EE 235, Winter 2018, Homework 8: Fourier Transforms, LTI Systems, and Filters
Due Wednesday February 21, 2018 in class via Canvas Submission
Write down *ALL* steps for full credit

HW8 Topics:

- Fourier Transforms: LTI
- LTI Filters

HW8 Course Learning Goals Satisfied:

- Goal 1: Perform convolutions for arbitrary and closed-form continuous-time signals
- Goal 2: Analyze LTI system given different system representations (including input-output equations, impulse response, frequency response) and translate between these representations.
- Goal 3: Use and understand standard EE terminology associated with filtering and LTI systems (e.g. LPF, HPF, impulse response, step response, etc.)

HW8 References: OWN Sections 4.4, 4.7, 3.9.2, 6.1, 6.2.0

HW8 Problems (Total = 84 pts):

1. *Review* (15 pts)

   (a) (5 pts) *LTI Systems.*
   Consider two LTI subsystems that are connected in cascade, where system T1 has step response $s_1(t) = u(t-2) - u(t-6)$ and system T2 has impulse response $h_2(t) = e^{-4t}u(t)$. Find the overall impulse response $h(t)$.

   Solution. $s_1(t) = u(t-2) - u(t-6)$
   $h_2(t) = e^{-4t}u(t)$
   $h(t) = \frac{ds_1(t)}{dt} * h_2(t) = [\delta(t-2) - \delta(t-6)] * e^{-4t}u(t)$
   $\boxed{h(t) = e^{-4(t-2)}u(t-2) - e^{-4(t-6)}u(t-6)}$

   (b) (5 pts) *Fourier Series.*
   The input signal $x(t)$ and the impulse response $h(t)$ of the system is given as follows:
   $x(t) = sin(2t)cos(t) - e^{j3t} + 2$ and $h(t) = \frac{sin(2t)}{t}$
   Using Fourier Series, find the output $y(t)$.

   Solution. *Find w_o and a_k*
   Rewrite $x(t)$: $x(t) = \frac{1}{2}sin(3t) + \frac{1}{2}sin(t) - e^{j3t} + 2$
   $T_o = LCM(\frac{2\pi}{3}, \frac{2\pi}{1}, \frac{2\pi}{3}) = 2\pi \rightarrow w_o = 1$
   Decompose + Inspect:
   $x(t) = \frac{1}{4j}e^{j3t} - \frac{1}{4j}e^{-j3t} + \frac{1}{4j}e^{jt} - \frac{1}{4j}e^{-jt} - e^{j3t} + 2e^{j0t}$
   $= (\frac{1}{4j} - 1)e^{j3t} - \frac{1}{4j}e^{-j3t} + \frac{1}{4j}e^{jt} - \frac{1}{4j}e^{-jt} + 2e^{j0t}$
   $a_3 = \frac{1}{4j} - 1,\ a_{-3} = -\frac{1}{4j},\ a_1 = \frac{1}{4j},\ a_{-1} = -\frac{1}{4j},\ a_0 = 2,\ a_k = 0$ *otherwise.*
   Find $H(jw)$
   $h(t) = \frac{sin(2t)}{t} = \frac{2sin(2t)}{2t} = 2sinc(2t)$
   $H(jw) = 2[\frac{\pi}{2}rect(\frac{w}{4})] = \pi rect(\frac{w}{4})$
   Find b_k: $b_k = a_kH(jkw_o) = a_kH(jk)$
   $b_3 = (\frac{1}{4j} - 1)(0) = 0$
   $b_{-3} = (-\frac{1}{4j})(0) = 0$
   $b_1 = (\frac{1}{4j})(\pi) = \frac{\pi}{4j}$
   $b_{-1} = (-\frac{1}{4j})(\pi) = -\frac{\pi}{4j}$

$b_0 = (2)(\pi) = 2\pi$

Find $y(t)$: synthesize!

$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_o t}$

$y(t) = \frac{\pi}{4j} e^{j(1)(1)t} - \frac{\pi}{4j} e^{j(-1)(1)t} + 2\pi e^{j(0)(1)t}$

$$\boxed{y(t) = \frac{\pi}{4j}(e^{jt} - e^{-jt}) + 2\pi(1) = \frac{\pi}{2} sin(t) + 2\pi}$$

(c) (5 pts) *Parseval's Theorem.*
Let's consider the system in Problem 1-(b). Using Parseval's Theorem, compute the power P_∞ of the output $y(t)$ and the energy E_∞ of the impulse response $h(t)$.

Solution. *For power P_∞ of the output $y(t)$:*
By Parseval's Theorem, $P_\infty = \sum_{k=-\infty}^{\infty} |b_k|^2$.

