Contents

Chapter 1

Matrices, Vectors, and Vector Calculus

We know the usual definition of sequences in $\mathbb{R}$. A sequence of vectors converges similarly, except that the norm used in the definition of convergence is now a vector norm, not simply absolute value. A sequence of vectors $\{\vec{a}_i\}$ in $\mathbb{R}^n$ converges to $\vec{a} \in \mathbb{R}^n$ if for any $\epsilon > 0$ there exists a $n \in \mathbb{N}$ such that

$$n > N \implies \|\vec{a}_i - \vec{a}\| < \epsilon$$

This definition can be related to the convergence of the vector components.

Prop 1.1. *Let $\{\vec{a}_i\}$ be a sequence of vectors (coordinate vectors) in $\mathbb{R}^n$. Then $\{\vec{a}_i\}$ converges to $\vec{a} \in \mathbb{R}^n$ if and only if each component $(\vec{a}_i)_j$ converges to $(\vec{a})_j$.*

Proof. Assume that we have the components converging. That is, for $\epsilon > 0$ we have some $N \in \mathbb{N}$ such that

$$n > N \implies \left|(\vec{a}_i)_j - (\vec{a})_j\right| < \epsilon/\sqrt{n}$$

We choose N to hold for every component i. This can be arranged by taking N to the the maximum of the N guaranteed for each individual component. Thus, we have, for $n > N$,

$$\begin{aligned}\|\vec{a}_i - \vec{a}\|^2 &= \sum_{j=1}^{n} \left|(\vec{a}_i)_j - (\vec{a})_j\right|^2 \\ &< \sum_{j=1}^{n} \epsilon^2/n \\ &= \epsilon\end{aligned}$$

so that $\{\vec{a}_i\}$ converges to $\vec{a}$.

Conversely, assume that $\{\vec{a}_i\}$ converges to $\vec{a}$. Let $\epsilon > 0$ be arbitrary and let N be the guaranteed natural number in the definition of vector sequence convergence.

The key is to note that for any $j = 1, 2, \ldots, n$ we have

$$\left|(\vec{a}_i)_j - (\vec{a})_j\right|^2 \leq \sum_{k=1}^{n} \left|(\vec{a}_i)_k - (\vec{a})_k\right|^2 < \epsilon^2$$

whenever $n > N$. Thus, we have in particular

$$n > N \implies \left|(\vec{a}_i)_j - (\vec{a})_j\right| < \epsilon$$

and so the component sequences converge as expected. ■

Problem 1.2. hi

Let's look at some interesting consequences.