

Observation 1. *For all matrices M and P, $P \leq M \Rightarrow P \preceq M$.*

Observation 2. *For all matrices M and P, if P is a permutation matrix, then $P \leq M \Leftrightarrow P \preceq M$.*

Proof. If we have $P \preceq M$, then there is a partitioning of M into rectangles and for each one-entry of P there is at least one one-entry in the corresponding rectangle of M. Since P is a permutation matrix, it is sufficient to take rows and columns having at least one one-entry in the right rectangle and we can always do so.

Together with Observation 1 this gives us the statement. □

Observation 3. *Let $M \in \{0,1\}^{m \times n}$ and $P \in \{0,1\}^{k \times l}$, $P \preceq M \Leftrightarrow P^T \preceq M^T$.*

Because of this observation we will usually only show results only for rows or columns and expect both to hold and only show results for $P \in \{0,1\}^{k \times l}$ but assume the symmetrical results for P^T.

2. Characterizations

Definition 3. A *walk* in a matrix M is a sequence of some of its entries, beginning in the top left corner and ending in the bottom right one. If an entry $M[i,j]$ is in the sequence, the next one is either $M[i+1,j]$ or $M[i,j+1]$. A *reverse walk* in M is a sequence of some of its entries, beginning in the top right corner and ending in the bottom left one. In an entry $M[i,j]$ is in the sequence, the next one is either $M[i+1,j]$ or $M[i,j-1]$.

Definition 4. We call a binary matrix M a *walking matrix* if there is a walk in M such that all one-entries of M are contained on the walk.

Definition 5. For $M \in \{0,1\}^{m\times n}$ and $r \in [m], c \in [n]$ we say $M[r,c]$ is

- *top-left empty* if $M[[r-1],[c-1]]$ is an empty matrix,
- *top-right empty* if $M[[r-1],[c+1,n]]$ is empty,
- *bottom-left empty* if $M[[r-1],[c+1,n]]$ is empty,
- *bottom-right empty* if $M[[r-1],[c+1,n]]$ is empty.

Definition 6. For $M \in \{0,1\}^{m\times n}$ and $M' \in \{0,1\}^{m\times l}$ we define $M \to M' \in \{0,1\}^{m\times(n+l)}$ to be a matrix created by extending M by adding columns of M'.

2.1 Empty rows and columns

Observation 4. *For any $P \in \{0,1\}^{k\times l}$ let $P' = P \oplus_h 0^{k\times 1}$, and for any $M \in \{0,1\}^{m\times n}$ let $M' = M \oplus_h 1^{m\times 1}$, then $P \preceq M \Leftrightarrow P' \preceq M'$.*

Proof. $\Rightarrow$ Clearly, we can map the last column of P' just to the last column of M' and then map $P'[[k],[l]]$ to $M'[[m],[n]]$ the same way P is mapped to M.

$\Leftarrow$ Holds trivially, because we can take the restriction of the mapping of P' to M' to get a mapping of P to M.

□

The same proof can be used for adding an empty column as the first column or an empty row as the first or the last row. Using induction we can easily show that a pattern P' is avoided by a matrix M' if and only if P is avoided by M where P is derived from P' by excluding all empty leading or ending rows and columns and M is derived from M' by excluding the same number of leading or ending rows and columns. Therefore, when characterizing matrices avoiding a forbidden pattern, we do not need to consider patterns having empty rows or columns on their boundary.

The following theorem shows what happens when we add empty columns in between two columns of a pattern that only has two columns. The proof comes in Chapter 3, where we introduce several tools useful for proving it.

Theorem 5. *For all $M \in \{0,1\}^{m\times n}$ it holds $M \in Av(P^l) \Leftrightarrow$ there exists $N \in \{0,1\}^{m\times(n-l)}$ such that $N \in Av(P)$ is inclusion maximal and M is a submatrix of $N \oplus_h 0^{m\times l}$ placed over $0^{m\times l} \oplus_h N$ with elementwise OR.*

Open questions

- insertion of an empty column in between all columns of P

Next, we characterize matrices avoiding some small patterns. Because of the above results, we also characterize some of their generalizations and we completely omit empty lines in them. If $P \not\preceq M$ then also $P^T \not\preceq M^T$ and this holds for all rotations and mirrors of P and M and so we only mention these symmetries and do not prove all characterizations one by one.

2.2 Patterns having two one-entries and their generalization

These are up to rotation and mirroring the only patterns having two one-entries and no empty lines:

$$P_1 = \begin{pmatrix} \bullet & \bullet \end{pmatrix} \quad P_2 = \begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$$

They can be generalized to:

$$P_1' = \begin{pmatrix} \bullet & \cdots & \bullet \end{pmatrix} \quad P_2' = \begin{pmatrix} & & \bullet \\ & \cdots & \\ \bullet & & \end{pmatrix}$$

Theorem 6. *For all matrices M: $P_1 \not\preceq M \Leftrightarrow M$ has at most one non-empty column.*

Proof. $\Leftarrow$ M having at most one non-empty column does not contain P_1.

$\Rightarrow$ When M has two columns c_1, c_2 having a one-entry $M[r_1, c_1], M[r_2, c_2]$ respectively, those give us a mapping of P_1.

□

Theorem 7. *Let $P_1' = \{1\}^{1\times k}$. For all matrices M: $P_1' \not\preceq M \Leftrightarrow M$ has at most $k-1$ non-empty columns.*

Proof. $\Leftarrow$ M having at most $k-1$ non-empty columns does not contain P_1'.

$\Rightarrow$ When M has k columns $c_1, c_2, \ldots, c_k$ each having a one-entry $M[r_1, c_1]$, $M[r_2, c_2], \ldots, M[r_k, c_k]$ respectively, those give us a mapping of P_1'.

□

Theorem 8. *For all matrices M: $P_2 \not\preceq M \Leftrightarrow M$ is a walking matrix.*

Proof. $\Leftarrow$ a walking matrix does not contain P_2.

$\Rightarrow$ When M is not a walking pattern then there are two one-entries that cannot be in the same walk and those give us a mapping of P_2.

□

Theorem 9. *Let $P_2' \in \{0,1\}^{k\times k}$. For all matrices M: $P_2' \not\preceq M \Leftrightarrow M$ contains one-entries in at most $k-1$ walks.*

Proof. $\Leftarrow$ M containing one-entries in at most $k-1$ walks does not contain P_2'.

$\Rightarrow$ When one-entries of M cannot fit into $k-1$ walks, then there are k one-entries where no pair can fit to a single walk and those giving us a mapping of P_2'.

$\square$

2.3 Patterns having three one-entries and their generalization

These are up to rotation and mirroring the only patterns having three one-entries and no empty lines that we did not characterize so far:

$$P_3 = \begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix} \quad P_4 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix} \quad P_5 = \begin{pmatrix} \bullet & & \bullet \\ & \bullet & \end{pmatrix} \quad P_6 = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \end{pmatrix}$$

Theorem 10. *For all matrices $M \in \{0,1\}^{m \times n}$: $P_3 \not\preceq M \Leftrightarrow$ there exist a row r and a column c such that (see Figure 2.1)*

- $M[[r-1],[c-1]]$ *is empty,*
- $M[[r-1],[c+1,n]]$ *is empty,*
- $M[[r+1,m],[c-1]]$ *is empty and*
- $M[[r,m],[c,n]]$ *is a walking matrix.*

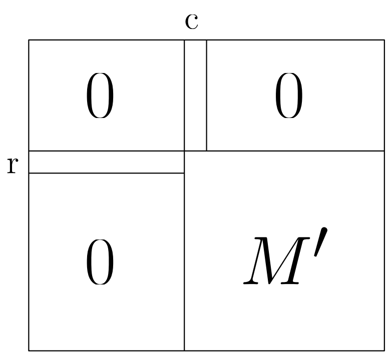


Figure 2.1: Characterization of matrices avoiding $\begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix}$ as an interval minor. Matrix M' is a walking matrix.

Proof. $\Rightarrow$ If M is a walking matrix then we set $r = c = 1$. Otherwise, there are one-entries $M[r, c']$ and $M[r', c]$ such that $r' < r$ and $c' < c$. If there is a one-entry in $M[[r-1],[c-1]]$, $M[[r-1], [c+1,n]]$ or $M[[r+1,m],[c-1]]$ then $P \preceq M$. If $M[[r,m],[c,n]]$ is not a walking matrix then it contains $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ and together with $M[r, c']$ it gives us the forbidden pattern.

$\Leftarrow$ For contradiction, assume that M described in Figure 2.1 contains P_3 as an interval minor. Without loss of generality we can assume $P_3[1,1]$ is mapped to the r-th row. But then both $P_3[1,2]$ and $P_3[2,1]$ need to be mapped to M' which is a contradiction with it being a walking matrix.

$\square$

Theorem 11. *For all matrices M: $P_4 \not\preceq M \Leftrightarrow$ for the top-left most reverse walk w in M such that there are no one-entries underneath it and for every one-entry $M[r,c]$ on w it holds $M[[r-1],[c-1]]$ is a walking matrix.*

Proof. $\Rightarrow$ For contradiction assume there are r, c such that $M[r,c]$ is a one-entry of w and $M[[r-1],[c-1]]$ is not a walking matrix. It means that $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \preceq M[[r-1],[c-1]]$ and together with $M[r,c]$ it gives us the forbidden pattern and a contradiction.

$\Leftarrow$ For contradiction let $P_4 \preceq M$ and consider a mapping of P_4, where $P_4[3,3]$ is mapped to $M[r,c]$ and there is no other one-entry in $M[[r,m],[c,n]]$. Clearly, $M[r,c]$ cannot lie on w, because then $M[[r],[c]]$ is a walking matrix and so $M[r,c]$ cannot be used to map $P_4[3,3]$. So $M[r,c]$ lies above w but that is a contradiction with w being top-left most reverse walk in M without one-entries underneath it.

$\square$

Theorem 12. *For all matrices M: $P_5 \not\preceq M \Leftrightarrow M = M_1 \rightarrow M_2$ where $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right) \not\preceq M_1$ and $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \not\preceq M_2$.*

Proof. $\Rightarrow$ Let $e = [r,c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right) \preceq M[[m],[c-1]]$, together with e it forms P_5. If $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \not\preceq M[[m],[c,n]]$ then we are done. Let us assume it is not the case and let $e_{1,1},\ e_{2,2}$ be any two one-entries forming the forbidden pattern. Symmetrically, let $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right) \preceq M[[m],[c]]$ and let $e_{1,2},\ e_{2,1}$ be any two one-entries forming the forbidden pattern. If we take $e_{1,1},\ e_{1,2}$ and $e_{2,1}$ or $e_{2,2}$ with bigger row, we get P_5 as an interval minor of M.

$\Leftarrow$ For contradiction, let us assume $P_5 \preceq M$. Let us look at the one-entry of M where $P_5[2,2]$ is mapped. If it is in M_1 then $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right) \preceq M_1$ and we get a contradiction. Otherwise we have $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right) \preceq M_2$ which is again a contradiction.

$\square$

Theorem 13. *For all matrices M: $P_6 \not\preceq M \Leftrightarrow$ for the top-right most walk w in M such that there are no one-entries underneath it and for every one-entry $M[r,c]$ on w there is at most one non-empty column in $M[[r-1],[c+1,n]]$.*

Proof. $\Rightarrow$ For contradiction assume that there is a one-entry of the walk $M[r,c]$ for which there are two non-empty columns in $M[[r-1],[c+1,m]]$. Then a one-entry from each of those columns and a one-entry in $M[r,c]$ together give us $P_6 \preceq M$ and a contradiction.

$\Leftarrow$ For contradiction let $P_6 \preceq M$. Without loss of generality $P_6[2,1]$ is mapped $M[r,c]$ which lies on w. But then $(\bullet\,\bullet) \preceq M[[r-1],[c+1,n]]$ which is a contradiction with it having one-entries in at most one column.

$\square$

2.4 Patterns having four one-entries

These are some of the patterns having four one-entries and no empty lines that we did not characterize so far:

$$P_7 = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} \quad P_8 = \begin{pmatrix} \bullet & \bullet & \bullet \\ & \bullet & \end{pmatrix} \quad P_9 = \begin{pmatrix} & \bullet & & \\ \bullet & & & \\ & & & \bullet \\ & & \bullet & \end{pmatrix}$$

TODO Still need to go through this and fix it.

