CHAPTER 32
No-Arbitrage Models of the Short Rate

Practice Questions

32.1
Equilibrium models usually start with assumptions about economic variables and derive the behavior of interest rates. The initial term structure is an output from the model. In a no-arbitrage model, the initial term structure is an input. The behavior of interest rates in a no-arbitrage model is designed to be consistent with the initial term structure.

32.2
No. The approach in Section 32.2 relies on the argument that, at any given time, all bond prices are moving in the same direction. This is not true when there is more than one factor.

32.3
Using the notation in the text, $s=3$, $T=1$, $L=100$, $K=87$, and

$$\sigma_P = \frac{0.015}{0.1}(1-e^{-2\times 0.1})\sqrt{\frac{1-e^{-2\times 0.1\times 1}}{2\times 0.1}} = 0.025886$$

From equation (31.6), $P(0,1)=0.94988$, $P(0,3)=0.85092$, and $h=1.14277$ so that equation (31.20) gives the call price as call price is

$$100\times 0.85092\times N(1.14277)-87\times 0.94988\times N(1.11688)=2.59$$

or $2.59.

32.4
As mentioned in the text, equation (32.10) for a call option is essentially the same as Black's model. By analogy with Black's formulas corresponding expression for a put option is

$$KP(0,T)N(-h+\sigma_P)-LP(0,s)N(-h)$$

In this case, the put price is

$$87\times 0.94988\times N(-1.11688)-100\times 0.85092\times N(-1.14277)=0.14$$

Since the underlying bond pays no coupon, put–call parity states that the put price plus the bond price should equal the call price plus the present value of the strike price. The bond price is 85.09 and the present value of the strike price is $87\times 0.94988=82.64$. Put–call parity is therefore satisfied:

$$82.64+2.59=85.09+0.14$$

32.5
As explained in Section 32.2, the first stage is to calculate the value of r at time 2.1 years which is such that the value of the bond at that time is 99. Denoting this value of r by r^*, we must solve

$$2.5A(2.1,2.5)e^{-B(2.1,2.5)r^*}+102.5A(2.1,3.0)e^{-B(2.1,3.0)r^*}=99$$

where the A and B functions are given by equations (31.7) and (31.8). In this case, A(2.1, 2.5) = 0.999685, A(2.1,3.0) = 0.998432, B(2.1,2.5) = 0.396027, and B(2.1, 3.0) = 0.88005, and Solver shows that $r^*=0.065989$. Since

$$2.5A(2.1,2.5)e^{-B(2.1,2.5)\times r^*}=2.434745$$

and

$$102.5A(2.1,3.0)e^{-B(2.1,3.0)\times r*}=96.56535$$

the call option on the coupon-bearing bond can be decomposed into a call option with a strike price of 2.434745 on a bond that pays off 2.5 at time 2.5 years and a call option with a strike price of 96.56535 on a bond that pays off 102.5 at time 3.0 years.

The options are valued using equation (32.10).

For the first option, L = 2.5, K = 2.434745, T = 2.1, and s = 2.5. Also, $A(0,T)$ = 0.991836, $B(0,T)$ = 1.99351, $P(0,T)$ = 0.880022 while $A(0,s)$ = 0.988604, $B(0,s)$ = 2.350062, and $P(0,s)$ = 0.858589. Furthermore, σ_P = 0.008176 and h = 0.223351. so that the option price is 0.009084.

For the second option L = 102.5, K = 96.56535, T = 2.1, and s = 3.0. Also, $A(0,T)$ = 0.991836, $B(0,T)$ = 1.99351, $P(0,T)$ = 0.880022 while $A(0,s)$ = 0.983904, $B(0,s)$ = 2.78584, and $P(0,s)$ = 0.832454. Furthermore σ_P = 0.018168 and h = 0.233343. so that the option price is 0.806105.

The total value of the option is therefore 0.0090084+0.806105=0.815189.

