Coalitions, Voting Power, and Computational Social Choice

Forming Coalitions: Coalitional Games

- *Coalitional games* model scenarios where agents *can benefit by cooperating*
- Issues in coalitional games (Sandholm et al, 1999):
  - Coalition Structure Generation
  - Teamwork
  - Dividing the benefits of cooperation

Coalition Structure Generation

- Deciding *in principle* who will work together
- The basic question:
  - Which coalition should I join?
- The result: *partitions* agents into disjoint *coalitions*
- The overall partition is a *coalition structure*

Solving the optimization problem of each coalition

- Deciding how to work together.
- Solving the “joint problem” of a coalition
- Finding how to maximize the utility of the coalition itself.
- Typically involves joint planning etc.

Dividing the Benefits

- Deciding “who gets what” in the payoff.
- Coalition members cannot ignore each other’s preferences, because members can *defect*: if you try to give me a bad payoff, I can always walk away
- We might want to consider issues such as *fairness* of the distribution

Formalizing Cooperative Scenarios

- A coalitional game:

  $$\langle Ag, v \rangle$$

  where
- $Ag = \{1, \ldots, n\}$ is a set of agents
- $v = 2^{Ag} \to \mathbb{R}$ is the characteristic function of the game
- Usual interpretation: if $v(C) = k$, then coalition C can cooperate in such a way they will obtain utility k, which may then be distributed among the team members

Which Coalition Should I Join?

- Most important question in coalitional games:
  is a coalition stable?
  that is,
  is it rational for all members of coalition to stay with the coalition, or could they benefit by defecting from it?
- (There is no point in me trying to join a coalition with you unless you want to form one with me, and vice versa.)
- Stability is a *necessary* but not *sufficient* condition for coalitions to form

The Core

- The *core* of a coalitional game is the set of *feasible* distributions of payoff to members of a coalition that *no* sub-coalition can reasonably object to
- An *outcome* for a coalition C in game $\langle < Ag, v \rangle >$ is a vector of payoffs to members of C, $\langle < x_1, \ldots, x_k \rangle >$ which represents *a feasible distribution of payoff to members of Ag*.
  Feasible means:

$$\nu(C) \geq \sum_{i \in C} x_i$$

Example

- If $v(\{1, 2\}) = 20$, then possible outcomes are <20, 0〉>, 〈<19, 1>, , 〈<18, 2〉>, ..., 〈<0, 20〉>. (Actually there will be infinitely many!)

Objections

- Intuitively, a coalition *C objects* to an outcome if there is some outcome *for them* that makes *all of them* strictly better off
- Formally, $C \subseteq Ag$ objects to an outcome $\langle <x_1, \ldots, x_n \rangle >$ for the grand coalition if there is some outcome $\langle <x'_1, , \ldots, x'_k>$ for C such that

$$x'_i > x_i \text{ for all } i \in C$$

- The idea is that an outcome is not going to happen if somebody objects to it!

The Core

- The *core* is the set of outcomes for the *grand coalition* to which *no* coalition objects
- If the core is *non-empty* then *the grand coalition is stable*, since nobody can benefit from defection
- Thus, asking

  is the grand coalition stable?

  is the same as asking:

  is the core non-empty?

Problems with the Core

- Sometimes, the core is empty; what happens then?
- Sometimes it is non-empty but isn't "fair".
  Suppose:
  $Ag = \{1, 2\}$, $v(\{1\}) = 5$, $v(\{2\}) = 5$, $v(\{1, 2\}) = 20$.
  Then outcome ⟨<20, 0⟩> (i.e., agent 1 gets everything) is *not* in the core, since the coalition {2} can object. (He can work on his own and do better.)
  However, outcome ⟨<15, 5⟩> *is* in the core: even though this seems unfair to agent 2, this agent has no objection.
- Why unfair? Because the agents are *identical*!

How To Share Benefits of Cooperation?

- The *Shapley value* is best known attempt to define how to divide benefits of cooperation fairly. It does this by taking into account *how much an agent contributes*.
- The Shapley value of agent *i* is the average amount that *i* is expected to contribute to a coalition.
- Axiomatically: a value which satisfies axioms:
  symmetry, *dummy player*, and *additivity*.

Shapley Defined

- Let $\delta_i(S)$ be the amount that i adds by joining $S \subseteq Ag$: $$\delta_i(S) = \nu(S \cup \{i\}) - \nu(S)$$

  …the *marginal contribution of i to S*
- Then the Shapley value for i, denoted Sh(S,i)/φ_i ,

  Sh(S,i)/ $$\varphi_i = \frac{\sum_{r \in R} \delta_i(S_i(r))}{|Ag|!}$$

  where R is the set of all orderings of Ag and $S_i(r)$ is the set of agents preceding i in ordering r.

