TEST #3 REVIEW : ANSWERS

#1.

$$\overset{k=1}{\left(\cancel{\frac{1}{5}}-\frac{1}{3}\right)}+\overset{k=2}{\left(\cancel{\frac{1}{7}}-\cancel{\frac{1}{5}}\right)}+\overset{k=3}{\left(\cancel{\frac{1}{9}}-\cancel{\frac{1}{7}}\right)}+\cdots$$

$$+\overset{k=n-2}{\left(\cancel{\frac{1}{2n-1}}-\cancel{\frac{1}{2n-3}}\right)}+\overset{k=n-1}{\left(\cancel{\frac{1}{2n+1}}-\cancel{\frac{1}{2n-1}}\right)}+\overset{k=n}{\left(\frac{1}{2n+3}-\cancel{\frac{1}{2n+1}}\right)}$$

$$S_n=-\frac{1}{3}+\frac{1}{2n+3}\ ;\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{-1}{3}+\frac{1}{2n+3}=\frac{-1}{3}$$

So, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2n+3}-\frac{1}{2n+1}\right)$ CONVERGES AND THE SUM IS $-1/3$

#2. $$\sum_{n=1}^{\infty}(-1)^{n-1}3^n 2^{1-n}=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot 3^{n-1}\cdot 3^{1}\cdot 2^{-(n-1)}$$

$$=\sum_{n=1}^{\infty}3\cdot\frac{(-1)^{n-1}\cdot 3^{n-1}}{2^{n-1}}=\sum_{n=1}^{\infty}3\left(\frac{-3}{2}\right)^{n-1}$$ THIS IS

GEOMETRIC w/ "a"$=3$ & $|r|=3/2>1$. DIVERGENT

#3. $$\sum_{n=1}^{\infty}\frac{n^2}{2n^2+1}\ ;\ \lim_{n\to\infty}\frac{n^2}{2n^2+1}=\frac{1}{2}\neq 0$$ DIVERGENT BY DIVERGENCE TEST

#4. $$\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{3^{n-1}}\cdot\frac{3}{3}=\sum_{n=0}^{\infty}\frac{3\cdot(-1)^n\cdot 2^n}{3^n}=\sum_{n=0}^{\infty}3\left(\frac{-2}{3}\right)^n$$

GEOMETRIC SERIES w/ "a"$=3$, $|r|=2/3<1$

CONVERGES ; $$\text{SUM}=\frac{3}{1-(-2/3)}=\frac{3}{5/3}=\frac{9}{5}$$

#5. $\sum_{n=1}^{\infty} n^{-\pi} = \sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$, $P = \pi > 1$ CONVERGENT "P"-SERIES

#6. $f(x) = \frac{\ln x}{x^2}$ AND IS P, C, D* ON $[2, \infty)$

$^* f'(x) = \frac{x^2 \cdot 1/x - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4} < 0$ ON

$\lim_{b\to\infty} \int_2^b \frac{\ln x}{x^2}\,dx \quad u = \ln x \quad du = 1/x\,dx \quad v = \int x^{-2}\,dx = -x^{-1}$

$\rightarrow \frac{-\ln x}{x} - \int \frac{-1}{x}\cdot\frac{1}{x}\,dx = -\frac{\ln x}{x} + \int x^{-2}\,dx = \frac{-\ln x}{x} - \frac{1}{x}$

$\rightarrow \lim_{b\to\infty}\left[-\frac{(1+\ln x)}{x}\right]_2^b \xrightarrow{L'} \lim_{b\to\infty}\left[-\frac{1/x}{1}\right]_2^b = \lim_{b\to\infty}\left[-\frac{1}{b}+\frac{1}{2}\right] = \frac{1}{2}$

So, $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ CONVERGES

#7. $f(x) = \frac{\tan^{-1} n}{n^2+1}$; P, C, D ON $[1, \infty)$ DECREASING BECAUSE $\tan^{-1} n$ IS BOUNDED

$\lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{x^2+1}\,dx \quad u = \tan^{-1} x \quad du = 1/x^2+1\,dx \rightarrow \lim_{b\to\infty}\int_1 u\,du$

