CHAPTER 30
Convexity, Timing, and Quanto Adjustments

Practice Questions

30.1
Suppose first that the correlation between the underlying asset price and interest rates is negative and we have a long forward contract. When interest rates increase, there will be a tendency for the asset price to decrease. The increase in interest rates means that an investor would like a positive payoff to be early and a negative one to be late. The negative correlation means that a negative payoff is more likely.
When interest rates decrease, there will be a tendency for the asset price to increase. The decrease in interest rates means that an investor would like a positive payoff to be late and a negative one to be early. The negative correlation means that a positive payoff is more likely.
This argument shows that a negative correlation works in the investor's favor. Similarly, a positive correlation works against the investor's interests.

30.2
(a) A convexity adjustment is necessary for the swap rate.
(b) No convexity or timing adjustments are necessary.

30.3
There are two differences. The discounting is done over a 1.0-year period instead of over a 1.25-year period. Also a convexity adjustment to the forward rate is necessary. From equation (30.2), the convexity adjustment is:

$$\frac{0.07^2 \times 0.2^2 \times 0.25 \times 1}{1 + 0.25 \times 0.07} = 0.00005$$

or about half a basis point.
In the formula for the caplet, we set $F_k = 0.07005$ instead of 0.07. This means that $d_1 = -0.5642$ and $d_2 = -0.7642$. The caplet price becomes

$$0.25 \times 10 e^{-0.065 \times 1.0}[0.07005 N(-0.5642) - 0.08 N(-0.7642)] = 0.0531$$

30.4
The convexity adjustment discussed in Section 30.1 leads to the instrument being worth an amount slightly different from zero. Define $G(y)$ as the value as seen in five years of a two-year bond with a coupon of 10% as a function of its yield.

$$G(y) = \frac{0.1}{1+y} + \frac{1.1}{(1+y)^2}$$

$$G'(y) = -\frac{0.1}{(1+y)^2} - \frac{2.2}{(1+y)^3}$$

$$G''(y) = \frac{0.2}{(1+y)^3} + \frac{6.6}{(1+y)^4}$$

It follows that $G'(0.1) = -1.7355$ and $G''(0.1) = 4.6582$ and the convexity adjustment that must be made for the two-year swap- rate is

$$0.5 \times 0.1^2 \times 0.2^2 \times 5 \times \frac{4.6582}{1.7355} = 0.00268$$

We can therefore value the instrument on the assumption that the swap rate will be 10.268% in five years. The value of the instrument is

$$\frac{0.268}{1.1^5} = 0.167$$

or \$0.167.

30.5

In this case, we have to make a timing adjustment as well as a convexity adjustment to the forward swap rate. For (a), equation (30.4) shows that the timing adjustment involves multiplying the swap rate by

$$\exp\left[-\frac{0.8 \times 0.20 \times 0.20 \times 0.1 \times 5}{1+0.1}\right] = 0.9856$$

so that it becomes $10.268 \times 0.9856 = 10.120$. The value of the instrument is

$$\frac{0.120}{1.1^6} = 0.068$$

or \$0.068.

For (b), equation (30.4) shows that the timing adjustment involves multiplying the swap rate by

$$\exp\left[-\frac{0.95 \times 0.2 \times 0.2 \times 0.1 \times 2 \times 5}{1+0.1}\right] = 0.9660$$

so that it becomes $10.268 \times 0.966 = 9.919$. The value of the instrument is now

$$-\frac{0.081}{1.1^7} = -0.042$$

or –\$0.042.

30.6

(a) The process for y is

$$dy = \alpha y\, dt + \sigma_y y\, dz$$

The forward bond price is $G(y)$. From Itô's lemma, its process is

$$d[G(y)] = [G'(y)\alpha y + \frac{1}{2}G''(y)\sigma_y^2 y^2]dt + G'(y)\sigma_y y\, dz$$

(b) Since the expected growth rate of $G(y)$ is zero

$$G'(y)\alpha y + \frac{1}{2}G''(y)\sigma_y^2 y^2 = 0$$

or

$$\alpha = -\frac{1}{2}\frac{G''(y)}{G'(y)}\sigma_y^2 y$$

(c) Assuming as an approximation that y always equals its initial value of y_0, this shows that the growth rate of y is

$$-\frac{1}{2}\frac{G''(y_0)}{G'(y_0)}\sigma_y^2 y_0$$

The variable y starts at y_0 and ends as y_T. The convexity adjustment to y_0 when we are calculating the expected value of y_T in a world that is defined by a numeraire equal to a zero-coupon bond maturing at time T is approximately $y_0 T$ times this or

$$-\frac{1}{2}\frac{G''(y_0)}{G'(y_0)}\sigma_y^2 y_0^2 T$$

This is consistent with equation (30.1).

