CHAPTER 26
Exotic Options

Practice Questions

26.1
A forward start option is an option that is paid for now but will start at some time in the future. The strike price is usually equal to the price of the asset at the time the option starts. A chooser option is an option where, at some time in the future, the holder chooses whether the option is a call or a put.

26.2
A floating lookback call provides a payoff of $S_T - S_{\min}$. A floating lookback put provides a payoff of $S_{\max} - S_T$. A combination of a floating lookback call and a floating lookback put therefore provides a payoff of $S_{\max} - S_{\min}$.

26.3
No, it is never optimal to choose early. The resulting cash flows are the same regardless of when the choice is made. There is no point in the holder making a commitment earlier than necessary. This argument applies when the holder chooses between two American options providing the options cannot be exercised before the 2-year point. If the early exercise period starts as soon as the choice is made, the argument does not hold. For example, if the stock price fell to almost nothing in the first six months, the holder would choose a put option at this time and exercise it immediately.

26.4
The payoffs from c_1, c_2, c_3, p_1, p_2, p_3 are, respectively, as follows:

$$\max(\bar{S} - K, 0)$$
$$\max(S_T - \bar{S}, 0)$$
$$\max(S_T - K, 0)$$
$$\max(K - \bar{S}, 0)$$
$$\max(\bar{S} - S_T, 0)$$
$$\max(K - S_T, 0)$$

The payoff from $c_1 - p_1$ is always $\bar{S} - K$; The payoff from $c_2 - p_2$ is always $S_T - \bar{S}$; The payoff from $c_3 - p_3$ is always $S_T - K$; It follows that

$$c_1 - p_1 + c_2 - p_2 = c_3 - p_3$$

or

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

26.5
Substituting for c, put-call parity gives

$$\max(c, p) = \max\left[p, p + S_1 e^{-q(T_2 - T_1)} - K e^{-r(T_2 - T_1)}\right]$$

$$= p + \max\left[0, S_1 e^{-q(T_2-T_1)} - Ke^{-r(T_2-T_1)}\right]$$

This shows that the chooser option can be decomposed into:

1. A put option with strike price K and maturity T_2; and
2. $e^{-q(T_2-T_1)}$ call options with strike price $Ke^{-(r-q)(T_2-T_1)}$ and maturity T_1.

26.6

Consider the formula for c_{do} when $H \geq K$

$$c_{\text{do}} = S_0 N(x_1) e^{-qT} - Ke^{-rT} N(x_1 - \sigma\sqrt{T}) - S_0 e^{-qT} (H/S_0)^{2\lambda} N(y_1)$$

$$+Ke^{-rT} (H/S_0)^{2\lambda-2} N(y_1 - \sigma\sqrt{T})$$

Substituting $H = K$ and noting that

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}$$

we obtain $x_1 = d_1$ so that

$$c_{\text{do}} = c - S_0 e^{-qT} (H/S_0)^{2\lambda} N(y_1) + Ke^{-rT} (H/S_0)^{2\lambda-2} N(y_1 - \sigma\sqrt{T})$$

The formula for c_{di} when $H \leq K$ is

$$c_{\text{di}} = S_0 e^{-qT} (H/S_0)^{2\lambda} N(y) - Ke^{-rT} (H/S_0)^{2\lambda-2} N(y - \sigma\sqrt{T})$$

Since $c_{\text{do}} = c - c_{\text{di}}$

$$c_{\text{do}} = c - S_0 e^{-qT} (H/S_0)^{2\lambda} N(y) + Ke^{-rT} (H/S_0)^{2\lambda-2} N(y - \sigma\sqrt{T})$$

From the formulas in the text $y_1 = y$ when $H = K$. The two expression for c_{do} are therefore equivalent when $H = K$

26.7

The option is in the money only when the asset price is less than the strike price. However, in these circumstances the barrier has been hit and the option has ceased to exist.