Lemma 14. *For any matrix M: $P_7 \not\preceq M \Rightarrow$ there exists a row r and a column c such that $M[r, c]$ is either*

1. *a one-entry and $(r, c) \in \{(1, 1), (1, n), (m, 1), (m, n)\}$ or*
2. *both top-left empty and bottom-right empty and $(r, c) \notin \{(1, n), (m, 1)\}$ or*
3. *both top-right empty and bottom-left empty and $(r, c) \notin \{(1, 1), (m, n)\}$.*

Proof. If there is a one-entry in any corner we are done. Otherwise, let A be a set of all top-left empty entries of M and B be a set of all bottom-right empty entries of M. If there is an entry $M[r, c] \in A \cap B$ different from $(1, n)$ and $(m, 1)$ we are done. Assume $A \cap B = \{(1, n), (m, 1)\}$. Since $(m, 1) \in A$, it also holds $(m - 1, 1) \in A$ and because it is not in the intersection we have $(m - 1, 1) \notin B$. This means $M[m - 1, 1]$ is not bottom-right empty; therefore there is a one-entry somewhere in $M[m, [2, n]]$. Moreover, no corner contains a one-entry so the is a one-entry in $M[m, [2, n - 1]]$. For simplicity, we will say that the last row in non-empty (knowing the corners are empty). Symmetrically, we also get that the first row is non-empty and both the first and the last columns are non-empty. If there is a one-entry $M[r_l, 1]$ in a different row than a one-entry $M[r_r, n]$ and at the same time a one-entry $M[1, c_t]$ in a different column than a one-entry $M[m, c_b]$ then these four one-entries form a mapping of the forbidden pattern P_7.

This is not true!!!

Without loss of generality assume there is only one one-entry in both the first and the last column and they are both in the same row r'. Let c' be a column such that there is a one-entry $M[1, c']$. Clearly, there is no other column that contains a one-entry above r', because we would again get a contradiction. Symmetrically, let c'' be the only column containing one-entries below r'. If $c' \geq c''$ we have that both $M[r', c']$ and $M[r', c'']$ are both top-left empty and bottom-right empty, which is a contradiction with $A \cap B = \{(1, n), (m, 1)\}$. Otherwise, $c' < c''$ and both $M[r', c']$ and $M[r', c'']$ are both top-right empty and bottom-left empty where $(r', c') \notin \{(1, 1), (m, n)\}$ which concludes the proof. □

Theorem 15. *For all matrices M: $P_7 \not\preceq M \Leftrightarrow M$ looks like one of the matrices in Figure 2.2, where $\begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix} \not\preceq M_1$, $\begin{pmatrix} & \bullet \\ \bullet & \bullet \end{pmatrix} \not\preceq M_2$, $\begin{pmatrix} \bullet & \bullet \\ & \bullet \end{pmatrix} \not\preceq M_3$ and $\begin{pmatrix} \bullet & \\ \bullet & \bullet \end{pmatrix} \not\preceq M_4$.*

Proof.

$\Rightarrow$ We proceed by induction on the size of M.

If $M \in \{0, 1\}^{2\times 2}$ then it either avoids $\begin{pmatrix} & \bullet \\ \bullet & \bullet \end{pmatrix}$ or $\begin{pmatrix} \bullet & \bullet \\ \bullet & \end{pmatrix}$ and we are done.

For bigger M there is, from Lemma 14, $M[r, c]$ satisfying some conditions. If there is a one-entry in any corner, we are done because the matrix cannot

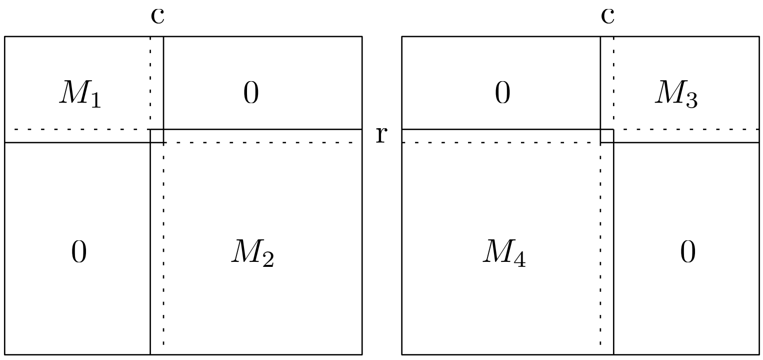


Figure 2.2: Characterization of matrices avoiding $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\bullet\end{smallmatrix}\right)$ as an interval minor.

contain one of the rotations of $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right)$. Otherwise, assume $M[r,c]$ is both top-right and bottom-left empty and $(r,c) \notin \{(1,n),(m,1)\}$. If M_1 is non-empty, then $\left(\begin{smallmatrix}&\bullet\\\bullet&\bullet\end{smallmatrix}\right) \npreceq M_2$. Symmetrically, $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right) \npreceq M_1$ if M_2 is non-empty. If one of them is empty, the other is a smaller matrix avoiding P as an interval minor and by induction hypothesis, it can be partitioned.

$\Leftarrow$ Without loss of generality, let us assume M looks like the left matrix in Figure 2.2. For contradiction, assume $P \preceq M$. In that case, we can partition M into four quadrants such that there is at least one one-entry in each of them. It does not matter where we partition it, every time we either get $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right) \preceq M_1$ or $\left(\begin{smallmatrix}&\bullet\\\bullet&\bullet\end{smallmatrix}\right) \preceq M_2$, which is a contradiction. $\square$

Lemma 16. *For all matrices M: $P_8 \npreceq M \Rightarrow M = M_1 \rightarrow M_2$ where*

1. $\left(\begin{smallmatrix}\bullet&\bullet\\&\bullet\end{smallmatrix}\right) \npreceq M_1$ *and* $\left(\begin{smallmatrix}&\bullet\\\bullet&\end{smallmatrix}\right) \npreceq M_2$ *or*

2. $\left(\begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}\right) \npreceq M_1$ *and* $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right) \npreceq M_2$.

Proof. Let $e = [r,c]$ be the top-most one-entry of M. If $\left(\begin{smallmatrix}\bullet&\bullet\\&\bullet\end{smallmatrix}\right) \preceq M[[m],[c-1]]$, together with e it would be the whole P_8. Symmetrically, $\left(\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}\right) \npreceq M[[m],[c+1,n]]$. For contradiction assume $\left(\begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}\right) \preceq M[[m],[c]]$ and let $e_{1,1},\ e_{2,2}$ (none of them equal to e) be any two one-entries forming the pattern. Symmetrically, assume $\left(\begin{smallmatrix}&\bullet\\\bullet&\end{smallmatrix}\right) \preceq M[[m],[c,n]]$ and let $e_{1,2},\ e_{2,1}$ be any two one-entries forming the pattern. Then $e_{1,1},\ e,\ e_{1,2}$ and $e_{2,1}$ or $e_{2,2}$ with bigger row give us mapping of P_8 to M. $\square$

Theorem 17. *For all matrices M: $P_8 \npreceq M \Leftrightarrow M$ is structured like the matrix in Figure 2.3 where* $\left(\begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}\right) \npreceq M_1$ *and* $\left(\begin{smallmatrix}&\bullet\\\bullet&\end{smallmatrix}\right) \npreceq M_2$.

Proof. $\Rightarrow$ From Lemma 16 we know $M = M_1' \rightarrow M_2'$ where $\left(\begin{smallmatrix}\bullet&\bullet\\&\bullet\end{smallmatrix}\right) \npreceq M_1'$ and $\left(\begin{smallmatrix}&\bullet\\\bullet&\end{smallmatrix}\right) \npreceq M_2'$. The second case can be dealt with symmetrically. From Theorem 10 we have that M_1' can be characterized exactly like $M[[m],[c_2-1]$ and $M[[m],[c_2,n]]$ forms a walking matrix. If there are two different columns having a one-entry above the r-th row, together with a one-entry in the r-th row between the columns c_1 and c_2 and a one-entry in the c_1-th column above the r-th row they form a mapping of P_8.

$\Leftarrow$ One-entry $P_8[2,2]$ can not be mapped anywhere but to the r-th row, but in that case there are at most two columns having one-entries above it.

$\square$

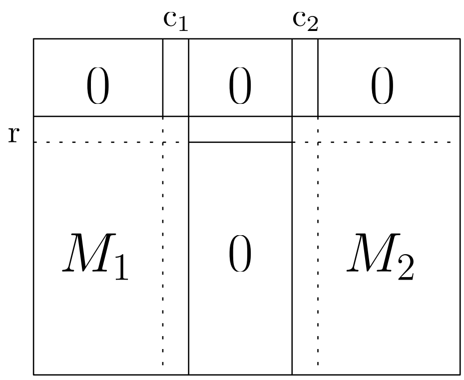


Figure 2.3: Characterization of matrices avoiding $\begin{pmatrix} \bullet & \bullet & \bullet \\ & \bullet & \end{pmatrix}$ as an interval minor.

2.5 Multiple patterns

Theorem 18. *Let* $P_{10} = \begin{pmatrix} \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{pmatrix}$ *and* $P_{11} = \begin{pmatrix} \circ & \bullet \\ \circ & \circ \\ \bullet & \circ \end{pmatrix}$, *then for all matrices* M*:* $\{P_{10}, P_{11}\} \not\preceq M \Leftrightarrow$ *for the top-right most walk* w *in* M *such that there are no one-entries underneath it each one-entry* $M[r, c]$ *is either on* w *or both* $M[r+1, c]$ *and* $M[r, c-1]$ *are on* w*.*

Proof. $\Rightarrow$ For contradiction assume there is a one-entry anywhere but on w or directly diagonally above any bottom-left corner of w. Then this one-entry together with at least one bottom-left corner of w give us P_{10} or P_{11} and a contradiction.

$\Leftarrow$ If we take any one-entry e, from the description of M there is no one-entry that creates P_{10} or P_{11} with e.

□

3. Pattern size constrains

In the previous chapter, we characterized matrices avoiding small patterns. Their structure is very dependent on the pattern and the results are hard to generalize for arbitrary patterns. In this chapter, we look for a more general property that restricts the complexity that a class of matrices can have.

Definition 7. For matrix $M \in \{0,1\}^{m \times n}$ a *one-interval* is a sequence of consecutive one-entries in a single line of M bounded by the edge of matrix or zero-entry from both sides. In the same spirit we define *zero-interval* to be an interval of consecutive zero-entries in a single line of M bounded by one-entry or the edge of matrix from both sides.

In the previous chapter, for pattern $P \in \{0,1\}^{k \times l}$ any inclusion maximal matrix M avoiding P as an interval minor has at most l zero-intervals in each row and at most k zero-intervals in each column. A natural question is whether the size of a pattern always bounds the number of zero-intervals of any inclusion maximal matrix that avoids it.

Let us present some useful notion. First of all, every time we speak about a *maximal* matrix of a class, we mean inclusion maximal – it has no zero-entry that can be changed to a one-entry so that it still belongs to the class. In terms of pattern avoidance, maximal matrices are those for which a change of a zero-entry creates a mapping of the pattern (or possibly many mappings).

Definition 8. Let P be a pattern, e a one-entry of P, M be a matrix avoiding P and zi be an arbitrary zero-interval of M. We say that *zi is usable for e* if there is a zero-entry contained in zi such that if we change it to a one-entry, it creates a mapping that uses the new one-entry to map e. Note that zi can be usable for many one-entries of P at the same time.

3.1 Inserting an empty column

We start by proving Theorem 5 stated in Section 2.1. Even before that we show an easy lemma to get familiar with notion of one-intervals.

Lemma 19. *Let $P \in \{0,1\}^{k \times 2}$ and let $M \in \{0,1\}^{m \times n}$ be a maximal matrix avoiding P, then M contains at most one one-interval in each row.*

Proof. For contradiction, assume there are several one-intervals in a row of M. Because M is maximal, changing any zero-entry e in between two consecutive one-intervals oi_1 and oi_2 creates a mapping of the forbidden pattern. Such a mapping uses the changed one-entry to map an element $P[r',1]$ or $P[r',2]$.

In the first case, the same mapping also works if we use any one-entry of oi_1 instead of e, which gives us $P \not\preceq M$ and therefore a contradiction. In the second case, the mapping can use any one-entry of oi_2 instead of e; therefore, we again get a contradiction with $P \not\preceq M$. Since e is not usable for any one-entry of P we can change it to a one-entry and get a contradiction with M being maximal. $\square$

Lemma 20. *Let $P \in \{0,1\}^{k\times 2}$ and for $l \geq 1$ let $P^l \in \{0,1\}^{k\times l+2}$ be a pattern created from P by adding a l new empty columns in between the two columns of P. If an $m \times n$ matrix $M \in Av(P^l)$ is maximal, then each row of M is either empty or it contains a single one-interval of length at least $l+1$.*

Proof. The same proof as in Lemma 19 can be used to show there is at most one one-interval in each row.

