Quantitative Interview Preparation

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Contents

1 Options Pricing & Financial Knowledge

1.1 Question List

1. What are the bid/offer quotes in the exchange and which one is higher?
2. A person who is short a commodity future are also given the optionality to decide delivery (how, where and when) and settlement method (either cash or physical). Is the future price worth more or less as compared with standard future contract (no optionality)?
3. If using forward and future for hedging purpose helps to mitigate risk and avoid losses, why doesn't everybody do it?
4. Explain the difference between convenience yield and cost-of-carry.
5. Which one of the following is easier to value: gold or coffee future?
6. Which of the following portfolio is expected to cost a higher premium (and why):
   - An option on a popular stock market index
   - A (weighted) portfolio comprising of options on each of the stocks included in the index
7. Discuss in detail the difference between American and European exercise type, in particular about the early exercise premium in American Options.
8. Describe the difference between a futures contract and a forward contract.
9. What is contango? Give an example.
10. What is backwardation? Give an example.
11. Describe put-call parity for options. Does it hold for American-style options?
12. You hold a call option with expiration date T on stock XYZ. The company announces a dividend distribution with ex-date *before* T. What happens to the value of the option? What if the ex-dividend date is *after* T?
13. What is implied volatility? Describe the general shape of the implied volatility curve for options with the same settlement date and different strikes, (a) in the case of equity indices, (b) in the case of an index of mining companies.
14. What is the delta of an option and what is it used for? Give an example.
15. What is a call (put) spread? Why would anyone buy/sell a call (put) spread?
16. What is a butterfly spread? Can a butterfly spread trade at a negative asking price?
17. Selling (naked) 10-year S&P500 index puts is a strategy that supports what view of the market?
18. Write a formula for currency forward rates and give an example.
19. A trade sells SGD for 1MM USD, and simultaneously enters into a 1-year forward contract to sell 1MM USD and buy SGD at the forward FX rate. What is the trader's PNL?
20. An up-and-out call should be more expensive if implied volatility suddenly rises. True or false?

21. In the context of a derivative contract, what are the differences between physical delivery and cash settlement, and when is one preferred to the other?
22. Explain the no-arbitrage argument behind the pricing mechanism of a forward contract.
23. What are the key distinguishing factors of commodities futures with respect to other types of asset classes?
24. How is spot discount factor related to spot LIBOR rate? How is forward discount factor related to forward LIBOR rate?
25. How is FRA different from a Eurodollar contract?
26. Describe the interest rate sensitivity (IR Delta) of a payer and receiver swaps.
27. What are the upper and lower boundaries for an European call/put option? For American call/put option?
28. Describe the concept of time value of an option. Can it ever be negative?
29. When is it optimal to exercise an American call option early?
30. When is it optimal to exercise an American put option early?
31. Describe the key features of a long gamma portfolio.
32. Describe the key features of a long vega portfolio.
33. Why can't we hedge gamma, vega or theta exposure using the underlying stock?
34. Describe the profile of delta, gamma and vega as an option approaches maturity.
35. How would you use vanilla European call/put options to replicate a digital cash-or-nothing option? A digital asset-or-nothing option?
36. Describe the vega profiles of a digital cash-or-nothing call and put options.
37. An up-and-out barrier call option $K = 100$ and $B = 98$ is worth 0. Why?
38. When would the vega of a barrier option becomes negative?
39. Explain what type of barrier combination give rise to an European option.
40. Explain the credit risk exposure profile for an interest rate swap under normal market scenario.
41. Why does it make more sense to measure the credit risks of corporate bonds by the spread between their yields vs. LIBOR interest rate swap instead of treasury bonds?
42. Why does high credit quality bonds have upward sloping credit curves while speculative grade bonds have downward sloping credit curves?
43. Describe the payoff of a credit default swap.
44. What is collateralisation and why is it important?
45. What is liquidity value adjustment? Credit Valuation Adjustment? Funding Valuation adjustment?

46. When will CVA and FVA be 0?

47. (a) What are the parameters affecting the price of an option?
    (b) Outline the steps used to derive the option pricing formula using binomial trees.
    (c) Outline the steps used to derive the option pricing formula in Black-Scholes approach.
    (d) Why does volatility appear in Black-Scholes but not in the binomial tree approach?

48. I have the performance data of 20 stocks in the past 5 years, consisting of 1000 readings for each stock. Explain how you would use principal component analysis to the data set.

49. (a) How does interest rate affect the price of a plain vanilla option, and why? (b) How does stock dividend affect the price of a plain vanilla option, and why?

50. A customer requested for the price of an option to a particular stock. How would you go about estimating the volatility of the stock to put into your model?

51. What are the differences between EWMA and GARCH? If I were to say GARCH is better would you agree, and why?

52. What does the term structure of interest rate looks like? As a bond portfolio manager, how would you trade if you know in advance that yield curve is going to invert?

53. How would you hedge your risk against parallel shift in yield curve?

54. What is the delta of an option? A plain vanilla call option strikes at the money. Assuming that interest rate is sufficiently small, estimate the delta.

55. Write down the Black-Scholes option pricing formula.

56. What is volatility smile and how does it come about?

57. How much stock should you long for one short position in a call option at the money 3 months away from maturity?

58. We are to enter into an interest rate swap contract. I am swapping floating rate for your fixed rate. The floating rate is expected to be continuously compounded at r1, r2, r3, ..., rN. What should your fixed rate be?

59. (a) What is the different between local volatility and implied volatility?
    (b) What is the major strength of the local volatility model?
    (c) What is the major weakness of the local volatiilty model?
    (d) How did SABR model overcome the weakness of the local volatility model?
    (e) What parameters affect the smile and skew in SABR model?
    (f) How do you hedge using local volatility model?
    (g) How do you hedge using SABR model?

60. Derive Black-Scholes' PDE.

61. Consider a digital cash-or-nothing call option that pays $1. Compare their prices in a world where the implied volatility skew for S&P500 is downward sloping and upward sloping.

62. How do you compute forward exchange rate given a spot exchange rate? How would you exploit arbitrage opportunity is someone quote you a forward exchange rate identical to the spot exchange rate in two different countries?

63. Explain Black-Scholes to someone who don't know math.

64. There are two portfolios of call options on the same underlying: 1) two European call with K = 100; 2) an European call with K = 90 and an European call with K= 110. Which one is more expensive? If someone is ready to trade this two portfolios at the same price, how would you exploit arbitrage opportunity?

65. S0 = 5. Su = 10, Sd = 1. r = 0. K = 3. What is the price of a call option at time 0?

66. Draw the delta, gamma, vega and theta against stock price.

67. There are two bonds with the same maturity in two countries: PA(0, T) and PB(0, T) in their respectively currency. The spot exchange rate is E0. How would you value the forward exchange rate, EF, at time T?

68. (a) Define an Asian option.
    (b) Is an Asian option typically more expensive or cheaper than say an European call?
    (c) An Asian call and an European call with 1 year to maturity for a slot of Vodafone share started out with the European call being more expensive than the Asian one. 0.5 year later, the Asian call is much more expensive than the European one. What do you think has happened?