$= \lim_{b\to\infty}\left[\frac{u^2}{2}\right] \rightarrow \lim_{b\to\infty}\left[\frac{(\tan^{-1} x)^2}{2}\right]_1^b = \lim_{b\to\infty}\left[\frac{(\tan^{-1} b)^2}{2} - \frac{(\tan^{-1} 1)^2}{2}\right]$

$= \frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2} = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}$ So, $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$ CONVERGES

#8. $\sum_{n=3}^{\infty} \frac{3^n}{2^n-4}$ $\quad \frac{3^n}{2^n-4} > \frac{3^n}{2^n}$; $\sum_{n=3}^{\infty} \left(\frac{3}{2}\right)^n$ <u>DIVERGES</u>

GEOMETRIC w/ $|r| = 3/2 > 1$

SO, BY COMPARISON TEST,

$\sum_{n=3}^{\infty} \frac{3^n}{2^n-4}$ ALSO <u>DIVERGES</u>.

#9. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$; $\frac{\cos^2 n}{n^2} \leq \frac{1}{n^2}$; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ <u>CONVERGES</u>

$P = 2 > 1$

SO, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ ALSO <u>CONVERGES</u>

#10. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2(n+3)}$, COMPARES TO $\frac{n^2}{n^3} = \frac{1}{n}$; $\sum_{n=1}^{\infty} \frac{1}{n}$; <u>DIV.</u>

$\lim_{n\to\infty} \frac{n^2+1}{n^2(n+3)} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3+n}{n^3+3n^2} = 1$ SO, SINCE

DIVERGES, SO DOES $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2(n+3)}$ BY L.C.T.

#11. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+n}}$; $\frac{n}{\sqrt{n^5+n}} < \frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}}$ FOR COMPARISON TEST

<u>OR L.C.T</u> $\lim_{n\to\infty} \frac{n}{\sqrt{n^5+n}} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^{5/2}}{\sqrt{n^5+n}} = 1$ SO, EITHER WAY

$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+n}}$ <u>CONVERGES</u> SINCE $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, $P = 3/2 > 1$ <u>CONVERGES</u>

#12. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{2n^2-1}$; $\lim_{n\to\infty} \frac{n^2}{2n^2-1} = \frac{1}{2} \neq 0$ DIVERGES BY A.S.T

#13. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot e^n}{\pi^{n+1}}$; $\lim_{n\to\infty} \frac{e^n}{\pi^{n+1}} = \lim_{n\to\infty} \frac{e^n}{\pi \cdot \pi^n} = \lim_{n\to\infty} \frac{1}{\pi}\left(\frac{e}{\pi}\right)^n$

SINCE $e < \pi$, $\lim_{n\to\infty} \frac{1}{\pi}\left(\frac{e}{\pi}\right)^n = 0$ ✓

ALSO; $a_n = \frac{e^n}{\pi^{n+1}} = \frac{1}{\pi}\left(\frac{e}{\pi}\right)^n > \frac{1}{\pi}\left(\frac{e}{\pi}\right)^{n+1}$ ← a_{n+1} FOR $n \geq 0$ AND SINCE $e < \pi$

BY A.S.T, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} e^n}{\pi^{n+1}}$ CONVERGES

#14. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$; $\cos n\pi = -1, 1, -1, 1, \ldots = (-1)^n$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ CONVERGENT $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

BY A.S.T $a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$ ✓

#15. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1} \to \sum_{n=1}^{\infty} \left|\frac{(-1)^n n}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$; $\frac{n}{n^2+1}$ COMPARES

TO $\frac{1}{n}$ & $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES ; $\lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = 1$

SO, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ ALSO DIVERGES BY L.C.T. BUT $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

CONVERGES BY A.S.T. SO, SERIES IS CONDITIONALLY CONVERGENT.
("SHOW THIS")

#16. $\sum_{n=1}^{\infty} \frac{\cos(n+1)}{n\sqrt{n}} \rightarrow \sum_{n=1}^{\infty} \left| \frac{\cos(n+1)}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{|\cos(n+1)|}{n\sqrt{n}}$

Now, $|\cos(n+1)| \leq 1$ so $\frac{|\cos(n+1)|}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$; $P = 3/2 > 1$ CONVERGES. So $\sum_{n=1}^{\infty} \frac{|\cos(n+1)|}{n\sqrt{n}}$

