CHAPTER 33
Modeling Forward Rates

Practice Questions

33.1
In a Markov model, the expected change and volatility of the short rate at time t depend only on the value of the short rate at time t. In a non-Markov model, they depend on the history of the short rate prior to time t.

33.2
Equation (33.1) becomes

$$dP(t,T) = r(t)P(t,T)\,dt + \sum_k v_k(t,T,\Omega_t)P(t,T)\,dz_k(t)$$

so that

$$d\ln[P(t,T_1)] = \left[r(t) - \sum_k \frac{v_k(t,T_1,\Omega_t)^2}{2}\right]dt + \sum_k v_k(t,T_1,\Omega_t)\,dz_k(t)$$

and

$$d\ln[P(t,T_2)] = \left[r(t) - \sum_k \frac{v_k(t,T_2,\Omega_t)^2}{2}\right]dt + v_k(t,T_2,\Omega_t)\,dz_k(t)$$

From equation (33.2)

$$df(t,T_1,T_2) = \frac{\sum_k [v_k(t,T_2,\Omega_t)^2 - v_k(t,T_1,\Omega_t)^2]}{2(T_2-T_1)}dt + \sum_k \frac{v_k(t,T_1,\Omega_t) - v_k(t,T_2,\Omega_t)}{T_2-T_1}dz_k(t)$$

Putting $T_1 = T$ and $T_2 = T + \Delta t$ and taking limits as Δt tends to zero, this becomes

$$dF(t,T) = \sum_k \left[v_k(t,T,\Omega_t)\frac{\partial v_k(t,T,\Omega_t)}{\partial T}\right]dt - \sum_k \left[\frac{\partial v_k(t,T,\Omega_t)}{\partial T}\right]dz_k(t)$$

Using $v_k(t,t,\Omega_t) = 0$

$$v_k(t,T,\Omega_t) = \int_t^T \frac{\partial v_k(t,\tau,\Omega_t)}{\partial \tau}d\tau$$

The result in equation (33.6) follows by substituting

$$s_k(t,T,\Omega_t) = \frac{\partial v_k(t,T,\Omega_t)}{\partial T}$$

33.3
Using the notation in Section 33.1, when s is constant,

$$v_T(t,T) = s \qquad v_{TT}(t,T) = 0$$

Integrating $v_T(t,T)$

$$v(t,T) = sT + \alpha(t)$$

for some function α. Using the fact that $v(T,T) = 0$, we must have

$$v(t,T) = s(T-t)$$

Using the notation from Chapter 32, in Ho–Lee $P(t,T) = A(t,T)e^{-r(T-t)}$. The standard deviation of the short rate is constant. It follows from Itô's lemma that the standard deviation of the bond price is a constant times the bond price times $T-t$. The volatility of the bond price is therefore constant times $T-t$. This shows that Ho–Lee is consistent with a constant s.

33.4

Using the notation in Section 33.1, when $v_T(t,T) = s(t,T) = \sigma e^{-a(T-t)}$

$$v_{TT}(t,T) = -a\sigma e^{-a(T-t)}$$

Integrating $v_T(t,T)$

$$v(t,T) = -\frac{1}{a}\sigma e^{-a(T-t)} + \alpha(t)$$

for some function α. Using the fact that $v(T,T) = 0$, we must have

$$v(t,T) = \frac{\sigma}{a}[1-e^{-a(T-t)}] = \sigma B(t,T)$$

Using the notation from Chapter 32, in Hull–White $P(t,T) = A(t,T)e^{-rB(t,T)}$. The standard deviation of the short rate is constant, σ. It follows from Itô's lemma that the standard deviation of the bond price is $\sigma P(t,T)B(t,T)$. The volatility of the bond price is therefore $\sigma B(t,T)$. This shows that Hull–White is consistent with $s(t,T) = \sigma e^{-a(T-t)}$.

