Let's investigate consistency and convergence.

We define the truncation error

$$\boxed{T_n = L_N\big(y(x_n)\big) - f_n}$$

(residual after plugging exact solution into discrete model)

and the global error

$$\boxed{e_n = y(x_n) - u_n.}$$

(discrepancy between exact and discrete solutions)

By the definition of u_n, we have

$$\boxed{L_N(e_n) = T_n}.$$

So to prove convergence ($e_n \to 0$) it's enough to prove consistency ($T_n \to 0$) and stability ($T_n \to 0 \Rightarrow e_n \to 0$).

This is another instance of the Lax-Richtmyer principle

Let's consider consistency first.

Theorem: (Consistency)

Suppose that $y : [0, X] \to \mathbb{R}$ is four times differentiable on $[0, X]$ with y'''' bounded. Then

$$T(h) := \max_{n=1,\dots,N-1} |T_n| = O(h^2) \quad \text{as } h \to 0.$$

Proof:

D^2 applied to $V_n = y(x_n)$

$$\max_{n=1,\dots,N-1} |T_n| = \max_{n=1,\dots,N-1} \left| -D^2(y(x_n)) + r_n y(x_n) - f_n \right|$$

$$= \max_{n=1,\dots,N-1} \left| y''(x_n) - D^2(y(x_n)) \right| \quad \text{(by ODE)}$$

$$= O(h^2) \text{ as } h \to 0. \quad \text{(by theoretical problem sheet 1)}$$

$\blacksquare$

So L_N is a 2nd order approximation of L.

Now for stability, we need the following:

Theorem (Discrete Maximum Principle)

Let $a_n, b_n, c_n > 0$ with $b_n \geq a_n + c_n$ for $n = 1, \dots, N-1$, and let $U_n \in \mathbb{R}$, $n = 0, \dots, N$ be such that

$$-a_n U_{n-1} + b_n U_n - c_n U_{n+1} \leq 0, \qquad n = 1, \dots, N-1.$$

Then

$$U_n \leq \max\{U_0, U_N, 0\}, \qquad n = 1, \dots, N-1.$$

Proof: See theoretical problem sheet 3.

Theorem: (stability)

$$\boxed{E(h) := \max_{n=0,\dots,N} |e_n| \leq \frac{X^2}{8} T(h)}.$$

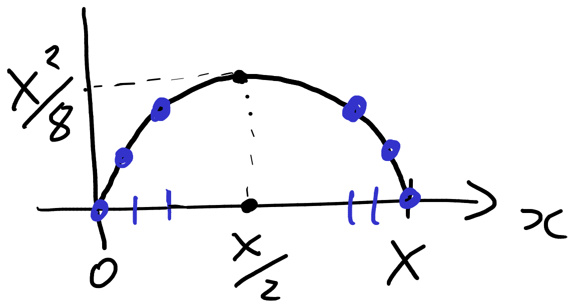
Proof:

Preliminary calculations:

Define the "companion function" $\varphi(x) = \frac{x(X-x)}{2}$.

This satisfies the BVP $\begin{cases} \varphi''(x) = -1, & x \in [0, X] \\ \varphi(0) = \varphi(X) = 0. \end{cases}$

Note also that $\varphi(x) \geq 0$ on $[0, X]$, and that $\varphi(x)$ attains its maximum value of $X^2/8$ at $x = X/2$.



The discrete samples $\varphi_n := \varphi(x_n)$ satisfy

$$\begin{cases} D^2 \varphi_n = -1, & n = 1, \ldots, N-1, \\ \varphi_0 = \varphi_N = 0, \end{cases}$$

and $0 \leq \varphi_n \leq X^2/8$ for $n = 0, \ldots, N$.

The point of introducing φ is that it allows us to bound global errors in terms of truncation errors. Specifically,

$$(*) \quad \boxed{|e_n| \le T(h)\,\varphi_n}, \qquad n = 0, \dots, N,$$

which (since $\varphi_n \le \frac{X^2}{8}$) proves the required stability bound

$$\boxed{E(h) \le \frac{X^2}{8}\,T(h)}.$$

To prove $(*)$, note that

$$L_N(\varphi_n) = 1 + r_n \varphi_n \ge 1, \qquad n = 1, \dots, N-1,$$

so that

$$\begin{aligned} L_N\left(\pm e_n - T(h)\varphi_n\right) &= \pm T_n - T(h)\,L_N(\varphi_n) && \left(\text{linearity of } L_N \text{ and } L_N(e_n) = T_n\right) \\ &\le \pm T_n - T(h) && \left(L_N(\varphi_n) \ge 1\right) \\ &\le 0 && \left(\text{def'n of } T(h)\right) \end{aligned}$$

Now, setting $a_n = c_n = \frac{1}{h^2}$, $b_n = \frac{2}{h^2} + r_n$ and $U_n = \pm e_n - T(h)\varphi_n$, we have

$$\boxed{L_N(\pm e_n - T(h)\varphi_n) = -a_n U_{n-1} + b_n U_n - c_n U_{n+1} \leq 0},$$

so that, by the DMAP,

$$\boxed{\pm e_n - T(h)\varphi_n \leq 0}, \quad n = 0, \ldots, N$$

(note that $U_0 = \pm e_0 - T(h)\varphi_0 = U_N = \pm e_N - T(h)\varphi_N = 0$ so $\max\{U_0, U_N, 0\} = 0$).

Since the inequality holds for both + and − signs, we have (*), as claimed. ∎

Corollary:

$$E(h) = \max_{n=0,\ldots,N} |e_n| = O(h^2) \text{ as } h \to 0$$

(the method is 2nd order convergent)

Proof: Simply combine consistency with stability!