CHAPTER 15
The Black-Scholes-Merton Model

Short Concept Questions

15.1 The return in a short period of time has constant drift and standard deviation.

15.2 See equation (15.3).

15.3 It goes down. It is proportional to one divided by the square root of time.

15.4 Weekends and holidays are ignored. Time is measured as the number of days when the market is open.

15.5 A position in the option is combined with a position in the underlying so that the portfolio has no risk over a short (theoretically infinitesimal) period of time.

15.6 See equations (15.20) and (15.21).

15.7 If we value a derivative in the risk-neutral world, we get the right price in all worlds.

15.8 In a warrant, the creator of the warrant issues new shares when the warrant is exercised. In an exchange-traded option, the shares used by the option seller to satisfy the option buyer must be purchased in the market.

15.9 The implied volatility is the volatility that makes the Black–Scholes–Merton price of an option equal to its market price. The implied volatility is calculated using an iterative procedure. A simple approach is the following. Suppose we have two volatilities one too high (i.e., giving an option price greater than the market price) and the other too low (i.e., giving an option price lower than the market price). By testing the volatility that is half way between the two, we get a new too-high volatility or a new too-low volatility. If we search initially for two volatilities, one too high and the other too low, we can use this procedure repeatedly to bisect the range and converge on the correct implied volatility. Other more sophisticated approaches (e.g., involving the Newton–Raphson procedure) are used in practice.

15.10 The stock price is reduced by the present value of the dividend.

Practice Questions

15.11
The Black–Scholes–Merton option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed.

15.12

The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility. In this problem, $\sigma = 0.3$ and, assuming 252 trading days in one year, $\Delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

15.13

In this case, $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$50N(-0.0917)e^{-0.1\times 0.25} - 50N(-0.2417)$$

$$= 50 \times 0.4634e^{-0.1\times 0.25} - 50 \times 0.4045 = 2.37$$

or $2.37.

15.14

In this case, we must subtract the present value of the dividend from the stock price before using Black–Scholes–Merton. Hence, the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667\times 0.1} = 48.52$$

As before, $K = 50$, $r = 0.1$, $\sigma = 0.3$, and $T = 0.25$. In this case,

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1\times 0.25} - 48.52N(-0.0414)$$

$$= 50 \times 0.5432e^{-0.1\times 0.25} - 48.52 \times 0.4835 = 3.03$$

or $3.03.

15.15

In this case, $\mu = 0.15$ and $\sigma = 0.25$. From equation (15.7), the probability distribution for the rate of return over a two-year period with continuous compounding is

$$\varphi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25^2}{2}\right)$$

that is,

$$\varphi(0.11875, 0.03125)$$

The expected value of the return is 11.875% per annum and the standard deviation is 17.7% per annum.

15.16

a) The required probability is the probability of the stock price being above \$40 in six months. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \varphi\left[\ln 38 + \left(0.16 - \frac{0.35^2}{2}\right)0.5, 0.35^2 \times 0.5\right]$$

that is,

$$\ln S_T \sim \varphi\left(3.687, 0.247^2\right)$$

Since $\ln 40 = 3.689$, we require the probability of $\ln(S_T) > 3.689$. This is

$$1 - N\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - N(0.008)$$

Since N(0.008) = 0.5032, the required probability is 0.4968.

b) In this case, the required probability is the probability of the stock price being less than \$40 in six months time. It is

$$1 - 0.4968 = 0.5032$$

15.17

From equation (15.3),

$$\ln S_T \sim \varphi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

95% confidence intervals for $\ln S_T$ are therefore,

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T are therefore,

$$e^{\ln S_0 + (\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}} \quad \text{and} \quad e^{\ln S_0 + (\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

that is,

$$S_0 e^{(\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}} \quad \text{and} \quad S_0 e^{(\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

15.18

This problem relates to the material in Section 15.3 and Business Snapshot 15.1. The statement is misleading in that a certain sum of money, say \$1,000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.