CONVERGES BY COMPARISON TEST. THEREFORE,

$\sum_{n=1}^{\infty} \frac{\cos(n+1)}{n\sqrt{n}}$ IS ABSOLUTELY CONVERGENT

#17. $\sum_{n=1}^{\infty} \frac{(-5)^{n-1}}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 5^{n-1}}{n^2 \cdot 3^n}$ RATIO TEST

$a_n = \frac{5^{n-1}}{n^2 \cdot 3^n}$, $a_{n+1} = \frac{5^n}{(n+1)^2 \cdot 3^{n+1}}$

$$\lim_{n\to\infty} \left| \frac{\cancel{5^n}^{\,5}}{(n+1)^2 \cdot \underset{3}{\cancel{3^{n+1}}}} \cdot \frac{n^2 \cdot \cancel{3^n}}{\cancel{5^{n-1}}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2 \cdot \frac{5}{3} = \frac{5}{3} > 1$$

DIVERGES BY RATIO TEST

#18. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1} \right)^n$; ROOT TEST $\rightarrow \lim_{n\to\infty} \sqrt[\cancel{n}]{\left(\frac{n}{2n+1} \right)^{\cancel{n}}}$

$= \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$ SERIES CONVERGES ABSOLUTELY BY ROOT TEST

#19. $\sum_{n=0}^{\infty} \frac{(-1)^n n! x^n}{2^n}$; RATIO TEST $a_n = \frac{n! x^n}{2^n}$

$a_{n+1} = \frac{(n+1)! x^{n+1}}{2^{n+1}} \rightarrow \lim_{n\to\infty} \left| \frac{\cancel{(n+1)!}^{(n+1)} \cancel{x^{n+1}}^{x}}{\cancel{2^{n+1}}_{2}} \cdot \frac{\cancel{2^n}}{\cancel{n!} \cancel{x^n}} \right|$

$= \lim_{n\to\infty} \frac{n+1}{2} \cdot |x| = \infty > 1$, FOR ALL "X" EXCEPT $x=0$

AT $x=0$, LIMIT $=0<1$

SO, SERIES CONVERGES ONLY AT $x=0$, $R=0$.

#20. $\sum_{n=0}^{\infty} \frac{(-1)^n (3x+2)^{2n}}{(2n)!}$; RATIO TEST , $a_n = \frac{(3x+2)^{2n}}{(2n)!}$

$a_{n+1} = \frac{(3x+2)^{2n+2}}{(2n+2)!}$; $\lim_{n\to\infty} \left| \frac{\cancel{(3x+2)^{2n+2}}^{(3x+2)^2}}{\cancel{(2n+2)!}_{(2n+2)(2n+1)}} \cdot \frac{\cancel{(2n)!}}{\cancel{(3x+2)^{2n}}} \right|$

$= \lim_{n\to\infty} \frac{(3x+2)^2}{(2n+2)(2n+1)} = 0 < 1$ FOR ALL "X", SO $R=\infty$, $(-\infty, \infty)$

#21. $\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n^3+n}$, RATIO TEST ; $a_n = \frac{(3x-1)^n}{n^3+n}$

$a_{n+1} = \frac{(3x-1)^{n+1}}{(n+1)^3+n+1} \rightarrow \lim_{n\to\infty} \left| \frac{\cancel{(3x-1)^{n+1}}}{(n+1)^3+n+1} \cdot \frac{n^3+n}{\cancel{(3x-1)^n}} \right|$

$= \lim_{n\to\infty} \frac{n^3+n}{n^3+3n^2+4n+2} \cdot |3x-1| = |3x-1| < 1$ FOR CONVERGENCE

$|3x-1| < 1 \rightarrow 3|x-1/3| < 1 \rightarrow |x-1/3| < 1/3$ $R = 1/3$

$-1/3 < x - 1/3 < 1/3 \rightarrow 0 < x < 2/3$

ENDPOINTS: $x=0$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+n}$ CONVERGES BY A.S.T

$x=2/3$: $\sum_{n=1}^{\infty} \frac{(1)^n}{n^3+n}$ CONVERGES BY COMP. TEST OR L.C.T.

