EE 235, Winter 2018, Homework 5: More Convolution and LTI Systems
Due Friday February 2, 2018 via Canvas Submission
Write down *ALL* steps for full credit

HW5 Topics:

- Convolution and Properties
- LTI System Properties

HW5 Course Learning Goals Satisfied:

- Goal 2: Understand the implications of different system properties and how to test for them.
- Goal 3: Perform convolutions for arbitrary and closed-form continuous-time signals.
- Goal 4: Analyze LTI systems given different system representations (including input-output equations and impulse response).

HW5 References:

- OWN Sections 2.2-2.3, 3.2

HW5 Problems (Total = 126 pts):

1. *Review*

   (a) *Partial Fraction Expansion.* In Ch.4 and Ch.9, you will need to be familiar with a method called partial fraction expansion (PFE). We will spend a few weeks reviewing this week starting with this homework.

   i. (2 pts) Let $X(s) = \frac{s+2}{s^2+7s+12}$. Using the cover-up method for partial fraction expansion (discussed in the supplementary notes), write $H(s)$ as a sum of partial fractions. Show that $X(s) = \frac{2}{s+4} - \frac{1}{s+3}$.

   Solution.
   $X(s) = \frac{s+2}{s^2+7s+12} = \frac{s+2}{(s+4)(s+3)} = \frac{A}{s+4} + \frac{B}{s+3}$
   $A = \left.\frac{s+2}{s+3}\right|_{s=-4} = \frac{-2}{-1} = 2$
   $B = \left.\frac{s+2}{s+4}\right|_{s=-3} = \frac{-1}{1} = -1$
   Therefore: $\boxed{X(s) = \frac{2}{s+4} - \frac{1}{s+3}}$

   ii. (4 pts) Let $X(s) = \frac{2+2s+s^2}{(1+s)(6+5s+s^2)}$. Using the cover-up method for partial fraction expansion (discussed in the supplementary notes), write $X(s)$ as a sum of partial fractions.

   Solution.
   $X(s) = \frac{2+2s+s^2}{(1+s)(2+s)(3+s)} = \frac{A}{1+s} + \frac{B}{2+s} + \frac{C}{3+s}$
   $X(s) = \frac{A}{1+s} + \frac{B}{2+s} + \frac{C}{3+s}$
   $A = \left.\frac{2+2s+s^2}{(2+s)(3+s)}\right|_{s=-1} = \frac{2-2+1}{(2-1)(3-1)} = \frac{1}{2}$
   $B = \left.\frac{2+2s+s^2}{(1+js)(3+s)}\right|_{s=-2} = \frac{2-4+4}{(1-2)(3-2)} = \frac{2}{(-1)(1)} = -2$
   $C = \left.\frac{2+2s+s^2}{(1+s)(2+s)}\right|_{s=-3} = \frac{2-6+9}{(1-3)(2-3)} = \frac{5}{(-2)(-1)} = \frac{5}{2}$
   Therefore: $\boxed{X(s) = \frac{1/2}{1+s} - \frac{2}{2+s} + \frac{5/2}{3+s}}$

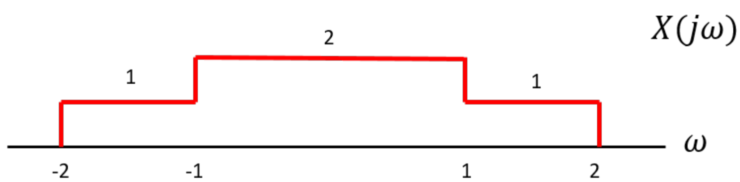
   (b) *Complex Numbers and Magnitude-Squared.* In Ch.3 and Ch.4, you will rely on the concept of magnitude-squared again that was discussed first in HW1. We will use this section to review that concept on different problem types.

i. (4 pts) Using properties of complex conjugates, evaluate the following:

$$\left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 + \left|\frac{1}{j}\right|^2 + \left|\frac{-1}{j}\right|^2 + \left|2e^{j\frac{\pi}{3}}\right|^2 + \left|2e^{-j\frac{\pi}{3}}\right|^2$$

Solution.
Since $\frac{1}{2}$ is a real number: $\left|\frac{1}{2}\right|^2 = (\frac{1}{2})^2 = \frac{1}{4}$
For the other terms, we will need to use complex conjugates:
$\left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 + \left|\frac{1}{j}\right|^2 + \left|\frac{-1}{j}\right|^2 + \left|2e^{j\frac{\pi}{3}}\right|^2 + \left|2e^{-j\frac{\pi}{3}}\right|^2 =$
$\frac{1}{4} + \frac{1}{4} + \left(\frac{1}{j}\right)\left(\frac{1}{-j}\right) + \left(\frac{-1}{j}\right)\left(\frac{-1}{-j}\right) + \left(2e^{j\frac{\pi}{3}}\right)\left(2e^{-j\frac{\pi}{3}}\right) + \left(2e^{-j\frac{\pi}{3}}\right)\left(2e^{j\frac{\pi}{3}}\right) =$
$\frac{1}{2} + 1 + 1 + 4 + 4 = \boxed{10.5}$

ii. (2 pts) Consider a complex function $X(j\omega)$ plotted below versus frequency ω

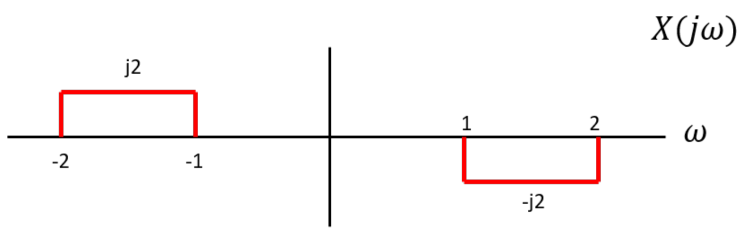


Using this information, evaluate the following integral: $\int\limits_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$

This integral type will be important for Ch.4. Show that this integral evaluates to 10.

Solution.
We need to split this integral and integral only the regions where the function is not 0
$\int\limits_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \int\limits_{-2}^{-1} |1|^2 \, d\omega + \int\limits_{-1}^{1} |2|^2 \, d\omega + \int\limits_{1}^{2} |1|^2 \, d\omega = 1 + 8 + 1 = \boxed{10}$

iii. (4 pts) Consider a complex function $X(j\omega)$ plotted below versus frequency ω



Using this information, evaluate the following integral: $\int\limits_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$

This integral type will be important for Ch.4.

