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On the calculation of a membership function for the solution of a fuzzy linear optimization problem

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Abstract

In the present paper the fuzzy linear optimization problem (with fuzzy coefficients in the objective function) is considered. Recent concepts of fuzzy solution to the fuzzy optimization problem based on the level-cut and the set of Pareto optimal solutions of a multiobjective optimization problem are applied. Chanas and Kuchta suggested one approach to determine the membership function values of fuzzy optimal solutions of the fuzzy optimization problem, which is based on calculating the sum of lengths of certain intervals. The purpose of this paper is to determine a method for realizing this idea. We derive explicit formulas for the bounds of these intervals in the case of triangular fuzzy numbers and show that only one interval needs to be considered.

Keywords: Fuzzy optimization; Multicriteria optimization; Pareto optimal solution; Membership function

1. Introduction

In many situations optimization problems with unknown or only approximately known data need to be solved e.g. the fuzzy multicommodity flow problem (compute optimal flows in a traffic network with fuzzy costs for passing streets, see e.g. [1]) or problems of optimal planning [2]. It seems reasonable to approach these problems within the framework of fuzzy set theory because continuous fuzzy numbers are particularly suited for describing such ambiguities.

Let us formulate a fuzzy optimization problem where the objective function has fuzzy values and the constraint function is a crisp one, i.e.:

$$\tilde{f}(x) \to \min,$$

$$g(x) \le 0. \tag{1}$$

Here $g = (g_1, \ldots, g_k) : \mathbb{R}^n \to \mathbb{R}^k$ is a crisp function and $\tilde{f}$ maps $\mathbb{R}^n$ to the space of fuzzy numbers.

The formulation of fuzzy optimization problems with no objective but fuzzy constraints can be found in [3,4].

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The linear case of (1) is represented by the following linear programming problem with fuzzy coefficients in the objective function:

$$\begin{aligned} &\tilde{c}^\top x \to \min, \\ &Ax \le b, \\ &x \ge 0. \end{aligned} \tag{2}$$

Many authors try to find a *single* best solution of the fuzzy optimization problem (see e.g. [5,6]). Those approaches are based on the extension principle suggested by Bellman and Zadeh [7]. We suggest to reflect the uncertainty in fuzzy optimization problems through (the existence of) a set of optimal solutions. Under the assumption that this set consists of more than one element, the decision-maker can improve a choice relying on some criteria that are not a priory defined in the optimization problem.

When the fuzzy optimization problem is solved it is natural to consider its solutions as fuzzy. Hence, a criterion for comparing the elements of the fuzzy set of optimal solutions of the fuzzy optimization problem is required. One possible choice is to compare values of the corresponding membership functions. One approach to calculate such a membership function is suggested by Chanas and Kuchta [8,9]. Knowledge of the membership function values of the elements of the set of fuzzy optimal solutions enables the decision-maker to make an *educated* choice between these solutions. Moreover, using our approach, a decision-maker can see a correlation among solutions and quantitatively measure the advantage of his choice over other solutions. The main aim of the present paper is to find the best realization of this idea based on modern solution algorithms [6,8–13].

To describe the membership function of the fuzzy linear objective function, α-cuts are used. It is assumed that its left- and right-hand side values are given by the functions $c_L(\alpha)$ and $c_R(\alpha)$ for $\alpha \in [0, 1]$. Following [14] we assume that $c_L(\alpha)$ is a bounded increasing function and $c_R(\alpha)$ is a bounded decreasing function of α. Moreover, it is obvious that $c_L(\alpha) \le c_R(\alpha)$ for all $\alpha \in [0, 1]$.

Then, using a suitable ordering of the intervals $\tilde{c}[\alpha] := [c_L(\alpha), c_R(\alpha)]$ for fixed level-cuts, the task of the fuzzy function minimization over a feasible set is transformed into a bicriterial optimization problem, which is solved by means of the scalarization technique.

Clearly, the approach of partial ordering the intervals may lead to situations of indecisiveness [15]. This reflects the noncomparability of the elements of the set of Pareto optimal solutions of the biobjective optimization problem. The computation of all such solutions is the basis of our approach to compute the membership function values of the fuzzy solutions of the initial problem. For this, we have to compute the membership function values for all feasible points with the presented algorithm.

As soon as a solution of the scalarized problem (with fixed α-cut) depends on a parameter λ, a variation of $\lambda \in [0, 1]$ gives the optimal solution points. The set of those points then represents a subset of a Pareto set. The solution of the fuzzy linear optimization problem (2) is defined through the Pareto-optimal solution of a bicriterial optimization problem. This will be described in Section 2.

