§ 5.7 - Systems of ODEs

In applications one commonly encounters systems of ODEs, giving IVPs of the form

$$\begin{cases} y'(t) = f(t, y(t)), & t > 0 \\ y(0) = y_0, \end{cases}$$

where $y : [0,\infty) \to \mathbb{R}^p$ for some $p \in \mathbb{N}$, $y_0 \in \mathbb{R}^p$ and $f : [0,\infty) \times \mathbb{R}^p \to \mathbb{R}^p$.

Classical examples include the SIR model for infectious diseases (particularly important in the current times!), Newton's laws of gravitation, and predator-prey models in ecology (see Examples 5.1.1 and 5.8.1).

SIR model for infectious diseases:

$$\begin{cases} \dfrac{dS}{dt} = -\beta I S \\ \dfrac{dI}{dt} = \beta I S - \gamma I \\ \dfrac{dR}{dt} = \gamma I \\ S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0, \quad \text{with } S_0 + I_0 + R_0 = 1. \end{cases}$$

Note: $p = 3$ here, with $y = (S, I, R)^T \in \mathbb{R}^3$, and $f(t, y) = (-\beta I S, \beta I S - \gamma I, \gamma I)^T$, which is Lipschitz w.r.t. S, I and R.

where S = proportion of population susceptible to infection

I = " " " infected, and infectious to others

R = " " " removed/resistent (either dead, or recovered and now immune)

Note: $\frac{d}{dt}(S + I + R) = 0$, so

$S(t) + I(t) + R(t) = 1$ for all $t \geq 0$.

SIR model for infectious diseases:

$$\begin{cases} \dfrac{dS}{dt} = -\beta I S \\ \dfrac{dI}{dt} = \beta I S - \gamma I \\ \dfrac{dR}{dt} = \gamma I \\ S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0, \quad \text{with } S_0 + I_0 + R_0 = 1. \end{cases}$$

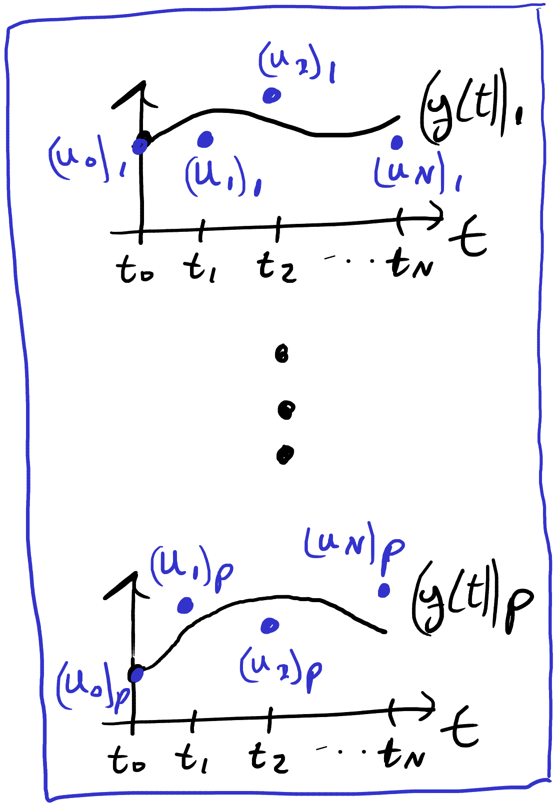
Here $1/\beta$ = typical time between contacts

and $1/\gamma$ = typical time until removal (death / recovery)

The ratio $\boxed{R_0 = \beta/\gamma}$ is called the <u>basic reproduction number</u>.

For COVID-19, $1/\gamma \approx 7$ days, while the value for β (and hence R_0) depends on social distancing/lockdown measures.

Much of our previous analysis extends to systems. In particular, all the one-step methods we studied have multi-dimensional versions, e.g.



FE:
$$\begin{cases} \underline{u}_0 = \underline{y}_0 \\ \underline{u}_{n+1} = \underline{u}_n + h\underline{f}(t_n, \underline{u}_n), \quad n = 0, 1, \ldots \end{cases}$$

BE:
$$\begin{cases} \underline{u}_0 = \underline{y}_0 \\ \underline{u}_{n+1} = \underline{u}_n + h\underline{f}(t_{n+1}, \underline{u}_{n+1}), \quad n = 0, 1, \ldots \end{cases}$$

CN:
$$\begin{cases} \underline{u}_0 = \underline{y}_0 \\ \underline{u}_{n+1} = \underline{u}_n + \frac{h}{2}\left(\underline{f}(t_n, \underline{u}_n) + \underline{f}(t_{n+1}, \underline{u}_{n+1})\right), \quad n = 0, 1, \ldots \end{cases}$$

Note that at each timestep t_n, the discrete solution $\underline{u}_n$ is a vector with p components, representing our approximation to the p components of $\underline{y}(t_n)$.

Consistency (and hence convergence) analysis can be carried out as before, except we need multi-dimensional Taylor series, which we won't cover in this course.

The conclusion is the same:

$$\boxed{\text{FE and BE are order 1, and CN is order 2}}$$

<u>Note</u>: to measure errors we'd need to choose an appropriate norm, e.g.

$$E(h) := \max_{n=0,\dots,N} \| \underline{u}_n - \underline{y}(t_n) \|_\infty = \max_{n=0,\dots,N} \max_{j=1,\dots,p} \left| (\underline{u}_n)_j - (\underline{y}(t_n))_j \right|.$$

or

$$E(h) := \max_{n=0,\dots,N} \| \underline{u}_n - \underline{y}(t_n) \|_2 = \max_{n=0,\dots,N} \left(\sum_{j=1}^{p} \left| (\underline{u}_n)_j - (\underline{y}(t_n))_j \right|^2 \right)^{1/2}.$$

For stability to perturbations (our more general form of absolute stability), we need to look at the Jacobian matrix

$$\frac{\partial f}{\partial y}(t,y) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(t,y) & \cdots & \frac{\partial f_1}{\partial y_p}(t,y) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial y_1}(t,y) & \cdots & \frac{\partial f_p}{\partial y_p}(t,y) \end{pmatrix}.$$

We denote the eigenvalues of $\frac{\partial f}{\partial y}(t,y)$ by $\{\lambda_j(t,y)\}_{j=1}^{p}$.

Suppose these are all distinct, real and negative.

Then BE and CN are both unconditionally stable to perturbations, and FE is stable to perturbations under the condition that

$$\boxed{h < \frac{2}{\max\limits_{t,y}\ \max\limits_{j=1,\dots,p} |\lambda_j(t,y)|} = \frac{2}{\max\limits_{t,y} \rho\left(\frac{\partial f}{\partial y}(t,y)\right)}}$$