69. The call options on the same underlying stock cost $11, $10 and $9 for a strike price of $8, $9 and $10 respectively. How would you trade?

70. (a) Describe the advantage of a long-gamma portfolio in detail.
    (b) In a long-gamma portfolio, you follow a buy-low sell-high strategy for the underlying stock. However, the theta is negative for this portfolio. Where are you losing money?
    (c) In a short-gamma portfolio, you follow a buy-high sell-low strategy for the underlying stock, which causes you to lose money. The theta is however positive for this portfolio. Where are you gaining money?
    (d) Would you prefer the realized volatility to be larger than the implied volatility if you are following a long-gamma strategy?

1.2 Answers To Selected Questions

1. Bid is the price where the exchange is happy to buy the asset, and offer is the price where the exchange is happy to sell the asset. Bid price has to be lower than offer price, otherwise you could arbitrage against the exchange by buying an asset from it at a lower price and selling it back to it again immediately at a higher price.

2. Option has positive non-zero value. Future with embedded optionality makes it more attractive for seller (short) and less attractive for buyer (long). To compensate for this, future price with embedded optionality tends to be lower.

3. A hedge not only cuts potential losses. It also cuts potential gain. The whole point of a hedge is to fix the payoff in the future today and thereby avoiding any risk exposure. If you want to participate in the upside and don't mind losing, then hedging is not for you.

4. Convenience yield measures the benefits and utility obtained from having ownership of the physical asset (difficult to quantify). On the other hand, cost-of-carry is the interest cost + storage cost − income earned (easy to quantify).

5. Coffee future is a commodity futures and need to account for convenience yield. As a result, gold future is the easier future to value.

6. The portfolio of options will cost more since the correlation among the stocks in the index offsets some volatility.

7.
   - The right to excercise early is what distinguishes American option from its European counterpart.
   - Most exchange-traded options are American style. Most OTC options are European style.
   - "Early exercise is not optimal" means that an option is worth more alive than dead. That is, you can get more by selling the option than you can by exercising it. If the underlying does not pay dividends, then early exercise is not optimal for vanilla American call: and thus American and European calls have equal value.
   - Early exercise of an American call option is optimal if, and only if, a dividend is about to be paid and it is large enough to replace the interest lost on the strike price, and the lost of the time value of the call. If the dividend is small, or the time to maturity is large, early exercise of an American call is unlikely to be optimal.
   - The benefit to early exercise of an American put is the ability to earn interest on the strike. The cost is that you give up any possible additional payoff.
   - The payoff to a put is bounded above by the strike (unlike a call which has unlimited upside). Therefore, if the stock price is low enough, then early exercise of an American put option may be optimal.
   - In general, it can be optimal to exercise a plain vanilla American put any time when it is deep in-the-money.

	American Call	American Put
Costs of early exercise	Lose interest on strike ☹ if $S \downarrow$	Lose dividends ☹ if $S \uparrow$
Benefits of early exercise	Capture dividends (Occasional benefit)	Earn interest on strike (Always available)
When to exercise early	Big dividend due Close to maturity Just before dividend payment	Small dividends remaining Far from maturity Deep in-the-money

2 Mathematics

2.1 Question List

1. Differentiate $f(x) = \alpha^x$ with respect to x.
2. Evaluate
$$1 + \frac{1}{1 + \frac{1}{1+\frac{1}{\cdots}}}$$
3. Evaluate the integral
$$\int \frac{1}{x} \ln(x) dx$$
4. Evaluate
$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$$
5. Evaluate
$$y = x^x, \quad \frac{dy}{dx} = ?$$
6. Evaluate
$$\int_0^1 a^x \; dx.$$
7. Prove that
$$1 + 2 + 3 + 4 + \cdots + N = \frac{N(N+1)}{2}.$$
8. Let $a, b \in \mathbb{R}$, and e denote the exponential function. Show that
$$\frac{e^a + e^b}{2} > e^{\frac{a+b}{2}}.$$
9. Evaluate
$$2 + \frac{2}{2 + \frac{2}{2+\frac{2}{2+\frac{2}{2+\frac{2}{\cdots}}}}}.$$
10. Which is greater, e^π or π^e?
11. Evaluate
$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}}}.$$
12. Suppose
$$x^{x^{x^{\cdots}}} = 2,$$
what is x? Suppose now that
$$x^{x^{x^{\cdots}}} = 4,$$
what is x? Discuss why do you get ins seemingly counter-intuitive results?

13. Solve the differential equation

$$y'' + y' + y = 10.$$

14. Evaluate

$$\int \ln(x)\, dx.$$

15. Evaluate

$$\int \frac{1}{x} \ln(x)\, dx.$$

16. Does the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ converge?

17. Prove that any differentiable function is continuous.

18. Prove that any differentiable even function $f : \mathbb{R} \to \mathbb{R}, f(x) = f(-x)$ must have zero gradient at $x = 0$.

    Form your arguments rigorously.

19. The temperature distribution on the earth is a 2-D continuous scalar field. When one travel one round trip around the earth, is there two points along the path travelled where the temperature is exactly identical? If the distance per round trip is L, is there two points $L/2$ apart where the temperature is exactly identical?

20. Prove that $\sqrt{5}$ is irrational.

2.2 Answers To Selected Questions

1.

$$\begin{aligned} y &= \alpha^x \\ \ln(y) &= x\ln(\alpha) \\ \frac{1}{y}\frac{dy}{dx} &= \ln(\alpha) \\ \frac{dy}{dx} &= \alpha^x \ln(\alpha) \end{aligned}$$

2. Using the recursive feature of the function,

$$\begin{aligned} x &= 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}} \\ x &= 1 + \frac{1}{x} \\ x^2 - x - 1 = 0 \\ x &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

Since the recursive function must evaluate to a value greater than 1, we pick the positive square root as solution $x = \frac{1+\sqrt{5}}{2}$.

3. Let $I = \int \frac{1}{x}\ln(x)dx$. Using integration by part, we have

$$\begin{aligned} \int \frac{1}{x}\ln(x)dx &= (\ln(x))^2 - \int \frac{1}{x}\ln(x)dx \\ I &= (\ln(x))^2 - I \\ I &= \frac{1}{2}(\ln(x))^2 \end{aligned}$$

4. Using the recursive feature,

$$\begin{aligned} x &= \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} \\ x &= \sqrt{1+x} \\ x^2 - x - 1 = 0 \\ x &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

So we select $x = \frac{1+\sqrt{5}}{2}$ as solution.

