Chapter 1 Curves

1.2 Parametrized Curves

Can we have a solution environment?

Exercise 1.2.1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Proof. We know that the familiar sine and cosine functions trace out a circle of unit radius when used as component functions of a plane vector. However, the usual ones run counterclockwise. We can modify this with an inclusion of a negative sign. Further, we need to shift the starting point, $t = 0$, so that the trace starts at the point $(0, 1)$. This gives:

$$\alpha(t) = \left(\cos\left(\frac{\pi}{2} + t\right), \sin\left(\frac{\pi}{2} - t\right)\right)$$

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Exercise 1.2.2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is a point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Assume $\alpha(t) \neq 0$ for any $t \in I$, and let $\alpha(t_0)$ be the point of the trace of α which is closest to the origin. That is, t_0 minimizes the scalar function $g(t) = |\alpha(t)|$. We know that this happens when the derivative of g is zero, i.e.,

$$\begin{aligned}
g'(t) &= \frac{d}{dt}|\alpha(t)| &(1)\\
&= \frac{d}{dt}\sqrt{x(t)^2 + y(t)^2} &(2)\\
&= \frac{1}{2\sqrt{x(t)^2 + y(t)^2}}\left[2x(t)x'(t) + 2y(t)y'(t)\right] &(3)\\
&= \frac{1}{\sqrt{x(t)^2 + y(t)^2}}\left[x(t)x'(t) + y(t)y'(t)\right] &(4)\\
&= \frac{(x(t), y(t)) \cdot (x'(t), y'(t))}{|\alpha(t)|} &(5)\\
&= \frac{\alpha(t) \cdot \alpha'(t)}{|\alpha(t)|} &(6)
\end{aligned}$$

which is well-defined at $t = t_0$ by our hypothesis. In order for this derivative to be 0 at $t = t_0$, the numerator must be 0. In the case where $\alpha'(t)$ is not the zero vector, this means that $\alpha(t_0) \cdot \alpha'(t_0) = 0$, so that the position vector and the velocity vector are orthogonal at this point of closest approach. This makes sense, since if the velocity was at all tangential to the position vector, there would be a small neighborhood about $t = t_0$ where the position vector moved even closer toward the origin, and this would be a contradiction.

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Exercise 1.2.3. A parametrized curve $\alpha(t)$ has the property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?

Proof. Let $\alpha(t) = (x(t), y(t))$ satisfy $\alpha''(t) = (x''(t), y''(t)) \equiv 0$. Then we have $x''(t) = 0$ and so $x(t) = at + b$ for constant scalars a, b; similarly, $y(t) = ct + d$ for constants c, d. Then we have

$$\alpha(t) = (x(t), y(t)) = (at + b, ct + d) = (a, c)t + (b, d)$$

which is a curve with a straight line trace and constant speed.

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Exercise 1.2.4. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Proof. Assume as in the problem statement that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. The first condition means that the velocity of the curve has no component in the direction of some fixed vector; this implies, since we are in $\mathbb{R}^3$, that our curve is restricted to moving in some plane which is normal to v. The second condition requires that the plane in question passes through the origin. From the standpoint of analysis, this seems sensible: the only way for α to become at all tangent to v would require α' being somewhat tangent to v for some moment of time. Explicitly, these conditions mean that, for $\alpha(t) = (x(t), y(t), z(t))$, we have

$$\alpha'(t) \cdot v = (x'(t), y'(t), z'(t)) \cdot v = 0$$

and working backward using the product rule, we can see that

$$\frac{d}{dt}[\alpha(t) \cdot v] = \alpha'(t) \cdot v = 0$$

so that

$$\alpha(t) \cdot v = k$$

for some constant k. However, we know that

$$\alpha(0) \cdot v = 0 = k$$

so that we have $\alpha(t) \cdot v = 0$ for all $t \in I$.

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Exercise 1.2.5. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$

Think of tangential velocity!

Proof. Assume that $|\alpha(t)| = v$ is a nonzero constant. Then

$$v^2 = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$$

and by differentiating both sides with respect to t, we obtain

$$0 = 2\alpha(t) \cdot \alpha'(t)$$

Since $\alpha'(t) \neq 0$, we know that we have α and α' always orthogonal.

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1.3 Regular Curves; Arc Length

Exercise 1.3.1. Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0$, $z = x$.