33.5

BGM is a similar model to HJM. It has the advantage over HJM that it involves forward rates that are readily observable. HJM involves instantaneous forward rates.

33.6

A ratchet cap tends to provide relatively low payoffs if a high (low) interest rate at one reset date is followed by a high (low) interest rate at the next reset date. High payoffs occur when a low interest rate is followed by a high interest rate. As the number of factors increase, the correlation between successive forward rates declines and there is a greater chance that a low interest rate will be followed by a high interest rate.

33.7

Equation (33.10) can be written

$$dF_k(t) = \zeta_k(t)F_k(t)\sum_{i=m(t)}^{k}\frac{\delta_i F_i(t)\zeta_i(t)}{1+\delta_i F_i(t)}dt + \zeta_k(t)F_k(t)\,dz$$

As δ_i tends to zero, $\zeta_i(t)F_i(t)$ becomes the standard deviation of the instantaneous t_i-maturity forward rate at time t. Using the notation of Section 33.1, this is $s(t,t_i,\Omega_t)$. As δ_i tends to zero

$$\sum_{i=m(t)}^{k}\frac{\delta_i F_i(t)\zeta_i(t)}{1+\delta_i F_i(t)}$$

tends to

$$\int_{\tau=t}^{t_k} s(t,\tau,\Omega_t)\,d\tau$$

Equation (33.10) therefore becomes

$$dF_k(t) = \left[s(t,t_k,\Omega_t)\int_{\tau=t}^{t_k} s(t,\tau,\Omega_t)\,d\tau\right]dt + s(t,t_k,\Omega_t)\,dz$$

This is the HJM result.

33.8
In a ratchet cap, the cap rate equals the previous reset rate, R, plus a spread. In the notation of the text it is $R_j + s$. In a sticky cap, the cap rate equal the previous capped rate plus a spread. In the notation of the text, it is $\min(R_j, K_j) + s$. The cap rate in a ratchet cap is always at least as great as that in a sticky cap. Since the value of a cap is a decreasing function of the cap rate, it follows that a sticky cap is more expensive.

33.9
When prepayments increase, the principal is received sooner. This increases the value of a PO. When prepayments increase, less interest is received. This decreases the value of an IO.

33.10
A bond yield is the discount rate that causes the bond's price to equal the market price. The same discount rate is used for all maturities. An OAS is the parallel shift to the Treasury zero curve that causes the price of an instrument such as a mortgage-backed security to equal its market price.

33.11
When there are p factors, equation (33.7) becomes

$$dF_k(t) = \sum_{q=1}^{p} \zeta_{k,q}(t) F_k(t) dz_q$$

Equation (33.8) becomes

$$dF_k(t) = \sum_{q=1}^{p} \zeta_{k,q}(t)[v_{m(t),q} - v_{k+1,q}]F_k(t)dt + \sum_{q=1}^{p} \zeta_{k,q}(t)(F_k(t)dz_q$$

Equation coefficients of dz_q in

$$\ln P(t,t_i) - \ln P(t,t_{i+1}) = \ln[1 + \delta_i F_i(t)]$$

Equation (33.9) therefore becomes

$$v_{i,q}(t) - v_{i+1,q}(t) = \frac{\delta_i F_i(t) \zeta_{i,q}}{1 + \delta_i F_i(t)}$$

Equation (33.15) follows.

33.12
From the equations in the text

$$s(t) = \frac{P(t,T_0) - P(t,T_N)}{\sum_{i=0}^{N-1} \tau_i P(t,T_{i+1})}$$

and

$$\frac{P(t,T_i)}{P(t,T_0)} = \prod_{j=0}^{i-1} \frac{1}{1 + \tau_j G_j(t)}$$

so that

$$s(t) = \frac{1 - \prod_{j=0}^{N-1} \frac{1}{1+\tau_j G_j(t)}}{\sum_{i=0}^{N-1} \tau_i \prod_{j=0}^{i} \frac{1}{1+\tau_j G_j(t)}}$$

