MATH0033 Numerical Methods, 2022-2023, David Hewett

Theoretical exercise sheet 2
Linear systems

Feedback from marking

GENERAL YOU NEED TO BE ABLE TO MULTIPLY, INVERT, AND FIND THE EIGENVALUES OF 2×2 and (SIMPLE) 3×3 MATRICES! Also, you need to know the definitions of the spectrum $\sigma(A)$ and the spectral radius $\rho(A)$. They are NOT the same thing. And don't forget absolute values, e.g. in the definition of $\rho(A)$, and when taking square roots, e.g. $\sqrt{\gamma^2} = |\gamma|$ for $\gamma \in \mathbb{R}$. (We always have in mind the principle branch of the square root.)

EXERCISE 1(*)
(a) and (b) Mostly fine.
(c) A lot of people didn't realise that $\sqrt{2/15} > 2/15$. (In general, $\sqrt{x} > x$ for $0 < x < 1$ and $\sqrt{x} < x$ for $x > 1$.)
(c) and (d) To check that both methods converge you could either show that A is SDD, or show that the spectral radii $\rho(B)$ for each method is less than 1. The spectral radius for GS is smaller than that for J. But this doesn't automatically mean the convergence is faster, with respect to some given norm (e.g. the 2-norm $\|\cdot\|_2$). If B were symmetric then $\rho(B)$ would be a convergence constant w.r.t. the 2-norm, since we would have the error bound $\|e^{k+1}\|_2 \le \rho(B)\|e^k\|_2$. But in our case B is NOT symmetric. So the spectral radius only gives you a "indication" of the convergence rate, not a rigorous error estimate. To say something rigorous about the convergence constant in the $\|\cdot\|_2$ norm you would need to compute $\|B\|_2$ by other means, which I didn't expect you to do. Instead, in part (d) you were asked to investigate the convergence rate in the $\|\cdot\|_\infty$ norm, by computing $\|B\|_\infty$ for the two methods. This is easy for this 2D example, using the formula from the notes. And then you definitely know $\|e^{k+1}\|_\infty \le \|B\|_\infty \|e^k\|_\infty$, so you can compare theoretical convergence rates by comparing the values of $\|B\|_\infty$. Some people omitted to do this. (Note: Gauss-Seidel still wins!)

EXERCISE 5(*)
(a) Make sure you remember to check ALL of the conditions in the definition of a norm. Re the norm equivalence, finding a value for C is not too hard, but c is a bit harder. For this, note that

$$\|\mathbf{x}\|_2 = \|A^{-1/2}A^{1/2}\mathbf{x}\|_2 \le \|A^{-1/2}\|_2\|A^{1/2}\mathbf{x}\|_2 = \|A^{-1/2}\|_2\|\mathbf{x}\|_A,$$

so one can take $c = 1/\|A^{-1/2}\|_2$. Some people expressed c and C in terms of the max and min eigenvalues of A, which was fine.
(b) Showing $A^{1/2}B_\alpha A^{-1/2} = B_\alpha$ is quite easy (just plug in the definition of B_α). But then to apply it, the correct argument is as follows: first note that $A^{1/2}B_\alpha A^{-1/2} = B_\alpha$ implies $A^{1/2}B_\alpha = B_\alpha A^{1/2}$, and then we find that

$$\begin{aligned}\|\mathbf{e}^{k+1}\|_A = \|B_\alpha \mathbf{e}^k\|_A = \|A^{1/2}B_\alpha \mathbf{e}^k\|_2 &= \|B_\alpha A^{1/2}\mathbf{e}^k\|_2 \\ &\le \|B_\alpha\|_2\|A^{1/2}\mathbf{e}^k\|_2 \\ &= \rho(B_\alpha)\|\mathbf{e}^k\|_A,\end{aligned}$$

Note that all but one of these steps is an EQUALITY. Only in bounding $\|B_\alpha A^{1/2}\mathbf{e}^k\|_2 \le \|B_\alpha\|_2\|A^{1/2}\mathbf{e}^k\|_2$ do we introduce an inequality.