EE 235, Winter 2018, Homework 6: Fourier Series
Due Friday February 9, 2018 via Canvas Submission
Write down $\boldsymbol{ALL}$ steps for full credit

HW6 Topics:

- Fourier Series: Analysis, Synthesis, Properties, and LTI

HW6 Course Learning Goals Satisfied:

- Goal 1: Describe signals in different domains (time, frequency, and Laplace) and map characteristics in one domain to those in another.
- Goal 4: Analyze LTI system given different system representations (including input-output equations, impulse response, frequency response) and translate between these representations.

HW6 References: OWN Sections 3.2 - 3.5, 3.8

HW6 Problems (Total = 104 pts):

1. *Review* (16 pts)

   (a) *Partial Fraction Expansion.* (2 pts)
   $X(s) = \frac{-5}{s^2+2s+2}$. Using the cover-up method, show that $X(s) = -\frac{\frac{5}{2}j}{s+1+j} + \frac{\frac{5}{2}j}{s+1-j}$.
   Solution. *Finding roots of denominator:*
   $s = -\frac{-2\pm\sqrt{4-4(2)(1)}}{2} = -1 \pm j$
   Rewriting $X(s)$ as partial fractions:
   $X(s) = \frac{-5}{s^2+2s+2} = \frac{-5}{(s+1+j)(s+1-j)} = \frac{A}{s+1+j} + \frac{B}{s+1-j}$
   Finding A:
   $A = \frac{-5}{(s+1+j)}\Big|_{s=-1-j} = \frac{-5}{-1-j+1-j} = \frac{5}{2j} = -\frac{5}{2}j$
   Finding B:
   $B = A^* = \frac{5}{2}j$

   (b) *Partial Fraction Expansion.* (4 pts)
   $X(s) = \frac{s+2}{s(s+1)^2(s+5)}$. Using the cover-up method.
   Solution. *Rewriting $X(s)$ as partial fractions:*
   $X(s) = \frac{A}{s} + \frac{B1}{s+1} + \frac{B2}{(s+1)^2} + \frac{C}{s+5}$
   Finding A:
   $A = \frac{s+2}{(s+1)^2(s+5)}\Big|_{s=0} = \frac{2}{1(5)} = \frac{2}{5}$
   Finding C:
   $C = \frac{s+2}{(s)(s+1)^2}\Big|_{s=-5} = \frac{-3}{(-5)(16)} = \frac{3}{80}$
   Finding B2:
   $B2 = \frac{s+2}{(s)(s+5)}\Big|_{s=-1} = \frac{1}{(-1)(4)} = -\frac{1}{4}$
   Finding B1:
   $B1 = \frac{d}{ds}\left[\frac{s+2}{(s)(s+5)}\right]\Big|_{s=-1} = \frac{(s^2+5s)(1)-(s+2)(2s+5)}{(s^2+5s)^2}\Big|_{s=-1} = \frac{(1-5)(1)-(-1+2)(-2+5)}{(1-5)^2} = \frac{-4-3}{16} = -\frac{7}{16}$

   (c) *Magnitude and Phase Equation.* (2 pts)
   Let $X = \frac{1}{\alpha-j\omega}$, where $\alpha > 0$. Evaluate the magnitude and phase of X. Show that $|X| = \frac{1}{\sqrt{\alpha^2+\omega^2}}$ and $\angle X = tan^{-1}(\frac{\omega}{\alpha})$.
   Solution. *Magnitude: we can evaluate the magnitude by taking the magnitude of the numerators and denominators separately*
   $|X| = \frac{|1|}{|\alpha-j\omega|} = \frac{1}{\sqrt{(\alpha^2+(-\omega)^2}} = \frac{1}{\sqrt{(\alpha^2+\omega^2)}}$

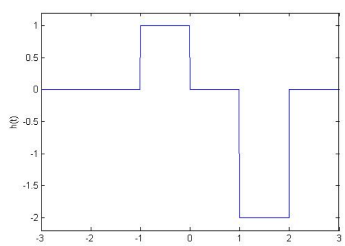
Phase: we can evaluate phase by taking the phase of the numerator minus the phase of the denominator
$\angle(X) = \angle(1) - \angle(\alpha - j\omega) = \arctan(\frac{0}{1}) - \arctan(\frac{-\omega}{\alpha}) = 0 - (-\arctan(\frac{\omega}{\alpha})) = \arctan(\frac{\omega}{\alpha})$.