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17. A function $f(x)$ is continuous at a if $f(a)$ is defined, it's limit at $x \to a$ exists and $\lim_{x\to a} f(x) = a$. If a function is differentiable at a then

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

Since it is given that $f'(a)$ exists, we can proceed as follow

$$\begin{aligned}\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\times(x-a) = f'(a)\times 0 = 0\\ \Rightarrow \lim_{x\to a} f(x) &= f(a). \quad \lhd\end{aligned}$$

18. Since $f(x)$ is differentiable, we know that $f'(0)$ exists, and we have

$$f'(0) = \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}.$$

Using the property of even functions, we see that

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{f(0-h)-f(0)}{-h}\\ \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{-h}\\ 2\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} &= 0 \quad\Rightarrow\quad f'(0) = 0.\end{aligned}$$

19. Two points on the surface of a sphere are called antipodal points if the line between them passes through the centre of the sphere.

Antipodal theorem states that there is always a pair of antipodal points on a sphere at which the temperature is the same. The proof can be constructed as follow: let θ denote the angle between a straight line passing through a moving point on the sphere and the centre of the sphere, and a straight line passing through a fixed point and and the centre of the sphere. Let f be the function returning the temperate on the moving point. Then we have

$$f:[0,2\pi]\to\mathbb{R}, f(0)=f(2\pi); \exists c\in[0,\pi]: f(c)=f(c+\pi).$$

If $f(0)=f(\pi)$, then $c=0$. Consider the case where $f(0)<f(\pi)$, define

$$g(\theta) = f(\theta) - f(\theta+\pi), \theta\in[0,\pi].$$

Since f is continuous on $[0, 2\pi]$, it follows that g is continuous on $[0, \pi]$, and

$$g(0) = f(0) - f(\pi) < 0$$
$$g(\pi) = f(\pi) - f(2\pi) = f(\pi) - f(0) > 0,$$

and so by Intermediate Value Theorem, $\exists c \in (0, \pi) : g(c) = 0$, in other words, $f(c) - f(c+\pi) = 0$. Consider the case where $f(0) > f(\pi)$, then define

$$g(0) = f(0) - f(\pi) > 0$$
$$g(\pi) = f(\pi) - f(2\pi) = f(\pi) - f(0) < 0,$$

and the same applies with Intermediate Value Theorem. $\lhd$

20. If root 5 were rational, then it can be expressed as $\frac{a}{b}$, where $a, b \in \mathbb{Z}$, and there's no common factor apart from 1. And so

$$\frac{a^2}{b^2} = 5 \quad \Rightarrow \quad a^2 = 5b^2.$$

This implies that a^2 is a multiple of 5. By the Fundamental Theorem of Arithmetic, a is also a multiple of 5 (the prime factorisation of a^2 is just a duplicated copy of a, no new prime factors is introduced). Hence we can let $a = 5a_0$, then

$$5b^2 = 25a_0^2 \quad \Rightarrow \quad b^2 = 5a_0^2,$$

and now b^2, and hence b, is a multiple of 5, which is a contradiction.

3 Probability

3.1 Question List

1. D'Alembert thought that in a coin-tossing game, after a long run of heads, a tail is more likely. What is wrong with this?

2. Leibniz thought that it is as easy to throw 12 with a pair of dice as to throw 11, what is wrong with this?

3. We toss a fair coin repeatedly until we get a head. What is the expected number of tosses?

4. We toss a fair coin repeatedly until we get 2 heads in a row. What is the expected number of tosses?

5. *Birthday Problem:* Find the probability that in a group of $k(< 365)$ people, at least 2 have the same birthday (ignoring leap year).

6. Three prisoners A, B and C under death sentence are kept in separate cells. The governor has randomly selected and pardoned one of them. When the warden is making his round, prisoner A tries to figure out who has been pardoned. The warden is not allowed to reveal the identity of the pardoned prisoner, so A said: "then tell me the name of one of the others who will be pardoned. If B is pardoned, tell me C's name. If C is pardoned tell me B's name. If I am pardoned, give me either B or C's name."

   The warden thought about this and gave him B's name. Prisoner A thinks that his chance of survival has risen from 1/3 to 1/2 (since either A or C will survive). Is this correct?

7. What is the probability that the first business day of the month is a Monday?

8. What is the probability that the last business day of the month is a Friday?

9. In Missouri, Ernest Pullen won a lottery in June for $1 million. In September, he won a second lottery for $2 million. Journalists called this a one in 1,000,000,000,000 event, since assuming

   - one winner per day
   - 1,000,000 player per day
   - probability of winning is one in 1,000,000
   - probability of winning twice is 1,000,000,000,000

   Is this reasoning correct?

10. 3 players. Throwing a coin. First one to throw a head wins. If first player throws tail, pass it on to next player etc. What is the probability of each player to win? (ans: 4/7, 2/7, 1/7)

11. There are 3 red balls and 2 blue balls. I put all of them randomly into two jars, with at least 2 balls in each jar. I then randomly select a jar, and pick up two balls. I show you the 2^{nd} one, which is blue; what is the probability of the first one being blue? If I show you the first one, which is red; what is the probability of the 2^{nd} one being blue?

12. I draw a number x from 0 to 1 randomly.

    (a) How does the probability distribution function looks like?

(b) What is the variance of x?

(c) A new number is defined as $z = x_1 + x_2$. What is the probability distribution function?

(d) What is the mean and variance of z?

13. You are standing on 0 on the natural number line. What is the expected number of steps required to reach 2?

14. A person is standing on the edge of a cliff. He has a 50% chance of moving to the left (death) and a 50% chance of moving to the right (away from the cliff).

    (a) What is his probability of survival?

    (b) What is the expected number of steps he has to take before his fall?

    (c) Explain qualitatively why he can take an infinite amount of steps before he fall.

15. I toss a fair coin 400 times. What is the probability of getting exactly 200 heads? 199 heads? 201 heads? What does the probability distribution function looks like?

16. I observed the mean temperature at a place for 110 days. For the first 100 days, the mean temperature is 80 degree. From day 101 to 105, the temperature is 85 degree, from day 106 to 110, the temperature is 90 degree. What is the median temperature?

17. I have a variable following a normal distributed profile. What is the probability of the variable > 5? (estimate)

18. Suppose $U \sim N(0,1)$, $V \sim N(0,1)$, and $U \perp V$. Given a constant $a \in \mathbb{R}$, find $\mathbb{E}[e^{aUV}]$.

19. Suppose there are 1000 coins, 999 are fair and 1 has 2 heads. You pick one from them and flip it and observe a head, what is the probability it's the unfair coin? If you flip it 10 times and all are heads, what is the probability then?

20. For a coin tossing game, the game terminates whenever you toss a head. If you toss 1 tail, you receive a payoff of \$2. If you toss 2 consecutive tails, you receive \$4. For 3 consecutive tails, you receive \$8, and so on. How much would you pay to play this game?

21. A random experiment has three outcomes, A, B and C, with probabilities p_A, p_B and p_C, respectively, where $p_C = 1 - p_A - p_B$. What is the probability that, in independent performances of the experiment, A will occur before B?

22. A random experiment has 3 possible outcomes: A, B, and C. The associated probabilities are $p_A = 0.2$ and $p_B = 0.4$. What is the probability that, in independent performances of the experiment, A will occur before B?

