CHAPTER 13
Binomial Trees

Short Concept Questions

13.1 Risk neutral valuation involves valuing a derivative assuming that all market participants are risk neutral (i.e., they do not require extra returns for taking risks). This is valid because it can be shown that the price of a derivative in the risk-neutral world is the same as its price in the real world.

13.2 We set up a portfolio consisting of the option and the underlying asset that has the same value on both tree branches. It value today can therefore be calculated by discounting this known value at the risk-free rate.

13.3 *u* and *d* are the proportional up and down movements on the binomial tree. To match volatility, we set $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$.

13.4 p is the probability of an up movement. See equation (13.17).

13.5 We do not have a measure of the option's systematic risk.

13.6 The delta of a stock option measures the sensitivity of the option price to the price of the stock when small changes are considered. Specifically, it is the ratio of the change in the price of the stock option to the change in the price of the underlying stock.

13.7 Girsanov's theorem shows that volatility does not change when we move from the real world to the risk neutral world.

13.8 The *a* in the equation for *p* (see equation 13.17) becomes $e^{(r\text{-}q)\Delta t}$ where *q* is the dividend yield.

13.9 Consider a portfolio consisting of:
-1: Call option
$+\Delta$: Shares
If the stock price rises to \$42, the portfolio is worth $42\Delta - 3$. If the stock price falls to \$38, it is worth 38Δ. These are the same when

$$42\Delta - 3 = 38\Delta$$

or $\Delta = 0.75$. The value of the portfolio in one month is 28.5 for both stock prices. Its value today must be the present value of 28.5, or $28.5e^{-0.08\times 0.08333} = 28.31$. This means that

$$-f + 40\Delta = 28.31$$

where f is the call price. Because $\Delta = 0.75$, the call price is $40\times 0.75 - 28.31 = \$1.69$. As an alternative approach, we can calculate the probability, p, of an up movement in a risk-neutral world. This must satisfy:

$$42p + 38(1-p) = 40e^{0.08\times 0.08333}$$

so that

$$4p = 40e^{0.08\times 0.08333} - 38$$

or $p = 0.5669$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[3 \times 0.5669 + 0 \times 0.4331]e^{-0.08 \times 0.08333} = 1.69$$

or \$1.69. This agrees with the previous calculation.

13.10 Consider a portfolio consisting of:

-1: Put option

$+\Delta$: Shares

If the stock price rises to \$55, this is worth 55Δ. If the stock price falls to \$45, the portfolio is worth $45\Delta - 5$. These are the same when

$$45\Delta - 5 = 55\Delta$$

or $\Delta = -0.50$. The value of the portfolio in six months is -27.5 for both stock prices. Its value today must be the present value of -27.5, or $-27.5e^{-0.1 \times 0.5} = -26.16$. This means that

$$-f + 50\Delta = -26.16$$

where f is the put price. Because $\Delta = -0.50$, the put price is \$1.16. As an alternative approach, we can calculate the probability, p, of an up movement in a risk-neutral world. This must satisfy:

$$55p + 45(1 - p) = 50e^{0.1 \times 0.5}$$

so that

$$10p = 50e^{0.1 \times 0.5} - 45$$

or $p = 0.7564$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[0 \times 0.7564 + 5 \times 0.2436]e^{-0.1 \times 0.5} = 1.16$$

or \$1.16. This agrees with the previous calculation.

Practice Questions

13.11

In this case, $u = 1.10$, $d = 0.90$, $\Delta t = 0.5$, and $r = 0.08$, so that

$$p = \frac{e^{0.08 \times 0.5} - 0.90}{1.10 - 0.90} = 0.7041$$

The tree for stock price movements is shown in Figure S13.1. We can work back from the end of the tree to the beginning, as indicated in the diagram, to give the value of the option as \$9.61. The option value can also be calculated directly from equation (13.10):

$$[0.7041^2 \times 21 + 2 \times 0.7041 \times 0.2959 \times 0 + 0.2959^2 \times 0]e^{-2 \times 0.08 \times 0.5} = 9.61$$

or \$9.61.

