*MATH0033 Numerical Methods*
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**Self-study problem sheet 2**

1. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. We admit without proof the fact that we can write
$$A = V\Sigma V^{\mathsf{T}}$$
where $V$ is a matrix whose columns are composed of the eigenvectors of $A$ satisfying the orthogonality condition $VV^{\mathsf{T}} = V^{\mathsf{T}}V = I$ and
$$\Sigma = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$
is the diagonal matrix with the eigenvalues $\{\lambda_j\}_{j=1}^n$ (which are all real) along the diagonal.

   (a) Show that $\|V\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

   (b) Show that the spectral radius satisfies
$$\rho(A) = \|A\|_2.$$

2. Let $\|\cdot\|_M$ be the matrix norm induced by a norm $\|\cdot\|_V$ on $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix.

   (a) Show that $K_M(A) \geq 1$

   (b) Show that $K_M(\alpha A) = K_M(A)$ for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

   (c) Show that $K_M(A^{-1}) = K_M(A)$.

   (d) Show that for any *singular* matrix $B$, we have the bound
$$\frac{\|A - B\|_M}{\|A\|_M} \geq \frac{1}{K_M(A)}.$$

**SOLUTIONS**

1. (a) We have by definition of the 2-norm $\|V\boldsymbol{x}\|_2^2 = (V\boldsymbol{x})^\mathsf{T} V\boldsymbol{x} = \boldsymbol{x}^\mathsf{T} V^\mathsf{T} V\boldsymbol{x} = \boldsymbol{x}^\mathsf{T}\boldsymbol{x} = \|\boldsymbol{x}\|_2^2$.

   (b) Since we have $\rho(A) \leq \|A\|_2$ from the lectures, it is enough to show that $\rho(A) \geq \|A\|_2$.
   For any $\boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, we have $\boldsymbol{y} = V\boldsymbol{x}$ for $\boldsymbol{x} = V^\mathsf{T}\boldsymbol{y} \neq \boldsymbol{0}$. Therefore

$$\frac{\|A\boldsymbol{y}\|_2}{\|\boldsymbol{y}\|_2} = \frac{\|AV\boldsymbol{x}\|_2}{\|V\boldsymbol{x}\|} = \frac{\|V\Sigma V^\mathsf{T} V\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} = \frac{\|V\Sigma\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} = \frac{\|\Sigma\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2}$$

   where we have used $\|V\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ and $\|V\Sigma\boldsymbol{x}\|_2 = \|\Sigma\boldsymbol{x}\|_2$ from (a) above. Therefore, we have

$$\|A\|_2 = \sup_{\boldsymbol{y}\in\mathbb{R}^n\setminus\{\boldsymbol{0}\}} \frac{\|A\boldsymbol{y}\|_2}{\|\boldsymbol{y}\|_2} = \sup_{\boldsymbol{x}\in\mathbb{R}^n\setminus\{\boldsymbol{0}\}} \frac{\|\Sigma\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} = \sup_{\boldsymbol{x}\in\mathbb{R}^n\setminus\{\boldsymbol{0}\}} \frac{\sqrt{\sum_{i=1}^n |\lambda_i|^2|x_i|^2}}{\sqrt{\sum_{i=1}^n |x_i|^2}}$$

   Clearly $\sum_{i=1}^n |\lambda_i|^2|x_i|^2 \leq \rho(A)^2 \sum_{i=1} |x_i|^2$ since $\rho(A) \geq |\lambda_i|$ for all eigenvalues $\lambda_i$ of $A$. From this we then get

$$\|A\|_2 \leq \rho(A)$$

   which completes the proof.

2. (a) Since $AA^{-1} = I$ we get $1 = \|I\|_M \leq \|A\|_M\|A^{-1}\|_M = K_M(A)$.

   (b) Since $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ we get $\|(\alpha A)\|_M\|(\alpha A)^{-1}\|)_M = \|A\|_M\|A^{-1}\|_M$. This gives the result.

   (c) This follows from $(A^{-1})^{-1} = A$.

   (d) $B$ is singular implies that there exists a vector $\boldsymbol{v} \neq 0$ such that $B\boldsymbol{v} = 0$. We then write $B = A - (A - B)$ to obtain

$$\boldsymbol{0} = B\boldsymbol{v} = A\boldsymbol{v} - (A - B)\boldsymbol{v} \iff A\boldsymbol{v} = (A - B)\boldsymbol{v} \iff \boldsymbol{v} = (I - A^{-1}B)\boldsymbol{v}.$$

   Therefore, by definition of the induced norm,

$$\|(I - A^{-1}B)\|_M \geq \frac{\|(I - A^{-1}B)\boldsymbol{v}\|_V}{\|\boldsymbol{v}\|_V} = \frac{\|\boldsymbol{v}\|_V}{\|\boldsymbol{v}\|_V} = 1.$$

   But then $\|(I - A^{-1}B)\|_M = \|A^{-1}(A - B)\|_M \leq \|A^{-1}\|_M\|A - B\|_M$, and thus

$$\frac{\|A - B\|_M}{\|A\|_M} \geq \frac{\|(I - A^{-1}B)\|_M}{\|A\|_M\|A^{-1}\|_M} \geq \frac{1}{K_M(A)}.$$