

递归部分

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- **Theorem** Suppose that we have a recurrence of the form

$$T(n) = aT(n/2) + n,$$

where a is a positive integer and $T(1)$ is nonnegative. Then we have the following **big Θ** bounds on the solution:

1. If $a < 2$, then $T(n) = \Theta(n)$.
2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$

Proof

We already proved Case 1 when $a = 1$ in Example 3.

(will not prove it for $1 < a < 2$)

We already proved Case 2 in Example 1.

We will now prove Case 3.



- $T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.

Iterating as in Example 5 gives

$$T(n) = a^i T\left(\frac{n}{2^i}\right) + \left(\frac{a^{i-1}}{2^{i-1}} + \frac{a^{i-2}}{2^{i-2}} + \cdots + \frac{a}{2} + 1\right) n$$

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

Work at
“bottom”

Iterated
Work

- n times the largest term in the geometric series is

$$n \left(\frac{a}{2}\right)^{\log_2 n - 1} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{n} = \frac{2}{a} \cdot a^{\log_2 n}$$

Notice that

$$a^{\log_2 n} = (2^{\log_2 a})^{\log_2 n} = (2^{\log_2 n})^{\log_2 a} = n^{\log_2 a}$$

So the total work is

$$a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

$$\Theta(n^{\log_2 a})$$

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