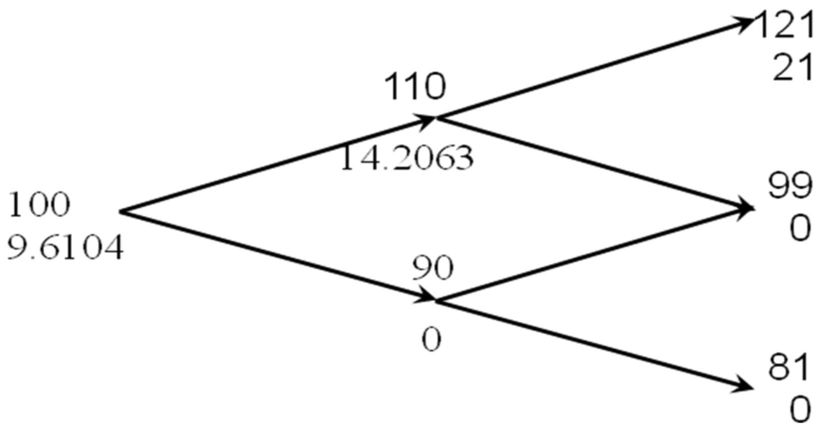


Figure S13.1: *Tree for Problem 13.11*

13.12
Figure S13.2 shows how we can value the put option using the same tree as in Problem 13.11. The value of the option is \$1.92. The option value can also be calculated directly from equation (13.10):

$$e^{-2\times 0.08\times 0.5}[0.7041^2\times 0+2\times 0.7041\times 0.2959\times 1+0.2959^2\times 19]=1.92$$

or \$1.92. The stock price plus the put price is $100+1.92=\$101.92$. The present value of the strike price plus the call price is $100e^{-0.08\times 1}+9.61=\$101.92$. These are the same, verifying that put–call parity holds.

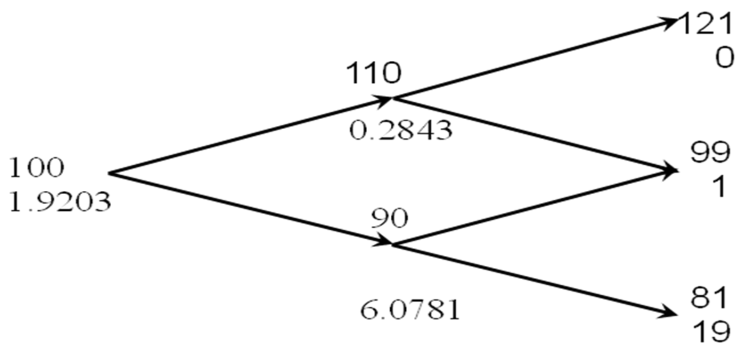


Figure S13.2: *Tree for Problem 13.12*

13.13
The riskless portfolio consists of a short position in the option and a long position in Δ shares. Because Δ changes during the life of the option, this riskless portfolio must also change.

13.14

At the end of two months, the value of the option will be either \$4 (if the stock price is \$53) or \$0 (if the stock price is \$48). Consider a portfolio consisting of:

$$+\Delta \quad : \quad \text{shares}$$

$$-1 \quad : \quad \text{option}$$

The value of the portfolio is either 48Δ or $53\Delta - 4$ in two months. If

$$48\Delta = 53\Delta - 4$$

that is,

$$\Delta = 0.8$$

the value of the portfolio is certain to be 38.4. For this value of Δ, the portfolio is therefore riskless. The current value of the portfolio is:

$$0.8 \times 50 - f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(0.8 \times 50 - f)e^{0.10 \times 2/12} = 38.4$$

that is

$$f = 2.23$$

The value of the option is therefore \$2.23.