(d) *Signal Properties.* (4 pts)
$x(t) = \cos(t + \frac{\pi}{3})$. Evaluate P_∞. Is $x(t)$ a power signal?
Show that the power of the signal is $P_\infty = \frac{1}{2}$, thus making $x(t)$ a power signal.
Hint: If $x(t)$ is periodic with fundamental period T_o, the power of $x(t)$, P_∞ is equivalent to the average power of $x(t)$ over any interval of length T_o: $P = \frac{1}{T_o}\int_0^{T_o} |x(t)|^2 dt$.

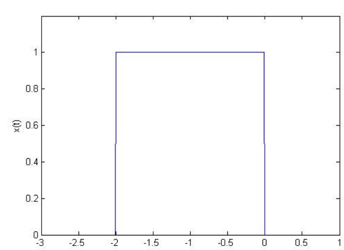
Solution. $P_\infty = \frac{1}{T_o}\int_0^{T_o} \cos^2(t + \frac{\pi}{3})dt$
$P_\infty = \frac{1}{2\pi}\int_0^{2\pi} \cos^2(t + \frac{\pi}{3})dt$
$P_\infty = \frac{1}{2\pi}\left[\frac{t}{2} + \frac{\cos(2(t+\frac{\pi}{3}))}{4}\right]\Big|_0^{2\pi}$
$P_\infty = \frac{1}{2\pi}\left[\frac{2\pi}{2} + \frac{\cos(2(2\pi+\frac{\pi}{3}))}{4} - 0 - \frac{\cos(2(0+\frac{\pi}{3}))}{4}\right]$
$P_\infty = \frac{1}{2}$
$P_\infty < \infty$, *so $x(t)$ is a power signal.*

(e) *Convolution.* (4 pts)
Let T denote an LTI system with impulse response $p(t+1) - 2p(t-1)$, then find and sketch $y(t) = T[p(-t/2)]$.
Hint: The pulse signal $p(t) = u(t) - u(t-1)$.

Solution. *(We have sketched all the different cases here, but you don't need to give all these sketches in your solution as long as you work out the convolution correctly.)*
The impulse response of the system is:



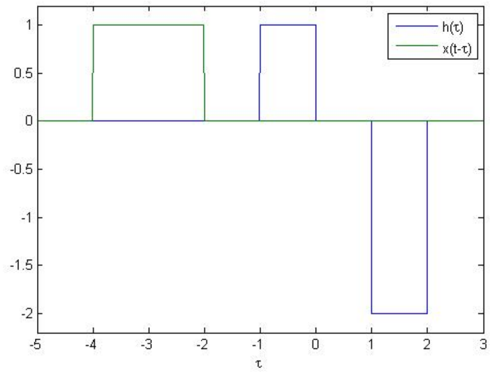
The input signal is:



The output signal is the convolution of the input signal and the impulse response:

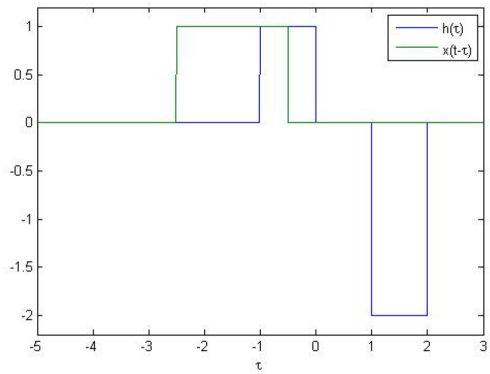
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} (p(\tau+1) - 2p(\tau-1))\, p(\frac{\tau - t}{2})d\tau$$

When $t < -3$, as shown in the following figure, $y(t) = 0$ because there is not any overlap region between $h(t)$ and $x(t)$.



