第六章 积分变换法

$$f(x) \xrightarrow{\text{transform}} G(\omega) = \int \underline{k(x,\omega)}\, f(x)\, dx$$

像函数　　核函数　　原函数

§6.1 Fourier变换

1. 以2l为周期的 Fourier 级数展开

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{where} \begin{cases} \cos nx = \dfrac{e^{inx} + e^{-inx}}{2} \\ \sin nx = \dfrac{e^{inx} - e^{-inx}}{2i} \end{cases}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{inx}, \quad C_n = \frac{a_n - ib_n}{2}, n \in \mathbb{Z}$$

$$= \frac{1}{2l} \int_{-l}^{l} f(\xi) e^{-i\omega_n x} d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi \right] e^{i\omega x} d\omega$$

令 $G(\omega) = \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$　正变换

则 $f(x) = \frac{1}{2\pi} \int G(\omega) e^{i\omega x} d\omega$　逆变换

2. 傅立叶变换的条件：

①在任一有限区间上满足 Dirichlet Conditions（充分不必要条件）

②在无界区间上绝对可积 $\int_{-\infty}^{\infty} |f(x)| dx$ 有界

3. 三维 Fourier 变换

对 $f(\vec{r}) = f(x,y,z) \xrightarrow{F} G(\vec{\omega}) = G(\omega_x, \omega_y, \omega_z)$,

$\vec{r} = (x,y,z)$　$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$　$d^3\vec{r} = dx\,dy\,dz$　$d^3\vec{\omega} = d\omega_x\, d\omega_y\, d\omega_z$

正变换：$\underset{\text{像函数}}{G(\vec{\omega})} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underset{\text{原函数}}{f(\vec{r})} \cdot \underset{\text{核函数}}{e^{-i\vec{\omega}\vec{r}}}\, \underset{dxdydz}{d^3\vec{r}}$

逆变换：$f(\vec{r}) = F^{-1}[G(\vec{\omega})] = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} G(\vec{\omega}) \cdot e^{i\vec{\omega}\vec{r}}\, dx\,dy\,dz$

4. Fourier 变换的性质

①线性性质：若 $f_1(x) \xrightarrow{F} F_1(\omega)$，$f_2(x) \xrightarrow{F} F_2(\omega)$

则对于 $\forall$ 常数 C_1, C_2，$C_1 f_1(x) + C_2 f_2(x) \xrightarrow{F} C_1 F_1(\omega) + C_2 F_2(\omega)$

②延迟性质：若 $f(x) \xrightarrow{F} F(\omega)$，则 $f(x) \cdot e^{i\omega_0 x} \xrightarrow{F} F(\omega - \omega_0)$

其中 ω_0 是实常数

③位移性质：若 $f(x) \xrightarrow{F} F(w)$

则 $f(x-x_0) \xrightarrow{F} e^{-iwx_0} F(w)$

④相似性质：若 $f(x) \xrightarrow{F} F(w)$

（放大性质） 则 $f(ax) \xrightarrow{F} \frac{1}{|a|} F(\frac{w}{a})$

⑤导数性质：若 $f(x) \xrightarrow{F} F(w)$

则 $f'(x) \xrightarrow{F} iwF(w)$，$f''(x) \xrightarrow{F} (iw)^2 F(w)$，

$\cdots\ f^{(n)}(x) \xrightarrow{F} (iw)^n F(w)$

⑥积分性质：若 $f(x) \xrightarrow{F} F(w)$，

则 $\int_{x_0}^{x} f(t)\,dt \xrightarrow{F} \frac{1}{iw} F(w)$

即 $g(x) = \int_{x_0}^{x} f(\xi)\,d\xi \xrightarrow{F} G(w) = \frac{1}{iw} F(w)$

$g'(x) = f(x) \xrightarrow{F} iwG(w) = F(w)$

⑦卷积性质（广义）

卷积定理：若 $f_1(x) \xrightarrow{F} F_1(w)$，$f_2(x) \xrightarrow{F} F_2(w)$，

则 $f_1(x) * f_2(x) \xrightarrow{F} F_1(w) F_2(w)$

且 $\frac{1}{2\pi} f_1(x) f_2(x) \xrightarrow{F} F_1(w) * F_2(w)$ 逆定理

⑧ $\int_{-\infty}^{\infty} |f^2(x)|\,dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F^2(w)|\,dw$