$P_\infty = |b_{-3}|^2 + |b_3|^2 + |b_{-1}|^2 + |b_1|^2 + |b_0|^2 = \boxed{\frac{33\pi^2}{8}}$

For energy E_∞ of the impulse response $h(t)$:
By Parserval's Theorem, $E_\infty = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(jw)|^2 dw = \frac{1}{2\pi}\int_{-2}^{2} |\pi|^2 dw = \frac{1}{2\pi}(2\pi^2 - (-2)\pi^2) = \boxed{2\pi}$.

2. *Fourier Transform: Frequency Response* (15 pts)

(a) (5 pts) Let's consider the LTI system with the impulse response $h(t) = 5e^{-3t}u(t)$. And the input to this LTI system is $x(t) = e^{-2t}u(t)$. Find $Y(jw)$ and then take the inverse transform to find $y(t)$.

Solution. $Y(j\omega) = H(j\omega)X(j\omega)$

$H(j\omega) = \frac{5}{j\omega+3}$

$X(j\omega) = \frac{1}{j\omega+2}$

$$\boxed{Y(j\omega) = \frac{5}{(j\omega+3)(j\omega+2)}}$$

By Partial Fraction Expansion,

$Y(j\omega) = \frac{5}{j\omega+2} - \frac{5}{j\omega+3}$

$$\boxed{y(t) = 5e^{-2t}u(t) - 5e^{-3t}u(t)}$$

(b) (5 pts) The impulse response and the output are given as follows:
$H(jw) = \frac{1}{5+jw}$ and $y(t) = e^{-4t}u(t) - e^{-5t}u(t)$.
Find input $x(t)$.

Solution. $Y(j\omega) = H(j\omega)X(j\omega)$

$X(j\omega) = \frac{Y(j\omega)}{H(j\omega)}$

$Y(j\omega) = \frac{1}{j\omega+4} - \frac{1}{j\omega+5} = \frac{j\omega+5-j\omega-4}{(j\omega+4)(j\omega+5)} = \frac{1}{(j\omega+4)(j\omega+5)}$

$X(j\omega) = \frac{\frac{1}{(j\omega+4)(j\omega+5)}}{\frac{1}{j\omega+5}} = \frac{1}{j\omega+4}$

$$\boxed{x(t) = e^{-4t}u(t)}$$

(c) (5 pts) Let's consider the LTI system with the impulse response $h(t) = \frac{4}{\pi} sinc(2(t-1))$.

i. Find the frequency response $H(jw)$.

Solution. *Let $h(t) = h_1(t-1)$, where $h_1(t) = \frac{4}{\pi} sinc(2t)$*

$h_1(t) = \frac{4}{\pi} sinc(2t)$

$H_1(j\omega) = 2rect(\frac{\omega}{2(2)}) = rect(\frac{\omega}{4})$

$H(j\omega) = e^{-j\omega} H_1(j\omega)$

$\boxed{H(j\omega) = 2e^{-j\omega} rect(\frac{\omega}{4})}$

ii. Find the output $y(t)$ when input is $x(t) = \sin(t)$.

Solution. $y(t) = h(t) * x(t)$
$Y(j\omega) = H(j\omega)X(j\omega)$
$x(t) = \sin(t)$
$X(j\omega) = \frac{\pi}{j}(\delta(\omega-1) - \delta(\omega+1))$
$Y(j\omega) = 2e^{-j\omega} rect(\frac{\omega}{4})(\frac{\pi}{j}(\delta(\omega-1) - \delta(\omega+1)))$
$Y(j\omega) = 2e^{-j\omega}(\frac{\pi}{j}(\delta(\omega-1) - \delta(\omega+1)))$
$Y(j\omega) = 2e^{-j\omega} Y_1(j\omega)$
$Y_1(j\omega) = \frac{2\pi}{j}(\delta(\omega-1) - \delta(\omega+1))$
$y_1(t) = 2\sin(t)$
$\boxed{y(t) = 2\sin(t-1)}$

3. *Fourier Transform: LTI Systems Described by LCCDE.* (32 pts)

(a) Consider the causal LTI system represented by its input-output relationship: $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = -x(t)$.

i. (4 pts) Find the frequency response $H(jw)$.

Solution. $(jw)^2 Y(jw) + 4(jw)Y(jw) + 3Y(jw) = -X(jw)$
$H(jw) = \boxed{\frac{-1}{3 + 4jw + (jw)^2}}$

ii. (4 pts) Find the impulse response $h(t)$.

