Coursework (2) for *Introductory Lectures on Optimization*

Xiaoyu Wang
3220104364

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Excercise 1. For the function $f(x) : \mathbb{R}^n \to \mathbb{R}^m$, please write down the zeroth-order Taylor expansion with an integral remainder term.

Solution of Excercise 1: The answer is as follows:

$$\mathbf{x} - \mathbf{x_0} = (x_1 - x_{01}, x_2 - x_{02}, \cdots, x_n - x_{0n})^T f(\mathbf{x}) = y_1, y_2, \cdots, y_m{}^T$$

when $m = 1$,we get the zeroth-order Taylor expansion with an integral remainder term:

$$f(\mathbf{x}) = f(\mathbf{x_0}) + \int_0^1 (\nabla f(\mathbf{x_0} + t(\mathbf{x} - \mathbf{x_0})))^T (\mathbf{x} - \mathbf{x_0})\mathrm{d}t$$

when $m > 1$,we get the zeroth-order Taylor expansion with an integral remainder term:

$$f(\mathbf{x}) = f(\mathbf{x_0}) + \int_0^1 J_f(\mathbf{x_0} + t(\mathbf{x} - \mathbf{x_0}))(\mathbf{x} - \mathbf{x_0})\mathrm{d}t$$

in the expression:

$$J_f(t) = \begin{pmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} & \cdots & \frac{\partial f_1}{\partial t_n} \\ \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} & \cdots & \frac{\partial f_2}{\partial t_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial t_1} & \frac{\partial f_m}{\partial t_2} & \cdots & \frac{\partial f_m}{\partial t_n} \end{pmatrix}$$

□

Excercise 2. Please write down the definition of the p-norm for a n-dimensional real vector.

Solution of Excercise 2: The p-norm of a n-dimensional real vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is defined as:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

□

Excercise 3. Please write down the definition of the matrix norms induced by vector p-norms.

Solution of Excercise 3: The matrix norm induced by a vector p-norm is defined as:

$$\|A\|_p = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$$

□

Excercise 4. Let A be an $n \times n$ symmetric matrix. Proof that A is positive semidefinite if and only if all eigenvalues of A are nonnegative. Moreover, A is positive definite if and only if all eigenvalues of A are positive.

Proof of Excercise 4: Proof:

1. Positive Semidefinite Condition:

   (a) If A is positive semidefinite, then all eigenvalues of A are nonnegative.

   $\because$ A is positive semidefinite
   $\therefore$ $\mathbf{x}^T A\mathbf{x} \geq 0$ for any nonzero vector $\mathbf{x} \in \mathbb{R}^n$.
   Let λ be anyone of arbitary eigenvalue of A, and v (nonzero vector) be the corresponding eigenvector, that is:

   $$\begin{gathered} Av = \lambda v \\ \therefore v^T Av = \lambda v^T v = \lambda\|v\|^2 \geq 0 \\ \because \|v\|^2 > 0 \\ \therefore \lambda \geq 0 \end{gathered}$$

   (b) If all eigenvalues of A are nonnegative, then A is positive semidefinite.

   Since A is an $n \times n$ symmetric matrix.
   A can be diagonalized by an orthogonal matrix P:

   $$A = P\Lambda P^{-1}$$

   where Λ is a diagonal matrix containing the eigenvalues of A and P is an orthogonal matrix $P^{-1} = P^T$.
   $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A,and $\lambda_i \geq 0$ for $1 \leq i \leq n$
   For any nonzero vector $\mathbf{x} \in \mathbb{R}^n$, we have:

   $$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T P\Lambda P^{-1}\mathbf{x} = \mathbf{x}^T P\Lambda P^T\mathbf{x}$$

   we set $\mathbf{y} = P^T\mathbf{x}$.

   $$\mathbf{x}^T P\Lambda P^T\mathbf{x} = \mathbf{y}^T\Lambda\mathbf{y} = \sum_{i=1}^{n} \lambda_i y_i^2$$

   Since $\lambda_i \geq 0$ for $1 \leq i \leq n$

   $$\therefore \mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n} \lambda_i y_i^2 \geq 0$$

   Therefore, A is positive semidefinite.

2. Positive Definite Condition:
   If we replace ≥ 0 with > 0 in the above proof, we can get the proof of the positive definite condition.

$\square$

Excercise 5. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and upper bounded. Show that f must be a constant function.

Proof of Excercise 5: Proof:
Since f is upper bounded, there exists a constant M such that

$$f(\mathbf{x}) \leq M \ for \ all \ \mathbf{x} \in \mathbb{R}^n$$

.

Since f is convex ,$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$$

We assume f is not a constant function, then there exists $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $f(\mathbf{x}) \neq f(\mathbf{y})$.
To simplify the proof,we assume $f(\mathbf{x}) > f(\mathbf{y})$,we set $\mathbf{x} = \frac{\mathbf{x}-(1-\lambda)\mathbf{y}}{\lambda}$,so we get:

$$f(\mathbf{x}) \leq \lambda f(\frac{\mathbf{x}-(1-\lambda)\mathbf{y}}{\lambda}) + (1-\lambda) f(\mathbf{y})$$

$$\therefore f(\frac{\mathbf{x}-(1-\lambda)\mathbf{y}}{\lambda}) \geq \frac{f(\mathbf{x}) - f(\mathbf{y})}{\lambda} + f(\mathbf{y})$$

$$\because \lim_{\lambda \to 0^+} \frac{f(\mathbf{x}) - f(\mathbf{y})}{\lambda} + f(\mathbf{y}) = +\infty$$

$$\therefore f(\frac{\mathbf{x}-(1-\lambda)\mathbf{y}}{\lambda}) \geq +\infty > M$$

This is a contradiction to the assumption that f is upper bounded.
So, f must be a constant function.

$\square$