EE 235, Winter 2018, Homework 10: Laplace Transforms
Due Friday March 9, 2018 via Canvas Submission
Write down *ALL* steps for full credit

HW10 Topics:

- Laplace Transform and Inverse Laplace Transform
- Laplace Transform ROC and Signal Properties
- Laplace Transform ROC and LTI System Properties

HW10 Course Learning Goals Satisfied:

- Goal 1: Describe signals in different domains and map characteristics in one domain to those in another.
- Goal 2: Understand the implications of different system properties and how to test for them.
- Goal 4: Analyze LTI systems given different system representations, and translate between these different representations.

HW10 References: OWN Sections 9.1 - 9.2, 9.5 - 9.6, 9.7

HW10 Problems (Total = 60 pts):

1. *Laplace Transform.* (10 pts)
Find the Laplace Transform of the following signals and sketch the corresponding pole-zero plot for each signal. In the plot, indicate the regions of convergence (ROC). Write $X(s)$ as a single fraction in the form of $\frac{N(s)}{D(s)}$.

   (a) (2 pts) $x(t) = e^{-4t}u(t) + e^{-6t}u(t)$. Show that $X(s) = \frac{2s+10}{(s+4)(s+6)}$. with ROC of $Re\{s\} > -4$.

   Solution.
   $x(t) = e^{-4t}u(t) + e^{-6t}u(t)$ $X(s) = \frac{1}{s+4} + \frac{1}{s+6} = \frac{2s+10}{(s+4)(s+6)}$
   poles: $s = -4,\ -6$
   zero: $s = -5$
   ROC: $Re\{s\} > -4$

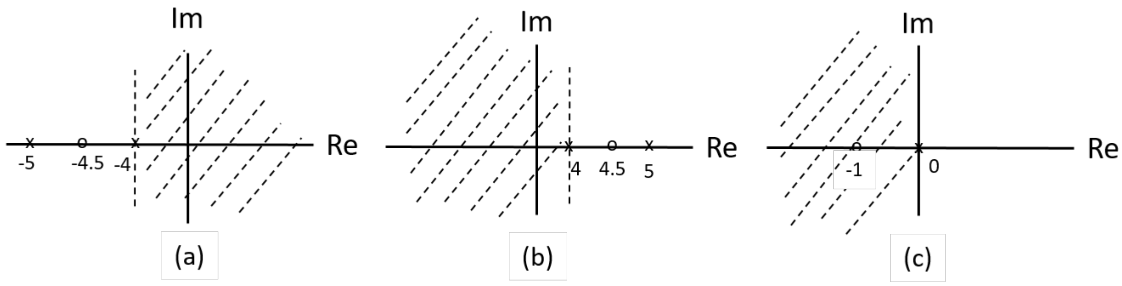
   (b) (4 pts) $x(t) = e^{4t}u(-t) + e^{8t}u(-t)$.

   Solution.
   $x(t) = e^{4t}u(-t) + e^{8t}u(-t)$ *Rewriting $x(t)$ in correct form*
   $x(t) = -[-e^{4t}u(-t)] - [-e^{8t}u(-t)]$
   $X(s) = -\frac{1}{s-4} - \frac{1}{s-8} = \frac{-2s+12}{(s-4)(s-8)}$
   poles: $s = 4,\ 8$
   zero: $s = 6$
   ROC: $Re\{s\} < 4$

   (c) (4 pts) $x(t) = \delta(t) - u(-t)$.

   Solution.
   $x(t) = \delta(t) + u(t)$ $X(s) = 1 + \frac{1}{s} = \frac{s+1}{s}$
   poles: $s = 0$
   zero: $s = -1$

ROC: Re{s} < 0



2. *Inverse Laplace Transform.* (14 pts)
   Find $x(t)$ for given $X(s)$ and ROC. Plot pole-zero plots.

   (a) (2 pts) $X(s) = \frac{1}{s^2+5s+6}$, ROC: $Re\{s\} > -2$. Show that $x(t)$ is $x(t) = e^{-2t}u(t) - e^{-3t}u(t)$.

   Solution.
   $X(s) = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)}$
   poles: $s = -2,\ -3,$ *no zero*
   Using partial fraction expansion, $X(s) = \frac{A}{S+2} + \frac{B}{s+3}$
   $A = 1$ *and* $B = -1$.
   Hence, $X(s) = \frac{1}{s+2} - \frac{1}{s+3}$
   Then, both are 'right' sided using ROC condition. $x(t) = e^{-2t}u(t) - e^{-3t}u(t)$

   (b) (4 pts) $X(s) = \frac{s-3}{s^2+5s+6}$, ROC: $-3 < Re\{s\} < -2$.

