Coursework (5) for *Introductory Lectures on Optimization*

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December 10, 2024

Excercise 1. Prove the following theorem:
Let $\|\cdot\|$ be a vector norm in $\mathbb{R}^n$, then

$$\partial \|\cdot\| = \left\{ V(\boldsymbol{x}) \triangleq \left\{ \boldsymbol{v} \in \mathbb{R}^n \middle| \langle \boldsymbol{v},\ \boldsymbol{x} \rangle = \|\boldsymbol{x}\|, \|\boldsymbol{v}\|_* \leq 1 \right\} \right\},$$

where $\|\boldsymbol{v}\|_*$ is the dual norm of $\|\cdot\|$, defined as

$$\|\boldsymbol{v}\|_* \triangleq \sup_{\|\boldsymbol{u}\| \leq 1} \langle \boldsymbol{v},\ \boldsymbol{u} \rangle.$$

Proof of Excercise 1: The answer is as follows:

We prove the theorem by the $V(\boldsymbol{x}) \subset \partial \|\boldsymbol{x}\|$ and $\partial \|\boldsymbol{x}\| \subset V(\boldsymbol{x})$

1. $V(\boldsymbol{x}) \subset \partial \|\boldsymbol{x}\|$

   Let $\boldsymbol{v} \in V(\boldsymbol{x})$, and $\forall \boldsymbol{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} \|\boldsymbol{x}\| + \langle \boldsymbol{v},\ \boldsymbol{y} - \boldsymbol{x} \rangle &= \|\boldsymbol{x}\| + \langle \boldsymbol{v},\ \boldsymbol{y} \rangle - \langle \boldsymbol{v},\ \boldsymbol{x} \rangle & (1) \\ &= \|\boldsymbol{x}\| + \langle \boldsymbol{v},\ \boldsymbol{y} \rangle - \|\boldsymbol{x}\| & (2) \\ &= \langle \boldsymbol{v},\ \boldsymbol{y} \rangle & (3) \\ &\leq \|\boldsymbol{y}\| \cdot \|\boldsymbol{v}\|_* & (4) \\ &\leq \|\boldsymbol{y}\| & (5) \end{aligned}$$

   (2)(5)'s reason is that $\boldsymbol{v} \in V(\boldsymbol{x})$.

   (4)'s reason is that **Holder's inequality**.

   That is $\|\boldsymbol{y}\| \geq \|\boldsymbol{x}\| + \langle \boldsymbol{v},\ \boldsymbol{y} - \boldsymbol{x} \rangle$, since of arbitrariness of $\boldsymbol{y}$, it is the definition of sub-gradient of $\|\cdot\|$ and $\boldsymbol{v} \in \partial \|\boldsymbol{x}\|$.

   So $V(\boldsymbol{x}) \subset \partial \|\boldsymbol{x}\|$.

2. $\partial \|\boldsymbol{x}\| \subset V(\boldsymbol{x})$

   Let $\boldsymbol{v} \in \partial \|\boldsymbol{x}\|$, and $\forall \boldsymbol{y} \in \mathbb{R}^n$, we have:

$$\|\boldsymbol{y}\| \geq \|\boldsymbol{x}\| + \langle \boldsymbol{v},\ \boldsymbol{y} - \boldsymbol{x} \rangle \tag{6}$$

$$\therefore \forall \boldsymbol{y} \in \mathbb{R}^n, \langle \boldsymbol{v},\ \boldsymbol{y} \rangle - \|\boldsymbol{y}\| \leq \langle \boldsymbol{v},\ \boldsymbol{x} \rangle - \|\boldsymbol{x}\| \tag{7}$$

$$\therefore \|\boldsymbol{v}\|_* \text{ Fenchel conjugate} \triangleq \sup_{\boldsymbol{y}} \{\langle \boldsymbol{v},\ \boldsymbol{y}\rangle - \|\boldsymbol{y}\|\} \leq \langle \boldsymbol{v},\ \boldsymbol{x}\rangle - \|\boldsymbol{x}\| \tag{8}$$

So we get:

$$\|\boldsymbol{v}\|_* = \begin{cases} 0 & \|\boldsymbol{v}\|_* \leq 1 \\ +\infty & \|\boldsymbol{v}\|_* > 1 \end{cases} \tag{9}$$

Since $\langle \boldsymbol{v},\ \boldsymbol{x}\rangle - \|\boldsymbol{x}\|$ is always finite, we have $\|\boldsymbol{v}\|_* \leq 1$.

$$\begin{aligned} 0 &\leq \langle \boldsymbol{v},\ \boldsymbol{x}\rangle - \|\boldsymbol{x}\| & (10)\\ &\leq \|\boldsymbol{x}\| \cdot \|\boldsymbol{v}\|_* - \|\boldsymbol{x}\| & (11)\\ &\leq \|\boldsymbol{x}\| \cdot 1 - \|\boldsymbol{x}\| \ \because \|\boldsymbol{v}\|_* \leq 1 & (12)\\ &= 0 & (13)\end{aligned}$$

So we have $\langle \boldsymbol{v},\ \boldsymbol{x}\rangle = \|\boldsymbol{x}\|$,that implies $\partial \|\boldsymbol{x}\| \subset V(\boldsymbol{x})$.

Therefore, we have $\partial \|\boldsymbol{x}\| = V(\boldsymbol{x})$.

$\square$

Excercise 2. Write down the subdifferentials of following functions.