Solution.
We need to split this integral and integral only the regions where the function is not 0
$\int\limits_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \int\limits_{-2}^{-1} |j2|^2 \, d\omega + \int\limits_{1}^{2} |-j2|^2 \, d\omega$
Note that $|j2|^2 = (j2)(-j2) = 4$ and $|-j2|^2 = (-j2)(j2) = 4$
Therefore:
$\int\limits_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \int\limits_{-2}^{-1} 4 d\omega + \int\limits_{1}^{2} 4 d\omega = 4 + 4 = \boxed{8}$

(c) *Complex Functions and Euler's Formula.* In Ch.3, we will need to be fluent in using different versions of Euler's formula in order to convert between sinusoids and complex exponentials. We will use this problem as an opportunity to do that. See Supplementary notes for a summary of Euler's formulas.

i. (2 pts) Using Euler's formulas, convert $f(t) = 2\cos(6t + \frac{\pi}{4}) + 3\sin(4t)$ into a sum of weighted complex exponential functions $e^{j\omega t}$ where ω can be any real number.
Your answer should be in the form of a sum of $Ae^{j\omega t}$ where A is a complex number.
Show that $f(t) = e^{j\frac{\pi}{4}}e^{j6t} + e^{-j\frac{\pi}{4}}e^{-j6t} + \frac{3}{2j}e^{j4t} - \frac{3}{2j}e^{-j4t}$

Solution. *Using Euler's formulas to rewrite cosine and sine into complex exponentials, we get:*
$f(t) = 2\left[\frac{1}{2}e^{j(6t+\frac{\pi}{4})} + \frac{1}{2}e^{-j(6t+\frac{\pi}{4})}\right] + 3\left[\frac{1}{2j}e^{j4t} - \frac{1}{2j}e^{-j4t}\right]$
$f(t) = e^{j(6t+\frac{\pi}{4})} + e^{-j(6t+\frac{\pi}{4})} + \frac{3}{2j}e^{j4t} - \frac{3}{2j}e^{-j4t}$
Pulling out the phase components so that we separate out $e^{j}\omega t$:

$$\boxed{f(t) = e^{j\frac{\pi}{4}}e^{j6t} + e^{-j\frac{\pi}{4}}e^{-j6t} + \frac{3}{2j}e^{j4t} - \frac{3}{2j}e^{-j4t}}$$

ii. (4 pts) Using Euler's formulas, convert $f(t) = 8\cos(\frac{\pi}{3}t) - 2\sin(\frac{3\pi}{4}t)$ into a sum of weighted complex exponential functions $e^{j\omega t}$ where ω can be any real number.
Your answer should be in the form of a sum of $Ae^{j\omega t}$ where A is a complex number.

Solution. *Using Euler's formulas to rewrite cosine and sine into complex exponentials, we get:*
$f(t) = 8\left[\frac{1}{2}e^{j\frac{\pi}{3}t} + \frac{1}{2}e^{-j\frac{\pi}{3}t}\right] - 2\left[\frac{1}{2j}e^{j\frac{3\pi}{4}t} - \frac{1}{2j}e^{-j\frac{3\pi}{4}t}\right]$

$$\boxed{f(t) = 4e^{j\frac{\pi}{3}t} + 4e^{-j\frac{\pi}{3}t} - \frac{1}{j}e^{j\frac{3\pi}{4}t} + \frac{1}{j}e^{-j\frac{3\pi}{4}t}}$$

iii. (2 pts) Using Euler's formulas, convert $f(t) = je^{j3t} + \frac{1}{4}e^{-j\frac{\pi}{3}}e^{j2t} - je^{-j3t} + \frac{1}{4}e^{j\frac{\pi}{3}}e^{-j2t}$ into a sum of sinusoidal functions of the form $Asin(\omega_o t + \theta)$ or $Acos(\omega_o t + \theta)$.
Show that $f(t) = -2\sin(3t) + \frac{1}{2}\cos(2t - \frac{\pi}{3})$

Solution. *Grouping together similar terms:*
$f(t) = je^{j3t} - je^{-j3t} + \frac{1}{4}e^{-j\frac{\pi}{3}}e^{j2t} + \frac{1}{4}e^{j\frac{\pi}{3}t}e^{-j2t}$
$f(t) = j[e^{j3t} - e^{-j3t}] + \frac{1}{4}[e^{-j\frac{\pi}{3}}e^{j2t} + e^{j\frac{\pi}{3}t}e^{-j2t}]$
Rewriting last terms as one exponential by adding exponents:
$f(t) = j[e^{j3t} - e^{-j3t}] + \frac{1}{4}[e^{j(2t-\frac{\pi}{3})} + e^{-j(2t-\frac{\pi}{3})}]$
Using Euler's formulas to convert to sinusoidals:

$$f(t) = j[2j\sin(3t)] + \frac{1}{4}[2\cos(2t - \frac{\pi}{3})] = \boxed{-2\sin(3t) + \frac{1}{2}\cos\left(2t - \frac{\pi}{3}\right)}$$

iv. (4 pts) Using Euler's formulas, convert $f(t) = je^{j\frac{\pi}{20}t} + 2e^{-j\frac{\pi}{3}}e^{j\frac{\pi}{4}t} - je^{-j\frac{\pi}{20}t} + 2e^{j\frac{\pi}{3}}e^{-j\frac{\pi}{4}t}$ into a sum of sinusoidal functions of the form $Asin(\omega_o t + \theta)$ or $Acos(\omega_o t + \theta)$.

Solution. $f(t) = je^{j\frac{\pi}{20}t} - je^{-j\frac{\pi}{20}t} + 2e^{-j\frac{\pi}{3}}e^{j\frac{\pi}{4}t} + 2e^{j\frac{\pi}{3}}e^{-j\frac{\pi}{4}t}$
$f(t) = je^{j\frac{\pi}{20}t} - je^{-j\frac{\pi}{2}t} + 2e^{j(\frac{\pi}{4}t-\frac{\pi}{3})} + 2e^{-j(\frac{\pi}{4}t-\frac{\pi}{3})}$