In Section 3 optimality conditions for the fuzzy linear optimization problem (2) are derived. A procedure to compute the membership function of each fuzzy solution is given in Section 4. For brevity, the discussions is limited to one element of the set of fuzzy optimal solutions, but can easily be extended.

The paper is concluded with a short discussion devoted to the wide class of triangular fuzzy numbers (that can be extended to the class of LR-numbers) in Section 5 and a numerical example in Section 6.

2. Fuzzy linear optimization problem

For simplicity instead of (2) we investigate the fuzzy linear optimization problem:

$$\begin{aligned} &\tilde{c}^\top x \to \min, \\ &Ax = b, \\ &x \ge 0 \end{aligned} \tag{3}$$

with an n-dimensional vector of decision variables x. Adding slack variables, it is easy to see that this problem is equivalent to (2). Here $\tilde{c}$ is a vector of fuzzy numbers, the constraint matrix is $A \in \mathbb{R}^{m\times n}$ and the right-hand side vector is $b \in \mathbb{R}^m$.

This problem is replaced with the minimization of the α-cuts on the feasible set (see e.g. [9,11,16,8] for the similar procedures for a single α-cut). Rommelfanger et al. [11] suggested an approach for determining a compromise solution of the linear optimization problem using a fixed aspiration level $\alpha \in (0, 1)$.

Thereby, an interval optimization problem is obtained:

$$\begin{aligned} &[c_L^\top(\alpha)x, c_R^\top(\alpha)x] \to \min,\\ &Ax = b,\\ &x \geq 0. \end{aligned} \tag{4}$$

To find an optimal solution of the problem (4) it is necessary to compare intervals in the objective function for different values of x. For the comparison of two different intervals $[a,b]$ and $[c,d]$ in $\mathbb{R}$ the following definition is adopted:

Definition 1 (*Chanas and Kuchta [9]*). An interval $[a, b]$ is smaller than an interval $[c, d]$, i.e. $[a, b] \prec [c, d]$, if $a \leq c$ and $b \leq d$ with at least one strong inequality.

Applying this notion to problem (3) it is easy to see that $\tilde{c}^\top x_1 \prec \tilde{c}^\top x_2$ iff

$$\begin{aligned} &c_L^\top(\alpha)x_1 \leq c_L^\top(\alpha)x_2,\\ &c_R^\top(\alpha)x_1 \leq c_R^\top(\alpha)x_2 \end{aligned} \tag{5}$$

with at least one strong inequality.

Using this ordering of intervals the task of finding an optimal solution of the interval optimization problem (4) reduces to the search of a solution of the following two-objective optimization problem with fixed α-cut:

$$\begin{aligned} &c_L^\top(\alpha)x \to \min,\\ &c_R^\top(\alpha)x \to \min,\\ &Ax = b,\\ &x \geq 0. \end{aligned} \tag{6}$$

Definition 2 (*Dempe and Ruziyeva [17]*). A feasible solution $\hat{x}$ of the fuzzy optimization problem 3 is optimal iff $\hat{x}$ is Pareto-optimal for (6).

It is well-known that in multiobjective optimization problems objective functions often conflict with each other. In general, no single solution will simultaneously minimize all scalar objective functions. Therefore, solutions of problem (6) are defined by means of the Pareto optimality concept [18]:

Definition 3. A feasible point $\widehat{x} \in X := \{x : Ax = b, x \geq 0\}$ is called Pareto optimal for (6) if there does not exist another feasible point $\check{x} \in X$ such that $c_L^\top(\alpha)\check{x} \leq c_L^\top(\alpha)\widehat{x}$ and $c_R^\top(\alpha)\check{x} \leq c_R^\top(\alpha)\widehat{x}$ with at least one strong inequality.

Let $\Psi(\alpha)$ denote the set of Pareto optimal solutions of problem (6) for a certain α-cut.

Now it is possible to rewrite Definition 2 as

Definition 4. A point $\widehat{x} \in X$ is called an optimal solution of the problem (3) on α-cut provided that $\widehat{x} \in \Psi(\alpha)$.

Note that, in general, this approach gives no unique optimal solution of the fuzzy optimization problem, since the Pareto optimal solutions of the problem (6) form a certain set in $\mathbb{R}^n$. This is related to the idea of Chanas and Kuchta

[9,8]. According to Ehrgott [18] and Zadeh [19], to compute all Pareto optimal solutions of the biobjective optimization problem it is sufficient to compute all optimal points of the scalarized problem:

$$f(x,\lambda)[\alpha] := \lambda c_L^\top(\alpha)x + (1-\lambda)c_R^\top(\alpha)x \to \min,$$

$$Ax = b,$$

$$x \geq 0 \tag{7}$$

with $0 < \lambda < 1$. For a fixed α-cut the optimal points of problem (7) form the sets of optimal solutions $\Psi_\alpha(\lambda)$, where λ is the so-called coefficient of scalarization.