   Solution.
   $X(s) = \frac{s-3}{s^2+5s+6}$
   poles: $s = -2,\ -3$
   zero: $s = 3$
   Using partial fraction expansion, $X(s) = \frac{-5}{s+2} + \frac{6}{s+3}$
   From ROC, $x(t) = 6e^{-3t}u(t) + 5e^{-2t}u(-t)$

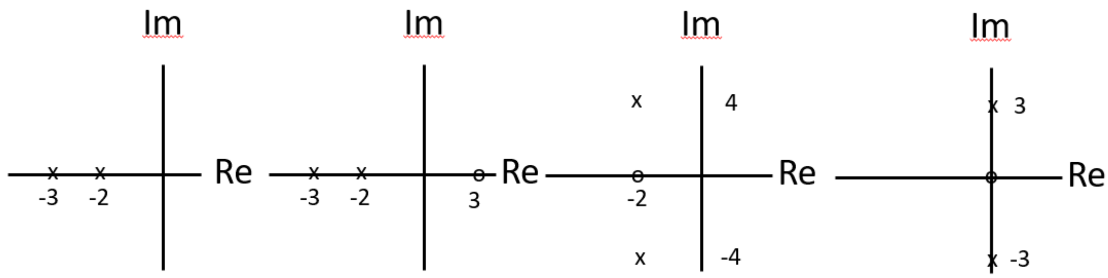
   (c) (4 pts) $X(s) = \frac{s+2}{s^2+4s+20}$, ROC: $Re\{s\} < -2$.

   Solution.
   $X(s) = \frac{s+2}{(s+2)^2+16}$, *ROC:* $Re\{s\} < -2$
   $X(s) = \frac{s+2}{(s+2)^2+(4)^2}$
   $a = 2$
   $w_o = 4$
   $Re\{s\} < -2 \rightarrow$ *left-sided*
   $x(t) = -e^{-2t}\cos(4t)u(-t)$

   (d) (4 pts) $X(s) = \frac{s}{s^2+9}$, ROC: $Re\{s\} < 0$.

   Solution.
   $X(s) = \frac{s}{s^2+9}$, *ROC:* $Re\{s\} < 0$
   $X(s) = \frac{s}{(s)^2+(3)^2}$
   $w_o = 3$
   $Re\{s\} < 0 \rightarrow$ *left-sided*

$x(t) = -\cos(3t)u(-t)$



(a) (b) (c) (d)

3. *ROC and Signal Properties.* (10 pts)

   (a) (5 pts) Suppose $x(t)$ is a real signal with rational Laplace transform $X(s)$ with the following properties:

      i. $X(s)$ has two poles and one zero, with one pole at $s = -1 - 2j$,

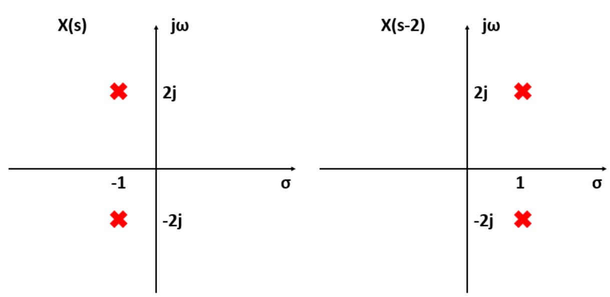
      ii. the Fourier transform of $e^{2t}x(t)$ does not exist,

      iii. $\int_{-\infty}^{\infty} x(t)dt = -2$,

      iv. $\int_{-\infty}^{\infty} e^{-t}x(t)dt = 0$.

   Find $x(t)$ (the time-domain signal).

   Solution. *Because $x(t)$ is real, from (i),*
   the two poles must be conjugate to each other, we can assume that
   $X(s) = \frac{k(s-a)}{(s+1-2j)(s+1+2j)} = \frac{k(s-a)}{(s+1)^2+4}$
   From (ii), We know that the ROC of $X(s-2)$ does not include the imaginary axis, so it must be



   right-sided.
   Thus, the ROC of $X(s)$ is also right-sided $Re\{s\} > -1$
   From (iii) and (iv),
   $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$
   When $s = 0$, $X(0) = \int_{-\infty}^{\infty} x(t)dt = -2 = \frac{-ka}{5}$
   When $s = 1$, $X(1) = \int_{-\infty}^{\infty} x(t)e^{-t}dt = 0 \Rightarrow a = 1 \Rightarrow k = 10$
   Thus, $X(s) = \frac{10(s-1)}{(s+1)^2+2^2} = \frac{10(s+1-2)}{(s+1)^2+2^2} = \frac{10(s+1)}{(s+1)^2+2^2} - \frac{10\cdot 2}{(s+1)^2+2^2}$
   $x(t) = \boxed{10e^{-t}cos(2t)u(t) - 10e^{-t}sin(2t)u(t)}$
   ROC: $Re\{s\} > -1$

