

Theoretical exercise sheet 2

Linear systems

Exercises 1 and 5 (marked *) to be submitted via *Crowdmark* in pdf format (either handwritten and scanned, or typeset using LaTeX).

Deadline: 23:59hrs Sunday 13th November

EXERCISE 1(*) Let

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

- (a) The Jacobi and Gauss-Seidel methods for the approximation of the solution to the linear system $A\mathbf{x} = \mathbf{b}$ can both be written in the form

$$P\mathbf{x}^{k+1} = N\mathbf{x}^k + \mathbf{b}.$$

Write down the matrices P and N for each of the two methods. What are the associated iteration matrices B_J and B_{GS} ?

- (b) Compute the vector \mathbf{x}^1 obtained after one iteration with the Jacobi method starting from the initial vector $\mathbf{x}^0 = \left(\frac{1}{2}, \frac{1}{2}\right)^T$.
- (c) Do both methods converge? Which gives the iteration matrix with the smallest spectral radius?
- (d) Prove that both methods converge linearly with respect to the norm $\|\cdot\|_\infty$, and compare the convergence constants.

EXERCISE 2 Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

- (a) Write the Gauss-Seidel method for the solution of the linear system $A\mathbf{x} = \mathbf{b}$. What is the associated iteration matrix B_{GS} ?
- (b) What can be said about the convergence of the Gauss-Seidel method for this system?
- (c) Now consider the iteration

$$D(\mathbf{x}^{k+1} - \mathbf{x}^k) = \alpha \mathbf{r}^k \quad k \geq 0,$$

where D is the diagonal of A and $\mathbf{r}^k = \mathbf{b} - A\mathbf{x}^k$ is the residual. This method is sometimes called the Jacobi over-relaxation (JOR) method; note that for $\alpha = 1$ the method is equivalent to the Jacobi method.

Write down the associated iteration matrix B_{JOR} , and find the optimal choice of α minimising the spectral radius $\rho(B_{JOR})$.

- (d) Starting from the initial vector $\mathbf{x}^{(0)} = (1, 1)^T$, compute the first iteration \mathbf{x}^1 of the JOR method using the optimal choice of α you derived in part (c).

EXERCISE 3 Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

- (a) Write down the Jacobi method for this system and establish if the method is convergent.
 (b) Now consider the iterative method defined by

$$L\mathbf{x}^{k+1} = L\mathbf{x}^k + \delta \mathbf{r}^k, \quad k \geq 0, \quad (1)$$

where $\delta > 0$ is a parameter, $\mathbf{r}^k = \mathbf{b} - A\mathbf{x}^k$ is the residual, and

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}.$$

Rewrite (1) in the form $\mathbf{x}^{k+1} = B_\delta \mathbf{x}^k + \mathbf{c}_\delta$, $k \geq 0$, giving explicit expressions for the matrix B_δ and the vector \mathbf{c}_δ .

- (c) For what values of the parameter $\delta > 0$ does the method (1) converge? Find the optimal value of δ which minimises the spectral radius $\rho(B_\delta)$.
 (d) Take $\delta = \frac{4}{3}$. By using the results from part (c), establish if (1) converges. If yes, which method do you expect to have the faster convergence, (1) or the Jacobi method? Why?

EXERCISE 4 Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} \gamma & \kappa & 0 \\ \kappa & \gamma & \beta \\ 0 & \beta & \gamma \end{pmatrix},$$

and $\kappa, \beta, \gamma \in \mathbb{R}$ are three real parameters.

(Warning: Before answering this question, make sure you know the difference between “necessary” and “sufficient” conditions.)

- (a) Without constructing the iteration matrices, give a *sufficient condition* on the coefficients κ, β and γ for the Jacobi and Gauss-Seidel methods to be convergent.
 (b) Write down the iteration matrix B_J for the Jacobi method and give a *necessary and sufficient* condition on κ, β, γ for the method to be convergent.
 (c) Now suppose we want to solve the system $A\mathbf{x} = \mathbf{b}$ using the JOR method (cf. Exercise 2)

$$D(\mathbf{x}^{k+1} - \mathbf{x}^k) = \alpha \mathbf{r}^k, \quad k \geq 0, \quad (2)$$

where D is the diagonal of A . This method corresponds to applying the stationary Richardson method (from lectures) to the modified system $D^{-1}A\mathbf{x} = D^{-1}\mathbf{b}$.

Determine for which γ, β, κ the matrix $D^{-1}A$ is symmetric positive definite. Assuming this holds, apply an appropriate theorem from lectures to determine for which α the method (2) converges, and compute the optimal value $\alpha = \alpha_*$ giving the fastest convergence, along with the associated error reduction factor $C > 0$ such that

$$\|\mathbf{x}^{k+1} - \mathbf{x}\|_2 \leq C \|\mathbf{x}^k - \mathbf{x}\|_2, \quad k \geq 0.$$

- (d) Given that $\|\mathbf{x}^0 - \mathbf{x}\|_2 = 1$, estimate the smallest integer for which $\|\mathbf{x}^k - \mathbf{x}\|_2 \leq \left(\frac{1}{2}\right)^9$, in the case when $\gamma = 1$, $\beta = 2^{-4}$, $\kappa = \sqrt{3}\beta$.

EXERCISE 5(*) Let A be a n -by- n symmetric positive definite matrix, and define the function $\|\mathbf{x}\|_A : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^T A \mathbf{x}}$.

- (a) Let $A^{1/2}$ denote the unique symmetric positive definite square root of A , which satisfies $(A^{1/2})^2 = A$ (you may assume without proof that $A^{1/2}$ exists). Show that $\|\mathbf{x}\|_A = \|A^{1/2}\mathbf{x}\|_2$, and use this fact to check that the function $\|\mathbf{x}\|_A$ defines a norm on \mathbb{R}^n . Determine constants $0 < c \leq C$ such that

$$c\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_A \leq C\|\mathbf{x}\|_2, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

- (b) Now consider the stationary Richardson method for the solution of the linear system $A\mathbf{x} = \mathbf{b}$, with iteration matrix $B_\alpha = I - \alpha A$. Show that $A^{1/2}B_\alpha = B_\alpha A^{1/2}$. Use this to prove that

$$\|\mathbf{e}^{k+1}\|_A \leq \rho(B_\alpha)\|\mathbf{e}^k\|_A,$$

where $\mathbf{e}^k = \mathbf{x} - \mathbf{x}^k$ denotes the solution error after the k th iteration.

EXERCISE 6 Consider the (unpreconditioned) gradient method for the solution of a linear system $A\mathbf{x} = \mathbf{b}$.

- (a) Show that the acceleration parameter α_k in the gradient method is the unique solution to the minimisation problem

$$\alpha_k = \arg \min \Phi(\mathbf{x}^k + \alpha \mathbf{r}^k),$$

where $\Phi(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T A \mathbf{y} - \mathbf{y}^T \mathbf{b}$ denotes the energy of the system $A\mathbf{x} = \mathbf{b}$.

- (b) Show that the residuals in the gradient method satisfy $(\mathbf{r}^{k+1}, \mathbf{r}^k) = 0$ for each k .
- (c) Give a geometric interpretation of the gradient method.