§6.2 Laplace变换

1. Laplace Transform：

对 $f(t)$ $t \in [0, +\infty)$ $(f(t) = 0,\ t<0)$，

有 $L(f(t)) = F(p) = \int_0^\infty f(t)\, e^{-pt}\,dt$

$f(t) = L^{-1}[F(p)] = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} F(p)\, e^{pt}\,dp$

常见结论：

$L[e^{at}] = \frac{1}{p-a}$，$L[1] = \frac{1}{p}$，$L[C] = \frac{C}{p}$ Const.

$L[e^{iwt}] = \frac{1}{p-iw}$，$L[e^{-iwt}] = \frac{1}{p+iw}$，

$L[\cos wt] = \frac{p}{p^2+w^2}$，$L[\sin wt] = \frac{w}{p^2+w^2}$

$L[t^n] = \frac{n!}{p^{n+1}}$ $(n \in \mathbb{Z})$

2. 常见性质

① 若 $f_1(t) \xrightarrow{L} F_1(p)$, $f_2(t) \xrightarrow{L} F_2(p)$

则 $L[C_1 f_1(t) + C_2 f_2(t)] = C_1 F_1(p) + C_2 F_2(p)$

② 若 $f(t) \xrightarrow{L} F(p)$, 则 $e^{p_0 t} f(t) \xrightarrow{L} F(p-p_0)$ 延迟性质

③ 若 $f(t) \xrightarrow{L} F(p)$, 则 $f(t-t_0) \xrightarrow{L} e^{-pt_0} F(p)$ 位移性质

④ 若 $f(t) \xrightarrow{L} F(p)$, 则 $f(at) \xrightarrow{L} \frac{1}{a} F(\frac{p}{a})$ 放大性质(相似性质)

⑤ 若 $f(t) \xrightarrow{L} F(p)$, 则 $f'(t) \xrightarrow{L} pF(p) - f(0)$

导数性质 $f''(t) \xrightarrow{L} p^2 F(p) - pf(0) - f'(0)$

$f^{(n)}(t) \xrightarrow{L} p^n F(p) - p^{n-1} f(0) - \cdots - p^0 f^{(n-1)}(0)$

⑥ $f_1(t) f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$ 其中 $f(t) = \begin{cases} 0 & t<0 \\ f(t) & t \geq 0 \end{cases}$

若 $f_1(t) \xrightarrow{L} F_1(p)$, $f_2(t) \xrightarrow{L} F_2(p)$, 卷积定理

则 $f_1(t) * f_2(t) \xrightarrow{L} F_1(p) F_2(p)$

逆定理 若 $f_1(t) \xrightarrow{L} F_1(p)$, $f_2(t) \xrightarrow{L} F_2(p)$,

则 $f_1 f_2 \xrightarrow{L} \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} F_1(q) F_2(p-q) dq$

⑦ 若 $f(t) \xrightarrow{L} F(p)$, 则 $\int_0^t f(\xi) d\xi \xrightarrow{L} \frac{1}{p} F(p)$ 积分性质

⑧ 若 $f(t) \xrightarrow{L} F(p)$, 则 $t^n f(t) \xrightarrow{L} (-1)^n \frac{d^n F(p)}{dp^n}$

⑨ 若 $f(t) \xrightarrow{L} F(p)$, 则 $\frac{1}{t} f(t) \xrightarrow{L} \int_p^\infty F(q) dq$

