CHAPTER 27
More on Models and Numerical Procedures

Practice Questions

27.1
It follows immediately from the equations in Section 27.1 that

$$p - c = Ke^{-rT} - S_0 e^{-qT}$$

in all cases.

27.2
In this case, $\lambda' = 0.3\times1.5=0.45$. The variable f_n is the Black–Scholes–Merton price when the variance rate is $0.25^2+0.25n = 0.0625+0.25n$ and the risk-free rate is $-0.1+n\times\ln(1.5)=-0.1+0.4055n$. A spreadsheet can be constructed to value the option using the first (say) 20 terms in the Merton expansion.

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} f_n$$

The result is 5.47 which is also the price given by DerivaGem.

27.3
With the notation in the text the value of a call option, c is

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} c_n$$

where c_n is the Black–Scholes–Merton price of a call option where the variance rate is

$$\sigma^2 + \frac{ns^2}{T}$$

and the risk-free rate is

$$r - \lambda k + \frac{n\gamma}{T}$$

where $\gamma = \ln(1+k)$. Similarly, the value of a put option p is

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} p_n$$

where p_n is the Black–Scholes–Merton price of a put option with this variance rate and risk-free rate. It follows that

$$p - c = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} (p_n - c_n)$$

From put–call parity

$$p_n - c_n = Ke^{(-r+\lambda k)T} e^{-n\gamma} - S_0 e^{-qT}$$

Because

$$e^{-n\gamma} = (1+k)^{-n}$$

it follows that

$$p - c = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T + \lambda k T} (\lambda' T / (1+k))^n}{n!} K e^{-rT} - \sum_{n=0}^{\infty} \frac{e^{-\lambda' T} (\lambda' T)^n}{n!} S_0 e^{-qT}$$

Using $\lambda' = \lambda(1+k)$ this becomes

$$\frac{1}{e^{\lambda T}} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} K e^{-rT} - \frac{1}{e^{\lambda' T}} \sum_{n=0}^{\infty} \frac{(\lambda' T)^n}{n!} S_0 e^{-qT}$$

From the expansion of the exponential function, we get

$$e^{\lambda T} = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!}$$

$$e^{\lambda' T} = \sum_{n=0}^{\infty} \frac{(\lambda' T)^n}{n!}$$

Hence,

$$p - c = K e^{-rT} - S_0 e^{-qT}$$

showing that put–call parity holds.

27.4

The average variance rate is

$$\frac{6 \times 0.2^2 + 6 \times 0.22^2 + 12 \times 0.24^2}{24} = 0.0509$$

The volatility used should be $\sqrt{0.0509} = 0.2256$ or 22.56%.

27.5

In a risk-neutral world, the process for the asset price exclusive of jumps is

$$\frac{dS}{S} = (r - q - \lambda k)\, dt + \sigma\, dz$$

In this case, $k = -1$ so that the process is

$$\frac{dS}{S} = (r - q + \lambda)\, dt + \sigma\, dz$$

The asset behaves like a stock paying a dividend yield of $q - \lambda$. This shows that, conditional on no jumps, call price

$$S_0 e^{-(q-\lambda)T} N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0 / K) + (r - q + \lambda + \sigma^2 / 2)T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

There is a probability of $e^{-\lambda T}$ that there will be no jumps and a probability of $1 - e^{-\lambda T}$ that there will be one or more jumps so that the final asset price is zero. It follows that there is a probability of $e^{-\lambda T}$ that the value of the call is given by the above equation and $1-e^{-\lambda T}$ that it will be zero. Because jumps have no systematic risk, it follows that the value of the call option is

$$e^{-\lambda T}\left[S_0 e^{-(q-\lambda)T}N(d_1)-Ke^{-rT}N(d_2)\right]$$

or

$$S_0 e^{-qT}N(d_1)-Ke^{-(r+\lambda)T}N(d_2)$$

This is the required result. The value of a call option is an increasing function of the risk-free interest rate (see Chapter 11). It follows that the possibility of jumps increases the value of the call option in this case.

