Mid-term Exam for *Introductory Lectures on Optimization*

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Excercise 1. Proof that if $f_i(\boldsymbol{x})$, $i \in I$, are convex, then

$$g(\boldsymbol{x}) = \max_{i \in I} f_i(\boldsymbol{x})$$

is also convex.

Proof of Excercise 1: The answer is as follows:

$\because \forall i \in I,\ f_i(\boldsymbol{x})$ is convex

$$\therefore \forall i \in I,\ \alpha \in [0,1], \boldsymbol{x}, \boldsymbol{y} \in \mathrm{dom} f_i, f_i(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha f_i(\boldsymbol{x}) + (1-\alpha) f_i(\boldsymbol{y})$$

$$\because g(\boldsymbol{x}) = \max_{i \in I} f_i(\boldsymbol{x})$$

$$f_i(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha f_i(\boldsymbol{x}) + (1-\alpha) f_i(\boldsymbol{y}) \le \alpha g(\boldsymbol{x}) + (1-\alpha) g(\boldsymbol{y})$$

So, we have $\forall i \in I,\ \alpha \in [0,1]$

$$g(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) = \max_{i \in I} f_i(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha g(\boldsymbol{x}) + (1-\alpha) g(\boldsymbol{y})$$

So $g(\boldsymbol{x})$ is also convex.

□

Excercise 2. Proof that

1. if f is a convex function on $\mathbb{R}^n$ and $F(\cdot)$ is a convex and non-decreasing function on $\mathbb{R}$, then $g(\boldsymbol{x}) = F(f(\boldsymbol{x}))$ is convex.
2. If $f_i, i = 1, \ldots, m$ are convex functions on $\mathbb{R}^n$ and $F(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$ is convex and non-decreasing (component-wise) in each argument, then

$$g(\boldsymbol{x}) = F(f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x}))$$

   is convex.

Proof of Excercise 2: The answer is as follows:

1. Proof of 1
   $\because f(\boldsymbol{x})$ is convex and F is non-decreasing

$$\therefore \forall \alpha \in [0,1]\ , \boldsymbol{x}, \boldsymbol{y} \in \mathrm{dom} f, f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y})$$

$$\therefore g(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) = F\left(\left(f\left(\alpha \boldsymbol{x} + (1-\alpha)\right)\boldsymbol{y}\right)\right) \le F\left(\alpha f(\boldsymbol{x}) + (1-\alpha)(f(\boldsymbol{y}))\right)$$

$\because F(\cdot)$ is convex

$$\begin{aligned} \therefore g(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) &\le F\left(\alpha f(\boldsymbol{x}) + (1-\alpha)(f(\boldsymbol{y}))\right) && (1)\\ &\le \alpha F\left(f(\boldsymbol{x})\right) + (1-\alpha)F\left(f(\boldsymbol{y})\right) && (2)\\ &= \alpha g(\boldsymbol{x}) + (1-\alpha)g(\boldsymbol{y}) && (3) \end{aligned}$$

$$\therefore \forall \alpha \in [0,1], \boldsymbol{x}, \boldsymbol{y} \in \mathrm{dom} f,\ g(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha g(\boldsymbol{x}) + (1-\alpha)g(\boldsymbol{y})$$

$$\therefore g(\boldsymbol{x}) \text{is convex.}$$

2. Proof of 2

   $\because f(\boldsymbol{x})$ is convex and F is non-decreasing

$$\therefore \forall \alpha \in [0,1], \boldsymbol{x}, \boldsymbol{y} \in \mathrm{dom} f, f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha f(\boldsymbol{x}) + (1-\alpha)f(\boldsymbol{y})$$

$$\begin{aligned} \therefore g(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) &= F(f_1(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}), \cdots, f_m(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y})) && (4)\\ &\le F(\alpha f_1(\boldsymbol{x}) + (1-\alpha)(f_1(\boldsymbol{y})), \cdots, \alpha f_m(\boldsymbol{x}) + (1-\alpha)(f_m(\boldsymbol{y}))) && (5) \end{aligned}$$

