INTRODUCTION TO PRINCIPAL COMPONENT ANALYSIS (PCA)

INT301 Bio-computation, Week 11, 2021

Review: Eigenvalues and Eigenvectors

- If v is a nonzero vector and λ is a number such that

$$\mathbf{Av = \lambda v}$$

then v is said to be an *eigenvector* of A with *eigenvalue* λ.

λ (eigenvalues)

<u>Example</u>

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A

V (eigenvectors)

Review: Eigenvalues and Eigenvectors

- **Eigenvectors** (for a square $m \times m$ matrix $\mathbf{S}$)

$$\mathbf{S}\mathbf{v} = \lambda\mathbf{v}$$

eigenvector $\mathbf{v} \in \mathbb{R}^m \neq \mathbf{0}$

eigenvalue $\lambda \in \mathbb{R}$

Example

$$\begin{pmatrix} 6 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- How many eigenvalues are there at most?

$$\mathbf{S}\mathbf{v} = \lambda\mathbf{v} \iff (\mathbf{S} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$$

only has a non-zero solution if $|\mathbf{S} - \lambda\mathbf{I}| = 0$

with **at most *m* distinct λ values**

Review: Eigenvalues and Eigenvectors

For symmetric matrices, eigenvectors for distinct eigenvalues are **orthogonal**

$$Sv_{\{1,2\}} = \lambda_{\{1,2\}} v_{\{1,2\}}, \text{ and } \lambda_1 \neq \lambda_2 \Rightarrow v_1 \bullet v_2 = 0$$

All eigenvalues of a real symmetric matrix are **real**.

All eigenvalues of a positive semidefinite matrix are **non-negative**

$$\overbrace{\forall w \in \Re^n, w^T S w \geq 0}, \text{ then if } \mathrm{Sv} = \lambda \mathrm{v} \Rightarrow \lambda \geq 0$$

Review: Eigenvalues and Eigenvectors

- Let $$S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$ Real, symmetric.
- Then $$S - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \Rightarrow (2-\lambda)^2 - 1 = 0.$$
- The eigenvalues are 1 and 3 (nonnegative, real).
- The eigenvectors are orthogonal (and real):

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Substitute these values and solve for eigenvectors.

Diagonal Decomposition

- Let $\mathbf{S} \in \mathbb{R}^{m \times m}$ be a **square** matrix with m **linearly independent eigenvectors**
- **Theorem**: Exists an **eigen decomposition**

diagonal

$$\mathbf{S} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}$$

Unique for distinct eigen-values

  - (matrix diagonalization theorem)
- Columns of **U** are **eigenvectors** of $\mathbf{S}$
- Diagonal elements of $\mathbf{\Lambda}$ are **eigenvalues** of $\mathbf{S}$

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m), \quad \lambda_i \geq \lambda_{i+1}$$

Diagonal Decomposition

Write *U* with the eigenvectors as columns: $U = \begin{bmatrix} v_1 & \ldots & v_m \end{bmatrix}$

Then, *SU* can be written

$$SU = S\begin{bmatrix} v_1 & \ldots & v_m \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \ldots & \lambda_m v_m \end{bmatrix} = \begin{bmatrix} v_1 & \ldots & v_m \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ldots & \\ & & \lambda_m \end{bmatrix}$$

Thus $SU=U\Lambda$, or $U^{-1}SU=\Lambda$

Therefore $S=U\Lambda U^{-1}$.

Diagonal Decomposition

Recall $S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \lambda_1 = 1, \lambda_2 = 3.$

The eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form $U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Inverting U, we have

$$U^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Recall $UU^{-1} = 1$.

