Exercise 3

Solution:

a)

It is **NOT** possible to use the bisection method to find both roots of the function $f(x)$, and we can only find the root from the right side.

Because for the right side root r_{right}, we can certainly get $f(a)f(b) \leq 0$, and the function f changes sign on any interval $[a, b]$ that contain the root r_{right}. Therefore, we can find the root r_{right} by applying the bisection method.

However, for the left side root r_{left}, we cannot find it using the bisection method. Because there is no sign changes on the interval $[a, b]$ that contain the root r_{left}, and $f(a)f(b) > 0$, so it is impossible to find the root r_{left}.

(Personally, I think there is still a slim possibility that we can find the root r_{left}. If we can get a perfect interval $(r_{left} - \delta, r_{left} + \delta)$ for $\delta \in \mathbb{R}$ initially or after some iterations, then x_k will be $\frac{(r_{left}-\delta)+(r_{left}+\delta)}{2} = r_{left}$, and $f(x_k) = f(r_{left}) = 0$, so that we can find the root r_{left}. But this probability is too small, so we only consider to find the root r_{right} in the following.)

Assume that the error between x_k and root α is e_k, and

$$|e_k| = |x_k - \alpha| \leq tol$$

According to the priori criteria, we can know that the bisection method produces iterates satisfy

$$|e_k| = |x_k - \alpha| \leq \frac{b-a}{2^{k+1}} = tol$$
$$k \geq log_2\Big(\frac{b-a}{tol}\Big) - 1$$

where a and b are the bound of interval.

To find the root α by bisection with a relative accuracy $tol = 10^{-10}$, suppose we have a starting interval $[-\pi, \pi]$, then we can get

$$k \geq log_2\Big(\frac{b-a}{tol}\Big) - 1$$
$$k \geq log_2\Big(\frac{\pi-(-\pi)}{10^{-10}}\Big) - 1$$
$$k \geq 34.87$$

Therefore, after 35 iterations, the bisection method can found the root α to the accuracy $tol = 10^{-10}$.

b)

The formula for the Newton iteration is:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, 2, \ldots$$

and the function f is:

$$f(x) = \frac{x}{2} - sinx + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

So, we can get $f'(x) = \frac{1}{2} - cosx$.

Therefore, the Newton's method for the problem is:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{\frac{x}{2} - sinx + \frac{\pi}{6} - \frac{\sqrt{3}}{2}}{\frac{1}{2} - cosx}$$

To determine the order of convergence for the Newton method, we should calculate the value p from:

$$||x_{n+1} - x|| \leq C||x_n - x||^p,\ p \geq 1$$

Suppose that this function f will finally converge to a root x, so $f(x) = 0$.

Since $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, we can get $x_{k+1} - x = x_k - x + \frac{f(x_k)}{f'(x_k)}$.

Next, according to the Taylor's theorem, we can get

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2 + \ldots + o(|x - x_k|^p)$$

Because Taylor approximation is accurate enough such that we can ignore higher order terms, so we neglect third and higher powers of $(x - x_k)$, and now get

$$f(x) \approx f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2$$

Since $f(x) = 0$, then we can get

$$\begin{aligned}
0 &= f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2 \\
f(x_k) + f'(x_k)(x - x_k) &= -\frac{f''(x_k)}{2}(x - x_k)^2 \\
\frac{f(x_k)}{f'(x_k)} + (x - x_k) &= -\frac{f''(x_k)}{2f'(x_k)}(x - x_k)^2 \\
x - x_k + \frac{f(x_k)}{f'(x_k)} &= -\frac{f''(x_k)}{2f'(x_k)}(x - x_k)^2 \\
x - \left[x_k - \frac{f(x_k)}{f'(x_k)}\right] &= -\frac{f''(x_k)}{2f'(x_k)}(x - x_k)^2
\end{aligned}$$

Since $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, the equation will become

$$x - x_{k+1} = -\frac{f''(x_k)}{2f'(x_k)}(x - x_k)^2$$

$$||x_{n+1} - x|| \leq -\frac{f''(x_k)}{2f'(x_k)}||x_n - x||^2$$

Therefore, we can get $p = 2$ from

$$||x_{n+1} - x|| \leq C||x_n - x||^2$$

So, for both of the two zeros, the order of convergence of the Newton method is 2.

c)

According to the Contraction mapping theorem, we can know that if there exists a constant $0 < \Lambda < 1$ such that $|\phi'(x^0)| \leq \Lambda$ for all $x^0 \in [\frac{\pi}{2}, \pi]$, and $\phi(x) \in [\frac{\pi}{2}, \pi]$ for all $x \in [\frac{\pi}{2}, \pi]$, then we can prove the point iteration converges linearly to α.

Since $\phi(x) = sinx + \frac{x}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)$, we can get

$$\phi'(x^0) = cosx^0 + \frac{1}{2}$$

According to the property of $cosx$, we can know that $\phi'(x^0)$ monotonic decreasing in interval $[\frac{\pi}{2}, \pi]$, and value range is $[\frac{1}{2}, -\frac{1}{2}]$. Thus, there exists a constant $0 < \Lambda < 1$ such that $|\phi'(x^0)| \leq \Lambda$ for all $x^0 \in [\frac{\pi}{2}, \pi]$.

Next, we should prove $\phi(x) \in [\frac{\pi}{2}, \pi]$ for all $x \in [\frac{\pi}{2}, \pi]$.

From $\phi'(x)$, we can calculate that when $x = \frac{2\pi}{3}$, $\phi'(x) = 0$. Because $\phi'(\frac{\pi}{2}) = \frac{1}{2}$, and $\phi'(\pi) = -\frac{1}{2}$, we can know that $\phi(x)$ monotonic increasing in interval $[\frac{\pi}{2}, \frac{2\pi}{3}]$, and monotonic decreasing in interval $[\frac{2\pi}{3}, \pi]$.

Thus, we can calculate that value range of $\phi(x)$.

$$\phi\left(\frac{2\pi}{3}\right) = sinx + \frac{x}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right) = \sqrt{3} + \frac{\pi}{6}$$

$$\phi\left(\frac{\pi}{2}\right) = sinx + \frac{x}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right) = \frac{2 + \sqrt{3}}{2} + \frac{\pi}{12}$$

$$\phi(\pi) = sinx + \frac{x}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Therefore, the value range of $\phi(x)$ is $[\frac{\pi}{3} + \frac{\sqrt{3}}{2}, \sqrt{3} + \frac{\pi}{6}]$, and $\frac{\pi}{3} + \frac{\sqrt{3}}{2} > \frac{\pi}{2}$, and $\sqrt{3} + \frac{\pi}{6} < \pi$.

So, the fixed point iteration converges linearly to α for every initial guess $x^0 \in [\frac{\pi}{2}, \pi]$, with $|x^{k+1} - \alpha| \leq \Lambda|x^k - \alpha|$.