   (b) (5 pts) Signal $x(t)$ has the following properties:

      i. $X(s)$ is rational with 1 zero and 2 poles

      ii. $x(t)$ is real

iii. $X(s)$ has its zero at s = 1 and a known pole at s = 1 - 2j

iv. Area under $x(t)$ is equal to 1

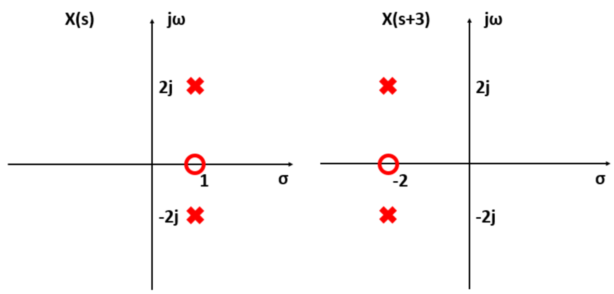
v. $e^{-3t}x(t)$ is absolutely integrable (so Fourier Transform of $e^{-3t}x(t)$ does exist)

Deduce the expression for signal $x(t)$.

Solution. *From (i) to (iii), we have* $X(s) = \frac{k(s-1)}{(s-1+2j)(s+1+2j)} = \frac{k(s-1)}{(s-1)^2+2^2}$

From (iv), we have $X(0) = \frac{k(-1)}{(-1)^2+4} = \frac{-k}{5} = 1 \Rightarrow k = -5$

From (v), since $e^{-3t}x(t)$ *is absolutely integrable, its ROC includes jw-axis, we know that the correct ROC is* $Re\{s\} > 1$



Thus, $X(s) = \frac{-5(s-1)}{(s-1)^2+4}$ *and the corresponding ROC is* $Re\{s\} > 1$ $x(t) = \boxed{-5e^t cos(2t)u(t)}$

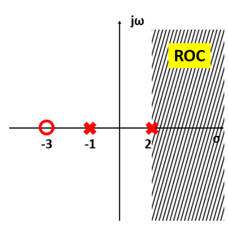
4. *ROC and LTI Systems.* (16 pts)

(a) (5 pts) Consider another system described by: $-2y(t) - \frac{dy(t)}{dt} + \frac{d^2y(t)}{dt} = 3x(t) + \frac{dx(t)}{dt}$.

Solution. *Find* $H(s)$*:* $-2Y(s) - sY(s) + s^2Y(s) = 3X(s) + sX(s)$

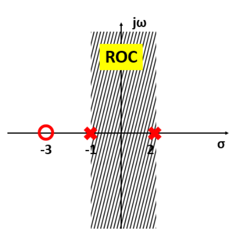
$H(s) = \frac{s+3}{s^2-s-2} = \frac{s+3}{(s-2)(s+1)}$

i. (1 pt) Specify the ROC corresponding to H(s) if it is known the system is causal. Also, sketch the pole-zero plot and associated ROC.



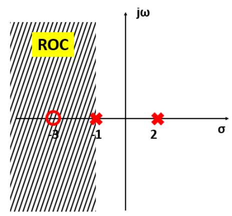
Solution. *ROC = at least* $Re\{s\} > -1 \cap Re\{s\} > 2 =$ *at least* $Re\{s\} > 2$

ii. (2 pts) Specify the ROC corresponding to H(s) if it is known the system is stable. Also, sketch the pole-zero plot and associated ROC.



Solution. *Stable* $\Rightarrow$ *must include jw-axis* $\Rightarrow -1 < Re\{s\} < 2$

iii. (2 pts) Specify the ROC corresponding to H(s) if it is known the system is left-sided. Also, sketch the pole-zero plot and associated ROC.



Solution. *Left-sided* $\Rightarrow$ *ROC left-sided* $Re\{s\} < -1$

(b) (5 pts) Let $H(s)$ represent the system function for a causal, stable LTI system. The input to the system consists of the sum of three terms, one of which is an impulse $\delta(t)$ and the second term is a complex exponential $e^{s_0 t}$, where s_0 is a complex constant. The output of the system is:

$$y(t) = e^{-t}u(t) + \frac{10}{34}e^{4t}\cos(3t) + \frac{6}{34}e^{4t}\sin(3t).$$

Determine $h(t)$ and s_0 consistent with this information.
Hint1: What happens when a complex exponential is passed through an LTI system?
Hint2: One thing to note is that the output is real even if s_0 was complex. This indicates that the third term consists of $e^{s_0^* t}$, where s_0^* is the complex conjugate of s_0.