1. $f(\boldsymbol{x}) = |\boldsymbol{x}|, \boldsymbol{x} \in \mathbb{R}^1$.
2. $f(\boldsymbol{x}) = \sum_{i=1}^m |\langle \boldsymbol{a}_i,\ \boldsymbol{x}\rangle - \boldsymbol{b}_i|$.
3. $f(\boldsymbol{x}) = \max_{1\leq i\leq n} \boldsymbol{x}^{(i)}$.
4. $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$.
5. $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 = \sum_{i=1}^n |\boldsymbol{x}^{(i)}|$.

Solution of Excercise 2: The answer is as follows:

1. $f(\boldsymbol{x}) = |\boldsymbol{x}|, \boldsymbol{x} \in \mathbb{R}^1$.

$$f(\boldsymbol{x}) = \max_{-1\leq t\leq 1} \boldsymbol{g} \cdot \boldsymbol{x} \Rightarrow \partial f(\boldsymbol{0}) = [-1, 1] \tag{14}$$

2. $f(\boldsymbol{x}) = \sum_{i=1}^m |\langle \boldsymbol{a}_i,\ \boldsymbol{x}\rangle - \boldsymbol{b}_i|$

   Denote

$$\begin{aligned} I_-(\boldsymbol{x}) &= \{i \mid \langle \boldsymbol{a}_i,\ \boldsymbol{x}_i\rangle - \boldsymbol{b}_i < 0\},\\ I_+(\boldsymbol{x}) &= \{i \mid \langle \boldsymbol{a}_i,\ \boldsymbol{x}_i\rangle - \boldsymbol{b}_i > 0\},\\ I_0(\boldsymbol{x}) &= \{i \mid \langle \boldsymbol{a}_i,\ \boldsymbol{x}_i\rangle - \boldsymbol{b}_i = 0\}.\end{aligned}$$

   Then

$$\partial f(\boldsymbol{x}) = \sum_{i\in I_+(\boldsymbol{x})} \boldsymbol{a}_i - \sum_{i\in I_-(\boldsymbol{x})} \boldsymbol{a}_i + \sum_{i\in I_0(\boldsymbol{x})} [-\boldsymbol{a}_i, \boldsymbol{a}_i].$$

3. $f(\boldsymbol{x}) = \max_{1\leq i\leq n} \boldsymbol{x}^{(i)}$

   Denote:$I(\boldsymbol{x}) = \{i \mid \boldsymbol{x}^{(i)} = f(\boldsymbol{x})\}$

Then we have:

$$\partial f(\boldsymbol{x}) = \begin{cases} \text{Conv}\{\boldsymbol{e}_i \mid 1 \leq i \leq n\} \equiv \Delta_n, & \boldsymbol{x} = \boldsymbol{0}, \\ \text{Conv}\{\boldsymbol{e}_i \mid i \in I(\boldsymbol{x})\}, & \boldsymbol{x} \neq \boldsymbol{0}. \end{cases}$$

4. $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$

$$\partial f(\boldsymbol{x}) = \begin{cases} B_2(0,1) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| \leq 1\}, & \boldsymbol{x} = 0, \\ \{\boldsymbol{x}/\|\boldsymbol{x}\|\}, & \boldsymbol{x} \neq 0. \end{cases}$$

5. $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 = \sum_{i=1}^n |\boldsymbol{x}^{(i)}|$

   Denote:

$$I_+(\boldsymbol{x}) = \{i \mid \boldsymbol{x}^{(i)} > 0\},$$
$$I_-(\boldsymbol{x}) = \{i \mid \boldsymbol{x}^{(i)} < 0\},$$
$$I_0(\boldsymbol{x}) = \{i \mid \boldsymbol{x}^{(i)} = 0\}.$$

   Then we have

$$\partial f(\boldsymbol{x}) = \begin{cases} B_\infty(0,1) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \max_{1 \leq i \leq n} |\boldsymbol{x}^{(i)}| \leq 1\}, & \boldsymbol{x} = 0, \\ \sum_{i \in I_+(\boldsymbol{x})} \boldsymbol{e}_i - \sum_{i \in I_-(\boldsymbol{x})} \boldsymbol{e}_i + \sum_{i \in I_0(\boldsymbol{x})} [-\boldsymbol{e}_i, \boldsymbol{e}_i], & \boldsymbol{x} \neq 0. \end{cases}$$

□

Excercise 3. Please write down three sequences and prove that they satisfy the following conditions:

$$h_k > 0, h_k \to 0, \sum_{k=0}^{\infty} h_k = \infty.$$

Solution of Excercise 3: The answer is as follows:

1. $h_k = \frac{1}{k+1}$

$$h_k = \frac{1}{k+1} > 0 \tag{15}$$
$$\lim_{k\to\infty} h_k = \lim_{k\to\infty} \frac{1}{k+1} = 0 \tag{16}$$
$$\sum_{k=0}^{\infty} h_k = \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty. \because \sum_{k=0}^{\infty} \frac{1}{k+1} \text{is } \textbf{harmonic progression}. \tag{17}$$

2. $h_k = \frac{1}{\sqrt{k+1}}$

$$h_k = \frac{1}{\sqrt{k+1}} > 0 \tag{18}$$
$$\lim_{k\to\infty} h_k = \lim_{k\to\infty} \frac{1}{\sqrt{k+1}} = 0 \tag{19}$$
$$\sum_{k=0}^{\infty} h_k = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}} \geq \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty. \tag{20}$$

3. $h_k = \frac{1}{\ln{(k+2)}}$

$$h_k = \frac{1}{\ln{(k+2)}} > 0 \tag{21}$$

$$\lim_{k\to\infty} h_k = \lim_{k\to\infty} \frac{1}{\ln{(k+2)}} = 0 \tag{22}$$

$$\sum_{k=0}^{\infty} h_k = \sum_{k=0}^{\infty} \frac{1}{\ln{(k+2)}} > \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty. \tag{23}$$

$\square$