

# Multivariable Calculus Notes

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# Contents

<b>1</b>	<b>Sequences</b>	<b>2</b>
1.1	Subsequences . . . . .	3
<b>2</b>	<b>Continuity</b>	<b>5</b>
<b>3</b>	<b>The Derivative</b>	<b>9</b>

# Chapter 1

## Sequences

We know the usual definition of sequences in  $\mathbb{R}$ . A sequence of vectors converges similarly, except that the norm used in the definition of convergence is now a vector norm, not simply absolute value. A sequence of vectors  $\{\vec{a}_i\}$  in  $\mathbb{R}^n$  converges to  $\vec{a} \in \mathbb{R}^n$  if for any  $\epsilon > 0$  there exists a  $n \in \mathbb{N}$  such that

$$n > N \implies \|\vec{a}_i - \vec{a}\| < \epsilon$$

This definition can be related to the convergence of the vector components.

**Prop 1.1.** *Let  $\{\vec{a}_i\}$  be a sequence of vectors (coordinate vectors) in  $\mathbb{R}^n$ . Then  $\{\vec{a}_i\}$  converges to  $\vec{a} \in \mathbb{R}^n$  if and only if each component  $(\vec{a}_i)_j$  converges to  $(\vec{a})_j$ .*

*Proof.* Assume that we have the components converging. That is, for  $\epsilon > 0$  we have some  $N \in \mathbb{N}$  such that

$$n > N \implies |(\vec{a}_i)_j - (\vec{a})_j| < \epsilon/\sqrt{n}$$

We choose  $N$  to hold for every component  $i$ . This can be arranged by taking  $N$  to be the maximum of the  $N$  guaranteed for each individual component. Thus, we have, for  $n > N$ ,

$$\begin{aligned} \|\vec{a}_i - \vec{a}\|^2 &= \sum_{j=1}^n |(\vec{a}_i)_j - (\vec{a})_j|^2 \\ &< \sum_{j=1}^n \epsilon^2/n \\ &= \epsilon \end{aligned}$$

so that  $\{\vec{a}_i\}$  converges to  $\vec{a}$ .

Conversely, assume that  $\{\vec{a}_i\}$  converges to  $\vec{a}$ . Let  $\epsilon > 0$  be arbitrary and let  $N$  be the guaranteed natural number in the definition of vector sequence convergence.

The key is to note that for any  $j = 1, 2, \dots, n$  we have

$$|(\tilde{a}_i)_j - (\tilde{a})_j|^2 \leq \sum_{k=1}^n |(\tilde{a}_i)_k - (\tilde{a})_k|^2 < \epsilon^2$$

whenever  $n > N$ . Thus, we have in particular

$$n > N \implies |(\tilde{a}_i)_j - (\tilde{a})_j| < \epsilon$$

and so the component sequences converge as expected. ■

Let's look at some interesting consequences.

**Prop 1.2.** *Let  $\tilde{x}_k \rightarrow \tilde{a}$ . Then  $\|\tilde{x}_k\| \rightarrow \|\tilde{a}\|$  and  $\vec{b} \cdot \tilde{x}_k \rightarrow \vec{b} \cdot \tilde{a}$ .*

*Proof.* The first can be proven by writing  $\tilde{x} = [x_1, x_2, \dots, x_n]$  and proceeding as usual with the usual 2-norm. More generally, though, we can use the (reverse) triangle inequality property of the norm to see that

$$|\|\tilde{x}_k\| - \|\tilde{a}\|| \leq \|\tilde{x}_k - \tilde{a}\|$$

which can be made smaller than arbitrary  $\epsilon > 0$  by proper choice of  $N$ . Thus, the sequence of norms converges as expected.

Next, we could let  $f(\tilde{x}) = \vec{b}^T \tilde{x}$  and use continuity of linear functions to prove the conclusion. Instead, just notice that, from a useful proposition in the next chapter, we have

$$\|\vec{b}^T \tilde{x}\| \leq \|\vec{b}\| \|\tilde{x}\|$$

so that we have

$$\|\vec{b}^T \tilde{x}_k - \vec{b}^T \tilde{a}\| = \|\vec{b}^T (\tilde{x}_k - \tilde{a})\| = \|\vec{b}\| \|\tilde{x}_k - \tilde{a}\|$$

Since  $\tilde{x}_k \rightarrow \tilde{a}$ , given arbitrary  $\epsilon > 0$ , there is some  $N$  such that  $k > N \implies \|\tilde{x}_k - \tilde{a}\| < \epsilon / \|\vec{b}\|$ . Thus, we have proven the statement. ■

## 1.1 Subsequences

A subsequence of a sequence  $\{\tilde{x}_k\}_k$  is a subset  $\{\tilde{x}_{k_i}\}_{k_i}$  where  $k_1 < k_2 < k_3 < \dots$  is an increasing sequence of natural numbers.