Then, $S = U\Lambda U^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Diagonal Decomposition

Let's divide ***U*** (and multiply ***U***$^{-1}$) by $\sqrt{2}$

Then, ***S=*** $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Q Λ ***(Q***$^{-1}$ ***= Q***^{T} ***)***

Symmetric Diagonal Decomposition

- **Theorem**: If $\mathbf{S} \in \mathbb{R}^{m \times m}$ is a **symmetric** matrix, there exists an **eigen decomposition**, where ***Q*** is **orthogonal:**

$$S = Q \Lambda Q^T$$

  - $\boldsymbol{Q^{-1} = Q^T}$
  - Columns of ***Q*** are normalized eigenvectors
  - Columns are orthogonal.
  - (everything is real)

Singular Value Decomposition

For an $m\times n$ matrix **A** of rank r, there exists a factorization (Singular Value Decomposition = **SVD**) as follows:

$$A = U\Sigma V^T$$

$m\times m$ | $m\times n$, nonnegative real | $n\times n$

The columns of U are orthogonal eigenvectors of AA^T.

The columns of V are orthogonal eigenvectors of A^TA.

Eigenvalues $\lambda_1 \ldots \lambda_r$ of AA^T are the eigenvalues of A^TA.

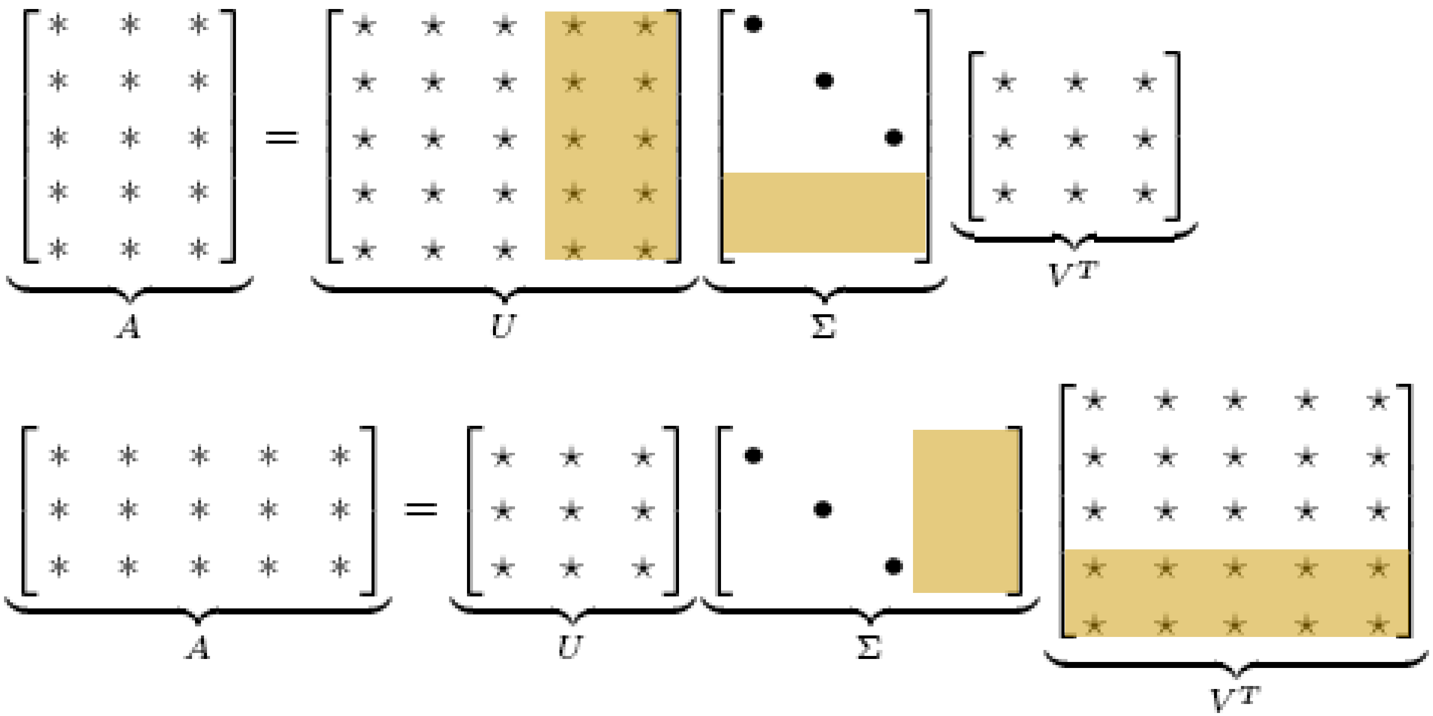
$$\sigma_i = \sqrt{\lambda_i}$$

$$\Sigma = diag(\sigma_1 \ldots \sigma_r)$$

Singular values.