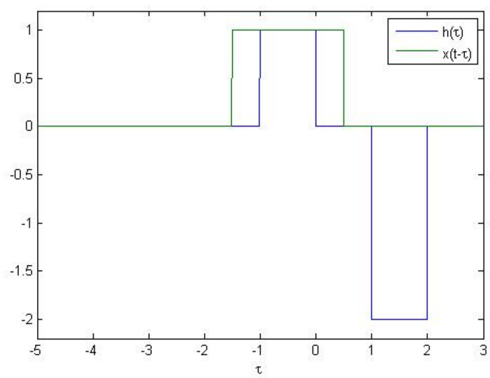
When $-3 \leq t < -2$, as shown in the following figure, the system output $y(t)$ is:

$$y(t) = \int_{-1}^{t} 1 d\tau = t + 3$$



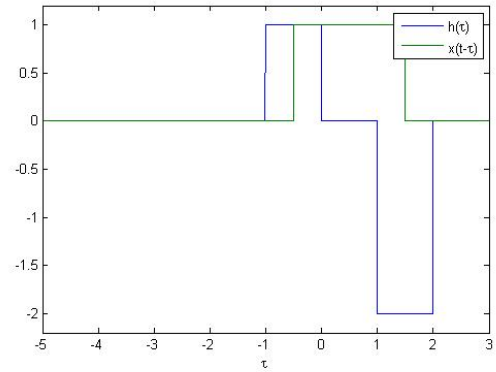
When $-2 \leq t < -1$, as shown in the following figure, the system output $y(t)$ is:

$$y(t) = \int_{-1}^{0} 1 d\tau = 1$$



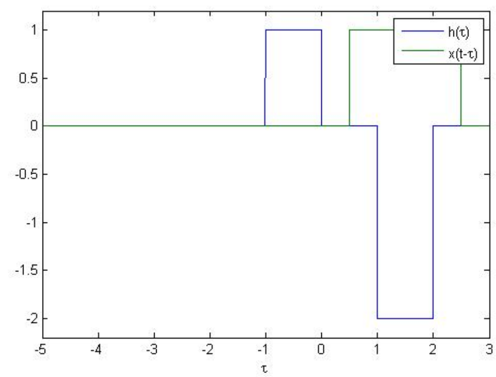
When $-1 \leq t < 0$, as shown in the following figure, the system output $y(t)$ is:

$$y(t) = \int_{t}^{0} 1 d\tau + \int_{1}^{t+2} -2 d\tau = -3t - 2$$



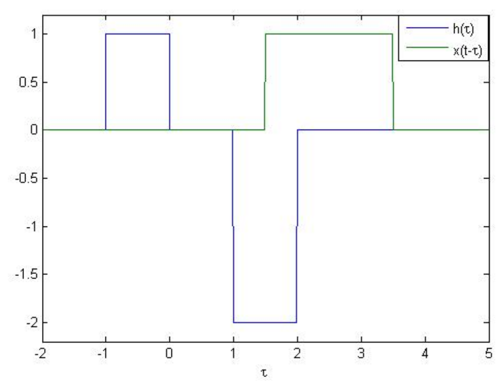
When $0 \leq t < 1$, as shown in the following figure, the system output $y(t)$ is:

$$y(t) = \int_1^2 -2d\tau = -2$$



When $1 \leq t < 2$, as shown in the following figure, the system output $y(t)$ is:

