CHAPTER 31
Equilibrium Models of the Short Rate

Practice Questions

31.1
In Vasicek's model, it stays at $0.01\sqrt{\Delta t}$. In the Rendleman and Bartter model, the coefficient of *dz* is proportional to the level of the short rate. When the short rate increases from 4% to 8%, the standard deviation in time Δt increases to $0.02\sqrt{\Delta t}$. In the Cox, Ingersoll, and Ross model, the coefficient of Δt is proportional to the square root of the short rate. When the short rate increases from 4% to 8%, the standard deviation in time Δt increases to $0.01414\sqrt{\Delta t}$.

31.2.
If the price of a traded security followed a mean-reverting or path-dependent process, there would be market inefficiency. The short-term interest rate is not the price of a traded security. In other words, we cannot trade something whose price is always the short-term interest rate. There is therefore no market inefficiency when the short-term interest rate follows a mean-reverting or path-dependent process. We can trade bonds and other instruments whose prices do depend on the short rate. The prices of these instruments do not follow mean-reverting or path-dependent processes.

31.3
In a one-factor model, there is one source of uncertainty driving all rates. This usually means that in any short period of time all rates move in the same direction (but not necessarily by the same amount). In a two-factor model, there are two sources of uncertainty driving all rates. The first source of uncertainty usually gives rise to a roughly parallel shift in rates. The second gives rise to a twist where long and short rates moves in opposite directions.

31.4
$$B(0,5)=\frac{1-e^{-0.1\times 5}}{0.1}=3.9347$$
$$A(0,5)=\exp\left[\frac{(B(0,5)-5)(0.1^2\times 0.03-0.01^2/2)}{0.1^2}-\frac{0.01^2\times B(0,5)^2}{4\times 0.1}\right]=0.9700$$

The price of the 5-year zero-coupon bond is

$$A(0,5)\,\mathrm{e}^{-B(0,5)\times 0.02}=0.8966$$

31.5
The risk-neutral process for the short rate is
$$dr=[0.1(0.03-r)]dt+0.07\sqrt{r}dz$$

The real world process for the short rate is

$$dr = [0.1(0.03 - r) + (-1)\sqrt{r} \times 0.07\sqrt{r}]dt + 0.07\sqrt{r}dz$$

or

$$dr = [0.17(0.0176 - r)]dt + 0.07\sqrt{r}dz$$

The risk-neutral process for a zero-coupon bond with a current maturity of 4 years is

$$dP(t,4) = rP(t,4)dt - 0.07\sqrt{r}B(t,4)P(t,4)dz$$

where

$$B(t,4) = \frac{2(e^{(4-t)\gamma} - 1)}{(\gamma + 0.1)(e^{(4-t)\gamma} - 1) + 2\gamma}$$

with

$$\gamma = \sqrt{0.1^2 + 2 \times 0.07^2} = 0.1407$$

In the real world, this becomes

$$dP(t,4) = \left[1 + 0.07B(t,4)\right]rP(t,4)dt - 0.07\sqrt{r}B(t,4)P(t,4)dz$$

31.6

The risk-neutral process for the short rate is

$$dr = a(b - r)dt + \sigma dz$$

The real world process is

$$dr = [a(b - r) + (\lambda_1 + \lambda_2 r)\sigma]dt + \sigma dz$$

or

$$dr = [(ab + \lambda_1\sigma) - r(a - \lambda_2\sigma)]dt + \sigma dz$$

or

$$dr = \left[(a - \lambda_2\sigma)\left(\frac{ab + \lambda_1\sigma}{a - \lambda_2\sigma} - r\right)\right]dt + \sigma dz$$

This shows that the reversion rate is $a - \lambda_2\sigma$ and the reversion level is

$$\frac{ab + \lambda_1\sigma}{a - \lambda_2\sigma}$$

31.7

The change $r_i - r_{i-1}$ is normally distributed with mean $a(b^* - r_{i-1})$ and variance $\sigma^2\Delta t$. The probability density of the observation is

$$\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left(\frac{r_i - r_{i-1} - a(b^\bullet - r_{i-1})}{2\sigma^2\Delta t}\right)$$

We wish to maximize

$$\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2 \Delta t}} \exp\left(\frac{r_i - r_{i-1} - a(b^{\bullet} - r_{i-1})}{2\sigma^2 \Delta t} \right)$$

Taking logarithms, this is the same as maximizing

$$\sum_{i=1}^{m} \left(-\ln(\sigma^2 \Delta t) - \frac{\left[r_i - r_{i-1} - a(b^{\bullet} - r_{i-1})\Delta t \right]^2}{\sigma^2 \Delta t} \right)$$

