Exercise 4

Solution:

a)

A point x is a fixed point of the function ϕ should satisfy

$$\phi(x) = x \quad \text{if and only if} \quad f(x) = 0$$

For $f(x) = x^3 - 2$, we can easily get when $x = \sqrt[3]{2}$, $f(x) = 0$.

Next, we should let $\phi(\sqrt[3]{2}) = \sqrt[3]{2}$.

$$\begin{aligned}
\phi(x) &= x(1 - \frac{\omega}{3}) + x^3(1-\omega) + \frac{2\omega}{3x^2} + 2(\omega - 1) \\
\phi(\sqrt[3]{2}) &= \sqrt[3]{2}(1 - \frac{\omega}{3}) + 2(1-\omega) + \frac{2\omega}{3(\sqrt[3]{2})^2} + 2(\omega - 1)
\end{aligned}$$

Since $\phi(\sqrt[3]{2}) = \sqrt[3]{2}$, we can get

$$\begin{aligned}
\sqrt[3]{2}(1 - \frac{\omega}{3}) + 2(1-\omega) + \frac{2\omega}{3(\sqrt[3]{2})^2} + 2(\omega - 1) &= \sqrt[3]{2} \\
\sqrt[3]{2}(1 - \frac{\omega}{3}) + 2(1-\omega) + \frac{2\omega}{3(\sqrt[3]{2})^2} - 2(1 - \omega) &= \sqrt[3]{2} \\
\sqrt[3]{2}(1 - \frac{\omega}{3}) + \frac{2\omega}{3(\sqrt[3]{2})^2} &= \sqrt[3]{2} \\
2(1 - \frac{\omega}{3}) + \frac{2\omega}{3} &= 2 \\
2 - \frac{2\omega}{3} + \frac{2\omega}{3} &= 2 \\
2 &= 2
\end{aligned}$$

Therefore, ω can be any value, and the root of $f(x) = 0$ is a fixed point.

b)

The method is locally convergent when $\phi : [a, b] \to \mathbb{R}$ is continuously differentiable and $\alpha \in (a, b)$ is a fixed point of ϕ such that $|\phi'(\alpha)| < 1$.

Therefore, for root x, we should find ω that makes $|\phi'(x)| < 1$.

Since $\phi(x) = x(1 - \frac{\omega}{3}) + x^3(1-\omega) + \frac{2\omega}{3x^2} + 2(\omega - 1)$, then we can get

$$\phi'(x) = (1 - \frac{\omega}{3}) + 3x^2(1-\omega) - \frac{4\omega}{3x^3}$$

Because the root $x = \sqrt[3]{2}$, and $|\phi'(x)| < 1$, so

$$-1 < \phi'(x) < 1$$

$$-1 < (1 - \frac{\omega}{3}) + 3x^2(1-\omega) - \frac{4\omega}{3x^3} < 1$$

$$-1 < 1 - \frac{\omega}{3} + 3(\sqrt[3]{2})^2(1-\omega) - \frac{4\omega}{6} < 1$$

$$-1 < 1 - \omega + 3(\sqrt[3]{2})^2(1-\omega) < 1$$

$$-1 < (1-\omega)[1 + 3(\sqrt[3]{2})^2] < 1$$

$$-\frac{1}{1+3(\sqrt[3]{2})^2} < 1 - \omega < \frac{1}{1+3(\sqrt[3]{2})^2}$$

$$-\frac{1}{1+3(\sqrt[3]{2})^2} - 1 < -\omega < \frac{1}{1+3(\sqrt[3]{2})^2} - 1$$

$$\frac{2+3(\sqrt[3]{2})^2}{1+3(\sqrt[3]{2})^2} > \omega > \frac{3(\sqrt[3]{2})^2}{1+3(\sqrt[3]{2})^2}$$

Therefore, for the value $\frac{3(\sqrt[3]{2})^2}{1+3(\sqrt[3]{2})^2} < \omega < \frac{2+3(\sqrt[3]{2})^2}{1+3(\sqrt[3]{2})^2}$, the method is locally convergent.

c)

The method of second order when satisfied $\phi : [a,b] \to \mathbb{R}$ is twice continuously differentiable and $\alpha \in (a,b)$ is a fixed point of ϕ satisfying $\phi'(\alpha) = 0$.

Therefore, for root x, we should find ω that makes $\phi'(x) = 0$.

Since $\phi'(x) = (1 - \frac{\omega}{3}) + 3x^2(1-\omega) - \frac{4\omega}{3x^3}$, and the root $x = \sqrt[3]{2}$, we can get

$$\phi'(x) = (1 - \frac{\omega}{3}) + 3x^2(1-\omega) - \frac{4\omega}{3x^3} = 0$$

$$\phi'(\sqrt[3]{2}) = (1 - \frac{\omega}{3}) + 3(\sqrt[3]{2})^2(1-\omega) - \frac{4\omega}{3(\sqrt[3]{2})^3} = 0$$

$$1 - \frac{\omega}{3} + 3(\sqrt[3]{2})^2(1-\omega) - \frac{2\omega}{3} = 0$$

$$1 - \omega + 3(\sqrt[3]{2})^2(1-\omega) = 0$$

$$\omega - 3(\sqrt[3]{2})^2(1-\omega) = 1$$

$$\omega\sqrt[3]{2} - 6(1-\omega) = \sqrt[3]{2}$$

$$\omega\sqrt[3]{2} + 6\omega = \sqrt[3]{2} + 6$$

$$\omega = 1$$

Therefore, for the value $\omega = 1$, the method of second order.

d)

Firstly, we want to find in which condition the method is of order higher than 2.

A Taylor expansion of $\phi(x_k)$ around $x = \alpha$ gives

$$\begin{aligned} x_{k+1} - \alpha &= \phi(x_k) - \phi(\alpha) \\ &= \phi'(\alpha)(x_k - \alpha) + \frac{\phi''(\alpha)}{2}(x_k - \alpha)^2 + \ldots + \frac{\phi^{(p)}(\xi_k)}{p!}(x_k - \alpha)^p \end{aligned}$$

for some point ξ_k between x_k and α. And we can know that if we want to let the method of order higher than 2, it should satisfying $\phi'(\alpha) = 0$, $\phi''(\alpha) = 0$, and $\phi^{(j)}$ is continuous.

Since $\phi'(x) = (1 - \frac{\omega}{3}) + 3x^2(1-\omega) - \frac{4\omega}{3x^3}$, then we can get

$$\phi''(x) = 6x(1-\omega) + \frac{4\omega}{x^4}$$

We already get when $\omega = 1$, $\phi'(x) = 0$ before. However, when $\omega = 1$, $\phi''(x) \neq 0$.

Therefore, there is no value of ω make the method is of order higher than 2.