Theoretical exercise sheet 1
Nonlinear equations

Feedback from marking

Below are some remarks about common problems people had with the exercises. Please ask me if you would like me to explain more, or if anything is confusing to you.

EXERCISE 3(*) - MARKED

(a) Most people correctly identified which root could be obtained by bisection, but not so many people gave a satisfactory explanation.

The positive root α_+ can be found by bisection. To prove this you needed to find an interval $[a, b]$ containing α_+ such that $f(a)f(b) < 0$ and that α_+ is the only root of f in $[a, b]$.

The negative root α_- cannot be found by bisection in general. This is because the negative root is actually a local maximum. You might have suspected this from the graph, but, as I hinted in the question, you needed to prove this. It's easy - just find the appropriate critical point of f (you can find it analytically!), show that this point is actually a root of f, and that f'' is negative there. The fact that it is a local maximum means that, for sufficiently small δ, $f(x) < 0$ for all x such that $|x-\alpha_-| < \delta$ and $x \neq \alpha_-$. So bisection cannot converge to this root for any starting interval $[a, b]$, except in special cases where $a = \alpha_-$ or $b = \alpha_-$, or $x^k = \alpha_-$ for some k. To prove this last statement you just have to argue by contradiction: Suppose bisection did converge to α_-, starting from some initial interval $[a, b]$ for which $f(a)f(b) \leq 0$, but that $\alpha_- \neq a$, $\alpha_- \neq b$ and $\alpha_- \neq x^k$ for any k. Then for any such δ satisfying the above criterion, one could find $k \in \mathbb{N}$ such that the interval $[a, b]$ obtained after the kth iteration of bisection would lie inside the interval $[\alpha_- - \delta, \alpha_- + \delta]$, which would give a contradiction as f does not change sign on this interval, so the interval $[a, b]$ could not possibly have arisen in the course of the bisection algorithm. (THIS IS MORE DETAIL THAN I WOULD EXPECT FROM THE AVERAGE STUDENT IN AN EXAM SITUATION, BUT STILL I HOPE IT IS HELPFUL IN SHOWING YOU HOW TO CONSTRUCT A RIGOROUS PROOF.)

(b) Mostly fine. But some students are confused about convergence orders. Look carefully at the definitions on p9 of the notes. "Linear convergence" is the another name for "convergence of order 1" or "first order convergent". "Quadratic convergence" is the another name for "convergence of order 2" or "second order convergent".

(c) Most people realised that you needed to apply the contraction mapping theorem here. But be careful - it is not enough to just prove that $|\phi'(x)| \leq \Lambda$ for some $0 < \Lambda < 1$ (e.g. $\Lambda = 1/2$ works) and all $x \in [\pi/2, \pi]$. You also need to prove that ϕ maps the interval $[\pi/2, \pi]$ to itself. How to do that? Try and find the maximum/minimum of ϕ on $[\pi/2, \pi]$ and show that they both lie in the interval $[\pi/2, \pi]$. Is ϕ monotonic on this interval?

Some people tried to prove convergence directly by using a Taylor expansion (without using the CMT), claiming that since $|\phi'(x)| \leq \Lambda < 1$ on $[\pi/2, \pi]$ the iterates satisfy a linear convergence type relation $|x^{k+1} - \alpha| \leq \Lambda|x^k - \alpha|$. This is certainly true for $k = 0$ if $x^0 \in [\pi/2, \pi]$. But how do you know it holds for $k > 0$? The problem is that x^{k+1} might lie outside the interval $[\pi/2, \pi]$. So this sort of direct approach is not valid, unless you also show that ϕ maps $[\pi/2, \pi]$ to $[\pi/2, \pi]$.

