MATH0033 Numerical Methods

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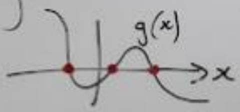
Drop-in : Office 4.12 , Tues 2pm

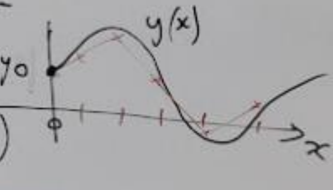
Course outline: numerical methods for solving

- nonlinear equations $\quad g(x) = 0$



- linear systems $\quad A\underline{x} = \underline{b}$
- differential equations

$$y'(x) = f(x, y(x))$$
$$y(0) = y_0$$



Engineers/scientists compute,
mathematicians analyse.

We'll ask:
- when does the method work / not work ?
- how does solution accuracy depend on computational effort (time, memory ...)

But, we'll also do some computations to see how things work in practice.

§2.1 Norms

Our "exact solution" and our numerical approximation will live in some normed vector space $(V, \|\cdot\|)$.

The norm gives us a way of measuring error/accuracy.

Def'n: Let V be a vector space over $\mathbb{R}$.
A norm is a function $\|\cdot\| : V \to \mathbb{R}$ s.t.

1. $\|x\| \geq 0 \quad \forall x \in V$ and $\|x\| = 0 \iff x = 0$.
2. $\|\lambda x\| = |\lambda| \|x\| \quad \forall x \in V \quad \forall \lambda \in \mathbb{R}$

(3) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$.

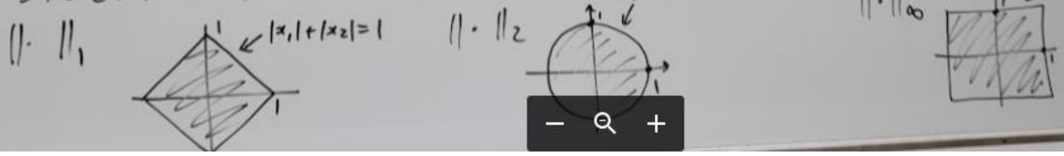
Examples: $V = \mathbb{R}^n$ with e.g. $\quad (x = (x_1, x_2, \dots, x_n)^T)$

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

$$\|x\|_\infty := \max_{i \in \{1, \dots, n\}} |x_i| .$$

Sketch: unit balls in $\mathbb{R}^2$:



Def'n: Two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if $\exists\, c, C > 0$ s.t.

$$c\|x\| \leq \|x\|' \leq C\|x\| \quad \forall x \in V.$$

Fact: On a finite-dimensional space, all norms are equivalent.

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§2.2 Errors and convergence

Given $x \in V$ and an "approximation" $\hat{x} \in V$, we define:

$$E_{abs} := \|\tilde{x} - x\| \qquad \text{(absolute error)}$$

$$E_{rel} := \frac{\|\tilde{x} - x\|}{\|x\|} \qquad \text{(relative error)}$$

When $V = \mathbb{R}$ and $\|\cdot\| = |\cdot|$, $-\log_{10}(E_{rel})$ tells you how many digits of accuracy you have.

$$\left(\text{e.g.} \quad \begin{aligned} x = \pi &= \boxed{3.14}159\dots \\ \tilde{x} &= \boxed{3.14}721\dots \end{aligned}\right)$$

Recall, we say a sequence (x_n) in $(V, \|\cdot\|)$ <u>converges</u> to $x \in V$ if $\|x_n - x\| \to 0$ as $n \to \infty$.

<u>Def'n</u>: We say (x_n) converges <u>linearly</u> to x if $\exists\ 0 < C < 1$ and $N \in \mathbb{N}$ s.t. $\|x_{n+1} - x\| \leq C \|x_n - x\|$ $\forall n \geq N$.

<u>Exercise</u>: Show that if this holds then $\exists\ \tilde{C} > 0$ s.t.

$$\|x_n - x\| \leq \tilde{C} C^n \quad \forall n \in \mathbb{N}.$$