27.6

Suppose that S_1 is the stock price at time t_1 and S_T is the stock price at time T. From equation (15.3), it follows that in a risk- neutral world:

$$\ln S_1-\ln S_0 \sim \varphi\left[\left(r_1-\frac{\sigma_1^2}{2}\right)t_1,\sigma_1^2 t_1\right]$$

$$\ln S_T-\ln S_1 \sim \varphi\left[\left(r_2-\frac{\sigma_2^2}{2}\right)t_2,\sigma_2^2 t_2\right]$$

Since the sum of two independent normal distributions is normal with mean equal to the sum of the means and variance equal to the sum of the variances,

$$\begin{aligned}\ln S_T-\ln S_0 &= (\ln S_T-\ln S_1)+(\ln S_1-\ln S_0)\\ &\sim \varphi\left(r_1t_1+r_2t_2-\frac{\sigma_1^2t_1}{2}-\frac{\sigma_2^2t_2}{2},\sigma_1^2t_1+\sigma_2^2t_2\right)\end{aligned}$$

(a) Because

$$r_1t_1+r_2t_2=\bar{r}T$$

and

$$\sigma_1^2t_1+\sigma_2^2t_2=\bar{V}T$$

it follows that:

$$\ln S_T-\ln S_0 \sim \varphi\left[\left(\bar{r}-\frac{\bar{V}}{2}\right)T,\bar{V}T\right]$$

(b) If σ_i and r_i are the volatility and risk-free interest rate during the i th subinterval $(i=1,2,3)$, an argument similar to that in (a) shows that:

$$\ln S_T - \ln S_0 \sim \varphi\left(r_1 t_1 + r_2 t_2 + r_3 t_3 - \frac{\sigma_1^2 t_1}{2} - \frac{\sigma_2^2 t_2}{2} - \frac{\sigma_3^2 t_3}{2}, \sigma_1^2 t_1 + \sigma_2^2 t_2 + \sigma_3^2 t_3\right)$$

where t_1, t_2 and t_3 are the lengths of the three subintervals. It follows that the result in (b) is still true.

(c) The result in (b) remains true as the time between time zero and time T is divided into more subintervals, each having its own risk-free interest rate and volatility. In the limit, it follows that, if r and σ are known functions of time, the stock price distribution at time T is the same as that for a stock with a constant interest rate and variance rate with the constant interest rate equal to the average interest rate and the constant variance rate equal to the average variance rate.

27.7

The equations are:

$$S(t+\Delta t) = S(t)\exp[(r - q - V(t)/2)\Delta t + \varepsilon_1\sqrt{V(t)\Delta t}]$$

$$V(t+\Delta t) - V(t) = a[V_L - V(t)]\Delta t + \xi\varepsilon_2 V(t)^\alpha\sqrt{\Delta t}$$

where ε_1 and ε_2 are samples from a standard normal distribution with a correlation equal to the correlation between *S* and *V*.

27.8

The IVF model is designed to match the volatility surface today. There is no guarantee that the volatility surface given by the model at future times will reflect the true evolution of the volatility surface.

27.9

The IVF model ensures that the risk-neutral probability distribution of the asset price at any future time conditional on its value today is correct (or at least consistent with the market prices of options). When a derivative's payoff depends on the value of the asset at only one time the IVF model therefore calculates the expected payoff from the asset correctly.

27.10

In this case, $S_0 = 1.6$, $r = 0.05$, $r_f = 0.08$, $\sigma = 0.15$, $T = 1.5$, $\Delta t = 0.5$. This means that