For contradiction assume there are at most $l > 0$ one-entries $M[\{r\},[c_1,c_2]]$ in row r:

- $c_1 = 1$: we can set $M[r,c_2+1] = 1$ and the matrix still avoids P^l, which is a contradiction with M being maximal.
- $c_2 = n$: symmetrically with the previous case this cannot happen.
- otherwise: let us choose e_l and e_r zero-entries in row r such that there are exactly l columns in between them and all one-entries of row r lie in between them. For contradiction, assume we can not change neither $e_l = M[r,c_l]$ nor $e_r = M[r,c_r]$ to a one-entry without creating the pattern. This means e_l is usable for some $P^l[r_1,1]$, let M_l be the corresponding mapping. At the same time e_r is usable for some $P^l[r_2,l+2]$ with M_r being the corresponding mapping. We show that the two mappings can be altered to find a mapping of P^l to M giving a contradiction. Without loss of generality in both mappings, empty columns of P are mapped exactly to l columns of M. We describe how to partition M into k rows. Consider Figure 3.1:
  - $r_1 \neq r_2$: Without loss of generality we assume $r_1 > r_2$. Let r_3 be the first row used to map r_1 in M_l and let r_4 be the last row used to map r_1 in M_r. From M_l being a mapping we know that the first $r_1 - 1$ rows can be mapped to rows $[1, r_3 - 1]$ and from M_r being a mapping we know that the last $k - r_1$ rows can be mapped to rows $[r_4+1, m]$. We can therefore use rows $[r_3, r_4]$ to map row r_1 without using e_l and e_r.
  - $r_1 = r_2$: We proceed similarly as in the previous case. Let r_3 and r_4 be the first and the last rows respectively used to map r_1 in M_l and let r_5 and r_6 be the first and the last rows respectively used to map r_1 in M_r. Without loss of generality let $r_3 < r_5$ and from M_l being a mapping we know that the first $r_1 - 1$ rows can be mapped to rows $[1, r_3 - 1]$. Again, without loss of generality let $r_4 < r_6$ and from M_r being a mapping we know that the last $k - r_1$ rows can be mapped to rows $[r_6 + 1, m]$. We can therefore use rows $[r_3, r_6]$ to map row r_1 without using e_l and e_r.

We showed that either e_l or e_r can be changed to a one-entry and since there is at most one one-interval, we can repeat the process until we get a one-interval of length $l + 1$.

□

Theorem 21. *Let $P \in \{0,1\}^{k\times 2}$ and for $l \geq 1$ let $P^l \in \{0,1\}^{k\times l+2}$ be a pattern created from P by adding a l new empty columns in between the two columns of P. For all $M \in \{0,1\}^{m\times n}$ it holds $M \in Av(P^l) \Leftrightarrow$ there exists $N \in \{0,1\}^{m\times(n-l)}$ such that $N \in Av(P)$ is inclusion maximal and M is a submatrix of elementwise OR of $N \oplus_h 0^{m\times l}$, $0^{m\times 1} \oplus_h N \oplus_h 0^{m\times(l-1)}, \dots,$ $0^{m\times(l-1)} \oplus_h N \oplus_h 0^{m\times 1}$, $0^{m\times l} \oplus_h N$.*

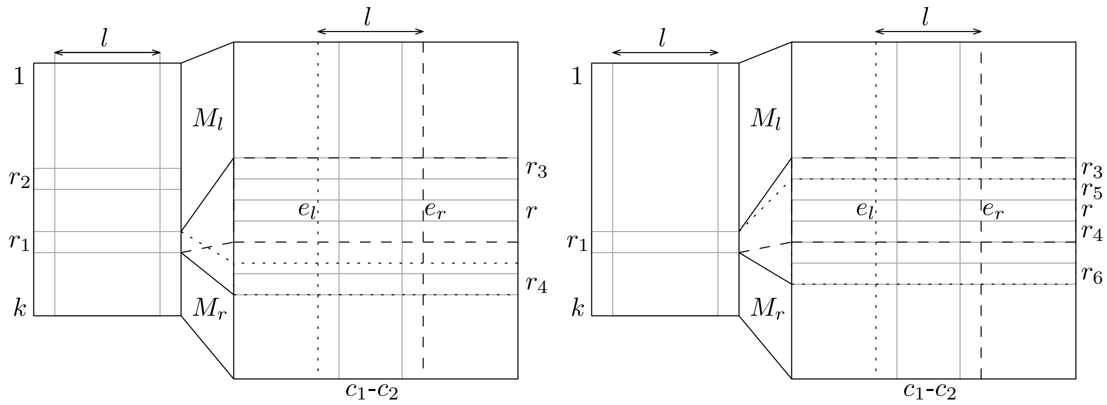


Figure 3.1: Dashed and dotted lines resembling two different mappings of a forbidden pattern, where two horizontal lines show the boundaries of the mapping of row r and the vertical lines show boundaries of the mapping of column c.

Proof. $\Rightarrow$ It suffices to only prove the statement for M that is inclusion maximal. To do so, we use Lemma 20. It says that each row of M contains either no one-entry or a single one-interval of length at least $l+1$. We consider N to be created from M by deleting the last l one-entries on each row and excluding the last l columns. Clearly, M is equal to elementwise OR of $N \oplus_h 0^{m\times l}$, $0^{m\times 1} \oplus_h N \oplus_h 0^{m\times(l-1)}, \ldots,$ $0^{m\times(l-1)} \oplus_h N \oplus_h 0^{m\times 1}$, $0^{m\times l} \oplus_h N$. If $P \preceq N$ then each mapping of P can be extended to a mapping of P^l to M by mapping each $P^l[i,1]$ to the same one-entry where $P[i,1]$ is mapped in $N \oplus_h 0^{m\times l}$ and mapping each $P^l[j, l+2]$ to the same one-entry where $P[j,2]$ is mapped in $0^{m\times l} \oplus_h N$.

$\Leftarrow$ Let M be equal to $N \oplus_h 0^{m\times l}$ placed over $0^{m\times l} \oplus_h N$ with elementwise OR. It suffices to show that it belongs to $Av(P^l)$. For contradiction, assume it does not. Then there is mapping of P^l to M and we can assume that one-entries of the first column of P^l are mapped to those one-entries of M created from $N \oplus_h 0^{m\times l}$. If there is such one-entry mapped to a one-entry of M not created from $N \oplus_h 0^{m\times l}$ we can take the first one-entry in the row instead. Symmetrically, all one-entries of the last column of P^l are mapped to one-entries created from $0^{m\times 1} \oplus_h N$. But then, then same one-entries of N can be used to map P, which is a contradiction with $N \in Av(P)$. $\square$

Observation 22. *Let $P \in \{0,1\}^{k\times l}$ and $M \in \{0,1\}^{m\times n}$ such that $P \not\preceq M$. Let $zi = M[\{r_1\},[c_1,c_2]]$ be a zero-interval of M usable for one-entry $e = P[r,c]$. If we change a zero-entry of zi and create a mapping of P that uses the changed entry to map e, then no such mapping can map column c outside of columns $[c_1,c_2]$.*

Proof. Since the changed entry is used to map e, clearly every mapping needs to use a column from $[c_1,c_2]$ to map column c. If for contradiction after a change of a zero-entry there was a mapping using columns outside $[c_1,c_2]$ then it without loss of generality uses c_1-1 but since it bounds zero-interval zi it is a one-entry and this one-entry can be used in the mapping instead of the changed entry, which gives us a contradiction with $P \not\preceq M$. $\square$

Lemma 23. *Let $P \in \{0,1\}^{k\times 2}$ and for $l \geq 1$ let $P^l \in \{0,1\}^{k\times l+2}$ be a pattern created from P by adding l new empty columns in between the two columns of P. If an $m \times n$ matrix $M \in Av(P^l)$ is maximal, then for each one-entry e of P there are at most k^2 zero-intervals usable for e.*

Proof. Given an arbitrary maximal matrix M avoiding P let us look at an arbitrary column c in M and an arbitrary one-entry e of P. Without loss of generality assume $e = P[r, 1]$. For contradiction, assume there are $k^2 + 1$ zero-intervals $zi_1, \ldots, zi_{k^2+1}$ in c usable for e.

- $P[r, 2] = 1$: Clearly, there is a one-entry next to each zi_j and if we combine each such entry with a one-entry bounding each zi_j we find a mapping of $\{1\}^{k^2\times 2}$, contradicting $P \not\preceq M$.
- $P[r, 2] = 0$: We define extended interval zi_i^* for each $i \in [t]$ to be the interval containing zi_i and also all element of c between zi_i and zi_{i+1}. Because of the pidgeon hole principle we find either k consecutive zi_i^* extended intervals such that there are no one-entries in columns $[c + l + 1, n]$ in the rows they cover or k extended intervals such that there is a one-entry next to each. Because each extended intervals contains a one-entry, in the second case we find $\left(\{1\}^{k\times 2}\right)^l$ as an intervals minor. In the first case, without loss of generality assume $P[r_1, 2] = 1$ and it is the minimum such $r_1 > r$. Also let $zi_{first}, \ldots, zi_{last}$ be the consecutive zero-intervals, which extended intervals have no one-entries next to them. Now consider the mapping of P created when a zero-entry of zi_{first} gets changed to a one-entry used to map e. Since $P[r_1, 2] = 1$ and there are no one-entries next to $zi_{first} - zi_{last}$, it has to be mapped to the rows of M passed the end of zi_{last}. This leaves k one-entries to be used to map potential one-entries in $P[[r, r_2 - 1], \{2\}]$ and so $P \preceq M$.

□

Corollary 24. *Let $P \in \{0,1\}^{k\times 2}$ and for $l \geq 1$ let $P^l \in \{0,1\}^{k\times l+2}$ be a pattern created from P by adding l new empty columns in between the two columns of P. Then P^l is bounded for any $l \geq 1$.*

3.2 Pattern complexity

We saw that for patterns having only two rows or columns we can indeed bound the number of one-intervals of maximal matrices avoiding them. On the other hand, already for a pattern of size 3×3 we show that there are maximal matrices with arbitrarily many one-intervals.

Definition 9. Let $\mathcal{P}$ be a class of patterns and for any $P \in \mathcal{P}$ let e be a one-entry of P. We define the *row-complexity of one-entry e* $r_{\mathcal{P}}(e)$ to be the supremum of the number of zero-intervals of a single row of any maximal matrix from $Av(\mathcal{P})$ that are usable for e. We say that e is *row-unbounded* in $\mathcal{P}$ if $r_{\mathcal{P}}(e) = \infty$ and *row-bounded* otherwise. Symmetrically we define the *column-complexity of one-entry e* $c_{\mathcal{P}}(e)$ to be the maximum number of zero-intervals of a single column of any maximal matrix from $Av(\mathcal{P})$ that are usable for e and say e is *column-unbounded* if it is infinite and *column-bounded* otherwise.

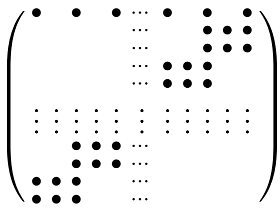
Definition 10. Let $\mathcal{P}$ be a class of patterns. We define the *row-complexity* of $\mathcal{P}$ $r_{\mathcal{P}} = sup\{r_{\mathcal{P}}(e) | e$ one-entry of $P \in \mathcal{P}\}$, the *column-complexity* of $\mathcal{P}$ $c_{\mathcal{P}} = sup\{c_{\mathcal{P}}(e) | e$ one-entry of $P \in \mathcal{P}\}$ and the *complexity* of $\mathcal{P}$ $comp_{\mathcal{P}} = sup\{r_{\mathcal{P}}, c_{\mathcal{P}}\}$. We say $\mathcal{P}$ is *unbounded* if $comp_{\mathcal{P}} = \infty$ and *bounded* otherwise.

The following observation follows directly the definition and we use it heavily throughout the chapter to break symmetries.

Observation 25. *For every $\mathcal{P}$, $P \in \mathcal{P}$, $P \in \{0,1\}^{k \times l}$, $r \in [k]$ and $c \in [l]$, if $P[r,c]$ is row-bounded in $Av(\mathcal{P})$ then $P^T[c,r]$ is column-bounded in $Av(\mathcal{P}^T)$.* $\square$

Theorem 26. *Let $P = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$. For every $n > 1$ there is a maximal matrix M avoiding P as an interval minor having n one-intervals (P is unbounded).*

Proof. Let M be a $(2n-1) \times (2n-1)$ matrix described by the picture:



We see that $P \not\preceq M$ because we always need to map $P[2,1]$ and $P[3,3]$ to just one "block" of one-entries which only leaves a zero-entry for $P[1,2]$.

When we change any zero-entry of the first row into a one-entry we get a matrix containing a minor of $\{1\}^{3\times 3}$; therefore, containing P as an interval minor. In case M is not maximal, we can add some more one-entries to make it maximal but it will still contain a row with n one-intervals. $\square$

Not only M is a maximal matrix avoiding P but it also avoids any $P' \in \{0,1\}^{3\times 3}$ such that $P \preceq P'$. Its rotations avoid rotations of P and we can deduce that a big portion of patterns of size 3×3 are unbounded. Moreover, the result can be generalized also for bigger matrices. The pattern is so important that we call it P_1 for the rest of the chapter.

Theorem 27. *For every P such that $P_1 \preceq P$ and every $n > 1$ there is a maximal matrix M avoiding P as an interval minor having n one-intervals.*

Proof. First, assume there is a mapping of P_1 into $P \in \{0,1\}^{k\times l}$ that assigns a one-entry of the first row to $P_1[1,2]$, a one-entry of the first column to $P_1[2,1]$ and a one-entry of the last row and column to $P_1[3,3]$. Then, we can construct a similar matrix as we did in the proof of Theorem 26 avoiding P that after changing any zero-entry of the first row contains the whole $\{1\}^{k\times l}$.