26.8

The argument is similar to that given in Chapter 11 for a regular option on a non-dividend-paying stock. Consider a portfolio consisting of the option and cash equal to the present value of the terminal strike price. The initial cash position is

$$Ke^{gT-rT}$$

By time τ ($0 \leq \tau \leq T$), the cash grows to

$$Ke^{-r(T-\tau)+gT} = Ke^{g\tau} e^{-(r-g)(T-\tau)}$$

Since $r > g$, this is less than $Ke^{g\tau}$ and therefore is less than the amount required to exercise the option. It follows that, if the option is exercised early, the terminal value of the portfolio is less than S_T. At time T the cash balance is Ke^{gT}. This is exactly what is required to exercise the option. If the early exercise decision is delayed until time T, the terminal value of the portfolio is therefore

$$\max[S_T, Ke^{gT}]$$

This is at least as great as S_T. It follows that early exercise cannot be optimal.

26.9

When the strike price of an option on a non-dividend-paying stock is defined as 10% greater that the stock price, the value of the option is proportional to the stock price. The same argument as that given in the text for forward start options shows that if t_1 is the time when the option starts and t_2 is the time when it finishes, the option has the same value as an option starting today with a life of $t_2 - t_1$ and a strike price of 1.1 times the current stock price.

26.10

Assume that we start calculating averages from time zero. The relationship between $A(t+\Delta t)$ and $A(t)$ is

$$A(t+\Delta t)\times(t+\Delta t) = A(t)\times t + S(t)\times\Delta t$$

where $S(t)$ is the stock price at time t and terms of higher order than Δt are ignored. If we continue to ignore terms of higher order than Δt, it follows that

$$A(t+\Delta t) = A(t)\left[1-\frac{\Delta t}{t}\right] + S(t)\frac{\Delta t}{t}$$

Taking limits as Δt tends to zero

$$dA(t) = \frac{S(t)-A(t)}{t}dt$$

The process for $A(t)$ has a stochastic drift and no dz term. The process makes sense intuitively. Once some time has passed, the change in S in the next small portion of time has only a second order effect on the average. If S equals A the average has no drift; if $S > A$ the average is drifting up; if $S < A$ the average is drifting down.

26.11

In an Asian option, the payoff becomes more certain as time passes and the delta always approaches zero as the maturity date is approached. This makes delta hedging easy. Barrier options cause problems for delta hedgers when the asset price is close to the barrier because delta is discontinuous.

26.12

The value of the option is given by the formula in the text

$$V_0 e^{-q_2 T} N(d_1) - U_0 e^{-q_1 T} N(d_2)$$

where

$$d_1 = \frac{\ln(V_0/U_0) + (q_1 - q_2 + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

In this case, $V_0 = 1{,}520$, $U_0 = 1600$, $q_1 = 0$, $q_2 = 0$, $T = 1$, and

$$\sigma = \sqrt{0.2^2 + 0.2^2 - 2\times 0.7\times 0.2\times 0.2} = 0.1549$$

Because $d_1 = -0.2536$ and $d_2 = -0.4086$, the option price is

$$1520N(-0.2536) - 1600N(-0.4086) = 61.54$$

or \$61.54.

26.13

No. If the future's price is above the spot price during the life of the option, it is possible that the spot price will hit the barrier when the futures price does not.

26.14

(a) The put–call relationship is

$$cc + K_1 e^{-rT_1} = pc + c$$

where cc is the price of the call on the call, pc is the price of the put on the call, c is the price today of the call into which the options can be exercised at time T_1, and K_1 is the exercise price for cc and pc. The proof is similar to that in Chapter 11 for the usual put–call parity relationship. Both sides of the equation represent the values of portfolios that will be worth $\max(c, K_1)$ at time T_1. Because

$$M(a,b;\rho) = N(a) - M(a,-b;-\rho) = N(b) - M(-a,b,;-\rho)$$

and

$$N(x) = 1 - N(-x)$$

we obtain

$$cc - pc = Se^{-qT_2} N(b_1) - K_2 e^{-rT_2} N(b_2) - K_1 e^{-rT_1}$$

Since

$$c = Se^{-qT_2} N(b_1) - K_2 e^{-rT_2} N(b_2)$$

put–call parity is consistent with the formulas.