31.8
The calculations are as follows:

Time, T_i	*Cash Flow, c_i*	*$B(0,T_i)$*	*$A(0,T_i)$*	*$P(0,T_i)$*	*Value of cash flow*	*Weight*	*Weight×$B(0,T_i)$*
0.5	1.5	0.4841	0.9998	0.9950	1.4925	0.0144	0.0070
1.0	1.5	0.9377	0.9993	0.9899	1.4849	0.0143	0.0134
1.5	1.5	1.3628	0.9984	0.9849	1.4773	0.0142	0.0194
2.0	101.5	1.7611	0.9972	0.9798	99.4536	0.9571	1.6856
Total					103.9083		1.7254

The bond price is 103.9083. The alternative duration measure is 1.7254. The percentage decrease in the bond price of a 0.0005 increase in r is estimated as 0.0005×1.7524 or 0.0863%. so that the bond price decreases by an amount 0.000863×103.9083= 0.0896. The new bond price is 103.8187**.** This is also the bond price we get when we calculate the $P(0,T_i)$ from r=1.05%.

31.9
In Vasicek's model, $a = 0.1$, $b = 0.1$, and $\sigma = 0.02$ so that

$$B(t, t+10) = \frac{1}{0.1}(1 - e^{-0.1\times 10}) = 6.32121$$

$$A(t,t+10) = \exp\left[\frac{(6.32121 - 10)(0.1^2 \times 0.1 - 0.0002)}{0.01} - \frac{0.0004 \times 6.32121^2}{0.4} \right]$$

$$= 0.71587$$

The bond price is therefore $0.71587 e^{-6.32121\times 0.1} = 0.38046$

In the Cox, Ingersoll, and Ross model, $a = 0.1$, $b = 0.1$ and $\sigma = 0.02/\sqrt{0.1} = 0.0632$. Also

$$\gamma = \sqrt{a^2 + 2\sigma^2} = 0.13416$$

Define

$$\beta = (\gamma + a)(e^{10\gamma} - 1) + 2\gamma = 0.92992$$

$$B(t,t+10)=\frac{2(e^{10\gamma}-1)}{\beta}=6.07650$$

$$A(t,t+10)=\left(\frac{2\gamma e^{5(a+\gamma)}}{\beta}\right)^{2ab/\sigma^2}=0.69746$$

The bond price is therefore $0.69746e^{-6.07650\times 0.1}$ = 0.37986.

31.10

(a) The risk neutral process for r has a drift rate which is $0.006/r$ higher than the real world process. The volatility is $0.01/r$. This means that the market price of interest rate risk is $-0.006/0.01$ or -0.6.

(b) The expected return on the bond in the risk-neutral world is the risk free rate of 4%. The volatility is $0.01\times B(0,5)$ where

$$B(0,5)=\frac{1-e^{-0.1\times 5}}{0.1}=3.935$$

i.e., the volatility is 3.935%.

(c) The process followed by the bond price in a risk-neutral world is

$$dP=0.04Pdt-0.03935\,Pdz$$

Note that the coefficient of dz is negative because bond prices are negatively correlated with interest rates. When we move to the real world the return increases by the product of the market price of dz risk and -0.03935. The bond price process becomes:

$$dP=[0.04+(-0.6\times -0.03935)]Pdt-0.03935Pdz$$

or

$$dP=0.06361\,Pdt-0.03935\,Pdz$$

The expected return on the bond increases from 4% to 6.361% as we move from the risk-neutral world to the real world.

31.11

(a) $\dfrac{\partial^2 P(t,T)}{\partial r^2}=B(t,T)^2A(t,T)e^{-B(t,T)r}=B(t,T)^2P(t,T)$

(b) A corresponding definition for $\hat{C}$ is

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial r^2}$$

(c) When $Q=P(t,T)$, $\hat{C}=B(t,T)^2$ For a coupon-bearing bond $\hat{C}$ is a weighted average of the $\hat{C}$'s for the constituent zero-coupon bonds where weights are proportional to bond prices.

