Numerical Methods

Recall: want to solve $A\underline{x} = \underline{b}$.

Direct methods:

If A is diagonal, inversion is trivial $\left((A^{-1})_{ii} = \frac{1}{a_{ii}} \right)$

– cost of $\underline{x} = A^{-1}\underline{b}$ is $O(n)$ as $n \to \infty$.

If A is upper or lower triangular, can solve in $O(n^2)$ cost.

In general, LU factorisation (Gauss elimination) costs $O(n^3)$ operations.

Only use pens labelled 'Drywipe Markers' on this board

Iterative methods:

Iterative methods: generate a sequence of approximations $\underline{x}^0, \underline{x}^1, \underline{x}^2, \ldots$ to $\underline{x}$.

indices not powers

Roughly speaking, cost $=$ (# iterations) $\times$ (cost per iteration).

We'd like cost to be small (at least smaller than $O(n^3)$).

Stationary iterative methods:

$$\boxed{\underline{x}^{k+1} = B\underline{x}^k + \underline{c}}, \quad k = 0, 1, 2, \ldots$$

$B, \underline{c}$ don't depend on k.

$B \in \mathbb{R}^{n \times n}$ iteration matrix

$\underline{c} \in \mathbb{R}^n$ iteration vector.

Iteration is called consistent if exact sol'n $\underline{x}$ of $A\underline{x} = \underline{b}$ satisfies $\boxed{\underline{x} = B\underline{x} + \underline{c}}$.

In this case, the error $\underline{e}^k := \underline{x} - \underline{x}^k \in \mathbb{R}^n$ satisfies

$$\underline{e}^{k+1} = B\underline{e}^k, \quad k = 0, 1, \ldots \qquad \text{(exercise: check!)}$$

So that

$$\underbrace{\underline{e}^k}_{\text{error after } k\text{th iteration}} = B^k \underbrace{\underline{e}^0}_{\text{initial error}}, \quad k = 0, 1, \ldots \qquad \text{(by induction)}$$

Theorem: A consistent stationary method is convergent for all initial $x^0 \in \mathbb{R}^n$ if and only if $\rho(B)<1$.

Proof: If $\rho(B)<1$ then $B^k \xrightarrow{k\to\infty} 0$, so $\|e^k\|_V \le \underbrace{\|B^k\|_M}_{\to 0}\|e^0\|_V \to 0$ for all x^0.

use $\lambda\in\sigma(B)$, $|\lambda|=\rho(B)$

If $\rho(B)\ge 1$ then $\exists\, x^0$ s.t. $e^k \not\to 0$.

To see this, recall that $\exists\, \hat{v} \in \mathbb{R}^n\setminus\{0\}$ and $\varepsilon>0$ s.t. $\|B^k\hat{v}\|_V \ge \varepsilon\, \underbrace{\rho(B)^k}_{\ge 1}\|\hat{v}\|_V$ $\forall k$.

Then $\|B^k \hat{v}\|_V \ge \varepsilon\|\hat{v}\|_V>0$ $(\rho(B)\ge 1)$, so choosing $e^0=\hat{v}$ (i.e. $x^0 = x-\hat{v}$) we have $e^k\not\to 0$.

What about convergence rate?

Theorem: If $\|\cdot\|_V$ is such that the induced norm $\|\cdot\|_M$ satisfies $\|B\|_M<1$, then we get linear convergence $(\forall x^0)$ and

$$\|e^k\|_V \le \left(\|B\|_M\right)^k\|e^0\|_V, \quad k=0,1,\dots$$

Proof: $\|e^k\|_V = \|B^k e^0\|_V \le \|B^k\|_M\|e^0\|_V \le \|B\|_M^k \|e^0\|$.

$$\boxed{x^{k+1}=Bx^k+c}$$

Corollary: If B is symmetric and $\rho(B)<1$ then we get linear convergence w.r.t. $\|\cdot\|_2$ (Euclidean norm on $\mathbb{R}^n$) with constant $C=\rho(B)$, so

$$\|e^k\|_2\le(\rho(B))^k\|e^0\|_2, \quad k=0,1,\dots$$

Simplest choice of B: $B=I-A$, $c=b$.

"Basic stationary method" (BSM): $\boxed{x^{k+1}=(I-A)x^k+b}$, $k=0,1,\dots$

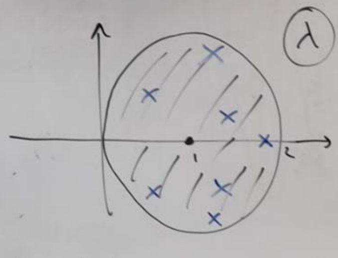
Clearly this is consistent with $Ax=b$.

$$Ax=b$$
$$x+Ax=x+b$$
$$x=(I-A)x+b$$

BSM is convergent $\forall x^0 \iff \rho(I-A)<1$,

i.e. $\sigma(A)$ is contained in the open ball of radius 1 around the point 1.

$\left(|1-\lambda|<1 \quad \forall \lambda\in\sigma(A)\right)$



If this fails, ~~we~~ want to try and modify B to shift the eigenvalues.