30.7

(a) In the traditional risk-neutral world, the process followed by S is

$$dS = (r-q)S\,dt + \sigma_S S\,dz$$

where r is the instantaneous risk-free rate. The market price of dz-risk is zero.

(b) In the traditional risk-neutral world for currency B, the process is

$$dS = (r - q + \rho_{QS}\sigma_S\sigma_Q)S\,dt + \sigma_S S\,dz$$

where Q is the exchange rate (units of A per unit of B), σ_Q is the volatility of Q and ρ_{QS} is the coefficient of correlation between Q and S. The market price of dz-risk is $\rho_{QS}\sigma_Q$.

(c) In a world that is defined by a numeraire equal to a zero-coupon bond in currency A maturing at time *T*,

$$dS = (r - q + \sigma_S\sigma_P)S\,dt + \sigma_S S\,dz$$

where σ_P is the bond price volatility. The market price of dz-risk is σ_P

(d) In a world that is defined by a numeraire equal to a zero-coupon bond in currency B maturing at time *T*,

$$dS = (r - q + \sigma_S\sigma_P + \rho_{FS}\sigma_S\sigma_F)S\,dt + \sigma_S S\,dz$$

where F is the forward exchange rate, σ_F is the volatility of F (units of A per unit of B, and ρ_{FS} is the correlation between F and S. The market price of dz-risk is $\sigma_P + \rho_{FS}\sigma_F$.

30.8

Define:

$P(t,T)$: Price in yen at time t of a bond paying 1 yen at time T

$E_T(.)$: Expectation in world that is defined by numeraire $P(t,T)$

F: Dollar forward price of gold for a contract maturing at time T

F_0: Value of F at time zero

σ_F: Volatility of F

G: Forward exchange rate (dollars per yen)

σ_G: Volatility of G

We assume that S_T is lognormal. We can work in a world that is defined by numeraire $P(t,T)$ to get the value of the call as

$$P(0,T)[E_T(S_T)N(d_1) - N(d_2)]$$

where

$$d_1 = \frac{\ln[E_T(S_T)/K] + \sigma_F^2 T/2}{\sigma_F \sqrt{T}}$$

$$d_2 = \frac{\ln[E_T(S_T)/K] - \sigma_F^2 T/2}{\sigma_F \sqrt{T}}$$

The expected gold price in a world that is defined by a numeraire equal to a zero-coupon dollar bond maturing at time T is F_0. It follows from equation (30.6) that

$$E_T(S_T) = F_0(1+\rho\sigma_F\sigma_G T)$$

Hence the option price, measured in yen, is

$$P(0,T)[F_0(1+\rho\sigma_F\sigma_G T)N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln[F_0(1+\rho\sigma_F\sigma_G T)/K] + \sigma_F^2 T/2}{\sigma_F \sqrt{T}}$$

$$d_2 = \frac{\ln[F_0(1+\rho\sigma_F\sigma_G T)/K] - \sigma_F^2 T/2}{\sigma_F \sqrt{T}}$$

30.9

(a) The value of the option can be calculated by setting $S_0 = 400$, $K = 400$, $r = 0.06$, $q = 0.03$, $\sigma = 0.2$, and $T = 2$. With 100 time steps, the value (in Canadian dollars) is 52.92.

(b) The growth rate of the index using the CDN numeraire is $0.06 - 0.03$ or 3%. When we switch to the USD numeraire, we increase the growth rate of the index by $0.4 \times 0.2 \times 0.06$ or 0.48% per year to 3.48%. The option can therefore be calculated using DerivaGem with $S_0 = 400$, $K = 400$, $r = 0.04$, $q = 0.04 - 0.0348 = 0.0052$, $\sigma = 0.2$, and $T = 2$. With 100 time steps, DerivaGem gives the value as 57.51.