So, $R=1/3$ AND $I: [0, 2/3]$

#22. $\sum_{n=1}^{\infty} \frac{e^n x^n}{n}$; RATIO TEST $a_n = \frac{e^n x^n}{n}$

$a_{n+1} = \frac{e^{n+1} x^{n+1}}{n+1} \rightarrow \lim_{n\to\infty} \left| \frac{\cancel{e^{n+1}}^{e} \cancel{x^{n+1}}^{x}}{n+1} \cdot \frac{n}{\cancel{e^n x^n}} \right|$

$= \lim_{n\to\infty} \frac{n}{n+1} \cdot e|x| = e|x| < 1$ FOR CONVERGENCE

$|x| < 1/e$, $R = 1/e$, $-1/e < x < 1/e$

ENDPOINTS: $x = -1/e$; $\sum_{n=1}^{\infty} \frac{e^n(-1/e)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ CONVERGES A.S.T.

$x = 1/e$; $\sum_{n=1}^{\infty} \frac{e^n(1/e)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES (HARMONIC)

So, $R = 1/e$, $I: [-1/e, 1/e)$

#23. $\frac{1}{1+3x} = \frac{1}{1-(-3x)} = 1 + (-3x) + (-3x)^2 + (-3x)^3 + (-3x)^4 + \cdots$

$= 1 - 3x + 3^2x^2 - 3^3x^3 + 3^4x^4 + \cdots = \sum_{n=0}^{\infty} (-1)^n 3^n x^n$

$-1 < "x" < 1 \rightarrow -1 < -3x < 1 \rightarrow 1/3 > x > -1/3 \rightarrow R = 1/3$; $I(-1/3, 1/3)$

$\#24.\quad e^{2x}, \; c=-1 \to e^{2x} = e^{2(x+1)-2} = e^{2(x+1)} \cdot e^{-2}$

$$= e^{-2}\left[1 + 2(x+1) + \frac{[2(x+1)]^2}{2!} + \frac{[2(x+1)]^3}{3!} + \dots\right]$$

$$= e^{-2}\left[1 + 2(x+1) + \frac{2^2(x+1)^2}{2!} + \frac{2^3(x+1)^3}{3!} + \dots\right]$$

$$= e^{-2}\sum_{n=0}^{\infty} \frac{2^n(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n(x+1)^n}{e^2 \cdot n!} \; ; \quad R=\infty, \; I:(-\infty,\infty)$$

$\#25.\quad f(x) = 1/x, \; c=1, \; n=5 \; [0.9, 1.1] \qquad f(1)=1$

$f'(x) = -1/x^2 \qquad f'(1) = -(1!)$

$f''(x) = 2/x^3 \qquad f''(1) = 2!$

$f'''(x) = -2\cdot3/x^4 \qquad f'''(1) = -(3!)$

$f^4(x) = 2\cdot3\cdot4/x^5 \qquad f^4(1) = 4!$

$f^5(x) = -2\cdot3\cdot4\cdot5/x^6 \qquad f^5(1) = -(5!)$

$f^6(x) = 2\cdot3\cdot4\cdot5\cdot6/x^7 = 6!/x^7$

$$P_5(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^4(1)(x-1)^4}{4!} + \frac{f^5(1)(x-1)^5}{5!}$$

$$= 1 - (x-1) + \frac{\cancel{2!}(x-1)^2}{\cancel{2!}} - \frac{\cancel{3!}(x-1)^3}{\cancel{3!}} + \frac{\cancel{4!}(x-1)^4}{\cancel{4!}} - \frac{\cancel{5!}(x-1)^5}{\cancel{5!}}$$

$$P_5(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-4)^4 - (x-1)^5$$

$$R_5(x) = \frac{f^{6}(z)(x-1)^6}{6!} = \frac{\cancel{6!}}{z^7} \cdot \frac{(x-1)^6}{\cancel{6!}} = \frac{(x-1)^6}{z^7}$$

FOR <u>LARGEST</u> ERROR, $x = 1.1$, $z = 0.9$

$$R_5(x) \le \frac{(1.1-1)^6}{(.9)^7} = \frac{(.1)^6}{(.9)^7} \approx .0000021$$