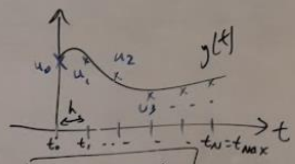
Numerical Methods

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in [0, t_{max}] \\ y(0) = y_0 \end{cases} \qquad (*)$$



$$\begin{cases} u_{n+1} = u_n + h\,\Psi(t_n, u_n, u_{n+1}, h) \\ u_0 = y_0 \end{cases} \qquad (**)$$

$$\boxed{u_n \simeq y(t_n)}$$

uniform mesh: $t_n = nh, \quad h = \frac{t_{max}}{N}$

Last time: "consistency" – how well do $(*)$ and $(**)$ agree locally? (i.e. on each timestep)

This time: towards global accuracy – stability.

Zero stability: How does method behave as $h \to 0$ with t_{max} fixed?

Consider the perturbed problem:

$$\begin{cases} v_{n+1} = v_n + h\,\Psi(t_n, v_n, v_{n+1}, h) + h\delta_{n+1}, \\ v_0 = y_0 + \delta_0, \end{cases} \qquad (***)$$

where $\delta_0, \delta_1, \ldots, \delta_N$ are some small perturbations. (e.g. rounding errors)

Def'n: We say $(**)$ is zero stable if $\exists\, h_* > 0$ and $C \geq 0$ s.t. if $\boxed{0 < h < h_*}$ then the solution of $(***)$ satisfies

$$\boxed{\max_{n=0,\ldots,N} |v_n - u_n| \leq C \max_{n=0,\ldots,N} |\delta_n|}$$

C does not depend on h so this gives good behavior as $h \to 0$.

Theorem: Suppose $\exists\, \hat{h} > 0$, $L_1 \geq 0$ and $L_2 \geq 0$ s.t.

$$|\Psi(t, u, v, h) - \Psi(t, u', v', h)| \leq L_1 |u - u'| + L_2 (v - v')$$

Uniform Lipschitz condition w.r.t. u and v.

for all $0 < h < \hat{h}$, all $t \in [0, t_{max}]$, and $u, u', v, v' \in \mathbb{R}$.

Then for any $\rho \in (0,1)$, the method $(**)$ is zero stable, with constants $\boxed{h_* = \min\left(\hat{h}, \frac{\rho}{L_2}\right)}$ and $\boxed{C = \left(1 + \frac{1}{L_1 + L_2}\right) e^{\frac{t_{max}(L_1 + L_2)}{1-\rho}}}$

Proof: Let $w_n = v_n - u_n$. Then w_n satisfies

$$\begin{cases} w_{n+1} = w_n + h\left(\Psi(t_n, v_n, v_{n+1}, h) - \Psi(t_n, u_n, u_{n+1}, h)\right) + h\delta_{n+1} \\ w_0 = \delta_0 \end{cases}$$

Applying Δ-ineq + Lipschitz assumption gives:

$$|w_{n+1}| \leq |w_n| + h\left|\Psi(\ \dots) - \Psi(\dots)\right| + h|\delta_{n+1}|$$
$$\leq |w_n| + h\left(L_1|w_n| + L_2|w_{n+1}|\right) + h|\delta_{n+1}|.$$

so $(1-hL_2)|w_{n+1}| \leq (1+hL_1)|w_n| + h\delta$, where $\delta := \max_n |\delta_n|$.

Now let $\rho \in (0,1)$ and assume $0 < h < \frac{\rho}{L_2}$. $(\text{so } 0 < hL_2 < \rho)$

Then $\boxed{|w_{n+1}| \leq a|w_n| + b\delta}$, where $a := \frac{1+hL_1}{1-hL_2}$, $b = \frac{h}{1-hL_2}$.

Hence $|w_n| \leq \left(a^n + b\frac{a^n - 1}{a-1}\right)\delta$ (proof by induction, cf. geometric series $\sum_{k=0}^{n-1} a^k = \frac{a^n-1}{a-1}$ etc.)

Hence $|w_n| \leq \left(1 + \frac{b}{a-1}\right)a^n\delta$, since $a > 1$ and $a^n - 1 \leq a^n$.

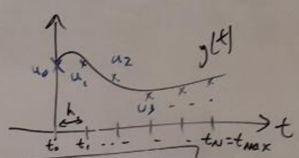
Now, using $(1+x)^n \leq e^{nx}$ for $x \geq 0$,

we get $a^n = \left(1 + \frac{h(L_1+L_2)}{1-hL_2}\right)^n \leq e^{\frac{nh(L_1+L_2)}{1-hL_2}} \leq e^{\frac{t_{max}(L_1+L_2)}{1-\rho}}$ using $hL_2 < \rho$ so $1 - hL_2 > 1-\rho$.