(b) The put–call relationship is

$$cp + K_1 e^{-rT_1} = pp + p$$

where cp is the price of the call on the put, pp is the price of the put on the put, p is the price today of the put into which the options can be exercised at time T_1, and K_1 is the exercise price for cp and pp. The proof is similar to that in Chapter 11 for the usual put–call parity relationship. Both sides of the equation represent the values of portfolios that will be worth $\max(p, K_1)$ at time T_1. Because

$$M(a,b;\rho) = N(a) - M(a,-b;-\rho) = N(b) - M(-a,b,;-\rho)$$

and

$$N(x) = 1 - N(-x)$$

it follows that

$$cp - pp = -Se^{-qT_2} N(-b_1) + K_2 e^{-rT_2} N(-b_2) - K_1 e^{-rT_1}$$

Because

$$p = -Se^{-qT_2} N(-b_1) + K_2 e^{-rT_2} N(-b_2)$$

put–call parity is consistent with the formulas.

26.15

As we increase the frequency, we observe a more extreme minimum which increases the value of a floating lookback call.

26.16

As we increase the frequency with which the asset price is observed, the asset price becomes more likely to hit the barrier and the value of a down-and-out call goes down. For a similar reason the value of a down-and-in call goes up. The adjustment mentioned in the text, suggested by Broadie, Glasserman, and Kou, moves the barrier further out as the asset price is observed less frequently. This increases the price of a down-and-out option and reduces the price of a down-and-in option.

26.17

If the barrier is reached, the down-and-out option is worth nothing while the down-and-in option has the same value as a regular option. If the barrier is not reached, the down-and-in option is worth nothing while the down-and-out option has the same value as a regular option. This is why a down-and-out call option plus a down-and-in call option is worth the same as a regular option. A similar argument cannot be used for American options.

26.18

This is a cash-or-nothing call. The value is $100N(d_2)e^{-0.08\times 0.5}$ where

$$d_2 = \frac{\ln(960/1000)+(0.08-0.03-0.2^2/2)\times 0.5}{0.2\times\sqrt{0.5}} = -0.1826$$

Since $N(d_2)=0.4276$, the value of the derivative is \$41.08.

26.19

This is a regular call with a strike price of \$20 that ceases to exist if the futures price hits \$18. With the notation in the text $H=18$, $K=20$, $S=19$, $r=0.05$, $\sigma=0.4$, $q=0.05$, $T=0.25$. From this $\lambda=0.5$ and

$$y = \frac{\ln[18^2/(19\times 20)]}{0.4\sqrt{0.25}} + 0.5\times 0.4\sqrt{0.25} = -0.69714$$

The value of a down-and-out call plus a down-and-in call equals the value of a regular call. Substituting into the formula given when $H<K$ we get $c_{\text{di}}=0.4638$. The regular Black–Scholes–Merton formula gives $c=1.0902$. Hence, $c_{\text{do}}=0.6264$. (These answers can be checked with DerivaGem.)

26.20

DerivaGem shows that the value is 53.38. Note that the Minimum to date and Maximum to date should be set equal to the current value of the index for a new deal. (See material on DerivaGem at the end of the book.)

26.21

We can use the analytic approximation given in the text.

$$M_1 = \frac{(e^{0.05\times 0.5}-1)\times 30}{0.05\times 0.5} = 30.378$$

Also $M_2 = 936.9$ so that $\sigma = 17.41\%$. The option can be valued as a futures option with $F_0 = 30.378$, $K=30$, $r=5\%$, $\sigma=17.41\%$, and $t=0.5$. The price is 1.637.

