CHAPTER 14
Wiener Processes and Itô's Lemma

Short Concept Questions

14.1 A Markov model is a model where only the current value of a variable is relevant for determining future movements. The past history is irrelevant.

14.2 A Wiener process has the properties that (a) the change in a short period of time is a sample from a standard normal distribution multiplied by the square root of the length of the time period, and (b) the changes in two different (nonoverlapping) periods are independent.

14.3 A Wiener process has a drift rate of zero and a variance rate per unit time of one. A generalized Wiener process has a constant (not necessarily zero) drift rate and a constant variance rate per unit time (not necessarily one).

14.4 A Wiener process is simulated by dividing time into a number of short periods and sampling from a standard normal distribution to determine movements in each of the short period.

14.5 In geometric Brownian motion, the coefficient of dt and dz in the stochastic process are proportional to the value of the variable.

14.6 Itô's lemma calculates the process followed by a function of a variable from the process followed by the variable itself.

14.7 A generalized Wiener process.

14.8 (a) negative, (b) zero, (c) positive.

14.9 Imagine that you have to forecast the future temperature from a) the current temperature, b) the history of the temperature in the last week, and c) a knowledge of seasonal averages and seasonal trends. If temperature followed a Markov process, the history of the temperature in the last week would be irrelevant.
To answer the second part of the question, you might like to consider the following scenario for the first week in May:

(i) Monday to Thursday are warm days; today, Friday, is a very cold day.
(ii) Monday to Friday are all very cold days.

What is your forecast for the weekend? If you are more pessimistic in the case of the second scenario, temperatures do not follow a Markov process.

14.10 Values of the Hurst parameter below 0.5 lead to negative correlations between movements in successive time periods. Values of the Hurst parameter above 0.5 lead to positive correlations between movements in successive time periods. As Figure 14.3 shows, the "noise" is greatest for low values of the Hurst exponent.

Practice Questions

14.11

The first point to make is that any trading strategy can, just because of good luck, produce above average returns. The key question is whether a trading strategy *consistently* outperforms the market when adjustments are made for risk. It is certainly possible that a trading strategy could do this. However, when enough investors know about the strategy and trade on the basis of the strategy, the profit will disappear.

As an illustration of this, consider a phenomenon known as the small firm effect. Portfolios of stocks in small firms appear to have outperformed portfolios of stocks in large firms when appropriate adjustments are made for risk. Research was published about this in the early 1980s and mutual funds were set up to take advantage of the phenomenon. There is some evidence that this has resulted in the phenomenon disappearing.

14.12

Suppose that the company's initial cash position is x. The probability distribution of the cash position at the end of one year is

$$\varphi(x+4\times 0.5, 4\times 4)=\varphi(x+2.0,16)$$

where $\varphi(m,v)$ is a normal probability distribution with mean m and variance v. The probability of a negative cash position at the end of one year is

$$N\left(-\frac{x+2.0}{4}\right)$$

where $N(x)$ is the cumulative probability that a standardized normal variable (with mean zero and standard deviation 1.0) is less than x. From the properties of the normal distribution

$$N\left(-\frac{x+2.0}{4}\right)=0.05$$

when:

$$-\frac{x+2.0}{4}=-1.6449$$

that is, when $x=4.5796$. The initial cash position must therefore be \$4.58 million.

14.13

(a) Suppose that X_1 and X_2 equal a_1 and a_2 initially. After a time period of length T, X_1 has the probability distribution

$$\varphi(a_1+\mu_1 T, \sigma_1^2 T)$$

and X_2 has a probability distribution

$$\varphi(a_2+\mu_2 T, \sigma_2^2 T)$$

From the property of sums of independent normally distributed variables, X_1+X_2 has the probability distribution

$$\varphi\left(a_1+\mu_1 T+a_2+\mu_2 T,\sigma_1^2 T+\sigma_2^2 T\right)$$

that is,

$$\varphi\left[a_1+a_2+(\mu_1+\mu_2)T,(\sigma_1^2+\sigma_2^2)T\right]$$

This shows that X_1+X_2 follows a generalized Wiener process with drift rate $\mu_1+\mu_2$ and variance rate $\sigma_1^2+\sigma_2^2$.

(b) In this case, the change in the value of X_1+X_2 in a short interval of time Δt has the probability distribution:

$$\varphi\left[(\mu_1+\mu_2)\Delta t,(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)\Delta t\right]$$

If μ_1, μ_2, σ_1, σ_2 and ρ are all constant, arguments similar to those in Section 14.2 show that the change in a longer period of time T is

$$\varphi\left[(\mu_1+\mu_2)T,(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)T\right]$$

The variable, X_1+X_2, therefore follows a generalized Wiener process with drift rate $\mu_1+\mu_2$ and variance rate $\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2$.