This λ can be explained referring to the decision rule of Hurwicz. The optimism/pessimism parameter $\lambda \in [0,1]$ reflects the attitude of the decision-maker. Because of this restriction for λ the weighted sum $\lambda c_L^\top(\alpha)x + (1-\lambda)c_R^\top(\alpha)x$ is a convex combination of the objective functions $c_L^\top(\alpha)x$ and $c_R^\top(\alpha)x$. Therefore, the weighting factor λ can be interpreted as the relative importance between two-objective functions of the biobjective optimization problem (6).

In general, each single optimization problem for some fixed α and λ determines an optimal solution set. The weighted sum method changes weights among the objective functions $c_L^\top(\alpha)x$ and $c_R^\top(\alpha)x$ systematically (e.g. by a predetermined step size in the hyper-ellipse approximation [20,21]). Also, the weight on each single-objective function may be adaptively determined (see, for instance, the adaptive weighted sum method [22]). Ideas to compute the set of Pareto optimal solutions can be found in e.g. [23,24,21]. Here we assume that Pareto optimal solutions are already determined.

As stated in [18,23], any optimal solution of (7) for $0 < \lambda < 1$ is Pareto optimal for (6). Hence, solutions of the problem (7) for different weight combinations produce a subset of the Pareto solutions. Vice-versa, for each Pareto optimal solution $\hat{x}$ of (6), there exists $0 < \lambda < 1$ such that $\hat{x} \in \Psi_\alpha(\lambda)$.

Discussions presented in this section could be extended under convexity assumptions to fuzzy nonlinear optimization problems (see [17,18]).

3. Optimality conditions

In this section the fuzzy linear optimization problem (3) and its optimality conditions are discussed. The idea resembles the approach to optimality conditions for the crisp linear optimization problem.

A feasible set in a decision space $X = \{x : Ax = b, x \geq 0\}$ is defined by the $m \times n$ constraint matrix A and the right-hand side vector $b \in \mathbb{R}^m$. Assume that $rank(A) = m$ and $b \geq 0$.

A nonsingular $m \times m$ submatrix A_B of A is called basic matrix, where B is a set of the columns of A defining A_B. B is called a basis. Let $N := \{1, \dots n\} \backslash B$ be a set of nonbasic column indices. A variable x_i and an index i are called basic if $i \in B$, nonbasic otherwise.

With the notion of a basis it is possible to split A, c and x into basis and nonbasis parts, using B and N as index sets. Let us write $A = (A_B, A_N)$, $c^\top = (c_B^\top, c_N^\top)$ and $x = (x_B, x_N)$.

Using those notations, consider the feasible set in the decision space $X = \{x : Ax = b, x \geq 0\} = \{x : A_B x_B + A_N x_N = b, x \geq 0\} = \{x = (x_B, x_N)^\top : x_B = A_B^{-1}b - A_B^{-1}A_N x_N \geq 0, x_N \geq 0\}$ under invertibility assumption of the matrix A_B.

Setting $\overline{x}_N = 0$ (and, therefore, $\overline{x}_B = A_B^{-1}b$), a basic solution can be obtained as $\overline{x} = (A_B^{-1}b, 0)$. If in addition $\bar{x}_B \geq 0$, it is called a basic feasible solution. Then, the basis B is also called feasible.

Each single basic feasible solution, i.e. each vertex of the convex polyhedron X, determines a corresponding matrix A_B, with A_B being nonsingular.

The well-known optimality condition in the crisp case of (3) reads as follows [25]:

$$c_B^\top A_B^{-1} A - c^\top \leq 0. \tag{8}$$

The optimality condition (8) can be reformulated as optimality condition for the interval optimization problem (4) with fixed level-cut:

$$\tilde{c}_B^\top[\alpha] A_B^{-1} A - \tilde{c}^\top[\alpha] \leq 0. \tag{9}$$

Then, taking into account all derivations of the reduction of the interval optimization problem (4) to the scalarized optimization problem (7), the optimality condition for the problem (7) can be defined using the auxiliary function:

$$h(\alpha,\lambda)=(\lambda c_L(\alpha)+(1-\lambda)c_R(\alpha))_B^\top A_B^{-1}A-(\lambda c_L(\alpha)+(1-\lambda)c_R(\alpha))^\top \leq 0. \tag{10}$$

It is easy to see that this vector function $h(\alpha,\lambda)=(h_1(\alpha,\lambda),\ldots,h_n(\alpha,\lambda))^\top$ is linear w.r.t. λ.