(We employ the convention that empty sums equal zero and empty products equal one.)
Equivalently

$$s(t)=\frac{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)]-1}{\sum_{i=0}^{N-1}\tau_i\prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]}$$

or

$$\ln s(t)=\ln\left\{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)]-1\right\}-\ln\left\{\sum_{i=0}^{N-1}\tau_i\prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]\right\}$$

so that

$$\frac{1}{s(t)}\frac{\partial s(t)}{\partial G_k(t)}=\frac{\tau_k\gamma_k(t)}{1+\tau_k G_k(t)}$$

where

$$\gamma_k(t)=\frac{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)]}{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)]-1}-\frac{\sum_{i=0}^{k-1}\tau_i\prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]}{\sum_{i=0}^{N-1}\tau_i\prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]}$$

From Ito's lemma, the q th component of the volatility of $s(t)$ is

$$\sum_{k=0}^{N-1}\frac{1}{s(t)}\frac{\partial s(t)}{\partial G_k(t)}\beta_{k,q}(t)G_k(t)$$

or

$$\sum_{k=0}^{N-1}\frac{\tau_k\beta_{k,q}(t)G_k(t)\gamma_k(t)}{1+\tau_k G_k(t)}$$

The variance rate of $s(t)$ is therefore,

$$V(t)=\sum_{q=1}^{p}\left[\sum_{k=0}^{N-1}\frac{\tau_k\beta_{k,q}(t)G_k(t)\gamma_k(t)}{1+\tau_k G_k(t)}\right]^2$$

33.13.

$$1+\tau_j G_j(t)=\prod_{m=1}^{M}[1+\tau_{j,m}G_{j,m}(t)]$$

so that

$$\ln[1+\tau_j G_j(t)]=\sum_{m=1}^{M}\ln[1+\tau_{j,m}G_{j,m}(t)]$$

Equating coefficients of dz_q

$$\frac{\tau_j\beta_{j,q}(t)G_j(t)}{1+\tau_j G_j(t)}=\sum_{m=1}^{M}\frac{\tau_{j,m}\beta_{j,m,q}(t)G_{j,m}(t)}{1+\tau_{j,m}G_{j,m}(t)}$$

If we assume that $G_{j,m}(t)=G_{j,m}(0)$ for the purposes of calculating the swap volatility, we see from equation (33.17) that the volatility becomes

$$\sqrt{\frac{1}{T_0}\int_{t=0}^{T_0}\sum_{q=1}^{p}\left[\sum_{k=n}^{N-1}\sum_{m=1}^{M}\frac{\tau_{k,m}\beta_{k,m,q}(t)G_{k,m}(0)\gamma_k(0)}{1+\tau_{k,m}G_{k,m}(0)}\right]^2 dt}$$

This is equation (33.19).

33.14

The two types of flexi caps mentioned are more difficult to value than the flexi cap considered in Section 33.2. There are two reasons for this.

(i) They are American-style. (The holder gets to choose whether a caplet is exercised.) This makes the use of Monte Carlo simulation difficult.

(ii) They are path dependent. In (a) the decision on whether to exercise a caplet is liable to depend on the number of caplets exercised so far. In (b) the exercise of a caplet is liable to depend on a decision taken some time earlier.

In practice, flexi caps are sometimes valued using a one-factor model of the short rate in conjunction with the techniques described in Section 27.5 for handling path-dependent derivatives.

The flexi cap in (b) is worth more than the flexi cap considered in Section 33.2. This is because the holder of the flexi cap in (b) has all the options of the holder of the flexi cap in the text and more. Similarly, the flexi cap in (a) is worth more than the flexi cap in (b). This is because the holder of the flexi cap in (a) has all the options of the holder of the flexi cap in (b) and more. We therefore expect the flexi cap in (a) to be the most expensive and the flexi cap considered in Section 33.2 to be the least expensive.