This can also be calculated directly from equations (13.2) and (13.3). $u = 1.06$, $d = 0.96$ so that

$$p = \frac{e^{0.10 \times 2/12} - 0.96}{1.06 - 0.96} = 0.5681$$

and

$$f = e^{-0.10 \times 2/12} \times 0.5681 \times 4 = 2.23$$

13.15

At the end of four months, the value of the option will be either \$5 (if the stock price is \$75) or \$0 (if the stock price is \$85). Consider a portfolio consisting of:

$$-\Delta \quad : \quad \text{shares}$$

$$+1 \quad : \quad \text{option}$$

(Note: The delta, Δ of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either -85Δ or $-75\Delta + 5$ in four months. If

$$-85\Delta = -75\Delta + 5$$

that is

$$\Delta = -0.5$$

the value of the portfolio is certain to be 42.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.5 \times 80 + f$$

where f is the value of the option. Since the portfolio is riskless

$$(0.5 \times 80 + f)e^{0.05 \times 4/12} = 42.5$$

that is

$$f = 1.80$$

The value of the option is therefore \$1.80.

This can also be calculated directly from equations (13.2) and (13.3). $u = 1.0625$, $d = 0.9375$

so that

$$p = \frac{e^{0.05\times 4/12} - 0.9375}{1.0625 - 0.9375} = 0.6345$$

$1 - p = 0.3655$ and

$$f = e^{-0.05\times 4/12} \times 0.3655 \times 5 = 1.80$$

13.16

At the end of three months the value of the option is either \$5 (if the stock price is \$35) or \$0 (if the stock price is \$45).

Consider a portfolio consisting of:

$$-\Delta \quad : \quad \text{shares}$$

$$+1 \quad : \quad \text{option}$$

(Note: The delta, Δ, of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either $-35\Delta + 5$ or -45Δ. If:

$$-35\Delta + 5 = -45\Delta$$

that is,

$$\Delta = -0.5$$

the value of the portfolio is certain to be 22.5. For this value of Δ, the portfolio is therefore riskless. The current value of the portfolio is

$$-40\Delta + f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(40 \times 0.5 + f) \times 1.02 = 22.5$$

Hence,

$$f = 2.06$$

i.e., the value of the option is \$2.06.

This can also be calculated using risk-neutral valuation. Suppose that p is the probability of an upward stock price movement in a risk-neutral world. We must have

$$45p + 35(1-p) = 40 \times 1.02$$

that is

$$10p = 5.8$$

Or,

$$p = 0.58$$

The expected value of the option in a risk-neutral world is:

$$0 \times 0.58 + 5 \times 0.42 = 2.10$$

This has a present value of

$$\frac{2.10}{1.02} = 2.06$$

This is consistent with the no-arbitrage answer.

13.17

A tree describing the behavior of the stock price is shown in Figure S13.3. The risk-neutral probability of an up move, p, is given by

$$p = \frac{e^{0.05\times 3/12} - 0.95}{1.06 - 0.95} = 0.5689$$

There is a payoff from the option of $56.18 - 51 = 5.18$ for the highest final node (which corresponds to two up moves) zero in all other cases. The value of the option is therefore

$$5.18 \times 0.5689^2 \times e^{-0.05 \times 6/12} = 1.635$$

This can also be calculated by working back through the tree as indicated in Figure S13.3. The value of the call option is the lower number at each node in the figure.

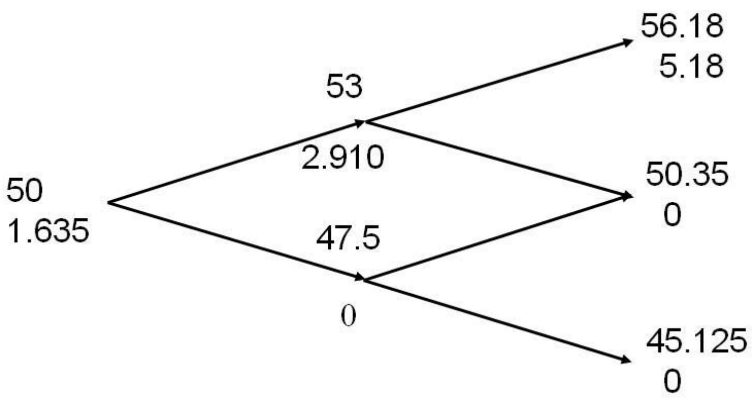


Figure S13.3: *Tree for Problem 13.17*

13.18

The tree for valuing the put option is shown in Figure S13.4. We get a payoff of $51 - 50.35 = 0.65$ if the middle final node is reached and a payoff of $51 - 45.125 = 5.875$ if the lowest final node is reached. The value of the option is therefore

$$(0.65 \times 2 \times 0.5689 \times 0.4311 + 5.875 \times 0.4311^2) e^{-0.05 \times 6/12} = 1.376$$

