Numerical Methods

Course feedback questionnaire on Moodle – please complete!

$$u_{n+1} = u_n + h\Psi(t_n, u_n, u_{n+1}, h)$$

Recap: One-step methods for ODE IVPs:

Method	Explicit/implicit	Zero stable?	Order (consistency and convergence)	Absolutely stable?
FE	Explicit	✓	1	Not A-stable, but abs. stable if $h < \frac{2}{\lvert\lambda\rvert}$
BE	Implicit	✓	1	A-stable.
CN	Implicit	✓	2	A-stable

Question: Is it possible to have an explicit method with order 2?

Answer: Yes – Runge-Kutta methods

Starting point: CN: $u_{n+1} = u_n + \frac{h}{2}\left(f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right)$

this is what makes CN implicit.

Try replacing u_{n+1} on RHS by a FE step:

$$\boxed{u_{n+1} = u_n + \frac{h}{2}\Big(f(t_n, u_n) + f\big(t_{n+1}, u_n + h f(t_n, u_n)\big)\Big)}.$$

This is called "Heun's method" or the "improved Euler method".

Claim: Heun's method is order 2 consistent.

Proof: Truncation error is: ($\frac{1}{h}\times$residual when we substitute $y(t_n)$ in place of u_n in method)

$$T_n = \frac{y(t_n+h) - y(t_n)}{h} - \frac{1}{2}\Big(f(t_n, y(t_n)) + f\big(t_n+h, y(t_n) + h f(t_n, y(t_n))\big)\Big).$$

Now Taylor expand:

$$y(t_n+h) = y(t_n) + y'(t_n)h + \frac{y''(t_n)h^2}{2} + \frac{y'''(\xi_n)h^3}{6}$$

$$= y(t_n) + y'(t_n)h + \frac{y''(t_n)h^2}{2} + O(h^3),$$

provided y is 3 times diff. with y'''' bounded.

for some ξ_n between t_n and t_{n+1}.

And

$$\overbrace{f\big(t_n+h, y(t_n) + h f(t_n, y(t_n))\big)}^{=: g(h)}$$

$$= f(t_n, y(t_n)) + h\left(\frac{\partial f}{\partial t}(t_n, y(t_n)) + f(t_n, y(t_n))\frac{\partial f}{\partial y}(t_n, y(t_n))\right) + R_2,$$

where $R_2 = O(h^2)$, provided f is twice diff. with bounded 2nd derivatives.

think of this as $g(h)$ and evaluate $g(h) = g(0) + g'(0)h + O(h^2)$.

need multi-dim chain rule to evaluate this.

Note that by differentiating the ODE $y'(t) = f(t, y(t))$ w.r.t. t we get $y''(t) = \frac{\partial f}{\partial t}(t, y(t)) + y'(t)\frac{\partial f}{\partial y}(t, y(t))$.

$$= \frac{\partial f}{\partial t} + f \cdot \frac{\partial f}{\partial y}.$$

Hence we have

$$T_n = \frac{\cancel{y(t_n)} + \cancel{y'(t_n)h} + \cancel{\frac{y''(t_n)h^2}{2}} + O(h^3) - \cancel{y(t_n)}}{h}$$

$$- \frac{1}{2}\left(\cancel{f(t_n, y(t_n))} + \cancel{f(t_n, y(t_n))} + \cancel{h\, y''(t_n)} + O(h^2)\right)$$

$$= O(h^2).$$ So the claim is proved.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[dr, "F"'] & \mathbb{R} \\ & \mathbb{R}^2 \arrow[u, "f"'] \end{tikzcd}$$

$g(t) = f(t, y(t))$

$g(t) = (f \circ F)(t)$

$F(t) = (t, y(t)) \in \mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$, $F: \mathbb{R} \to \mathbb{R}^2$

$$g'(t) = (\underbrace{f \circ F}_{})'(t)$$

$$= \underbrace{F'(t)}_{=\begin{pmatrix} F_1' \\ F_2' \end{pmatrix} = \begin{pmatrix} 1 \\ y' \end{pmatrix}} \cdot \underbrace{\nabla f(F(t))}_{=\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial y}\right)}$$

Exercise: Check that Heun's method is zero stable under the assumption that f is uniformly Lipschitz. (Hint: use our general theorem, and try to find a formula for the constant L, in terms of L_f)

Exercise: Check that Heun's method is absolutely stable under the same condition $\left(h < \frac{2}{|\lambda|}\right)$ as FE.