32.6

Put–call parity shows that:

$$c+I+PV(K)=p+B_0$$

or

$$p=c+PV(K)-(B_0-I)$$

where c is the call price, K is the strike price, I is the present value of the coupons, and B_0 is the bond price. In this case $c=0.8152$, $PV(K)=99\times P(0,2.1)=87.1222$, $B_0-I=2.5\times P(0,2.5)+102.5\times P(0,3)=87.4730$ so that the put price is

$$0.8152+87.1222-87.4730=0.4644$$

32.7

Using the notation in the text $P(0,T)=e^{-0.1\times1}=0.9048$ and $P(0,s)=e^{-0.1\times5}=0.6065$. Also

$$\sigma_P=\frac{0.01}{0.08}(1-e^{-4\times0.08})\sqrt{\frac{1-e^{-2\times0.08\times1}}{2\times0.08}}=0.0329$$

and $h=-0.4192$ so that the call price is

$$100\times0.6065N(h)-68\times0.9048N(h-\sigma_P)=0.439$$

32.8

This problem is similar to Problem 32.5. The difference is that the Hull–White model, which fits an initial term structure, is used instead of Vasicek's model where the initial term structure is determined by the model.

The yield curve is flat with a continuously compounded rate of 5.9118%.

As explained in Section 32.2, the first stage is to calculate the value of r at time 2.1 years which is such that the value of the bond at that time is 99. Denoting this value of r by r^*, we must solve

$$2.5A(2.1,2.5)e^{-B(2.1,2.5)r^*}+102.5A(2.1,3.0)e^{-B(2.1,3.0)r^*}=99$$

where the A and B functions are given by equations (32.7) and (32.8). In this case, A(2.1, 2.5) = 0.999732, A(2.1,3.0) = 0.998656, B(2.1,2.5) = 0.396027, and B(2.1, 3.0) = 0.88005. and Solver shows that $r*=0.066244$. Since

$$2.5A(2.1,2.5)e^{-B(2.1,2.5)\times r^*} = 2.434614$$

and

$$102.5A(2.1,3.0)e^{-B(2.1,3.0)\times r^*} = 96.56539$$

the call option on the coupon-bearing bond can be decomposed into a call option with a strike price of 2.434614 on a bond that pays off 2.5 at time 2.5 years and a call option with a strike price of 96.56539 on a bond that pays off 102.5 at time 3.0 years.

The options are valued using equation (32.10).

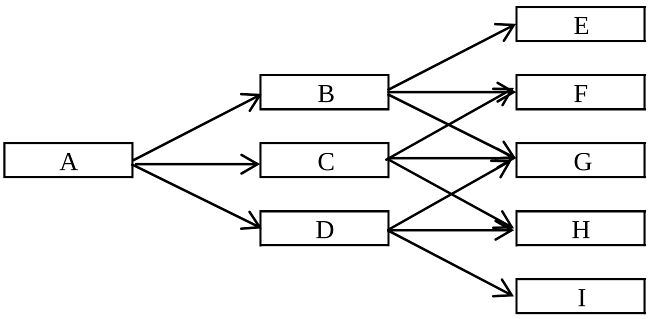
For the first option, $L = 2.5$, $K = 2.434614$, $T = 2.1$, and $s = 2.5$. Also,
$P(0,T) = \exp(-0.059118 \times 2.1) = 0.88325$ and $P(0,s) = \exp(-0.059118 \times 2.5) = 0.862609$.
Furthermore, $\sigma_P = 0.008176$ and $h = 0.353374$. so that the option price is 0.010523.

For the second option, $L = 102.5$, $K = 96.56539$, $T = 2.1$, and $s = 3.0$. Also,
$P(0,T) = \exp(-0.059118 \times 2.1) = 0.88325$ and $P(0,s) = \exp(-0.059118 \times 3.0) = 0.837484$.
Furthermore, $\sigma_P = 0.018168$ and $h = 0.363366$. so that the option price is 0.934074.

The total value of the option is therefore 0.010523 + 0.934074 = 0.944596.

32.9

The time step, Δt, is 1 so that $\Delta r = 0.015\sqrt{3} = 0.02598$. Also $j_{\max} = 4$ showing that the branching method should change four steps from the center of the tree. With only three steps, we never reach the point where the branching changes. The tree is shown in Figure S32.1.