If you are not convinced by the above, here is a (hopefully) enlightening example. Let

$$\phi(x) = \begin{cases} -\frac{x^2}{4}, & x \geq 0, \\ 32x^2, & x < 0. \end{cases} \tag{1}$$

which has a unique fixed point at $x = 0$ (convince yourself by sketching the graph of ϕ). The function ϕ is continuously differentiable (you might be worried about what happens at $x = 0$, but here $\phi'(x) = 0$ whichever side we approach from). And on the interval $[-\frac{1}{128}, 1]$ (which contains the fixed point) it is certainly true that $|\phi'(x)| \leq 1/2 < 1$. But the resulting fixed point iteration $x^{(k+1)} = \phi(x^{(k)})$ will not converge for every $x^{(0)} \in [-\frac{1}{128}, 1]$. Indeed, for $x^{(0)} = 1$ we get $x^{(1)} = -1/4$, $x^{(2)} = 2$, $x^{(3)} = -1$, $x^{(4)} = 32$, ..., which diverges. The problem is that ϕ does not map the interval $[-\frac{1}{128}, 1]$ to itself.
[Note that this example doesn't contradict Theorem 3.3.8 in the notes, because that theorem doesn't say anything about arbitrary initial guesses in $[a, b]$, it just says that if x^0 is close enough to α we get convergence.]

EXERCISE 4(*) - NOT MARKED

This question was just a quick-fire test of your understanding of fixed point theory. Since I didn't specify an interval in which the initial guesses should lie, you can assume that by "convergent" I mean "locally convergent", i.e. "convergent for a sufficiently good initial guess". That is, you don't need to apply the CMT directly - you can use Theorems 3.3.8 and 3.3.10.
Part (a) was simpler than some people tried to make it. The real root of $f(x) = 0$ can be found by hand to be $x = 2^{1/3}$. You then just had to check for which ω it holds that $\phi(2^{1/3}) = 2^{1/3}$. For $x \neq 0$ the equation $\phi(x) = x$ is equivalent to $(x^3 - 2)(3(1 - w)x^2 - w) = 0$. So it's clear that $x = 2^{1/3}$ is a fixed point of ϕ for every $\omega \in \mathbb{R}$. That was all you needed to check to answer the question.
Some of you noticed that for certain values of ω there are other fixed points. (Note that you didn't need to find these to answer the question, so the following is "extra material"). Specifically, for $0 < \omega < 1$ there are two other fixed points at $x = \pm\sqrt{\omega/(3(1 - \omega))}$. So for $0 < \omega < 1$ the fixed point formulation is not consistent with $f(x) = 0$ on the whole of $\mathbb{R}$ (since ϕ has up to three real fixed points but $f(x) = 0$ has only one real root). But it is still consistent with $f(x) = 0$ on a sufficiently small interval surrounding $x = 2^{1/3}$. This is usually all we can hope for in general, and usually all we need in practice for "local" convergence (i.e. convergence to a particular fixed point for a sufficiently good initial guess).
Note that I wrote "up to three real fixed points" above because in the special case $\omega = 1/(1 + 3^{-1}2^{-2/3}) \approx 0.83$ the fixed point $x = \sqrt{\omega/(3(1 - \omega))}$ coincides with the original fixed point $x = 2^{1/3}$, so there are actually only two distinct fixed points in this special case. But in general for $0 < \omega < 1$ there are three.
In part (d), from Theorem 3.3.10 of the notes we can see that if ϕ is twice continuously differentiable at the fixed point $2^{1/3}$ (which it is in this case) and the fixed point iteration has faster than second order convergence, then it must be true that $\phi''(2^{1/3}) = 0$. (How to prove this? Try assuming that you have convergence of order $p > 2$. What does this tell you about the limit of the ratio $|x_{k+1} - \alpha|/|x_k - \alpha|^p$, and hence of the ratio $|x_{k+1} - \alpha|/|x_k - \alpha|^2$, as $k \to \infty$?) But note that according to our definitions (on p9 of notes), if a method is of order $p > 2$ then it must also be of second order (and first order). So even if you find a ω for which $\phi''(2^{1/3}) = 0$, you also have to check that $\phi'(2^{1/3}) = 0$ for that choice of ω. (In fact it is not possible to satisfy both these conditions.)