Singular Value Decomposition

- Illustration of SVD dimensions and



Singular Value Decomposition

Let $$A=\begin{bmatrix}1 & -1\\ 0 & 1\\ 1 & 0\end{bmatrix}$$

Thus m=3, n=2. Its SVD is

$$\begin{bmatrix}0 & 2/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \sqrt{3}\\ 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

Dimensionality Reduction

- One approach to deal with high dimensional data is by reducing their dimensionality.
- Project high dimensional data onto a lower dimensional subspace using <u>linear</u> or <u>non-linear</u> transformations.

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_N \end{bmatrix} \xrightarrow{\textbf{Reduce dimensionality}} \mathrm{y} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_k \end{bmatrix} \quad (K<<N)$$

- Linear transformations are simple to compute

$$\underset{k \times 1}{Y} = \underset{k \times d}{U} \; \underset{d \times 1}{X} \qquad (b_i = u_i^t a_i) \qquad (k<<d)$$

Dimensionality Reduction

- **Find a basis in a low dimensional sub-space:**
  - Approximate vectors by projecting them in a low dimensional sub-space:

(1) Original space representation:

$$x = a_1 v_1 + a_2 v_2 + \ldots + a_N v_N \qquad \boxed{\begin{bmatrix} a_1 \\ a_2 \\ \ldots \\ a_N \end{bmatrix}}$$

where $\boxed{v_1, v_2, \ldots, v_n}$ is a base in the original N-dimensional space

(2) Lower-dimensional sub-space representation:

$$\hat{x} = b_1 u_1 + b_2 u_2 + \ldots + b_K u_K \qquad \boxed{\begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_k \end{bmatrix}}$$

where $\boxed{u_1, u_2, \ldots, u_K}$ is a base in the K-dimensional sub-space (K<N)

Dimensionality Reduction

- If K=N, then $\hat{x} = x$
- Example (K=N):

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{(standard basis)}$$

$$x_v = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3v_1 + 3v_2 + 3v_3$$

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(some other basis)}$$

$$x_u = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 0u_1 + 0u_2 + 3u_3$$

thus, $x_v = x_u$

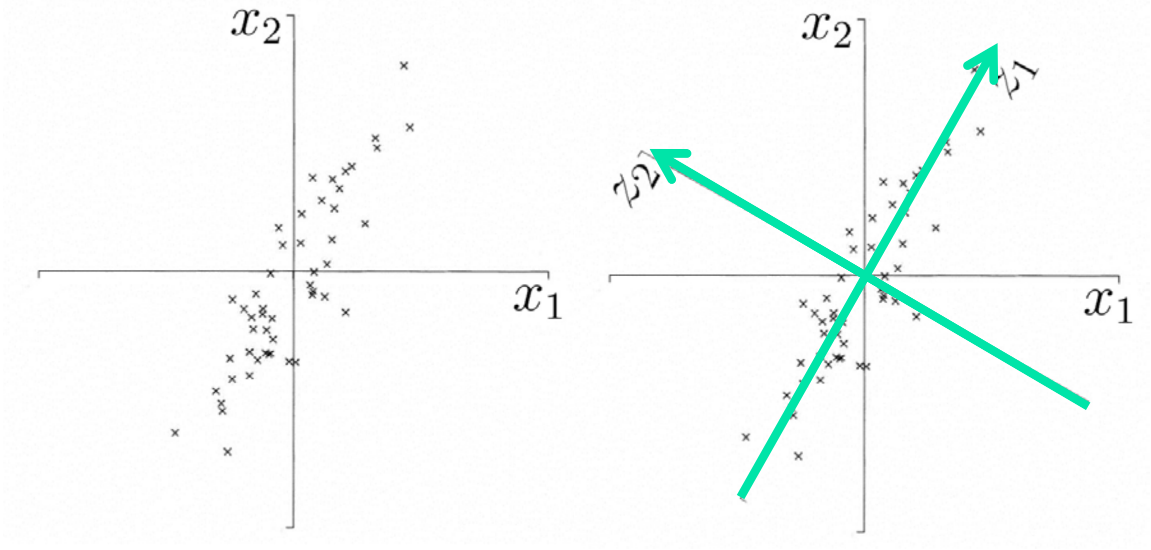
Principal Component Analysis (PCA)

- Each dimensionality reduction technique finds an *appropriate transformation* by satisfying certain criteria (e.g., *information loss, data discrimination*, etc.)