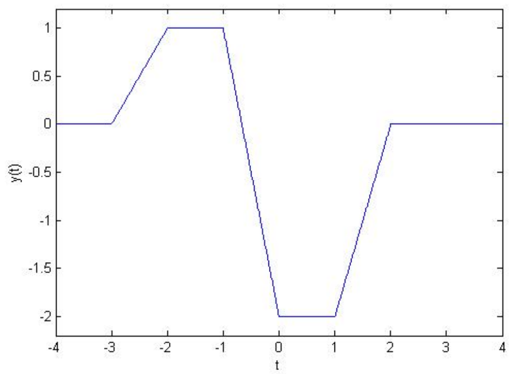
$$y(t) = \int_t^2 -2d\tau = 2t - 4$$



When $t \geq 2$, $y(t) = 0$ because there is not any overlap between the input signal and the system *impulse response.*

$$y(t) = T[p(-t/2)] = \begin{cases} t+3, & -3 \le t < -2 \\ 1, & -2 \le t < -1 \\ -3t-2, & -1 \le t < 0 \\ -2, & 0 \le t < 1 \\ 2t-4, & 1 \le t < 2 \\ 0, & \text{o.w.} \end{cases}$$

The plot is sketched as follows:



2. *LTI System Description* (14 pts)
   Consider the following three LTI systems:

   - T_1: Has impulse response $h_1(t) = \begin{cases} 2, & -1 < t < 1 \\ 0, & otherwise \end{cases}$
   - T_2: Has step response $s_2(t) = 2u(t-1)$
   - T_3: Has input-output relationship $y_3(t) = x_3(t+2)$

   (a) (4 pts) What is the step response $s_1(t)$ of system T_1?

   Solution. $s_1(t) = \int_{-\infty}^{t} h_1(\tau)d\tau$
   Case 1: $t < -1$
   $s_1(t) = \int_{-\infty}^{t} h_1(\tau)d\tau$
   Since $h_1(\tau) = 0$ *from* $\tau = -\infty$ *to* $\tau = t$ *for this case,* $s_1(t) = 0$
   Case 2: $-1 < t < 1$
   $s_1(t) = \int_{-\infty}^{t} h_1(\tau)d\tau$
   Since $h_1(\tau) = 2$ *only from* $\tau = -1$ *to* $\tau = t$ *for this case,* $s_1(t) = \int_{-1}^{t} 2d\tau = 2t+2$
   Case 3: $t > 1$
   $s_1(t) = \int_{-\infty}^{t} h_1(\tau)d\tau$
   Since $h_1(\tau) = 2$ *only from* $\tau = -1$ *to* $\tau = 1$ *for this case,* $s_1(t) = \int_{-1}^{1} 2d\tau = 4u(t)$

   (b) (4 pts) What is impulse response $h_2(t)$ of system T_2?

   Solution. $h_2(t) = \frac{ds_2(t)}{dt} = \frac{2u(t-1)}{dt} = 2\delta(t-1)$

   (c) (2 pts) What is the corresponding input-output relationship for system T_2? Show that $y_2(t) = 2x_2(t-1)$.

   Solution. *We can find the input-output relationship using convolution:*
   $y(t) = x_2(t) * h(t) = x_2(t) * 2\delta(t-1) = 2x_2(t-1)$.

   (d) (4 pts) What is the impulse response $h_3(t)$ of system T_3? What is the corresponding step response $s_3(t)$?

Solution. *Using the definition of the impulse response: Let* $x_3(t) = \delta(t)$
$y(t) = T[\delta(t)] = \delta(t+2)$.
Therefore, $h_3(t) = \delta(t+2)$.
For step response:
$s_3(t) = \int_{-\infty}^{t} h_3(\tau)d\tau = \int_{-\infty}^{t} \delta(\tau+2)d\tau = u(t+2) - u(-\infty+2)$
Therefore, $s_3(t) = u(t+2)$.

3. *LTI System Interconnection and Properties* (14 pts)
Consider the same LTI systems in Problem 2. Answer the following questions.

