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Chapter 1

Vector Spaces

The concept of a vector space leads naturally to that of a tensor. Tensors generalize vectors.

A vector U is simply an element of a set $\mathcal{V}$ called a (linear) vector space. A vector space is a specific type of set along with two operations

$$+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

and

$$\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$$

which satisfies certain axioms:

- $U + V \in \mathcal{V}$ for all $U, V \in \mathcal{V}$.
- $aU \in \mathcal{V}$ for all $c \in \mathbb{R}, U \in \mathcal{V}$.
- There is some $0 \in \mathcal{V}$ such that $0 + U = U + 0 = U \in \mathcal{V}$.
- For any $U \in \mathcal{V}$, there exists $-U \in \mathcal{V}$ such that $U + (-U) = (-U) + U = 0$.
- $(U + V) + W = U + (V + W)$ for all $U, V, W \in \mathcal{V}$.

Since vector spaces are dominated by the concept of linearity, we often like to express vectors as linear combinations of other vectors:

$$q = aw + bp + cv + \cdots$$

From linear algebra, we know that the number of basis vectors for a particular vector space is unique (well-defined) and is called the dimension of that space.

Picking a basis is often required for computations, but it is not necessary always. Things can be done in a basis-free way.

Two vector spaces are isomorphic if we can establish a bijective linear map between the two.

Vector spaces are a very elementary concept. Many things in mathematics and physics are vector spaces with additional structures on top.

We will write basis vectors as

$$e_0, e_1, e_2, \ldots$$

and write arbitrary vector A as

$$A = A^0 e_0 + A^1 e_1 + A^2 e_2 + A^3 e_3 = A^\mu e_\mu$$

1.1 Mappings

Consider two vector spaces $\mathcal{V}$ and $\mathcal{W}$ with bases $\{e_\mu\}$ and $\{f_\mu\}$, respectively. A map between the vector spaces can be written

$$\Lambda : \mathcal{V} \to \mathcal{W}$$

or in bracket notation as

$$\langle \Lambda, \cdot \rangle : \mathcal{V} \to \mathcal{W}$$

When we restrict ourselves to linear maps, we have

$$\begin{aligned} \langle \Lambda, A \rangle &= \langle \Lambda, A^\mu e_\mu \rangle \\ &= A^\mu \langle \Lambda, e_\mu \rangle \end{aligned}$$

which only depends on the action of Λ on the basis vectors of $\mathcal{V}$. Thus, Λ is completely defined if we know how it operates on all basis vectors.

1.2 Dual Space

Consider real vector space $\mathcal{V}$ with basis $\{e_\mu\}$. We can map $\mathcal{V}$ to any other vector space $\mathcal{W}$ with basis $\{f_\mu\}$. One particular choice for $\mathcal{W}$ is the underlying set of scalars, $\mathcal{R}$ (which is, itself, a vector space).

The *dual space* to real vector space $\mathcal{V}$ is the vector space $\mathcal{V}^*$ consisting of linear maps $\Lambda : \mathcal{V} \to \mathbb{R}$. It is a vector space when we define addition and scalar multiplication in the obvious way. As a vector space, it has its own basis, which we will write as $\{e^\mu\}$.

Note that we automatically obtain the dual space from only the original vector space; no additional structure is needed to obtain this.

There is no immediate relation between the bases of the vector space and the dual space; there is no canonical mapping from a basis vector to a dual basis vector. However, we make the convenient choice of dual basis which satisfies

$$\langle e^\mu, e_\nu \rangle = \delta^\mu_\nu$$

Prop 1.1. *Let $\mathcal{V}$ be a finite-dimensional vector space with basis $\{e_1, \ldots, e_n\}$. Then the set $\{e^1, \ldots, e^n\}$ of covectors defined by*

$$\langle e^\mu, e_\nu \rangle = \delta^\mu_\nu$$

forms a basis for the dual space to $\mathcal{V}$, $\mathcal{V}^*$. In particular,

$$\dim \mathcal{V} = \dim \mathcal{V}^*$$

Proof. Existence of this basis is guaranteed, since given a basis for $\mathcal{V}$, we are defining the linear functionals $\{e_\nu\}$.

Linear independence is easy to prove. Assume

$$A_\mu e^\mu = 0$$

is a trivial covector and let it operate on the basis for V:

$$0 = \langle A_\mu e^\mu, e_\nu \rangle = A_\mu \langle e^\mu, e_\nu \rangle = A_\mu \delta^\mu_\nu = A_\nu$$

so that every component of the trivial covector is 0. Hence, the dual basis vectors $\{e^\nu\}$ are a linearly independent set.

