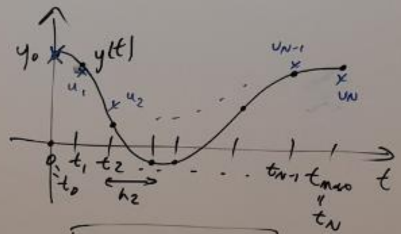
Numerical Methods

Last time: finite difference methods for ODE IVPs:

$$\begin{cases} y'(t) = f(t, y(t)), & 0 < t < t_{max} \\ y(0) = y_0 \end{cases}$$



One-step method (general form):

$$\begin{cases} u_0 = y_0 \\ u_{n+1} = u_n + h_n \Psi(t_n, u_n, u_{n+1}, h_n) \end{cases}$$

$\boxed{h_n = t_{n+1} - t_n}$

$\boxed{u_n \approx y(t_n)}$

Examples: FE, BE, CN.

Warning: other authors might use different definitions.

Truncation error:

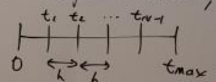
$$T_n := \frac{y(t_{n+1}) - y(t_n)}{h_n} - \Psi(t_n, y(t_n), y(t_{n+1}), h_n), \quad n = 0, 1, \dots, N-1.$$

$$= \frac{1}{h_n}\Big(\underbrace{y(t_{n+1}) - y(t_n) - h_n \Psi(t_n, y(t_n), y(t_{n+1}), h_n)}_{\text{residual when we plug exact solution } y(t_n) \text{ into the recurrence relation for } u_n}\Big)$$

IMPORTANT: $y(t_n)$ not u_n

For simplicity, let's assume henceforth that we have a uniform mesh, with

$$h_n = h = \frac{t_{max}}{N} \quad \text{for all } n = 0, \dots, N-1.$$



so $t_n = nh$, $n = 0, \dots, N$.

Def'n: We say the method is uniformly consistent if

$$T(h) := \max_{n \in \{0, \dots, N-1\}} |T_n|$$

tends to zero as $h \to 0$. (i.e. method approximates the ODE in the limit $h \to 0$ i.e. $N \to \infty$ with t_{max} fixed)

We say the method is (consistent) of order $p \in \mathbb{N}$ if $T(h) = O(h^p)$ as $h \to 0$. (i.e. as $N \to \infty$)

Example: FE (forward Euler) is consistent of order 1. (for sufficiently smooth y)

Proof: For ease of notation, write y_n for $y(t_n)$ (not the same as u_n) and y'_n for $y'(t_n)$ etc.

$$T_n = \left(\frac{y(t_{n+1}) - y(t_n)}{h} - f(t_n, y(t_n))\right)$$

By Taylor's theorem,

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + h\underbrace{y'(t_n)}_{\substack{= f(t_n, y(t_n)) \\ \text{by ODE.}}} + \frac{h^2}{2} y''(\xi_n) \qquad \text{for some } \xi_n \in [t_n, t_{n+1}]$$

Hence $|T_n| = \frac{h}{2} |y''(\xi_n)|$

Assuming y is twice differentiable on $[0, t_{max}]$ and y'' is bounded,

$$\boxed{T(h) \leq Ch}, \quad \text{where } C = \frac{1}{2} \sup_{\xi \in [0, t_{max}]} |y''(\xi)|.$$

i.e. $T(h) = O(h)$ as $h \to 0$.

<u>Example:</u> BE is also consistent of order 1 (under same assumptions on y)

<u>Proof:</u> Now $T_n = \frac{y_{n+1} - y_n}{h} - f(t_{n+1}, y_{n+1})$

Expand y_{n+1} as before, but also need to expand this term.

Trick: $f(t_{n+1}, y_{n+1}) = y'_{n+1}$ (by ODE)

$$= y'(t_n + h)$$

$$= y'(t_n) + h y''(\eta_n) \qquad \text{for some } \eta_n \in (t_n, t_{n+1})$$

Hence $|T_n| = h \left| \frac{y''(\xi_n)}{2} - y''(\eta_n) \right| \underset{\Delta\text{-ineq.}}{\leq} Ch$ where $C = \frac{3}{2} \sup_{\xi \in [0, t_{max}]} |y''(\xi)|$.

<u>Example:</u> CN is consistent of order 2 (for sufficiently smooth y)

with $T(h) \leq Ch^2$, $C = \frac{5}{12} \sup_{\xi \in [0, t_{max}]} |y'''(\xi)|$

(need y three times differentiable with y''' bounded)

– for proof see notes (need to expand to a higher order)

Note: smoothness of y is a property of the IVP:

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = y_0. \end{cases}$$

We won't prove any smoothness results, but you need to state your smoothness assumptions clearly.