Proof. The tangent lines to α at t have direction vector

$$\alpha'(t) = \left(3, 6t, 6t^2\right)$$

and so a particular tangent line corresponding to t has the form

$$L(\lambda) = \alpha(t) + \lambda\alpha'(t) = \left(3t + 3\lambda, 3t^2 + 6t\lambda, 2t^3 + 6\lambda t^2\right)$$

where λ is the line parameter. Next, the line $y = 0$, $z = x$ can be parametrized as

$$M(\omega) = (\omega, 0, \omega) = \omega(1, 0, 1)$$

The angle between the tangent lines L and the line M can be computed by the inner product between their direction vectors:

$$\cos\theta = \frac{(1,0,1)\cdot(3,6t,6t^2)}{|(1,0,1)|\,|(3,6t,6t^2)|} = \frac{3+6t^2}{\sqrt{2}\sqrt{3^2+(6t)^2+(6t^2)^2}} = \frac{3+6t^2}{\sqrt{2}\sqrt{(3+6t^2)^2}} = \frac{1}{\sqrt{2}}$$

which is independent of t, concluding the proof. ∎

Exercise 1.3.2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a *cycloid*.

(a) Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.

(b) Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Proof. (a) Picture a circular disk which begins rolling along the positive x axis after starting with its center positioned above the origin. Consider the point (x, y) which lies at the origin and is attached to the radius of the disk. As the disk rolls, this point spins along its edge; it moves under the rotation of the disk and also under the rightward motion of the disk.

After the disk rolls through an angle of θ, the center of the disk has moved $r\theta$ to the right, which is equal to the arc length the point has moved. Thus, the motion of the center of the disk is given by

$$C(\theta) = (r\theta, r)$$

We can figure out the position of our point (x, y) by thinking of triangles relative to the center of the disk. The x position of the point is $x(\theta) = r\theta - r\sin\theta$, and the y coordinate is $y(\theta) = r - r\cos\theta$. Thus, the parametrization is

$$\alpha(t) = (r(\theta - \sin\theta), r(1 - \cos\theta))$$

The singular points are those which have zero velocity:

$$\alpha'(t) = r(1 - \cos\theta, \sin\theta) = (0,0)$$

and from our rudimentary knowledge of sine and cosine functions, the coordinates are 0 precisely when $\theta = 2n\pi$, i.e., when the disk has rolled through exactly an integer number of rotations.

(b) From part (a), we need to integrate the arc length over $\theta \in [0, 2\pi]$, which is a single complete rotation of the disk. The integrand is

$$\begin{aligned} |\alpha'(\theta)| &= \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta} \\ &= r\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} \\ &= r\sqrt{2 - 2\cos\theta} \\ &= 2r\sin\frac{\theta}{2} \end{aligned}$$

by the law of cosines. The integral is

$$\begin{aligned} L &= \int_0^{2\pi} |\alpha'(\theta)| \, d\theta \\ &= 2r\int_0^{2\pi} \sin\frac{\theta}{2} \, d\theta \\ &= -4r\cos\frac{\theta}{2}\bigg|_0^{2\pi} \\ &= 8r \end{aligned}$$

and so the arc length of a cycloid corresponding to a circle of radius $r = 1$ is simply 8.

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Exercise 1.3.3. Let $0A = 2a$ be the diameter of a circle S^1 and $0y$ and AV be the tangents to S^1 at 0 and A, respectively. A half-line r is drawn from 0 which meets the circle S^1 at C and the line AV at B. On $0B$ mark off the segment $0p = CB$. If we rotate r about 0, the point p will describe a curve called the *cissoid of Diocles*. By taking $0A$ as the x axis and $0Y$ as the y axis, prove that

(a) The trace of

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right), \; t \in \mathbb{R}$$

is the cissoid of Diocles ($t = \tan\theta$).

(b) The origin $(0,0)$ is a singular point of the cissoid.

(c) As $t \to \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \to 0$, $2a$. Thus, as $t \to \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an *asymptote* to the cissoid.

Exercise 1.3.4. Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha'(t)$. The trace of α is called the *tractrix*. Show that

(a) α is a differentiable parametrized curve, regular except at $t = \pi/2$.

