Theoretical exercise sheet 1
Nonlinear equations

Exercises 3 and 4 (marked *) to be submitted via *Crowdmark* in pdf format (either handwritten and scanned, or typeset using LaTeX).

EXERCISE 1 Let $\hat{x} \neq 0$ be an approximation of a non-zero quantity x. In this question we study the relationship between the relative error

$$e = \frac{|x - \hat{x}|}{|x|},$$

and the error normalised using the approximation $\hat{x}$,

$$\hat{e} = \frac{|x - \hat{x}|}{|\hat{x}|}.$$

Assuming that $\hat{e} < 1$, show that there exist quantities $f_1(\hat{e})$ and $f_2(\hat{e})$ (depending on $\hat{e}$) such that

$$f_1(\hat{e}) \leq e \leq f_2(\hat{e}).$$

EXERCISE 2 Consider the finite difference approximations

$$Du(x) = \frac{u(x+h) - u(x)}{h} \approx u'(x)$$

and

$$D^2u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \approx u''(x).$$

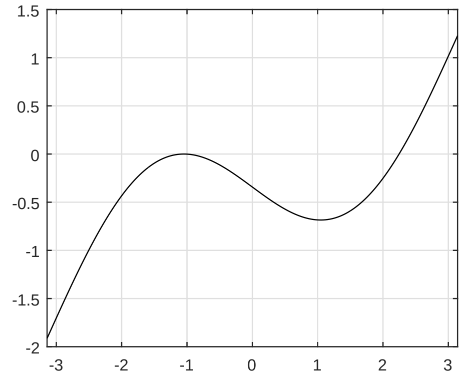
Use Taylor's theorem to show that, if u is sufficiently smooth on some interval $[x-h_0, x+h_0]$,

$$\begin{aligned} |u'(x) - Du(x)| &\leq C_1(u, h_0)h, \qquad && 0 < h < h_0, \\ |u''(x) - D^2u(x)| &\leq C_2(u, h_0)h^2, \qquad && 0 < h < h_0. \end{aligned}$$

For each of the estimates you should state clearly the smoothness assumption required for your proof to be valid, and detail the dependence of the constants $C_1(u, h_0)$ and $C_2(u, h_0)$ on u and h_0.

EXERCISE 3(*)

Suppose we wish to compute the zeros of the function $f(x) = \frac{x}{2} - \sin x + \frac{\pi}{6} - \frac{\sqrt{3}}{2} = 0$, which has exactly two roots in the interval $[-\pi, \pi]$. The following figure shows the graph of f:



(a) Is it possible to apply the bisection method to compute both roots? Why? (You may use the graph above to gain intuition, but be sure to justify your answer with some calculations.) For the root(s) which can be found by bisection, estimate the number of iterations necessary to compute the root(s) to a relative accuracy $tol = 10^{-10}$, having chosen a suitable starting interval.

(b) Write down Newton's method for the problem $f(x) = 0$ in this case (i.e. substitute the definition of f into the formula for the Newton iteration). Determine the order of convergence of the Newton method for the approximation of each of the two zeros.

(c) Now consider the fixed point method $x^{k+1} = \phi(x^k)$, with

$$\phi(x) = \sin x + \frac{x}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right) ,$$

for the computation of the root $\alpha > 0$. Prove that the fixed point iteration converges linearly to α for every initial guess $x^0 \in [\frac{\pi}{2}, \pi]$, with

$$|x^{k+1} - \alpha| \leq \Lambda |x^k - \alpha| ,$$

for some constant $0 < \Lambda < 1$ which you should specify.

EXERCISE 4(*)

Suppose we wish to compute the real root of $f(x) = x^3 - 2$ using the fixed point iteration $x^{k+1} = \phi(x^k)$, with

$$\phi(x) = x\left(1 - \frac{\omega}{3}\right) + x^3(1 - \omega) + \frac{2\omega}{3x^2} + 2(\omega - 1), \tag{1}$$

where $\omega \in \mathbb{R}$ is a real parameter.

(a) For what values of ω is the root of $f(x) = 0$ a fixed point of (1)?

(b) For what values of ω is the method locally convergent?

(c) For what values of ω is the method of second order?

(d) Is there a value of ω such that the method is of order higher than 2?

EXERCISE 5

We wish to solve the equation $f_a(x) = 0$, where $a \in \mathbb{R}$ is a parameter and $f_a(x) = (1-a)x + ax^3$. We will study the fixed point iteration $x^{k+1} = \phi(x^k)$, where the function $\phi(x) = ax(1-x^2)$.

(a) Show that the fixed point method is consistent (this is easy!) and that $\phi : [0,1] \to [0,1]$ provided $0 \leq a \leq \frac{3\sqrt{3}}{2}$.

(b) Find the positive values of a such that the fixed point iteration converges to the root $\alpha_1 = 0$ for any initial guess $x^0 \in [0,1]$.

(c) Find a condition on a under which a second zero $\alpha_2 > 0$ exists in the interval $[0,1]$ and determine for which a the fixed point iteration can approximate α_2.

(d) For what value of a can the fixed point iteration approximate α_2 with second order convergence?

EXERCISE 6

Consider the following nonlinear system:

$$\begin{cases} -\frac{1}{81}\cos x_1 - x_1 + \frac{1}{9}(x_2)^2 + \frac{1}{3}\sin x_3 = 0, \\ \frac{1}{3}\sin x_1 - x_2 + \frac{1}{3}\cos x_3 = 0, \\ -\frac{1}{9}\cos x_1 + \frac{1}{3}x_2 + \frac{1}{6}\sin x_3 - x_3 = 0, \end{cases} \tag{2}$$

for the unknowns x_1, x_2, x_3.

(a) Write down Newton's method for the system (2) in the form

$$[D\mathbf{f}(\mathbf{x}^{(k)})](\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}) = -\mathbf{f}(\mathbf{x}^{(k)}), \qquad k = 0, 1, \ldots$$

giving explicit expressions for the function $\mathbf{f}$ and the Jacobian matrix $D\mathbf{f}$.

(b) Using the initial data $\mathbf{x}^{(0)} = (\pi, 0, \pi/2)^T$, write down the linear system by which we may compute $\mathbf{x}^{(1)}$.

(c) Consider the fixed point iteration $\mathbf{x}^{n+1} = \mathbf{\Psi}(\mathbf{x}^n)$ where

$$\Psi(\mathbf{x}) = \Psi((x_1, x_2, x_3)^T) = \begin{pmatrix} -\frac{1}{81}\cos x_1 + \frac{1}{9}(x_2)^2 + \frac{1}{3}\sin x_3 \\ \frac{1}{3}\sin x_1 + \frac{1}{3}\cos x_3 \\ -\frac{1}{9}\cos x_1 + \frac{1}{3}x_2 + \frac{1}{6}\sin x_3 \end{pmatrix}.$$

Analyse the convergence of the method to the fixed point $\alpha = (0, 1/3, 0)^T$.