$$u = e^{0.15\sqrt{0.5}} = 1.1119$$

$$d = \frac{1}{u} = 0.8994$$

$$a = e^{(0.05-0.08)\times 0.5} = 0.9851$$

$$p = \frac{a-d}{u-d} = 0.4033$$

$$1 - p = 0.5967$$

The option pays off

$$S_T - S_{\min}$$

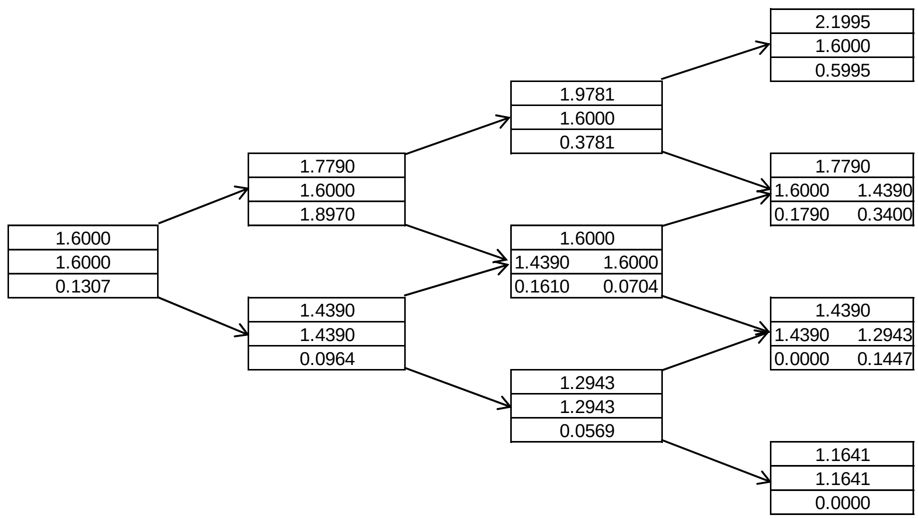

Figure S27.1: *Binomial tree for Problem 27.10*

The tree is shown in Figure S27.1. At each node, the upper number is the exchange rate, the middle number(s) are the minimum exchange rate(s) so far, and the lower number(s) are the value(s) of the option. The tree shows that the value of the option today is 0.131.

27.11

As v tends to zero, the value of g becomes T with certainty. This can be demonstrated using the GAMMADIST function in Excel. By using a series expansion for the $\ln$ function, we see that ω becomes $-\theta T$. In the limit, the distribution of $\ln S_T$ therefore has a mean of $\ln S_0 + (r-q)T$ and a standard deviation of $\sigma\sqrt{T}$ so that the model becomes geometric Brownian motion.

27.12

In this case, $S_0 = 40$, $K = 40$, $r = 0.1$, $\sigma = 0.35$, $T = 0.25$, $\Delta t = 0.08333$. This means that

$$u = e^{0.35\sqrt{0.08333}} = 1.1063$$

$$d = \frac{1}{u} = 0.9039$$

$$a = e^{0.1 \times 0.08333} = 1.008368$$

$$p = \frac{a-d}{u-d} = 0.5161$$

$$1 - p = 0.4839$$

The option pays off

$$40 - \overline{S}$$

where $\overline{S}$ denotes the geometric average. The tree is shown in Figure S27.2. At each node, the upper number is the stock price, the middle number(s) are the geometric average(s), and the lower number(s) are the value(s) of the option. The geometric averages are calculated using the first, the last and all intermediate stock prices on the path. The tree shows that the value of the option today is \$1.40.

