CHAPTER 22
Value at Risk

Practice Questions

22.1
The standard deviation of the daily change in the investment in each asset is $1,000. The variance of the portfolio's daily change is

$$1,000^2 + 1,000^2 + 2 \times 0.3 \times 1,000 \times 1,000 = 2,600,000$$

The standard deviation of the portfolio's daily change is the square root of this or $1,612.45. The standard deviation of the 5-day change is

$$1,612.45 \times \sqrt{5} = \$3,605.55$$

Because $N^{-1}(0.01) = 2.326$, 1% of a normal distribution lies more than 2.326 standard deviations below the mean. The 5-day 99 percent value at risk is therefore 2.326×3,605.55 = $8,388. The 5-day 99% ES is

$$\frac{3605.55 \times e^{-2.326^2/2}}{\sqrt{2\pi} \times 0.01} = 9,617$$

22.2
The three alternative procedures mentioned in the chapter for handling interest rates when the model building approach is used to calculate VaR involve (a) the use of the duration model, (b) the use of cash flow mapping, and (c) the use of principal components analysis. When historical simulation is used, we need to assume that the change in the zero-coupon yield curve between Day m and Day $m+1$ is the same as that between Day i and Day $i+1$ for different values of i. In the case of a LIBOR, the zero curve is usually calculated from deposit rates, Eurodollar futures quotes, and swap rates. We can assume that the percentage change in each of these between Day m and Day $m+1$ is the same as that between Day i and Day $i+1$. In the case of a Treasury curve, it is usually calculated from the yields on Treasury instruments. Again, we can assume that the percentage change in each of these between Day m and Day $m+1$ is the same as that between Day i and Day $i+1$.

22.3
The approximate relationship between the daily change in the portfolio value, ΔP, and the daily change in the exchange rate, ΔS, is

$$\Delta P = 56 \Delta S$$

The percentage daily change in the exchange rate, Δx, equals $\Delta S / 1.5$. It follows that

$$\Delta P = 56 \times 1.5 \Delta x$$

or

$$\Delta P = 84 \Delta x$$

The standard deviation of Δx equals the daily volatility of the exchange rate, or 0.7 percent. The standard deviation of ΔP is therefore $84 \times 0.007 = 0.588$. It follows that the 10-day 99 percent VaR for the portfolio is

$$0.588 \times 2.33 \times \sqrt{10} = 4.33$$

22.4
The relationship is

$$\Delta P = 56 \times 1.5\Delta x + \frac{1}{2} \times 1.5^2 \times 16.2 \times \Delta x^2$$

or

$$\Delta P = 84\Delta x + 18.225\Delta x^2$$

22.5
The factors calculated from a principal components analysis are uncorrelated. The daily variance of the portfolio is

$$6^2 \times 20^2 + 4^2 \times 8^2 = 15,424$$

and the daily standard deviation is $\sqrt{15,424} = \$124.19$. Since $N(-1.282) = 0.9$, the 5-day 90% value at risk is (assuming factors are normally distributed)

$$124.19 \times \sqrt{5} \times 1.282 = \$356.01$$

22.6
The linear model assumes that the percentage daily change in each market variable has a normal probability distribution. The historical simulation model assumes that the probability distribution observed for the percentage daily changes in the market variables in the past is the probability distribution that will apply over the next day.

22.7
The forward contract can be regarded as the exchange of a foreign zero-coupon bond for a domestic zero-coupon bond. Each of these can be mapped in zero-coupon bonds with standard maturities.

22.8
Value at risk is the loss that is expected to be exceeded $(100 - X)\%$ of the time in N days for specified parameter values, X and N. Expected shortfall is the expected loss conditional that the loss is greater than the Value at Risk.

22.9
The change in the value of an option is not linearly related to the change in the value of the underlying variables. When the change in the values of underlying variables is normal, the change in the value of the option is non-normal. The linear model assumes that it is normal and is, therefore, only an approximation.

22.10
The contract is a long position in a sterling bond combined with a short position in a dollar bond. The value of the sterling bond is $1.53e^{-0.05\times0.5}$ or \$1.492 million. The value of the dollar bond is $1.5e^{-0.05\times0.5}$ or \$1.463 million. The variance of the change in the value of the contract in one day is

$$1.492^2 \times 0.0006^2 + 1.463^2 \times 0.0005^2 - 2 \times 0.8 \times 1.492 \times 0.0006 \times 1.463 \times 0.0005$$
$$= 0.000000288$$

The standard deviation is therefore \$0.000537 million. The 10-day 99% VaR is $0.000537 \times \sqrt{10} \times 2.33 = \$0.00396$ million.