(b) The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Proof. (a) Note that the functions $\cos \frac{t}{2}$ and $\sin \frac{t}{2}$ are nonzero and positive on the interval $(0, \pi)$. Thus,$\tan \frac{t}{2}$ and the log are well-defined on this interval. Further, the derivative is

$$\alpha'(t) = \left(\cos t, -\sin t + \frac{2}{\tan \frac{t}{2}} \sec^2 \frac{t}{2}\right) \tag{7}$$

$$= \left(\cos t, -\sin t + 2/\cos \frac{t}{2} \sin \frac{t}{2}\right) \tag{8}$$

$$\tag{9}$$

This derivative can be zero only if both components are. The first component, $\cos t$, is zero only when $t = \pi/2$. However, the second component at $t = \pi/2$ is

$$-\sin \frac{\pi}{2} + \frac{2}{\cos \frac{\pi}{4} \sin \frac{\pi}{4}} = -1 + 1 = 0$$

and so $\alpha'(t) = 0$ only at $t = \pi/2$, and so α is not regular there.

(b) The tangent line at any point of the tractrix is given by

$$T(\lambda) = \alpha(t) + \lambda \alpha'(t)$$

The length of the desired segment we seek will be given by the Euclidean distance (the straight line distance) between the point $\alpha(t)$ and the point where $T(\lambda)$ intersects the y axis.

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Exercise 1.3.5. Let $\alpha : (-1, +\infty) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right).$$

Prove that:

(a) For $t = 0$, α is tangent to the x axis.

(b) As $t \to +\infty$, $\alpha(t) \to (0, 0)$ and $\alpha'(t) \to (0, 0)$.

(c) Take the curve with the opposite orientation. Now, as $t \to -1$, the curve and its tangent approach the line $x + y + a = 0$

The figure obtained by completing the trace of α in such a way that it becomes symmetric relative to the line $y = x$ is called the *folium of Descartes*.

Proof. (a) The tangent vector to α is

$$\alpha'(t) = 3a\left(\frac{1-2t^3}{(1+t^3)^2}, \frac{2t-t^4}{(1+t^3)^2}\right)$$

At $t = 0$, this is

$$\alpha'(0) = 3a(1, 0)$$

which is easily parallel to the x axis, an so α is tangent to the x axis since $\alpha(0) = (0, 0)$.

(b) These limits are easy to see from the definition of α and the computation of α' as both ratios of polynomials in t with higher order powers in the denominator; from basic calculus, these ratios all tend toward 0.

(c)

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Exercise 1.3.6. Let $\alpha(t) = \left(ae^{bt}\cos t, ae^{bt}\sin t\right)$, $t \in \mathbb{R}$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.

(a) Show that as $t \to +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of α is called the *logarithmic spiral*).

(b) Show that $\alpha'(t) \to (0, 0)$ as $t \to +\infty$ and that

$$\lim_{t\to+\infty}\int_{t_0}^{t} |\alpha'(t)|\, dt$$

is finite; that is, α has finite arc length in $[t_0, \infty)$.

Proof. (a) We can factor some things out of the vector as

$$\alpha(t) = ae^{bt}(\cos t, \sin t)$$

so that ae^{bt} is a scalar factor and $(\cos t, \sin t)$ is the usual circular motion. Hence, α is roughly circular motion, but the magnitude of $\alpha(t)$ goes as

$$|\alpha| = ae^{bt} \to 0,$$

thus $\alpha(t)$ approaches 0.

(b) The derivative of α can be computed as

$$\begin{aligned}\alpha'(t) &= \left(abe^{bt}\cos t - ae^{bt}\sin t, abe^{bt}\sin t + ae^{bt}\cos t\right)\\ &= ae^{bt}\left(b\cos t - \sin t, b\sin t + \cos t\right)\end{aligned}$$

which has magnitude

$$\begin{aligned}\left|\alpha'(t)\right|^2 &= a^2e^{2bt}\left[b^2\cos^2 t + \sin^2 t - 2b\cos t\sin t + b^2\sin^2 t + \cos^2 t + 2b\cos t\sin t\right]\\ &= a^2e^{2bt}\left(b^2+1\right) \to 0\end{aligned}$$

and so $\alpha'(t) \to 0$ as $t \to +\infty$. The arc length integral can be easily computed from this:

$$\begin{aligned}\int_0^{+\infty} \left|\alpha'(t)\right|\, dt &= a\sqrt{b^2+1}\int_0^{+\infty} e^{bt}\, dt\\ &= \frac{a}{b}\sqrt{b^2+1}\; e^{bt}\Big|_0^{+\infty}\\ &= -\frac{a}{b}\sqrt{b^2+1}\end{aligned}$$

which is certainly finite (and also positive since $b < 0$).

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Exercise 1.3.7.