The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

15.19

a) At time *t*, the expected value of $\ln S_T$ is from equation (15.3),

$$\ln S + (\mu - \sigma^2/2)(T - t)$$

In a risk-neutral world, the expected value of $\ln S_T$ is therefore,

$$\ln S + (r - \sigma^2 / 2)(T - t)$$

Using risk-neutral valuation, the value of the derivative at time t is

$$e^{-r(T-t)}[\ln S + (r - \sigma^2 / 2)(T - t)]$$

b) If

$$f = e^{-r(T-t)}[\ln S + (r - \sigma^2 / 2)(T - t)]$$

then

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}[\ln S + (r - \sigma^2 / 2)(T - t)] - e^{-r(T-t)}\left(r - \sigma^2 / 2\right)$$

$$\frac{\partial f}{\partial S} = \frac{e^{-r(T-t)}}{S}$$

$$\frac{\partial^2 f}{\partial S^2} = -\frac{e^{-r(T-t)}}{S^2}$$

The left-hand side of the Black–Scholes–Merton differential equation is

$$e^{-r(T-t)}\left[r \ln S + r(r - \sigma^2 / 2)(T - t) - (r - \sigma^2 / 2) + r - \sigma^2 / 2\right]$$

$$= e^{-r(T-t)}\left[r \ln S + r(r - \sigma^2 / 2)(T - t)\right]$$

$$= rf$$

Hence, the differential equation is satisfied.

15.20

If $G(S,t) = h(t,T)S^n$ then $\partial G / \partial t = h_t S^n$, $\partial G / \partial S = hnS^{n-1}$, and $\partial^2 G / \partial S^2 = hn(n-1)S^{n-2}$ where $h_t = \partial h / \partial t$. Substituting into the Black–Scholes–Merton differential equation, we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$

The equation

$$h(t,T) = e^{[0.5\sigma^2 n(n-1) + r(n-1)](T-t)}$$

satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively, we can solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The solution to this is

$$\ln h = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)](T - t)$$

or

$$h(t,T) = e^{[0.5\sigma^2 n(n-1) + r(n-1)](T-t)}$$

15.21

In this case, $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$

$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$\begin{aligned}&52N(0.5365) - 50e^{-0.12\times 0.25}N(0.3865)\\ &= 52\times 0.7042 - 50e^{-0.03}\times 0.6504\\ &= 5.06\end{aligned}$$

or $5.06.

15.22

In this case, $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2)\times 0.5}{0.35\sqrt{0.5}} = 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$\begin{aligned}&70e^{-0.05\times 0.5}N(0.0809) - 69N(-0.1666)\\ &= 70e^{-0.025}\times 0.5323 - 69\times 0.4338\\ &= 6.40\end{aligned}$$

or $6.40.

15.23

Using the notation of Section 15.12, $D_1 = D_2 = 1$, $K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1\times 0.1667}) = 1.07$, and $K(1 - e^{-r(t_2 - t_1)}) = 65(1 - e^{-0.1\times 0.25}) = 1.60$. Since

$$D_1 < K(1 - e^{-r(T-t_2)})$$

and

$$D_2 < K(1 - e^{-r(t_2 - t_1)})$$

It is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

15.24

In the case, $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be calculated using an iterative procedure.

A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation, the implied volatility is about 0.396 or 39.6% per annum.

The implied volatility can also be calculated using DerivaGem. Select equity as the Underlying Type in the first worksheet. Select Black–Scholes European as the Option Type. Input stock price as 15, the risk-free rate as 5%, time to exercise as 0.25, and exercise price as 13. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Select the implied volatility button. Input the Price as 2.5 in the second half of the option data table. Hit the *Enter* key and click on calculate. DerivaGem will show the volatility of the option as 39.64%.

15.25

(a) Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

(b)

$$N'(d_1)=N'(d_2+\sigma\sqrt{T-t})$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{d_2^2}{2}-\sigma d_2\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)\right]$$

$$=N'(d_2)\exp\left[-\sigma d_2\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)\right]$$

Because

$$d_2=\frac{\ln(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

it follows that

$$\exp\left[-\sigma d_2\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)\right]=\frac{Ke^{-r(T-t)}}{S}$$

As a result,

$$SN'(d_1)=Ke^{-r(T-t)}N'(d_2)$$

which is the required result.

