EXERCISE 2(*) Consider the following differential equation:

$$\begin{cases} y'(t) = -C\arctan(ky), & t > 0 \\ y(0) = y_0 \ , \end{cases} \tag{1}$$

where C and k are given real positive constants.

(a) Write the backward Euler scheme for solving (1) in the form

$$u_{n+1} = g(u_n, u_{n+1}, h), \tag{2}$$

specifying the function g, where h is the timestep and u_n the approximation of $y(t_n)$.

(b) For each timestep one has to solve the nonlinear equation (2). Interpret this equation as a fixed point problem for the computation of u_{n+1}, and determine a condition on h which guarantees that there exists a unique fixed point to which the fixed point iteration $x^{k+1} = \phi(x^k)$ converges for any initial guess.

(c) Write down the Newton iteration for the solution of the nonlinear equation (2).

Sol :

a), To get a backward Euler scheme, we should follow the form :

$$\begin{cases} u_0 = y_0 , \\ u_{n+1} = u_n + h_n f(t_{n+1}, u_{n+1}) , \quad n = 0, \dots, N-1 \end{cases}$$

The initial value problem is considered as the following form:

$$\begin{cases} y'(t) = f(t, y(t)) \quad \text{for } t > 0 \\ y(0) = y_0 \end{cases}$$

And according to the differential equation given by (1),

$$\begin{cases} y'(t) = -C\arctan(ky) \quad t > 0 \\ y(0) = y_0 \end{cases}$$

So, we can know that $f(t, y(t)) = -C\arctan(k\,y(t))$.

Therefore, we can get

$$
\begin{aligned}
u_{n+1} &= u_n + h_n f(t_{n+1}, u_{n+1}) \\
&= u_n + h_n \cdot [-C \arctan(k \cdot u_{n+1})] \\
&= u_n - h_n \cdot C \arctan(k \cdot u_{n+1})
\end{aligned}
$$

So, the function $g = u_n - h_n \cdot C \arctan(k \cdot u_{n+1})$.

b). To solve the nonlinear equation by fixed point method, we should interpret the equation as a fixed point problem.

Since $u_{n+1} = u_n - h_n \cdot C \arctan(k \cdot u_{n+1})$,
So $f(u_{n+1}) = u_{n+1} - u_n + h_n \cdot C \arctan(k \cdot u_{n+1})$

Then, according to $\phi(x) = x - \beta f(x)$, we take $\beta = 1$, so,

$$\phi(u_{n+1}) = u_n - h_n \cdot C \arctan(k \cdot u_{n+1}) = u_{n+1},$$

$$\phi'(u_{n+1}) = -\frac{h_n \cdot C \cdot k}{1 + (k \cdot u_{n+1})^2}.$$

According to the Theorem 3.3.8, if ϕ is continuously differentiable and $|\phi'(x)| < 1$, then there exists a unque fixed point and iteration converges for any initial guess.

So, $-1 < \phi'(u_{n+1}) < 1$

$$\Rightarrow \quad -1 < -\frac{h_n \cdot C \cdot k}{1+(k\cdot U_{n+1})^2} < 1$$

$$\Rightarrow \quad -1-(k\cdot U_{n+1})^2 < -h_n \cdot C \cdot k < 1+(k\cdot U_{n+1})^2$$

$$\Rightarrow \quad -1-(k\cdot U_{n+1})^2 < h_n \cdot C \cdot k < 1+(k\cdot U_{n+1})^2$$

$$\Rightarrow \quad \frac{-1-(k\cdot U_{n+1})^2}{C\cdot k} < h_n < \frac{1+(k\cdot U_{n+1})^2}{C\cdot k}$$

Therefore, for condition $\frac{-1-(k\cdot U_{n+1})^2}{C\cdot k} < h < \frac{1+(k\cdot U_{n+1})^2}{C\cdot k}$, there exists a unque fixed point and iteration converges for any initial guess.

C), Since the Newton's method is:

$$X_{k+1} = X_k - \frac{f(x_k)}{f'(x_k)},$$

So, $$U_{n+1} = U_n - \frac{f(U_n)}{f'(U_n)}$$

Because $f(U_n) = U_n - U_{n-1} + h_{n-1} \cdot C \arctan(k\cdot U_n)$.

So, $$f'(U_n) = 1 + \frac{h_{n-1}\cdot C \cdot k}{1+(k\cdot U_n)^2}$$

Thus, $$U_{n+1} = U_n - \frac{U_n - U_{n-1} + h \cdot C \cdot \arctan(k\cdot U_n)}{1+\frac{h\cdot C\cdot k}{1+(k\cdot U_n)^2}}$$