Numerical Methods

Last time: def'n of normed vector space $(V, \|\cdot\|)$

and of convergence $x_n \to x \iff \|x_n - x\| \to 0$.

Def'n.: • A sequence x_n tending to x conveges **linearly** if $\exists\, N \in \mathbb{N}$ and $0 < C < 1$

s.t. $\|x_{n+1} - x\| \leq C\,\|x_n - x\|$, $n \geq N$.

(In this case, $\exists\, \tilde{C} > 0$ s.t. $\|x_n - x\| \leq \tilde{C}\, C^n$, $n \in \mathbb{N}$.)

exercise: find $\tilde{C}$ "a priori error bound"

Note: $0 < C < 1 \Rightarrow C^n \to 0$ as $n \to \infty$.

Exercise: explain why we don't need $C<1$ here

• x_n conveges **quadratically** to x if $\exists\, N \in \mathbb{N}$ and $C > 0$

s.t. $\|x_{n+1} - x\| \leq C\,\|x_n - x\|^2$, $n \geq N$.

not necessarily integer!

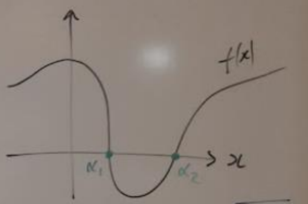
• x_n converges with **order $q > 1$** to x if $\exists\, N \in \mathbb{N}$ and $C > 0$

s.t. $\|x_{n+1} - x\| \leq C\,\|x_n - x\|^q$, $n \geq N$.

§3: Nonlinear equations (root-finding)

Given $f: \mathbb{R} \to \mathbb{R}$, find $\alpha \in \mathbb{R}$ s.t. $f(\alpha) = 0$.

(Can rephrase any equation $F(x) = G(x)$ in this way by setting $f(x) := F(x) - G(x)$.)



e.g. $\alpha_{\pm} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For simple functions (e.g. low order polynomials) we might have a **closed form solution**. But in general we need a numerical method to generate an approximate solution.

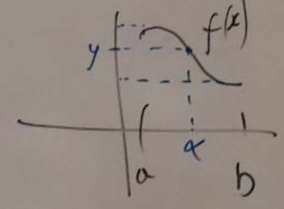
Existence of roots?

Recall <u>Intermediate Value Theorem</u>:

If $f:[a,b] \to \mathbb{R}$ is continuous then f is bounded and attains its bounds, and for every $y \in \mathbb{R}$ s.t.

$$\boxed{\min_{x\in[a,b]} f(x) \leqslant y \leqslant \max_{x\in[a,b]} f(x)}$$

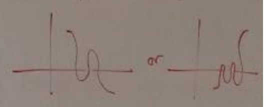
$\exists\, \alpha \in [a,b]$ s.t. $\boxed{f(\alpha) = y}$



<u>Corollary</u>: If $f:[a,b]\to\mathbb{R}$ is continuous and $\boxed{f(a)f(b) \leqslant 0}$ then $\exists\, \alpha \in [a,b]$ s.t. $\boxed{f(\alpha) = 0}$.

< here means $f(a)$ and $f(b)$ have opposite signs.

<u>Proof</u> – see notes.



<u>Note</u>: this gives existence but not uniqueness. There might be more than one root in $[a,b]$.

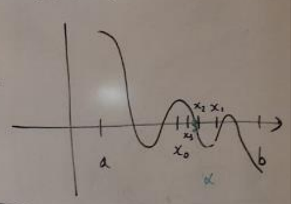
This leads naturally to a numerical method:

<u>Bisection method</u>: Assume $f:[a,b]\to\mathbb{R}$ is continuous and $f(a)f(b) \leqslant 0$.

- Set $k = 0$
- If $f(a)f(b) = 0$ then either $f(a) = 0$ or $f(b) = 0$. STOP.

  If not:
- Set $x_k = \dfrac{a+b}{2}$ ←
- If $f(x_k) = 0$, STOP

  If not,
  - if $f(a)f(x_k) < 0$, redefine $b = x_k$, increment k and go to here. ($\exists$ root in $[a, x_k]$)
  - if $f(a)f(x_k) > 0$, redefine $a = x_k$, increment k and go to here. ($\exists$ root in $[x_k, b]$)

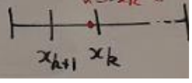


Theorem: If $f:[a,b]\to\mathbb{R}$ is continuous and $f(a)f(b)\leq 0$ then the bisection method converges to some $\alpha\in[a,b]$ with $f(\alpha)=0$, and

$$(*)\qquad \boxed{|x_k-\alpha| \leq \frac{b-a}{2^{k+1}}} = \underbrace{\frac{b-a}{2}}_{\hat{c}}\cdot\Big(\underbrace{\frac{1}{2}}_{c}\Big)^k$$

Proof: Straightforward from def'n of algorithm.