$\because F(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$ is convex

$$\begin{aligned} \therefore g(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) &\le F(\alpha f_1(\boldsymbol{x}) + (1-\alpha)(f_1(\boldsymbol{y})), \cdots, \alpha f_m(\boldsymbol{x}) + (1-\alpha)(f_m(\boldsymbol{y}))) && (6)\\ &\le \alpha F(f_1(\boldsymbol{x}), \cdots, f_m(\boldsymbol{x})) + (1-\alpha)F(f_1(\boldsymbol{y}), \cdots, f_m(\boldsymbol{y})) && (7)\\ &= \alpha g(\boldsymbol{x}) + (1-\alpha)g(\boldsymbol{y}) && (8) \end{aligned}$$

$$\therefore \forall \alpha \in [0,1], \boldsymbol{x}, \boldsymbol{y} \in \mathrm{dom} f,\ g(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha g(\boldsymbol{x}) + (1-\alpha)g(\boldsymbol{y})$$

$$\therefore g(\boldsymbol{x}) \text{is convex.}$$

$\square$

Excercise 3. Proof that if $f(\boldsymbol{x}, \boldsymbol{y})$ is convex in $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n$ and Y is a convex set, then

$$g(\boldsymbol{x}) = \inf_{\boldsymbol{y} \in Y} f(\boldsymbol{x}, \boldsymbol{y})$$

is convex.

Proof of Excercise 3: The answer is as follows:

For $\boldsymbol{x} \in \mathbb{R}^n$,there exists a sequence $\{\boldsymbol{y}_n\}$ such that:

$$\inf_{\boldsymbol{y} \in Y} f(\boldsymbol{x}, \boldsymbol{y}) = \lim_{n \to \infty} f(\boldsymbol{x}, \boldsymbol{y}_n)$$

So we assume $\boldsymbol{x_1}, \boldsymbol{x_2} \in \mathbb{R}^n$, $\boldsymbol{y}_n^{(\boldsymbol{x_1})}, \boldsymbol{y}_n^{(\boldsymbol{x_2})}$:

$$\inf_{\boldsymbol{y} \in Y} f(\boldsymbol{x_1}, \boldsymbol{y}) = \lim_{n \to \infty} f(\boldsymbol{x_1}, \boldsymbol{y}_n^{(\boldsymbol{x_1})})$$

$$\inf_{\boldsymbol{y}\in Y} f(\boldsymbol{x_2}, \boldsymbol{y}) = \lim_{n\to\infty} f(\boldsymbol{x_2}, \boldsymbol{y}_n^{(\boldsymbol{x_2})})$$

now we $\forall\alpha \in [0, 1]$, we have:

$$\begin{aligned}
\alpha g(\boldsymbol{x_1}) + (1-\alpha)g(\boldsymbol{x_2}) &= \alpha \inf_{\boldsymbol{y}\in Y} f(\boldsymbol{x_1}, \boldsymbol{y}) + (1-\alpha) \inf_{\boldsymbol{y}\in Y} f(\boldsymbol{x_2}, \boldsymbol{y}) && (9)\\
&= \alpha \lim_{n\to\infty} f(\boldsymbol{x_1}, \boldsymbol{y_n^{(x_1)}}) + (1-\alpha) \lim_{n\to\infty} f(\boldsymbol{x_2}, \boldsymbol{y_n^{(x_2)}}) && (10)\\
&= \lim_{n\to\infty} (\alpha f(\boldsymbol{x_1}, \boldsymbol{y_n^{(x_1)}}) + (1-\alpha) f(\boldsymbol{x_2}, \boldsymbol{y_n^{(x_2)}})) && (11)
\end{aligned}$$

Since of $f(\boldsymbol{x}, \boldsymbol{y})$ is convex, we have:

$$\begin{aligned}
\alpha g(\boldsymbol{x_1}) + (1-\alpha)g(\boldsymbol{x_2}) &= \lim_{n\to\infty} (\alpha f(\boldsymbol{x_1}, \boldsymbol{y_n^{(x_1)}}) + (1-\alpha) f(\boldsymbol{x_2}, \boldsymbol{y_n^{(x_2)}})) && (12)\\
&\geq \lim_{n\to\infty} f(\alpha\boldsymbol{x_1} + (1-\alpha)\boldsymbol{x_2}, \alpha\boldsymbol{y_n^{(x_1)}} + (1-\alpha)\boldsymbol{y_n^{(x_2)}}) && (13)
\end{aligned}$$