Example: Shapley value

S	$v(S)$
()	0
(1)	1
(2)	3
(12)	6

Example: Shapley value

$$
\begin{aligned}
Sh(\{1,2\},1) &= \frac{1}{2}\cdot(v(1)-v()+v(21)-v(2)) \\
&= \frac{1}{2}\cdot(1-0+6-3)=2 \\
Sh(\{1,2\},2) &= \frac{1}{2}\cdot(v(12)-v(1)+v(2)-v()) \\
&= \frac{1}{2}\cdot(6-1+3-0)=4
\end{aligned}
$$

Representing Coalitional Games

- It is important for an agent to know (for example) whether the core of a coalition is non-empty…so, how hard is it to decide this?
- Problem: naive, obvious representation of coalitional game is *exponential* in the size of *Ag*
- Now such a representation is:
  – utterly infeasible in practice; and
  – so large that it renders comparisons to this input size meaningless: stating that we have an algorithm that runs in (say) time *linear* in the size of such a representation means it runs in time *exponential* in the size of *Ag*

How to Represent Characteristic Functions?

- Two approaches to this problem:
  - try to find a *complete* representation that is succinct in “most” cases
  - try to find a representation that is not complete but is always succinct
  - A common approach:
    interpret characteristic function over combinatorial structure

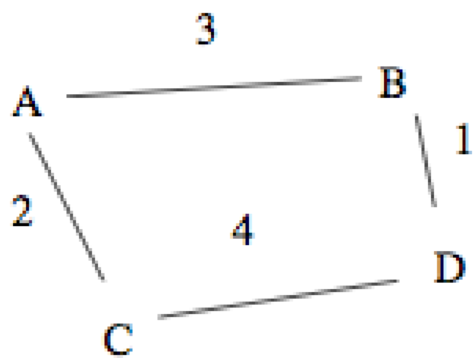
Representation 1: Induced Subgraph

- Represent v as an undirected graph on Ag, with integer weights $w_{i,j}$ between nodes $i, j \in Ag$
- Value of coalition C then:

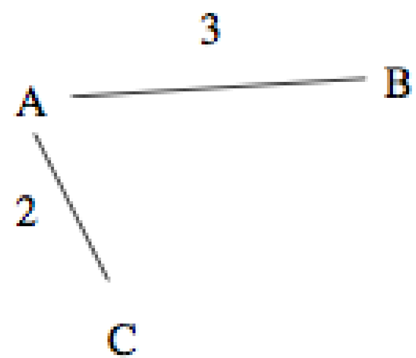
$$\nu(C) = \sum_{\{i,j\} \subseteq Ag} w_{i,j}$$

i.e., the value of a coalition $C \subseteq Ag$ is the weight of the subgraph induced by C

Representation 1: Induced Subgraph



the original graph defining v



subgraph induced by {A,B,C}
giving v({A,B,C}) = 3 + 2 = 5

Representation 2: Marginal Contribution Nets

(Ieong & Shoham, 2005)

- Characteristic function represented as rules:

  pattern $\rightarrow$ value

Representation 2: Marginal Contribution Nets

- Pattern is conjunction of agents, a rule *applies* to a group of agents *C* if *C* is a superset of the agents in the pattern
- Value of a coalition is then sum over the values of all the rules that apply to the coalition
- Example:

$$\begin{aligned} a \wedge b &\longrightarrow 5 \\ b &\longrightarrow 2 \end{aligned}$$

  We have: $v(\{a\}) = 0$, $v(\{b\}) = 2$, and $v(\{a, b\}) = 7$
- We can also allow negations in rules (agent not present)

Example: Marginal Contribution Nets

Consider the marginal contribution net:

$$
\begin{array}{lcl}
a & \longrightarrow & 2 \\
a \wedge b & \longrightarrow & 7 \\
b & \longrightarrow & 3 \\
c & \longrightarrow & 4 \\
b \wedge c & \longrightarrow & -3
\end{array}
$$

(a) Let ν be the characteristic function defined by these rules. Give the values of the following:

1. $\nu(\varnothing)$
2. $\nu(\{a\})$
3. $\nu(\{b\})$
4. $\nu(\{a, b\})$
5. $\nu(\{a, b, c\})$

Example: Marginal Contribution Nets

Answer:

$$v(\varnothing) = 0, v(\{a\}) = 2, v(\{b\}) = 3, v(\{a,b\}) = 12, v(\{a,b,c\}) = 13.$$