23. Let X denote a random variable with a mean of μ. Discuss whether the following inequality is always true:

$$\mathbb{E}[X^2] \geq \mu^2$$

24. *The Lift Problem* A lift has three occupants: A, B and C, and there are three possible floors: 1, 2 and 3, at which they can get out. Assuming that each person acts independently of the others and that each person is equally likely to get out at each floor, calculate the probability that exactly one person will get out at each floor.

25. It rains with 50% probability in Singapore. The weather forecaster makes a prediction of the weather every day and is correct $\frac{2}{3}$ of the time, i.e.

    - the probability that it rains, given that she has predicted rain, is $\frac{2}{3}$
    - the probability that it does not rain, given that she has predicted that it won't rain, is also $\frac{2}{3}$.

    If the forecaster predicts rain, Mr. Tee brings his umbrella. If the forecaster predicts no rain, he takes it with probability $\frac{1}{3}$. Determine

    (a) the probability that Mr. Tee has no umbrella, given that it rains.

    (b) the probability that it doesn't rain, given that he brings his umbrella.

    (c) the probability that he has an umbrella, given that it doesn't rain.

26. Assume that E and F are two events with positive probabilities. Show that if $\mathbb{P}(E|F) = \mathbb{P}(E)$, then $\mathbb{P}(F|E) = \mathbb{P}(F)$.

27. If $\mathbb{P}(\bar{B}) = \frac{1}{4}$, and $\mathbb{P}(A|B) = \frac{1}{2}$, determine $\mathbb{P}(A \cap B)$.

28. Prove the following relationship for any three events A, B, and C, each having positive probability:

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|A \cap B).$$

29. Prove that if A and B are independent, so are

    (a) A and $\bar{B}$.

    (b) $\bar{A}$ and $\bar{B}$.

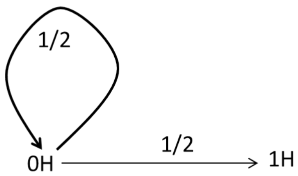
30. Suppose that A and B are events such that $\mathbb{P}(A|B) = \mathbb{P}(B|A)$, $\mathbb{P}(A \cup B) = 1$, and $\mathbb{P}(A \cap B) > 0$. Prove that $\mathbb{P}(A) > \frac{1}{2}$.

3.2 Answers To Selected Questions

1. Every coin toss is independent.

2. Only $\{6,6\}$ yields 12 but $\{5,6\}$ and $\{6,5\}$ both yield 11, so getting 11 is twice as likely as getting 12.

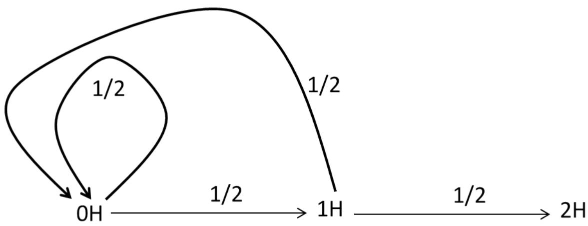
3. Let N_1 denote the expected number of fair coin tosses to get 1 head, given the diagram below:



we can work out that

$$\begin{aligned} & N_1 = \frac{1}{2} \times 1 + \frac{1}{2} \times (1 + N_1) \\ \Rightarrow \quad & N_1 = 2. \quad \lhd \end{aligned}$$

4. Similar to the approach in the previous question, let N_2 denote the expected number of fair coin tosses to get 2 heads, given the relationship:



we can work out that

$$\begin{aligned} & N_2 = \frac{1}{2} \times (1 + N_2) + \frac{1}{2} \times \frac{1}{2} \times (2 + N_2) + \frac{1}{2} \times \frac{1}{2} \times 2 \\ \Rightarrow \quad & N_2 = 6. \quad \lhd \end{aligned}$$

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21. Let D denote the event where A occurs before B. Condition on the result of the first trial (A_1, B_1 or C_1). Thus

$$\begin{aligned}\mathbb{P}(D) &= \mathbb{P}(D|A_1)\mathbb{P}(A_1) + \mathbb{P}(D|B_1)\mathbb{P}(B_1) + \mathbb{P}(D|C_1)\mathbb{P}(C_1)\\ &= \mathbb{P}(D|A_1)p_A + \mathbb{P}(D|B_1)p_B + \mathbb{P}(D|C_1)p_C.\end{aligned}$$

It should be clear that: $\mathbb{P}(D|A_1) = 1$, $\mathbb{P}(D|B_1) = 0$ and $\mathbb{P}(D|C_1) = \mathbb{P}(D)$, since this is the case the problem after the first trial is exactly as at the start (in view of the independence of the trials). So we have

$$\mathbb{P}(D) = p_A + \mathbb{P}(D)p_C \quad \Rightarrow \quad \mathbb{P}(D) = \frac{p_A}{p_A + p_B}. \quad \triangleleft$$

22. $\frac{1}{3}$.

23. Yes, variance is always ≥ 0.

24. **Solution I: conditional approach** Let F_i denote the event that one person gets off at floor i, and A_i denote the event that A gets off at floor i (events B_i and C_i are defined similarly). Then the required probability is

$$\mathbb{P}(F_1 \cap F_2 \cap F_3) = \mathbb{P}(F_1)\mathbb{P}(F_2|F_1)\mathbb{P}(F_3|F_1 \cap F_2).$$

Now,

$$\begin{aligned}\mathbb{P}(F_1) &= \mathbb{P}((A_1 \cap \bar{B}_1 \cap \bar{C}_1) \cup (\bar{A}_1 \cap B_1 \cap \bar{C}_1) \cup (\bar{A}_1 \cap \bar{B}_1 \cap C_1))\\ &= 3 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^3 = \frac{4}{9}.\end{aligned}$$

By similar argument,

$$\begin{aligned}\mathbb{P}(F_2|F_1) &= 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}\\ \mathbb{P}(F_3|F_1 \cap F_2) &= 1.\end{aligned}$$

So, the required probability is

$$\frac{4}{9} \times \frac{1}{2} \times 1 = \frac{2}{9}. \quad \triangleleft$$

Solution II: indicator approach Let

$$X = \begin{cases} 1, & \text{if exactly one person gets off at each floor} \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y_i = \begin{cases} 1, & \text{if no one gets off at floor } i \\ 0, & \text{otherwise} \end{cases}$$