4. Membership function

Consider the fuzzy linear programming problem (with fuzzy coefficients in the objective function) (3). An optimal solution $\bar{x}$ of this problem is defined in [8] as a fuzzy set in the set of feasible solutions with the membership function equal to the geometric measure of the set of all $\alpha \in [0,1]$ such that $\bar{x}$ is an optimal solution of the interval optimization problem (4). As soon as the fuzzy solution has a grade of membership in the set of optimal solutions [8,9], it is necessary to compute the value of the membership function exactly. That is an aim of the present paper.

In view of the fact that the interval optimization problem (4) transforms into the bicriterial parametric optimization problem (6) it is possible to determine for each basic solution, which is efficient for at least one value of α, the whole set of the α-cuts, for which it remains efficient. This set can be composed of a certain number of subintervals $[\alpha_{s-1},\alpha_s]\subseteq[0,1]$ for $s=1,\ldots,k$ for a fixed solution $\bar{x}$. Then, according to [8], the membership function can be obtained as $\mu(\bar{x})=\sum_{s=1}^{k}[\alpha_s-\alpha_{s-1}]$.

Now, using the linearization method, described e.g. in [18], the determination of the set of all α such that $\bar{x}\in\Psi(\alpha)$ involves determining the set of all α such that $\exists\lambda=\lambda(\alpha)\in[0,1]:\bar{x}$ is optimal for problem (7). As the basic solutions define the vertices of the polyhedron of the feasible set, let us compute them using the Simplex-method or an other appropriate method, (see e.g. [26,27]).

For all $i=1,\ldots,n$ let us solve $h_i(\alpha,\lambda)=0$ for a fixed α-cut. This means that for the basic solutions there are n equations instead of the same number of inequalities (10), i.e.

$$h_i(\alpha,\lambda)=[(\lambda c_L(\alpha)+(1-\lambda)c_R(\alpha))_B^\top A_B^{-1}A-(\lambda c_L(\alpha)+(1-\lambda)c_R(\alpha))^\top]_i=0. \tag{11}$$

For defining signs of $h_i(\alpha,\lambda)$ for all $i=1,\ldots,n$, let us reorder the functions h_i such that $h_1,\ldots,h_t$ are decreasing and $h_{t+1},\ldots,h_n$ are increasing functions w.r.t. λ. For this it is enough to calculate the signs of the derivatives

$$h_i'(\alpha,\cdot)=[(c_L(\alpha)-c_R(\alpha))_B^\top A_B^{-1}A-(c_L(\alpha)-c_R(\alpha))^\top]_i \tag{12}$$

for all $i=1,\ldots,n$.

For the fixed α let us denote by λ_i the root of the function $h_i(\alpha,\lambda)$. It is easy to compute $l.h.s.(\alpha)=\max\{\lambda_1,\ldots,\lambda_t\}$ and $r.h.s.(\alpha)=\min\{\lambda_{t+1},\ldots,\lambda_n\}$ and, thus, to define an interval $I(\alpha)=[l.h.s.(\alpha),r.h.s.(\alpha)]$ for which (10) holds. This interval is moving when α is changing. The interval $I(\alpha)$ is presented in Fig. 1 for a fixed α so that the feasible solution $\bar{x}$ is Pareto optimal. In Fig. 1, k $(1\leq k\leq t)$ is an index of a decreasing function $h_k(\alpha,\lambda)$ such that $l.h.s.(\alpha)=\lambda_k$ and p $(t+1\leq p\leq n)$ is an index of an increasing function $h_p(\alpha,\lambda)$ such that $r.h.s.(\alpha)=\lambda_p$.

Note, that emptiness of $I(\alpha)$ means that the chosen solution $\bar{x}$ is nonoptimal for (7). Therefore, for this α the set of Pareto optimal solutions (for short, the Pareto set) for the biobjective optimization problem (6) $\Psi(\alpha)$ does not include $\bar{x}$. This means, that the set of optimal solutions of problem (3) does not contain $\bar{x}$.

