EXERCISE 1(*) Let

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

(a) The Jacobi and Gauss-Seidel methods for the approximation of the solution to the linear system $A\mathbf{x} = \mathbf{b}$ can both be written in the form

$$P\mathbf{x}^{k+1} = N\mathbf{x}^k + \mathbf{b}.$$

Write down the matrices P and N for each of the two methods. What are the associated iteration matrices B_J and B_{GS}?

(b) Compute the vector $\mathbf{x}^1$ obtained after one iteration with the Jacobi method starting from the initial vector $\mathbf{x}^0 = \left(\frac{1}{2}, \frac{1}{2}\right)^T$.

(c) Do both methods converge? Which gives the iteration matrix with the smallest spectral radius?

(d) Prove that both methods converge linearly with respect to the norm $\|\cdot\|_\infty$, and compare the convergence constants.

Sol:

a), Since the Jacobi and Gauss-Seidel methods both decompose the matrix A as:

$$A = L + D + U$$

and $A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$,

So, $L = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

For Jacobi method, it takes the preconditioner to be $P = D$, and $N = -(L + U)$.

Therefore: $P = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

$$N = -\left[\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}.$$

For Gauss-Seidel method, it uses the preconditioner $P = L + D$, and $N = -U$.

Therefore, $P = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix}$,

$$N = -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

So, according to the above results, we can get that:

$$B_J = I - D^{-1}A = -D^{-1}(L+U)$$
$$= -\left[\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}^{-1}\right] \cdot \left[\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$$
$$= -\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} \\ -\frac{2}{5} & 0 \end{pmatrix}$$

$$B_{GS} = I - (L+D)^{-1}A = -(L+D)^{-1}U$$
$$= -\left[\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\right]^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$= -\begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$= -\begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{2}{15} & \frac{1}{5} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} \\ 0 & \frac{2}{15} \end{pmatrix}.$$

b). According to the A obtained above, we can see that A have all non-zero diagonal elements. So, the iteration step for Jacobi method can written as:

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j\neq 1, j\neq i}^{n} a_{ij} x_j^k\right), \quad i = 1, \dots, n.$$

Since, $x_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$,

So we can get:

$$x' = \left(\frac{1}{3}, \frac{1}{5}\right) \cdot \begin{pmatrix} 4 - (1 \times \frac{1}{2}) \\ 7 - (2 \times \frac{1}{2}) \end{pmatrix} = \left(\frac{1}{3}, \frac{1}{5}\right) \cdot \begin{pmatrix} \frac{7}{2} \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{6} \\ \frac{6}{5} \end{pmatrix}$$

Therefore, we can get that $x' = \left(\frac{7}{6}, \frac{6}{5}\right)^T$.

C). According to the Theorem 4.9.2, we can know that, if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \text{ for } i = 1, \dots, n$$

then the Jacobi method and the Gauss-Seidel method are both convergent.

So, when $i = 1$, $|a_{ii}| = 3$, $\sum_{j=1, j\neq i}^{n} |a_{ij}| = 1$, $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$;

and when $i = 2$, $|a_{ii}| = 5$, $\sum_{j=1, j\neq i}^{n} |a_{ij}| = 2$, $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$.

Therefore the two methods are both converge.

Moreover, as we calculated before:

$$B_J = \begin{pmatrix} 0 & -\frac{1}{3} \\ -\frac{2}{5} & 0 \end{pmatrix}, \quad B_{GS} = \begin{pmatrix} 0 & -\frac{1}{3} \\ 0 & \frac{2}{15} \end{pmatrix}$$

According to the Definition 4.4.1, the spectral radius $\rho(A)$ of a matrix A is:

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$

So, the eigenvalues of B_J is:

$$|A - \lambda I| = \left| \begin{pmatrix} -\lambda & -\frac{1}{3} \\ -\frac{2}{5} & -\lambda \end{pmatrix} \right| = 0,$$

$$= \left| \lambda^2 - \frac{2}{15} \right| = 0, \text{ so } \lambda_J = \sqrt{\frac{2}{15}} \Rightarrow \rho(B_J) = \sqrt{\frac{2}{15}}.$$

And, the eigenvalues of B_{GS} is:

$$|A - \lambda I| = \left| \begin{pmatrix} -\lambda & -\frac{1}{3} \\ 0 & \frac{2}{15} - \lambda \end{pmatrix} \right| = 0,$$

$$= \left| -\lambda \cdot \left(\frac{2}{15} - \lambda\right) \right| = 0$$

$$= \left| -\frac{2\lambda}{15} + \lambda^2 \right| = 0$$

So $\lambda_{GS} = 0$ or $\lambda_{GS} = \frac{2}{15} \Rightarrow \rho(B_{GS}) = \frac{2}{15} > \rho(B_J)$.

Therefore, the Jacobi method gives the iteration matrix with the smallest spectral radius $\rho(B_J)$.

d). For the Jacobi method, since the entries (b_{ij}) of iteration matrix B_J are given by $b_{ij} = -a_{ij}/a_{ii}$ for $i \neq j$ and $b_{ii} = 0$ for each i. So, we can get:

$$\|B_J\|_\infty = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^{n} |b_{ij}| = \max_{i\in\{1,\dots,n\}} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} = \max_{i\in\{1,\dots,n\}} \frac{1}{|a_{ii}|} \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|.$$

Since $A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$, we can get that:

$\|B_J\|_\infty = \max\left\{0, \frac{1}{3}, \frac{2}{5}, 0\right\} = \frac{2}{5} < 1$.

According to Lemma 4.4.2, $\|B_J\|_\infty < 1$ implies $\rho(B_J) < 1$, so the Jacobi method converges. And according to Theorem 4.6.2, we can know that the Jacobi method converges linearly with respect to $\|\cdot\|_\infty$ with a convergence constant $\|B_J\|_\infty = \frac{2}{5}$.

In the same way, we can get that:

$\|B_{GS}\|_\infty = \max\left\{0, \frac{1}{3}, 0, \frac{2}{15}\right\} = \frac{1}{3} < 1$.

Therefore, both methods converge linearly with respect to the $\|\cdot\|_\infty$, and the convergence constant of Jacobi method is $\frac{2}{5}$, and the convergence constant of Gauss-Seidel method is $\frac{1}{3}$. The Gauss-Seidel method has smaller convergence constant.