(c)

$$d_1=\frac{\ln\frac{S}{K}+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$=\frac{\ln S-\ln K+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Hence,

$$\frac{\partial d_1}{\partial S}=\frac{1}{S\sigma\sqrt{T-t}}$$

Similarly,

$$d_2=\frac{\ln S-\ln K+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S}=\frac{1}{S\sigma\sqrt{T-t}}$$

Therefore,

$$\frac{\partial d_1}{\partial S}=\frac{\partial d_2}{\partial S}$$

(d)

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

From (b),

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence,

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

Since

$$d_1 - d_2 = \sigma\sqrt{T-t}$$

$$\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = \frac{\partial}{\partial t}(\sigma\sqrt{T-t})$$

$$= -\frac{\sigma}{2\sqrt{T-t}}$$

Hence,

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

(e) From differentiating the Black–Scholes–Merton formula for a call price, we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{dS}$$

From the results in (b) and (c), it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

(f) Differentiating the result in (e) and using the result in (c), we obtain

$$\frac{\partial^2 c}{\partial S^2} = N'(d_1)\frac{\partial d_1}{\partial S}$$

$$= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$$

From the results in d) and e)

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

$$+rSN(d_1) + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$$

$$= r[SN(d_1) - Ke^{-r(T-t)}N(d_2)]$$

$$= rc$$

This shows that the Black–Scholes–Merton formula for a call option does indeed satisfy the Black–Scholes–Merton differential equation.

(g) Consider what happens in the formula for c in part (d) as t approaches T. If $S > K$,

d_1 and d_2 tend to infinity and $N(d_1)$ and $N(d_2)$ tend to 1. If $S < K$, d_1 and d_2 tend to minus infinity and $N(d_1)$ and $N(d_2)$ tend to zero. It follows that the formula for c tends to $\max(S-K,0)$.

15.26

The Black–Scholes–Merton formula for a European call option is

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

so that

$$c + Ke^{-rT} = S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT}$$

or

$$c + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT}[1 - N(d_2)]$$

or

$$c + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT} N(-d_2)$$

The Black–Scholes–Merton formula for a European put option is

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

so that

$$p + S_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1) + S_0$$

or

$$p + S_0 = Ke^{-rT} N(-d_2) + S_0[1 - N(-d_1)]$$

or

$$p + S_0 = Ke^{-rT} N(-d_2) + S_0 N(d_1)$$

This shows that the put–call parity result

$$c + Ke^{-rT} = p + S_0$$

holds.

15.27

Using DerivaGem, we obtain the following table of implied volatilities:

Stock Price	*Maturity = 3 months*	*Maturity = 6 months*	*Maturity = 12 months*
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

To calculate the first number, select equity as the Underlying Type in the first worksheet. Select Black–Scholes European as the Option Type. Input stock price as 50, the risk-free rate as 5%, time to exercise as 0.25, and exercise price as 45. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Select the implied volatility button. Input the Price as 7.0 in the second half of the option data table. Hit the *Enter* key and click on calculate. DerivaGem will show the volatility of the option as 37.78%. Change the strike price and time to exercise and recompute to calculate the rest of the numbers in the table.

The option prices are not exactly consistent with Black–Scholes–Merton. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have higher implied volatilities than high strike price options on the same stock. This phenomenon is discussed in Chapter 20.

15.28
Black's approach in effect assumes that the holder of option must decide at time zero whether it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T. In fact, the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n, it can still be exercised at time T.
It appears that Black's approach should understate the true option value. This is because the holder of the option has more alternative strategies for deciding when to exercise the option than the two strategies implicitly assumed by the approach. These alternative strategies add value to the option.
However, this is not the whole story! The standard approach to valuing either an American or a European option on a stock paying a single dividend applies the volatility to the stock price less the present value of the dividend. (The procedure for valuing an American option is explained in Chapter 21.) Black's approach when considering exercise just prior to the dividend date applies the volatility to the stock price itself. Black's approach therefore assumes more stock price variability than the standard approach in some of its calculations. In some circumstances, it can give a higher price than the standard approach.