To prove that the dual basis spans the dual space, first note that, if so, we could write arbitrary dual vector $A \in \mathcal{V}^*$ as $A_\mu e^\mu$ and operate on basis vector e_ν to obtain $\langle A_\mu e^\mu, e_\nu \rangle = A_\nu$ so that

$$A_\mu e^\mu = \langle A_\nu e^\nu, e_\mu \rangle e^\mu = A(e_\mu) e^\mu$$

That is, the coefficients of the dual vector in the dual basis would be just the result of the dual vector applied to the basis vectors of $\mathcal{V}$. To show that this expansion is correct, we must show that

$$A(V) = A(e_\mu) \langle e^\mu, V \rangle$$

for arbitrary vector $V = V^\lambda e_\lambda \in \mathcal{V}$. Then

$$A(V) = A\left(V^\lambda e_\lambda\right) = V^\lambda A(e_\lambda)$$

by linearity of A. The right-hand side gives

$$A(e_\mu) \langle e^\mu, V \rangle = V^\lambda A(e_\mu) \delta^\mu_\lambda = V^\lambda A(e_\lambda)$$

and so we have $A = A_\mu e^\mu$ where $A_\mu = A(e_\mu)$, thus the dual basis spans $\mathcal{V}^*$. This choice of dual basis is arbitrary, yet convenient.

As a linearly independent spanning set for $\mathcal{V}^*$, the dual basis is truly a basis for this vector space. There are the same number of basis vectors in the dual basis as in the original basis for $\mathcal{V}$, and so the spaces have equal dimension.

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Since the dual space is itself a vector space, we can imagine maps which send the dual space vectors to the real numbers: this is the "double-dual" space $\mathcal{V}^{**}$. Though it seems that we could repeat this construct to obtain infinitely many such spaces, an observation simplifies the situation. Note that if Λ is a dual vector and $A^\mu e_\mu$ is a vector, then we write the linear transformation as

$$\langle \Lambda, A^\mu e_\mu \rangle$$

and we think of this as the operation of the linear function

$$\langle \Lambda, \cdot \rangle : \mathcal{V} \to \mathbb{R}$$

However, notice that we can alternatively think of this as the operation of the function

$$\langle \cdot, A^\mu e_\mu \rangle : \mathcal{V}^* \to \mathbb{R}$$

which is a linear function. Thus, we have $\mathcal{V} \subset \mathcal{V}^* *$.

Theorem 1.2. *A finite-dimensional vector space $\mathcal{V}$ and its double-dual space $\mathcal{V}^{**}$ are isomorphic.*

Proof. Let $V \in \mathcal{V}$ be an arbitrary vector and define the map $\phi_V : \mathcal{V}^* \to \mathbb{R}$ as

$$\phi_V(\Lambda) = \langle \Lambda, V \rangle$$

We claim that the mapping $V \mapsto \phi_V$ is a vector space isomorphism, i.e., a linear bijection, between $\mathcal{V}$ and $\mathcal{V}^{**}$. First, note that $\phi_V \in \mathcal{V}^{**}$.

Next, assume $\phi_V = \phi_W$ for some $V, W \in \mathcal{V}$. That is, for arbitrary dual vector Λ we have

$$\phi_V(\Lambda) = \langle \Lambda, V \rangle = \langle \Lambda, W \rangle = \phi_W(\Lambda)$$

In particular, let Λ be the projection onto the i-th basis vector, π_i (certainly a linear mapping from $\mathcal{V}$ to $\mathbb{R}$). Then

$$V^i = \langle \pi_i, V \rangle = \langle \pi_i, W \rangle = W^i$$

We can do this for every basis vector, and conclude that $V = W$. Hence, our mapping is injective.

Last, let $\Psi \in \mathcal{V}^{**}$ be an arbitrary double-dual vector. We must show that $\Psi = \Phi_V$ for some $V \in \mathcal{V}$. This is the direct approach....

Instead, note that since a vector space and its dual space have equal dimension, it follows that $\mathcal{V}$ and $\mathcal{V}^{**}$ have equal dimension. ∎

Chapter 2

Tensor Spaces

Given a vector space $\mathcal{V}$, we automatically obtain a dual vector space $\mathcal{V}^*$.

We can naturally start taking Cartesian products:

$$\mathcal{V} \times \mathcal{V} = \{(V, W) \mid V, W \in \mathcal{V}\}$$