Let P be an arbitrary pattern containing P_1. Let $P[r_1,c_1], P[r_2,c_2]$ and $P[r_3,c_3]$ be one-entries that can be used to map $P_1[1,2], P_1[2,1]$ and $P_1[3,3]$ respectively. Then we take a submatrix $P' := P[[r_1,r_3],[c_2,c_3]]$. Such a pattern fulfills assumptions of the more restricted case stated at the beginning of the proof and we can find a maximal matrix M' avoiding P' having n one-intervals. We construct M from M' by simply adding new rows and columns, all containing one-entries. We add $r_1 - 1$ rows in front of the first row and $k - r_3$ rows behind the last row. We also add $c_2 - 1$ columns in front of the first column and $l - c_3$

columns behind the last column. Constructed matrix M avoids pattern P as its submatrix P' cannot be mapped to M'. At the same time, any change of a zero-entry of the r_1-th row of M to a one-entry creates a copy of $1^{k \times l}$ so the changed matrix contains P. Constructed M can be seen in Figure 3.2.

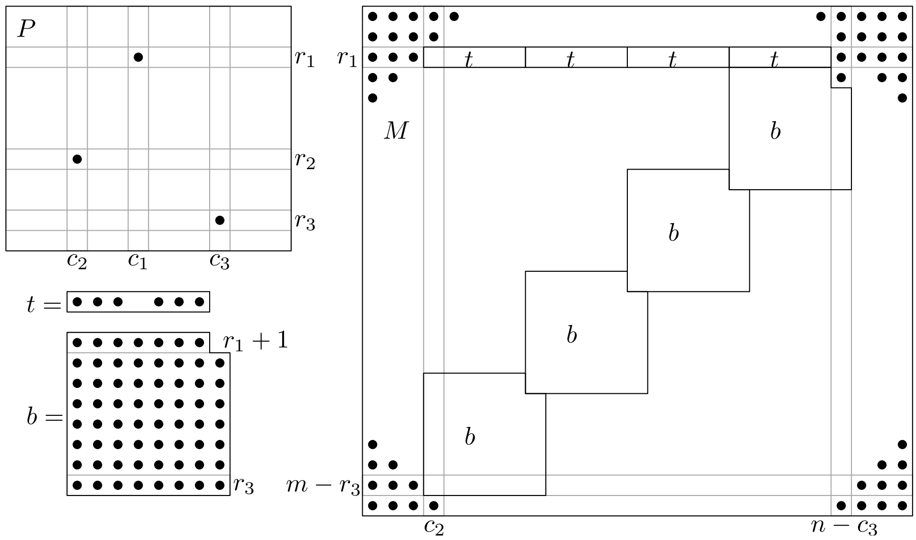


Figure 3.2: Structure of a maximal matrix avoiding P that has arbitrarily many one-intervals.

□

What makes it even more interesting is that any pattern avoiding all rotations of P_1 is already bounded. To prove that we need a few partial results.

Theorem 28. *Let P be a pattern avoiding all rotations of P_1, then P:*

1. *contains one non-empty line or*
2. *contains two non-empty lines or*
3. *contains three non-empty lines or*
4. *avoids* $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$.

Proof. Assume P has four one-entries that do not share any row or column. Then those one-entries induce a 4×4 permutation inside P and because P does not contain any rotation of P_1, the induced permutation is either 1234 or 4321. Without loss of generality, assume it is the first case and denote the one-entries by e_1, e_2, e_3 and e_4.

For contradiction with the statement, assume P also contains $P' = \left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$. Clearly, no one-entry from e_1, e_2, e_3 and e_4 can be part of any mapping of P' because it would induce a mapping of a rotation of P_1.

Let $e_2 = P[r_2, c_2]$ and $e_3 = P[r_3, c_3]$. Submatrix $P[[r_2], [c_2, l]]$ avoids P'; otherwise, together with e_1 it would give us a rotated copy of P_1. Symmetrically, $P[[r_3, k], [c_3]]$ does not contain P'. Also, $P[[r_3 - 1], [c_3 - 1]]$ and $P[[r_2 + 1, k], [c_2 + 1, l]]$ are empty; otherwise, they would together with e_2 and e_3 give us a rotation of P_1. Up to rotation, the only possible way to have $P' \preceq P$ is that $P'[1, 1]$ lies in $P[[r_3 - 1], [c_2, c_3 - 1]]$ but then this entry together with e_1 and e_3 give us a rotation of P_1 which is a contradiction. □

Now comes the hard part. For each group of patterns, we need to prove they are bounded.

Lemma 29. *Let $P \in \{0,1\}^{k\times l}$ be a pattern having only one non-empty line. Then for every maximal matrix $M \in \{0,1\}^{m\times n}$ avoiding P the number of one-intervals in each row and column is bounded by $k+l$.*

Proof. Without loss of generality let the non-empty line of P be a row r. Since M is maximal, $M[[r-1],[n]]$ and $M[[m-r+1,m],[n]]$ contain no zero-entry and each of their rows contains just one interval of one-entries. If we look at any other row, it cannot contain k one-entries, so the maximum number of one-intervals is $k-1$.

Let us look on an arbitrary column c of M. If there is at least one one-entry in $M[[r,m-r],c]$ then because M is maximal, the whole column is made of one-entries. Otherwise, there are two intervals of one-entries – $M[[r-1],c]$ and $M[[m-r,m],c]$. □

Lemma 30. *Let $P \in \{0,1\}^{k\times l}$ be a pattern having two non-empty lines. Then for every maximal matrix $M \in \{0,1\}^{m\times n}$ avoiding P the number of one-intervals in each row and column is bounded by $2k^3+2l^3+1$.*

Proof. First we assume the two non-empty lines of P are rows $r_1 < r_2$ (or symmetrically columns). From Observation 4 and maximality of M we have that $M[[r_1-1],[n]]$ and $M[[m-r_2+1,m],[n]]$ contain no zero-entry. Therefore, we may restrict ourselves to the case when $r_1 = 1$ and $r_2 = k$. From Lemma 23 we know that every maximal N avoiding $P[\{r_1,r_2\},[n]]$ has at most $2k^3+1$ one-intervals in each row and at most 1 one-interval in each column. From Theorem 5 we also know that for given M there is a maximal N avoiding $P[\{r_1,r_2\},[n]]$ such that M is a submatrix of shifted and OR-ed copies of N. Since M is maximal, it is equal to those shifted and OR-ed copies of N and because the number of one-intervals of N is bounded, so is the number of one-intervals of M.

Let the two non-empty lines of P be row r and column c. Because of symmetry, we only show the bound for rows. Let us take an arbitrary row of M an look at its zero-intervals. For every one-entry e of the pattern except those in the r-th row, there is at most one zero-interval usable for e. For contradiction, assume there are two such zero-intervals zi_1 and zi_2. Let Figure 3.3 illustrate the situation where dashed and dotted lines form mappings of the minor P to M when a zero-entry of zi_1 and zi_2 respectively is changed to a one-entry. When we take the outer two vertical and horizontal lines, we get a mapping of P that can use an existing one-entry in between zi_1 and zi_2 to map e. This gives us a contradiction with $P \not\preceq M$.

For a one-entry $e = P[r,c']$, if $c' \le c$ then there must be less than c' one-entries before any zero-intervals usable for e; otherwise, we could map $P[r,[1,c']]$ just to the single row of M. It follows that e is row-bounded. Symmetrically, the same holds in case $c' > c$. □

To make the analysis of the last two groups of patterns easier, we introduce three helpful lemmata.

Lemma 31. *Let $P \in \{0,1\}^{k\times l}$ be a pattern structured like one of the matrices in Figure 3.4. Then every one-entry in $P[\{r_2\},[c_1,c_2]]$ is row-bounded.*

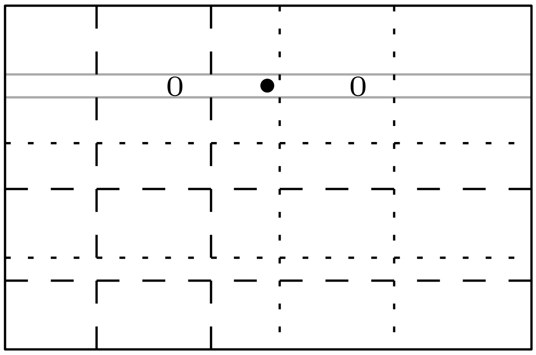


Figure 3.3: Dashed and dotted lines resembling two different mappings of a forbidden pattern, where two horizontal lines show the boundaries of the mapping of row r and the vertical lines show boundaries of the mapping of column c.

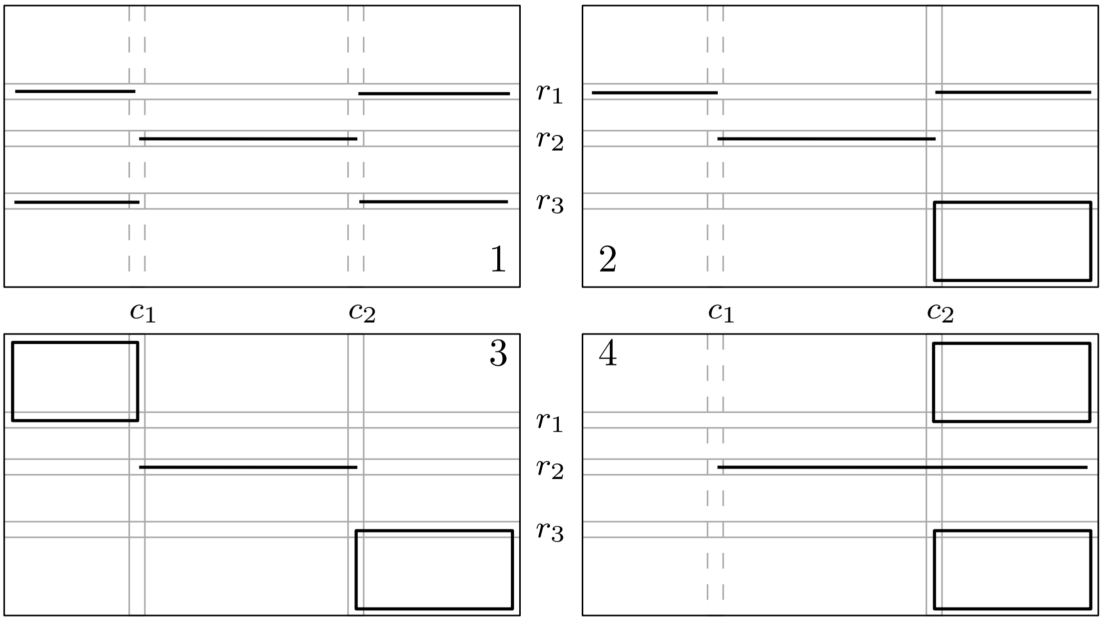


Figure 3.4: Patterns for which one-entries in row r_2 and columns c_1 to c_2 are row-bounded. One-entries may only be in the areas enclosed by bold lines.

Proof. Let P be the first described pattern and let $k' = c_2 - c_1$. We show that for each one-entry e from row r_2 and every M maximal matrix avoiding P there is at most k' zero-intervals for which it is usable. For contradiction assume there is a row r with $k'+1$ zero-intervals usable for e. It follows that there are at least k' one-entries in between two most distant zero-intervals z_1 and z_2. Therefore, the whole row r_2 can be mapped just to r. Since changing a zero-entry of z_1 to a one-entry to which e can be mapped creates a partitioning of M where all one-entries from columns 1 to c_1 are mapped to columns up to z_1 and similarly all one-entries from columns c_2 to l can be mapped to columns from and past z_2, we can simply map empty rows from $r_1 + 1$ to $r_3 - 1$ around row r and use the rest to map rows r_1 and r_2. Described partitioning gives us $P \preceq M$ and a contradiction. We can see the partitioning in Figure 3.5.

Proofs of cases two and three are similar to the first one and we skip them.