**Prop 1.3.** *If a sequence converges, then any subsequence converges. If every subsequence of a sequence converges, then the sequence converges.*

*Proof.* The converse is trivial, since the sequence itself is a subsequence. Note that it is not the case that a single convergent subsequence implies convergence of the sequence.

Let  $\{\tilde{x}_{k_i}\}_{k_i}$  be a subsequence of convergent sequence  $\{\tilde{x}_k\}_k$  which has limit  $\tilde{a}$ .

Then for arbitrary  $\epsilon > 0$ , there is an  $N$  such that

$$k > N \implies \|\vec{x}_k - \vec{a}\| < \epsilon$$

Since  $\{\vec{x}_{k_i}\}_{k_i}$  is a subsequence, we then have

$$i > N \implies \|\vec{x}_{k_i} - \vec{a}\| < \epsilon$$

■

# Chapter 2

## Continuity

**Prop 2.1.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map represented by the matrix  $A \in \mathbb{R}^{m \times n}$ . Then we have

$$\|T(h)\| = \|Ah\| \leq M \|h\|$$

for some constant  $M$ .

*Proof.* Note that this is just a statement about matrix-vector multiplication. Let  $[Ah]_i$  denote the  $i$ th entry of the matrix product  $Ah \in \mathbb{R}^m$ . That is,

$$[Ah]_i = \sum_{j=1}^n A_{ij} h_j$$

Then let

$$r = \max_{i,j} |A_{ij}|$$

Also note that since

$$\|h\|^2 = \sum_{j=1}^n |h_j|^2$$

we have

$$|h_j| \leq \|h\|$$

Taken together, we have

$$[Ah]_i = \sum_{j=1}^n A_{ij} h_j \leq \sum_{j=1}^n |A_{ij}| |h_j| \leq rn \|h\|$$

so that

$$\|Ah\|^2 = \sum_{i=1}^m [Ah]_i^2 \leq \sum_{i=1}^m r^2 n^2 \|h\|^2 = mr^2 n^2 \|h\|^2$$

and so we have

$$\|Ah\| \leq rn\sqrt{m} \|h\|$$

Letting  $M = rn\sqrt{m}$ , the conclusion holds. ■

**Lemma 2.2.** *Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map represented by the matrix  $A \in \mathbb{R}^{m \times n}$ . Then for  $a \in \mathbb{R}^n$ ,  $T$  is continuous at  $a$ .*

*Proof.* We must show that

$$\lim_{h \rightarrow 0} \|T(a+h) - T(a)\| = 0$$

Note that, by linearity,

$$\|T(a+h) - T(a)\| = \|T(a) + T(h) - T(a)\| = \|T(h)\| \leq M \|h\|$$

and the right-hand side clearly goes to 0 as  $h \rightarrow 0$ . ■

The next seems to be an alternative approach. For arbitrary linear map  $A \in \mathbb{R}^{m \times n}$ , we can define a matrix norm

$$\|A\| := \sum_{i=1, \dots, m} \sum_{j=1, \dots, n} A_{i,j}^2$$

This is technically the square of the element-wise matrix  $p$ -norm:

$$\|A\|_p = \left( \sum_{\text{rows}} \sum_{\text{cols}} A_{i,j}^p \right)^{1/p}$$

Then we can look at bounds on matrix-vector and matrix-matrix products. In fact, the following proposition is a generalization of the one above.

**Prop 2.3.** *Let  $A \in \mathbb{R}^{m \times n}$ ,  $B \in \mathbb{R}^{n \times p}$ , and  $\vec{b} \in \mathbb{R}^n$ . Then*

$$\|A\vec{b}\| \leq \|A\| \|\vec{b}\|$$

$$\|AB\| \leq \|A\| \|B\|$$

*Proof.* For the first inequality, notice that the magnitude on the left is a vector norm. That is,

$$\|A\vec{b}\|^2 = \sum_{i=1}^m (A\vec{b})_i^2$$

and that each entry in this vector is the inner product (usual Euclidean dot product) of  $\vec{b}$  with a row of  $A$ :

$$(A\vec{b})_i = A_{i*} \vec{b}$$

Since this is an inner product, we may apply the Cauchy-Schwarz inequality to it and obtain

$$(A\vec{b})_i = A_{i*} \vec{b} \leq |A_{i*} \vec{b}| \leq \|A_{i*}\| \|\vec{b}\|$$

Taking this in the original vector magnitude, we have

$$\begin{aligned}
\|A\vec{b}\|^2 &= \sum_{i=1}^m (A\vec{b})_i^2 \\
&\leq \sum_{i=1}^m \|A_{i*}\|^2 \|\vec{b}\|^2 \\
&= \left( \sum_{i=1}^m \|A_{i*}\|^2 \right) \|\vec{b}\|^2 \\
&= \|A\|^2 \|\vec{b}\|^2
\end{aligned}$$

and so  $\|A\vec{b}\| \leq \|A\| \|\vec{b}\|$  as we wanted to prove.