Then $X = (1 - Y_1)(1 - Y_2)(1 - Y_3)$ and

$$\begin{aligned} \mathbb{P}(\text{one person gets off at each floor}) &= \mathbb{P}(X = 1) \\ &= \mathbb{E}[X] \\ &= \mathbb{E}[1 - (Y_1 + Y_2 + Y_3) + (Y_1Y_2 + Y_1Y_3 + Y_2Y_3) - Y_1Y_2Y_3] \\ &= 1 - p_1 - p_2 - p_3 + p_{12} + p_{13} + p_{23} - p_{123}, \end{aligned}$$

where

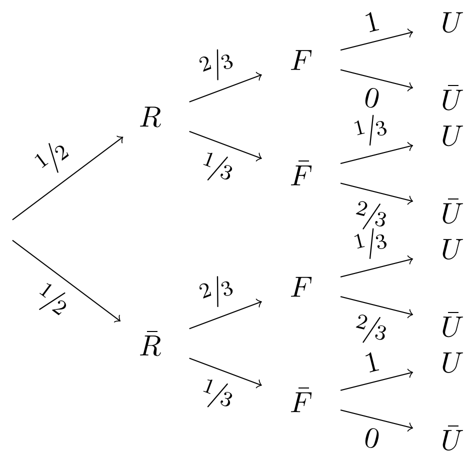
$$\begin{aligned} p_i &= \mathbb{P}(Y_i = 1) = \left(\frac{2}{3}\right)^3, \qquad i = 1, 2, 3 \\ p_{ij} &= \mathbb{P}(Y_i = 1, Y_j = 1) = \left(\frac{1}{3}\right)^3, \qquad i \neq j \\ p_{123} &= \mathbb{P}(Y_1 = 1, Y_2 = 1, Y_3 = 1) = 0. \end{aligned}$$

So the required probability is

$$1 - 3 \times \left(\frac{2}{3}\right)^3 + 3 \times \left(\frac{1}{3}\right)^3 = \frac{2}{9}. \quad \triangleleft$$

25. Let R denote the event that it rains, F denote the event that the forecast is correct, and U denote the event that Mr. Tee brings an umbrella.

<u>Method I: Binomial Tree</u>



(a) Using the binomial tree, we can work out that

$$\mathbb{P}(\bar{U}|R) = \frac{\mathbb{P}(\bar{U} \cap R)}{\mathbb{P}(R)} = \frac{\frac{1}{2} \times \frac{1}{3} \times \frac{2}{3}}{\frac{1}{2}} = \frac{2}{9} \approx 0.222 \quad \triangleleft$$

(b) Using the binomial tree, we can work out that

$$\begin{aligned}\mathbb{P}(\bar{R}|U) = \frac{\mathbb{P}(\bar{R}\cap U)}{\mathbb{P}(U)} &= \frac{\frac{1}{2}\times\frac{2}{3}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}\times 1}{\frac{1}{2}\times\frac{2}{3}\times 1+\frac{1}{2}\times\frac{1}{3}\times\frac{1}{3}+\frac{1}{2}\times\frac{2}{3}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}\times 1} \\ &= \frac{5}{12} \approx 0.41667 \quad \triangleleft\end{aligned}$$

(c) Using the binomial tree, we can work out that

$$\mathbb{P}(U|\bar{R}) = \frac{\mathbb{P}(U\cap\bar{R})}{\mathbb{P}(\bar{R})} = \frac{\frac{1}{2}\times\frac{2}{3}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}\times 1}{\frac{1}{2}} = \frac{5}{9}. \quad \triangleleft$$

Method II: Analytical

(a)

$$\begin{aligned}\mathbb{P}(\bar{U}|R) &= \mathbb{P}(\bar{U}|R\cap F)\mathbb{P}(F|R) + \mathbb{P}(\bar{U}|R\cap\bar{F})\mathbb{P}(\bar{F}|R) \\ &= 0\times\frac{2}{3}+\frac{2}{3}\times\frac{1}{3} = \frac{2}{9}.\end{aligned}$$

(b) The probability of bringing an umbrella is

$$\begin{aligned}\mathbb{P}(U) &= \mathbb{P}(U|R)\mathbb{P}(R) + \mathbb{P}(U|\bar{R})\mathbb{P}(\bar{R}) \\ &= \Big[\mathbb{P}(U|R\cap F)\mathbb{P}(F|R) + \mathbb{P}(U|R\cap\bar{F})\mathbb{P}(\bar{F}|R)\Big]\mathbb{P}(R) \\ &\quad + \Big[\mathbb{P}(U|\bar{R}\cap F)\mathbb{P}(F|\bar{R}) + \mathbb{P}(U|\bar{R}\cap\bar{F})\mathbb{P}(\bar{F}|\bar{R})\Big]\mathbb{P}(\bar{R}) \\ &= \left[1\times\frac{2}{3}+\frac{1}{3}\times\frac{1}{3}\right]\times\frac{1}{2} + \left[\frac{1}{3}\times\frac{2}{3}+1\times\frac{1}{3}\right]\times\frac{1}{2} \\ &= \frac{2}{3}.\end{aligned}$$

The probability that Mr Tee doesn't have an umbrella, given that it is raining, is:

$$\begin{aligned}\mathbb{P}(\bar{U}|R) &= \mathbb{P}(\bar{U}|R\cap F)\mathbb{P}(F|R) + \mathbb{P}(\bar{U}|R\cap\bar{F})\mathbb{P}(\bar{F}|R) \\ &= 0\times\frac{2}{3}+\frac{2}{3}\times\frac{1}{3} = \frac{2}{9}.\end{aligned}$$

Hence, we have

$$\mathbb{P}(\bar{R}|U) = \frac{\mathbb{P}(U|\bar{R})\mathbb{P}(\bar{R})}{\mathbb{P}(U)} = \frac{\frac{5}{9}\times\frac{1}{2}}{\frac{2}{3}} = \frac{5}{12}.$$

(c)

$$\begin{aligned}\mathbb{P}(U|\bar{R}) &= \mathbb{P}(U|\bar{R}\cap F)\mathbb{P}(F|\bar{R}) + \mathbb{P}(U|\bar{R}\cap\bar{F})\mathbb{P}(\bar{F}|\bar{R}) \\ &= \frac{1}{3}\times\frac{2}{3}+1\times\frac{1}{3} = \frac{5}{9}.\end{aligned}$$

26. Given that $\mathbb{P}(E|F) = \mathbb{P}(E)$, we have

$$\begin{aligned}\mathbb{P}(E\cap F) &= \mathbb{P}(E|F)\mathbb{P}(F) \\ &= \mathbb{P}(E)\mathbb{P}(F).\end{aligned}$$

Similarly, we could have written

$$\mathbb{P}(E \cap F) = \mathbb{P}(F|E)\mathbb{P}(E),$$

substituting, we obtain

$$\begin{aligned} & \mathbb{P}(E)\mathbb{P}(F) = \mathbb{P}(F|E)\mathbb{P}(E) \\ \Rightarrow \quad & \mathbb{P}(F|E) = \mathbb{P}(F) \quad \triangleleft \end{aligned}$$

27. We have

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A|B)\mathbb{P}(B) \\ &= \mathbb{P}(A|B)(1 - \mathbb{P}(\bar{B})) \\ &= \frac{1}{2} \times \frac{3}{4} = \frac{3}{8} = 0.375 \quad \triangleleft \end{aligned}$$

28.

29.