- The goal of PCA is to reduce the dimensionality of the data while ***retaining as much as possible of the variation present in the dataset***.

Principal Component Analysis (PCA)

- **Motivation**
  - Find bases which has high variance in data
  - Encode data with small number of bases with low MSE



- First PC is direction of maximum variance
- Subsequent PCs are orthogonal to 1st PC and describe maximum residual variance

Principal Component Analysis (PCA)

Assume that $E[\mathbf{x}] = \mathbf{0}$ $\quad a = \mathbf{x}^T\mathbf{q} = \mathbf{q}^T\mathbf{x}$ $\quad \|\mathbf{q}\| = (\mathbf{q}^T\mathbf{q})^{1/2} = 1$

$$\Rightarrow \sigma^2 = E[a^2] - E[a]^2 = E[a^2]$$
$$= E[(\mathbf{q}^T\mathbf{x})(\mathbf{x}^T\mathbf{q})] = \mathbf{q}^T E[\mathbf{x}\mathbf{x}^T]\mathbf{q} = \mathbf{q}^T\mathbf{R}\mathbf{q}$$

Find q's maximizing variance!!

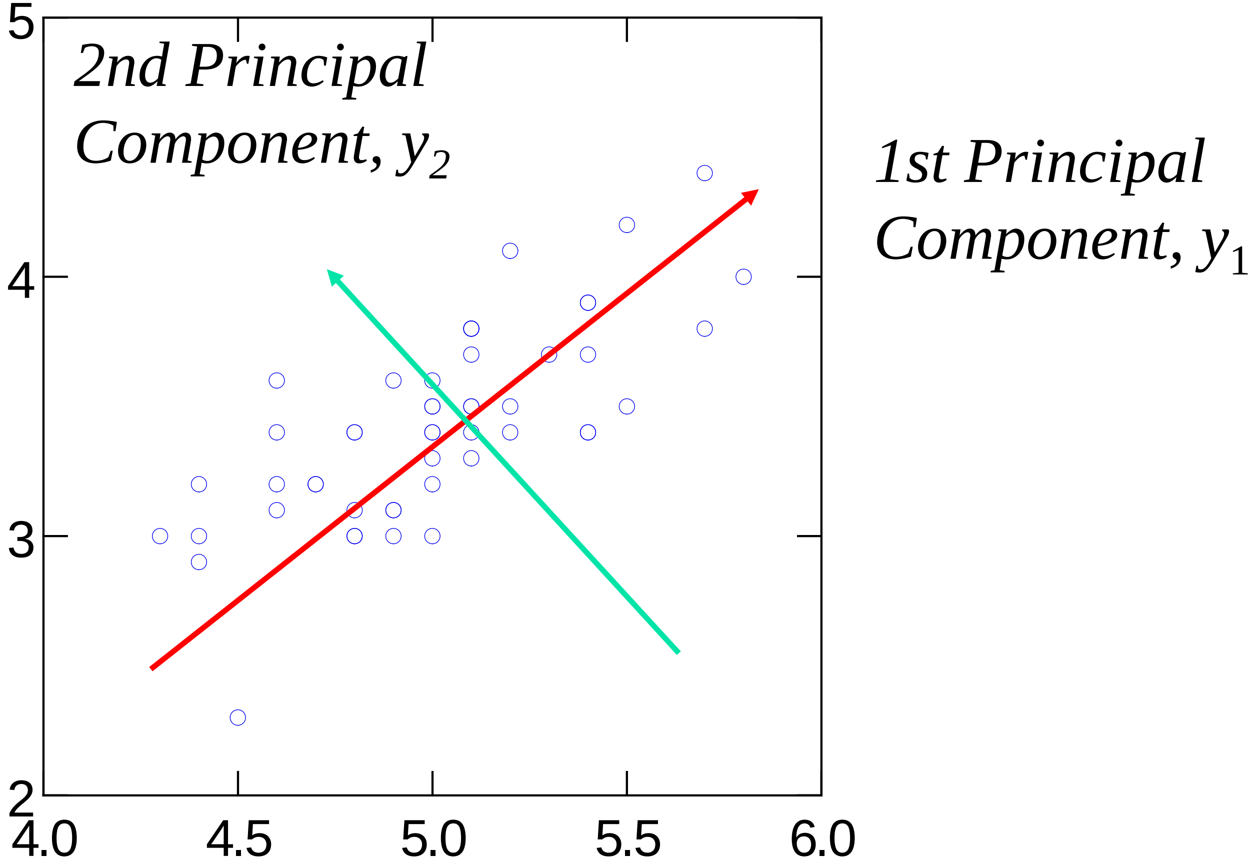
It can be shown that variance is maximized when q is the principal component of R.