The next inequality is even more general and follows from the matrix-vector product case. Note that every column of the matrix product  $AB$  is of the form  $A\vec{b}$  where  $\vec{b} = B_{*j}$ . Thus

$$\begin{aligned}
\|AB\|^2 &= \sum_{i=1}^m \sum_{j=1}^n (AB)_{i,j}^2 \\
&= \sum_{j=1}^n |AB_{*j}|^2 \\
&\leq \sum_{j=1}^n \|A\|^2 \|B_{*j}\|^2 \\
&= \|A\|^2 \left( \sum_{j=1}^n \|B_{*j}\|^2 \right) \\
&= \|A\|^2 \|B\|^2
\end{aligned}$$

and so

$$\|AB\| \leq \|A\| \|B\|$$

as desired. ■

The second inequality defines the matrix norm  $\|\cdot\|$  (and more generally the matrix 2-norm) as a *submultiplicative* norm. With these general properties in hand, we can easily verify continuity of a linear function.

*Proof.* The linear function  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  will be continuous if we have

$$\lim_{x \rightarrow a} T(x) = T(a)$$

Let  $\epsilon > 0$  be arbitrary. Then

$$|\vec{x} - \vec{a}| < \epsilon / \|A\|$$



implies that

$$\begin{aligned}\|T(\vec{x}) - T(\vec{a})\| &= \|A\vec{x} - A\vec{a}\| \\ &= \|A(\vec{x} - \vec{a})\| \\ &\leq \|A\| \|\vec{x} - \vec{a}\| \\ &< \epsilon\end{aligned}$$

Note that linearity of  $T$  allowed us to express its action as a matrix multiplication, and the associative (linearity) property of matrix multiplication allowed us to “pull it off” of the  $\vec{x}$  and  $\vec{a}$  and onto their vector difference. Then, the previous proposition allowed us to bound this matrix-vector product by the norm of the vector and a quantity which measures in some sense the size of the matrix  $A$ . ■

# Chapter 3

## The Derivative

Let

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

be an arbitrary function. Then the derivative of  $f$  at  $a \in \mathbb{R}^n$  is the unique linear map

$$\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

such that

$$\lim_{h \rightarrow 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0$$

The norms are required since all quantities are vectors. Note that since  $\lambda$  is a linear map, it has a matrix representation, written  $Df(a)$  and called the derivative of  $f$  at  $a$ . The action of  $\lambda$  on  $h$  is

$$\lambda(h) = Df(a)h$$

i.e., the usual matrix multiplication.

**Prop 3.1.** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $a \in \mathbb{R}^n$ , then this derivative is unique.*

*Proof.* Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $a \in \mathbb{R}^n$  and assume there are two derivatives  $\lambda$  and  $\mu$ . Then

$$\|\lambda(h) - \mu(h)\| \leq \|\lambda(h) - f(a+h) + f(a)\| + \|\mu(h) - f(a+h) + f(a)\|$$

and upon dividing by  $\|h\|$  and taking the limit  $h \rightarrow 0$ , we obtain

$$\lim_{h \rightarrow 0} \frac{\|\lambda(h) - \mu(h)\|}{\|h\|} = 0$$

Choosing arbitrary  $x \in \mathbb{R}^n \setminus \{0\}$ , note that  $tx \rightarrow 0$  when  $t \rightarrow 0$  ( $t \in \mathbb{R}$ ). Then

$$0 = \lim_{t \rightarrow 0} \frac{\|\lambda(tx) - \mu(tx)\|}{\|tx\|} = \lim_{t \rightarrow 0} \frac{|t| \|\lambda(x) - \mu(x)\|}{|t| \|x\|} = \frac{\|\lambda(x) - \mu(x)\|}{\|x\|}$$

and so we have  $\lambda(x) = \mu(x)$  for arbitrary  $x \in \mathbb{R}^n$ . Thus, the derivative is unique. ■

**Prop 3.2.** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $a \in \mathbb{R}^n$ , then it is continuous at  $a$ .*

*Proof.*

$$\|f(a+h) - f(a)\| \leq \|f(a+h) - f(a) - \lambda(h)\| + \|\lambda(h)\|$$

so that

$$\|f(a+h) - f(a)\| \leq \|h\| \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} + \|\lambda(h)\|$$

By taking the limit of both sides as  $h \rightarrow 0$ , the first term on the right-hand side is 0, and the second term, containing the linear  $\lambda$ , goes to 0 as well. Specifically, from a property of linear maps proved in the previous chapter, we have

$$\|\lambda(h)\| \leq M \|h\|$$

for some constant  $M$  which certainly tends toward 0. Specifically, we have shown that

$$\|\lambda(h)\| \leq \|Df(a)\| \|h\|$$

where  $\|Df(a)\|$  is the squared entry-wise matrix norm of the derivative (matrix) of  $f$  at  $a$ , which certainly exists (is finite). Thus, we have

$$\lim_{h \rightarrow 0} \|f(a+h) - f(a)\| = 0,$$

establishing continuity of  $f$  at  $a$ . ■