22.11
If we assume only one factor, the model is

$$\Delta P = -1.99 f_1$$

The standard deviation of f_1 is 11.54. The standard deviation of ΔP is therefore $1.99 \times 11.54 = 22.965$ and the 1-day 99 percent value at risk is $22.965\times2.326=53.42$. If we assume three factors, our exposure to the third factor is

$$10 \times (0.376) + 4 \times (0.006) - 8 \times (-0.332) - 7 \times (-0.349) + 2 \times (-0.153) = 8.58$$

The model is therefore,

$$\Delta P = -1.99f_1 - 3.06f_2 + 8.58f_3$$

The variance of ΔP is

$$1.99^2 \times 11.54^2 + 3.06^2 \times 3.55^2 + 8.58^2 \times 1.78^2 = 878.62$$

The standard deviation of ΔP is the square root of this or 29.64 and the 1-day 99% value at risk is $29.64\times 2.326 = \$68.95$.

The example illustrates that the relative importance of different factors depends on the portfolio being considered. Normally, the second factor is more important than the third, but in this case it is less important.

22.12

The delta of the options is the rate of change of the value of the options with respect to the price of the asset. When the asset price increases by a small amount, the value of the options decrease by 30 times this amount. The gamma of the options is the rate of change of their delta with respect to the price of the asset. When the asset price increases by a small amount, the delta of the portfolio decreases by five times this amount.

By entering 20 for S, 1% for the volatility per day, −30 for delta, −5 for gamma, and recomputing, we see that $E(\Delta P) = -0.10$, $E(\Delta P^2) = 36.03$, and $E(\Delta P^3) = -32.415$. The 1-day, 99% VaR given by the software for the quadratic approximation is 14.5. This is a 99% 1-day VaR. The VaR is calculated using the formulas in footnote 9 and the results in Technical Note 10.

22.13

Define σ as the volatility per year, $\Delta\sigma$ as the change in σ in one day, and Δw and the proportional change in σ in one day. We measure in σ as a multiple of 1% so that the current value of σ is $1\times\sqrt{252} = 15.87$. The delta-gamma-vega model is

$$\Delta P = -30\Delta S - .5\times5\times(\Delta S)^2 - 2\Delta\sigma$$

or

$$\Delta P = -30\times20\Delta x - 0.5\times5\times20^2(\Delta x)^2 - 2\times15.87\Delta w$$

which simplifies to

$$\Delta P = -600\Delta x - 1{,}000(\Delta x)^2 - 31.74\Delta w$$

The change in the portfolio value now depends on two market variables. Once the daily volatility of σ and the correlation between σ and S have been estimated, we can estimate moments of ΔP and use a Cornish–Fisher expansion.

22.14

The 95% one-day VaR is the 25th worst loss. This is \$163,620. The 95% one-day ES is the average of the 25 largest losses. It is \$323,690. The 97% one-day VaR is the 15th worst loss. This is \$229,683. The 97% one-day ES is the average of the 15 largest losses. It is \$415,401. The model building approach gives the 95% one-day VaR as \$197,425 and the 95% one-day ES as \$247,579. The model building approach gives the 97% one-day VaR as \$225,744 and the 95% one-day ES as \$272,226.

22.15

For the historical simulation approach, in the "Scenarios" worksheet the portfolio investments are changed to 2,500 in cells L2:O2. The losses are then sorted from the largest to the smallest. The fifth worst loss is \$394,437. This is the one-day 99% VaR. The average of the five worst losses is \$633,716. This is the one-day 99% ES. For the model building approach, we make a similar change to the equal weights sheet and find that the value at risk is \$275,757 while the expected shortfall is \$315,926.

22.16

The change in the value of the portfolio for a small change Δy in the yield is approximately $-DB\Delta y$ where D is the duration and B is the value of the portfolio. It follows that the standard deviation of the daily change in the value of the bond portfolio equals $DB\sigma_y$ where σ_y is the standard deviation of the daily change in the yield. In this case, $D = 5.2$, $B = 6{,}000{,}000$, and $\sigma_y = 0.0009$ so that the standard deviation of the daily change in the value of the bond portfolio is

$$5.2 \times 6{,}000{,}000 \times 0.0009 = 28{,}080$$

The 20-day 90% VaR for the portfolio is $1.282 \times 28{,}080 \times \sqrt{20} = 160{,}990$ or \$160,990. This approach assumes that only parallel shifts in the term structure can take place. Equivalently, it assumes that all rates are perfectly correlated or that only one factor drives term structure movements. Alternative more accurate approaches described in the chapter are (a) cash flow mapping, and (b) a principal components analysis.