15.29
With the notation in the text

$$D_1 = D_2 = 1.50, \quad t_1 = 0.3333, \quad t_2 = 0.8333, \quad T = 1.25, \quad r = 0.08 \quad \text{and} \quad K = 55$$

$$K\left[1-e^{-r(T-t_2)}\right] = 55(1-e^{-0.08\times 0.4167}) = 1.80$$

Hence,

$$D_2 < K\left[1-e^{-r(T-t_2)}\right]$$

Also,

$$K\left[1-e^{-r(t_2-t_1)}\right] = 55(1-e^{-0.08\times 0.5}) = 2.16$$

Hence,

$$D_1 < K\left[1-e^{-r(t_2-t_1)}\right]$$

It follows from the conditions established in Section 15.12 that the option should never be exercised early.
The present value of the dividends is

$$1.5e^{-0.3333\times 0.08} + 1.5e^{-0.8333\times 0.08} = 2.864$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.864 = 47.136, \quad K = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$$

$$d_1 = \frac{\ln(47.136/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, \quad N(d_2) = 0.3692$$

and the call price is

$$47.136 \times 0.4783 - 55e^{-0.08\times 1.25} \times 0.3692 = 4.17$$

or $4.17.

15.30

The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T. In a risk neutral world

$$\ln S_T \sim \varphi\left[\ln S_0 + (r - \sigma^2/2)T, \sigma^2 T\right]$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$1 - N\left[\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N\left[\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N(d_2)$$

The expected value at time T in a risk neutral world of a derivative security which pays off \$100 when $S_T > K$ is therefore

$$100N(d_2)$$

From risk neutral valuation, the value of the security at time zero is

$$100e^{-rT}N(d_2)$$

15.31

If the perpetual American put is exercised when $S=H$, it provides a payoff of $(K-H)$. We obtain its value, by setting $Q=K-H$ in equation (15.17), as

$$V = (K-H)\left(\frac{S}{H}\right)^{-2r/\sigma^2} = (K-H)\left(\frac{H}{S}\right)^{2r/\sigma^2}$$

Now

$$\frac{dV}{dH} = -\left(\frac{H}{S}\right)^{2r/\sigma^2} + \frac{K-H}{S}\left(\frac{2r}{\sigma^2}\right)\left(\frac{H}{S}\right)^{2r/\sigma^2-1}$$

$$= \left(\frac{H}{S}\right)^{2r/\sigma^2}\left(-1 + \frac{2r(K-H)}{H\sigma^2}\right)$$

$$\frac{d^2V}{dH^2} = -\frac{2rK}{H^2\sigma^2}\left(\frac{H}{S}\right)^{2r/\sigma^2} + \left(-1 + \frac{2r(K-H)}{H\sigma^2}\right)\frac{2r}{\sigma^2 S}\left(\frac{H}{S}\right)^{2r/\sigma^2-1}$$

dV/dH is zero when

$$H = \frac{2rK}{2r+\sigma^2}$$

and, for this value of H, d^2V/dH^2 is negative indicating that it gives the maximum value of V.

The value of the perpetual American put is maximized if it is exercised when S equals this value of H. Hence the value of the perpetual American put is

$$(K-H)\left(\frac{S}{H}\right)^{-2r/\sigma^2}$$

when H=2rK/(2r+σ^2). The value is

$$\frac{\sigma^2 K}{2r+\sigma^2}\left(\frac{S(2r+\sigma^2)}{2rK}\right)^{-2r/\sigma^2}$$

This is consistent with the more general result produced in Chapter 26 for the case where the stock provides a dividend yield.

15.32

The answer is no. If markets are efficient, they have already taken potential dilution into account in determining the stock price. This argument is explained in Business Snapshot 15.3.