Also, note that $1 + \frac{b}{a-1} = 1 + \frac{\cancel{h}}{\cancel{1-hL_2}}\,\frac{\cancel{1-hL_2}}{\cancel{h}(L_1+L_2)} = 1 + \frac{1}{L_1+L_2}$.

So we're done.

Examples: If f is uniformly Lipschitz on $\mathbb{R}$,



i.e. $|f(t,v) - f(t,w)| \leq L_f|v-w|$ $\forall t \in [0, t_{max}]$ $\forall v, w \in \mathbb{R}$.

$\boxed{u_n \approx y(t_n)}$

Then FE, BE and CN are all zero stable, since they satisfy the Lipschitz condition in the theorem, with

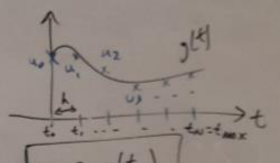
$\boxed{FE}$: $L_1 = L_f$, $L_2 = 0$ explicit

$\boxed{BE}$: $L_1 = 0$, $L_2 = L_f$ fully implicit

$\boxed{CN}$: $L_1 = L_2 = L_f/2$ semi-implicit

Exercise - check this!

<u>Convergence:</u>



$\boxed{u_n \approx y(t_n)}$

Let $e_n := y(t_n) - u_n$.

<u>Def'n:</u> We say the one-step method is uniformly convergent if and only if

$\|\cdot\|_\infty$

$$E(h) := \max_{n=0,\dots,N} |e_n|$$

so $h \to 0 \iff N \to \infty$

tends to zero as $h \to 0$ (with t_{max} fixed).

We say method is convergent of order p if $E(h) = O(h^p)$ as $h \to 0$.

i.e. $E(h) \leq C h^p$ for some $C > 0$ for suf. small h.

Proof of convergence for FE, BE and CN will follow by "Lax-Richtmyer equivalence principle", which in general terms is "consistency + stability $\iff$ convergence".

so IVP is well-posed.

<u>Theorem:</u> Let f and y_0 satisfy the conditions of Picard's Theorem. Suppose the one step method $(**)$ is <u>zero stable</u> and <u>uniformly consistent</u>. Then the method is uniformly convergent, and

$$E(h) = O(T(h)) \quad \text{as } h \to 0.$$

$T(h) = \max_n |T_n|$ (truncation error)

Furthermore, consistency of order p implies convergence of order p.

$$T(h) = O(h^p) \Rightarrow E(h) = O(h^p).$$

<u>Proof:</u> Set $v_n := y(t_n)$. Then v_n satisfies $(***)$ with

$v_0 = y(t_0) = y(0) = y_0$

$\delta_0 = 0$ and $\delta_{n+1} = T_n$, $n = 0, \dots, N-1$.

def'n of T_n. ($\frac{1}{h}$ times the residual when we plug $y(t_n)$ into one-step method)

Then by zero-stability (with same h_*, C), we have

$$E(h) = \max_n |e_n| = \max_n |y(t_n) - u_n| = \max_n |v_n - u_n| \leq C\,T(h)$$

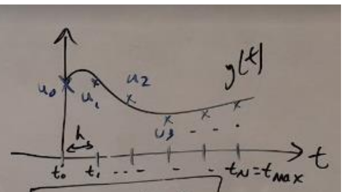
for $0 < h < h_*$.

$\delta = T(h) = \max_n |T_n|$ here

By uniform consistency, $T(h) \to 0$ as $h \to 0$, so $E(h) \to 0$ as $h \to 0$, as claimed. ∎

Examples:

If f is uniformly Lipschitz and y is sufficiently smooth, then:



$\boxed{u_n \approx y(t_n)}$

- FE is order 1 convergent $\left(E(h) = O(h)\right)$
- BE - - - - - - - - - - (- - - -)
- CN is order 2 convergent $\left(E(h) = O(h^2)\right)$

- just combine our previous results.

Absolute stability:

Our analysis gave $E(h) \leq C\,T(h)$ for suff. small h.

Explicitly, we obtained a constant C that was proportional to $e^{\frac{t_{max}(L_1 + L_2)}{1-S}}$.

As $t_{max} \to \infty$, this $\to \infty$ exponentially fast!

This suggests that for small but fixed $h > 0$, we might see large errors as $t_{max} \to \infty$ (i.e. $N \to \infty$), making it difficult to obtain long-time behavior of the true solution $y(t)$.

For some methods this is genuinely a problem.

But for "absolutely stable" methods, it's not.

(in such cases the bound $E(h) \leq C\,T(h)$ is overly pessimistic in the limit as $t_{max} \to \infty$.)