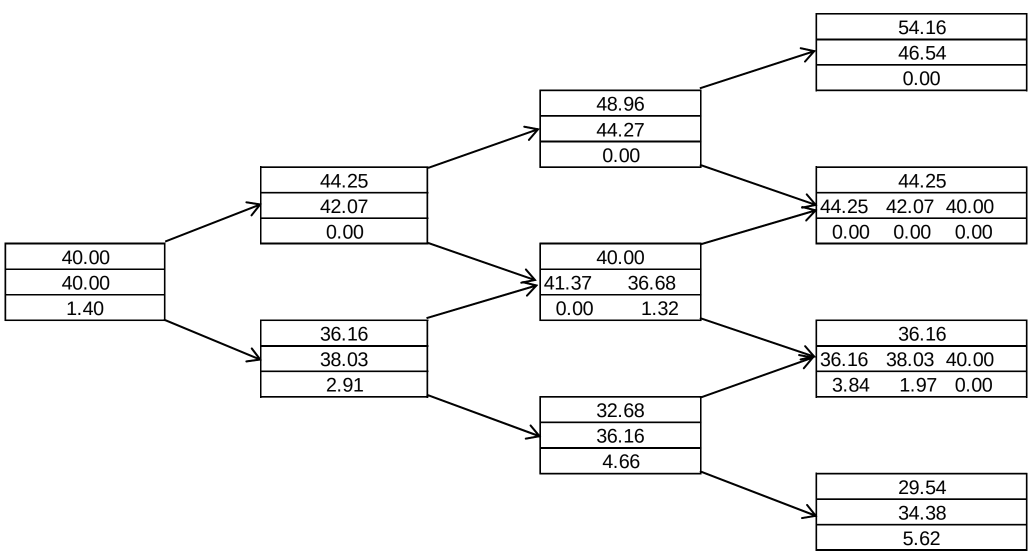

Figure S27.2: *Binomial tree for Problem 27.12*

27.13

As mentioned in Section 27.5, for the procedure to work it must be possible to calculate the value of the path function at time $\tau + \Delta t$ from the value of the path function at time τ and the value of the underlying asset at time $\tau + \Delta t$. When S_{ave} is calculated from time zero until the end of the life of the option (as in the example considered in Section 27.5), this condition is satisfied. When it is calculated over the last three months, it is not satisfied. This is because, in order to update the average with a new observation on S, it is necessary to know the observation on S from three months ago that is now no longer part of the average calculation.

27.14

We consider the situation where the average at node X is 53.83. If there is an up movement to node Y, the new average becomes:

$$\frac{53.83\times 5+54.68}{6}=53.97$$

Interpolating, the value of the option at node Y when the average is 53.97 is

$$\frac{(53.97-51.12)\times 8.635+(54.26-53.97)\times 8.101}{54.26-51.12}=8.586$$

Similarly, if there is a down movement the new average will be

$$\frac{53.83\times 5+45.72}{6}=52.48$$

In this case, the option price is 4.416. The option price at node X when the average is 53.83 is therefore:

$$8.586\times 0.5056+4.416\times 0.4944)e^{-0.1\times 0.05}=6.492$$

27.15

Under the least squares approach, we exercise at time $t=1$ in paths 4, 6, 7, and 8. We exercise at time $t=2$ for none of the paths. We exercise at time $t=3$ for path 3. Under the exercise boundary parameterization approach, we exercise at time $t=1$ for paths 6 and 8. We exercise at time $t=2$ for path 7. We exercise at time $t=3$ for paths 3 and 4. For the paths sampled, the exercise boundary parameterization approach gives a higher value for the option. However, it may be biased upward. As mentioned in the text, once the early exercise boundary has been determined in the exercise boundary parameterization approach, a new Monte Carlo simulation should be carried out.

27.16

If the average variance rate is 0.06, the value of the option is given by Black–Scholes with a volatility of $\sqrt{0.06}=24.495\%$; it is 12.460. If the average variance rate is 0.09, the value of the option is given by Black-Scholes with a volatility of $\sqrt{0.09}=30.000\%$; it is 14.655. If the average variance rate is 0.12, the value of the option is given by Black–Scholes–Merton with a volatility of $\sqrt{0.12}=34.641\%$; it is 16.506. The value of the option is the Black–Scholes–Merton price integrated over the probability distribution of the average variance rate. It is

$$0.2 \times 12.460 + 0.5 \times 14.655 + 0.3 \times 16.506 = 14.77$$

27.17

Suppose that there are two horizontal barriers, H_1 and H_2, with $H_1 < H_2$ and that the underlying stock price follows geometric Brownian motion. In a trinomial tree, there are three possible movements in the asset's price at each node: up by a proportional amount u; stay the same; and down by a proportional amount d where $d = 1/u$. We can always choose u so that nodes lie on both barriers. The condition that must be satisfied by u is

$$H_2 = H_1 u^N$$

or

$$\ln H_2 = \ln H_1 + N \ln u$$

for some integer N.

When discussing trinomial trees in Section 21.4, the value suggested for u was $e^{\sigma\sqrt{3\Delta t}}$ so that $\ln u = \sigma\sqrt{3\Delta t}$. In the situation considered here, a good rule is to choose $\ln u$ as close as possible to this value, consistent with the condition given above. This means that we set

$$\ln u = \frac{\ln H_2 - \ln H_1}{N}$$

where

$$N = \text{int}\left[\frac{\ln H_2 - \ln H_1}{\sigma\sqrt{3\Delta t}} + 0.5\right]$$

and $\text{int}(x)$ is the integral part of x. This means that nodes are at values of the stock price equal to H_1, $H_1 u$, $H_1 u^2$, …, $H_1 u^N = H_2$

Normally, the trinomial stock price tree is constructed so that the central node is the initial stock price. In this case, it is unlikely that the current stock price happens to be $H_1 u^i$ for some i. To deal with this, the first trinomial movement should be from the initial stock price to $H_1 u^{i-1}$, $H_1 u^i$ and $H_1 u^{i+1}$ where i is chosen so that $H_1 u^i$ is closest to the current stock price. The probabilities on all branches of the tree are chosen, as usual, to match the first two moments of the stochastic process followed by the asset price. The approach works well except when the initial asset price is close to a barrier.