30. First, we note that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A).$$

Since it is given that $\mathbb{P}(A|B) = \mathbb{P}(B|A)$, we have

$$\mathbb{P}(B) = \mathbb{P}(A) = p.$$

Next, we use the collectively exhaustive probability of events A and B to write

$$\begin{aligned} \mathbb{P}(A \cup B) = 1 &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ 1 &= p + p - \mathbb{P}(A \cap B) \\ 1 &= 2p - \mathbb{P}(A \cap B) \\ \Rightarrow \quad 1 &< 2p \qquad \because \mathbb{P}(A \cap B) > 0 \end{aligned}$$

Hence we conclude that $\mathbb{P}(A) = \mathbb{P}(B) > \frac{1}{2}$. $\triangleleft$

4 Stochastic Calculus

4.1 Question List

1. Consider the following 2 options with the same maturity:

   (a) Pays one dollar if the stock ends up greater than \$100 on maturity;

   (b) Pays one dollar once the stock hits \$100 before maturity, and the contract terminates.

   What is the relationship between the prices of the two options?

2. Show that

$$\mathbb{E}[W_t^4] = 3t^2.$$

3. Show that

$$\mathbb{E}_s[W_t^2] = W_s^2 + (t-s)$$

4. Consider the mean reverting process

$$dx_t = \lambda(\theta - x_t)dt + \sigma dW_t,$$

   where λ is the mean reversion speed, θ is the long run mean, σ is the volatility and $W_t \sim N(0,t)$ is a standard Brownian motion. What is the half-life of x_t decaying to the long run mean?

5. Discuss what will happen when the volatility tends to infinity for a call option when the underlying asset follows

   - a geometric Brownian motion
   - an arithmetic Brownian motion

6. If I write $dS = rSdt + \sigma SdW_t$, what is the numeraire of this world?

7. What is the drift rate for a stock price if $\mathbb{E}\left[\frac{B_t}{S_t}\right]$ is to be a Martingale, where B is a zero bond given by $dB = rBdt$.

8. If S is log-normally distributed, derive the general equation for $\mathbb{E}[S^n]$.

9. If Q is normally distributed as $N(0,1)$; compute $\mathbb{E}[Q]$, $\mathbb{E}[Q^2]$, $\mathbb{E}[e^Q]$.

10. If the stock price motion can be described by the following equation:

$$\log\frac{S_{i+1}}{S_i} = -\frac{\sigma^2}{2}\Delta t + \sigma Z_t\sqrt{\Delta t}, \quad i = 0, 1, \cdots, N-1,$$

   find S_N.

11. Write down the Black-Scholes partial differential equation. Now change the variable S in this partial differential equation into Q, where $Q = \log(S)$.

12. In a stochastic volatility model, how does the volatility of volatility affects the smile? How does the correlation affects the smile? How does a power stock process affects the smile?

13. A stock price is currently at \$0. If the volatility is 40%, and the drift rate is 0.04, what is the probability of the stock price reaching \$10?

14. Derive the Black-Scholes formula for a call option which pays $(S_T^2 - K)^+$ on maturity.

15. Suppose the spot stock price is \$100. After one period, the price will be either \$110 or \$90. Assuming zero interest rate, price an ATM call option on the stock.

16. (a) What is the probability measure and the choice of numeraire when we value a swaption using the Swap Market Model?

    (b) How do you value a swaption under the risk-neutral measure associated with the zero-coupon discount bond maturing at the same time as the maturity of the swaption as numeraire (T-forward measure)?

17. Compute $\mathbb{E}[W_t^4]$, where W_t denote a standard Brownian motion.

18. Consider the stock price process S_t modelled as either

$$dS_t = rS_t dt + \sigma S_t dW_t$$

or

$$dS_t = \sigma S_0 dW_t$$

Derive a valuation formula for a European call option, and discuss what happens to the option value when volatility tends towards ∞?

19. Find the mean and variance of X_t if

    (a) $dX_t = \mu dt + \sigma dW_t$

    (b) $dX_t = \mu X_t dt + \sigma X_t dW_t$

    (c) $dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$

20. (a) Show that $X_t = e^{W_t}$ is a solution to the stochastic differential equation $dX_t = \frac{1}{2}X_t dt + X_t dW_t$.

    (b) $X_t = \frac{W_t}{1+t}$ is a solution to the stochastic differential equation $dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dW_t$.

21. (a) Solve the following stochastic differential equation

$$dX_t = X_t dt + dW_t$$

using the integrating factor e^{-t} (i.e. considering the process dY_t where $Y_t = e^{-t}X_t$).

    (b) Solve the following stochastic differential equation

$$dX_t = -X_t dt + e^{-t} dW_t.$$

4.2 Answers to Selected Questions

1. This is a question on the reflection principle. For a symmetrical random walk, once it hit a particular level H, we can construct a shadow path that, upon arriving at H following the original trajectory, is thereafter reflected as the mirror image of the original trajectory. Both trajectories will cause the second option to pay out one dollar, but only one of them will cause the first option to pay out one dollar. Since the random walk is symmetrical, it is equally likely that the first option will end up in the money vs. out of the money. So the first option is approximately half the price of the second option.

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14. Under Black-Scholes formulation, we have the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a standard Brownian motion under the risk-neutral measure associated with the risk-free bond B_t as numeraire. Solving the sde, we obtain

$$\begin{aligned} S_T &= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T} \\ S_T^2 &= S_0^2 e^{\left(2r-\sigma^2\right)T+2\sigma W_T} \end{aligned}$$

The option is in-the-money when

$$\begin{aligned} S_T^2 &> K \\ S_0^2 e^{\left(2r-\sigma^2\right)T+2\sigma W_T} &> K \\ \Rightarrow \quad x &> \frac{\log \frac{K}{S_0^2} - (2r-\sigma^2)T}{2\sigma\sqrt{T}} = x^*. \end{aligned}$$

And so

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}^*[(S_T^2 - K)^+] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left[S_0^2 e^{(2r-\sigma^2)T+2\sigma\sqrt{T}x} - K \right] e^{-\frac{x^2}{2}}\, dx \\ &= S_0^2 e^{(r+\sigma^2)T}\Phi(-x^* + 2\sigma\sqrt{T}) - Ke^{-rT}\Phi(-x^*). \quad \lhd \end{aligned}$$

15. The binomial tree is as follow:

$$\begin{tikzcd} & 90 \\ 100 \arrow[ur] \arrow[dr] & \\ & 110 \end{tikzcd}$$

Under the martingale valuation framework, we have

$$\frac{S_0}{B_0} = \mathbb{E}^*\left[\frac{S_1}{B_1}\right].$$

Since $r = 0$, we have $B_0 = B_1$. Hence

$$\begin{aligned} S_0 = \mathbb{E}^*[S_1] &= p^* \times 110 + (1 - p^*) \times 90 \\ \Rightarrow \quad p^* &= \frac{1}{2}. \end{aligned}$$

Using the risk-neutral probabilities, we can value the call option as follows:

$$\begin{aligned} \frac{c_0}{B_0} &= \mathbb{E}^*\left[\frac{c_1}{B_1}\right] \\ \Rightarrow \quad c_0 &= p^* \times (110 - 100)^* + (1 - p^*)(90 - 100)^* = 5. \quad \triangleleft \end{aligned}$$

