EXERCISE 5(*) Let A be a n-by-n symmetric positive definite matrix, and define the function $\|\mathbf{x}\|_A : \mathbb{R}^n \to \mathbb{R}$ by $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^T A\mathbf{x}}$.

(a) Let $A^{1/2}$ denote the unique symmetric positive definite square root of A, which satisfies $(A^{1/2})^2 = A$ (you may assume without proof that $A^{1/2}$ exists). Show that $\|\mathbf{x}\|_A = \|A^{1/2}\mathbf{x}\|_2$, and use this fact to check that the function $\|\mathbf{x}\|_A$ defines a norm on $\mathbb{R}^n$. Determine constants $0 < c \leq C$ such that

$$c\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_A \leq C\|\mathbf{x}\|_2, \qquad \forall \mathbf{x} \in \mathbb{R}^n.$$

(b) Now consider the stationary Richardson method for the solution of the linear system $A\mathbf{x} = \mathbf{b}$, with iteration matrix $B_\alpha = I - \alpha A$. Show that $A^{1/2}B_\alpha = B_\alpha A^{1/2}$. Use this to prove that

$$\|\mathbf{e}^{k+1}\|_A \leq \rho(B_\alpha)\|\mathbf{e}^k\|_A,$$

where $\mathbf{e}^k = \mathbf{x} - \mathbf{x}^k$ denotes the solution error after the kth iteration.

Sol:

a), Firstly, we should show that $\|x\|_A = \|A^{\frac{1}{2}}x\|_2$.

Since $\|A^{\frac{1}{2}}x\|_2 = \sqrt{(A^{\frac{1}{2}}x)\cdot(A^{\frac{1}{2}}x)^T}$,

and $\|x\|_A = \sqrt{x^T A x} = \sqrt{x^T[(A^{\frac{1}{2}})^T A^{\frac{1}{2}}]x}$

$= \sqrt{(A^{\frac{1}{2}}\cdot x)^T\cdot(A^{\frac{1}{2}}\cdot x)} = \|A^{\frac{1}{2}}x\|_2$

Next, we should check the function $\|x\|_A$ defines a norm on $\mathbb{R}^n$.

If $\|x\|_A : \mathbb{R}^n \to \mathbb{R}$ is a norm on $\mathbb{R}^n$, it must satisfy the following conditions:

1, (i) $\|x\|_A \geqslant 0$, $\forall x \in \mathbb{R}^n$, and (ii) $\|x\|_A = 0$ if and only if $x = 0$;

Check: Since $A^{\frac{1}{2}}$ is symmetric positive matrix, so if $x \in \mathbb{R}^n$ and $x \neq 0$, we must have $A^{\frac{1}{2}}x \neq 0$, and $(A^{\frac{1}{2}}x)^2 \geqslant 0$; And only $x = 0$, $A^{\frac{1}{2}}x = 0$.

2, $\|\alpha x\|_A = |\alpha|\,\|x\|_A$ $\forall \alpha \in \mathbb{R}$ and $\forall x \in \mathbb{R}^n$;

Check: $\|\alpha x\|_A = \|\alpha \cdot A^{\frac{1}{2}}x\|_2 = |\alpha|\,\|A^{\frac{1}{2}}x\|_2 = |\alpha|\,\|x\|_A$

3, $\|x + w\|_A \leqslant \|x\|_A + \|w\|_A$, $\forall x, w \in \mathbb{R}^n$;

Check: $\|x+w\|_A = \|A^{\frac{1}{2}}(x+w)\|_2 \leqslant \|A^{\frac{1}{2}}x\|_2 + \|A^{\frac{1}{2}}w\|_2 = \|x\|_A + \|w\|_A$

Therefore, the function $\|x\|_A$ defines a norm on $\mathbb{R}^n$.

Finally, the constants $0 < c \leq C$ such that

$$c\|x\|_2 \leq \|x\|_A \leq C\|x\|_2, \ \forall x \in R^n$$

Since the function $\|x\|_A$ defines a norm, and $\|x\|_A = \|A^{\frac{1}{2}}x\|_2$, we can get that:

$$\|cx\|_2 \leq \|A^{\frac{1}{2}}x\|_2 \leq \|Cx\|_2$$

Because A is a n-by-n symmetric positive define matrix, so $A^{\frac{1}{2}} > 0$.

Therefore, $0 < c = C = A^{\frac{1}{2}}$.

b), Firstly, we should to prove that $A^{\frac{1}{2}}B_\alpha = B_\alpha A^{\frac{1}{2}}$,

Since $B_\alpha = I - \alpha A$, so we can get:

$$A^{\frac{1}{2}}B_\alpha = B_\alpha A^{\frac{1}{2}} \Rightarrow A^{\frac{1}{2}}(I - \alpha A) = (I - \alpha A) \cdot A^{\frac{1}{2}}$$

$$\Rightarrow A^{\frac{1}{2}} - A^{\frac{1}{2}}\alpha A = A^{\frac{1}{2}} - \alpha A A^{\frac{1}{2}}$$

Because relaxation parameter $\alpha \in R$, so,

$$\Rightarrow A^{\frac{1}{2}} - \alpha A^{\frac{1}{2}}A = A^{\frac{1}{2}} - \alpha A A^{\frac{1}{2}}$$

$$\Rightarrow A^{\frac{1}{2}} - \alpha I = A^{\frac{1}{2}} - \alpha I$$

Therefore, $A^{\frac{1}{2}}B_\alpha = B_\alpha A^{\frac{1}{2}}$.

Next, we need to prove that $\|e^{k+1}\|_A \leq \rho(B_\alpha)\|e^k\|_A$.

According to the question, we can get that:

$$\|e^{k+1}\|_2 = \|B_\alpha \cdot e^k\|_2 \leq \|B_\alpha\|_2 \cdot \|e^k\|_2$$

Then, we multipy $\|A^{\frac{1}{2}}\|_2$ to both sides.

$$\|A^{\frac{1}{2}}\|_2 \, \|e^{k+1}\|_2 \leq \|A^{\frac{1}{2}}\|_2 \cdot \|B_\alpha\|_2 \cdot \|e^k\|_2$$

Because $A^{\frac{1}{2}} B_\alpha = B_\alpha A^{\frac{1}{2}}$, Next, we can get:

$$\|A^{\frac{1}{2}} e^{k+1}\|_2 \leq \|B_\alpha\|_2 \cdot \|A^{\frac{1}{2}} \cdot e^k\|_2$$

From Lemma 4.4.3, we can know that $\|B_\alpha\|_2 = \rho(B_\alpha)$.

So, $\|e^{k+1}\|_A \leq \rho(B_\alpha) \|e^k\|_A$.