(a) (2 pts) Is system T_2 BIBO stable? Using the impulse response test that we discussed in lecture, show that T_2 is BIBO stable with $\int_{-\infty}^{\infty} |h(t)|dt = 2$.

Solution. $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} |2\delta(t-1)|dt = \int_{-\infty}^{\infty} 2\delta(t-1)dt = 2\int_{-\infty}^{\infty} \delta(t-1)dt = 2(1) = 2$
Because the integral is finite, that means T_2 *is* $\boxed{BIBOstable}$.

(b) (4 pts) Suppose T_1 and T_3 are connected in parallel. Is the overall system causal? Use the impulse response test that we discussed in lecture.

Solution. *In parallel,* $h(t) = h_1(t) + h_3(t) = \delta(t+2) + \begin{cases} 2, & -1 < t < 1 \\ 0, & otherwise \end{cases}$

Because $h(t) \neq 0$ *for all* $t < 0$, *the overall system is* $\boxed{\textit{not causal}}$.

(c) (4 pts) Suppose T_2 and T_3 are connected in parallel. Find the overall impulse response $h(t)$ and then find the overall output $y(t)$ when the input is $x(t) = \sin(t^2)$.

Solution. *In parallel,* $h(t) = h_2(t) + h_3(t) = 2\delta(t-1) + \delta(t+2)$
Computing the output:
$y(t) = x(t) * h(t) = \sin(t^2) * [2\delta(t-1) + \delta(t+2)] = \boxed{2\sin(t^2 - 2t + 1) + \sin(t^2 + 4t + 4)}$.

(d) (4 pts) Suppose T_1 and T_3 are connected in series. Is the system causal? Is the system BIBO stable? Use the impulse response tests.

Solution. *In series,* $h(t) = h_1(t) * h_3(t) = h_1(t) * \delta(t+2) = \begin{cases} 2, & -3 < t < -1 \\ 0, & otherwise \end{cases}$

For causality:
Because $h(t) \neq 0$ *for all* $t < 0$, *the overall system is* $\boxed{\textit{not causal}}$.
For stability:
$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-3}^{-1} |2|dt = 4$
Because the integral is finite, this system is $\boxed{BIBO\ stable}$

4. *Fourier Series: Synthesis.* (4 pts)

(a) (2 pts) $w_o = \frac{\pi}{4}$ and $a_0 = -2$, $a_1 = a_{-1} = 1$, $a_2 = a_{-2}^* = \frac{1}{j}$, $a_3 = 2$, and $a_{-3} = -2$.
Using the synthesis equation, convert back to the time domain and show that:
$x(t) = -2[1 - \cos(\frac{\pi}{4}t) - \sin(\frac{\pi}{2}t) - 2j\sin(\frac{3\pi}{4}t)]$

Solution. $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$
$x(t) = -2e^{-j\frac{3\pi}{4}t} - \frac{1}{j}e^{-j\frac{\pi}{2}t} + e^{j\frac{\pi}{4}t} - 2e^0 + e^{-j\frac{\pi}{4}t} + \frac{1}{j}e^{j\frac{\pi}{2}t} + 2e^{j\frac{3\pi}{4}t}$
$x(t) = -2 + (e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}) + \frac{1}{j}(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}) + 2(e^{j\frac{3\pi}{4}t} - e^{-j\frac{3\pi}{4}t})$
$x(t) = -2[1 - \cos(\frac{\pi}{4}t) - \sin(\frac{\pi}{2}t) - 2j\sin(\frac{3\pi}{4}t)]$

(b) (2 pts) $w_o = \frac{\pi}{4}$ and $a_1 = \frac{1}{2j}e^{j\frac{\pi}{4}}$, $a_{-1} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}$.
Using the synthesis equation, convert back to the time domain and show that:
$x(t) = \sin(\frac{\pi}{4}t + \frac{\pi}{4})$