$\because Y$ is a convex set, we have:

$$\begin{aligned}
\alpha g(\boldsymbol{x_1}) + (1-\alpha)g(\boldsymbol{x_2}) &\geq \lim_{n\to\infty} f(\alpha\boldsymbol{x_1} + (1-\alpha)\boldsymbol{x_2}, \alpha\boldsymbol{y_n^{(x_1)}} + (1-\alpha)\boldsymbol{y_n^{(x_2)}}) && (14)\\
&\geq \inf_{\boldsymbol{y}\in Y} f(\alpha\boldsymbol{x_1} + (1-\alpha)\boldsymbol{x_2}, \boldsymbol{y}) && (15)\\
&= g(\alpha\boldsymbol{x_1} + (1-\alpha)\boldsymbol{x_2}) && (16)
\end{aligned}$$

$$\therefore \forall\alpha \in [0, 1], \boldsymbol{x_1}, \boldsymbol{x_2} \in \mathbb{R}^n,\ \alpha g(\boldsymbol{x_1}) + (1-\alpha)g(\boldsymbol{x_2}) \geq g(\alpha\boldsymbol{x_1} + (1-\alpha)\boldsymbol{x_2})$$

$$\therefore g(\boldsymbol{x})\text{is convex.}$$

□

Excercise 4. Proof that the following univariate functions are in the set of $\mathcal{F}^1(\mathbb{R})$:

$$\begin{aligned}
&f(x) = e^x,\\
&f(x) = |x|^p,\ p > 1,\\
&f(x) = \frac{x^2}{1+|x|},\\
&f(x) = |x| - \ln(1 + |x|).
\end{aligned}$$

Proof of Excercise 4: The answer is as follows:

if univariate functions f are in the set of $\mathcal{F}^1(\mathbb{R})$ if and only if:

$\forall x, y \in \mathbb{R}$

$$\langle \nabla f(x) - \nabla f(y), x - x\rangle \geq 0 \tag{17}$$

That is:

$$\left(f^{'}(x) - f^{'}(y)\right)(x - y) \geq 0 \tag{18}$$

1. $f(x) = e^x$

$$f^{'}(x) = e^x$$

Because e^x is a increasing function, so we have:

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (e^x - e^y)(x-y) \geq 0 \tag{19}$$

So $f(x) = e^x$ is in the set of $\mathcal{F}^1(\mathbb{R})$

2. $f(x) = |x|^p, p > 1$

$$f'(x) = \begin{cases} p(x^{p-1}) & x > 0 \\ 0 & x = 0 \\ -p|x|^{p-1} & x < 0 \end{cases} \tag{20}$$

we can merge the above equation into:

$$f'(x) = \begin{cases} p(x^{p-1}) & x \geq 0 \\ -p|x|^{p-1} & x < 0 \end{cases} \tag{21}$$

Because $p > 1$, so x^p,x^{p-1} are a increasing function, so we have:

(a) when $x \geq 0, y \geq 0$:

$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = p(x^{p-1} - y^{p-1})(x-y) \geq 0$

(b) when $x \geq 0, y < 0$

$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = p(x^{p-1} + |y|^{p-1})(x-y) \geq 0$

(c) when $x < 0, y \geq 0$

$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = p(-|x|^{p-1} - y^{p-1})(x-y) \geq 0$

(d) when $x < 0, y < 0$

$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = p(-|x|^{p-1} + |y|^{p-1})(x-y) \geq 0$

So we have

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) \geq 0$$

So $f(x) = |x|^p, p > 1$ is in the set of $\mathcal{F}^1(\mathbb{R})$

3. $f(x) = \frac{x^2}{1+|x|}$

$$f^{'}(x) = \begin{cases} 1 - \frac{1}{(1+|x|)^2} & x > 0 \\ 0 & x = 0 \\ \frac{1}{(1+|x|)^2} - 1 & x < 0 \end{cases} \tag{22}$$

we can merge the above equation into:

$$f^{'}(x) = \begin{cases} 1 - \frac{1}{(1+|x|)^2} & x \geq 0 \\ \frac{1}{(1+|x|)^2} - 1 & x < 0 \end{cases} \tag{23}$$