It turns out that $\nexists$ an explicit A-stable method!

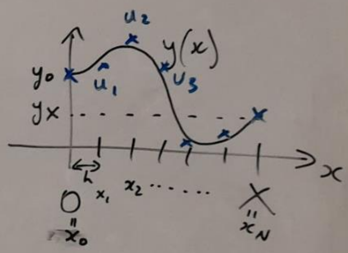
§5.9 Boundary value problems:

Now consider a second order linear ODE with 2 BCs:

$$\underbrace{-y''(x) + r(x)y(x)}_{=: L(y)(x)} = f(x), \quad x \in (0, X).$$

$$y(0) = y_0$$
$$y(X) = y_X.$$

linear differential operator. $(Ly = -y'' + ry)$



Here X is the length of the interval, y_0, y_X are given, and r and f are given continuous functions on $[0, X]$ with $r(x) \geq 0$ on $[0, X]$.

Let's try a finite difference approximation, using a uniform mesh

$$x_n = nh, \quad n = 0, \dots, N, \quad h = \frac{X}{N}.$$

We approximate $\boxed{y(x_n) \approx u_n}$, where u_n satisfies the following problem:

$$(*) \quad \begin{cases} \underbrace{-D^2 u_n + r_n u_n}_{=: L_N(u_n)} = f_n, & n = 1, \dots, N-1 \\ u_0 = y_0, \quad u_N = y_N. \end{cases}$$

motivation: $y''(t_n) \approx \dfrac{y'(t_n + \frac{h}{2}) - y'(t_n - \frac{h}{2})}{h}$, $\quad y'(t_n + \frac{h}{2}) \approx \dfrac{y(t_n+h) - y(t_n)}{h}$, $\quad y'(t_n - \frac{h}{2}) \approx \dfrac{y(t_n) - y(t_n-h)}{h}$

Here $r_n := r(x_n)$, $f_n := f(x_n)$ and

$$D^2 u_n = \frac{u_{n-1} - 2u_n + u_{n+1}}{h^2}, \quad n = 1, \dots, N-1$$

(see Th. Prob. Sheet 1)

Provided that u is four times diff. with u'''' bounded,

$$\max_{n=1,\dots,N-1} \left| u''(x_n) - D^2 u(x_n) \right| = O(h^2) \quad \text{as } h \to 0.$$

We'll use this when we analyse the truncation error later.

The system $(*)$ can be written as a linear system

$$A \underline{u} = \underline{b}$$

of size $(N-1)$-by-$(N-1)$, where $\boldsymbol{u} = (u_1, \dots, u_{N-1})^T$,

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & -1 & 0 \\ \vdots & & & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} + \begin{pmatrix} r_1 & 0 & \cdots & & \cdots & 0 \\ 0 & r_2 & 0 & & & \vdots \\ \vdots & 0 & \ddots & \ddots & & \\ & & \ddots & & 0 & \vdots \\ \vdots & & & 0 & r_{N-2} & 0 \\ 0 & \cdots & & \cdots & 0 & r_{N-1} \end{pmatrix},$$

and

$$\boldsymbol{b} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-2} \\ f_{N-1} \end{pmatrix} + \frac{1}{h^2}\begin{pmatrix} u_0 \\ 0 \\ \vdots \\ 0 \\ u_N \end{pmatrix}.$$

Note that the matrix A describes the action of the discrete operator L_N and the boundary conditions are incorporated into the first and last elements of the vector $\boldsymbol{b}$.

Note that A is symmetric and diagonally dominant by rows, but not strictly diagonally dominant unless $r_n > 0$ for all n. That A is invertible follows from the fact that it is positive definite, which can be checked by noting that, for any $\mathbf{0} \neq \boldsymbol{x} = (x_1, \dots, x_{N-1})^T \in \mathbb{R}^{N-1}$,

$$\boldsymbol{x}^T A \boldsymbol{x} \geq x_1^2 + x_{N-1}^2 + \sum_{n=2}^{N-1} (x_n - x_{n-1})^2 > 0. \tag{5.56}$$

Hence the numerical solution u_n is uniquely defined.