Numerical Methods

Last time: matrix norms.

Given a norm $\|\cdot\|_V$ on $\mathbb{R}^n$, the induced matrix norm is

$$\boxed{\|A\|_M := \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|A\underline{x}\|_V}{\|\underline{x}\|_V}} \qquad \left(= \sup_{\substack{\underline{x} \in \mathbb{R}^n \\ \|\underline{x}\|_V = 1}} \|A\underline{x}\|_V \right)$$

In particular, for each p-norm $\|\cdot\|_p$ on $\mathbb{R}^n$ $\left(\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right)$

we get an induced matrix p-norm

$$\|A\|_p = \sup_{\substack{x \in \mathbb{R}^n \\ \underline{x} \neq 0}} \frac{\|A\underline{x}\|_p}{\|\underline{x}\|_p}.$$

[So when you see e.g. $\|A\|_2$ it means the induced 2-norm,

<u>not</u> the Frobenius norm $\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$]

<u>Example</u>: The induced ∞-norm $\|A\|_\infty := \sup_{\substack{\underline{x} \in \mathbb{R}^n \\ \underline{x} \neq \underline{0}}} \frac{\|A\underline{x}\|_\infty}{\|\underline{x}\|_\infty}$ can be evaluated as

$$\boxed{\|A\|_\infty = \underbrace{\max_{i \in \{1, \dots, n\}} \sum_{j=1}^{n} |a_{ij}|}_{=: S}}, \qquad A \in \mathbb{R}^{n \times n}.$$

← maximum row sum of $|a_{ij}|$

<u>Proof</u>: To see that $\|A\|_\infty \leq S$, note that for $\underline{x} \in \mathbb{R}^n$

$$\|A\underline{x}\|_\infty = \max_i \left| \sum_j a_{ij} x_j \right| \underset{\Delta\text{-ineq}}{\leq} \max_i \sum_j |a_{ij}||x_j| \underset{|x_j| \leq \|\underline{x}\|_\infty}{\leq} \left(\max_i \sum_j |a_{ij}| \right) \|\underline{x}\|_\infty = S \|\underline{x}\|_\infty.$$

Hence $\frac{\|A\underline{x}\|_\infty}{\|\underline{x}\|_\infty} \leq S$ for $\underline{x} \neq 0$, so $\|A\|_\infty \leq S$.

To see that $\|A\|_\infty \geq S$, let $i_* \in \{1, \dots, n\}$ be such that $S = \sum_{j=1} |a_{i_*j}|$.

Let $\underline{x} \in \mathbb{R}^n$ be defined by $x_j = \begin{cases} 1, & \text{if } a_{i_*j} \geq 0 \\ -1, & \text{if } a_{i_*j} < 0 \end{cases}$.
$= (x_1, \dots, x_n)^T$

$$A = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix}$$

Then $\|A\underline{x}\|_\infty \geq \left| \sum_{j=1}^{n} a_{i_*j} x_j \right| = \sum_{j=1}^{n} |a_{i_*j}| = S$.

Also, $\|\underline{x}\|_\infty = 1$, so $\dfrac{\|Ax\|_\infty}{\|x\|_\infty} \geq S$.

Hence $\|A\|_\infty \geq S$.

Eigenvalues satisfy the polynomial equation

$$\det(A - \lambda I) = 0$$

determinant sometimes written $|A - \lambda I|$

$A\underline{v} = \lambda \underline{v}$
$\Leftrightarrow (A - \lambda I)\underline{v} = \underline{0}$
$\underline{v} \neq 0 \Leftrightarrow A - \lambda I$ is singular
$\Leftrightarrow \det(A - \lambda I) = 0$.

[I expect you to be able to calculate eigenvalues of 2×2 and 3×3 matrices by hand]

e.g. $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \underbrace{(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0}_{\text{quadratic eqn for } \lambda}$

<u>Lemma</u>: Let $\|\cdot\|_V$ be a vector norm on $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n \times n}$ and let $\lambda \in \sigma(A)$.

Then there exists $\hat{\underline{v}} \in \mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$ s.t.

$$\boxed{\frac{\|A\hat{\underline{v}}\|_V}{\|\hat{\underline{v}}\|_V} \geq |\lambda|} \quad \text{and} \quad \boxed{\frac{\|A^k \hat{\underline{v}}\|_V}{\|\hat{\underline{v}}\|} \geq \varepsilon |\lambda|^k}.$$

<u>Proof</u>: Full proof is in the notes, but is unassessed.

If $\exists$ real eigenvector $\underline{v} \in \mathbb{R}^n \setminus \{\underline{0}\}$ then $\|A^k \underline{v}\|_V = \|\lambda^k \underline{v}\|_V \ \forall k$ $= |\lambda^k| \|\underline{v}\|_V$

so $\dfrac{\|A^k \underline{v}\|_V}{\|\underline{v}\|_V} = |\lambda|^k \ \forall k$, so can take $\hat{\underline{v}} = \underline{v}$ in this case.

But if eigenvectors are complex we have to work harder – see notes.

Def'n The spectral radius of $A \in \mathbb{R}^{n \times n}$ is defined to be

$$\boxed{\rho(A) := \max_{\lambda \in \sigma(A)} |\lambda|}$$

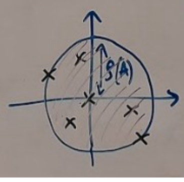
Note $|\cdot|$ here

ρ "rho" $\neq$ σ "sigma"

$\rho(A)$ is a non-negative real number.

$\sigma(A)$ is a subset of $\mathbb{C}$ with $\leq n$ elements.