Solution. *Recall that $y(t) = x(t) * h(t)$ for an LTI system. Also recall that $h(t) * \delta(t) = h(t)$. Since the input $x(t)$ has a $\delta(t)$ in it, it must be that one of the terms in $y(t)$ has a $h(t)$. We claim that $h(t) = e^{-t}u(t)$. To verify this claim, we would need to prove that the output is indeed the convolution of the input and $h(t)$. Note that $H(s) = \frac{1}{s+1}$. Also when $e^{s_0 t}$ is passed through an LTI system, we get as the output $H(s_0)e^{s_0 t}$. We don't know what s_0 is - But the output has an e^{4t} term in it. We can thus guess that $Re(s_0) = 4$. Let $s_0 = \alpha + j\beta$. We need to figure out α, β. We have that,*

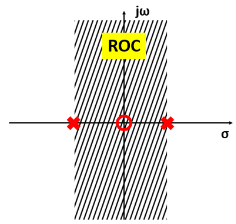
$$\begin{aligned}
(\delta(t) + e^{s_0 t} + e^{-s_0 t}) * h(t) &= h(t) + H(s_0)e^{s_0 t} + H(s_0^*)e^{s_0^* t} \\
&= h(t) + \tfrac{1}{s_0+1}e^{s_0 t} + \tfrac{1}{s_0^*+1}e^{s_0^* t} \\
&= h(t) + e^{\alpha t}\left(\tfrac{1}{(\alpha+1)+j\beta}e^{j\beta t} + \tfrac{1}{(\alpha+1)-j\beta}e^{-j\beta t}\right) \\
&= h(t) + e^{\alpha t}\left(\tfrac{1}{(\alpha+1)+j\beta}\tfrac{(\alpha+1)-j\beta}{(\alpha+1)-j\beta}e^{j\beta t} + \tfrac{1}{(\alpha+1)-j\beta}\tfrac{(\alpha+1)+j\beta}{(\alpha+1)+j\beta}e^{-j\beta t}\right) \\
&= h(t) + e^{\alpha t}\left(\tfrac{(\alpha+1)-j\beta}{(\alpha+1)^2+\beta^2}e^{j\beta t} + \tfrac{(\alpha+1)+j\beta}{(\alpha+1)^2+\beta^2}e^{-j\beta t}\right).
\end{aligned} \tag{1}$$

Clearly $\alpha = 4$, since e^{4t} shows up in the output, $y(t)$. Also note that $(\alpha+1)^2 + \beta^2$ shows up in the denominator of both the second term and the third term. Since the denominator in the given problem is 34, we have that $\beta^2 = 34 - (4+1)^2 = 34 - 25 = 9$. Thus, $\beta = 3$. It can now be verified that the last equation in (1) does simplify to $y(t)$ as given in the problem. Hence, we have that $h(t) = \boxed{e^{-t}u(t)}$. Also, $s_0 = \boxed{4+3j}$.

(c) For each of the cases below, draw a pole-zero plot and ROC for a system that matches the description:

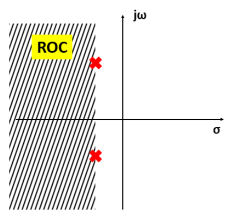
i. (2 pts) The system is stable; the impulse response is two-sided; and the system frequency response approaches a non-zero constant at high frequencies.

Solution. *The system is stable* $\Rightarrow$ *The plot contains jw axis*
The impulse response is two-sided $\Rightarrow$ *The signal is two-sided*
The system frequency response approaches a non-zero constant at high frequencies $\Rightarrow$ *Highpass filter,* $H(0) = 0$



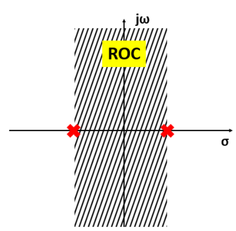
ii. (2 pts) The system is not stable and contains complex poles in the left half plane.

Solution. *The system is not stable* $\Rightarrow$ *The plot does not contain jw axis*
The system contains complex poles in the left half plane $\Rightarrow$ *The signal is left-sided*



iii. (2 pts) The impulse-response is two-sided and absolutely integrable, and the system behaves like a lowpass filter.

Solution. *The impulse-response is two-sided* $\Rightarrow$ *The signal is two-sided*
The system is absolutely integrable $\Rightarrow$ *The system is stable* $\Rightarrow$ *The plot contains jw axis*
The system behaves like a lowpass filter $\Rightarrow H(0) \neq 0$



5. *Homework Self-Reflection*
(10 pts) After completing your homework, go to the following link to rate your skill or concept understanding level for each item listed. Y our self-reflection must be completed by the due date. All submissions are time-stamped, so please give yourself plenty of time to complete and submit your self-reflection.
http://bit.ly/2FFYTAd