22.17

An approximate relationship between the daily change in the value of the portfolio, ΔP and the proportional daily change in the value of the asset Δx is

$$\Delta P = 10 \times 12 \Delta x = 120 \Delta x$$

The standard deviation of Δx is 0.02. It follows that the standard deviation of ΔP is 2.4. The 1-day 95% VaR is $2.4 \times 1.65 = \$3.96$. The quadratic relationship is

$$\Delta P = 10 \times 12 \Delta x + 0.5 \times 10^2 \times (-2.6) \Delta x^2$$

or

$$\Delta P = 120 \Delta x - 130 \Delta x^2$$

This could be used in conjunction with Monte Carlo simulation. We would sample values for Δx and use this equation to convert the Δx samples to ΔP samples.

22.18

The cash flows are as follows:

Year	1	2	3	4	5
2-yr bond	5	105			
3-yr bond	5	5	105		
5-yr bond	−5	−5	−5	−5	−105
Total	5	105	100	−5	−105
Present Value	4.756	95.008	86.071	−4.094	−81.774
Impact of 1bp change	−0.0005	−0.0190	−0.0258	0.0016	0.0409

The duration relationship is used to calculate the last row of the table. When the one-year rate increases by one basis point, the value of the cash flow in year 1 decreases by 1 × 0.0001 × 4.756 = 0.0005; when the two year rate increases by one basis point, the value of the cash flow in year 2 decreases by 2×0.0001×95.008 = 0.0190; and so on.

The sensitivity to the first factor is
−0.0005 × 0.083 − 0.0190 × 0.210 − 0.0258 × 0.286+ 0.0016 × 0.336 + 0.0409 × 0.386 or 0.004915. (We assume that PC1 for 4 years is the average of that for 3 and 5 years.) Similarly, the sensitivity to the second and third factors are 0.007496 and −0.02148. Assuming one factor, the standard deviation of the one-day change in the portfolio value is 0.004915 × 11.54 = 0.05672. The 20-day 95% VaR is therefore 0.05672 × 1.645 $\sqrt{20}$ = 0.417. Assuming two factors, the variance of the one-day change in the portfolio value is

$$0.004915^2 \times 11.54^2 + 0.007496^2 \times 3.55^2 = 0.003925$$

so that the standard deviation is 0.06267.
The 20-day 95% VaR is therefore 0.06267 × 1.645 $\sqrt{20}$ = 0.461.
Assuming three factors, the variance of the one-day change in the portfolio value is

$$0.004915^2 \times 11.54^2 + 0.007496^2 \times 3.55^2 + 0.02148^2 \times 1.78^2 = 0.005387$$

so that the standard deviation is 0.07339.
The 20-day 95% VaR is therefore 0.07339 × 1.645 $\sqrt{20}$ = 0.540.

22.19
This assignment is useful for consolidating students' understanding of alternative approaches to calculating VaR, but it is calculation intensive. Realistically, students need some programming skills to make the assignment feasible. My answer follows the usual practice of assuming that the 10-day 99% value at risk is $\sqrt{10}$ times the 1-day 99% value at risk. Some students may try to calculate a 10-day VaR directly, which is fine.

(a) From DerivaGem, the values of the two option positions are –5.413 and –1.014. The deltas are –0.589 and 0.284, respectively. An approximate linear model relating the change in the portfolio value to proportional change, Δx_1, in the first stock price and the proportional change, Δx_2, in the second stock price is

$$\Delta P = -0.589 \times 50 \Delta x_1 + 0.284 \times 20 \Delta x_2$$

or

$$\Delta P = -29.45 \Delta x_1 + 5.68 \Delta x_2$$

The daily volatility of the two stocks are $0.28/\sqrt{252} = 0.0176$ and $0.25/\sqrt{252} = 0.0157$, respectively. The one-day variance of ΔP is

$$29.45^2 \times 0.0176^2 + 5.68^2 \times 0.0157^2 - 2 \times 29.45 \times 0.0176 \times 5.68 \times 0.0157 \times 0.4 = 0.2396$$

The one day standard deviation is, therefore, 0.4895 and the 10-day 99% VaR is $2.33 \times \sqrt{10} \times 0.4895 = 3.61$.

(b) In the partial simulation approach, we simulate changes in the stock prices over a one-day period (building in the correlation) and then use the quadratic approximation to calculate the change in the portfolio value on each simulation trial. The one percentile point of the probability distribution of portfolio value changes turns out to be 1.22. The 10-day 99% value at risk is, therefore, $1.22\sqrt{10}$ or about 3.86.

(c) In the full simulation approach, we simulate changes in the stock price over one-day (building in the correlation) and revalue the portfolio on each simulation trial. The results are very similar to (b) and the estimate of the 10-day 99% value at risk is about

3.86.

22.20

If the loss has mean μ and standard deviation σ, VaR with 99% confidence is $\mu + 2.326\sigma$.
ES with 97.5% confidence is

$$\mu + \frac{\sigma e^{-1.96^2/2}}{\sqrt{2\pi} \times 0.025} = \mu + 2.337\sigma$$