This can also be calculated by working back through the tree as indicated in Figure S13.4. The value of the put plus the stock price is

$$1.376 + 50 = 51.376$$

The value of the call plus the present value of the strike price is

$$1.635 + 51 e^{-0.05 \times 6/12} = 51.376$$

This verifies that put–call parity holds.

To test whether it is worth exercising the option early, we compare the value calculated for the option at each node with the payoff from immediate exercise. At node C, the payoff from immediate exercise is $51 - 47.5 = 3.5$. Because this is greater than 2.8664, the option should be exercised at this node. The option should not be exercised at either node A or node B.

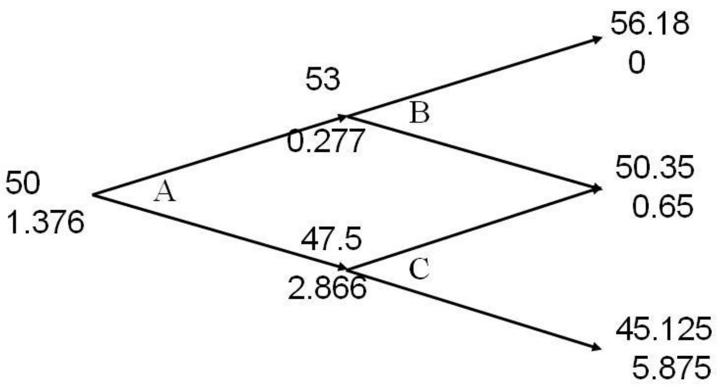


Figure S13.4: *Tree for Problem 13.18*

13.19
This problem shows that the valuation procedures introduced in the chapter can be used for derivatives other than call and put options.
At the end of two months, the value of the derivative will be either 529 (if the stock price is 23) or 729 (if the stock price is 27). Consider a portfolio consisting of:

$$+\Delta \quad : \quad \text{shares}$$

$$-1 \quad : \quad \text{derivative}$$

The value of the portfolio is either $27\Delta - 729$ or $23\Delta - 529$ in two months. If

$$27\Delta - 729 = 23\Delta - 529$$

that is,

$$\Delta = 50$$

the value of the portfolio is certain to be 621. For this value of Δ, the portfolio is therefore riskless. The current value of the portfolio is:

$$50 \times 25 - f$$

where f is the value of the derivative. Since the portfolio must earn the risk-free rate of interest

$$(50 \times 25 - f)e^{0.10 \times 2/12} = 621$$

that is

$$f = 639.3$$

The value of the option is therefore \$639.3.
This can also be calculated directly from equations (13.2) and (13.3). $u = 1.08$, $d = 0.92$ so that

$$p = \frac{e^{0.10 \times 2/12} - 0.92}{1.08 - 0.92} = 0.6050$$

and

$$f = e^{-0.10 \times 2/12}(0.6050 \times 729 + 0.3950 \times 529) = 639.3$$

13.20
In this case,

$$a = e^{(0.05 - 0.08) \times 1/12} = 0.9975$$

$$u = e^{0.12\sqrt{1/12}} = 1.0352$$

$$d = 1/u = 0.9660$$

$$p = \frac{0.9975 - 0.9660}{1.0352 - 0.9660} = 0.4553$$

13.21

$$u = e^{0.30\times\sqrt{0.1667}} = 1.1303$$
$$d = 1/u = 0.8847$$
$$p = \frac{e^{0.30\times 2/12} - 0.8847}{1.1303 - 0.8847} = 0.4898$$

The tree is given in Figure S13.5. The value of the option is \$4.67. The initial delta is 9.58/(88.16 – 69.01) which is almost exactly 0.5 so that 500 shares should be purchased.