Solution. $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$
$x(t) = -\frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{4}t} + \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j\frac{\pi}{4}t}$
$x(t) = \frac{1}{2j}e^{j(\frac{\pi}{4}+\frac{\pi}{4}t)} - \frac{1}{2j}e^{-j(\frac{\pi}{4}+\frac{\pi}{4}t)}$
$x(t) = \sin(\frac{\pi}{4}t + \frac{\pi}{4})$

5. *(Fourier Series: Analysis)* (20 pts) In the following problems, we practice analyzing signals and computing their Fourier Series coefficients:

(a) Consider the continuous-time signal $x(t) = 2cos(3t)sin(t) - je^{-j8t} + e^{j6(t-3)}$.

i. (4 pts) Show the Fourier Series representation of $x(t)$ by finding the fundamental frequency ω_0 and its Fourier Series coefficients a_k.
Hint: $cos(A)sin(B) = \frac{1}{2}[sin(A+B) - sin(A-B)]$

Solution. $x(t) = 2cos(3t)sin(t) - je^{j8t} + e^{j6(t-3)} = sin(4t) - sin(2t) - je^{j8t} + e^{j6(t-3)}$
$T_0 = LCM(\frac{2\pi}{4}, \frac{2\pi}{2}, \frac{2\pi}{8}, \frac{2\pi}{6}) = \pi$
$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{pi} = 2.$
$x(t) = \frac{1}{2j}e^{j4t} - \frac{1}{2j}e^{-j4t} - \frac{1}{2j}e^{j2t} + \frac{1}{2j}e^{-j2t} - je^{-j8t} + e^{-j18}e^{j6t}.$
Hence,
$a_{-4} = -j, a_{-2} = -\frac{1}{2j}, a_{-1} = \frac{1}{2j}, a_1 = -\frac{1}{2j}, a_2 = \frac{1}{2j}, a_3 = e^{-j18}, a_k = 0, otherwise.$

ii. (2 pts) Using the result of the previous part, find the power of $x(t)$ using Parseval's Theorem.

Solution. $P_\infty = \sum_{k=-\infty}^{\infty} |a_k|^2 = 1 + \frac{1}{4} + \frac{1}{4} + 1 + \frac{1}{4} + \frac{1}{4} = 3$

(b) Consider a signal $x(t) = \sin(3\pi t) + \cos(2\pi t)$. Answer the following questions about $x(t)$.

i. (1 pt) What the fundamental frequency ω_0 of this signal?
Solution. $T_0 = LCM(\frac{2\pi}{3\pi}, \frac{2\pi}{2\pi}) = 2$. *Therefore* $w_0 = 2\pi/2 = \pi$ *(rad/s).*

ii. (1 pt)Express $x(t)$ as a sum of complex exponentials using Euler's formula.
Solution. $x(t) = \frac{1}{2j}e^{j3\pi t} - \frac{1}{2j}e^{-j3\pi t} + \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}$.

iii. (2 pts) What are the coefficients a_k in the Fourier Series representation of $x(t)$?
Solution. *Synthesis equation:* $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$.
From solution to part (b), and the fact that $\omega_0 = \pi$, *we have:*
$a_3 = 1/2j$, $a_{-3} = -1/2j$, $a_2 = 1/2$, *and* $a_{-2} = 1/2$. *All other* a_k*'s are 0.*

iv. (1 pt) The DC component of a signal is defined as its mean value. What is the DC component of this signal?
Solution. $a_0 = 0$ *so the DC component is 0.*

(c) Consider the periodic signal described by $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-2k) - \frac{1}{4}\sum_{k=-\infty}^{\infty} \delta(t+2k-1)$. Answer the following questions about this signal.

i. (1 pt) Find the fundamental frequency, ω_0.
Solution. *Period* $= 2$, $\omega_0 = \frac{2\pi}{2} = \pi$.

ii. (4 pts) Find the DC value c_0, and the Fourier series coefficients c_k for $k \neq 0$. Specify the value of c_k for even k and odd k.