With this notation it is possible to rewrite the optimality condition (10) as $|I(\alpha)|>0$ and to compute a membership function of the fuzzy solution $\bar{x}$:

$$\mu(\bar{x})=card\{\alpha|\bar{x}\in\Psi(\alpha)\}=card\{\alpha||I(\alpha)|>0\}=card\{\alpha|r.h.s.(\alpha)-l.h.s.(\alpha)>0\}. \tag{13}$$

Note that if a vertex x_1 is an optimal solution of the biobjective optimization problem (6) for $\alpha\in[\underline{\alpha}_1,\overline{\alpha}_1]$ and a neighboring vertex x_2 is also optimal for problem (6) for $\alpha\in[\underline{\alpha}_2,\overline{\alpha}_2]$, then each point x on the face in-between these two optimal solutions is also Pareto optimal for this problem for $\alpha\in[\underline{\alpha}_1,\overline{\alpha}_1]\bigcap[\underline{\alpha}_2,\overline{\alpha}_2]$. It is essential to note that if for some α there exist a λ such that x_1 is optimal for (7) and there exist no λ for x_2, then the face in-between these two vertices is not a subset of the set of Pareto optimal solutions.

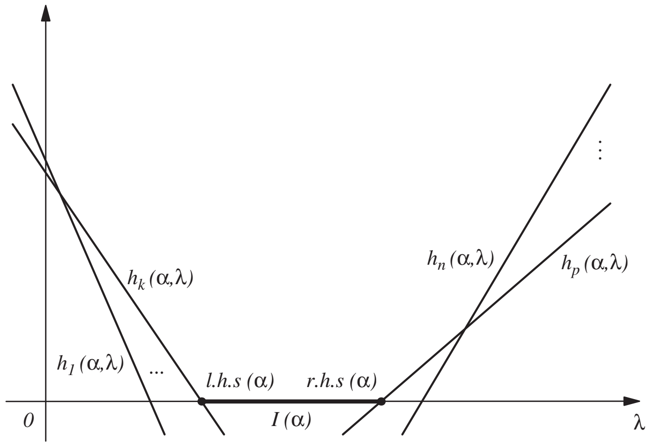

Fig. 1. Interval of satisfied optimality conditions (α is fixed).

5. The special case of triangular fuzzy numbers

Consider now a subclass of LR-numbers—the continuous triangular fuzzy numbers that are represented by a triple (c_L, c_T, c_R) [28]. In this case $c_L(\alpha) = (c_T - c_L)\alpha + c_L$ and $c_R(\alpha) = (c_T - c_R)\alpha + c_R$.

Assume that the fuzzy solution $\bar{x}$ is optimal for the fuzzy linear optimization problem (3). Let us compute the membership function of $\bar{x}$.

Consider the optimality conditions (10). As soon as all components of the vector function $h(\alpha, \lambda) = (h_1(\alpha, \lambda), \ldots, h_n(\alpha, \lambda))^\top$ are equal to zero (analogously to (11)) for $i \in B$, it makes sense to check the optimality condition (10) for $i \in N$. Let us rewrite (10) in terms of α componentwise. Note that $h(\alpha, \lambda)$ is linear w.r.t. α. Denoting through $Num_i(\lambda) := [c^\top(\lambda)]_i - [c_B^\top(\lambda)A_B^{-1}A]_i$ (numerator) and $Den_i(\lambda) := [(c_T^\top - c^\top(\lambda))_B A_B^{-1}A]_i - [c_T^\top]_i + [c^\top(\lambda)]_i$ (denominator), where $c(\lambda) = \lambda c_L + (1-\lambda)c_R$ and $i \in N$, the condition $z_i^+(\lambda) := (Num_i(\lambda)/Den_i(\lambda)) \geq \alpha$ is derived for $\lambda \in [([c_R]_i - [c_T]_i)/([c_R]_i - [c_L]_i), 1]$. For $\lambda \in [0, ([c_R]_i - [c_T]_i)/([c_R]_i - [c_L]_i)]$ optimality condition (10) results in $z_i^-(\lambda) := (Num_i(\lambda)/Den_i(\lambda)) \leq \alpha$. Since $0 \leq \alpha \leq 1$, it is obvious that it only makes sense to compute $z_i^-(\lambda)$ and $z_i^+(\lambda)$ if $Den_i(\lambda)Num_i(\lambda) \geq 0$.

Let us start the computation of the interval $I(\alpha)$ with the lower bound. We solve a following optimization problem for each $i \in N$:

$$z_i^-(\lambda) \to \min,$$

$$\lambda \in \left[0, \frac{[c_R]_i - [c_T]_i}{[c_R]_i - [c_L]_i}\right] \tag{14}$$

and denote an optimal function value through z_i^{*-}.

This means that if for all $i \in N$ the inequality $\alpha \geq z_i^{*-}$ holds, then $\exists \lambda \in [0, ([c_R]_i - [c_T]_i)/([c_R]_i - [c_L]_i)]$ such that the solution $\bar{x}$ belongs to $\Psi_\alpha(\lambda)$. Therefore, if for any $0 \leq \alpha \leq 1$ inequality $\alpha \geq \max\limits_{i \in N} z_i^{*-}$ holds, then $\bar{x} \in \Psi(\alpha)$ and the set of all such α is convex.