(a) when $x \geq 0, y \geq 0$

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (\frac{1}{(1+|x|)^2} - \frac{1}{(1+|y|)^2})(x-y) \geq 0$$

(b) when $x \geq 0, y < 0$

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (2 - \frac{1}{(1+|x|)^2} - \frac{1}{(1+|y|)^2})(x-y)$$

since of $x > y$ and $\frac{1}{(1+|x|)^2} \leq 1$,so we get:

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (2 - \frac{1}{(1+|x|)^2} - \frac{1}{(1+|y|)^2})(x-y) \geq 0$$

(c) when $y \geq 0, x < 0$

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (\frac{1}{(1+|x|)^2} + \frac{1}{(1+|y|)^2} - 2)(x-y)$$

since of $y > x$ and $\frac{1}{(1+|x|)^2} \leq 1$,so we get:

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (\frac{1}{(1+|x|)^2} + \frac{1}{(1+|y|)^2} - 2)(x-y) \geq 0$$

(d) when $x < 0, y < 0$

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (\frac{1}{(1+|y|)^2} - \frac{1}{(1+|x|)^2})(x-y) \geq 0$$

So we have

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) \geq 0$$

So $f(x) = \frac{x^2}{1+|x|}$ is in the set of $\mathcal{F}^1(\mathbb{R})$

4. $f(x) = |x| - ln(1+|x|)$

$$f^{'}(x) = \begin{cases} 1 - \frac{1}{1+|x|} & x > 0 \\ 0 & x = 0 \\ \frac{1}{1+|x|} - 1 & x < 0 \end{cases} \tag{24}$$

we can merge the above equation into:

$$f^{'}(x) = \begin{cases} 1 - \frac{1}{1+|x|} & x \geq 0 \\ \frac{1}{1+|x|} - 1 & x < 0 \end{cases} \tag{25}$$

(a) when $x \geq 0, y \geq 0$

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (\frac{1}{1+|x|} - \frac{1}{1+|y|})(x-y) \geq 0$$

(b) when $x \geq 0, y < 0$

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (2 - \frac{1}{1+|y|} - \frac{1}{1+|x|})(x-y)$$

since of $x > y$ and $f(x) = \frac{1}{1+|x|} \leq 1$,so we get:

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (2 - \frac{1}{1+|y|} - \frac{1}{1+|x|})(x-y) \geq 0$$

(c) when $x < 0, y \geq 0$:

$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (\frac{1}{1+|x|} - \frac{1}{1+|y|})(x-y) \geq 0$

since of $y > x$ and $f(x) = \frac{1}{1+|x|} \leq 1$,so we get:

$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (\frac{1}{1+|x|} - \frac{1}{1+|y|})(x-y) \geq 0$

(d) when $x < 0, y < 0$:

$\left(f^{'}(x) - f^{'}(y)\right)(x-y) = (\frac{1}{1+|y|} - \frac{1}{1+|x|})(x-y) \geq 0$

So we have

$$\left(f^{'}(x) - f^{'}(y)\right)(x-y) \geq 0$$

So $f(x) = |x| - ln(1+|x|)$ is in the set of $\mathcal{F}^1(\mathbb{R})$

□

Excercise 5. For $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$ and function $\phi(\boldsymbol{y}) = f(\boldsymbol{y}) - \langle \nabla f(\boldsymbol{x}_0),\ \boldsymbol{y} \rangle$, prove that $\phi \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$, and its optimal point is $\boldsymbol{y}^* = \boldsymbol{x}_0$.

Proof of Excercise 5: The answer is as follows:

To prove that $\phi \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$, we need to prove that $\phi(\boldsymbol{y})$ is convex and $\phi(y) \in C_L^{1,1}(\mathbb{R}^n)$

According to the definition of $\phi(\boldsymbol{y}) = f(\boldsymbol{y}) - \langle \nabla f(\boldsymbol{x}_0),\ \boldsymbol{y} \rangle$, we have:

$$\nabla \phi(\boldsymbol{y}) = \nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x_0}) \tag{26}$$

1. $\phi(y)$ is convex.

