CHAPTER 23
Estimating Volatilities and Correlations

Practice Questions

23.1
Define u_i as $(S_i - S_{i-1})/S_{i-1}$, where S_i is value of a market variable on day i. In the EWMA model, the variance rate of the market variable (i.e., the square of its volatility) calculated for day n is a weighted average of the u_{n-i}^2's ($i = 1, 2, 3, \ldots$). For some constant λ ($0 < \lambda < 1$), the weight given to u_{n-i-1}^2 is λ times the weight given to u_{n-i}^2. The volatility estimated for day n, σ_n, is related to the volatility estimated for day $n-1$, σ_{n-1}, by

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

This formula shows that the EWMA model has one very attractive property. To calculate the volatility estimate for day n, it is sufficient to know the volatility estimate for day $n-1$ and u_{n-1}.

23.2
The EWMA model produces a forecast of the daily variance rate for day n which is a weighted average of (i) the forecast for day $n-1$, and (ii) the square of the proportional change on day $n-1$. The GARCH (1,1) model produces a forecast of the daily variance for day n which is a weighted average of (i) the forecast for day $n-1$, (ii) the square of the proportional change on day $n-1$, and (iii) a long run average variance rate. GARCH (1,1) adapts the EWMA model by giving some weight to a long run average variance rate. Whereas the EWMA has no mean reversion, GARCH (1,1) is consistent with a mean-reverting variance rate model.

23.3
In this case, $\sigma_{n-1} = 0.015$ and $u_n = 0.5/30 = 0.01667$, so that equation (23.7) gives

$$\sigma_n^2 = 0.94 \times 0.015^2 + 0.06 \times 0.01667^2 = 0.0002281$$

The volatility estimate on day n is therefore $\sqrt{0.0002281} = 0.015103$ or 1.5103%.

23.4
Reducing λ from 0.95 to 0.85 means that more weight is put on recent observations of u_i^2 and less weight is given to older observations. Volatilities calculated with $\lambda = 0.85$ will react more quickly to new information and will "bounce around" much more than volatilities calculated with $\lambda = 0.95$.

23.5
The volatility per day is $30/\sqrt{252} = 1.89\%$. There is a 99% chance that a normally distributed variable will be within 2.57 standard deviations. We are therefore 99% confident that the daily change will be less than $2.57 \times 1.89 = 4.86\%$.

23.6

The weight given to the long-run average variance rate is $1-\alpha-\beta$ and the long-run average variance rate is $\omega/(1-\alpha-\beta)$. Increasing ω increases the long-run average variance rate; increasing α increases the weight given to the most recent data item, reduces the weight given to the long-run average variance rate, and increases the level of the long-run average variance rate. Increasing β increases the weight given to the previous variance estimate, reduces the weight given to the long-run average variance rate, and increases the level of the long-run average variance rate.

23.7

The proportional daily change is $-0.005/1.5000=-0.003333$. The current daily variance estimate is $0.006^2=0.000036$. The new daily variance estimate is

$$0.9\times 0.000036+0.1\times 0.003333^2=0.000033511$$

The new volatility is the square root of this. It is 0.00579 or 0.579%.

23.8

With the usual notation u_{n-1} = 20/3040 = 0.006579 so that the new variance is

$$0.000002 + 0.06 \times 0.006579^2 + 0.92 \times 0.01^2 = 0.00009660$$

so that σ_n = 0.00983. The new volatility estimate is therefore 0.983% per day.

23.9

(a) The volatilities and correlation imply that the current estimate of the covariance is $0.25\times 0.016\times 0.025=0.0001$.

(b) If the prices of the assets at close of trading are \$20.5 and \$40.5, the proportional changes are $0.5/20=0.025$ and $0.5/40=0.0125$. The new covariance estimate is

$$0.95\times 0.0001+0.05\times 0.025\times 0.0125=0.0001106$$

The new variance estimate for asset A is

$$0.95\times 0.016^2+0.05\times 0.025^2=0.00027445$$

so that the new volatility is 0.0166. The new variance estimate for asset B is

$$0.95\times 0.025^2+0.05\times 0.0125^2=0.000601562$$

so that the new volatility is 0.0245. The new correlation estimate is

$$\frac{0.0001106}{0.0166\times 0.0245}=0.272$$

23.10

The long-run average variance rate is $\omega/(1-\alpha-\beta)$ or $0.000004/0.03=0.0001333$. The long-run average volatility is $\sqrt{0.0001333}$ or 1.155%. The equation describing the way the variance rate reverts to its long-run average is equation (23.13)