第七章
格林函数法

§7.1
Green函数

1. Green函数

影响函数　传播子函数　相对于一个点源的响应

$G(M,M_0) \sim M_0(x_0,y_0,z_0) \longrightarrow M(x,y,z) \sim G(x,y,z;x_0,y_0,z_0)$

2. δ函数

$\delta(x) \xrightarrow{\text{推广}} \delta(x,y),\ \delta(x,y,z)$

$$\begin{cases} \delta(x) = \begin{cases} \infty & x=0 \\ 0 & x\neq 0 \end{cases} \\ \int_{-\infty}^{\infty} \delta(x)\,dx = 1 \end{cases} \qquad \delta(x)=\delta(-x) \text{ —— 偶函数}$$

① 积分性质：$\int_{x_0-a}^{x_0+a} f(x)\,\delta(x-x_0)\,dx = \int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx$

$= f(x_0)$

$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$

② 导数性质：　设 $\delta'(x) = \frac{d\delta(x)}{dx}$，

则有 $\int_{-\infty}^{\infty} f(x)\,\delta'(x-x_0)\,dx = -f'(x_0)$

③ 多阶导：$\int_{-\infty}^{\infty} f(x)\,\delta^{(n)}(x-x_0)\,dx = (-1)^n f^{(n)}(x_0) \qquad n=0,1,2,3,\cdots$

④ 级数：$\delta[\varphi(x)] = \sum_{i=1}^{n} \frac{1}{|\varphi'(x_i)|}\,\delta(x-x_i)$

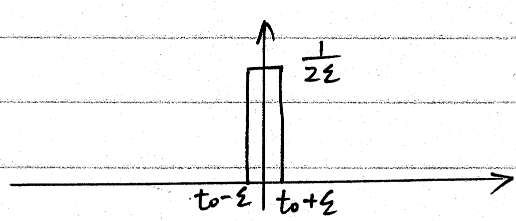
条件：$\varphi(x)$ 可导，$\varphi(x_i)=0$ 且 x_i 是 $\varphi(x)$ 的单根，$i=1,2,\cdots,n$

⑤ $\delta(x^2-a^2) = \frac{1}{2|a|}[\delta(x+a)+\delta(x-a)]$　实 $a\neq 0$

3. δ函数的若干等价表达式

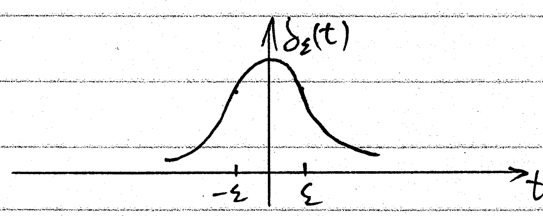
① $\delta_\varepsilon(t-t_0) = \begin{cases} \frac{1}{2\varepsilon} & |t-t_0|<\varepsilon \\ 0 & \text{其它} \end{cases}$

$\delta(t-t_0) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t-t_0)$



② $\delta_\varepsilon(t) = \frac{\varepsilon}{\pi(\varepsilon^2+t^2)}$

$\delta(t) = \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\pi(\varepsilon^2+t^2)}$



③ $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\omega t} \qquad \delta(t) \xrightarrow{F} 1$

$\delta(t) = \lim_{a\to\infty} \delta_a(t) = \lim_{a\to\infty} \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega t}\,d\omega$

④ $\delta(t) = \lim_{a\to\infty}\left(\frac{\sin at}{\pi t}\right)$

⑤ $\frac{dH(t)}{dt} = \delta(t)$

⑥ $F[H(t)] = \frac{1}{iw} + \pi\delta(w)$

§7.2
Green function
Method

1. 为了解 Poisson 方程 $\begin{cases} \nabla^2 u = -h(M) \\ [\alpha\frac{\partial u}{\partial n} + \beta u]_\sigma = g(M) \end{cases}$ $u = u(M) = u(x,y,z)$ 场点, $M=(x,y,z)$,