Let us look on the fourth case. For i-th one-entry in row r_2 (ordered from

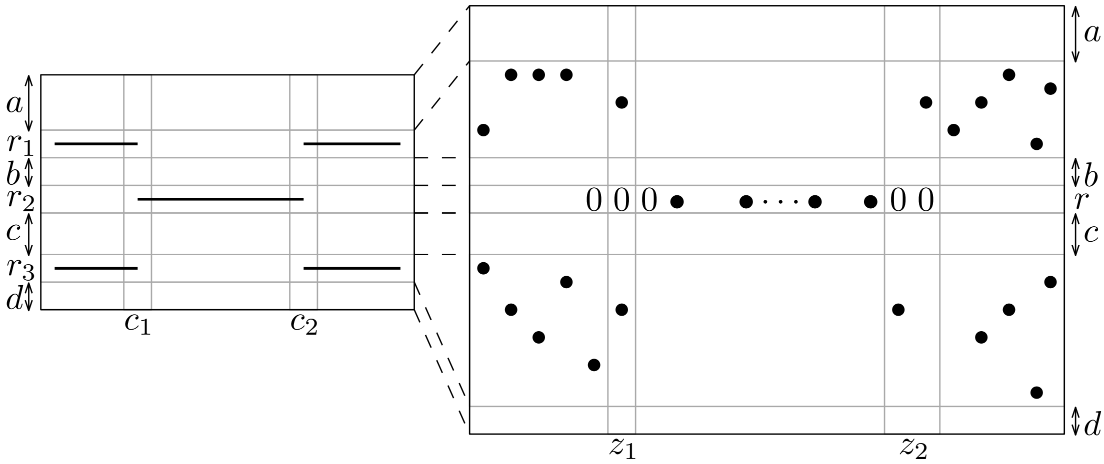


Figure 3.5: Mapping of a pattern into a matrix only using one line to map an empty line of the pattern and only using one line to map row r_2.

left to right and only considering those in columns c_1 to c_2) no zero-interval of a maximal matrix avoiding the pattern cannot have i one-entries to the left of it and so each such one-entry is bounded by $i \geq l$.

It is important to realize we could not have used the same proof we used for the first three cases also for the fourth case, because we can never rely on the fact a mapping of P only uses one row of M to map row r_2. This is because in the fourth case, unlike the first three, there are also potential one-entries in $P[\{r_2\}, [c_2, l]]$. ☐

Lemma 32. *Let $P \in \{0,1\}^{k \times l}$ be a pattern structured like one of the matrices in Figure 3.6. Then every one-entry in $P[[r_1+1, r_2-1], \{c\}]$ is row-bounded.*

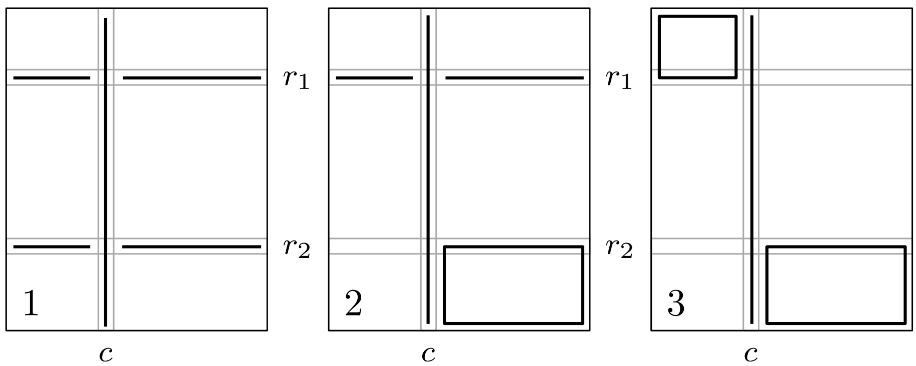


Figure 3.6: Patterns for which one-entries in column c and rows $r_1 + 1$ to $r_2 - 1$ are row-bounded. One-entries may only be in the areas enclosed by bold lines.

Proof. Let P be the first described pattern. We show that for each one-entry from $P[[r_1+1, r_2-1], \{c\}]$ and every M maximal matrix avoiding P there is at most one zero-interval for which it is usable. For contradiction assume there is a row r with two zero-intervals z_1 and z_2 usable for e. Look at Figure 3.7 and let the dashed partitioning be a mapping of P to M when a zero-entry of z_1 is changed to a one-entry used to map e and let the dotted partitioning be a mapping of P to M when a zero-entry of z_2 is changed to a one-entry used to map e. If we map column c to where it is mapped in both mappings together and map rows r_1 and r_2 as suggested in the picture, we get a partitioning of P inside M and so a contradiction with $P \not\prec M$.

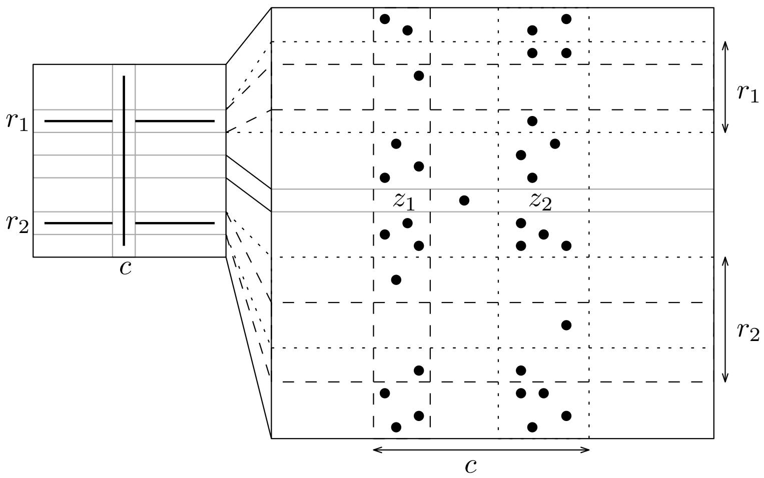


Figure 3.7:

Proofs of cases two and three are similar to the first one and we skip them. $\square$

Lemma 33. *Let P be a pattern and c be its first non-empty column. Then every one-entry from c is row-bounded.*

Proof. The result follows immediately from the fourth case of Lemma 31. $\square$

Lemma 34. *Let $P \in \{0,1\}^{k\times l}$ be a pattern avoiding $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ (or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$). Then for every maximal matrix $M \in \{0,1\}^{m\times n}$ avoiding P the number of one-intervals in each row and column is bounded.*

Proof. From Theorem 8 we know that P is a walking pattern. Every one-entry of P satisfies either conditions of the third case of Lemma 31 or it satisfies conditions of the third case of Lemma 32 and therefore is row-bounded. To prove it is also column-bounded, we use Observation 25. $\square$

Lemma 35. *Let $P \in \{0,1\}^{k\times l}$ be a pattern having three non-empty lines and avoiding all rotations of P_1. Then for every maximal matrix $M \in \{0,1\}^{m\times n}$ avoiding P the number of one-intervals in each row and column is bounded.*

Proof. First of all, if P avoids $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$ we can use Lemma 34. Therefore, we assume it contains both.

Let us prove that each pattern having one-entries in three rows is bounded. If all one-entries are in up to two columns then we are again done. Therefore, P has one-entries in at least three columns and it contains a three by three permutation matrix as a submatrix. Since rotations of P_1 are avoided, only feasible permutations are 123 and 321 and without loss of generality we assume the first case. In Figure 3.8 we see the structure of each such pattern. Capital letters stand for one-entries of the permutation, letters $a - f$ stand each for a potential one-entry and greek letters stand each for a potential sequence of one-entries and zero-entries. Everything else is zero. Not all one-entries can be present at the same time, because that would create a mapping of P_1 or its rotation but we also need to find $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix}$. The following analysis only uses hereditary arguments. This means that if we prove P is bounded, we also prove that each submatrix of P is bounded. With this in mind, we restrict ourselves to maximal patterns.

- γ contains a one-entry $\Rightarrow f = 0 \Rightarrow$ because $\begin{pmatrix}\bullet & \\ & \bullet\end{pmatrix}$ needs to be there it holds $a = 1 \Rightarrow \alpha = 0$
  - $d = 1 \Rightarrow b = 0,\ \beta = 0,\ e = 0,\ c = ?$:
    Lemma 31 (case 4): one-entries in c, C, γ are row-bounded.
    Lemma 33: a and A are row-bounded.
    Lemma 32 (case 1): d and B are row-bounded.

    Lemma 33: all one-entries except for B are column-bounded.
    Lemma 31 (case 1): B is column-bounded.
  - $d = 0$
    - $c = 1 \Rightarrow \beta = 0,\ e = 0,\ b = ?$:
      Lemma 31 (case 4): one-entries in c, C, γ are row-bounded.
      Lemma 33: a, b, A are row-bounded.
      Lemma 31 (case 1): B is row-bounded.

      Lemma 33: one-entries in the first and the third non-empty rows are column-bounded.
      Lemma 31 (case 2): b, B are column-bounded.
    - $c = 0 \Rightarrow$ in the maximal case $b = 1,\ e = 1,\ \gamma$ contains a one-entry:
      Lemma 31 (case 4): one-entries in c, C, γ are row-bounded.
      Lemma 33: one-entries in the first non-empty column are row-bounded.
      Lemma 31 (case 1): one-entries in the middle non-empty row are row-bounded.

      Lemma 33: one-entries in the first and the third non-empty rows are column-bounded.
      Lemma 32 (case 2): one-entries in the middle non-empty row are column-bounded.
- $\gamma = 0$
  - α contains a one-entry $\Rightarrow a = 0,\ b = 0$:
    Every such pattern has already been dealt with as we can rotate it by 180 degrees, map A and α to γ, map d to C and so on.
  - $\alpha = 0$:
    Without loss of generality, we can assume that $a = 1$, because there needs to be $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ and if we set $a = 0$, it must hold $f = 1$ and then we can just rotate the pattern by 180 degrees and get the case $a = 1$.
    - $d = 1 \Rightarrow b = 0,\ e = 0,\ \beta = 0,\ c = ?,\ f = ?$:
      Lemma 33: a, f, A and C are row-bounded.
      Lemma 32 (case 1): c, d and B are row-bounded.

Lemma 33: one-entries in the first and third non-empty rows are column-bounded.
Lemma 31 (case 1): B is column-bounded.

* $d = 0$
  - $e = 1 \Rightarrow c = 0,\ b = ?,\ f = ?$:
    Since $\alpha = 0$ it follows that if there is a one-entry in β only if it can be in e.
    Lemma 33: a, f, A and C are row-bounded.
    Lemma 31 (case 1): one-entries in b, e, B and β are row-bounded.

    Lemma 33: a, f, A and C are column-bounded.
    Lemma 32 (case 1): one-entries in b, e, B and β are column-bounded.
  - $e = 0$:
    We can assume $c = 1$ as else e can be 1 and we have already dealt with that case. We can also assume $b = 1$ since otherwise, we would have a submatrix of the case dealt with when $d = 1$:
    Lemma 33: a, b and A are row-bounded.
    Lemma 31 (case 4): c and C are row-bounded.
    Lemma 31 (case 1): B is row-bounded.

    Because the pattern is symmetric, it is also column-bounded.

The same analysis also proves that if the pattern with the same restrictions only has three non-empty columns then it is bounded.

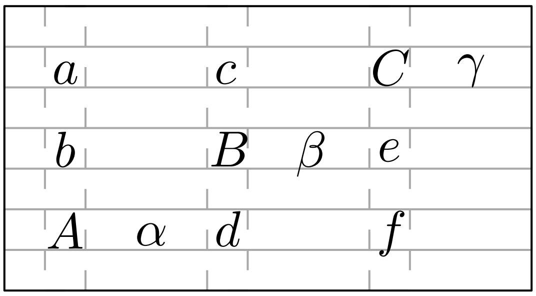


Figure 3.8: Structure of a pattern only having three non-empty rows and avoiding all rotations of P_1.

Let us now look at the case where all one-entries of the pattern are in either one of two rows r_1, r_2 or in column c_1. Without loss of generality, we again assume permutation 123 is present and we distinguish three cases. Consider Figure 3.9:

- C lies in column c_1
  - $a = 1 \Rightarrow b = 0,\ \alpha = 0$ and everything else can be one:
    Lemma 31 (case 2): one-entries in a, c, B and β are row-bounded.
    Lemma 33: all other one-entries are row-bounded.

Lemma 31 (case 4): one-entries in c, C and γ are column-bounded.
Lemma 32 (case 1): one-entries in a, c, B and β are column-bounded.
Lemma 33: d and A are column-bounded.

- $a = 0$ and everything else can be one:
  Lemma 31 (case 4): one-entries in b, A and α are row-bounded.
  Lemma 31 (case 2): one-entries in c, B and β are row-bounded.
  Lemma 33: one-entries in c, d, C and γ are row-bounded.

  Lemma 31 (case 4): one-entries in c, C and γ are column-bounded.
  Lemma 32 (case 2): one-entries in c, B and β are column-bounded.
  Lemma 33: one-entries in b, d, A and α are column-bounded.

- B lies in column c_1
  - $a = 1 \Rightarrow \alpha = 0$
    * $d = 1 \Rightarrow \gamma = 0$:
      Lemma 32 (case 1): all one-entries in column c_1 are row-bounded.
      Lemma 33: all other one-entries are row-bounded.

      Lemma 31 (case 1): all one-entries in column c_1 are column-bounded.
      Lemma 33: all other one-entries are column-bounded.
    * $d = 0$:
      Lemma 32 (case 1): all one-entries in column c_1 are row-bounded.
      Lemma 33: a and A are row-bounded.
      Lemma 31 (case 4): one-entries in C and γ are row-bounded.

      Lemma 31 (case 1): all one-entries in column c_1 are column-bounded.
      Lemma 33: all other one-entries are column-bounded.
  - $a = 0$
    * $d = 1 \Rightarrow \gamma = 0$:
      Lemma 32 (case 1): all one-entries in column c_1 are row-bounded.
      Lemma 33: d and C are row-bounded.
      Lemma 31 (case 4): one-entries in A and α are row-bounded.