(d)

$$\Delta P(t,T)=\frac{\partial P(t,T)}{\partial r}\Delta r+\frac{1}{2}\frac{\partial^2 P(t,T)}{\partial r^2}\Delta r^2+\dots$$
$$=-B(t,T)P(t,T)\Delta r+\frac{1}{2}B(t,T)^2P(t,T)\Delta r^2+\dots$$

31.12

The risk-neutral process for the short rate is

$$dr = [0.15(0.025 - r)]dt + 0.012dz$$

The real world process for the short rate is

$$dr = [0.15(0.025 - r) - 0.20 \times 0.012]dt + 0.012dz$$

or

$$dr = [0.15(0.009 - r)]dt + 0.012dz$$

The risk-neutral process for a zero-coupon bond with a current maturity of 3 years is

$$dP(t,3) = rP(t,3)dt - 0.012B(\mathrm{t},3)P(t,3)dz$$

where

$$B(t,3) = \frac{1 - e^{-0.15(3-t)}}{0.15}$$

In the real world this becomes

$$dP(t,3) = [rP(t,3) + 0.2 \times 0.012B(t,3)P(t,3)]dt - 0.012B(t,3)P(t,3)dz$$

or

$$dP(t,3) = [r + 0.0024B(t,3)]P(t,3)dt - 0.012B(t,3)P(t,3)dz$$

31.13

The risk-neutral process for the short rate is

$$dr = a(b - r)dt + \sigma\sqrt{r}dz$$

The real world process is

$$dr = [a(b - r) + (\lambda_1 / \sqrt{r} + \lambda_2 \sqrt{r})\sigma\sqrt{r}]dt + \sigma\sqrt{r}dz$$

or

$$dr = [(ab + \lambda_1\sigma) - r(a - \lambda_2\sigma)]dt + \sigma\sqrt{r}dz$$

or

$$dr = [(a - \lambda_2\sigma)\left(\frac{ab + \lambda_1\sigma}{a - \lambda_2\sigma} - r\right)]dt + \sigma\sqrt{r}dz$$

This shows that the reversion rate is $a - \lambda_2\sigma$ and the reversion level is

$$\frac{ab + \lambda_1\sigma}{a - \lambda_2\sigma}$$

31.14
The differential equation satisfied by the bond price $P(t,T)$ is

$$\frac{\partial f}{\partial t}+(u-ar)\frac{\partial f}{\partial r}-bu\frac{\partial f}{\partial u}+\frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial r^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 f}{\partial u^2}+\sigma_1\sigma_2\frac{\partial^2 f}{\partial r\partial u}=rf$$

$$f=A(t,T)e^{-B(t,T)r-C(t,T)u}$$

satisfies this if

$$A_t-AB_tr-AC_tu-(u-ar)AB+buAC+\frac{1}{2}\sigma_1^2B^2+\frac{1}{2}\sigma_2^2C^2+\sigma_1\sigma_2BC=rA$$

or if

$$A_t+\frac{1}{2}\sigma_1^2B^2+\frac{1}{2}\sigma_2^2C^2+\sigma_1\sigma_2BC=0$$

$$B_t-aB+1=0$$

$$C_t+B-bC=0$$

The equation

$$B_t-aB+1=0$$

is satisfied by

$$B(t,T)=\frac{1-e^{-a(T-t)}}{a}$$

The equation

$$\mathrm{C}_t+\frac{1-e^{-a(T-t)}}{a}-bC=0$$

is satified by

$$C(t,T)=\frac{1}{a(a-b)}e^{-a(T-t)}-\frac{1}{b(a-b)}e^{-b(T-t)}+\frac{1}{ab}$$

31.15 (Excel file)
See Excel worksheet. Both approaches give a=0.136, b^*=0.0168, and σ=0.0119.

31.16 (Excel file)
In the case of the CIR model, the change $r_i - r_{i-1}$ is normally distributed with mean $a(b - r_{i-1})\Delta t$ and variance $\sigma^2 r_{i-1}\Delta t$ and the maximum likelihood function becomes

$$\sum_{i=1}^{m}\left(-\ln(\sigma^2 r_{i-1}\Delta t)-\frac{\left[r_i-r_{i-1}-a(b-r_{i-1})\Delta t\right]^2}{\sigma^2 r_{i-1}\Delta t}\right)$$

The Excel worksheet shows that the best fit (real world) parameters are a^* = 0.201, b*=0.025, and σ = 0.077. The estimated real world process is therefore:

$$dr=0.201(0.025-r)+0.077\sqrt{r}dz$$

The short rate reverts to 2.5% with a 20% reversion rate. These parameters are not unreasonable and the likelihood is higher than for Vasicek's model (see Problem 31.16). However, the market price of risk is estimated as $-0.0266\sqrt{r}$ which, although it has the right sign is very small indicating very little difference between real world and risk neutral processes.