   $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$, since of its convexity, we have:

   $$\langle \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \geq 0$$

   So we can get:

   $$\langle \nabla \phi(\boldsymbol{x}) - \nabla \phi(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle = \langle \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \geq 0 \tag{27}$$

   So $\phi(x)$ is convex.

2. $\phi(y) \in C_L^{1,1}(\mathbb{R}^n)$

   firstly, $\phi(y)$ is continuously differentiable:

   $$\because f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$$

   $$\therefore f(\boldsymbol{x}) \text{ is continuously differentiable}$$

   $$\therefore \nabla f(\boldsymbol{x})\text{exists}$$

   $$\therefore \nabla \phi(\boldsymbol{y}) = \nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x_0}) \text{ is continuously differentiable}$$

   $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$, since of its Lipschitz continuity, we have:

   secondly, $\phi(y)$ satisfies the Lipschitz continuous with constant L.

   $$\because f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$$

$$\therefore |\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|^2$$
$$\therefore |\nabla \phi(\boldsymbol{x}) - \nabla \phi(\boldsymbol{y})| = |\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|^2$$
$$\therefore \phi(y) \in C_L^{1,1}(\mathbb{R}^n)$$

So $\phi(y) \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$

We can easily find that $\nabla \phi(\boldsymbol{x_0}) = \nabla f(\boldsymbol{x_0}) - \nabla f(\boldsymbol{x_0}) = 0$.

Since of its convexity, $y^* = \boldsymbol{x_0}$ is optimal point.

□

Excercise 6. Proof that, for $f : \mathbb{R}^n \to \mathbb{R}$ and α from $[0, 1]$, if

$$\begin{aligned}\alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) \geq\ & f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \\ & + \frac{\alpha(1-\alpha)}{2L} \left\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\right\|^2,\end{aligned}$$

then $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$.

Proof of Excercise 6: The answer is as follows:

To prove that $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$, we need to prove that f is convex and $f \in C_L^{1,1}(\mathbb{R}^n)$

1. f is convex

$$\begin{aligned}\alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) &\geq f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) + \frac{\alpha(1-\alpha)}{2L} \left\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\right\|^2 \\ &\geq f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y})\end{aligned}$$

   So f is convex.

2. $f \in C_L^{1,1}(\mathbb{R}^n)$

$$\alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) \geq f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) + \frac{\alpha(1-\alpha)}{2L} \left\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\right\|^2$$
$$f(\boldsymbol{y}) \geq \frac{f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) - \alpha f(\boldsymbol{x})}{1-\alpha} + \frac{\alpha}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2 \tag{28}$$

$$\begin{aligned}\therefore f(\boldsymbol{y}) &= \lim_{\alpha \to 1} f(\boldsymbol{y}) && (29)\\ &\geq \lim_{\alpha \to 1} \left(\frac{f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) - \alpha f(\boldsymbol{x})}{1-\alpha} + \frac{\alpha}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2 \right) && (30)\\ &\geq \lim_{\alpha \to 1} \left(\frac{f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) - f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{x})}{1-\alpha} \right) + \frac{1}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2 && (31)\\ &\geq f(\boldsymbol{x}) + \lim_{\alpha \to 1} \left(\frac{f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) - f(\boldsymbol{x})}{1-\alpha} \right) + \frac{1}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2 && (32)\\ &\geq f(\boldsymbol{x}) + \lim_{\alpha \to 1} \left(\frac{\langle \boldsymbol{x} - \boldsymbol{y},\ \nabla f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \rangle}{-1} \right) + \frac{1}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2 && (33)\\ &\geq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle + \frac{1}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2 && (34)\end{aligned}$$

swap $\boldsymbol{x}, \boldsymbol{y}$ we can get:

$$f(\boldsymbol{x}) \geq f(\boldsymbol{y}) + \langle \nabla f(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle + \frac{1}{2L} \|\nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x})\|^2 \tag{35}$$