      Lemma 31 (case 1): all one-entries in column c_1 are column-bounded.
      Lemma 33: all other one-entries are column-bounded.
    * $d = 0$:
      Lemma 32 (case 1): all one-entries in column c_1 are row-bounded.
      Lemma 31 (case 4): one-entries in A, C, α and γ are row-bounded.

      Lemma 31 (case 1): all one-entries in column c_1 are column-bounded.
      Lemma 33: all other one-entries are column-bounded.

- A lies in column c_1:
  This is the first situation rotated by 180 degrees.

The same analysis also proves that if one-entries of a pattern with the same restrictions are in one row or two columns then the pattern is bounded. $\square$

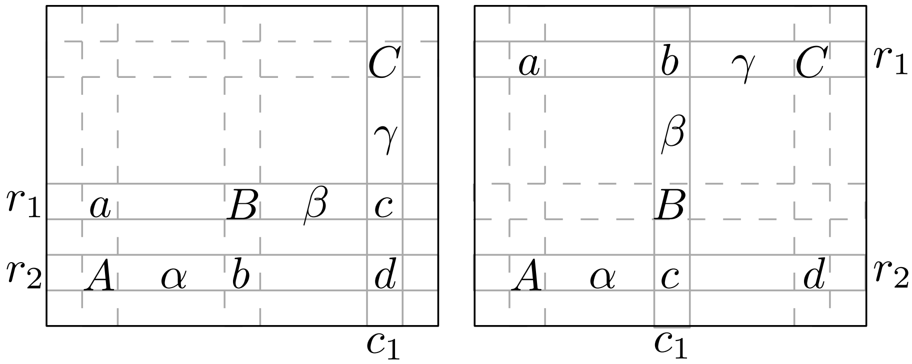


Figure 3.9: Structure of a pattern only having one-entries in two rows and one column that avoids all rotations of P_1.

Combining all the lemmata we finally get the following result.

Theorem 36. *Let P be a pattern avoiding all rotations of $P_1 = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}$, then P is bounded.* $\square$

3.3 Complexity of one-entries

So far we have been working with the whole patterns and determining their complexity. To make the results even more general, we can analyze the complexity of each one-entry.

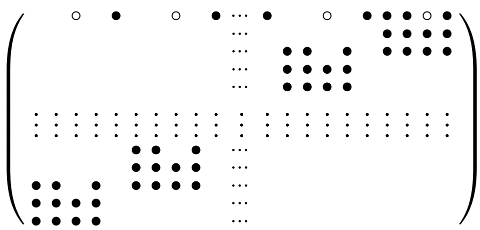
Lemma 37. *Let $P \in \{0,1\}^{k \times l}$ be a pattern such that all its one-entries are either in rows $r_1 < r_2$ or in $P[[r_2], \{c\}]$. Then $P[r_1, c]$ is row-bounded.*

Proof. We prove there are at most k^4 zero-intervals usable for $P[r_1, c]$ in each row of any maximal matrix M avoiding P. For contradiction, let there be more than k^4 of them $(zi_1, \ldots, zi_{k^4})$ in some row and for each of them, consider the top most row r'_j used to map r_2-th row of P in a mapping created when a zero-entry of zi_j is changed to a one-entry used to map $P[r_1, c]$. Then pairs $[zi_1, r'_1], [zi_2, r'_2], \ldots, [zi_{k^4}, r'_{k^4}]$ form a sequence of distinct pairs and thanks to the Pidgeonhole principle, there is a subsequence of length at least k^2 such that the values of r'_j are either non-increasing or non-decreasing. Without loss of generality, assume they are non-decreasing and let $zi'_1, \ldots, zi'_{k^2}$ be their corresponding zero-intervals.

What if $P[r_2, c] = 0$? TODO $\square$

Lemma 38. *Let $P = \begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & & \bullet & \end{pmatrix}$. For every $n > 1$ there is a maximal matrix M avoiding P as an interval minor having n zero-intervals usable for $P[1, 3]$.*

Proof. Let M be a $(2n-1) \times (2n-1)$ matrix described by the picture:

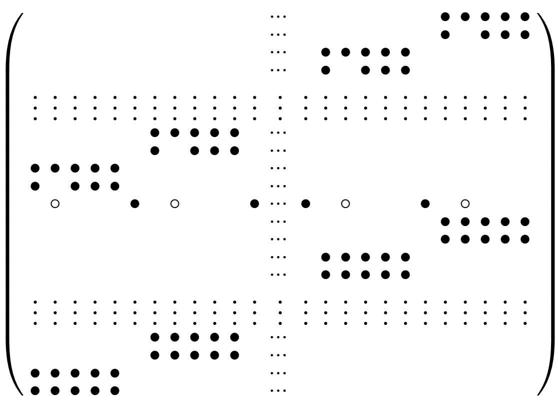


We see that $P \not\preceq M$ because we always need to map $P[2,1]$ and $P[3,3]$ to just one "block" of one-entries of M which only leaves a zero-entry where we need to map $P[1,3]$ or $P[2,4]$.

When we change any marked zero-entry of the first row into a one-entry we get a matrix containing a minor of $\{1\}^{3\times 4}$; therefore, containing P as an interval minor. In case M is not maximal, we can add some more one-entries to make it maximal but it will still contain a row with n one-intervals. $\square$

Lemma 39. *Let* $P = \begin{pmatrix} \bullet & & & & \\ & \bullet & & \bullet & \\ & & \bullet & & \\ & & & & \bullet \end{pmatrix}$ *and* $P' = \begin{pmatrix} \bullet & & & & \\ & \bullet & & \bullet & \\ & & & & \bullet \\ & & \bullet & & \end{pmatrix}$. *For every $n > 1$ there is a maximal matrix M avoiding both P and P' as an interval minor having n zero-intervals usable for $P[2,2]$ and $P'[2,2]$. Moreover, for every pattern containing P or P' as a submatrix, the one-entry that can be used to map $P[2,2]$ or $P'[2,2]$ is also row-unbounded.*

Proof. Let M be a $(2n-1) \times (2n-1)$ matrix described by the picture:



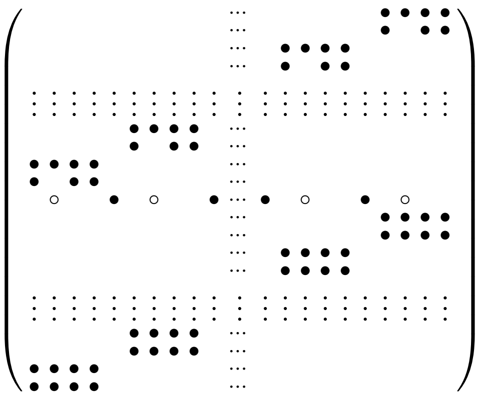
We see that $P \not\preceq M$.

When we change any marked zero-entry of the middle row into a one-entry we get a matrix containing $\{1\}^{4\times 5}$; therefore, both containing P and P' as an interval minor. In case M is not maximal, we can add some more one-entries to make it maximal but it will still contain a row with n one-intervals.

TODO general argument for bigger patterns. $\square$

Lemma 40. *Let* $P = \begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \\ & \bullet & & \\ & & & \bullet \end{pmatrix}$ *and* $P' = \begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \\ & & & \bullet \\ & \bullet & & \end{pmatrix}$. *For every $n > 1$ there is a maximal matrix M avoiding both P and P' as an interval minor having n zero-intervals usable for $P[2,2]$ and $P'[2,2]$.*

Proof. Let M be a $(2n-1)\times(2n-1)$ matrix described by the picture:



We see that $P \not\preceq M$.

When we change any marked zero-entry of the middle row into a one-entry we get a matrix containing $\{1\}^{4\times 5}$; therefore, containing P as an interval minor. In case M is not maximal, we can add some more one-entries to make it maximal but it will still contain a row with n one-intervals. $\square$

Theorem 41. *Let P be a pattern. Any one-entry $P[r,c]$ is row-unbounded if (and only if) there is a trivially unbounded one-entry $P[r,c']$ and we cannot apply the fourth case of Lemma 31 nor Lemma 37 to $P[r,c]$.*

Proof. Without loss of generality, let $P[r,c']$ be part of mapping of P_1, where $P_1[1,2]$ is mapped to it. Let $P_1[2,1]$ be mapped to $P[r_2,c_2]$ and $P_1[3,3]$ be mapped to $P[r_3,c_3]$. We go through all potential one-entries $P[r,c]$ and show that either we can use one of the lemmata mentioned in the statement or the one-entry is row-unbounded.

- $c < c_2$: If there is no one-entry in $P[[r-1],[c-1]]$ nor $P[[r+1,k],[c-1]]$, then the fourth case of Lemma 31 can be used for $P[r,c]$. Otherwise, first consider there is a one-entry in $P[[r-1],[c-1]]$, then we can use the construction from Lemma **??**. In the last case, assume there is a one-entry $P[r',c']$ in $P[[r+1,k],[c-1]]$, then if $r'! = r_2$, entries $P[r,c], P[r',c']$ and $P[r_2,c_2]$ form either P_1 or P_2 and $P[r,c]$ is trivially row-unbounded. If $r' = r_2$, then we use $P[r,c], P[r',c']$ and $P[r_3,c_3]$ to again find either P_1 or P_2 and $P[r,c]$ is trivially row-unbounded once again.

- $c = c_2$: If there is no one-entry in $P[[r-1],[c-1]]$ nor $P[[r+1,k],[c-1]]$, then the fourth case of Lemma 31 can be used for $P[r,c]$. Otherwise, first assume there is a one-entry in $P[[r-1],[c-1]]$, then we can use the construction from Lemma 39. In the last case, assume there is a one-entry $P[r',c']$ in $P[[r+1,k],[c-1]]$, then if $r'! = r_3$, entries $P[r,c], P[r',c']$ and $P[r_3,c_3]$ form either P_1 or P_2 and $P[r,c]$ is trivially row-unbounded. If $r' = r_3$, then what?

  Cannot just use lemma even if it was proved.

  TOOD

- $c_2 < c < c_3$: In this case $P[r,c]$ is trivially unbounded as together with $P[r_2,c_2]$ and $P[r_3,c_3]$ it forms P_1.

- $c = c_3$: If there is no one-entry in $P[[r-1],[c+1,l]]$ nor $P[[r+1,k],[c+1,l]]$, then the fourth case of Lemma 31 can be used for $P[r,c]$. Otherwise, first consider there is a one-entry in $P[[r-1],[c+1,l]]$, then we can use the construction from Lemma 39. In the last case, assume there is a one-entry $P[r',c']$ in $P[[r+1,k],[c-1]]$, then if $r'! = r_2$, entries $P[r,c], P[r',c']$ and $P[r_2,c_2]$ form either P_1 or P_2 and $P[r,c]$ is trivially row-unbounded. If $r' = r_2$, then we use the construction from Lemma 38 to show $P[r,c]$ is row-unbounded once again.

- $c > c_3$: There are three cases to go through and we can handle them the same way as we did in case $c < c_2$.

□

3.4 Chain rules

In this section, we study what happens when we combine multiple classes that are bounded or unbounded.

Theorem 42. *Let $\mathcal{P}$ and $\mathcal{Q}$ be classes of patterns. If both $\mathcal{P}$ and $\mathcal{Q}$ are bounded then $Av(\mathcal{P} \cup \mathcal{Q})$ is bounded.*

Proof. We show $comp_{\mathcal{P}\cup\mathcal{Q}} \leq comp_{\mathcal{P}} + comp_{\mathcal{Q}} = C$.

For contradiction, let M be a maximal matrix avoiding $\mathcal{P} \cup \mathcal{Q}$ having at least $C+1$ zero-intervals in a single row (or column). Without loss of generality it means there is more than $comp_{\mathcal{P}}$ zero-intervals usable for one-entries of the patterns from $\mathcal{P}$. Not let us change some zero-entries of M to one-entries to get $M' \in Av(\mathcal{P})$. Clearly, it still contains more than $comp_{\mathcal{P}}$ zero-intervals usable for one-entries of the patterns from $\mathcal{P}$, which is a contradiction with the definition of $comp_{\mathcal{P}}$.

Similarly, the same inequality holds also for the column-complexity of $\mathcal{P} \cup \mathcal{Q}$ and so the union is bounded. □

Using induction, we can show that also a union of a finite number of bounded classes of finite sizes is bounded. Interestingly enough, unbounded classes are not closed the same way.

Theorem 43. *For every $1 \leq i < j \leq 4]$ is $\{P_i, P_j\}$ bounded.*

Proof. Due to symmetries it is enough to only consider $i = 1$ and $j = [1,2]$.

