Numerical Methods

$f(x)=0$

Last time: Newton's method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

as a fixed point method for $\phi(x) = x - \dfrac{f(x)}{f'(x)}$.

We saw it was locally quadratically convergent provided $f'(\alpha) \neq 0$.

Comment: when $f'(\alpha) = 0$, only get linear convergence in general. Need to check details carefully.

E.g. is $\phi(x)$ still continuous at $x = \alpha$?

To evaluate $\lim\limits_{x \to \alpha} x - \dfrac{f(x)}{f'(x)}$ (assuming $f'(x) \neq 0$ near α, except at α)

try Taylor expanding: $$f(x) = \cancelto{0}{f(\alpha)} + \cancelto{0}{f'(\alpha)}(x-\alpha) + \dots + \cancelto{0}{\frac{f^{(p-1)}(\alpha)}{(p-1)!}}(x-\alpha)^{p-1} + \frac{f^{(p)}(\alpha)}{p!}(x-\alpha)^p + o((x-\alpha)^p)$$

where p is the smallest integer s.t. $f^{(p)}(\alpha) \neq 0$.

Also $$f'(x) = \cancelto{0}{f'(\alpha)} + \cancelto{0}{f''(\alpha)}(x-\alpha) + \dots + \cancelto{0}{\frac{f^{(p-1)}(\alpha)}{(p-2)!}}(x-\alpha)^{p-2} + \frac{f^{(p)}(\alpha)}{(p-1)!}(x-\alpha)^{p-1} + o((x-\alpha)^{p-1})$$

So $$\frac{f(x)}{f'(x)} = \frac{\frac{f^{(p)}(\alpha)(x-\alpha)^p}{p!} + o((x-\alpha)^p)}{\frac{f^{(p)}(\alpha)(x-\alpha)^{p-1}}{(p-1)!} + o((x-\alpha)^{p-1})} = \frac{(x-\alpha) + o(x-\alpha)}{p + o(1)} = \frac{(x-\alpha)}{p} + o(x-\alpha).$$

$\left(\text{Since } \dfrac{1}{1+o(1)} = 1 + o(1)\right)$

Hence $\lim\limits_{x \to \alpha} x - \dfrac{f(x)}{f'(x)} = \alpha$.

Newton is fast but requires knowledge of f'.

To avoid evaluating $f'(x_k)$ we could replace it by the approximation $f'(x_k) \approx \dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$.

This leads to the secant method:

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad k = 1, 2, \dots$$

This is a 2nd order recurrence - we need 2 initial guesses, x_0 and x_1

Fact (without proof): The secant method is order p convergent, for $p = \frac{1+\sqrt{5}}{2} \approx 1.63$.

An even simpler method, sometimes used in practice, is the chord method, which is a fixed point method for

$$\phi(x) = x - \frac{f(x)}{q},$$

$$f(x) = 0$$
$$-\frac{f(x)}{q} = 0$$
$$x - \frac{f(x)}{q} = x$$

for some $q \in \mathbb{R}$, $q \neq 0$.

This is only linearly convergent in general. Need $\overbrace{\left|1 - \frac{f'(\alpha)}{q}\right|}^{=|\phi'(\alpha)|} < 1$ for local linear convergence.

§3.8 Stopping criteria

Given an iterative method producing a sequence (x_k) converging to a real number α, we'd like to know how large k needs to be to ensure

$$|e_k| := |x_k - \alpha| \leq tol,$$

where tol is some user-prescribed tolerance.

If we have an a priori error bound, we can use that:

e.g. for bisection, $|e_k| \leq \frac{b-a}{2^{k+1}}$

so $|e_k| \leq tol$ provided $\frac{b-a}{2^{k+1}} \leq tol$

i.e. $k \geq \frac{\log \frac{b-a}{tol}}{\log 2} - 1$.

Similarly, if we're using any linearly convergent method for which $|e_k| \leq C\Lambda^k$ for some $0 < \Lambda < 1$.