And we plus above two equations, we can get:

$$\begin{aligned} \frac{1}{L} \|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2 &\leq \langle \nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle && (36) \\ &\leq \|\nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x})\| \|\boldsymbol{y} - \boldsymbol{x}\| && (37) \end{aligned}$$

So we get:

$$|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})| \leq L \|\boldsymbol{y} - \boldsymbol{x}\|$$

So, $f \in C_L^{1,1}(\mathbb{R}^n)$

So, $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$

□

Excercise 7. Proof that, for $f : \mathbb{R}^n \to \mathbb{R}$ and α from $[0, 1]$, if

$$\begin{aligned} 0 &\leq \alpha f(\boldsymbol{x}) + (1 - \alpha) f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1 - \alpha) \boldsymbol{y}) \\ &\leq \alpha (1 - \alpha) \frac{L}{2} \|\boldsymbol{x} - \boldsymbol{y}\|^2, \end{aligned}$$

then $f \in \mathcal{F}_L^{\mathbf{1},\mathbf{1}}(\mathbb{R}^n)$.

Proof of Excercise 7: The answer is as follows:

We make equivalent transformation for the inequality given above:

$$f(\boldsymbol{y}) \leq \frac{f(\alpha \boldsymbol{x} + (1 - \alpha) \boldsymbol{y}) - \alpha f(\boldsymbol{x})}{1 - \alpha} + \frac{\alpha L}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 \tag{38}$$

$$\begin{aligned} f(\boldsymbol{y}) &= \lim_{\alpha \to 1} f(\boldsymbol{y}) && (39) \\ &\leq \lim_{\alpha \to 1} \left(\frac{f(\alpha \boldsymbol{x} + (1 - \alpha) \boldsymbol{y}) - \alpha f(\boldsymbol{x})}{1 - \alpha} + \frac{\alpha L}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 \right) && (40) \\ &\leq \lim_{\alpha \to 1} \left(\frac{f(\alpha \boldsymbol{x} + (1 - \alpha) \boldsymbol{y}) - f(\boldsymbol{x}) + (1 - \alpha) f(\boldsymbol{x})}{1 - \alpha} \right) + \frac{L}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 && (41) \\ &\leq f(\boldsymbol{x}) + \lim_{\alpha \to 1} \left(\frac{\langle \boldsymbol{x} - \boldsymbol{y},\ \nabla f(\alpha \boldsymbol{x} + (1 - \alpha) \boldsymbol{y}) \rangle}{-1} \right) + \frac{L}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 && (42) \\ &\leq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle + \frac{L}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 && (43) \end{aligned}$$

Simliarly,we can get:

$$
\begin{aligned}
f(\boldsymbol{y}) &= \lim_{\alpha\to 1} f(\boldsymbol{y}) && (44)\\
&\geq \lim_{\alpha\to 1} \frac{f(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y})-\alpha f(\boldsymbol{x})}{1-\alpha} && (45)\\
&\geq \lim_{\alpha\to 1} \left(\frac{f(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y})-f(\boldsymbol{x})+(1-\alpha)f(\boldsymbol{x})}{1-\alpha}\right) && (46)\\
&\geq f(\boldsymbol{x}) + \lim_{\alpha\to 1} \left(\frac{\langle \boldsymbol{y}-\boldsymbol{x},\ \nabla f(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y})\rangle}{-1}\right) && (47)\\
&\geq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{y}), \boldsymbol{x}-\boldsymbol{y}\rangle && (48)
\end{aligned}
$$

That is:

$$
f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{y}), \boldsymbol{x}-\boldsymbol{y}\rangle \leq f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x}\rangle + \frac{L}{2}\|\boldsymbol{y}-\boldsymbol{x}\|^2 \tag{49}
$$

$$
0 \leq f(\boldsymbol{y}) - f(\boldsymbol{x}) - \langle \nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x}\rangle \leq \frac{L}{2}\|\boldsymbol{y}-\boldsymbol{x}\|^2 \tag{50}
$$

Acctually, it follows from the definition of convex functions and Lemma (1.2.3) of Nesterov[2003].It is the equivalent of $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$.

So $f \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$.

$\square$