Solution. $H(jw) = \frac{-1}{(3+jw)(1+jw)} = \frac{A}{3+jw} + \frac{B}{1+jw}$
$A = \frac{1}{3+jw}|_{jw=-1} = \frac{1}{3-1} = 0.5$
$B = \frac{1}{1+jw}|_{jw=-3} = \frac{1}{1-3} = -0.5$
$\therefore h(t) = \boxed{0.5e^{-3t}u(t) - 0.5e^{-t}u(t)}$

iii. (4 pts) Find the output $y(t)$ when $x(t) = e^{-2t}u(t)$.

Solution. $X(jw) = \frac{1}{2+jw}$
$Y(jw) = X(jw)H(jw) = \frac{-1}{(3+jw)(2+jw)(1+jw)}$
$Y(jw) = \frac{A}{3+jw} + \frac{B}{2+jw} + \frac{C}{1+jw}$
$A = \frac{-1}{(2+jw)(1+jw)}|_{jw=-3} = \frac{-1}{(2-3)(1-3)} = \frac{-1}{(-1)(-2)} = -0.5$
$B = \frac{-1}{(3+jw)(1+jw)}|_{jw=-2} = \frac{-1}{(3-2)(1-2)} = \frac{-1}{-1} = 1$
$C = \frac{-1}{(3+jw)(2+jw)}|_{jw=-1} = \frac{-1}{(3-1)(2-1)} = \frac{-1}{(2)(1)} = -0.5$
Hence, $y(t) = -0.5e^{-3t}u(t) + e^{-2t}u(t) - 0.5e^{-t}u(t)$.

(b) A causal LTI system is described by the following differential equation:

$$\frac{dy(t)}{dt} + 4y(t) = 9x(t).$$

i. (4 pts) Find the frequency response $H(j\omega)$ of this system.

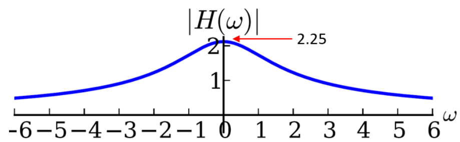
Solution.

$$j\omega Y(j\omega) + 4Y(j\omega) = 9X(j\omega)$$
$$(j\omega + 4)Y(j\omega) = 9X(j\omega)$$
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \boxed{\frac{9}{4 + j\omega}}$$

ii. (4 pts) Find the magnitude of the frequency response, $|H(j\omega)|$.

Solution. $|H(j\omega)| = \boxed{\dfrac{9}{\sqrt{16 + \omega^2}}}$

iii. (4 pts) Sketch the magnitude of the frequency response (for both positive and negative ω).



Solution.

iv. (4 pts) Classify this system as low-pass/high-pass/band-pass/band-stop.

Solution. $\boxed{\textit{Lowpass}}$

v. (4 pts) Find the impulse response $h(t)$ of this system.

Solution. $h(t) = \boxed{9e^{-4t}u(t)}$

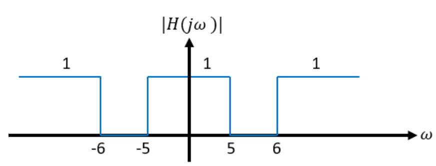
4. *Fourier Transforms: LTI Filters.* (12 pts)

An LTI system is described by the following equation: $y(t) = x(t) - x(t) * h_1(t)$, where $h_1(t)$ is an ideal BPF with gain A $= 1$ and cutoff frequencies $w_l = 5$ and $w_u = 6$.

(a) (4 pts) What is the overall frequency response $H(jw)$ in terms of $H_1(jw)$?

Solution. $y(t) = x(t) - x(t) * h_1(t)$
$Y(j\omega) = X(j\omega) - X(j\omega)H_1(j\omega) = (1 - H_1(j\omega))X(j\omega) = H(j\omega)X(j\omega)$
$H(j\omega) = \boxed{1 - H_1(j\omega)}$

(b) (4 pts) Sketch $|H(jw)|$.



Solution.

(c) (4 pts) Classify filter type of this system. Show, or explain why, this is a bandstop filter (BSF).

Solution. *This system is a* $\boxed{\textit{bandstop filter}}$ *because as can be seen from the figure above, the magnitude response of filter is zero for* $-6 \le \omega \le -5$ *and* $5 \le \omega \le 6$.

5. *Homework Self-Reflection*
(10 pts) After completing your homework, go to the following link to rate your skill or concept understanding level for each item listed. Your self-reflection must be completed by the due date. All submissions are time-stamped, so please give yourself plenty of time to complete and submit your self-reflection.
http://bit.ly/2G8x0zL