Principal component q can be obtained by **eigenvector decomposition**:

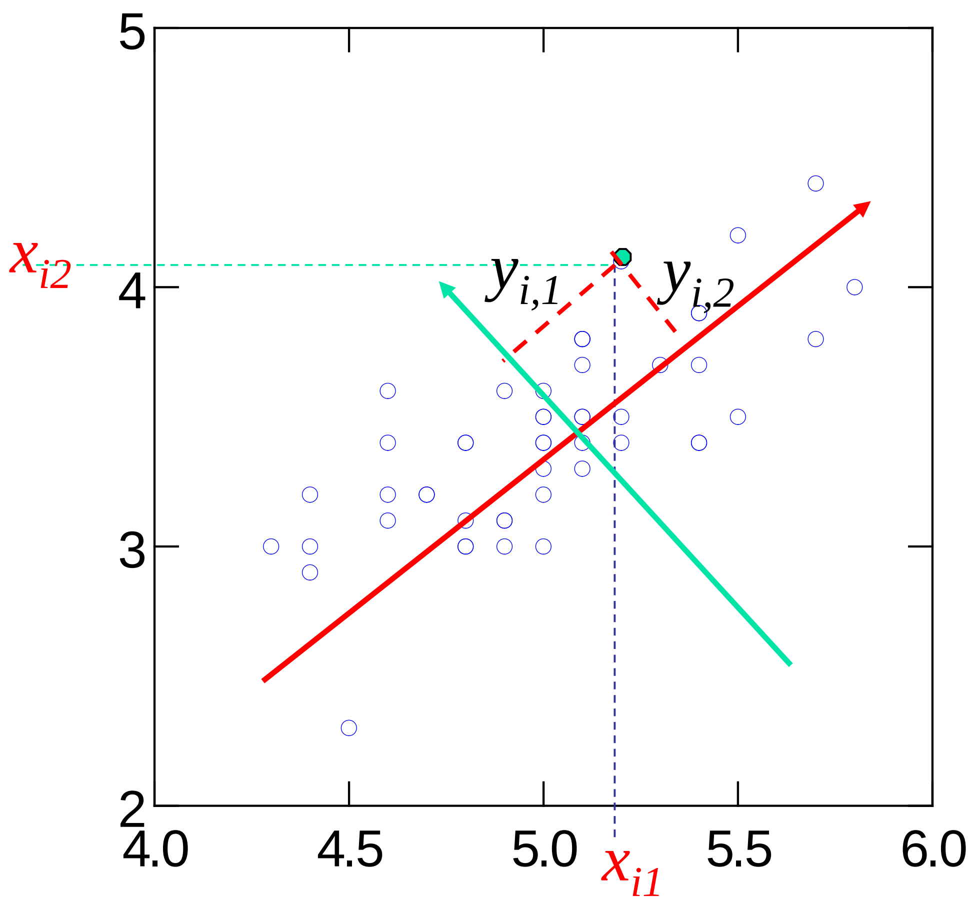
$$\mathbf{R} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T, \quad \mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_j, \ldots, \mathbf{q}_m], \boldsymbol{\Lambda} = diag[\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots, \lambda_m]$$

$$\Leftrightarrow \mathbf{R}\mathbf{q}_j = \lambda_j \mathbf{q}_j \quad j = 1, 2, \ldots, m \quad \Rightarrow \quad \mathbf{R}\mathbf{q} = \lambda\mathbf{q}$$

Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



Principal Component Analysis (PCA)

- Advantage
  - Reduce the dimension of the original data
    - reduce time consumption in the training process, and improve efficiency
  - Discard some information of the original data
    - if this information is noise

Principal Component Analysis (PCA)

- Limitation
  - Discard some information of the original data
    - if the discarded information is important, it is not appropriate to apply PCA
  - The meaning of the principal component
    - PC or basis may not be interpretable
  - Linear model of PCA
    - not suitable for nonlinear problem
  - Assume first PC has higher importance

Case study: Eigenface

Face image: high-dimensional vector

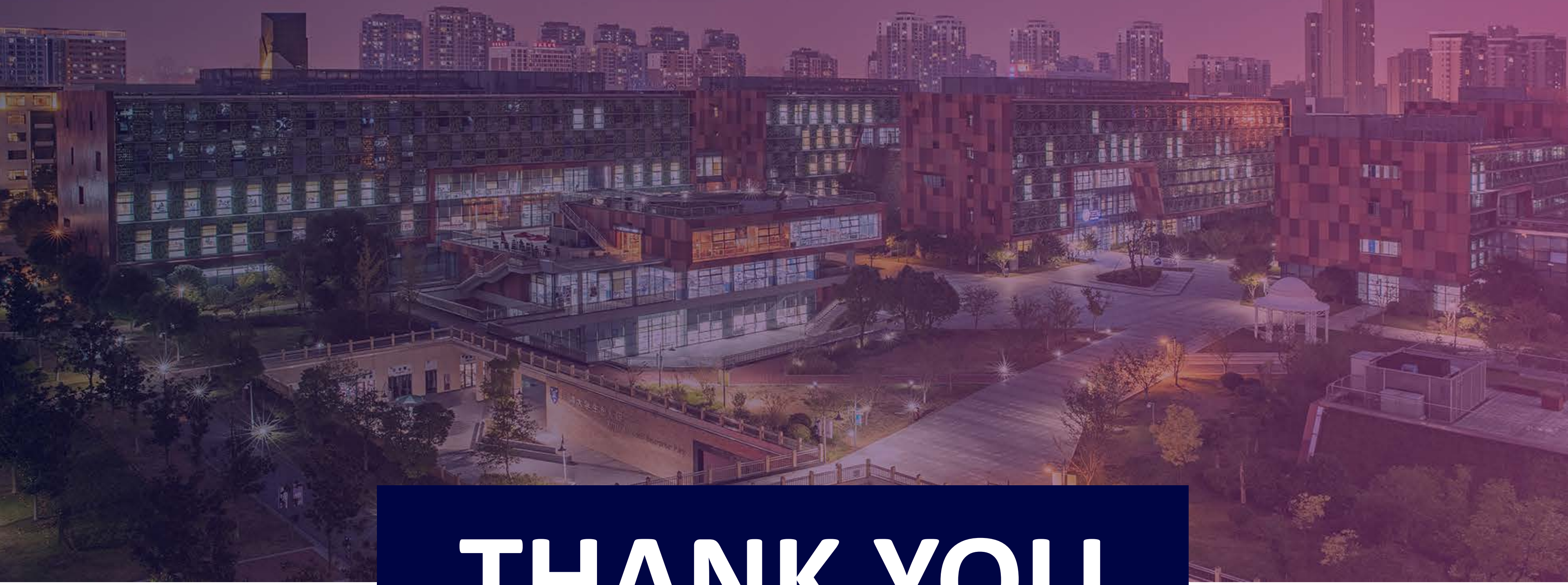
face image = linear combination of eigenvectors

The eigenvectors can be viewed as images.

Eigenfaces:

b_1 b_2 b_3 b_4

Original face image



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