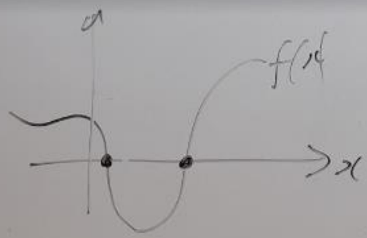
Numerical Methods

Root-finding: $f(x) = 0$



Last time: bisection method.

This time: fixed point methods.

Idea: first find ϕ s.t. $\boxed{f(x) = 0} \iff \boxed{\phi(x) = x}$

Then construct a sequence of approximations to a solution of this via the fixed point iteration $x_{k+1} = \phi(x_k)$, $k = 0, 1, 2, \ldots$

When does this converge?

(contraction mapping)

Def'n: $\phi : [a,b] \to \mathbb{R}$ is called a strict contraction if $\exists \Lambda \in (0,1)$ s.t.

$$|\phi(x) - \phi(x')| \leq \Lambda |x - x'|, \quad \forall x, x' \in [a,b].$$

Lemma: Let $\phi : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and that $\exists \Lambda \in (0,1)$ s.t. $|\phi'(x)| \leq \Lambda \quad \forall x \in [a,b]$. Then ϕ is a strict contraction with constant Λ.

Proof: Let $x, x' \in [a,b]$. By MVT $|\phi(x) - \phi(x')| = |\phi'(\xi)(x - x')| = |\phi'(\xi)||x - x'| \leq \Lambda |x - x'|$.

Theorem (C.M.T.)

Let $\phi : [a,b] \to \mathbb{R}$ be a strict contraction with constant $\Lambda \in (0,1)$.
Suppose that $\phi(x) \in [a,b]$ for all $x \in [a,b]$. (i.e. $\phi : [a,b] \to [a,b]$)
Then
(i) ϕ has a unique fixed point $\alpha \in [a,b]$ s.t. $\phi(\alpha) = \alpha$.
(ii) the iteration $x_{k+1} = \phi(x_k)$ converges to α for any $\underbrace{x_0}_{\text{initial guess}} \in [a,b]$.
(iii) the convergence is linear, with

$$|x_{k+1} - \alpha| \leq \Lambda |x_k - \alpha|, \quad \forall k \geq 0.$$

Recall: (iii) $\Rightarrow$ $\underbrace{|x_k-\alpha| \le \Lambda^k |x_0-\alpha|}_{\text{a priori error bound.}}$, $k \ge 0$.

Note: α is unknown, but we can bound $|x_0-\alpha| \le b-a$ to get a fully explicit bound $|x_k-\alpha| \le \Lambda^k(b-a)$.

Proof: ϕ contraction $\Rightarrow$ ϕ continuous, so existence of α follows from Brouwer fixed point theorem.

To show uniqueness, suppose $\alpha_1 \neq \alpha_2$ are two fixed points. i.e. $\phi(\alpha_1)=\alpha_1$, and $\phi(\alpha_2)=\alpha_2$.

Then $|\alpha_1-\alpha_2| = |\phi(\alpha_1)-\phi(\alpha_2)| \le \Lambda|\alpha_1-\alpha_2| < |\alpha_1-\alpha_2|$ since $\alpha_1 \neq \alpha_2$ and $\Lambda \in (0,1)$.

Contradiction.

Suppose assumptions of the CMT hold.

Remark: If $\phi:[a,b]\to\mathbb{R}$ is continuously differentiable, and $x_k \neq \alpha$ for any k, then

$$\lim_{k\to\infty} \frac{x_{k+1}-\alpha}{x_k-\alpha} = \phi'(\alpha).$$

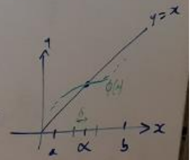
Proof: By the MVT $\exists\ \xi_k$ between α and x_k s.t.

$$x_{k+1}-\alpha \overset{\text{def'n of } x_{k+1}}{\underset{\text{fixed point}}{=}} \phi(x_k)-\phi(\alpha) = \phi'(\xi_k)(x_k-\alpha)$$

so $\dfrac{x_{k+1}-\alpha}{x_k-\alpha} = \phi'(\xi_k) \to \phi'(\alpha)$ by this, the continuity of ϕ', and the fact that $x_k \to \alpha$.

The CMT is a global convergence result. (Convergence for any initial guess in $[a,b]$).
Here's a "local" result:

$\leftarrow$ ϕ doesn't have to be a contraction on $[a,b]$.



Theorem: Let $\phi : [a,b] \to \mathbb{R}$ be continuously differentiable, and let $\alpha \in [a,b]$ be a fixed point of ϕ (i.e. $\phi(\alpha) = \alpha$)

Suppose that $|\phi'(\alpha)| < 1$.

Then $\exists\ \delta > 0$ s.t. $\forall\ x_0 \in [\alpha - \delta, \alpha + \delta]$ the iteration $x_{k+1} = \phi(x_k)$ converges linearly to α, with

$$\lim_{k \to \infty} \frac{x_{k+1} - \alpha}{x_k - \alpha} = \phi'(\alpha).$$

Proof: (sketch!)

Step 1: Use continuity of ϕ' to find $\delta > 0$

s.t. $|\phi'(x)| \leq \dfrac{1 + |\phi'(\alpha)|}{2} < 1 \quad \forall x \in [\alpha - \delta, \alpha + \delta]$.

Step 2: Apply C.M.T. on this interval.

(need to check that $\phi : [\alpha - \delta, \alpha + \delta] \to [\alpha - \delta, \alpha + \delta]$

– for this use M.V.T.
+ fixed point property.)