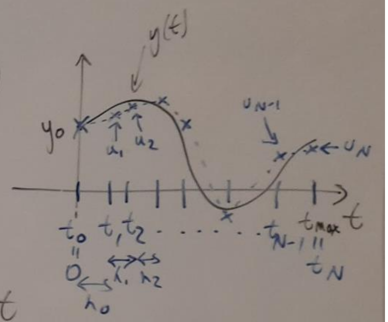
Numerical Methods

§5 - Ordinary differential equations (ODEs)



Initial value problem (IVT):

$\frac{dy}{dt}$ $\begin{cases} y'(t) = f(t, y(t)) , & 0<t \\ y(0) = y_0 . \end{cases}$

Here $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is assumed continuous, $y_0 \in \mathbb{R}$,

and $y: \mathbb{R}_+ \to \mathbb{R}$ is the solution to be determined.

For a numerical solution, truncate $t>0$ to $0<t<t_{max}$, and discretize $[0, t_{max}]$ into N subintervals with mesh points

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = t_{max} .$$

Aim to approximate $y(t)$ by a set of $N+1$ real numbers

$u_0, u_1, \ldots, u_N$ such that $u_n \simeq y(t_n)$.

How to choose $\{u_n\}$?

They should satisfy a discrete approximation of the IVP.

We're going to study "finite difference" methods.

Forward Euler method:

Replace $y'(t_n)$ by $\frac{u_{n+1} - u_n}{h_n}$, $h_n = t_{n+1} - t_n$,

and replace $f(t_n, y(t_n))$ by $f(t_n, u_n)$, giving:

(Motivation: $y'(t_n) \simeq \frac{y(t_n + h_n) - y(t_n)}{h_n}$ when h_n is small)

$$\text{FE} \begin{cases} u_0 = y_0 \\ u_{n+1} = u_n + h_n f(t_n, u_n) & n \geq 0 \end{cases} .$$

"Explicit" because u_{n+1} doesn't appear inside f.

So given u_n we have a formula for u_{n+1}.

Backward Euler method:

Like FE, but replace $f(t_n, y(t_n))$ by $f(t_{n+1}, u_{n+1})$ instead of $f(t_n, u_n)$:

$$\text{BE} \begin{cases} u_0 = y_0 \\ u_{n+1} = u_n + h_n f(t_{n+1}, u_{n+1}), \quad n \geq 0 \end{cases}$$

"Implicit" because to find u_{n+1} from u_n we have to solve a nonlinear equation $g(u_{n+1}) = 0$, where $\boxed{g(u) := u - u_n - h_n f(t_{n+1}, u)}$.

– can use e.g. bisection, fixed point methods, Newton, secant, chord, ...

Crank-Nicolson / trapezium rule method

Average of FE and BE:

$$\begin{cases} u_0 = y_0 \\ u_{n+1} = u_n + \frac{h_n}{2}\left(f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right), \quad n \geq 0. \end{cases}$$

Implicit, like BE.

(could use e.g. FE or BE as initial guess for nonlinear solve)

General "one-step method":

$$\begin{cases} u_0 = y_0 \\ u_{n+1} = u_n + h_n \Psi(t_n, u_n, u_{n+1}, h_n), \quad n \geq 0 \quad (n = 0, 1, \ldots, N-1) \end{cases}$$

where Ψ is an "increment function".

FE : $\Psi(t_n, u_n, u_{n+1}, h_n) = f(t_n, u_n)$

$t_{n+1} = t_n + h_n$

BE : " $= f(t_{n+1}, u_{n+1})$

CN : " $= \frac{1}{2}\left(f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right)$

General "one-step method":

$$\begin{cases} u_0 = y_0 \\ u_{n+1} = u_n + h_n \Psi(t_n, u_n, u_{n+1}, h_n), \quad n \geq 0 \quad (n = 0, 1, \dots, N-1) \end{cases}$$

where Ψ is an "increment function".

FE: $\Psi(t_n, u_n, u_{n+1}, h_n) = f(t_n, u_n)$

BE: " $= f(t_{n+1}, u_{n+1})$ $\quad t_{n+1} = t_n + h_n$

CN: " $= \frac{1}{2}\left(f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right)$

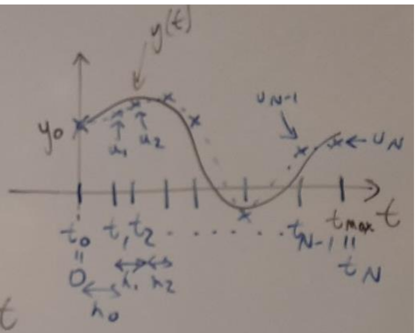
Note: in Süli/Mayers they don't include u_{n+1} in increment function, so for implicit methods Ψ is defined implicitly.

