EXERCISE 3(*) For the numerical solution of the Cauchy problem

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in (0, T], \quad 0 < T < +\infty, \\ y(0) = y_0, \end{cases}$$

where f is assumed to be uniformly Lipschitz continuous with respect to its second argument, consider the following method for $n \geq 0$, with $u_0 = y_0$, and uniform step size $h = T/N$ for some $N \in \mathbb{N}$:

$$\begin{cases} u_{n+\frac{1}{2}} = u_n + \frac{h}{2} f(t_n, u_n), \\ p_n = f(t_n + \frac{h}{2}, u_{n+\frac{1}{2}}), \\ u_{n+1} = u_n + h p_n. \end{cases} \tag{3}$$

(a) Is the method given by (3) explicit or implicit ?

(b) Write the method in the form

$$u_{n+1} = u_n + h\Psi(t_n, u_n, u_{n+1}; h)$$

for an appropriate increment function Ψ.

(c) Prove that the method is zero-stable. Reference carefully any theorems you use from lectures.

(d) Write down the formula for the truncation error of the method, and show that the method is consistent, with $T_n = O(h^2)$ as $h \to 0$ for y and f sufficiently smooth. Under what smoothness assumptions on y and f is your analysis valid?

Hint: You may find it helpful to recall the "generalised chain rule"

$$\frac{d}{dh} F(T(h), Y(h)) = T'(h) \frac{\partial F}{\partial t}(T(h), Y(h)) + Y'(h) \frac{\partial F}{\partial y}(T(h), Y(h)).$$

(e) By quoting an appropriate theorem from lectures, combine the results of parts (c) and (d) to prove that the method is second order convergent.

Sol:

a), Since $\begin{cases} U_{n+\frac{1}{2}} = U_n + \frac{h}{2} f(t_n, U_n), \\ P_n = f(t_n + \frac{h}{2}, U_{n+\frac{1}{2}}) \\ U_{n+1} = U_n + h p_n \end{cases}$

So, $U_{n+1} = U_n + h \cdot f(t_n + \frac{h}{2}, U_{n+\frac{1}{2}})$
$= U_n + h \cdot f(t_n + \frac{h}{2}, U_n + \frac{h}{2} \cdot f(t_n, U_n))$

The method given by (3) is "explicit", because the right-hand side is independent of U_{n+1}.

b), From question (a), we can get that:

$\Psi(t_n, U_n, U_{n+1}; h) = f(t_n + \frac{h}{2}, U_n + \frac{h}{2} \cdot f(t_n, U_n))$

c). According to the Theorem 5.5.7, we can know that if Φ is uniformly Lipschitz, then the method is zero-stable.

Due to $|\Phi(t,u,v;h) - \Phi(t,u',v';h)| \leq L_1|u-u'| + L_2|v-v'|$

So, $$|\Phi(t,u,w;h) - \Phi(t,u',w';h)|$$
$$= |f(t+\tfrac{h}{2}, u+\tfrac{h}{2}\cdot f(t,u)) - f(t+\tfrac{h}{2}, u'+\tfrac{h}{2}\cdot f(t,u'))|$$
$$\leq L_f\,|u+\tfrac{h}{2}\cdot f(t,u) - u' - \tfrac{h}{2}\cdot f(t,u')|$$
$$\leq L_f\,(1+\tfrac{h}{2}\cdot L_f)\cdot|u-u'|$$

where L_f is the Lipschitz constant of function f, and Lipschitz $L_\Phi = L_f\,(1+\frac{h}{2}\cdot L_f)$.

Therefore, Φ is uniformly Lipschitz, so the method is zero-stable.

d). According to the definition of truncation error, we can set:

$$T_n = \frac{y(t_{n+1}) - y(t_n)}{h_n} - \Phi(t_n, y(t_n), y(t_{n+1}); h_n),$$
$$= \frac{y(t_{n+1}) - y(t_n)}{h_n} - f(t_n+\tfrac{h}{2}, y(t_n)+\tfrac{h}{2}\cdot f(t_n, y(t_n)))$$

Next, we should prove that the method is consistent with $T_n = O(h^2)$.

Thus, according to the truncation error T_n we got above, we can get the following equations by Taylor's Theorem:

$$\frac{y(t_{n+1}) - y(t_n)}{h_n} = y'(t_n) + \frac{h}{2} \cdot y''(t_n) + O(h^2)$$

and:

$$y(t_{n+1}) = y(t_n) + h \cdot y'(t_n) + \frac{h^2}{2} \cdot y''(t_n) + O(h^2)$$

$$U_{n+\frac{1}{2}} = U_n + \frac{h}{2} \cdot f(t_n, y(t_n)) = y(t_n) + \frac{h}{2} \cdot y'(t_n)$$

Therefore,

$$\Phi(t_n, y(t_n), y(t_{n+1}); h_n) = f(t_n + \tfrac{h}{2}, y(t_n) + \tfrac{h}{2} \cdot f(t_n, y(t_n)))$$
$$= y'(t_{n+\frac{1}{2}}) = y'(t_n + \tfrac{h}{2}) = y'(t_n) + \frac{h}{2} \cdot y''(t_n) + O(h^2)$$

$$T_n = \frac{y(t_{n+1}) - y(t_n)}{h_n} - f(t_n + \tfrac{h}{2}, y(t_n) + \tfrac{h}{2} \cdot f(t_n, y(t_n)))$$
$$= y'(t_n) + \frac{h}{2} \cdot y''(t_n) - y'(t_n) - \frac{h}{2} \cdot y''(t_n) + O(h^2)$$
$$= O(h^2)$$

Thus, so the method is consistent with $T_n = O(h^2)$.

Moreover, to make the analysis valid, the y''' and all terms in Taylor series are exists and bounded.

e), Due to the Theorem 5.5.10, if the one-step method is zero-stable for the IVP and is consistent of order $p > 0$ (namely $T(h) = O(h^p)$), then the method has convergence order p.

Thus, we already got the method is zero-stable in parts (c), and got $T(h) = O(h^2)$ in parts (d).
Therefore, the method is second order convergent.