$$E[\sigma_{n+k}^2]=V_L+(\alpha+\beta)^k(\sigma_n^2-V_L)$$

In this case,

$$E[\sigma_{n+k}^2]=0.0001333+0.97^k(\sigma_n^2-0.0001333)$$

If the current volatility is 20% per year, $\sigma_n=0.2/\sqrt{252}=0.0126$. The expected variance rate in 20 days is

$$0.0001333+0.97^{20}(0.0126^2-0.0001333)=0.0001471$$

The expected volatility in 20 days is therefore $\sqrt{0.0001471} = 0.0121$ or 1.21% per day.

23.11

Using the notation in the text $\sigma_{u,n-1} = 0.01$ and $\sigma_{v,n-1} = 0.012$ and the most recent estimate of the covariance between the asset returns is $\text{cov}_{n-1} = 0.01 \times 0.012 \times 0.50 = 0.00006$. The variable $u_{n-1} = 1/30 = 0.03333$ and the variable $v_{n-1} = 1/50 = 0.02$. The new estimate of the covariance, cov_n, is

$$0.000001 + 0.04 \times 0.03333 \times 0.02 + 0.94 \times 0.00006 = 0.0000841$$

The new estimate of the variance of the first asset, $\sigma_{u,n}^2$ is

$$0.000003 + 0.04 \times 0.03333^2 + 0.94 \times 0.01^2 = 0.0001414$$

so that $\sigma_{u,n} = \sqrt{0.0001414} = 0.01189$ or 1.189%. The new estimate of the variance of the second asset, $\sigma_{v,n}^2$ is

$$0.000003 + 0.04 \times 0.02^2 + 0.94 \times 0.012^2 = 0.0001544$$

so that $\sigma_{v,n} = \sqrt{0.0001544} = 0.01242$ or 1.242%. The new estimate of the correlation between the assets is therefore 0.0000841/(0.01189 × 0.01242) = 0.569.

23.12

The FTSE expressed in dollars is XY where X is the FTSE expressed in sterling and Y is the exchange rate (value of one pound in dollars). Define x_i as the proportional change in X on day i and y_i as the proportional change in Y on day i. The proportional change in XY is approximately $x_i + y_i$. The standard deviation of x_i is 0.018 and the standard deviation of y_i is 0.009. The correlation between the two is 0.4. The variance of $x_i + y_i$ is therefore

$$0.018^2 + 0.009^2 + 2 \times 0.018 \times 0.009 \times 0.4 = 0.0005346$$

so that the volatility of $x_i + y_i$ is 0.0231 or 2.31%. This is the volatility of the FTSE expressed in dollars. Note that it is greater than the volatility of the FTSE expressed in sterling. This is the impact of the positive correlation. When the FTSE increases, the value of sterling measured in dollars also tends to increase. This creates an even bigger increase in the value of FTSE measured in dollars. Similarly, for a decrease in the FTSE.

23.13

Continuing with the notation in Problem 23.12, define z_i as the proportional change in the value of the S&P 500 on day i. The covariance between x_i and z_i is $0.7 \times 0.018 \times 0.016 = 0.0002016$. The covariance between y_i and z_i is $0.3 \times 0.009 \times 0.016 = 0.0000432$. The covariance between and z_i equals the covariance between x_i and z_i plus the covariance between $x_i + y_i$ y_i and z_i. It is

$$0.0002016 + 0.0000432 = 0.0002448$$

The correlation between $x_i + y_i$ and z_i is

$$\frac{0.0002448}{0.016 \times 0.0231} = 0.662$$

Note that the volatility of the S&P 500 drops out in this calculation.

23.14

$$\sigma_n^2 = \omega V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

so that

$$\sigma_n^2 = (1-\alpha-\beta)V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$
$$\sigma_n^2 - \sigma_{n-1}^2 = (1-\alpha-\beta)(V_L - \sigma_{n-1}^2) + \alpha(u_{n-1}^2 - \sigma_{n-1}^2)$$

The variable u_{n-1}^2 has a mean of σ_{n-1}^2 and a variance of

$$E(u_{n-1})^4 - [E(u_{n-1}^2)]^2 = 2\sigma_{n-1}^4$$

The standard deviation of u_{n-1}^2 is $\sqrt{2}\sigma_{n-1}^2$.