26.22

The price of a regular European call option is 7.116. The price of the down-and-out call option is 4.696. The price of the down-and-in call option is 2.419.
The price of a regular European call is the sum of the prices of down-and-out and down-and-in options.

26.23

When $r = q$ in the expression for a floating lookback call in Section 26.11 $a_1 = a_3$ and $Y_1 = \ln(S_0 / S_{\min})$ so that the expression for a floating lookback call becomes

$$S_0 e^{-qT} N(a_1) - S_{\min} e^{-rT} N(a_2)$$

As q approaches r in Section 26.13, we get

$$M_1 = S_0$$

$$M_2 = \frac{2e^{\sigma^2 T} S_0^2}{\sigma^4 T^2} - \frac{2S_0^2}{T^2} \frac{1+\sigma^2 T}{\sigma^4}$$

A proof of these results requires L'Hopital's rule where to get the limit of 0/0 we differentiate the numerator and denominator.

26.24

In this case, DerivaGem shows that $Q(K_1) = 0.1772$, $Q(K_2) = 1.1857$, $Q(K_3) = 4.9123$, $Q(K_4) = 14.2374$, $Q(K_5) = 45.3738$, $Q(K_6) = 35.9243$, $Q(K_7) = 20.6883$, $Q(K_8) = 11.4135$, $Q(K_9) = 6.1043$. $\hat{E}(\bar{V}) = 0.0502$. The value of the variance swap is \$0.51 million.

26.25

When q=0, $w=r-\sigma^2/2$ so that α_1=1 and $\alpha_2=2r/\sigma^2$. This is consistent with the results for perpetual derivatives in Section 15.6.

26.26

The price of the option is 3.528.

(a) The option price is a humped function of the stock price with the maximum option price occurring for a stock price of about \$57. If you could choose the stock price there would be a trade off. High stock prices give a high probability that the option will be knocked out. Low stock prices give a low potential payoff. For a stock price less than \$57, delta is positive (as it is for a regular call option); for a stock price greater than \$57, delta is negative.

(b) Delta increases up to a stock price of about 45 and then decreases. This shows that gamma can be positive or negative.

(c) The option price is a humped function of the time to maturity with the maximum option price occurring for a time to maturity of 0.5 years. This is because too long a time to maturity means that the option has a high probability of being knocked out; too short a time to maturity means that the option has a low potential payoff. For a time to maturity less than 0.5 years, theta is negative (as it is for a regular call option); for a time to maturity greater than 0.5 years, the theta of the option is positive.

(d) The option price is also a humped function of volatility with the maximum option price being obtained for a volatility of about 20%. This is because too high a volatility means that the option has a high probability of being knocked out; too low volatility means that the option has a low potential payoff. For volatilities less than 20%, vega is positive (as it is for a regular option); for volatilities above 20% vega is negative.

26.27

(a) Both approaches use a one call option with a strike price of 50 and a maturity of 0.75. In the first approach, the other 15 call options have strike prices of 60 and equally spaced times to maturity. In the second approach, the other 15 call options have strike prices of 60, but the spacing between the times to maturity decreases as the maturity of the barrier option is approached. The second approach to setting times to maturity produces a better hedge. This is because the chance of the barrier being hit at time t is an increasing function of t. As t increases, it therefore becomes more important to replicate the barrier at time t.

(b) By using either trial and error or the Solver tool, we see that we come closest to matching the price of the barrier option when the maturities of the third and fourth options are changed from 0.25 and 0.5 to 0.39 and 0.65.

(c) To calculate delta for the two 16-option hedge strategies, it is necessary to change the last argument of EPortfolio from 0 to 1 in cells L42 and X42. To calculate delta for the barrier option, it is necessary to change the last argument of BarrierOption in cell F12 from 0 to 1. To calculate gamma and vega, the arguments must be changed to 2 and 3, respectively. The delta, gamma, and vega of the barrier option are –0.0221, –0.0035, and –0.0254. The delta, gamma, and vega of the first 16-option portfolio are –0.0262, –0.0045, and –0.1470. The delta, gamma, and vega of the second 16-option portfolio are –0.0199, –0.0037, and –0.1449. The second of the two 16-option portfolios provides Greek letters that are closest to the Greek letters of the barrier option. Interestingly, neither of the two portfolios does particularly well on vega.