16. (a) Consider a swap that starts at T_n and ends at T_N, the payments are made at T_{n+1}, T_{n+2}, $\cdots$, T_N. Under the Swap Market Model, we model the swap rate using the following stochastic differential equation

$$dS_{n,N} = \sigma_{n,N} S_{n,N} dW_t^{n+1,N},$$

where $W_t^{n,N}$ is a standard Brownian motion under $\mathbb{Q}^{n+1,N}$, the risk-neutral measure associated with the numeraire $P_{n+1,N}(t) = \sum_{i=n+1}^{N} \Delta_{i-1} D_i(t)$. The payoff of a payer swap is given by

$$V^{flt}(t) - V^{fix}(t) = P_{n+1,N}(t)(S_{n,N}(t) - K).$$

Using martingale valuation framework, the Swap Market Model formula for a payer swaption is given by

$$\begin{aligned} \frac{V^{pay}(0)}{P_{n+1,N}(0)} &= \mathbb{E}^{n+1,N}\left[\frac{V^{pay}(T)}{P_{n+1,N}(T)}\right] \\ V^{pay}(0) &= P_{n+1,N}(0)\mathbb{E}^{n+1,N}[(S_{n,N}(T) - K)^+]. \quad \triangleleft \end{aligned}$$

(b) Suppose we were to proceed using the risk-neutral measure associated with the risk-free bond as numeraire, the payer swaption formula would look as follow:

$$\begin{aligned} V^{pay}(0) &= P_{n+1,N}(0)\mathbb{E}^{n+1,N}[(S_{n,N}(T) - K)^+] \\ &= P_{n+1,N}(0)\mathbb{E}^T\left[(S_{n,N}(T) - K)^+ \frac{d\mathbb{Q}^{n+1,N}}{d\mathbb{Q}^T}\right] \\ &= P_{n+1,N}(0)\mathbb{E}^T\left[(S_{n,N}(T) - K)^+ \frac{P_{n+1,N}(T)/P_{n+1,N}(0)}{D(T,T)/D(0,T)}\right] \\ &= D(0,T)\mathbb{E}^T[P_{n+1,N}(T)(S_{n,N}(T) - K)^+]. \end{aligned}$$

With the PVBP $P_{n+1,N}$ in the expectation, we will not be able to obtain an analytical solution. Choosing the appropriate numeraire often allows us to simplify the valuation formula. $\triangleleft$

17. Since $W_t \sim N(0,t)$, we have

$$\mathbb{E}[W_t^4] = \mathbb{E}[N(0,t)^4] = \mathbb{E}[(\sqrt{t})^4 N(0,1)^4] = t^2\mathbb{E}[N(0,1)^4] = 3t^2. \quad \lhd$$

Expectation of the powers of standard normal random variable are given by:

$$\mathbb{E}[N(0,1)] = 0, \quad \mathbb{E}[N(0,1)^2] = 1, \quad \mathbb{E}[N(0,1)^3] = 0, \quad \mathbb{E}[N(0,4)^4] = 3.$$

18. Using the geometric Brownian motion process, we obtain the Black-Scholes formula

$$c_{BS} = S_0\Phi\left(\frac{\log\frac{S_0}{K} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

When volatility tends to infinity, we have

$$\begin{aligned}
\lim_{\sigma\to\infty} c_{BS} &= \lim_{\sigma\to\infty}\left[S_0\Phi\left(\frac{\log\frac{S_0}{K} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right] \\
&= \lim_{\sigma\to\infty}\left[S_0\Phi\left(\frac{\log\frac{S_0}{K}+rT}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K}+rT}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right)\right] \\
&= S_0\Phi(0+\infty) - Ke^{-rT}\Phi(0-\infty) \\
&= S_0 \times 1 - Ke^{-rT}\times 0 = S_0.
\end{aligned}$$

The no-arbitrage price boundary for an European call option is the stock price, so this result is what we would expect.

On the other hand, using the arithmetic Brownian process, we obtain the Bachelier's formula:

$$c_B = e^{-rT}\left[(S_0-K)\Phi\left(\frac{S_0-K}{\sigma\sqrt{T}S_0}\right) + \sigma\sqrt{T}S_0\phi\left(\frac{S_0-K}{\sigma\sqrt{T}S_0}\right)\right].$$

When volatility tends to infinity, we have

$$\begin{aligned}
\lim_{\sigma\to\infty} c_B &= \lim_{\sigma\to\infty} e^{-rT}\left[(S_0-K)\Phi\left(\frac{S_0-K}{\sigma\sqrt{T}S_0}\right) + \sigma\sqrt{T}S_0\phi\left(\frac{S_0-K}{\sigma\sqrt{T}S_0}\right)\right] \\
&= \lim_{\sigma\to\infty} e^{-rT}\left[(S_0-K)\Phi(0) + \sigma\sqrt{T}S_0\phi(0)\right] \\
&= e^{-rT}\left[(S_0-K)\times 0.5 + \lim_{\sigma\to\infty}\sigma\sqrt{T}S_0 \times \frac{1}{\sqrt{2\pi}}\right] \\
&= \infty. \quad \lhd
\end{aligned}$$

As far as the distribution of the underlying asset is concerned, the two main advantages of a lognormal model over a normal model is 1) no density for negative asset value, and 2) large volatility does not cause derivative prices to grow without bound. $\lhd$

19. (a) This is an arithmetic Brownian motion. The solution is given by:

$$X_T = X_0 + \mu T + \sigma W_T.$$

Hence

$$\begin{aligned}
\mathbb{E}[X_T] &= X_0 + \mu T \\
Var(X_T) &= Var(\sigma W_T) = \sigma^2 T.
\end{aligned}$$

(b) This is a geometric Brownian motion. The solution is given by:

$$X_T = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}.$$

Hence

$$\begin{aligned}\mathbb{E}[X_T] &= X_0 e^{\mu T} \\ Var(X_T) &= \mathbb{E}[X_T^2] - \mathbb{E}[X_T]^2 = X_0^2 e^{(2\mu+\sigma^2)T} - X_0^2 e^{2\mu T}.\end{aligned}$$

(c) This is an Ornstein-Uhlenbeck mean reverting process. The solution is given by:

$$X_T = X_0 e^{-\kappa T} + \theta(1 - e^{-\kappa T}) + \sigma \int_0^T e^{\kappa(t-T)}\, dW_t.$$

Hence

$$\begin{aligned}\mathbb{E}[X_T] &= X_0 e^{-\kappa T} + \theta(1 - e^{-\kappa T}) \\ Var(X_T) &= \mathbb{E}\left[\sigma^2 \int_0^T e^{2\kappa(t-T)}\, dt\right] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa T}). \quad \triangleleft\end{aligned}$$