So $\rho(A)$ is the radius of the smallest closed ball centred at 0 containing $\sigma(A)$



λ

$\times$ = eigenvalues of A.

$\rho(A)$ is a semi-norm but not a norm. (i.e. satisfies all properties of a norm except $\rho(A) = 0 \not\Rightarrow A = 0$)

Lemma:

- For any $\|\cdot\|_M$ be induced by some $\|\cdot\|_V$

  Then $\rho(A) \leq \|A\|_M \quad \forall A \in \mathbb{R}^{n \times n}$

- Given $A \in \mathbb{R}^{n \times n}$ and $\varepsilon > 0$ $\exists \|\cdot\|_{V,A,\varepsilon}$ for which the induced norm $\|\cdot\|_{M,A,\varepsilon}$ satisfies

  $$\|A\|_{M,A,\varepsilon} \leq \rho(A) + \varepsilon.$$

  A, ε subscripts indicate that $\|\cdot\|_{V,A,\varepsilon}$ depends on both A and ε.

Hence $\rho(A) = \inf_{\|\cdot\|_M} \|A\|_M$ infimum over all induced matrix norms $\|\cdot\|_M$.

Proof: First part: Let $\lambda \in \sigma(A)$ be such that $|\lambda| = \rho(A)$.

Then by previous Lemma $\exists\, \hat{\underline{v}} \in \mathbb{R}^n \setminus \{\underline{0}\}$ s.t. $\dfrac{\|A\hat{\underline{v}}\|_V}{\|\hat{\underline{v}}\|_V} \geq |\lambda|$

Then $$\rho(A) = |\lambda| \leq \frac{\|A\hat{\underline{v}}\|_V}{\|\hat{\underline{v}}\|_V} \leq \sup_{\substack{\underline{v} \in \mathbb{R}^n \\ \underline{v} \neq \underline{0}}} \frac{\|A\underline{v}\|_V}{\|\underline{v}\|_V} = \|A\|_M.$$

Second part – beyond scope of course.
See books on "matrix analysis".

For symmetric real matrices, $\rho(A)$ is actually a norm.

Lemma: If $A \in \mathbb{R}^{n\times n}$ is symmetric then

$$\rho(A) = \|A\|_2 \qquad \left(= \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2} \right)$$

Euclidean norm on $\mathbb{R}^n$.

No proof given here, but this is a fact you should know.

(think about diagonalisation ...)

Eigenvalues are connected to the behaviour of powers of A:

Lemma: Let $A \in \mathbb{R}^{n\times n}$. Then $\boxed{\lim_{k\to\infty} A^k = 0 \iff \rho(A) < 1}$ ✱✱✱

i.e. $\|A^k\|_M \xrightarrow{k\to\infty} 0$ for any $\|\cdot\|_M$.

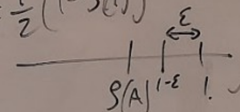
"convergent matrices"

Proof: Suppose $A^k \xrightarrow{k\to\infty} 0$. Let $\|\cdot\|_v$ be a vector norm and $\|\cdot\|_M$ its induced matrix norm. Then

$$(\rho(A))^k \le \rho(A^k) \overset{\text{previous lemma}}{\le} \|A^k\|_M \to 0. \qquad \text{So } \rho(A) < 1.$$

easy to prove! In fact we have equality!

Now suppose $\rho(A) < 1$. Choose $\varepsilon > 0$ s.t. $\rho(A) < 1 - \varepsilon$. (e.g. take $\varepsilon = \frac{1}{2}(1 - \rho(A))$)



Then by the previous lemma $\exists$ vector norm $\|\cdot\|_v$ and induced norm $\|\cdot\|_M$ s.t. $\|A\|_M \le \rho(A) + \varepsilon < 1$.

But then $\|A^k\|_M \le (\|A\|_M)^k \to 0$ as $k \to \infty$.

properties of induced norms
$\|AB\|_M \le \|A\|_M \|B\|_M$
plus induction

Def'n: Let $\|\cdot\|_M$ be induced by $\|\cdot\|_V$.
Given $A \in \mathbb{R}^{n\times n}$, the <u>condition number</u> of A w.r.t. $\|\cdot\|_M$ is defined by

$$K_M(A) := \|A\|_M \|A^{-1}\|_M . \qquad \left(K_M(A) := \infty \text{ if } A \text{ is singular} \right)$$

Facts:
- $K_M(A) \geq 1$ (proof - exercise: write $I = A A^{-1}$)
- $\frac{1}{K_M(A)}$ is the distance in $\|\cdot\|_M$ from A to the closest singular matrix.

§4.3 Eigenvalues and spectral radius

Given $A \in \mathbb{R}^{n\times n}$, a complex number $\lambda \in \mathbb{C}$ is called an <u>eigenvalue</u> of A if $\exists\, \underline{v} \in \mathbb{C}^n \setminus \{0\}$ such that $\boxed{A\underline{v} = \lambda \underline{v}}$.

"eigenvector" associated to λ (there may be more than one)

We call

$$\sigma(A) = \{ \lambda \in \mathbb{C} \text{ s.t. } \lambda \text{ is an eigenvalue of } A \}$$

the <u>spectrum</u> of A.

$K_M(A)$ indicates how easy it is to solve $A\underline{x} = \underline{b}$ in general.
$K_M(A) \simeq 1$ means "well-conditioned" - easier to solve $A\underline{x} = \underline{b}$.
$K_M(A) \gg 1$ means "ill-conditioned" - harder . . . .