27.18

In this case, $\Delta t = 0.5$, $\lambda = 0.03$, $\sigma = 0.25$, $r = 0.06$ and $q = 0$ so that u=1.1934, d=0.8380, $a = 1.0305$, p_u = 0.5767, p_d = 0.4084, and the probability on default branches is 0.0149. This leads to the tree shown in Figure S27.3. The bond is called at nodes B and D and this forces exercise. Without the call the value at node D would be 142.92, the value at node B would be 122.87, and the value at node A would be 108.29. The value of the call option to the bond issuer is therefore 108.29 −106.31 = 1.98.

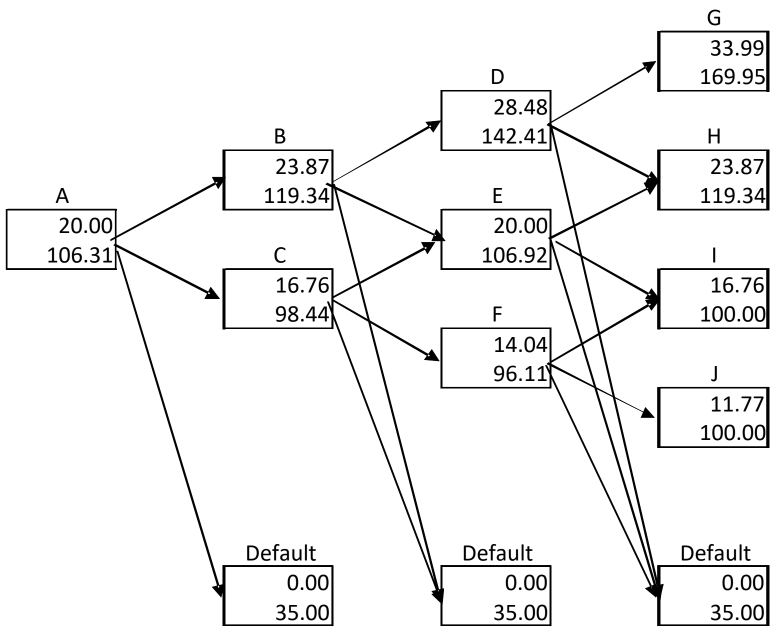

Figure S27.3: *Tree for Problem 27.18*

27.19

Using three-month time steps, the tree parameters are Δt=0.25, $u = 1.1052$, $d = 0.9048$, $a = 1.0050$, $p = 0.5000$. The tree is shown in Figure S27.4. The alternative minimum values of the stock price are shown in the middle box at each node. The value of the floating lookback option is 40.47.) DerivaGem shows that the value given by the analytic formula is 53.38. This is higher than the value given by the tree because the tree assumes that the stock price is observed only three times when the minimum is calculated.