14.14

The change in S during the first three years has the probability distribution

$$\varphi(2\times3,\ 9\times3)=\varphi(6,\ 27)$$

The change during the next three years has the probability distribution

$$\varphi(3\times3,\ 16\times3)=\varphi(9,\ 48)$$

The change during the six years is the sum of a variable with probability distribution $\varphi(6,27)$ and a variable with probability distribution $\varphi(9,48)$. The probability distribution of the change is therefore

$$\varphi(6+9,27+48)$$

$$=\varphi(15,75)$$

Since the initial value of the variable is 5, the probability distribution of the value of the variable at the end of year six is

$$\varphi(20,75)$$

14.15

From Itô's lemma

$$\sigma_G G=\frac{\partial G}{\partial S}\sigma_S S$$

Also the drift of G is

$$\frac{\partial G}{\partial S}\mu S+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2$$

where μ is the expected return on the stock. When μ increases by $\lambda\sigma_S$, the drift of G increases by

$$\frac{\partial G}{\partial S}\lambda\sigma_S S$$

or

$$\lambda\sigma_G G$$

The growth rate of G, therefore, increases by $\lambda\sigma_G$.

14.16
Define S_A, μ_A and σ_A as the stock price, expected return and volatility for stock A. Define S_B, μ_B and σ_B as the stock price, expected return and volatility for stock B. Define ΔS_A and ΔS_B as the change in S_A and S_B in time Δt. Since each of the two stocks follows geometric Brownian motion,

$$\Delta S_A = \mu_A S_A \Delta t + \sigma_A S_A \varepsilon_A \sqrt{\Delta t}$$

$$\Delta S_B = \mu_B S_B \Delta t + \sigma_B S_B \varepsilon_B \sqrt{\Delta t}$$

where ε_A and ε_B are independent random samples from a normal distribution.

$$\Delta S_A + \Delta S_B = (\mu_A S_A + \mu_B S_B)\Delta t + (\sigma_A S_A \varepsilon_A + \sigma_B S_B \varepsilon_B)\sqrt{\Delta t}$$

This *cannot* be written as

$$\Delta S_A + \Delta S_B = \mu(S_A + S_B)\Delta t + \sigma(S_A + S_B)\varepsilon\sqrt{\Delta t}$$

for any constants μ and σ. (Neither the drift term nor the stochastic term correspond.) Hence, the value of the portfolio does not follow geometric Brownian motion.

14.17
In

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant when both are expressed as a proportion (or as a percentage) of the stock price.
In

$$\Delta S = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant in absolute terms. For example, if the expected growth rate is \$5 per annum when the stock price is \$25, it is also \$5 per annum when it is \$100. If the standard deviation of weekly stock price movements is \$1 when the price is \$25, it is also \$1 when the price is \$100.
In

$$\Delta S = \mu S \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

the expected increase in the stock price is a constant proportion of the stock price while the variability is constant in absolute terms.
In

$$\Delta S = \mu \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

the expected increase in the stock price is constant in absolute terms while the variability of the proportional stock price change is constant.
The model

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

is the most appropriate one since it is most realistic to assume that the expected *percentage return* and the variability of the *percentage return* in a short interval are constant.

14.18
The drift rate is $a(b-r)$. Thus, when the interest rate is above b the drift rate is negative and, when the interest rate is below b, the drift rate is positive. The interest rate is therefore continually pulled towards the level b. The rate at which it is pulled toward this level is a. A volatility equal to c is superimposed upon the “pull” or the drift.
Suppose $a = 0.4$, $b = 0.1$ and $c = 0.15$ and the current interest rate is 20% per annum. The interest rate is pulled towards the level of 10% per annum. This can be regarded as a long run average. The current drift is -4% per annum so that the expected rate at the end of one year is about 16% per annum. (In fact, it is slightly greater than this, because as the interest rate decreases, the “pull” decreases.) Superimposed upon the drift is a volatility of 15% per annum.

