MATH0033 Numerical Methods
Self-study problem sheet Week 1

These problems are non-assessed and are intended to help you test your understanding of the course material. Solutions are provided, but I strongly recommend you try the problems yourself first.

1. Show the equivalence of norms

$$\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_p \le n^{\frac{1}{p}}\|\boldsymbol{x}\|_\infty \quad \forall \boldsymbol{x} \in \mathbb{R}^n$$

2. Use the equivalence of norms above to show that

$$\lim_{p\to\infty}\|\boldsymbol{x}\|_p = \|\boldsymbol{x}\|_\infty \quad \forall \boldsymbol{x} \in \mathbb{R}^n$$

3. Suppose that $V = \mathbb{R}^n$ with $n \ge 2$. Show that $\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ does not define a norm when $p < 1$.

4. Let $C([0,1])$ denote the space of continuous functions on the closed bounded interval $[0,1]$. Define, for all $f \in C([0,1])$,

$$\|f\|_\infty = \max_{x\in[0,1]}|f(x)|, \qquad \|f\|_* = \max_{x\in[0,1]}|xf(x)|$$

Show that $\|\cdot\|_*$ defines a norm on $C([0,1])$. Is $\|\cdot\|_*$ equivalent to $\|\cdot\|_\infty$ on $C([0,1])$? If not, why not?

5. **Linear convergence.** Suppose that $\boldsymbol{x}_n \to \boldsymbol{x}$ as $n \to \infty$ and that the rate of convergence is linear with $\|\boldsymbol{x} - \boldsymbol{x}_{n+1}\| \le C\|\boldsymbol{x} - \boldsymbol{x}_n\|$ for all $n \ge N$, and $C \in (0,1)$. Show that for all n sufficiently large

$$\|\boldsymbol{x} - \boldsymbol{x}_n\| \le C_0 C^n,$$

for some constant C_0 that you should determine.

SOLUTIONS

1. Recall that $\|\boldsymbol{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ and $\|\boldsymbol{x}\|_\infty = \max_{1\le i\le n}|x_i|$. Clearly we then have $\|\boldsymbol{x}\|_\infty^p \le \sum_{i=1}^n |x_i|^p$ since the sum must include the term with maximum absolute value. Taking p-roots on both sides implies that $\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_p$. For the other inequality, we bound every term in the sum inside $\sum_{i=1}^n |x_i|^p$ by $\|\boldsymbol{x}\|_\infty^p$ to obtain
$$\sum_{i=1}^n |x_i|^p \le n\|\boldsymbol{x}\|_\infty^p \implies \|\boldsymbol{x}\|_p \le n^{\frac{1}{p}}\|\boldsymbol{x}\|_\infty.$$

2. The limit $\lim_{p\to\infty}\|\boldsymbol{x}\|_p = \|\boldsymbol{x}\|_\infty$ holds trivially for $\boldsymbol{x} = 0$, so we need only consider the case $\boldsymbol{x} \neq 0$. Taking the equivalence of norms above and dividing by $\|\boldsymbol{x}\|_\infty$ (which is nonzero for $\boldsymbol{x} \neq 0$), we get
$$1 \le \frac{\|\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_\infty} \le n^{\frac{1}{p}}$$
Since $n^{1/p} \to 1$ as $p \to \infty$, we conclude that $\frac{\|\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_\infty} \to 1$, and thus we have the limit $\|\boldsymbol{x}\|_p \to \|\boldsymbol{x}\|_\infty$ as $p \to \infty$.

3. Suppose that $p < 1$, and let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ denote the first two unit vectors. Then
$$\|\boldsymbol{e}_1 + \boldsymbol{e}_2\|_p = (1+1)^{\frac{1}{p}} = 2^{\frac{1}{p}} > 2$$
where the last inequality follows from the fact that $1/p > 1$. On the other hand, $\|\boldsymbol{e}_1\|_p + \|\boldsymbol{e}_2\|_p = 1 + 1 = 2$. Therefore for $p < 1$ the triangle inequality does not hold since
$$\|\boldsymbol{e}_1 + \boldsymbol{e}_2\|_p \not\le \|\boldsymbol{e}_1\|_p + \|\boldsymbol{e}_2\|_p$$
So $\|\cdot\|_p$ does not define a norm when $p < 1$.

4. To check that $\|\cdot\|_*$ is a norm on $C([0,1])$ we must check all the conditions of the norm. The only condition that is not immediately obvious is the positive definiteness condition $\|f\|_* = 0 \implies f = 0$ (recall here $f = 0$ means that $f(x) = 0$ for all $x \in [0,1]$). Suppose that $\|f\|_* = 0$, then we have $|xf(x)| = 0$ for all $x \in [0,1]$. This allows us to conclude that $|f(x)| = 0$ for all $x \in (0,1]$, i.e. excluding $x = 0$. Therefore $f(x) = 0$ for all $x \in (0,1]$. But since f is continuous on $[0,1]$ we also deduce by continuity then that $f(0) = 0$ also, and thus $f = 0$.

   The norms $\|\cdot\|_\infty$ and $\|\cdot\|_*$ are not equivalent on $C([0,1])$. To show this, let $f_k(x)$, $k \in \mathbb{N}$, be defined by
$$f_k(x) = \begin{cases} 1 - kx & \text{for } x \in [0,1/k] \\ 0 & \text{for} x \in (1/k, 1] \end{cases}$$
   It follows that $f_k \in C([0,1])$ for all $k \in \mathbb{N}$. However
$$\|f_k\|_\infty = 1 \quad \forall k \in \mathbb{N},$$

whereas

$$\|f_k\|_* = \max_{x\in[0,1]} |xf_k(x)| = \max_{x\in[0,1/k]} |x - kx^2| = \frac{1}{2k} - \frac{k}{4k^2} = \frac{1}{4k} \to 0 \quad \text{as } k \to \infty.$$

Therefore, there is no constant $C > 0$ such that

$$\|f\|_\infty \leq C\|f\|_* \quad \forall f \in C([0,1]),$$

which shows that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_*$ are not equivalent.

5. By induction, we have for all $n \geq N$

$$\|\boldsymbol{x} - \boldsymbol{x}_n\| \leq C\|\boldsymbol{x} - \boldsymbol{x}_{n-1}\| \leq C^2\|\boldsymbol{x} - \boldsymbol{x}_{n-2}\| \leq \ldots \leq C^{n-N}\|\boldsymbol{x} - \boldsymbol{x}_N\|$$

Therefore, we get the required bound by setting $C_0 = C^{-N}\|\boldsymbol{x} - \boldsymbol{x}_N\|$.