30.10

(a) We require the expected value of the Nikkei index in a dollar risk-neutral world. In a yen risk-neutral world, the expected value of the index is $20{,}000e^{(0.02-0.01)\times 2} = 20{,}404.03$. In a dollar risk-neutral world, the analysis in Section 30.3 shows that this becomes

$$20{,}404.03e^{0.3\times 0.20\times 0.12\times 2} = 20{,}699.97$$

The value of the instrument is therefore,

$$20{,}699.97e^{-0.04\times 2} = 19{,}108.48$$

(b) An amount SQ yen is invested in the Nikkei. Its value in yen changes to

$$SQ\left(1+\frac{\Delta S}{S}\right)$$

In dollars this is worth

$$SQ\frac{1+\Delta S/S}{Q+\Delta Q}$$

where ΔQ is the increase in Q. When terms of order two and higher are ignored, the dollar value becomes

$$S(1+\Delta S/S-\Delta Q/Q)$$

The gain on the Nikkei position is therefore $\Delta S - S\Delta Q/Q$

When SQ yen are shorted the gain in dollars is

$$SQ\left(\frac{1}{Q}-\frac{1}{Q+\Delta Q}\right)$$

This equals $S\Delta Q/Q$ when terms of order two and higher are ignored. The gain on the whole position is therefore ΔS as required.

(c) In this case, the investor invests \$20,000 in the Nikkei. The investor converts the funds to yen and buys 100 times the index. The index rises to 20,050 so that the investment becomes worth 2,005,000 yen or

$$\frac{2,005,000}{99.7}=20,110.33$$

dollars. The investor therefore gains \$110.33. The investor also shorts 2,000,000 yen. The value of the yen changes from \$0.0100 to \$0.01003. The investor therefore loses $0.00003\times 2,000,000=60$ dollars on the short position. The net gain is 50.33 dollars. This is close to the required gain of \$50.

(d) Suppose that the value of the instrument is V. When the index changes by ΔS yen the value of the instrument changes by

$$\frac{\partial V}{\partial S}\Delta S$$

dollars. We can calculate $\partial V/\partial S$. Part (b) of this question shows how to manufacture an instrument that changes by ΔS dollars. This enables us to delta-hedge our exposure to the index.

30.11

To calculate the convexity adjustment for the five-year rate, define the price of a five year bond, as a function of its yield as

$$G(y)=e^{-5y}$$

$$G'(y)=-5e^{-5y}$$

$$G''(y)=25e^{-5y}$$

The convexity adjustment is

$$0.5\times 0.08^2\times 0.25^2\times 4\times 5=0.004$$

Similarly, for the two year rate the convexity adjustment is

$$0.5\times 0.08^2\times 0.25^2\times 4\times 2=0.0016$$

We can therefore value the derivative by assuming that the five year rate is 8.4% and the two-year rate is 8.16%. The value of the derivative is

$$0.24e^{-0.08\times 4}=0.174$$

If the payoff occurs in five years rather than four years, it is necessary to make a timing adjustment. From equation (30.4) this involves multiplying the forward rate by

$$\exp\left[-\frac{1\times 0.25\times 0.25\times 0.08\times 4\times 1}{1.08}\right]=0.98165$$

The value of the derivative is

$$0.24\times 0.98165e^{-0.08\times 5}=0.158.$$

30.12

(a) In this case, we must make a convexity adjustment to the forward swap rate. Define

$$G(y)=\sum_{i=1}^{6}\frac{4}{(1+y/2)^{i}}+\frac{100}{(1+y/2)^{6}}$$

so that

$$G'(y)=-\sum_{i=1}^{6}\frac{2i}{(1+y/2)^{i+1}}+\frac{300}{(1+y/2)^{7}}$$

$$G''(y)=\sum_{i=1}^{6}\frac{i(i+1)}{(1+y/2)^{i+2}}+\frac{1050}{(1+y/2)^{8}}$$

$G'(0.08)=-262.11$ and $G''(0.08)=853.29$ so that the convexity adjustment is

$$\frac{1}{2}\times 0.08^{2}\times 0.18^{2}\times 10\times\frac{853.29}{262.11}=0.00338$$

The adjusted forward swap rate is $0.08+0.00338=0.08338$ and the value of the derivative in millions of dollars is

$$\frac{0.08338\times 100}{1.03^{20}}=4.617$$

(b) When the swap rate is applied to a yen principal, we must make a quanto adjustment in addition to the convexity adjustment. From Section 30.3, this involves multiplying the forward swap rate by $e^{-0.25\times 0.12\times 0.18\times 10}=0.9474$. (Note that the correlation is the correlation between the dollar per yen exchange rate and the swap rate. It is therefore -0.25 rather than $+0.25$.) The value of the derivative in millions of yen is

$$\frac{0.08338\times 0.9474\times 100}{1.01^{20}}=6.474$$