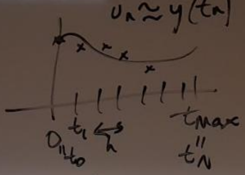
Numerical Methods.

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$u_n \simeq y(t_n)$.



solved by
$$\begin{cases} u_{n+1} = u_n + h\,\Phi(t_n, u_n, u_{n+1}, h) & (*) \\ u_0 = y_0 \end{cases}$$

Last time: zero-stability and convergence (t_{max} fixed, $h \to 0$, so $N \to \infty$)

This time: Absolute stability (h fixed, $t_{max} \to \infty$, so $N \to \infty$)

– important to ensure accurate long-time solution behaviour.

Consider the model problem
$$\begin{cases} y'(t) = \lambda y(t), & t > 0 \\ y(0) = 1 \end{cases} \quad (**)$$
$f(t, y(t)) = \lambda y(t)$.

Exact solution: $y(t) = e^{\lambda t}$, $t > 0$.
When $\lambda \in \mathbb{C}$ with $Re[\lambda] < 0$, $y(t) = \underbrace{e^{Re[\lambda]t}}_{\to 0 \text{ since } Re[\lambda] < 0} \underbrace{e^{i Im[\lambda] t}}_{\leq 1} \to 0$ as $t \to \infty$.

Def'n: The region of absolute stability for the one-step method $(*)$ is the set of all $h\lambda \in \mathbb{C}$ for which, when applied to $(**)$,

$$U_n \to 0 \quad \text{as} \quad n \to \infty.$$

The method $(*)$ is said to be A-stable if this region contains $\{z \in \mathbb{C} : Re[z] < 0\}$

To check abs. stability for a given method, often it's possible to compute U_n explicitly and study its behavior as $n \to \infty$.

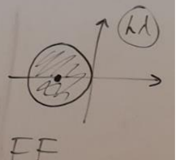
Examples:

Forward Euler: $u_0 = 1$, $u_{n+1} = u_n + h f(t_n, u_n) = u_n + h\lambda u_n = (1 + h\lambda) u_n$.

(By induction) Solution is $u_n = (1 + h\lambda)^n$

Hence $u_n \xrightarrow{n\to\infty} 0 \iff |1 + h\lambda| < 1$

$\iff h\lambda \in \{z \in \mathbb{C} : \underbrace{|1 + z|}_{\text{distance between } z \text{ and } -1} < 1\}$ ($|z - (-1)|$)



So this ball is the region of absolute stability for FE.
So FE is not A-stable.

In particular, if $\lambda < 0$ then FE is abs. stable ($u_n \to 0$) if and only if $\boxed{h < \frac{2}{|\lambda|}}$

If $h \geq \frac{2}{|\lambda|}$ the method is unstable and we could have $u_n \to \infty$ as $n \to \infty$.

So you have to be careful when using FE!

<u>Backward Euler / Crank Nicolson</u>

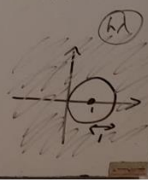
By contrast, BE and CN are A-stable.

E.g. for BE, get $u_{n+1} = u_n + hf(t_n + h, u_{n+1}) = u_n + h\lambda u_{n+1}$

so $(1 - h\lambda) u_{n+1} = u_n$ i.e. $u_{n+1} = \frac{1}{1-h\lambda} u_n$.

Also, $u_0 = 1$, so (by induction) $u_n = \left(\frac{1}{1-h\lambda}\right)^n \to 0 \iff |1 - h\lambda| > 1$

which includes the left half-plane.



While abs. stability is defined in terms of a very simple model problem, it actually says something about how the method performs more generally.

Return to general case
$$\begin{cases} y'(t) = f(t, y(t)), & t > 0 \\ y(0) = y_0 \end{cases}$$

Let's consider the behavior of $w_n = v_n - u_n$, where v_n satisfies the perturbed version of (*) as in zero-stability,

i.e.
$$\begin{cases} v_0 = y_0 + \delta_0 \\ v_{n+1} = v_n + h\Phi(t_n, v_n, v_{n+1}, h) + h\delta_{n+1} \end{cases}$$
perturbations.

We can ask: when does $|w_n|$ stay bounded as $n \to \infty$ (with h fixed).

Focus on FE only, and assume f is diff. w.r.t. y, with
$$-\alpha \leq \frac{\partial f}{\partial y} \leq -\beta < 0 \quad \text{for some } 0 < \beta < \alpha, \ \forall t, y.$$

Then by the MVT we have:
$$w_{n+1} = w_n + h\big(f(t_n, v_n) - f(t_n, u_n)\big) + h\delta_{n+1}$$
$$= w_n + h \underbrace{\frac{\partial f}{\partial y}(t_n, \xi_n)}_{=:\lambda_n} w_n + h\delta_{n+1}, \quad \text{some } \xi_n \text{ between } u_n \text{ and } v_n.$$

i.e. $w_{n+1} = (1 + h\lambda_n) w_n + h\delta_{n+1}$. then $h < \frac{2}{|\lambda_n|}$ $\forall n$.

Can then check that if $\boxed{h < \frac{2}{\alpha}}$ then $w_n = v_n - u_n$ is bounded as $n \to \infty$.

This is the condition for abs. stability for the model problem $\begin{cases} y'(t) = -\alpha y(t) \\ y(0) = 1 \end{cases}$.