EXERCISE 5(*) Consider a two-by-two system of differential equations

$$\begin{cases} \mathbf{w}'(t) = A\mathbf{w}(t), \\ \mathbf{w}(0) = \mathbf{w}_0, \end{cases} \qquad \mathbf{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \tag{4}$$

where A is a given 2×2 matrix.

(a) Write down the forward Euler method, the backward Euler method and the Crank-Nicolson method for the system (4).

(b) Suppose that you are given the diagonalization $A = VDV^{-1}$, where D is a diagonal matrix containing the eigenvalues d_1 and d_2 of A, and V is an invertible matrix whose columns are the corresponding eigenvectors. Show how the problem (4) and the three schemes in part (a) can be rewritten in terms of the diagonal matrix D and the transformed unknowns $\mathbf{x} = V^{-1}\mathbf{w}$ and $\mathbf{x}^n = V^{-1}\mathbf{w}^n$.

(c) Using the transformations in (b), determine for which step sizes $h > 0$ the three schemes are absolutely stable, under the assumption that the eigenvalues satisfy $d_1, d_2 < 0$.

(d) Now consider the particular system of differential equations:

$$\begin{cases} w_1'(t) = w_2(t), & t > 0, \\ w_2'(t) = -\lambda\, w_1(t) - \mu\, w_2(t), & t > 0, \\ w_1(0) = w_{1,0}, \\ w_2(0) = w_{2,0}, \end{cases} \tag{5}$$

where λ and μ are two positive real numbers such that $\mu^2 - 4\lambda > 0$.

Write the system (5) in the form (4), specifying the matrix A. Use your results in (c) to determine the stability of the three schemes in this case. What is the stability condition for the forward Euler method in the special case $\lambda = 6$ and $\mu = 5$?

Sol:

a), Since $\begin{cases} w'(t) = Aw(t), \\ w(0) = w_0, \end{cases}$ $w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$, A is 2×2 matrix,

So, $y(t, w(t)) = A \cdot w(t)$.

Forward Euler method:

$$\begin{cases} w^0 = w_0 \\ w^{n+1} = w^n + h_n \cdot A \cdot w^n \end{cases}$$

Backward Euler method:

$$\begin{cases} w^0 = w_0 \\ w^{n+1} = w^n + h_n \cdot A \cdot w^{n+1} \end{cases}$$

Crank-Nicolson method:

$$\begin{cases} W^0 = W_0 \\ W^{n+1} = W^n + \frac{h_n}{2} \cdot (A \cdot W^n + A \cdot W^{n+1}) \end{cases}$$

b), we should find a $y(t, x(t))$ to instand of $y(t, w(t))$,
Due to $A = VDV^{-1}$, $x = V^{-1}w$, and $x^n = V^{-1}w^n$.
So, $Vx = w$

Since $w'(t) = A \cdot w(t)$.
So, $w'(t) = Vx' = A \cdot Vx = VDV^{-1}Vx = VDx$,
Thus, $x' = Dx \Rightarrow x'(t) = D \cdot x(t)$

Forward Euler method:

$$\begin{cases} x^0 = V^{-1} \cdot w_0 \\ x^{n+1} = x^n + h_n \cdot D \cdot x^n \end{cases}$$

Backward Euler method:

$$\begin{cases} x^0 = V^{-1} \cdot w_0 \\ x^{n+1} = x^n + h_n \cdot D \cdot x^{n+1} \end{cases}$$

Crank-Nicolson method:

$$\begin{cases} x^0 = V^{-1} \cdot w_0 \\ x^{n+1} = x^n + \frac{h_n}{2} \cdot (D \cdot x^n + D \cdot x^{n+1}) \end{cases}$$

C), According to the concept of absolutely stable.

$$y'(t) = \lambda \cdot y(t)$$

Since we have $x'(t) = D \cdot x(t)$ in parts (b), so $\lambda = D$.

For forward Euler method, we have:

$$x^{n+1} = x^n + h_n \cdot D \cdot x^n = x^n \cdot (1 + h \cdot \lambda) = (1 + h \cdot \lambda)^{n+1}, \quad n \geq 0.$$

So, $\lim_{n \to \infty} x^n = 0$, if and only if $|1 + h\lambda| < 1$, so $h < 0$.
The forward Euler method is not absolutely stable.

For backward Euler method, we have:

$$x^{n+1} = x^n + h_n \cdot D \cdot x^{n+1}$$

$$\Rightarrow (1 - h \cdot \lambda) \cdot x^{n+1} = x^n$$

$$\Rightarrow x^{n+1} = \left(\frac{1}{1 - h \cdot \lambda}\right) \cdot x^n = \left(\frac{1}{1 - h \cdot \lambda}\right)^{n+1}$$

So, $\lim_{n \to \infty} x^n = 0$, if and only if $\left|\frac{1}{1 - h \cdot \lambda}\right| < 1$, so $h > 0$.
The backward Euler method is absolutely stable.

For Crank-Nicolson method, we have:

$$x^{n+1} = x^n + \frac{h_n}{2} \cdot (D \cdot x^n + D \cdot x^{n+1})$$

$$\Rightarrow x^{n+1} = \left[\frac{1 + \frac{h \cdot \lambda}{2}}{1 - \frac{h \cdot \lambda}{2}}\right] \cdot x^n = \left[\frac{1 + \frac{h \cdot \lambda}{2}}{1 - \frac{h \cdot \lambda}{2}}\right]^{n+1}$$

So, $\lim_{n \to \infty} x^n = 0$, if and only if $\frac{|1 + h \cdot \lambda / 2|}{|1 - h \cdot \lambda / 2|} < 1$,
The Crank-Nicolson method is absolutely stable.

d). Since $\begin{cases} w'(t) = Aw(t), \\ w(0) = w_0, \end{cases}$ $w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$, A is 2×2 matrix,

and $\begin{cases} w_1'(t) = w_2(t), & t>0 \\ w_2'(t) = -\lambda w_1(t) - \mu \cdot w_2(t), & t>0 \end{cases}$

we can know that

$$A = \begin{bmatrix} 0 & 1 \\ -\lambda & -\mu \end{bmatrix}$$