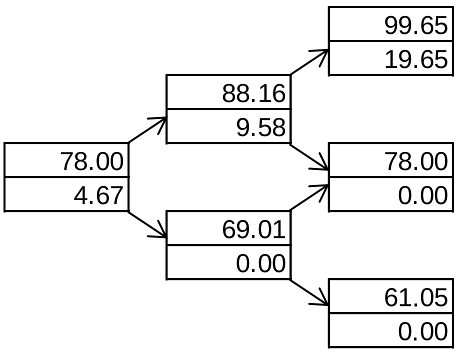


Figure S13.5: *Tree for Problem 13.21*

13.22

$$u = e^{0.18\times\sqrt{0.5}} = 1.1357$$
$$d = 1/u = 0.8805$$
$$p = \frac{e^{(0.04-0.025)\times 0.5} - 0.8805}{1.1357 - 0.8805} = 0.4977$$

The tree is shown in Figure S13.6. The option is exercised at the lower node at the six-month point. It is worth 78.41.

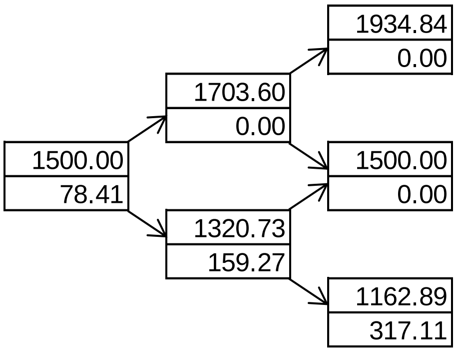
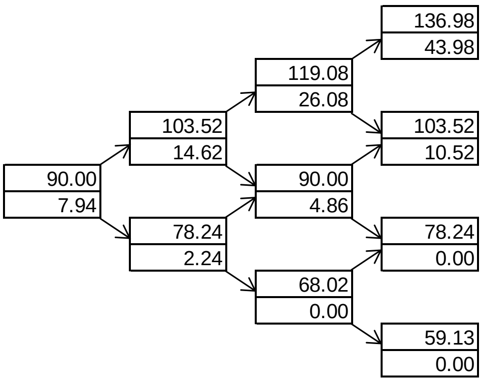


Figure S13.6: *Tree for Problem 13.22*

13.23

$$u = e^{0.28\times\sqrt{0.25}} = 1.1503$$
$$d = 1/u = 0.8694$$
$$u = \frac{1-0.8694}{1.1503-0.8694} = 0.4651$$

The tree for valuing the call is in Figure S13.7a and that for valuing the put is in Figure S13.7b. The values are 7.94 and 10.88, respectively.



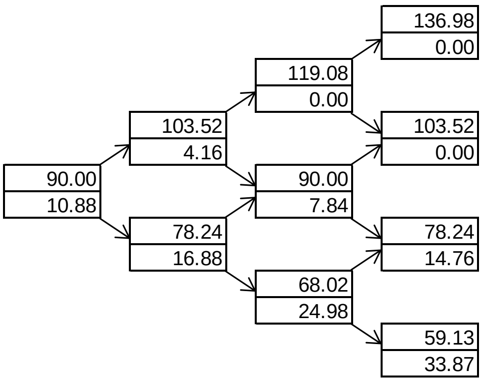


Figure S13.7a: *Call*

Figure S13.7b: *Put*

13.24

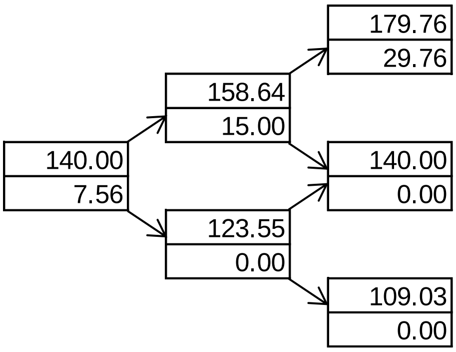
(a) $u = e^{0.25\times\sqrt{0.25}}$ = 1.1331. The percentage up movement is 13.31%.