14.19
If $G(S,t) = S^n$ then $\partial G/\partial t = 0$, $\partial G/\partial S = nS^{n-1}$, and $\partial^2 G/\partial S^2 = n(n-1)S^{n-2}$. Using Itô’s lemma

$$dG = [\mu n G + \frac{1}{2}n(n-1)\sigma^2 G]\,dt + \sigma n G\,dz$$

This shows that $G = S^n$ follows geometric Brownian motion where the expected return is

$$\mu n + \frac{1}{2}n(n-1)\sigma^2$$

and the volatility is $n\sigma$. The stock price S has an expected return of μ and the expected value of S_T is $S_0 e^{\mu T}$. The expected value of S_T^n is

$$S_0^n e^{[\mu n + \frac{1}{2}n(n-1)\sigma^2]T}$$

14.20
The process followed by B, the bond price, is from Itô’s lemma,

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right]dt + \frac{\partial B}{\partial x}sxdz$$

Since

$$B = e^{-x(T-t)}$$

the required partial derivatives are:

$$\frac{\partial B}{\partial t} = xe^{-x(T-t)} = xB$$
$$\frac{\partial B}{\partial x} = -(T-t)e^{-x(T-t)} = -(T-t)B$$
$$\frac{\partial^2 B}{\partial x^2} = (T-t)^2 e^{-x(T-t)} = (T-t)^2 B$$

Hence,

$$dB = \left[-a(x_0 - x)(T-t) + x + \frac{1}{2}s^2x^2(T-t)^2\right]Bdt - sx(T-t)Bdz$$

14.21 (Excel Spreadsheet)

The process is

$\Delta S = 0.09 \times S \times \Delta t + 0.20 \times S \times \varepsilon \times \sqrt{\Delta t}$

where Δt is the length of the time step (=1/12) and ε is a random sample from a standard normal distribution.

14.22

(a) With the notation in the text

$$\frac{\Delta S}{S} \approx \varphi(\mu\Delta t, \sigma^2 \Delta t)$$

In this case, $S = 50$, $\mu = 0.16$, $\sigma = 0.30$ and $\Delta t = 1/365 = 0.00274$. Hence,

$$\frac{\Delta S}{50} \sim \varphi(0.16 \times 0.00274, 0.09 \times 0.00274)$$
$$= \varphi(0.00044, 0.000247)$$

and

$$\Delta S \sim \varphi(50 \times 0.00044, 50^2 \times 0.000247)$$

that is,

$$\Delta S \sim \varphi(0.022, 0.6164)$$

(a) The expected stock price at the end of the next day is therefore 50.022.

(b) The standard deviation of the stock price at the end of the next day is . $\sqrt{0.6154} = 0.785$

(c) 95% confidence limits for the stock price at the end of the next day are:

$$50.022 - 1.96 \times 0.785 \quad \text{and} \quad 50.022 + 1.96 \times 0.785$$

that is,

$$48.48 \quad \text{and} \quad 51.56$$

Note that some students may consider one trading day rather than one calendar day. Then $\Delta t = 1/252 = 0.00397$. The answer to (a) is then 50.032. The answer to (b) is 0.945. The answers to part (c) are 48.18 and 51.88.

14.23
The process followed by B, the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right]dt + \frac{\partial B}{\partial x}sxdz$$

In this case,

$$B = \frac{1}{x}$$

so that

$$\frac{\partial B}{\partial t} = 0; \quad \frac{\partial B}{\partial x} = -\frac{1}{x^2}; \quad \frac{\partial^2 B}{\partial x^2} = \frac{2}{x^3}$$

Hence,

$$dB = \left[-a(x_0 - x)\frac{1}{x^2} + \frac{1}{2}s^2x^2\frac{2}{x^3}\right]dt - \frac{1}{x^2}sxdz$$

$$= \left[-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right]dt - \frac{s}{x}dz$$

The expected instantaneous rate at which capital gains are earned from the bond is therefore,

$$-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

The expected interest per unit time is 1. The total expected instantaneous return is therefore,

$$1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

When expressed as a proportion of the bond price this is

$$\left(1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right)\Big/\left(\frac{1}{x}\right)$$

$$= x - \frac{a}{x}(x_0 - x) + s^2$$

14.24 (See Excel Worksheet)
The processes are:

$$\Delta S_A = 0.11 \times S_A \times \Delta t + 0.25 \times S_A \times \varepsilon_A \times \sqrt{\Delta t}$$
$$\Delta S_B = 0.15 \times S_B \times \Delta t + 0.30 \times S_B \times \varepsilon_B \times \sqrt{\Delta t}$$

Where Δt is the length of the time step (=1/252) and the ε's are correlated samples from standard normal distributions.

14.25
In (a) markets are not efficient (unless H=0.5) but in (b) they may be efficient.