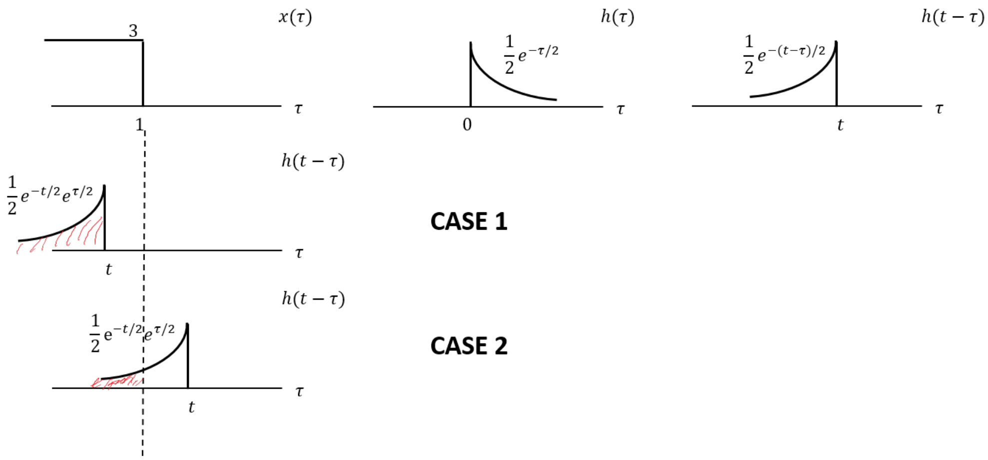
$$f(t) = j[2j\sin(\frac{\pi}{20}t)] + 2[2\cos(\frac{\pi}{4}t - \frac{\pi}{3})] = \boxed{-2\sin\left(\frac{\pi}{20}t\right) + 4\cos\left(\frac{\pi}{4}t - \frac{\pi}{3}\right)}$$

2. *Convolution* Evaluate the output $y(t)$ of an LTI system for each input signal $x(t)$ and impulse response $h(t)$ below:

(a) (2 pts) $x(t) = 3[1 - u(t-1)]$ and $h(t) = \frac{1}{2}e^{-t/2}u(t)$

Show that $y(t) = \begin{cases} 3, & t < 1 \\ 3e^{-(t-1)/2}, & t > 1 \end{cases}$

Solution. $y(t) = \int\limits_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$



Case 1: $t < 1$

$$y(t) = \int_{-\infty}^{t} (3)(\tfrac{1}{2}e^{-t/2}e^{\tau/2})d\tau = \tfrac{3}{2}e^{-t/2}\int_{-\infty}^{t} e^{\tau/2}d\tau = 3e^{-t/2}[e^{t/2} - 0] = 3$$

Case 2: $t > 1$

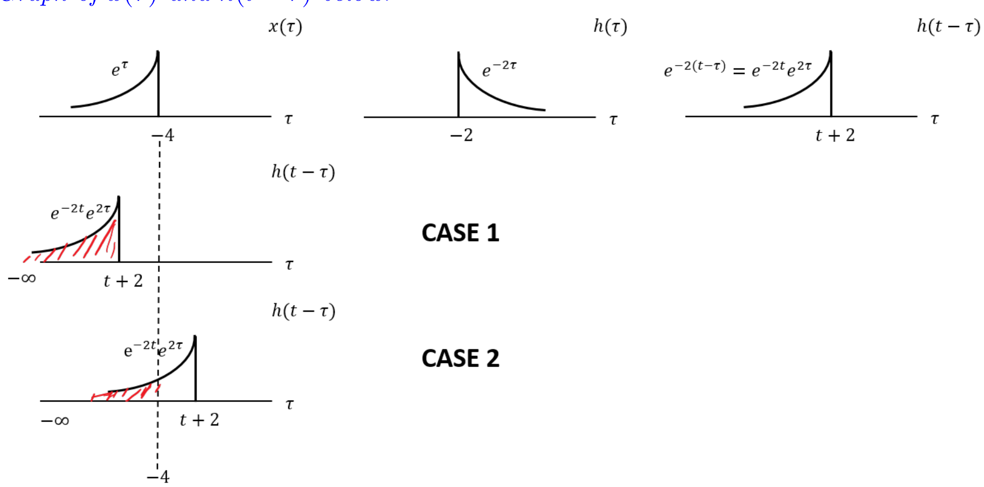
$$y(t) = \int_{-\infty}^{1} (3)(\tfrac{1}{2}e^{-t/2}e^{\tau/2})d\tau = \tfrac{3}{2}e^{-t/2}\int_{-\infty}^{1} e^{\tau/2}d\tau = 3e^{-t/2}[e^{1/2} - 0] = 3e^{-(t-1)/2}$$

In summary: $\boxed{y(t) = \begin{cases} 3, & t < 1 \\ 3e^{-(t-1)/2}, & t > 1 \end{cases}}$

(b) (4 pts) $x(t) = e^{t}u(-4-t)$ and $h(t) = e^{-2t}u(t+2)$.

Solution. $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Graph of $x(\tau)$ *and* $h(t-\tau)$ *below:*



Case 1: $-\infty < t < -6$

Overlap between $\tau = -\infty$ *and* $\tau = t+2$

$$y(t) = \int_{-\infty}^{t+2} (e^{\tau})(e^{-2t}e^{2\tau})d\tau = e^{-2t}\int_{-\infty}^{t+2} e^{3\tau}d\tau = \tfrac{1}{3}e^{-2t}[e^{3(t+2)} - 0] = \tfrac{1}{3}e^{t+6}$$

Case 2: $-6 < t < \infty$

Overlap between $\tau = -6$ *and* $\tau = \infty$

$$y(t) = \int_{-\infty}^{-4} (e^{\tau})(e^{-2t}e^{2\tau})d\tau = e^{-2t}\int_{-\infty}^{-4} e^{3\tau}d\tau = \tfrac{1}{3}e^{-2t}[e^{-12} - 0] = \tfrac{1}{3}e^{-2t-12}$$