Node	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
r	10.00%	12.61%	10.01%	7.41%	15.24%	12.64%	10.04%	7.44%	4.84%
p_u	0.1667	0.1429	0.1667	0.1929	0.1217	0.1429	0.1667	0.1929	0.2217
p_m	0.6666	0.6642	0.6666	0.6642	0.6567	0.6642	0.6666	0.6642	0.6567
p_d	0.1667	0.1929	0.1667	0.1429	0.2217	0.1929	0.1667	0.1429	0.1217

Figure S32.1: *Tree for Problem 32.9*

32.10

A two-year zero-coupon bond pays off \$100 at the ends of the final branches. At node B, it is worth $100e^{-0.12\times 1} = 88.69$. At node C, it is worth $100e^{-0.10\times 1} = 90.48$. At node D, it is worth $100e^{-0.08\times 1} = 92.31$. It follows that at node A, the bond is worth

$$(88.69\times 0.25+90.48\times 0.5+92.31\times 0.25)e^{-0.1\times 1}=81.88$$

or $81.88

32.11

A two-year zero-coupon bond pays off $100 at time two years. At node B, it is worth $100e^{-0.06937}$=93.30. At node C, it is worth $100e^{-0.05205}$ = 94.93. At node D, it is worth $100e^{-0.03473}$=96.59. It follows that at node A, the bond is worth

$$(93.30\times 0.167+94.93\times 0.666+96.59\times 0.167)e^{-0.0382\times 1}=91.37$$

or $91.37. Because $91.37=100e^{-0.04512\times 2}$, the price of the two-year bond agrees with the initial term structure.

32.12

An 18-month zero-coupon bond pays off $100 at the final nodes of the tree. At node E, it is worth $100e^{-0.088\times 0.5}=95.70$. At node F, it is worth $100e^{-0.0648\times 0.5}=96.81$. At node G, it is worth $100e^{-0.0477\times 0.5}=97.64$. At node H, it is worth $100e^{-0.0351\times 0.5}=98.26$. At node I, it is worth $100e^{0.0259\times 0.5}=98.71$. At node B, it is worth

$$(0.118\times 95.70+0.654\times 96.81+0.228\times 97.64)e^{-0.0564\times 0.5}=94.17$$

Similarly, at nodes C and D, it is worth 95.60 and 96.68. The value at node A is therefore

$$(0.167\times 94.17+0.666\times 95.60+0.167\times 96.68)e^{-0.0343\times 0.5}=93.92$$

The 18-month zero rate is $0.08-0.05e^{-0.18\times 1.5}=0.0418$. This gives the price of the 18-month zero-coupon bond as $100e^{-0.0418\times 1.5}=93.92$ showing that the tree agrees with the initial term structure.

32.13

The calibration of a one-factor interest rate model involves determining its volatility parameters so that it matches the market prices of actively traded interest rate options as closely as possible.

32.14

From equation (32.6)

$$P(t,t+\Delta t)=A(t,t+\Delta t)e^{-r(t)B(t,t+\Delta t)}$$

Also

$$P(t,t+\Delta t)=e^{-R(t)\Delta t}$$

so that

$$e^{-R(t)\Delta t}=A(t,t+\Delta t)e^{-r(t)B(t,t+\Delta t)}$$

or

$$e^{-r(t)B(t,T)}=\frac{e^{-R(t)B(t,T)\Delta t/B(t,t+\Delta t)}}{A(t,t+\Delta t)^{B(t,T)/B(t,t+\Delta t)}}$$

Hence equation (31.15) is true with

$$\hat{B}(t,T)=\frac{B(t,T)\Delta t}{B(t,t+\Delta t}$$

and

$$\hat{A}(t,T)=\frac{A(t,T)}{A(t,t+\Delta t)^{B(t,T)/B(t,t+\Delta t)}}$$

or

$$\ln \hat{A}(t,T) = \ln A(t,T) - \frac{B(t,T)}{B(t,t+\Delta t)} \ln A(t,t+\Delta t)$$

32.15

Using 10 time steps:

(a) The implied value of σ is 1.12%.