(b) d = 1/u = 0.8825. The percentage down movement is 11.75%.

(c) The probability of an up movement is $(e^{0.04\times 0.25)} - .8825)/(1.1331 - .8825) = 0.5089$.

(d) The probability of a down movement is0.4911.

The tree for valuing the call is in Figure S13.8a and that for valuing the put is in Figure S13.8b. The values are 7.56 and 14.58, respectively.



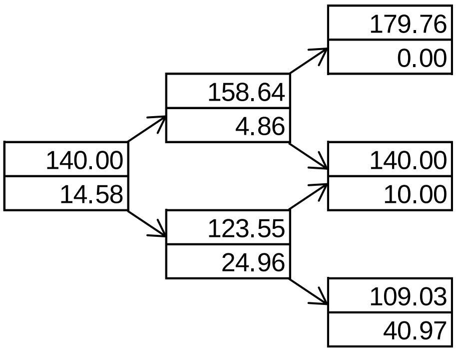


Figure S13.8a: *Call* **Figure S13.8b**: *Put*

13.25

The delta for the first period is 15/(158.64 – 123.55) = 0.4273. The trader should take a long position in 4,273 shares. If there is an up movement, the delta for the second period is 29.76/(179.76 – 140) = 0.7485. The trader should increase the holding to 7,485 shares. If there is a down movement, the trader should decrease the holding to zero.

13.26

At the end of six months, the value of the option will be either \$12 (if the stock price is \$60) or \$0 (if the stock price is \$42). Consider a portfolio consisting of:

$$+\Delta \quad : \quad \text{shares}$$

$$-1 \quad : \quad \text{option}$$

The value of the portfolio is either 42Δ or $60\Delta - 12$ in six months. If

$$42\Delta = 60\Delta - 12$$

that is,

$$\Delta = 0.6667$$

the value of the portfolio is certain to be 28. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.6667 \times 50 - f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(0.6667 \times 50 - f)e^{0.12\times 0.5} = 28$$

that is,

$$f = 6.96$$

The value of the option is therefore \$6.96.

This can also be calculated using risk-neutral valuation. Suppose that p is the probability of an upward stock price movement in a risk-neutral world. We must have

$$60p + 42(1-p) = 50 \times e^{0.06}$$

that is,

$$18p = 11.09$$

or:

$$p = 0.6161$$

The expected value of the option in a risk-neutral world is:

$$12 \times 0.6161 + 0 \times 0.3839 = 7.3932$$

This has a present value of

$$7.3932e^{-0.06} = 6.96$$

Hence, the above answer is consistent with risk-neutral valuation.

13.27

a. A tree describing the behavior of the stock price is shown in Figure S13.9. The risk-neutral probability of an up move, p, is given by

$$p=\frac{e^{0.12\times 3/12}-0.90}{1.1-0.9}=0.6523$$

Calculating the expected payoff and discounting, we obtain the value of the option as

$$[2.4\times 2\times 0.6523\times 0.3477+9.6\times 0.3477^2]e^{-0.12\times 6/12}=2.118$$

The value of the European option is 2.118. This can also be calculated by working back through the tree as shown in Figure S13.9. The second number at each node is the value of the European option.

b. The value of the American option is shown as the third number at each node on the tree. It is 2.537. This is greater than the value of the European option because it is optimal to exercise early at node C.

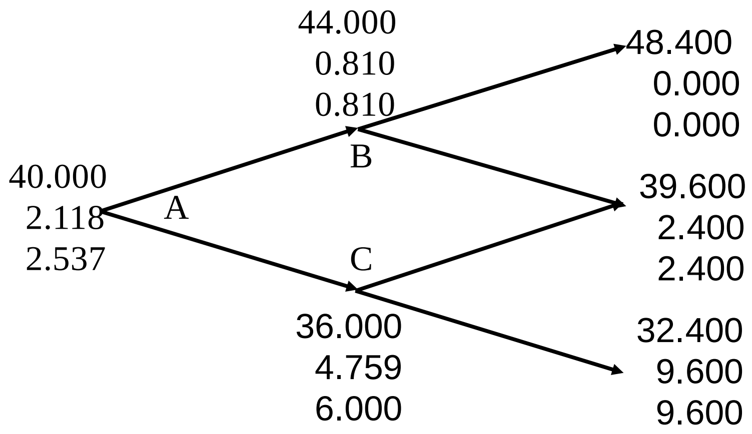


Figure S13.9: *Tree to evaluate European and American put options in Problem 13.27. At each node, upper number is the stock price, the next number is the European put price, and the final number is the American put price.*