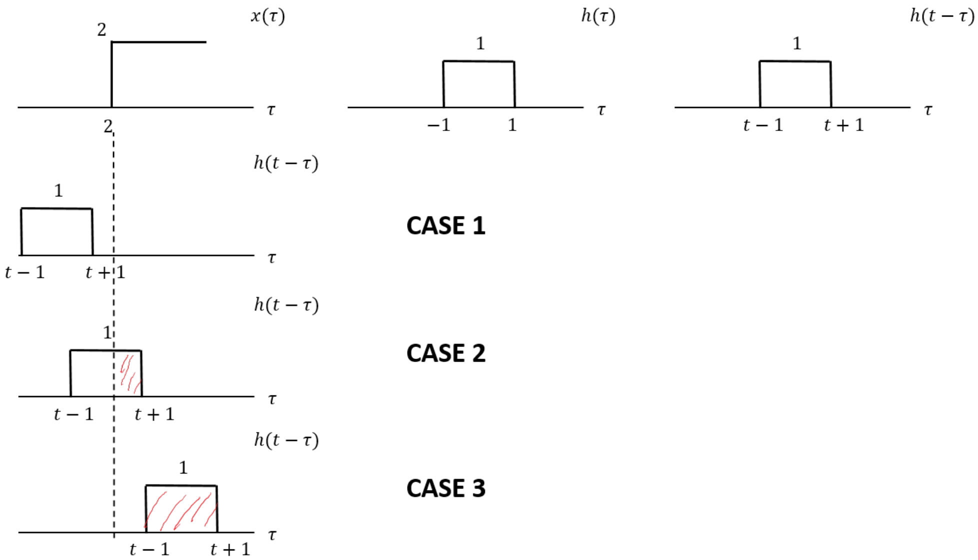
Summarizing output:

$$\boxed{y(t) = \begin{cases} \frac{1}{3}e^{-2t-12}, & t > -6 \\ \frac{1}{3}e^{t+6}, & t < -6 \end{cases}}$$

(c) (2 pts) $x(t) = 2u(t-2)$, $h(t) = u(t+1) - u(t-1)$

Show that $y(t) = \begin{cases} 0, & t < 1 \\ 2[t-1], & 1 < t < 3 \\ 4, & t > 3 \end{cases}$

Solution. $y(t) = \int\limits_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$



Case 1: $t < 1$

No overlap $\longrightarrow y(t) = 0$

Case 2: $1 < t < 3$

$y(t) = \int\limits_{2}^{t+1} (2)(1)d\tau = 2[t+1-2] = 2[t-1]$

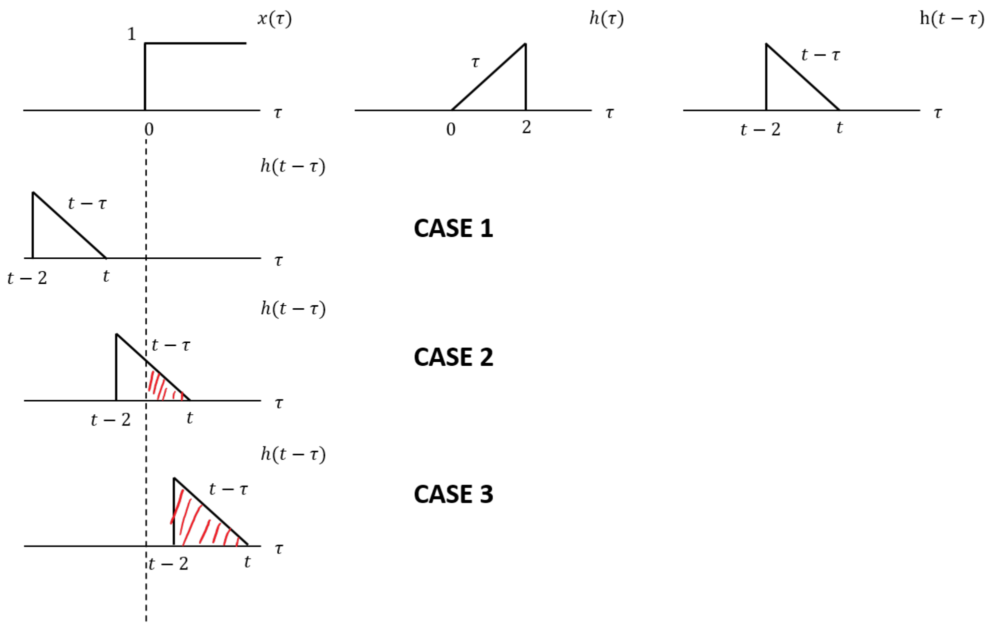
Case 3: $t > 3$

$y(t) = \int\limits_{t-1}^{t+1} (2)(1)d\tau = 2[t+1-t+1] = 4$

In summary: $\boxed{y(t) = \begin{cases} 0, & t < 1 \\ 2[t-1], & 1 < t < 3 \\ 4, & t > 3 \end{cases}}$

(d) (4 pts) $x(t) = u(t)$, $h(t) = t[u(t) - u(t-2)]$.

Solution. $y(t) = \int\limits_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$



Case 1: $-\infty < t < 0$

No overlap $\rightarrow y(t) = 0$

Case 2: $0 < t < 2$

$$y(t) = \int_0^t (t-\tau)(1)d\tau = t\tau - \tfrac{1}{2}\tau^2\Big|_0^t = t^2 - \tfrac{1}{2}t^2 = \tfrac{1}{2}t^2$$

Case 3: $2 < t < 8$

$$y(t) = \int_{t-2}^{t} (t-\tau)(1)d\tau = t\tau - \tfrac{1}{2}\tau^2\Big|_{t-2}^{t} = [t^2 - \tfrac{1}{2}t^2] - [t(t-2) - \tfrac{1}{2}(t-2)^2]$$

$$y(t) = \tfrac{1}{2}t^2 - [t^2 - 2t - \tfrac{1}{2}(t^2 - 4t + 4)] = \tfrac{1}{2}t^2 - [t^2 - 2t - \tfrac{1}{2}t^2 + 2t - 2] = 2$$

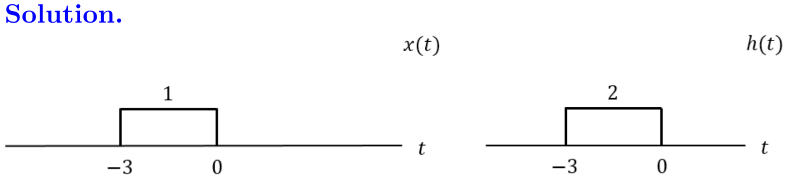
In summary: $\boxed{y(t) = \begin{cases} 2, & t > 2 \\ \frac{1}{2}t^2, & 0 < t < 2 \\ 0, & t < 0 \end{cases}}$

3. *Convolution Properties*

   (a) Given an input $x(t) = u(t+3) - u(t)$ and impulse response $h(t) = 2[u(t+3) - u(t)]$, answer the following questions:

      i. (2 pts) Without calculating output $y(t)$, deduce the start time, end time, and time width of $y(t)$.
      Hint: Plot out $x(t)$ and $h(t)$.
      Show that start = -6, end = 0, width = 6

Solution.