26.28

A natural approach is to attempt to replicate the option with positions in:

(a) A European call option with strike price 1.00 maturing in two years.
(b) A European put option with strike price 0.80 maturing in two years.
(c) A European put option with strike price 0.80 maturing in 1.5 years.
(d) A European put option with strike price 0.80 maturing in 1.0 years.
(e) A European put option with strike price 0.80 maturing in 0.5 years.

The first option can be used to match the value of the down-and-out-call for $t = 2$ and $S > 1.00$. The others can be used to match it at the following (t, S) points: (1.5, 0.80), (1.0, 0.80), (0.5, 0.80), (0.0, 0.80). Following the procedure in the text, we find that the required positions in the options are as shown in the following table.

Option Type	*Strike Price*	*Maturity (yrs)*	*Position*
Call	1.0	2.00	+1.0000
Put	0.8	2.00	−0.1255
Put	0.8	1.50	−0.1758
Put	0.8	1.00	−0.0956
Put	0.8	0.50	−0.0547

The values of the options at the relevant (t,S) points are as follows:

	Value initially	*Value at (1.5,0.8)*	*Value at (1.0,0.8)*	*Value at (0.5,0.8)*	*Value at (0,0.8)*
Option (a)	0.0735	0.0071	0.0199	0.0313	0.0410
Option (b)	0.0736	0.0568	0.0792	0.0953	0.1079
Option (c)	0.0603		0.0568	0.0792	0.0953
Option (d)	0.0440			0.0568	0.0792
Option (e)	0.0231				0.0568

The value of the portfolio initially is 0.0482. This is only a little less than the value of the down-and-out-option which is 0.0488. This example is different from the example in the text in a number of ways. Put options and call options are used in the replicating portfolio. The value of the replicating portfolio converges to the value of the option from below rather than from above. Also, even with relatively few options, the value of the replicating portfolio is close to the value of the down-and-out option.

26.29

In this case,

$M_1 = (900e^{(0.05-0.03)\times 0.25} + 900e^{(0.05-0.03)\times 0.50} + 900e^{(0.05-0.03)\times 0.75} + 900e^{(0.05-0.03)\times 1})/4 = 917.07$

and a more complicated calculation involving 16 terms shows that M_2=907,486.6

so that the option can be valued as an option on futures where the futures price is 917.07 and volatility is $\sqrt{\ln(907{,}486.6/917.07^2}$ or 27.58%. The value of the option is 103.13. DerivaGem gives the price as 86.77 (set option type =Asian). The higher price for the first option arises because the average is calculated from prices at times 0.25, 0.50, 0.75, and 1.00. The mean of these times is 0.625. By contrast, the corresponding mean when the price is observed continuously is 0.50. The later a price is observed the more uncertain it is and the more it contributes to the value of the option.

26.30

For the regular option, the theoretical price is about $240,000. For the average price option, the theoretical price is about 115,000. My 20 simulation runs (40 outcomes because of the antithetic calculations) gave results as shown in the following table.

	Regular Call	*Ave Price Call*
Ave Hedging Cost	247,628	114,837
SD Hedging Cost	17,833	12,123
Ave Trading Vol (20 wks)	412,440	291,237
Ave Trading Vol (last 10 wks)	187,074	51,658

These results show that the standard deviation of using delta hedging for an average price option is lower than that for a regular option. However, using the criterion in Chapter 19 (standard deviation divided by value of option) hedge performance is better for the regular option. Hedging the average price option requires less trading, particularly in the last 10 weeks. This is because we become progressively more certain about the average price as the maturity of the option is approached.