20. Apply Itôs formula.

21. (a) Let $Y_t = e^{-t} X_t = f(t, X_t)$, where the function f is defined as $f : \mathbb{R}^2 \to \mathbb{R},\ f(t,x) = e^{-t}x$. We have the partial derivatives

$$\frac{\partial f}{\partial t} = -e^{-t}x, \qquad \frac{\partial f}{\partial x} = e^{-t}, \qquad \frac{\partial^2 f}{\partial x^2} = 0.$$

By Itô's formula, we obtain

$$dY_t = -e^{-t} X_t dt + e^{-t}\left(X_t dt + dW_t\right) = e^{-t} dW_t.$$

Integrating both sides from 0 to T, we have

$$\begin{aligned}\int_0^T dY_u &= \int_0^T e^{-u}\, dW_u \\ Y_T - Y_0 &= \int_0^T e^{-u}\, dW_u \\ e^{-T} X_T &= X_0 + \int_0^T e^{-u}\, dW_u \\ X_T &= X_0 e^T + \int_0^T e^{T-u}\, dW_u. \quad \triangleleft\end{aligned}$$

(b) Using the integrating factor e^t, we consider the process $Y_t = e^t X_t = f(t, X_t)$, where the function f is defined as $f : \mathbb{R}^2 \to \mathbb{R},\ f(t,x) = e^t x$. We have the partial derivatives

$$\frac{\partial f}{\partial t} = e^t x, \qquad \frac{\partial f}{\partial x} = e^t, \qquad \frac{\partial^2 f}{\partial x^2} = 0.$$

By Itô's formula, we obtain

$$dY_t = Y_t dt + e^t dX_t = Y_t dt + e^t(-X_t dt + e^{-t} dW_t) = dW_t.$$

Integrating both sides from 0 to T, we have

$$\begin{aligned}Y_T &= Y_0 + W_T \\ e^T X_T &= X_0 + W_T \\ X_T &= e^{-T} X_0 + e^{-T} W_T. \quad \triangleleft\end{aligned}$$

5 Brain Teasers

5.1 Question List

1. You have a flock of sheep and I have a flock of sheep. If I gave you a sheep you would have twice as much as I have. If you gave me one, both would have the same size. How many sheeps do each of us have? (5 and 7)

2. You have a bar of gold which you use to pay someone for his/her work. The person works for you for 7 days and you need to make a fair payment every day. You are only allowed to cut the gold bar twice. How do you do it?

3. A, B, C and D are to cross a bridge. The speed they can cross the bridge is 1, 2, 5 and 10 minutes, respectively. There is only 1 torch, and only two persons can cross the bridge together (the torch has to be carried by the person crossing the bridge). How quickly can they get everybody across the bridge?

4. You have 2 ropes that burn for an hour each. How do you time 45 minutes?

5. There are 100 prisoners. Each of the have a hat which is black or white. But they can't see they own hat. They're going to be called one by one, and they have to tell a color. If a prisoner says the color of his hat his saved. Otherwise they kill him. Which strategy should they adopt to maximize the number of prisoners that survive?

6. 100 lions and a lamb are in a cage. The logic of the lions is as follow:

   - to survive
   - to eat the lamb
   - if a lion eats a lamb, it becomes a lamb as well

   What is going to happen?

7. Suppose in a game of Roulette, the probability of a 'red' outcome or a 'black' outcome is similar, since there is an equal number of either. If someone approaches you to bet separately from the gambling table, where you win \$60 when it turns up red and lose \$40 when it turns up black, how should you play it? (You're allowed to bet with the house as well in addition to playing with this person for each game)

8. Estimate the number of opticians in Great Britain.

9. Five pirates chance upon an island and find a treasure chest containing 100 gold coins. The pirates are perfectly rational but not malicious. It is ordained that pirate 1 is first given the chance to decide how the gold should be divided among the five. Then the remaining pirates get to vote on his decision and if the vote goes against him or is a tie, then he is killed and thrown off the ship. If that happens, then pirate 2 gets to decide and the vote is taken again. And so on. So, how should pirate 1 propose to divide the gold?

10. A national park is surrounded a road that makes a complete round trip. Along the road there scatters 5 petrol stations at random positions. The total amount of petrol available at these stations is only sufficient for one to make 1 round trip along the road. The distribution of petrol is not uniform. Prove that there exists a starting point that allows you to make 1 complete round trip. (Your car starts with its fuel tank empty)

11. A and B shares a cup of water. A drinks half of it, and then B drinks the remaining half, then A drinks the remaining half, and so on. What is the amount drank by A and B?

12. There are 8 balls, one of which is lighter than the rest. How would you weight it in 3 times? What about 9 balls? What is the maximum amount of ball you can distinguish in 3 weighting?

5.2 Answers to Selected Questions

6 C++ & Algorithm

6.1 Question List

1. What's the difference between single quote and double quote?
2. What's the difference between lvalue and rvalue.
3. Explain bisection method and Newton's method for root solving. Compare their relative performances, and describe the motivation for the Brent algorithm.
4. Compare gradient descend method against Gauss-Newton algorithm for function minimisation.
5. I have a blackbox that gives the values of a function, give me an algorithm to find the zero, and prove it.
6. What is the result of integrating 1/x from 0 to 1 using Monte carlo method?
7. Given normal deviate random number generator from 0 to 1, how would you generate Gaussian random number?
8. Write a C++ code that (i) reverse the input string; (ii) copy the input string to another string; (iii) count the number of bit '1' in a given number expressed in binary form; (iv) print out all possible permutation of an input string.
9. I want to implement a number sorting algorithm that arranges an array of N random integers into ascending order. Tell me the easiest algorithm you could think of that does this job. What is the speed of this algorithm? (Quadratic N). Let's say I now introduce a more efficient algorithm that works as thus: Label the array of random integers from 1, 2, 3, ..., n. I'll start by dividing the array from the middle (n/2) into two separate arrays, then the numbers greater than the n/2 one will go into the 2nd array, while the numbers lesser than the n/2 one will go into the 1st. Then the next step is to divide the two arrays into four from their corresponding middle point, and then all numbers greater than the middle point goes to one array and lesser to another, and so on. What is the speed of this algorithm?
10. Consider the following code segment:

    ```
    class A
    {
      ... // Actual implementation of the class
    }
    int a = 100;
    A *ptr = new A [a]; ...  delete ptr;
    ```

    (a) What does the code above do?

    (b) There is an error in the above code (actually in the last line of the code), can you identify it and tell me how this error affects the memory allocation.
11. Implement a C++ function that returns the value of

$$\frac{1}{a} \times (e^{-at} - 1), \quad a \in \mathbb{R}, \quad t \in \mathbb{R}^+.$$

12. Implement a C++ function that checks if a given string (`char* sentence`) contains a certain string of characters (`char* check`).

13. Implement a C++ function '`int Fibonacci(int i)`', which returns the value of the ith term in the Fibonacci series. Is this the optimized algorithm?

14. What is the best algorithm to perform a search for an element through an array and returns true if there is a match but false otherwise? What is the speed of this algorithm?

6.2 Answers to Selected Questions