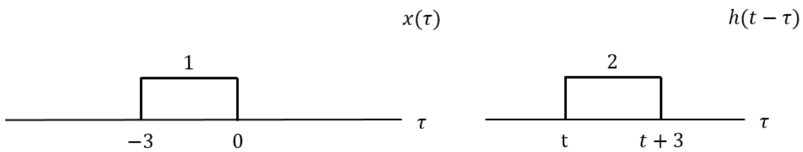
$Start = -3 + (-3) = \boxed{-6}$

$End = 0 + 0 = \boxed{0}$

$Width = 3 + 3 = \boxed{6}$

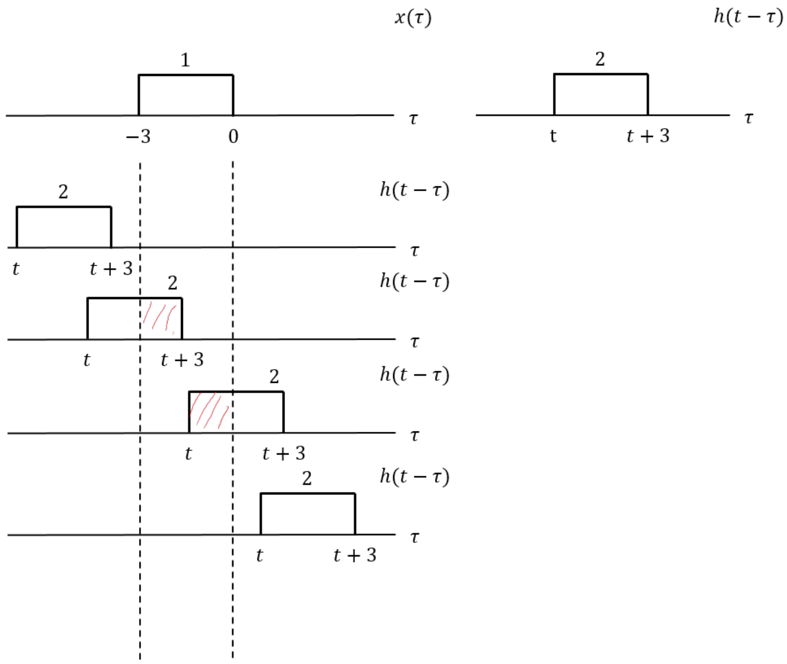
ii. (2 pts) Draw a rough sketch of $x(\tau)$ vs. τ and $h(t-\tau)$ vs. τ. Make sure to clearly label all critical times and amplitudes on your plots.

Solution.



iii. (2 pts) Without calculating the output, what are the critical time regions for output $y(t)$? For each time region, specify whether the output $y(t)$ is CONSTANT, INCREASING, or DECREASING? Make sure to JUSTIFY YOUR ANSWERS.
Hint: There should be 4 critical regions.

Solution.



Case 1: $t < -6$*, CONSTANT b/c no overlap*
Case 2: $-6 < t < -3$*, INCREASING b/c growing overlap*
Case 3: $-3 < t < 0$*, DECREASING b/c shrinking overlap*
Case 4: $t > 0$*, CONSTANT b/c no overlap*

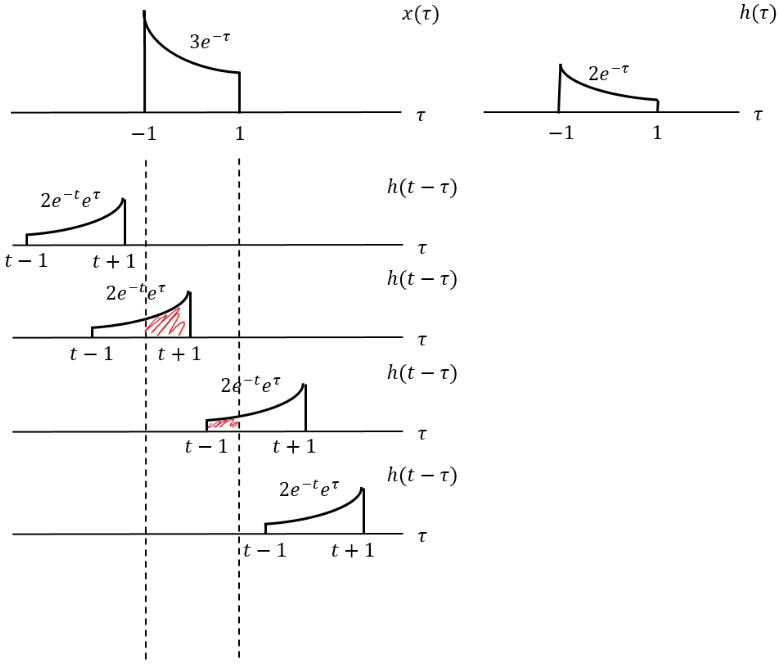
(b) Given an input $x(t) = 3e^{-t}[u(t+1)-u(t-1)]$ and impulse response $h(t) = 2e^{-t}[u(t+1)-u(t-1)]$, answer the following questions:

i. (4 pts) Without calculating output $y(t)$, deduce the start time, end time, and time width of $y(t)$

Solution. $x(t)$ *range: [-1, 1]*
$h(t)$ *range: [-1, 1]*
$Start = -1 - 1 = \boxed{-2}$
$End = 1 + 1 = \boxed{2}$
$Start = 2 + 2 = \boxed{4}$

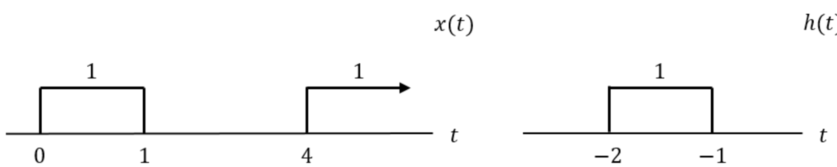
ii. (4 pts) Without calculating the output, what are the critical time regions for output $y(t)$? For each time region, specify whether the output $y(t)$ is CONSTANT, INCREASING, or DECREASING? Make sure to JUSTIFY YOUR ANSWERS.

Solution.