The inverse problem for computing the upper bound of the interval for α is given by analogy:

$$z_i^+(\lambda) \to \max,$$

$$\lambda \in \left[\frac{[c_R]_i - [c_T]_i}{[c_R]_i - [c_L]_i}, 1\right]. \tag{15}$$

Let the optimal function value of this problem be z_i^{*+} for each $i \in N$.

Then, using similar discussions, an optimality condition that guarantees that $\bar{x}$ is a solution of (6), i.e. $\bar{x} \in \Psi(\alpha)$, is that inequality $\alpha \leq \min_{i \in N} z_i^{*+}$ holds for any $0 \leq \alpha \leq 1$.

Combining (14) and (15) it is easy to derive the following result:

$$\bar{x} \in \Psi(\alpha) \Leftrightarrow \max_{i \in N}\{0, z_i^{*-}\} \le \alpha \le \min_{i \in N}\{z_i^{*+}, 1\}.$$

Using (13) the membership function value of the fuzzy solution $\bar{x}$ can be obtained as

$$\mu(\bar{x}) = \min_{i \in N}\{z_i^{*+}, 1\} - \max_{i \in N}\{0, z_i^{*-}\}. \tag{16}$$

6. Example

To complete the discussion it is interesting to explain the results by giving a special example—the traffic problem with fuzzy cost coefficients.

Consider a road system as a traffic network $G = (V, E)$ consisting of a node set V for junctions and an edge set E (that we define through vertices) containing all streets connecting the junctions. The streets may have different capacities.

Assume that v units of a certain good should be transported with minimal overall costs from the origin $s \in V$ to the destination $d \in V$ through the traffic network G. The problem is to compute optimal amounts of transported goods on the streets of the network. The travel costs for traversing a street usually are not known exactly that motivates us to assume that all travel costs have fuzzy values.

Let x_{kl} denote the amount of transported units over the edge $(k, l) \in E$ that connects two vertices $k, l \in V$. Let O_k (I_k) denote the set of all edges leaving (entering) the node k, i.e. such designations are connected to the words *out* and *in*.

Assume that the flow x_{kl} on the edge (k, l) is bounded by the capacity u_{kl}. This is expressed by inequality (18) given below. The constraint (19) is used to guarantee that the total incoming flow is equal to the total outgoing flow. Moreover, the outgoing flow in the origin equals to v (see (20)):

$$\widetilde{f}(x) = \sum_{k,l \in V} \widetilde{c}_{kl} x_{kl} \to \min, \tag{17}$$

$$x_{kl} \le u_{kl} \qquad \forall k, l \in V, \tag{18}$$

$$\sum_{k \in I_l} x_{kl} - \sum_{i \in O_l} x_{li} = 0, \quad \forall\, l \in V \setminus \{s, d\}, \tag{19}$$

$$\sum_{k \in I_s} x_{ks} - \sum_{i \in O_s} x_{si} = -v, \tag{20}$$

$$x_{kl} \ge 0. \tag{21}$$

This problem is a special case of the problem (3), where (17) reflects the total fuzzy cost and $Ax = b$ is an abbreviation of the constraints (18)–(20). Inequality (21) insures non-negativity of the flow.

In the following a numerical example is considered. A schematic illustration is given in Fig. 2. Let $\widetilde{f}(x)$ be the total fuzzy cost that we have to minimize:

$$\widetilde{f}(x) = \widetilde{3}x_{12} + \widetilde{8}x_{13} + \widetilde{7}x_{23} + \widetilde{7}x_{24} + \widetilde{3}x_{34} \to \min$$

with demand

$$\begin{aligned}
&x_{12} + x_{13} = 90,\\
&x_{24} + x_{34} = 90,\\
&x_{12} = x_{23} + x_{24},\\
&x_{13} + x_{23} = x_{34},
\end{aligned}$$

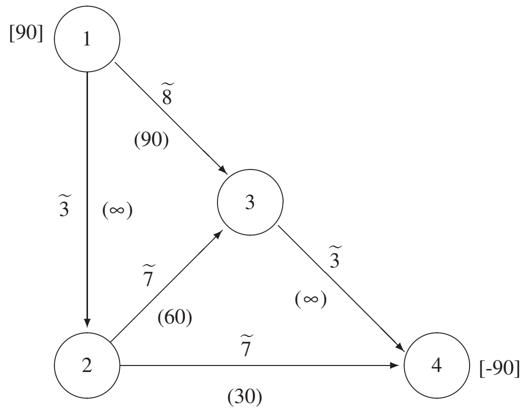

Fig. 2. The example of the traffic network.

and capacity

$$
\begin{aligned}
&0 \le x_{12} \le 90,\\
&0 \le x_{13} \le 90,\\
&0 \le x_{23} \le 60,\\
&0 \le x_{24} \le 30,\\
&0 \le x_{34} \le 90.
\end{aligned}
$$

Suppose that continuous triangular fuzzy numbers (c_L, c_T, c_R) [28] are used: $\widetilde{3} = (1, 3, 5)$, $\widetilde{7} = (5, 7, 9)$ and $\widetilde{8} = (0, 8, 16)$.