- $\{P_1, P_2\}$ is row-bounded: from Lemma 33 we have that one-entries $P_1[2,1], P_1[3,3], P_2[2,3]$ and $P_3[3,1]$ are row-bounded. For $P_1[1,2]$ and $P_2[1,2]$ we prove there are at most two zero-intervals usable for each of them. Otherwise, if there are three zero-intervals $z_1 < z_2 < z_3$ usable for $P_1[1,2]$ then the one-entries used to map $P_1[2,1]$ and $P_1[3,3]$ in a mapping created when a zero-entry of z_1 changes to one-entry used to map $P_1[1,2]$ together with a one-entry in between z_2 and z_3 give us a mapping of P_2 to M. Symmetrically, the same goes for $P_2[1,2]$ and z_3'.

- $\{P_1, P_2\}$ is column-bounded: from Lemma 33 combined with Observation 25 we have that one-entries $P_1[1,2], P_1[3,3], P_2[1,2]$ and $P_3[3,1]$ are column-bounded. For $P_1[2,1]$ and $P_2[2,3]$ we prove there are at most two zero-intervals usable for each of them. Otherwise, if there are three zero-intervals $z_1 < z_2 < z_3$ (from top down) usable for $P_[2,1]$ then the one-entries used to map $P_1[1,2]$ and $P_1[3,3]$ in a mapping created when a zero-entry of z_1 changes to one-entry used to map $P_1[1,2]$ together with a one-entry in between z_2 and z_3 give us a mapping of P_2 to M. Symmetrically, the same goes for $P_2[2,3]$ and z_3'.

- $\{P_1, P_3\}$ is row-bounded: we can use the same proof as when showing that $\{P_1, P_2\}$ is column-bounded.

- $\{P_1, P_3\}$ is column-bounded: we can use the same proof as when showing that $\{P_1, P_2\}$ is row-bounded.

□

We prove even stronger result by using a well known fact from the theory of ordered sets.

Fact 44 (Higman's lemma)**.** *Let A be a finite alphabet and A^* be a set of finite sequences over A. Then A^* is well quasi ordered with respect to the subsequence relation.*

Theorem 45. $\sigma = Av\left(\begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}, \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix}, \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}\right)$ *is bounded. Moreover, every subclass is bounded.*

Proof. From Theorem 28 we know that elements of σ fall into finitely many classes. For each we need to prove that it is bounded and also that it does not contain an infinite anti-chain. Knowing that we use Theorem 42 to obtain the result. Let us consider an m by n matrix $M \in \sigma$:

- M only contains up to three non-empty rows (columns):
  Clearly, if M is maximal then it contains three rows made of one-entries and everything else is zero, so the number of one-intervals is bounded by three.

  We use words over alphabet $A = \{a, b, c, d, e, f, g, h, i, j\}$ to describe each M as follows. Let $r_1 < r_2 < r_3$ be the non-empty rows (if less then three are non-empty we choose extra values arbitrarily). We define $w_M \in A^*$ as follows. First, we use letter g r_1 times, letter h $r_2 - r_1$ times, letter i $r_3 - r_2$ times and letter j $m - r_3$ times to describe the number of rows of M. Then we describe columns from the first one to the last one as follows. For each 0 in r_1 we use letter a and for 1, we use ab. For each 0 in r_2 we use letter c and for 1, we use cd. For each 0 in r_3 we use letter e and for 1, we use ef.

  If we have $w_M, w_{M'} \in A^*$ such that w_M is a subsequence of $w_{M'}$ then we want to show that M is an interval minor of M'. Let r_1, r_2, r_3 and r_1', r_2', r_3' be the non-empty rows of M and M' respectively. Since the number of leading letters g is not bigger in w_M, M does not have more empty rows before r_1 than M' does before r_1' and similarly it has at most as many empty rows in between r_1, r_2 and r_2, r_3 and after r_3.

Now consider there is ab in w_M and it corresponds to some $a \dots b$ in $w_{M'}$. We can always assume that in $w_{M'}$ the "a" is the one exactly before b. It can only happen that $abcdeface$ is a subsequence of $\mathbf{ab}cea\mathbf{cd}eac\mathbf{eface}$ if the bold letters are used and since they correspond to one-entries lying in the following columns, this indeed corresponds to an interval minor (but it clearly does not have to mean that M is a submatrix of M').

From Fact 44 we have that A^* is well ordered which means that matrices having at most three non-empty rows (columns) are well ordered (the construction can be extended to every fixed number of non-empty rows) and so they does not have an infitely long anti-chain.

- one-entries of M lie in at most two rows and one column (or vice versa): The number of one-intervals of any such maximal M is bounded by two.

  We use words over alphabet $A = \{a, b, c, d, e, f, g\}$ and for non-empty rows r_1, r_2 and column c_1 we define w_M as follows. We first encode each column in such a way that for each 0 in r_1 we use letter a and for 1, we use ab. For each 0 in r_2 we use letter c and for 1, we use cd. Right before and after the description of column c_1 we put letter g. Next we encode each row in such a way that for each 0 in c_1 we use letter e and for each 1 letters ef. Right before and after the descriptions of rows r_1 and r_2 we again place letter g.

  Because of the distinct letters for encoding rows and columns we can apply the same analysis as we did in the previous case and since entries at $M[r_1, c_1]$ and $M[r_2, c_1]$ are separated from the rest by a special letter g there is no way to find a one-entry if it is not there.

- M avoids $\left(\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}\right)$ (or $\left(\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix}\right)$):
  From Theorem 8 we know M is a walking matrix and any such maximal matrix only contains at most one one-intervals in each row and column.

  We use words over alphabet $A = \{a, b, c, d\}$ and encode M as follows. We choose an arbitrary walk of M containing all one-entries and index its entries as $w_1 \dots w_{m+n-1}$. Starting from w_1 we encode w_i so that a stands for 0 and ab for 1 if w_{i+1} lies in the same row as w_i and we use c for 0 and cd for 1 if w_{i+1} lies in the same column as w_i.

In the construction of words corresponding to matrices, we only made sure that $w_M \subseteq w_{M'} \Rightarrow M \preceq M'$ and the other impication does not hold. A different construction may lead to equivalence, but that is not necessary for our result.

We now use distinct alphabets to discribe different classes and when we given a potentialy infinite class of matrices from σ, we know that inside each class there is at most finite number of minimal matrices such that all of the rest contain a smaller one inside. Using induction on Theorem 42, we have that each class is bounded and by applying induction with Theorem 42 once again we get that the union of the classes is also bounded. □

Observation 46. *There exists a non-trivial bounded pattern P having an unbounded subset of $Av(P)$.*

Proof. Let $P = I_n$ (identity matrix) for $n > 3$. From Lemma 34 we have that P is bounded. On the other hand, $Av(I_n, P_1)$ is unbounded, because the construction used in the proof of Theorem 26 also works for this class. $\square$

We define matrices to be bounded if they are both row-bounded and column-bounded. From what we proved so far, we see that a pattern P is row-bounded if and only of it is column-bounded. But once we look at collections of patterns, this does not have to be true.

Lemma 47. *There exists a class of patters $\mathcal{P}$, which is row-bounded but column-unbounded.*

Proof. Let $\mathcal{P} = \left\{ P = \begin{pmatrix} & \bullet & \\ \bullet & & \\ & \bullet & \\ & & \bullet \end{pmatrix}, I_4 = \begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix} \right\}$. We can use the same construction as we did in Theorem 26, just transposed, to prove $Av(\mathcal{P})$ is column-unbounded.

To prove that $\mathcal{P}$ is row-bounded, we take any M maximal avoiding $\mathcal{P}$ and look at an arbitrary row. In Lemma 34 we proved that patterns avoiding $\begin{pmatrix} & \bullet \\ \bullet & \end{pmatrix}$ are bounded and so every one-entry of I_4 is row-bounded. We need to proof the same for P. Using Lemma 33, $P[2,1]$ and $P[4,3]$ are row-bounded. Using the first case of Lemma 32, $P[3,2]$ is row-bounded. We prove that there are at most two zero-intervals usable for $P[1,2]$. For contradiction, let there be three – $z_1 < z_2 < z_3$. It means there are at least two one-entries $e_1 < e_2$ in between them. Now consider the partitioning of P into M when a zero-entry of z_3 is changed to one-entry used to map $P[1,2]$. Clearly, the one-entry used for mapping $P[2,1]$ lies under the left one-entry e bounding z_3 or in a latter column; otherwise we could use e to map $P[1,2]$ and find the pattern in M. It may happen $e = e_2$, but still e_1 and the one-entries used for mapping $P[2,1], P[3,2]$ and $P[4,3]$ together give us a mapping of I_4 and so a contradiction with $M \in Av(\mathcal{P})$. It means that each one-entry of P is also row-bounded and $Av(\mathcal{P})$ is row-bounded. $\square$

Open questions:

- $Av\left(\begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}\right)$ hereditary

4. Extremal function

Notation 6. Let M be a matrix. We denote $|M|$ the weight of M, the number of one-entries in M.

Usually $|M|$ stands for a determinant of matrix M. However, in this paper we do not work with determinants at all so the notation should not lead to misunderstanding.

Definition 11. For a matrix P we define $Ex(P,m,n) := \max\{|M| | M \in \{0,1\}^{m\times n},\ P \not\leq M\}$. We denote $Ex(P,n) := Ex(P,n,n)$.

Definition 12. For a matrix P we define $Ex_{\preceq}(P,m,n) := max\{|M| | M \in \{0,1\}^{m\times n},\ P \not\preceq M\}$. We denote $Ex_{\preceq}(P,n) := Ex_{\preceq}(P,n,n)$.

Observation 48. *For all P,m,n; $Ex_{\preceq}(P,m,n) \leq Ex(P,m,n)$.*

Observation 49. *If $P \in \{0,1\}^{k\times l}$ has a one-entry at position $[a,b]$, then*

$$Ex(P,m,n) \geq \begin{cases} m\cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \textit{otherwise.} \end{cases}$$

Observation 50. *The same holds for $Ex_{\preceq}(P,m,n)$.*

Definition 13. $P \in \{0,1\}^{k\times l}$ is *(strongly) minimalist* if

$$Ex(P,m,n) = \begin{cases} m\cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Definition 14. $P \in \{0,1\}^{k\times l}$ is *weakly minimalist* if

$$Ex_{\preceq}(P,m,n) = \begin{cases} m\cdot n & k > m \vee l > m \\ (k-1)n + (l-1)m - (k-1)(l-1) & \text{otherwise.} \end{cases}$$

Observation 51. *If P is strongly minimalist, then P is weakly minimalist.*

4.1 Known results

Fact 52.
1. *$(\bullet)$ is strongly minimalist.*
2. *If $P \in \{0,1\}^{k\times l}$ is strongly minimalist and there is a one-entry in the last row in the c-th column, then $P' \in \{0,1\}^{k+1\times l}$, which is created from P by adding a new row having a one-entry only in the c-th column, is strongly minimalist.*
3. *If P is strongly minimalist, then after changing a one-entry into a zero-entry it is still strongly minimalist.*

Fact 53. *Let $P = \begin{pmatrix} \bullet & \cdots & \bullet \\ \bullet & \cdots & \bullet \end{pmatrix}$ have l columns, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{2\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{weight of } M[[i], \{j\}] > 0 \wedge \text{weight of } M[[i+1,m], \{j\} > 0]\}$. Clearly $|A_i| \leq l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M appears in at least $b_j - 1$ of sets A_i, $0 \leq i \leq m-2$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 1) + n \leq \sum_{i=0}^{m-2} |A_i| + n \leq (l-1)(m-1) + n$$

□

This result is indeed very important because it shows that there are matrices like $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, which are weakly minimalist, although it is known they are not strongly minimalist.

Fact 54. *Let* $P = \begin{pmatrix} \bullet & \cdots & \bullet \\ \bullet & \cdots & \bullet \\ \bullet & \cdots & \bullet \end{pmatrix}$ *have l columns, then P is weakly minimalist.*

Proof. Let $M \in \{0,1\}^{m\times n}$ be a matrix avoiding $P = \{1\}^{3\times l}$ as an interval minor and $A_i = \{j \in [n] | \text{ weight of } M[[i-1], \{j\}] > 0 \wedge \text{ weight of } M[[i+1,m], \{j\} > 0 \wedge M[i,j] \text{ one-entry}]\}$. Clearly $|A_i| \leq l-1$, otherwise $P \preceq M$. Let b_j denote the number of one-entries in the j-th column. Each column j of M (for which $b_j \geq 2$) appears in exactly $b_j - 2$ of sets A_i, $1 \leq i \leq m-1$. It follows that

$$\text{weight of } M = \sum_{j=0}^{n} b_j = \sum_{j=0}^{n} (b_j - 2) + 2n \leq \sum_{i=1}^{m-2} |A_i| + 2n \leq (l-1)(m-2) + 2n$$

□

5. Operations with matrices

Notation 7. When speaking about a class of matrices, unless stated otherwise, we always expect the class to be closed under minors. Also, all classes discussed are non-trivial. That means that there is at least one matrix of size 2 by 1 and at least one matrix of size 1 by 2 in each class. Moreover, at least one matrix is non-empty.

Definition 15. Let $\mathcal{F}$ be any class of forbidden matrices. We denote by $Av(\mathcal{F})$ the set of all matrices that avoid every $F \in \mathcal{F}$ as an interval minor.