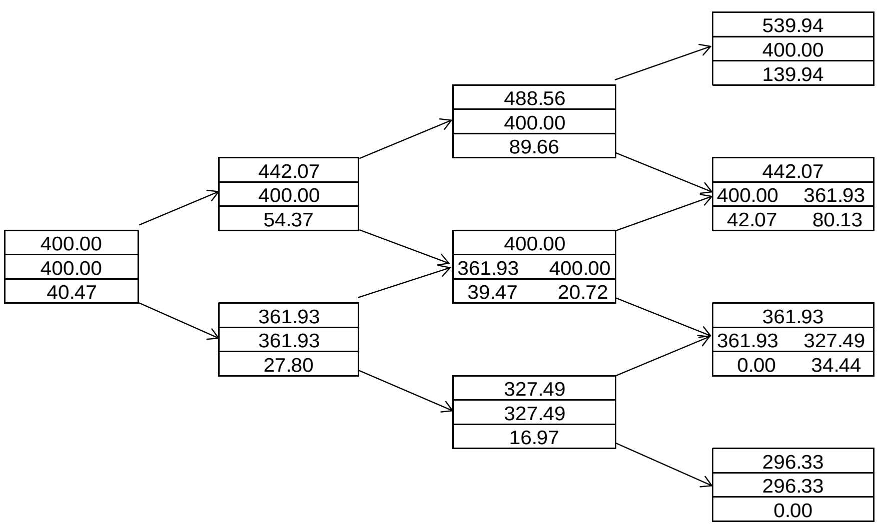

Figure S27.4: *Tree for Problem 27.19*

27.20

We construct a tree for $Y(t)=G(t)/S(t)$ where $G(t)$ is the minimum value of the index to date and $S(t)$ is the value of the index at time t. The tree is shown in Figure S27.5. It values the option in units of the stock index. This means that we value an instrument that pays off $1-Y(t)$. The tree shows that the value of the option is 0.1012 units of the stock index or 400×0.1012 or 40.47 dollars, as given by Figure S27.5.

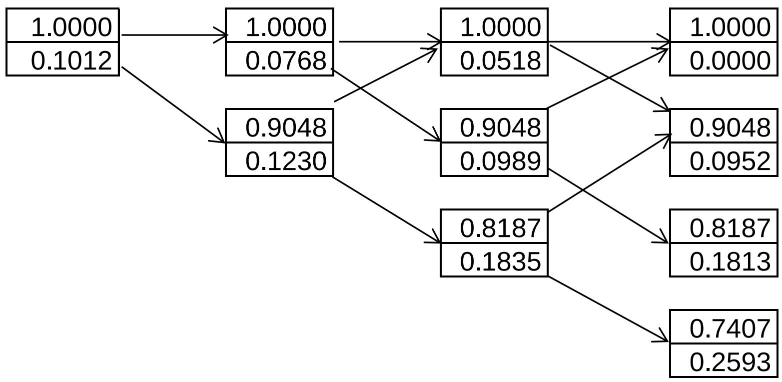

Figure S27.5 *Tree for Problem 27.20*

27.21

(a) The six-month call option with a strike price of 1.05 should be valued with a volatility of 13.4% and is worth 0.01829. The call option with a strike price of 1.10 should be valued with a volatility of 14.3% and is worth 0.00959. The bull spread is therefore worth $0.01829-0.00959=0.00870$.

(b) We now ask what volatility, if used to value both options, gives this price. Using the DerivaGem Application Builder in conjunction with Goal Seek, we find that the answer is 11.42%.

(c) Yes, this does support the contention at the beginning of the chapter that the correct volatility for valuing exotic options can be counterintuitive. We might reasonably expect the volatility to be between 13.4% (the volatility used to value the first option) and 14.3% (the volatility used to value the second option). 11.42% is well outside this range. The reason why the volatility is relatively low is as follows. The option provides the same payoff as a regular option with a 1.05 strike price when the asset price is between 1.05 and 1.10 and a lower payoff when the asset price is over 1.10. The implied probability distribution of the asset price (see Figure 20.2) is less heavy than the lognormal distribution in the 1.05 to 1.10 range and heavier than the lognormal distribution in the >1.10 range. This means that using a volatility of 13.4% (which is the implied volatility of a regular option with a strike price of 1.05) will give a price that is too high.

(d) The bull spread provides a payoff at only one time. It is therefore correctly valued by the IVF model.

27.22

Consider first the least squares approach. At the two-year point, the option is in the money for paths 1, 3, 4, 6, and 7. The five observations on S are 1.08, 1.07, 0.97, 0.77, and 0.84. The five continuation values are 0, $0.10e^{-0.06}$, $0.21e^{-0.06}$, $0.23e^{-0.06}$, $0.12e^{-0.06}$. The best fit continuation value is

$$-1.394+3.795S-2.276S^2$$

The best fit continuation values for the five paths are 0.0495, 0.0605, 0.1454, 0.1785, and 0.1876. These show that the option should be exercised for paths 1, 4, 6, and 7 at the two-year point.