15.33

The Black–Scholes–Merton price of the option is given by setting $S_0 = 50$, $K = 50$, $r = 0.05$, $\sigma = 0.25$, and $T = 5$. It is 16.252. From an analysis similar to that in Section 15.10, the cost to the company of the options is

$$\frac{10}{10+3}\times 16.252 = 12.5$$

or about \$12.5 per option. The total cost is therefore 3 million times this or \$37.5 million. If the market perceives no benefits from the options, the stock price will fall by \$3.75.

15.34

(a) $0.18/\sqrt{252} = 1.13\%$

(b) $0.18/\sqrt{52} = 2.50\%$

(c) $0.18/\sqrt{12} = 5.20\%$

15.35

In this case, $S_0 = 50$, $\mu = 0.18$ and $\sigma = 0.30$. The probability distribution of the stock price in two years, S_T, is lognormal and is, from equation (15.3), given by:

$$\ln S_T \sim \varphi\left[\ln 50 + \left(0.18 - \frac{0.09}{2}\right)2,\ 0.3^2 \times 2\right]$$

that is

$$\ln S_T \sim \varphi(4.18,\ 0.42^2)$$

The mean stock price is from equation (15.4)

$$50e^{0.18\times 2} = 50e^{0.36} = 71.67$$

and the standard deviation is from equation (15.5)

$$50e^{0.18\times 2}\sqrt{e^{0.09\times 2}-1} = 31.83$$

95% confidence intervals for $\ln S_T$ are:

$$4.18 - 1.96\times 0.42 \quad \text{and} \quad 4.18 + 1.96\times 0.42$$

that is

$$3.35 \quad \text{and} \quad 5.01$$

These correspond to 95% confidence limits for S_T of

$$e^{3.35} \quad \text{and} \quad e^{5.01}$$

that is

$$28.52 \quad \text{and} \quad 150.44$$

15.36 (Excel file)

The calculations are shown in the table below:

$$\sum u_i = 0.09471 \quad \sum u_i^2 = 0.01145$$

and an estimate of standard deviation of weekly returns is:

$$\sqrt{\frac{0.01145}{13} - \frac{0.09471^2}{14 \times 13}} = 0.02884$$

The volatility per annum is therefore $0.02884\sqrt{52} = 0.2079$ or 20.79%. The standard error of this estimate is

$$\frac{0.2079}{\sqrt{2 \times 14}} = 0.0393$$

or 3.9% per annum.

Week	*Closing Stock Price ($)*	*Price Relative* $= S_i / S_{i-1}$	*Weekly Return* $u_i = \ln(S_i / S_{i-1})$
1	30.2		
2	32.0	1.05960	0.05789
3	31.1	0.97188	–0.02853
4	30.1	0.96785	–0.03268
5	30.2	1.00332	0.00332
6	30.3	1.00331	0.00331
7	30.6	1.00990	0.00985
8	33.0	1.07843	0.07551
9	32.9	0.99697	–0.00303
10	33.0	1.00304	0.00303
11	33.5	1.01515	0.01504
12	33.5	1.00000	0.00000
13	33.7	1.00597	0.00595
14	33.5	0.99407	–0.00595
15	33.2	0.99104	–0.00900

15.37

The easiest way of proving this is to note that

$$\max(V-K, 0) - \max(K-V, 0) = V-K$$

so that

$$E[\max(K-V, 0)] = E[\max(V-K, 0)] - E(V) + K$$
$$= E(V)N(d_1) - KN(d_2) - E(V) + K$$

Because $1-N(d_2) = N(-d_2)$ and $1-N(d_1) = N(-d_1)$, this immediately gives the required result. (It can also be proved in the same way as the first result is proved in the appendix.)

Because

$$p = e^{-rT}\hat{E}\left[\max(K - S_T, 0)\right]$$

and

$$\hat{E}(S_T) = S_0 e^{rT}$$

The Black–Scholes–Merton pricing formula for a put option follows.