Case 1: $t < -2$, CONSTANT b/c no overlap
Case 2: $-2 < t < 0$, INCREASING b/c growing overlap
Case 3: $0 < t < 2$, DECREASING b/c shrinking overlap
Case 4: $t > 2$, CONSTANT b/c no overlap

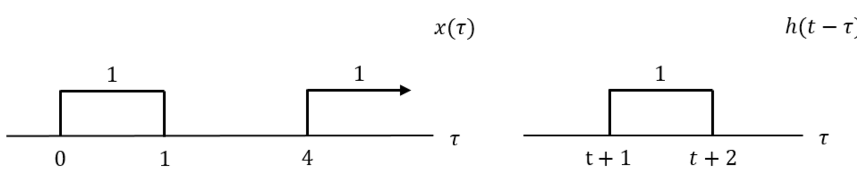
(c) Suppose an LTI system has the following input signal $x(t)$ and impulse response $h(t)$ below:



Answer the following questions:

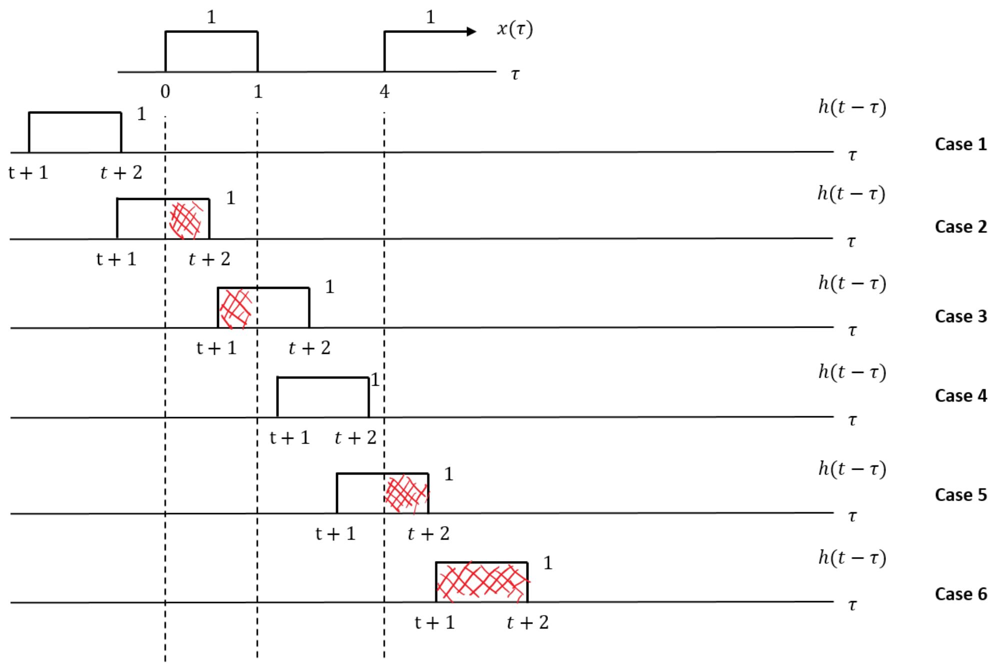
i. (4 pts) Draw a rough sketch of $x(\tau)$ vs. τ and $h(t-\tau)$ vs. τ. Make sure to clearly label all critical times and amplitudes on your plots.

Solution.



ii. (4 pts) Without calculating the output, what are the critical time regions for output $y(t)$?

For each time region, specify whether the output $y(t)$ is CONSTANT, INCREASING, or DECREASING? Make sure to JUSTIFY YOUR ANSWERS.



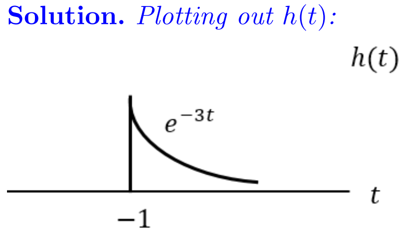
Solution.
Identifying critical time regions:
Case 1: $-\infty < t < -2$, *CONSTANT b/c no overlap*
Case 2: $-2 < t < -1$, *INCREASING b/c growing overlap*
Case 3: $-1 < t < 0$, *DECREASING b/c shrinking overlap*
Case 4: $0 < t < 2$, *CONSTANT b/c no overlap*
Case 5: $2 < t < 3$, *INCREASING b/c increasing overlap*
Case 6: $3 < t < \infty$, *CONSTANT b/c same amount of overlap*

4. *LTI System Properties*

   (a) Suppose you are given the following impulse response of an LTI system. Using your causality test from Ch.2 and last week's lectures, is this LTI system causal or not? Justify your answer. *Hint: it might help to plot out* $h(t)$.

      i. (2 pts) $h(t) = e^{-3t}u(1+t)$
      Show that this system IS NOT causal.

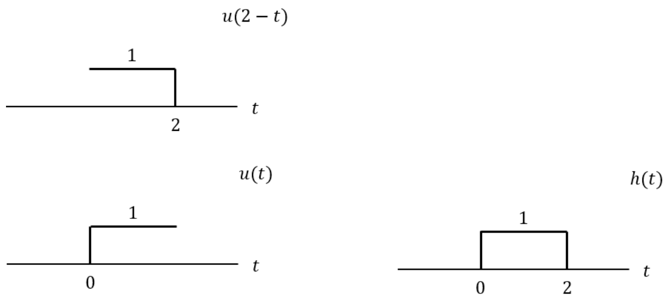
      Solution. *Plotting out* $h(t)$*:*



      Since $h(t) \neq 0$ *for all* $t < 0$*, this system IS NOT causal.*

      ii. (2 pts) $h(t) = u(2-t)u(t)$
      Show that this system IS causal.

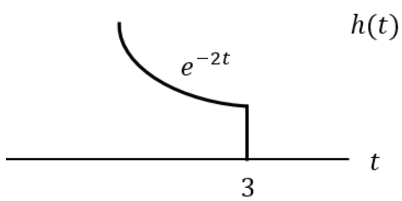
Solution. *Plotting out $h(t)$:*



Since $h(t) = 0$ for all $t < 0$, this system IS causal.

iii. (4 pts) $h(t) = e^{-2t}u(-t + 3)$

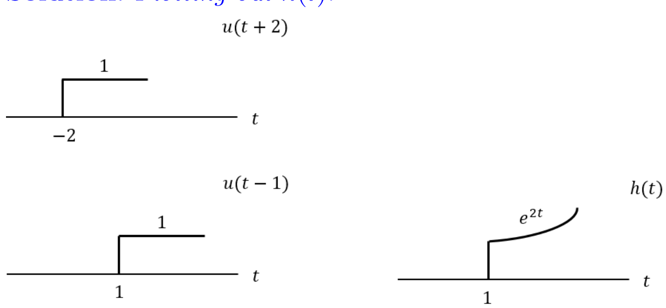
Solution. *Plotting out $h(t)$:*



Since $h(t) \neq 0$ for all $t < 0$, this system IS NOT causal.

iv. (4 pts) $h(t) = e^{2t}u(t + 2)u(t - 1)$

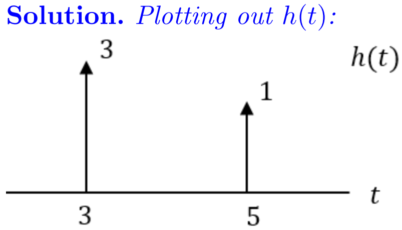
Solution. *Plotting out $h(t)$:*



Since $h(t) = 0$ for all $t < 0$, this system IS causal.

v. (2 pts) $h(t) = \delta(t - 5) + 3\delta(t - 3)$
Show that this system IS causal.