26.31

The value of the option is 1093. It is necessary to change cells F20 and F46 to 0.67. Cells G20 to G39 and G46 to G65 must be changed to calculate delta of the compound option. Cells H20 to H39 and H46 to H65 must be changed to calculate gamma of the compound option. Cells I20 to I40 and I46 to I66 must be changed to calculate the Black–Scholes price of the call option expiring in 40 weeks. Similarly, cells J20 to J40 and J46 to J66 must be changed to calculate the delta of this option; cells K20 to K40 and K46 to K66 must be changed to calculate the gamma of the option. The payoffs in cells N9 and N10 must be calculated as MAX(I40-0.015,0)*100,000 and MAX(I66-0.015,0)*100,000. Delta plus gamma hedging works relatively poorly for the compound option. On 20 simulation runs the cost of writing and hedging the option ranged from 200 to 2,500.

26.32

a) The outperformance certificate is equivalent to a package consisting of:

(i) A zero coupon bond that pays off S_0 at time T.

(ii) A long position in k one-year European call options on the stock with a strike price equal to the current stock price.

(iii) A short position in k one-year European call options on the stock with a strike price equal to M.

(iv) A short position in one European one-year put option on the stock with a strike price equal to the current stock price.

b) In this case, the present value of the four parts of the package are:

(i) $50e^{-0.05\times 1}=47.56$

(ii) $1.5\times 5.0056 = 7.5084$

(iii) $-1.5\times 0.6339 = -0.9509$

(iv) -4.5138

The total of these is 47.56 +7.5084 – 0.9509−4.5138 = 49.6. This is less than the initial investment of 50.

26.33

In this case, $F_0 = 1022.55$ and DerivaGem shows that $Q(K_1) = 0.0366$, $Q(K_2) = 0.2858$, $Q(K_3) = 1.5822$, $Q(K_4) = 6.3708$, $Q(K_5) = 30.3864$, $Q(K_6) = 16.9233$, $Q(K_7) = 4.8180$, $Q(K_8) = 0.8639$, and $Q_9 = 0.0863$. $\hat{E}(\bar{V}) = 0.0661$. The value of the variance swap is \$2.09 million.

26.34

With the notation in the text, a regular call option with strike price K_2 plus a binary call option that pays off $K_2 - K_1$ is a gap call option that pays off $S_T - K_1$ when $S_T > K_2$.

26.35

Suppose that there are n periods each of length τ, the risk-free interest rate is r, the dividend yield on the index is q, and the volatility of the index is σ. The value of the investment is

$$e^{-rn\tau} Q\hat{E}\left[\prod_{i=1}^{n} \max(1+R_i, 1)\right]$$

where R_i is the return in period i and as usual $\hat{E}$ denotes expected value in a risk-neutral world. Because (assuming efficient markets) the returns in successive periods are independent, this is

$$e^{-rn\tau} Q\prod_{i=1}^{n} \{\hat{E}[\max(1+R_i, 1)]\}$$
$$= e^{-rn\tau} Q\prod_{i=1}^{n} \left\{\hat{E}\left[1+\max\left(\frac{S_i - S_{i-1}}{S_{i-1}},\ 0\right)\right]\right\}$$

where S_i is the value of the index at the end of the ith period.

From Black–Scholes–Merton, the risk-neutral expectation at time $(i-1)\tau$ of $\max(S_i - S_{i-1}, 0)$ is

$$e^{(r-q)\tau} S_{i-1} N(d_1) - S_{i-1} N(d_2)$$

where

$$d_1 = \frac{(r - q + \sigma^2 / 2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{(r - q - \sigma^2 / 2)\tau}{\sigma\sqrt{\tau}}$$

The value of the investment is therefore,

$$e^{-rn\tau} Q\left[1 + e^{(r-q)\tau} N(d_1) - N(d_2)\right]^n$$