There are six paths at the one-year point for which the option is in the money. These are paths 1, 4, 5, 6, 7, and 8. The six observations on S are 1.09, 0.93, 1.11, 0.76, 0.92, and 0.88. The six continuation values are $0.05e^{-0.06}$, $0.16e^{-0.06}$, 0, $0.36e^{-0.06}$, $0.29e^{-0.06}$, and 0. The best fit continuation value is

$$2.055-3.317S+1.341S^2$$

The best fit continuation values for the six paths are 0.0327, 0.1301, 0.0253, 0.3088, 0.1385, and 0.1746. These show that the option should be exercised at the one-year point for paths 1, 4, 6, 7, and 8. The value of the option if not exercised at time zero is therefore

$$\frac{1}{8}(0.04e^{-0.06}+0+0.10e^{-0.18}+0.20e^{-0.06}+0+0.37e^{-0.06}+0.21e^{-0.06}+0.25e^{-0.06})$$

or 0.136. Exercising at time zero would yield 0.13. The option should therefore not be exercised at time zero and its value is 0.136.

Consider next the exercise boundary parameterization approach. At time two years, it is optimal to exercise when the stock price is 0.84 or below. At time one year, it is optimal to exercise whenever the option is in the money. The value of the option assuming no early exercise at time zero is therefore

$$\frac{1}{8}\left(0.04e^{-0.06}+0+0.10e^{-0.18}+0.20e^{-0.06}+0.02e^{-0.06}\right)$$

$$+0.37e^{-0.06}+0.21e^{-0.06}+0.25e^{-0.06})$$

or 0.139. Exercising at time zero would yield 0.13. The option should therefore not be exercised at time zero. The value at time zero is 0.139. However, this tends to be high. As explained in the text, we should use one Monte Carlo simulation to determine the early exercise boundary. We should then carry out a new Monte Carlo simulation using the early exercise boundary to value the option.

27.23 (Excel file)

See Excel file. The results show that the SABR model can fit a wide variety of smiles.

27.24

(a) In this case, $\Delta t = 1$, $\lambda = 0.02/0.7 = 0.02857$, $\sigma = 0.25$, $r = 0.05$, $q = 0$, u=1.2840, d=0.7788, $a = 1.0513$, p_u = 0.5827, p_d = 0.3891, and the probability of a default is 0.0282. The calculations are shown in Figure S27.6. The values at the nodes include the value of the coupon paid just before the node is reached. The value of the convertible is 108.33.

(b) The value if there is no conversion calculated from the same tree is 94.08. The value of the conversion option is therefore 14.25.

(c) If it is called at node D just before the coupon payment the bond is converted but the coupon payment is not received, this reduces the value at the node to 144.26. Calling at node B will lead to conversion reducing the value to $115.36. The value of the bond at node A is then 102.20.

(d) A dividend payment would affect the way the tree is constructed as described in Chapter 21.

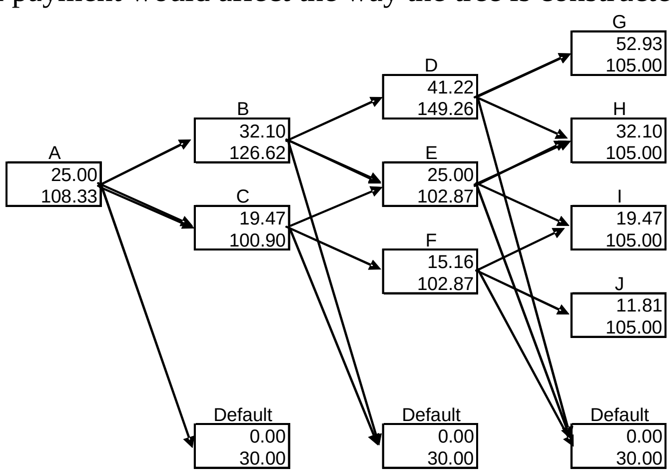

Figure S27.6: *Tree for Problem 27.24*

27.25

Suppose that U is the value if there is an up movement and D is the value if there is a down movement. Because the value is zero in the event of a default, the text shows that the value at a node is

$$\left[\left(\frac{e^{(r-q)\Delta t} - de^{-\lambda\Delta t}}{u-d}\right)U + \left(\frac{ue^{-\lambda\Delta t} - e^{(r-q)\Delta t}}{u-d}\right)D\right]e^{-r\Delta t}$$

This is the same as

$$= \left[\left(\frac{e^{(r+\lambda-q)\Delta t} - d}{u-d}\right)U + \left(\frac{u - e^{(r+\lambda-q)\Delta t}}{u-d}\right)D\right]e^{-(r+\lambda)\Delta t}$$

which proves the result.