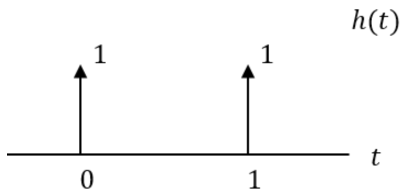
Solution. *Plotting out $h(t)$:*



Since $h(t) = 0$ for all $t < 0$, this system IS causal.

vi. (4 pts) $h(t) = \frac{d}{dt}[u(t - 1) + u(t)]$

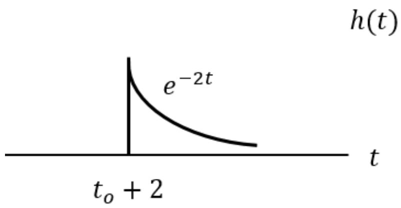
Solution. *Rewriting $h(t)$ using properties of the unit step: $h(t) = \delta(t - 1) + \delta(t)$*
Plotting out $h(t)$:



Since $h(t) = 0$ for all $t < 0$, this system IS causal.

(b) (4 pts) Suppose an LTI system has the impulse response $h(t) = e^{-t/2}u(t - t_o - 2)$. For what values of t_o will the system be causal?

Solution. *Drawing a general plot of $h(t)$: note the critical point of $h(t)$ is at $t - t_o - 2 = 0 \longrightarrow t = t_o + 2$*



We require that $h(t) = 0$ for $t < 0$ in order to be causal, so that means we need $t_o + 2 \geq 0$, so we need $\boxed{t_o \geq -2}$

(c) Suppose you are given the following impulse response of an LTI system. Using your stability test from Ch.2 and last week's lectures, is this LTI system stable or not? Justify your answer by computing or finding an upper bound to $\int_{-\infty}^{\infty} |h(t)|dt$.

i. (2 pts) $h(t) = e^{t/2}u(3 - t)$

Show that $\int_{-\infty}^{\infty} |h(t)|dt = 2e^{3/2}$, and hence we conclude the system IS stable.

Solution.

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} \left|e^{t/2}u(-t+3)\right| dt = \int_{-\infty}^{3} \left|e^{t/2}\right| dt = \int_{-\infty}^{3} e^{t/2}dt = 2[e^{3/2} - e^{-\infty}] = 2e^{3/2}$$

Since $\int_{-\infty}^{\infty} |h(t)|dt < \infty$, we can conclude that this system IS stable.

ii. (4 pts) $h(t) = e^{t}u(t + 1)$

Solution.

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} |e^{t}u(t+1)|\, dt = \int_{-1}^{\infty} |e^{t}|\, dt = \int_{-1}^{\infty} e^{t}dt = e^{\infty} - e^{-1} \to \infty$$

Since $\int_{-\infty}^{\infty} |h(t)|dt \to \infty$, we can conclude that this system IS NOT stable.

iii. (2 pts) $h(t) = e^{-(t-1)/3}u(1 - t)$

Show that $\int_{-\infty}^{\infty} |h(t)|dt \to \infty$, and hence we conclude the system IS NOT stable.

Solution.

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} \left|e^{-(t-1)/3}u(1-t)\right| dt = \int_{-\infty}^{1} \left|e^{-(t-1)/3}\right| dt = \int_{-\infty}^{1} e^{-(t-1)/3}dt = \int_{-\infty}^{1} e^{-t/3}e^{1/3}dt = -\tfrac{1}{3}e^{1/3}[e^{-1/3} - e^{\infty}] \to \infty$$

Since $\int_{-\infty}^{\infty} |h(t)|dt \to \infty$, we can conclude that this system IS NOT stable.

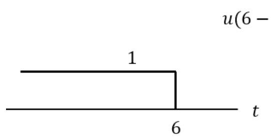
iv. (4 pts) $h(t) = e^{3t}u(-3-t)$

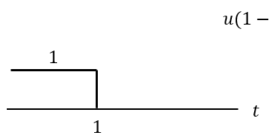
Solution.
$\int\limits_{-\infty}^{\infty} |h(t)|dt = \int\limits_{-\infty}^{\infty} \left|e^{3t}u(-3-t)\right| dt = \int\limits_{-\infty}^{-3} \left|e^{3t}\right| dt = \int\limits_{-\infty}^{-3} e^{3t}dt = \frac{1}{3}[e^{-9} - e^{-\infty}] = \frac{1}{3}e^{-9}$

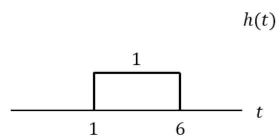
Since $\int\limits_{-\infty}^{\infty} |h(t)|dt < \infty$*, we can conclude that this system IS stable.*

v. (4 pts) $h(t) = u(6-t) - u(1-t)$
Hint: sketch out $h(t)$ *to help you reduce the integral.*

Solution.
$\int\limits_{-\infty}^{\infty} |h(t)|dt = \int\limits_{-\infty}^{\infty} |u(6-t) - u(1-t)|\, dt$
Drawing out $h(t)$*, we get:*







Hence, we can reduce the integral as follows:
$\int\limits_{1}^{6} |1|\, dt = 5$

Since $\int\limits_{-\infty}^{\infty} |h(t)|dt < \infty$*, we can conclude that this system IS stable.*

vi. (2 pts) $h(t) = 2e^{-3t}\sin(5t)[u(t-1) - u(t-3)]$
Show that $\int\limits_{-\infty}^{\infty} |h(t)|dt \leq \frac{2}{3}[e^{-3} - e^{-9}]$, and hence we conclude the system IS stable.