Note: Bisection method is not linearly convergent even though $(*)$ holds. $\left(\dfrac{\|x_{n+1}-x\|}{\|x_n-x\|}\right.$ could be arbitrarily large!$\left.\right)$



Bisection is nice as it only requires continuity of f.
But it's quite slow.
We can find faster methods, but in general they require more smoothness of f.

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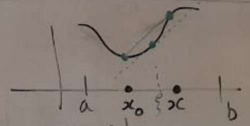
§ 2.4



Mean value theorem: Let $f:(a,b)\to\mathbb{R}$ be differentiable, then for $x,x_0\in(a,b)$ with $x\neq x_0$, $\exists\,\xi$ between x and x_0 s.t.

$$\boxed{f(x)=f(x_0)+f'(\xi)(x-x_0)}.$$

Taylor's Theorem: Let $f:(a,b)\to\mathbb{R}$ be k-times differentiable on (a,b) for some $k\in\mathbb{N}$. Then for $x,x_0\in(a,b)$ with $x\neq x_0$,

$$f(x)=\underbrace{\sum_{j=0}^{k}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j}_{\text{Taylor polynomial}}+R_k(x;x_0),$$

where $R_k(x;x_0)=o\left(|x-x_0|^k\right)$ as $x\to x_0$.

"asymptotic notation"
$\left(F(x)=o(G(x))\text{ as } x\to x_0 \text{ means } \dfrac{F(x)}{G(x)}\to 0 \text{ as } x\to x_0\text{, assuming } G(x)\neq 0\right)$

more generally, see notes.

If f is $(k+1)$-times differentiable on (a,b) then $\exists\,\xi$ between x_0 and x s.t. $R_k(x;x_0)=\dfrac{f^{(k+1)}(\xi)}{(k+1)!}(x-x_0)^{k+1}$.

Further, if $|f^{(k+1)}(x)|\leq M$ $\forall x\in(a,b)$ then
$$|R_k(x,x_0)|\leq\frac{M}{(k+1)!}|x-x_0|^{k+1}=O\left(|x-x_0|^{k+1}\right)$$

$\left(F(x)=O(G(x))\text{ as } x\to x_0 \text{ means } \exists\, C>0 \text{ s.t. } |F(x)|\leq C|G(x)| \text{ for } |x-x_0| \text{ suff. small}\right)$

§3.3 Fixed point methods:

Suppose we can find $\phi : [a,b] \to \mathbb{R}$ s.t.

$$\phi(x) = x \iff f(x) = 0 .$$

Then we say $\phi(x) = x$ is a fixed point reformulation of $f(x) = 0$

Simple example that always holds: $\phi(x) = x - \beta f(x)$
for any $0 \neq \beta \in \mathbb{R}$.

But other choices are often possible.

Example: $f(x) = \sin 2x - 1 + x = 0$

Two choices are:
- $\phi_1(x) = x$ where $\phi_1(x) = 1 - \sin 2x$.
- $\phi_2(x) = x$ where $\phi_2(x) = \frac{1}{2}\arcsin(1-x)$ on $[0,2]$.

we know roots of f lie in this interval since $f(x) = 0 \iff 1 - x = \sin 2x \iff x = \underbrace{1 - \sin 2x}_{\in [0,2]}$

Existence of fixed points?

Brouwer's fixed point theorem: Suppose $\phi : [a,b] \to \mathbb{R}$ is continuous and that $\phi(x) \in [a,b]$ for all $x \in [a,b]$. Then there exists $\alpha \in [a,b]$ s.t. $\boxed{\phi(\alpha) = \alpha}$

(Note: α may not be unique!).

Proof: Let $f(x) = x - \phi(x)$. Then since $\phi : [a,b] \to [a,b]$,
$f(a) = a - \phi(a) \leq 0$ and $f(b) = b - \phi(b) \geq 0$, so $f(a)f(b) \leq 0$,
and by previous corollary $\exists\, \alpha \in [a,b]$ s.t. $f(\alpha) = 0$, i.e. $\phi(\alpha) = \alpha$. ∎

To approximate fixed points we can try the fixed point iteration:

$$\boxed{x_{k+1} = \phi(x_k)}, \quad k \in \mathbb{N}_0, \quad \text{given some initial guess } x_0 \in [a,b].$$

i.e. $x_1 = \phi(x_0)$, $x_2 = \phi(x_1) = \phi(\phi(x_0))$, etc.

This produces a sequence $x_0, x_1, x_2, \dots$ which we hope converges to a fixed point of ϕ.