Observation 55. *Let $\mathcal{T} = Av(\mathcal{F})$ for some $\mathcal{F}$. Then $\mathcal{T}$ is closed under minors.*

Observation 56. *Let $\mathcal{M}$ be a finite class of matrices. There exists a finite set $\mathcal{F}$ such that $\mathcal{M} = Av(\mathcal{F})$.*

Definition 16. For matrices $A \in \{0,1\}^{m \times n}$ and $B \in \{0,1\}^{k \times l}$ we define their *direct sum* as a matrix $C := A \nearrow B \in \{0,1\}^{m+k \times n+l}$ such that $D[[k+1, m+k], [n]] = A$, $D[[k], [n+1, n+l]] = B$ and the rest is empty. Symmetrically, we define $D := A \searrow B \in \{0,1\}^{m+k \times n+l}$ such that $C[[m], [n]] = A$, $C[[m+1, m+k], [n+1, n+l]] = B$ and the rest is empty.

Theorem 57. $Av((\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix})) = (Av((\begin{smallmatrix} \bullet & \\ & \end{smallmatrix})) \searrow Av((\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix})) \searrow Av((\begin{smallmatrix} & \\ & \bullet \end{smallmatrix}))) \cup$
$\cup\, (Av((\begin{smallmatrix} & \\ \bullet & \end{smallmatrix})) \nearrow Av((\begin{smallmatrix} \bullet & \\ & \bullet \end{smallmatrix})) \nearrow Av((\begin{smallmatrix} & \bullet \\ & \end{smallmatrix}))).$

Proof. If follows from Theorem 15 and $Av((\begin{smallmatrix} \bullet & \bullet \\ \bullet & \end{smallmatrix})) = Av((\begin{smallmatrix} \bullet & \\ & \end{smallmatrix})) \searrow Av((\begin{smallmatrix} & \bullet \\ \bullet & \end{smallmatrix}))$. $\square$

Notation 8. Let $\mathcal{M}$ be a class of matrices. Denote by $Cl(\mathcal{M})$ a set containing each $M \in \mathcal{M}$ closed under direct sum and minors.

Definition 17. Let $M \in \{0,1\}^{m \times n}$ be a matrix. We call an element $M[r,c]$ an *articulation* of M if both $M[[r-1], [c-1]]$ and $M[[r+1, m], [c+1, n]]$ are empty.

Lemma 58. *Let $M \in \{0,1\}^{k \times l}$, then for all $X \in \{0,1\}^{m \times n}$ it holds $X \in Cl(M) \Leftrightarrow$ there exists a sequence of articulations of X such that each matrix in between two consecutive articulations of X is a minor of $(\,1\,) \nearrow M \nearrow (\,1\,)$.*

Proof. $\Rightarrow$

$\Leftarrow$

$\square$

Theorem 59. *For all $M \in \{0,1\}^{k \times l}$ there exists $\mathcal{F}$ finite such that $Cl(M) = Av(\mathcal{F})$.*

Proof. Using Lemma 58 $\square$

Theorem 60. $Cl((\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})) = Av\left((\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right).$

Proof. $\subseteq$

$\supseteq$

$\square$

Theorem 61. $Cl\left(\left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)\right) = Av\left(\left(\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&0\\0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\\0&1\end{smallmatrix}\right)\right).$

We can generalize direct sum to allow the matrices to overlap.

Definition 18. TODO $A \oplus_{k\times l} B$

Theorem 62. *Let $\mathcal{C}$ be any class of matrices such that*

- *$\mathcal{C}$ is closed under deleting of one-entries and*
- *$\mathcal{C}$ is closed under the direct sum with $k \times l$ overlap and*
- *there is any $M \in \{0,1\}^{m\times n}$ in $\mathcal{C}$*

then $\mathcal{C}$ is also closed under direct sum with $m - 2k \times n - 2l$ overlap.

Proof. Choose any two $A, B \in \mathcal{C}$ and $C\mathcal{C}$ such that $C \in \{0,1\}^{m\times n}$. Let $D \in \mathcal{C}$ denote the direct sum with $k \times l$ overlap of A and C. Finally, let E be the direct sum with $k \times l$ overlap of D and B. It has the same size as F, the direct sum with $m - 2k \times n - 2l$ overlap of A and B, which set of one-entries is also a subset of one-entries of $E \in \mathcal{C}$; therefore $F \in \mathcal{C}$. □

Theorem 63. *Let $\mathcal{C}$ be any class of matrices that is hereditary according to interval minors then for all m, n, k, l if $\mathcal{C}$ is closed under the direct sum with $m \times n$ overlap then is is also closed under the direct sum with $m + k \times n + l$ overlap.*

Proof. For contradiction, assume there are $A, B \in \mathcal{C}$ such that $A \oplus_{m+k\times n+l} B \notin \mathcal{C}$. □

Observation 64. *There is a $\mathcal{C}$ hereditary according to submatrices such that it is closed under the direct sum but it is not closed under the direct sum with 1×1 overlap.*

Proof. Let $\mathcal{C}$ be a class of all matrices obtained by applying the direct sum on $\left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right)$. Clearly, it is closed under the direct sum. On the other hand, $\left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right) \oplus_{1\times 1} \left(\begin{smallmatrix} & \bullet\\ \bullet & \end{smallmatrix}\right) = \begin{pmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \end{pmatrix} \notin \mathcal{C}$. □

Notation 9. We define $\mathrm{Av}(M)$ to be a class of all matrices avoiding M as

We state following characterization only for the direct sum with 1×1 overlap but, because of Theorem 63, it also holds for any other size of overlap.

Theorem 65. *Let M be a matrix. There are M_1, M_2 proper submatrices of M such that $M = M_1 \oplus_{1\times 1} M_2 \Leftrightarrow$ Av(M) is not closed under the direct sum with 1×1 overlap.*

Proof. $\Rightarrow$

$\Leftarrow$

□

Observation 66. *Let M be a matrix. There are M_1, M_2 proper submatrices of M such that $M = M_1 \oplus_{1\times 1} M_2 \Leftrightarrow$ exists r, c such that either*

1. $M[r,c]$ *is a one-entry and* $(r,c) \in \{(1,1),(m,n)\}$ *or*

2. $M[r,c]$ *is both top-right and bottom-left empty and* $(r,c) \notin \{(1,1),(m,n)\}$

Definition 19. Let F be a matrix. We denote $\mathcal{R}(F)$ to be a set of all minimal (relating to minors) matrices F' such that $F \preceq F'$ and F' is not a direct sum with 1×1 overlap of proper submatrices of F'. For a class of matrices $\mathcal{F}$ let $\mathcal{R}(\mathcal{F})$ denote a set of all minimal (relating to minors) matrices from the set $\bigcup_{F\in\mathcal{F}} \mathcal{R}(F)$.

Theorem 67. *Let* $\mathcal{T}$ *and* $\mathcal{F}$ *be classes of matrices such that* $\mathcal{T} = Av(\mathcal{F})$*, then* $Cl(\mathcal{T}) = Av(\mathcal{R}(\mathcal{F}))$.

Proof. Need to change the proof a bit probably after changing the statement

- $\subseteq$ Instead of proving $M \in Cl(\mathcal{T}) \Rightarrow M \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ we show $M \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F)) \Rightarrow M \notin Cl(\mathcal{T})$. Assume $M \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$. It follow from the definition that $M \in \bigcup_{F\in\mathcal{F}} \mathcal{R}(F)$, in particular, $M \in \mathcal{R}(F)$ for some $f \in \mathcal{F}$. Because of the definition of $\mathcal{R}(F)$, M is not a direct sum with 1×1 overlap of proper submatrices of M which means, according to Observation 66, there are no non-trivial articulations and both top-right and bottom-left corners are empty. For contradiction, assume $M \in Cl(\mathcal{T})$, then, according to a generalization of Lemma 58, there exists a sequence of articulations of M such that each matrix in between two consecutive articulations of M is a minor of $\begin{pmatrix} 1 \end{pmatrix} \nearrow T \nearrow \begin{pmatrix} 1 \end{pmatrix}$ for some $T \in \mathcal{T}$. Since M has only trivial articulations and they are both empty, it holds $M \preceq T$ and because of the choice of M is also holds $M \preceq F$ for some $F \in \mathcal{F}$ which together give us a contradiction with $\mathcal{T} = Av(\mathcal{F})$.

- $\supseteq$ First of all, $Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ is closed under the direct sum with 1×1 overlap. For contradiction, assume there are $M_1, M_2 \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ but $M = M_1 \nearrow_1 M_2 \notin Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$. Then there exists $F' \in \mathcal{R}(F)$ for some $F \in \mathcal{F}$ such that $F' \preceq M$. Because F' is not a direct sum with 1×1 overlap of proper submatrices of F', it follows that either $F' \preceq M_1$ or $F' \preceq M_2$ and since $F \preceq F'$ we have a contradiction.

  Now that we know that both sides are closed under the direct sum with 1×1 overlap it sufficient to show that the inclusion holds for any $M \in Av(\bigcup_{F\in\mathcal{F}} \mathcal{R}(F))$ that is not a direct sum with 1×1 overlap of proper submatrices of M. Such M does not contain (again from Observation 66) any non-trivial articulation and those trivial ones are empty. Because of that it holds $F \not\preceq M$ for every $F \in \mathcal{F}$; otherwise either $M \in \mathcal{R}(F)$ or its minor would be there. Therefore $M \in \mathcal{T}$ and also $M \in Cl(\mathcal{T})$.

□

Definition 20. Let T be a class of matrices. The *basis* of T is a set of all minimal (relating to minors) matrices that do not belong to T.

Corollary 68. *Let* $\mathcal{T}$ *and* $\mathcal{F}$ *be classes of matrices such that* $\mathcal{T} = Av(\mathcal{F})$*, then* $\mathcal{R}(\mathcal{F})$ *is a basis of* $Cl(\mathcal{T})$.

Proof. The proof follow directly from Theorem 67. □

A natural question follows, whether the closure under direct sum of a class with finite basis has final basis. We prove that this is not the case.

Definition 21. Let $Nucleus_1 = (\bullet)$ and for $n > 1$ let $Nucleus_n \in \{0,1\}^{n \times n+1}$ be a matrix described by the examples:

$Nucleus_2 = \begin{pmatrix} \bullet & \bullet & \\ & \bullet & \bullet \end{pmatrix}$, $Nucleus_3 = \begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & \bullet & \end{pmatrix}$, $Nucleus_4 = \begin{pmatrix} & & \bullet & \bullet & \\ & \bullet & & & \bullet \\ \bullet & & & \bullet & \\ & \bullet & \bullet & & \end{pmatrix}$,

$Nucleus_5 = \begin{pmatrix} & & & \bullet & \bullet & \\ & & \bullet & & & \bullet \\ & \bullet & & & \bullet & \\ \bullet & & & \bullet & & \\ & \bullet & \bullet & & & \end{pmatrix}$, $Nucleus_n = \begin{pmatrix} & & & & & \bullet & \bullet & \\ & & & & \bullet & & & \bullet \\ & & & \bullet & & & \bullet & \\ & & \cdots & & & \cdots & & \\ & \bullet & & & \bullet & & & \\ \bullet & & & \bullet & & & & \\ & \bullet & \bullet & & & & & \end{pmatrix}$.

Definition 22. Let $Candy_{k,n,l}$ be a matrix given by $I_k \nearrow_{1\times 2} Nucleus_n \nearrow_{1\times 2} I_l$, where I_k, I_l are unit matrices of sizes $k \times k$ and $l \times l$ respectively.

$$Candy_{4,1,4} \begin{pmatrix} & & & \bullet & & & \\ & & & & \bullet & & \\ & & & & & \bullet & \\ \bullet & & & \bullet & & & \bullet \\ & \bullet & & & & & \\ & & \bullet & & & & \\ & & & \bullet & & & \end{pmatrix} \quad Candy_{4,4,4} \begin{pmatrix} & & & & & \bullet & & & \\ & & & & & & \bullet & & \\ & & & & & & & \bullet & \\ & & & & \bullet & \bullet & & & \bullet \\ & & & \bullet & & & \bullet & & \\ & & \bullet & & & \bullet & & & \\ \bullet & & & \bullet & \bullet & & & & \\ & \bullet & & & & & & & \\ & & \bullet & & & & & & \\ & & & \bullet & & & & & \end{pmatrix}$$

Theorem 69. *There exists a matrix F such that $\mathcal{R}(F)$ is infinite.*

Proof. □

Corollary 70. *There exists a class of matrices $\mathcal{C}$ having a finite basis such that $Cl(\mathcal{C})$ has an infinite basis.*

Proof. From Theorem 69, we have a matrix F for which $\mathcal{R}(F)$ is infinite. Let $\mathcal{C} = Av(F)$. Clearly, $\mathcal{C}$ has a finite basis. On the other hand, from Theorem 67 we have $Cl(\mathcal{C}) = Av(\mathcal{R}(F))$ and $\mathcal{R}(F)$ is infinite from the choice of F. □