引入 Green func, 使之 satisfies $\begin{cases} \nabla^2 G(M,M_0) = -\delta(M,M_0) = -\delta^3(\vec{r}-\vec{r_0}) \\ \qquad = -\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \\ [\alpha\frac{\partial G}{\partial n} + \beta G]_\sigma = 0 \end{cases}$

①格林第一公式

当 u, v 在 τ 内有连续二阶导数, 在其边界 σ 有连续一阶导数,

则有 $\oiint_\sigma u\nabla v\cdot d\vec{\sigma} = \iiint_\tau \nabla(u\nabla v)d\tau$

$= \iiint_\tau u\Delta v\, dt + \iiint_\tau \nabla u\nabla v\, d\tau$

②格林第二公式

$\iiint_\tau (u\Delta v - v\Delta u)d\tau = \oiint_\sigma (u\nabla v - v\nabla u)d\vec{\sigma}$

$\iiint_{\tau-\tau_0} (u\Delta G - G\Delta u)\,d\tau = \oiint_{\sigma+\sigma_0}(u\nabla G - G\nabla u)\,d\vec{\sigma}$

其中 τ_0 包围 M_0 的无限小球形区域

$\oiint_{\sigma+\sigma_0}\left[G(M,M_0)\frac{\partial u}{\partial n} - u(M)\frac{\partial G(M,M_0)}{\partial n}\right]d\sigma = \iiint_{\tau-\tau_0}[u(M)\delta(M-M_0) - G(M,M_0)h(M)]d\tau$

$= -\iiint_{\tau-\tau_0} G(M,M_0)h(M)d\tau$

$u(M_0) = \iiint_\tau G(M,M_0)h(M)d\tau + \iint_\sigma G(M,M_0)\frac{\partial u}{\partial n}d\sigma - \iint_\sigma u(M)\frac{\partial}{\partial n}G(M,M_0)d\sigma$

③格林互易定理 $\quad G(M,M_0) = G(M_0,M)$

$\to u(M) = \iiint_\tau G(M,M_0)h(M_0)d\tau_0 + \iint_\sigma G(M,M_0)\frac{\partial u}{\partial n_0}d\sigma_0 - \iint_\sigma u(M_0)\frac{\partial}{\partial n_0}G(M,M_0)d\sigma_0$

where $M = (x,y,z) \in \tau$

$M_0 = (x_0, y_0, z_0) \in \tau \in \sigma$

2. 无界区域的 Green function

① $\Delta G = -\delta(x-x_0, y-y_0, z-z_0)$

选定 (x_0, y_0, z_0) 为球极坐标原点，球半径 $r=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$

$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dG}{dr}) = -\delta(r)$

当 $r\neq 0$, $\frac{d}{dr}(r^2\frac{dG}{dr}) = 0$, we have $\begin{cases} r^2\frac{dG}{dr} = C_1 \to \frac{dG(r)}{dr} = \frac{C_1}{r^2} \\ G(r) = \frac{-C_1}{r} + C_0，由 G(\infty)=0 \to C_0=0 \\ G(r) = -C_1/r \end{cases}$

当 $r=0$，利用散度定理，$\Delta = \nabla\cdot\nabla$, we have:

$$\iiint_\tau \nabla\cdot(\nabla u)d\tau = \oiint_s \nabla u\, d\vec{s},$$

$$\text{then } \iiint \nabla(\nabla G)d\tau = \lim_{\varepsilon\to 0}\oiint \nabla G\, d\vec{s} = -1$$

$$\nabla G = \frac{dG}{dr}, \quad G = -\frac{1}{4\pi} \to G(r) = \frac{1}{4\pi r} = \frac{\varepsilon_0}{4\pi\varepsilon_0 r}, \quad \rho = \varepsilon_0\delta(x-x_0, y-y_0, z-z_0)$$