Let us now compose the constraint matrix

$$
A = \begin{array}{c} \begin{array}{ccccc} x_{12} & x_{13} & x_{23} & x_{24} & x_{34} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix} \end{array}.
$$

This optimization problem with fuzzy coefficients in the objective has two different basic fuzzy solutions: $x_1 = (30, 60, 0, 30, 60)$ and $x_2 = (90, 0, 60, 30, 60)$. The Pareto set is a linear combination of these points.

The membership function for the fuzzy solution x_1 was computed with the method described in Section 5, with the basis $B = \{1, 2, 5\}$, i.e.

$$
B_A = \begin{array}{c} \begin{array}{ccc} x_{13} & x_{24} & x_{34} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \end{array}
$$

and $N = \{3, 4\}$. Let us perform the computations for $i = 3, 4$: $Den_3(\lambda) = 18 - 14\lambda$ and $Num_3(\lambda) = 8\lambda - 2$; $Den_4(\lambda) = 26 - 20\lambda$ and $Num_4(\lambda) = 12\lambda - 7$.

Further, $Den_3 > 0$ for all $\lambda \in [0, 1]$ and $Num_3 \ge 0$ for $\lambda \ge 0.25$. Therefore we consider only $z_3^+ = (8\lambda - 2)/(18 - 14\lambda)$ for $\lambda \in [0.25, 1]$ and $z_3^- := 0$. It is easy to see that $Den_4 > 0$ for all $\lambda \in [0, 1]$ and $Num_4 \ge 0$ when $\lambda \ge 7/12$. Again, it is interesting to compute the upper bound $z_4^+ = (12\lambda - 7)/(26 - 20\lambda)$ for $\lambda \in (7/12, 1]$ and $z_4^- := 0$. Then $z^{*+} = \min\{\max_{\lambda\in[1/4,1]}\{z_3^+\}, \max_{\lambda\in[7/12,1]}\{z_4^+\}\} = 5/6 \approx 0.833$ and $z^{*-} = \max\{z_3^-, z_4^-\} = 0$.

According to (16) the membership function for the fuzzy optimal solution $x_1 = (30, 60, 0, 30, 60)$ is equal to $\mu(x_1) = \min\{z^{*+}, 1\} - \max\{0, z^{*-}\} = z^{*+} - 0 \approx 0.833$.

The computations of the Pareto set of optimal solutions of the problem and membership function for the chosen optimal solution were performed with MATLAB 7.9.0(R2009b).

7. Conclusions

In the present paper special attention is given to the computation of the membership function of the fuzzy solution of the fuzzy linear optimization problem. The fuzzy linear optimization problem is solved by its reformulation into a related biobjective optimization problem. This problem, in turn, is solved by methods of the multiobjective optimization problem's scalarization technique [18]. Elements of the Pareto set of the biobjective optimization problem are interpreted as optimal solutions of the initial fuzzy optimization problem. The membership function value of such a solution equals to the cardinality of all α such that this solution is Pareto optimal for the problem obtained for the α-cut of the fuzzy objective function's coefficients. Formulas for computing this membership function value are given.

The theory for continuous triangular fuzzy numbers could be extended to the general LR-numbers. For this, formulas for computing left- and right-hand side functions $c_L(\alpha)$, $c_R(\alpha)$ need to be used (see [8,29]). But this functions are no longer linear w.r.t. α. Therefore, the computation of $z_i^+(\lambda)$ and $z_i^-(\lambda)$ is cumbersome.

The proposed method is illustrated using the example of a traffic problem with given crisp capacities and demands and fuzzy costs.

Under convexity assumptions the discussions could be extended to fuzzy nonlinear optimization problems. This leads to a more complicated formula for the membership function and is a topic of future research.

References

[1] S. Dempe, D. Fanghänel, T. Starostina, Optimal toll charges: fuzzy optimization approach, in: F. Heyde, A. Löhne, C. Tammer (Eds.), Methods of Multicriteria Decision – Theory and Applications, Shaker Verlag, Aachen, 2009, pp. 29–45.