Solution.
$\int\limits_{-\infty}^{\infty} |h(t)|dt = \int\limits_{-\infty}^{\infty} \left|2e^{-3t}\sin(5t)[u(t-1) - u(t-3)]\right| dt = \int\limits_{1}^{3} \left|2e^{-3t}\sin(5t)\right| dt = \int\limits_{1}^{3} \left|2e^{-3t}\right| |\sin(5t)|\, dt =$
$\int\limits_{1}^{3} e^{-3t}dt = \int\limits_{1}^{3} 2e^{-3t}\, |\sin(5t)|\, dt$
Since $|\sin(5t)| \leq 1$*, we can upper bound this integral:*
$\int\limits_{1}^{3} 2e^{-3t}\, |\sin(5t)|\, dt \leq \int\limits_{1}^{3} 2e^{-3t}(1)dt \leq \frac{2}{3}[e^{-9} - e^{-3}] \leq \frac{2}{3}[e^{-3} - e^{-9}]$

Since $\int\limits_{-\infty}^{\infty} |h(t)|dt < \infty$*, we can conclude that this system IS stable.*

vii. (4 pts) $h(t) = e^{(j-1)t}\cos(2\pi t)u(t-1)$

Solution.
$\int\limits_{-\infty}^{\infty} |h(t)|dt = \int\limits_{-\infty}^{\infty} \left|e^{(j-1)t}\cos(2\pi t)u(t-1)\right| dt = \int\limits_{1}^{\infty} \left|e^{(j-1)t}\cos(2\pi t)\right| dt = \int\limits_{1}^{\infty} \left|e^{jt}e^{-t}\right| |\cos(2\pi t)|\, dt =$
$\int\limits_{1}^{\infty} \left|e^{jt}\right| \left|e^{-t}\right| |\cos(2\pi t)|\, dt$
Since $|\cos(2\pi t)| \leq 1$*, we can upper bound this integral:*

$\int_1^{\infty} |e^{jt}| |e^{-t}| |\cos(2\pi t)| \, dt \le \int_1^{\infty} |e^{jt}| |e^{-t}| \, dt$

Using the equation $|x|^2 = xx^*$, *we can rewrite this as* $|x| = \sqrt{xx^*}$ *and use that for the term* $|e^{jt}|$ *as follows:* $|e^{jt}| = \sqrt{e^{jt}e^{-jt}} = \sqrt{1} = 1$

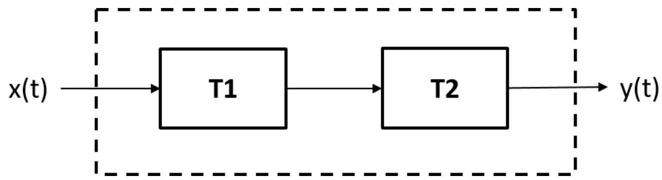
$\int_1^{\infty} |e^{jt}| |e^{-t}| \, dt = \int_1^{\infty} e^{-t} dt = e^{-1} - e^{-\infty} = e^{-1}$

Therefore: $\int_{-\infty}^{\infty} |h(t)| dt \le e^{-1}$

Since $\int_{-\infty}^{\infty} |h(t)| dt < \infty$, *we can conclude that this system IS stable.*

5. *LTI System Descriptions and Interconnections*

(a) Consider the following interconnection of LTI systems $T1$ and $T2$:



Suppose we are given the following descriptions of each system below:

- $T1 : y_1(t) = 3x_1(t+3) - 2x_1(t-1)$
- $T2 : h_2(t) = \delta(t-5)$

Answer the following questions:

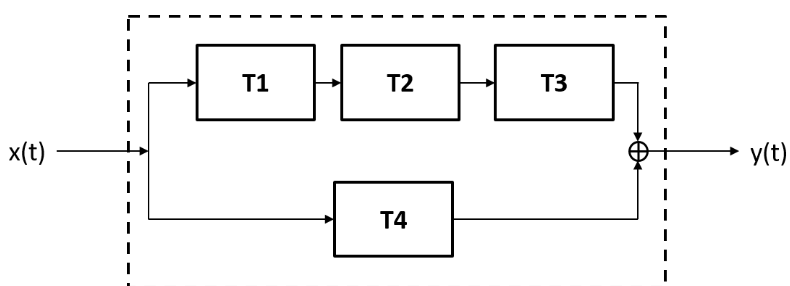
i. (2 pts) What is the impulse response $h_1(t)$ of system $T1$?
Show that $h_1(t) = 3\delta(t+3) - 2\delta(t-1)$

Solution.
$h_1(t) = T1[\delta(t)] = \boxed{3\delta(t+3) - 2\delta(t-1)}$

ii. (2 pts) Using rules for LTI system interconnections, what is the overall impulse response $h(t)$ of these two systems?
Show that $h(t) = 3\delta(t-2) - 2\delta(t-6)$

Solution.
Systems are connected in series, so $h(t) = h_1(t) * h_2(t)$
$h(t) = h_1(t) * \delta(t-5) = h_1(t-5) = 3\delta(t-5+3) - 2\delta(t-5-1) = \boxed{3\delta(t-2) - 2\delta(t-6)}$

(b) (4 pts) Consider the following interconnection of LTI systems $T1$, $T2$, $T3$, and $T4$:



Suppose we are given the following descriptions of each system below:

- $T1: h_1(t) = 4\delta(t+1)$
- $T2: y_2(t) = x(t-2)$
- $T3: y_3(t) = \int\limits_{-\infty}^{t+3} x(\tau - 1)d\tau$
- $T4: h_4(t) = 2u(1+t)$

Using rules for LTI system interconnections, what is the overall impulse response $h(t)$ of these four systems?

Solution. *Overall impulse response can be calculated as follows:* $h(t) = h_1(t) * h_2(t) * h_3(t) + h_4(t)$
Finding $h_2(t)$*:*
$h_2(t) = T2[\delta(t)] = \delta(t-2)$
Finding $h_3(t)$*:*
$h_3(t) = T3[\delta(t)] = \int\limits_{-\infty}^{t+3} \delta(\tau - 1)d\tau = u(\tau - 1)|_{-\infty}^{t+3} = u(t+3-1) - u(-\infty) = u(t+2)$
Therefore:
$h(t) = 4\delta(t+1) * \delta(t-2) * u(t+2) + 2u(1+t) = (4\delta(t+1) * \delta(t-2)) * u(t+2) + 2u(1+t) = 4\delta(t-1) * u(t+2) + 2u(1+t) = 4u(t+1) + 2u(t+1) = \boxed{6u(t+1)}$

6. *Homework Self-Reflection* (10 pts) After completing your homework, go to the following link to rate your skill or concept understanding level for each item listed. Your self-reflection must be completed by the due date. All submissions are time-stamped, so please give yourself plenty of time to complete and submit your self-reflection.

   `http://bit.ly/2Fjw3Uz`