(b) The value of the American option is 0.595.

(c) The implied value of σ is 18.45% and the value of the American option is 0.595. The two models give the same answer providing they are both calibrated to the same European price.

(d) We get a negative interest rate if there are 10 down moves. The probability of this is 0.16667 × 0.16418 × 0.16172 × 0.15928 × 0.15687 × 0.15448 × 0.15212 × 0.14978 × 0.14747 × 0.14518 = 8.3×10^{-9}

(e) The calculation is

$$0.164179 \times 1.7075 \times e^{-0.05288\times0.1} = 0.2789$$

32.16

With 100 time steps, the lognormal model gives prices of 5.585, 2.443, and 0.703 for strike prices of 95, 100, and 105. With 100 time steps, the normal model gives prices of 5.508, 2.522, and 0.895 for the three strike prices, respectively. The normal model gives a heavier left tail and a less heavy right tail than the lognormal model for interest rates. This translates into a less heavy left tail and a heavier right tail for bond prices. The arguments in Chapter 20 show that we expect the normal model to give higher option prices for high strike prices and lower option prices for low strike. This is indeed what we find.

32.17

(a) The results are shown in Figure S32.3.

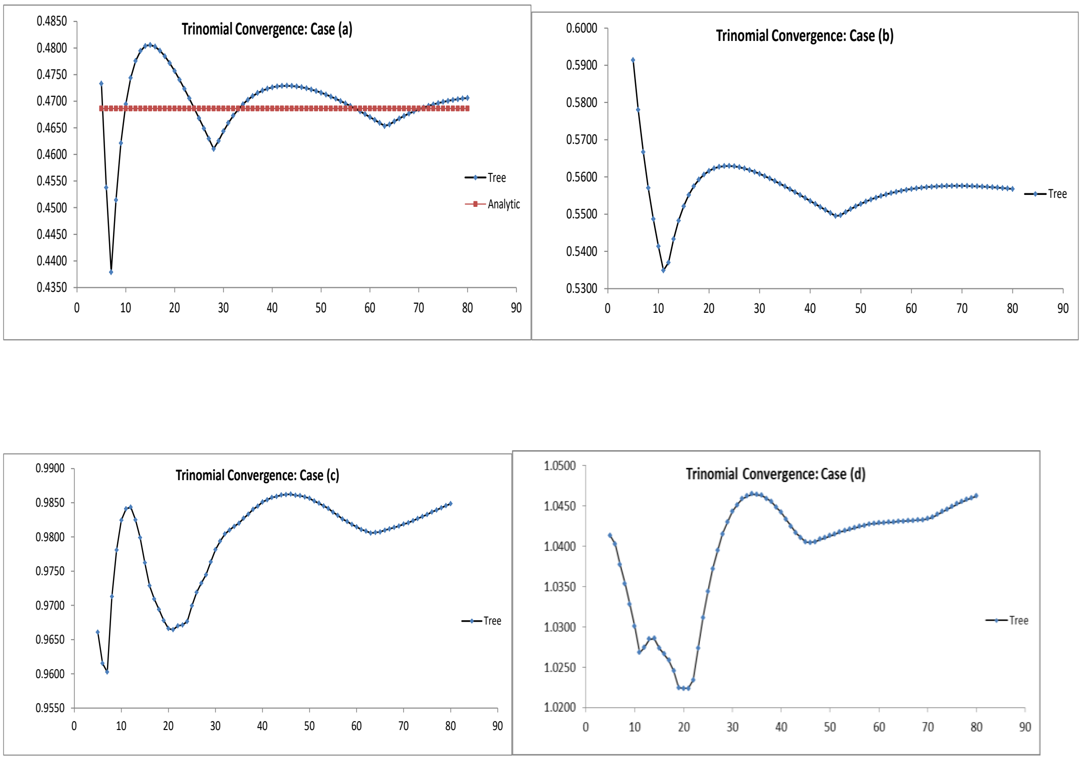


Figure S32.2: *Tree for Problem 32.17*