We can write $\Delta V = \sigma_n^2 - \sigma_{n-1}^2$ and $V = \sigma_{n-1}^2$. Substituting for u_{n-1}^2 into the equation for $\sigma_n^2 - \sigma_{n-1}^2$, we get

$$\Delta V = a(V_L - V) + Z$$

where Z is a variable with mean zero and standard deviation $\alpha\sqrt{2}V$. This equation defines the change in the variance over one day. It is consistent with the stochastic process

$$dV = a(V_L - V)dt + \alpha\sqrt{2}Vdz$$

or

$$dV = a(V_L - V)dt + \xi Vdz$$

when time is measured in days.

Discretizing the process, we obtain

$$\Delta V = a(V_L - V)\Delta t + \xi V\varepsilon\sqrt{\Delta t}$$

where ε is a random sample from a standard normal distribution.

Note that we are not assuming Z is normally distributed. It is the sum of many small changes $\xi V\varepsilon\sqrt{\Delta t}$.

When time is measured in years,

$$\Delta V = a(V_L - V)252\Delta t + \xi V\varepsilon\sqrt{252}\sqrt{\Delta t}$$

and the process for V is

$$dV = 252a(V_L - V)\,dt + \xi V\sqrt{252}\,dz$$

23.15 See Excel file

The worksheet monitors variances and covariances using EWMA setting initial values equal to the value calculated in the usual way from data. VaR is $302,459 while ES is $346,516.

23.16

The parameter λ is in cell N3 of the EWMA worksheet for the previous problem is changed to 0.97. VaR increases to $393,300 and ES increases to $450,590.

23.17

The proportional change in the price of gold is $-4/600 = -0.00667$. Using the EWMA model, the variance is updated to

$$0.94 \times 0.013^2 + 0.06 \times 0.00667^2 = 0.00016153$$

so that the new daily volatility is $\sqrt{0.00016153} = 0.01271$ or 1.271% per day. Using GARCH (1,1), the variance is updated to

$$0.000002 + 0.94 \times 0.013^2 + 0.04 \times 0.00667^2 = 0.00016264$$

so that the new daily volatility is $\sqrt{0.00016264} = 0.1275$ or 1.275% per day.

23.18

The proportional change in the price of silver is zero. Using the EWMA model, the variance is updated to

$$0.94 \times 0.015^2 + 0.06 \times 0 = 0.0002115$$

so that the new daily volatility is $\sqrt{0.0002115} = 0.01454$ or 1.454% per day. Using GARCH (1,1), the variance is updated to

$$0.000002 + 0.94 \times 0.015^2 + 0.04 \times 0 = 0.0002135$$

so that the new daily volatility is $\sqrt{0.0002135} = 0.01461$ or 1.461% per day. The initial covariance is $0.8 \times 0.013 \times 0.015 = 0.000156$. Using EWMA, the covariance is updated to

$$0.94 \times 0.000156 + 0.06 \times 0 = 0.00014664$$

so that the new correlation is $0.00014664/(0.01454 \times 0.01271) = 0.7934$. Using GARCH (1,1), the covariance is updated to

$$0.000002 + 0.94 \times 0.000156 + 0.04 \times 0 = 0.00014864$$

so that the new correlation is $0.00014864/(0.01461 \times 0.01275) = 0.7977$.

For a given α and β, the ω parameter defines the long run average value of a variance or a covariance. There is no reason why we should expect the long run average daily variance for gold and silver should be the same. There is also no reason why we should expect the long run average covariance between gold and silver to be the same as the long run average variance of gold or the long run average variance of silver. In practice, therefore, we are likely to want to allow ω in a GARCH(1,1) model to vary from market variable to market variable. (Some instructors may want to use this problem as a lead in to multivariate GARCH models.)

23.19 (Excel file)

In the spreadsheet, the first 25 observations on $(v_i-\beta_i)^2$ are ignored so that the results are not unduly influenced by the choice of starting values. The best values of λ for EUR, CAD, GBP and JPY were found to be 0.947, 0.898, 0.950, and 0.984, respectively. The best values of λ for S&P500, NASDAQ, FTSE100, and Nikkei225 were found to be 0.874, 0.901, 0.904, and 0.953, respectively.

23.20 (Excel file)

As the spreadsheets show, the optimal value of λ in the EWMA model is 0.958 and the log likelihood objective function is 11,806.4767. In the GARCH (1,1) model, the optimal values of ω, α, and β are 0.0000001330, 0.04447, and0.95343, respectively. The long-run average daily volatility is 0.7954% and the log likelihood objective function is 11,811.1955.