[2] S.A. Orlovski, Mathematical programming problems with fuzzy parameters, in: J. Kacprzyk, R.R. Yager (Eds.), Management Decision Support Systems using Fuzzy Sets and Possibility Theory, Verlag TÜV, Köln, 1985, pp. 136–145.

[3] M. Delgado, J.L. Verdegay, M.A. Vila, A general model for fuzzy linear programming, Fuzzy Sets and Systems 29 (1989) 21–29.

[4] H. Tanaka, H. Ichihashi, K. Asai, A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers, Control and Cybernetics 13 (1984) 185–194.

[5] J. Verdegay, Fuzzy mathematical programming, Fuzzy Information and Decision Processes (1982) 231–237.

[6] F. Jiménez, J. Cadenas, G. Sánchez, A. Gómez-Skarmeta, J. Verdegay, Multi-objective evolutionary computation and fuzzy optimization, International Journal of Approximate Reasoning 43 (2006) 59–75.

[7] R. Bellman, L. Zadeh, Decision making in a fuzzy environment, Management Science 17 (1970) 141–164.

[8] S. Chanas, D. Kuchta, Linear programming problem with fuzzy coefficients in the objective function, in: M. Delgado, J. Kacprzyk, J.L. Verdegay, M.A. Vila (Eds.), Fuzzy Optimization, Physica-Verlag, Heidelberg, 1994, pp. 148–157.

[9] S. Chanas, D. Kuchta, Multiobjective programming in optimization of interval objective functions – a generalized approach, European Journal of Operational Research 94 (1996) 594–598.

[10] S. Chanas, The use of parametric programming in fuzzy linear programming, Fuzzy Sets and Systems 11 (1983) 243–251.

[11] H. Rommelfanger, R. Hanuscheck, J. Wolf, Linear programming with fuzzy objectives, Fuzzy Sets and Systems 29 (1989) 31–48.

[12] H.-J. Zimmermann, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems 1 (1978) 45–55.

[13] J. Cadenas, J. Verdegay, Towards a new strategy for solving fuzzy optimization problems, Fuzzy Optimization and Decision Making 8 (2009) 231–244.

[14] M. Panigrahi, G. Panda, S. Nanda, Convex fuzzy mapping with differentiability and its application to fuzzy optimization, European Journal of Operational Research 185 (2008) 47–62.

[15] A. Sengupta, T. Pal, On comparing interval numbers, European Journal of Operational Research 127 (2000) 28–43.

[16] H.-J. Zimmermann, Fuzzy Set Theory and its Applications, Kluwer Academic Publishers, Dordrecht, 1991.

[17] S. Dempe, A. Ruziyeva, The Karush–Kuhn–Tucker Optimality Conditions in Fuzzy Optimization, Technical Report, TU Bergakademie Freiberg, 2010-06.

[18] M. Ehrgott, Multicriteria Optimization, Springer Verlag, Berlin, 2005.

[19] L. Zadeh, Optimality and non-scalar-valued performance criteria, IEEE Transactions on Automatic Control 8 (1963) 59–60.

[20] G.M. Fadel, Y. Li, Approximating the Pareto curve to help solve biobjective design problems, Structural and Multidisciplinary Optimization 23 (2002) 280–296.

[21] Y. Li, G.M. Fadel, M. Wiecek, V.Y. Blouin, Minimum effort approximation of the Pareto space of convex bi-criteria problems, Optimization and Engineering 4 (2003) 231–261.

[22] I.Y. Kim, O.L. Weck, Adaptive weighted sum method for bi-objective optimization: Pareto front generation, Structural and Multidisciplinary Optimization 29 (2005) 149–158.

[23] C. Audet, G. Savard, W. Zghal, Multiobjective optimization though a series of single-objective formulations, SIAM Journal of Optimization 19 (2008) 188–210.
[24] J. Fliege, Gap-free computation of Pareto-points by quadratic scalarizations, Mathematical Methods of Operations Research 59 (2004) 69–89.
[25] D. Bertsimas, J. Tsitsiklis, Introduction to Linear Optimization, Athena Scientific, Massachusetts, 1997.
[26] M. Dyer, L. Proll, An algorithm for determining all extreme points of a convex polytope, Mathematical Programming 12 (1977) 81–96.
[27] D. Avis, K. Fukuda, A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra, Technical Report B-237, Tokyo Institute of Technology, Department of Information Science, 1992.
[28] J. Buckley, Joint solution to fuzzy programming problems, Fuzzy Sets and Systems 72 (1995) 215–220.
[29] S. Chanas, Fuzzy programming in multiobjective linear programming – a parametric approach, Fuzzy Sets and Systems 29 (1989) 303–313.