Questions we'll investigate:

- Does $u_n \to y(t_n)$ as $h_n \to 0$? ("convergence")

  (For fixed t_{max}, $\frac{1}{h_n}$ is a measure of computational cost)
- If so, how fast? ($O(h_n)$? $O(h_n^2)$?)
- Is u_n "stable" to perturbations (e.g. rounding errors) both as $h_n \to 0$ and as $t_{max} \to \infty$.

Before we go further with discretization, let's return briefly to our initial value problem (IVT):

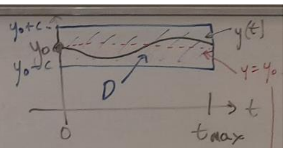


$\frac{dy}{dt}$

$$\begin{cases} y'(t) = f(t, y(t)), \quad 0 < t \\ y(0) = y_0 \end{cases}$$

Here $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is assumed continuous, $y_0 \in \mathbb{R}$, and $y: \mathbb{R}_+ \to \mathbb{R}$ is the solution to be determined.

What can we say about "well-posedness" of the problem? i.e. when does a unique solution $y(t)$ exist?



__Theorem (Picard)__ (see Süli / Mayers):

Suppose $f: D \to \mathbb{R}$ is continuous on $D = [0, t_{max}] \times [y_0 - c, y_0 + c]$ for some $t_{max} > 0$ and $c > 0$, and $\exists L > 0$ s.t.

$$(*) \quad \boxed{|f(t,v) - f(t,w)| \le L|v-w|} \quad \forall t \in [0, t_{max}],\ \forall v, w \in [y_0 - c, y_0 + c].$$

Lipschitz continuity w.r.t. 2nd argument

Suppose also that $K := \max_{t\in[0,t_{max}]} |f(t,y_0)| \le \dfrac{cL}{e^{Lt_{max}} - 1}$. $(**)$

Then $\exists$ a unique continuously differentiable function $y: [0, t_{max}] \to [y_0 - c, y_0 + c]$ s.t. $y(0) = y_0$ and $y'(t) = f(t, y(t))$ for $t \in [0, t_{max}]$.

Note: result is "stronger" than that proved in Analysis 4, which just provided well-posedness on "some" interval $t \in (-\delta, \delta)$, without telling us what δ is.

__Remark__: If $f(t,y)$ is differentiable w.r.t. y and $\frac{\partial f}{\partial y}$ is bounded in D, then $(*)$ holds with $L = \max_{(t,y)\in D} \left|\frac{\partial f}{\partial y}(t,y)\right|$, by MVT (Check!)

__Examples__:

- $y'(t) = \overbrace{3y - 3t}^{f(t,y)}$, $t > 0$ LINEAR!
  $y(0) = 1$

Here $|f(t,v) - f(t,w)| = 3|v-w|$ so $(*)$ holds with $L = 3$ and t_{max}, c orbitrary.

Given $t_{max} > 0$, can satisfy $(**)$ by taking $c > (1 + t_{max})(e^{3t_{max}} - 1)$.

So $y(t)$ is uniquely defined on $[0, \infty)$. Exact soln: $y(t) = \frac{2}{3}e^{3t} + t + \frac{1}{3}$, $t > 0$. (Check!)

- $y'(t) = \underbrace{\sqrt[3]{y(t)}}_{f(t,y) = y^{1/3}} = (y(t))^{1/3}$, $t > 0$, with $y(0) = 0$.

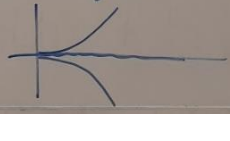
Now $f(t,y)$ is not Lipschitz in any neighbourhood of $(0,0)$, as $|\underbrace{f(0,0)}_{=0} - \underbrace{f(0,w)}_{=w^{1/3}}| = |w^{1/3}|$ which is not $\le L|w|$ for any L.

$\left(\text{since } \dfrac{|w^{1/3}|}{|w|} = |w|^{-2/3} \to \infty \text{ as } w \to 0\right)$

So Picard doesn't apply!

In fact uniqueness fails for this problem.

There are 3 different solutions: $y(t) \equiv 0$ (constant function zero), $y(t) = \pm\sqrt{\dfrac{8t^3}{27}}$.



• $y'(t) = 1 + (y(t))^2$, $t > 0$, $y(0) = 0$.

Now $f(t,y) = 1 + y^2$, so $|f(t,v) - f(t,w)| = |v^2 - w^2| = |v+w||v-w| \leq L|v-w|$ for $L = 2c$.

Also $f(t,0) = 1$, so Picard gives existence + uniqueness if e.g. $2c^2 \geq e^{2ct_{max}} - 1$.

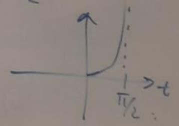
For $t_{max} < t_* \approx 0.569$ this can be satisfied by choosing c appropriately.

But for $t_{max} > t_*$ it can't.

Suggests IVP may only have a <u>local</u> solution (i.e. for suff. small t_{max})

This true: exact solution is $y(t) = \tan(t)$, $0 < t < \pi/2$

which blows up as $t \to \pi/2$



(Note $\pi/2 > t_*$ so theory isn't sharp)