Coursework (4) for *Introductory Lectures on Optimization*

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Excercise 1. Prove the following theorem:
for any $\boldsymbol{x}_0 \in \text{dom } f$, all vectors $\boldsymbol{g} \in \partial f(\boldsymbol{x}_0)$ are supporting to the level set $\mathcal{L}_f(f(\boldsymbol{x}_0))$:

$$\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x}\rangle \geq 0, \quad \forall \boldsymbol{x} \in \mathcal{L}_f(f(\boldsymbol{x}_0)) \equiv \{\boldsymbol{x} \in \text{dom } f : f(\boldsymbol{x}) \leq f(\boldsymbol{x}_0)\}.$$

Proof of Excercise 1: The answer is as follows:

$$\because \boldsymbol{g} \in \partial f(\boldsymbol{x}_0)$$
$$\therefore \forall \boldsymbol{x} \in \text{dom } f, f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g},\ \boldsymbol{x} - \boldsymbol{x}_0\rangle$$
$$\because \boldsymbol{x} \in \mathcal{L}_f(f(\boldsymbol{x}_0)) \Rightarrow f(\boldsymbol{x}) \leq f(\boldsymbol{x}_0)$$

So we have:

$$f(\boldsymbol{x}_0) \geq f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g},\ \boldsymbol{x} - \boldsymbol{x}_0\rangle \tag{1}$$

Thus we have $\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x}\rangle \geq 0$.It's the result we want to prove.

$$\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x}\rangle \geq 0, \quad \forall \boldsymbol{x} \in \mathcal{L}_f(f(\boldsymbol{x}_0)) \equiv \{\boldsymbol{x} \in \text{dom } f : f(\boldsymbol{x}) \leq f(\boldsymbol{x}_0)\}.$$

□

Excercise 2. Prove the following theorem:
let $Q \subseteq \text{dom } f$ be a closed convex set, $\boldsymbol{x}_0 \in Q$ and

$$\boldsymbol{x}^* = \text{argmin}\{f(\boldsymbol{x}) | \boldsymbol{x} \in Q\}.$$

Then for any $g \in \partial f(\boldsymbol{x}_0)$ we have $\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x}^*\rangle \geq 0$.

Proof of Excercise 2: The answer is as follows:

$$\because \boldsymbol{x}^* = \text{argmin}\{f(\boldsymbol{x}) | \boldsymbol{x} \in Q\}, \boldsymbol{x}_0 \in Q$$
$$\therefore f(\boldsymbol{x}_0) \geq f(\boldsymbol{x}^*)$$
$$\because \boldsymbol{g} \in \partial f(\boldsymbol{x}_0) \Rightarrow f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g},\ \boldsymbol{x} - \boldsymbol{x}_0\rangle, \boldsymbol{x}^* \in Q$$
$$\therefore f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g},\ \boldsymbol{x}^* - \boldsymbol{x}_0\rangle$$

So we have:

$$f(\boldsymbol{x}_0) \geq f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g},\ \boldsymbol{x}^* - \boldsymbol{x}_0\rangle \tag{2}$$

Thus we have $\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x}^* \rangle \geq 0$.It's the result we want to prove.

$$\text{for any } g \in \partial f(\boldsymbol{x}_0) \text{ we have } \langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x}^* \rangle \geq 0$$

□

Excercise 3. Prove the following theorem:
let f be closed and convex. Assume that it is differentiable on its domain. Then $\partial f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}$ for any $\boldsymbol{x} \in \operatorname{int}(\operatorname{dom} f)$.

Proof of Excercise 3: The answer is as follows:

As the Lacture10 says $\forall \boldsymbol{p} \in \mathbb{R}^n$, we have:

$$f^{'}(\boldsymbol{x};\boldsymbol{p}) = \langle \nabla f(\boldsymbol{x}), \boldsymbol{p} \rangle = \lim_{t \to 0} \frac{f(\boldsymbol{x} + t\boldsymbol{p}) - f(\boldsymbol{x})}{t} \tag{3}$$

$f^{'}(\boldsymbol{x};\boldsymbol{p})$ is called the directional derivative of f at $\boldsymbol{x}$.

$\because f^{'}(\boldsymbol{x};\boldsymbol{p}) = \max\{\langle \boldsymbol{g}, \boldsymbol{p} \rangle \,|\boldsymbol{g} \in \partial f(\boldsymbol{x})\} \,\forall \boldsymbol{x} \in \operatorname{int}(\operatorname{dom} f)$

$$\langle \nabla f(\boldsymbol{x}), \boldsymbol{p} \rangle = f^{'}(\boldsymbol{x};\boldsymbol{p}) \geq \langle \boldsymbol{g}, \boldsymbol{p} \rangle$$
$$\langle \nabla f(\boldsymbol{x}) - \boldsymbol{g}, \boldsymbol{p} \rangle \geq 0, \forall \boldsymbol{p} \in \mathbb{R}^n$$

Let $\boldsymbol{p} = \boldsymbol{g} - \nabla f(\boldsymbol{x})$

$$\therefore \|\boldsymbol{g} - \nabla f(\boldsymbol{x})\|^2 \leq 0 \tag{4}$$
$$\therefore \|\boldsymbol{g} - \nabla f(\boldsymbol{x})\|^2 = 0 \Rightarrow \boldsymbol{g} = \nabla f(\boldsymbol{x})$$

So we have $\boldsymbol{g} = \nabla f(\boldsymbol{x})$.

When $\boldsymbol{p}$ is arbitrary, $\boldsymbol{g}$ is arbitrary,so we can get $\partial f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}$ for any $\boldsymbol{x} \in \operatorname{int}(\operatorname{dom} f)$.

□