13.28

Trial and error shows that immediate early exercise is optimal when the strike price is above 43.2. This can be also shown to be true algebraically. Suppose the strike price increases by a relatively small amount q. This increases the value of being at node C by q and the value of being at node B by $0.3477e^{-0.03}q=0.3374q$. It therefore increases the value of being at node A by

$$(0.6523\times 0.3374q+0.3477q)e^{-0.03}=0.551q$$

For early exercise at node A, we require $2.537+0.551q<2+q$ or $q>1.196$. This corresponds to the strike price being greater than 43.196.

13.29

(a) This problem is based on the material in Section 13.8. In this case, $\Delta t = 0.25$ so that $u = e^{0.30\times\sqrt{0.25}} = 1.1618$, $d = 1/u = 0.8607$, and

$$p = \frac{e^{0.04\times0.25} - 0.8607}{1.1618 - 0.8607} = 0.4959$$

(b) and (c) The value of the option using a two-step tree as given by DerivaGem is shown in Figure S13.10 to be 3.3739. To use DerivaGem choose the first worksheet, select Equity as the underlying type, and select Binomial European as the Option Type. After carrying out the calculations, select Display Tree.
(d) With 5, 50, 100, and 500 time steps the value of the option is 3.9229, 3.7394, 3.7478, and 3.7545, respectively.

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Values in red are a result of early exercise.

Strike price = 40
Discount factor per step = 0.9900
Time step, dt = 0.2500 years, 91.25 days
Growth factor per step, a = 1.0101
Probability of up move, p = 0.4959
Up step size, u = 1.1618
Down step size, d = 0.8607

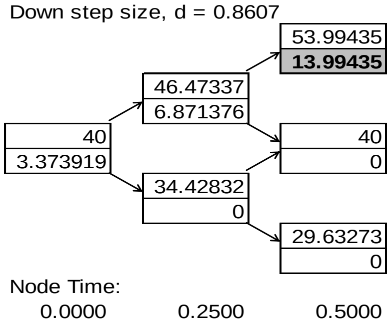


Figure S13.10: *Tree produced by DerivaGem to evaluate European option in Problem 13.29*

13.30

(a) In this case, $\Delta t = 0.25$ and $u = e^{0.40\times\sqrt{0.25}} = 1.2214$, $d = 1/u = 0.8187$, and

$$p = \frac{e^{0.1\times0.25} - 0.8187}{1.2214 - 0.8187} = 0.4502$$

(b) and (c) The value of the option using a two-step tree is 4.8604.
(d) With 5, 50, 100, and 500 time steps, the value of the option is 5.6858, 5.3869, 5.3981, and 5.4072, respectively.

13.31

The value of the put option is

$$(0.5503 \times 0 + 0.4497 \times 3)e^{-0.04 \times 3/12} = 1.3357$$

The expected payoff in the real world is

$$(0.6206 \times 0 + 0.3794 \times 3) = 1.1199$$

The discount rate R that should be used in the real world is therefore given by solving

$$1.3357 = 1.1199e^{-0.25R}$$

The solution to this is $R = -0.704$. The discount rate is −70.4%.

The underlying stock has positive systematic risk because its expected return is higher than the risk free rate. This means that the stock will tend to do well when the market does well. The call option has a high positive systematic risk because it tends to do very well when the market does well. As a result, a high discount rate is appropriate for its expected payoff. The put option is in the opposite position. It tends to provide a high return when the market does badly. As a result, it is appropriate to use a highly negative discount rate for its expected payoff.