② 二维无界区域的格林函数 —— 基本解

$\Delta_2 G(x,y) = \nabla^2 G = -\delta(x-x_0, y-y_0)$

取 (x_0, y_0) 为极坐标原点，$r=\sqrt{(x-x_0)^2+(y-y_0)^2}$

$$\frac{1}{r}\frac{d}{dG}(r\frac{dG}{dr}) = -\delta(r)$$

当 $r\neq 0$ 时，$G(r) = C_1 \ln r$

对于包围 (x_0, y_0) 的无穷小圆区域，曲面积分，

according to 散度定理 $\iint_s \nabla\cdot\nabla G\, d\vec{s} = \oint_c \nabla G\, d\vec{l}$,

$\nabla G = \frac{d}{dr}G(r)$ 得 $C_1 = -\frac{1}{2\pi}$, thus $G(r) = -\frac{1}{2\pi}\ln r = \frac{1}{2\pi}\ln\frac{1}{r}$

§7.3 特殊边界的 Green func. 的电像法

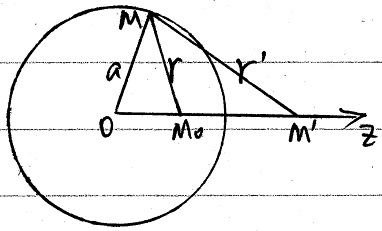
狄氏边界条件

$\begin{cases} \Delta G = -\delta(x-x_0, y-y_0, z-z_0) \\ G|_\sigma = 0 \end{cases}$ $\quad (x,y,z)\in\tau \begin{cases} \Delta g = 0 \\ g|_\sigma = -F|_\sigma = -\frac{1}{4\pi r}\Big|_{(x,y,z)\in} \end{cases}$

令 $G = F(x,y,z) + g(x,y,z)$,

we have $\begin{cases} \Delta F = -\delta(x-x_0, y-y_0, z-z_0) \\ 无界区域 \end{cases} \Rightarrow F = \frac{1}{4\pi r}$

1. 球面边界的电像法



M'关于球面与M_0共轭，要使 $\triangle OMM_0 \sim \triangle OM'M$,

we should have $\frac{1}{r} = \frac{a/\rho_0}{r'}$ $r = MM_0,\ r' = M'M,\ \rho_0 = OM$

i.e. $\frac{\varepsilon_0}{4\pi\varepsilon_0 r} = \frac{a/\rho_0 \cdot \varepsilon_0}{4\pi\varepsilon_0 r'}$

then $q' = \frac{-a}{\rho_0} q_0 \iff \begin{cases} q_0 = \varepsilon_0 \\ q' = \frac{-a}{\rho_0}\varepsilon_0 \end{cases}$

$\therefore a^2 = \rho'\rho_0,\ \rho' = a^2/\rho_0$

$$G(M) = F + g = \frac{\varepsilon_0}{4\pi\varepsilon_0 r} + \frac{q'}{4\pi\varepsilon_0 r'} = \frac{1}{4\pi r} - \frac{a/\rho_0}{4\pi r'}$$

2. 二维 圆形边界 Poisson Eqn. 狄氏格林函数

$$\begin{cases} \Delta_2 G(M) = -\delta(M - M_0) = -\delta(x - x_0, y - y_0) & (M(x,y)\text{在圆内}) \\ G|_l = 0 & (\lambda = \varepsilon_0) \end{cases}$$

$$G = F + g \quad \begin{cases} \Delta g = 0 \\ g|_l = -F|_l \end{cases} \quad \begin{cases} \Delta F = -\delta(x - x_0, y - y_0) \\ \text{无界区间} \end{cases}$$

$$F = -\frac{1}{2\pi}\ln r = \frac{\varepsilon_0}{2\pi\varepsilon_0}\ln\frac{1}{r} + C$$

$$G(M)\Big|_{M\text{在圆内}} = \frac{1}{2\pi}\ln\frac{1}{r} - \frac{1}{2\pi}\ln\frac{1}{r'} - \frac{1}{2\pi}\ln\frac{a}{\rho_0}$$