Solution. *We have,*

$$c_0 = \frac{1}{2}\int_{-1/2}^{3/2}(\delta(t) - \frac{1}{4}\delta(t-1))dt = \frac{1}{2}(1 - 1/4) = 3/8.$$

For a general coefficient c_k, we have:

$$\begin{aligned} c_k &= \frac{1}{2}\int_{-1/2}^{3/2}(\delta(t) - \frac{1}{4}\delta(t-1))e^{-jk\pi t}dt \\ &= \frac{1}{2}(\int_{-1/2}^{3/2}\delta(t)e^{-jk\pi(0)}dt - \frac{1}{4}\int_{-1/2}^{3/2}\delta(t-1)e^{-jk\pi(-1)}dt \\ &= \frac{1}{2}(1 - \frac{1}{4}e^{-jk\pi}) = \frac{1}{2}(1 - \frac{1}{4}(-1)^k) = \begin{cases} 3/8 & \text{if } k \text{ even} \\ 5/8 & k \text{ odd} \end{cases}. \end{aligned}$$

(d) (4 pts) Consider a periodic signal of the form $x(t) = \begin{cases} 1 & \text{if } 0 \leq t < 4 \\ -1 & 4 \leq t < 8 \end{cases}$, with period $T = 8$. Compute the Fourier series coefficients of this signal *using analysis formulas*.

Solution. *We have:*

$$\begin{aligned} a_k &= \frac{1}{8}\int_0^4 e^{-jk(\pi/4)t}dt - \frac{1}{8}\int_4^8 e^{-jk(\pi/4)t}dt \\ &= \frac{1}{8}\frac{e^{-jk(\pi/4)t}}{(-jk(\pi/4))}|_0^4 - \frac{1}{8}\frac{e^{-jk(\pi/4)}}{-jk(\pi/4)}|_4^8. \\ &= \frac{1}{2\pi jk}(1 - e^{-jk\pi}) + \frac{1}{2j\pi k}(e^{-2jk\pi} - e^{-jk\pi}) \\ &= \frac{1}{\pi jk}(1 - e^{-jk\pi}) \\ &= \frac{2e^{-jk(\pi/2)}}{2jk\pi}(e^{jk(\pi/2)} - e^{-jk(\pi/2)}) \\ &= \frac{2e^{-jk(\pi/2)}}{\pi k}\sin(k(\pi/2)) \\ &= e^{-jk\pi/2}\frac{\sin(k\pi/2)}{\pi k/2}. \end{aligned}$$

6. *Fourier Series: Properties* (6 pts)

(a) Let $x(t)$ be a continuous-time periodic signal with fundamental frequency ω and Fourier coefficients a_k. Express the Fourier series coefficients of the following signals (call them b_k) in terms of a_k. *Justify all your statements.* Use the properties from Table 3.1 in the textbook.

i. (2 pts) $y(t) = x(2-t) + x(t-1)$.

Solution. *Apply time reversal and time shift properties to get $b_k = a_k e^{-jk\omega} + a_{-k}e^{-2jk\omega}$.*

ii. (2 pts) $y(t) = \text{Even}(x(t))$.

Solution. *We have, $y(t) = \frac{1}{2}(x(t) + x(-t))$. By properties of amplitude scaling and time reversal, this means we have $b_k = \frac{1}{2}(a_k + a_{-k})$.*

iii. (2 pts) $y(t) = x(3t-1)$.

Solution. *By time-shift property, $b_k = e^{-jk\omega}a_k$. Note, even though the Fourier series coefficients only have a phase shift, the overall Fourier series representation also encodes the new frequency of the signal.*

7. *Fourier Series: Parseval's Relation and LTI* (30 pts)

(a) (2 pts) Suppose a signal $x(t)$ has Fourier Series representation $\omega_o = 2, a_5 = a_{-5} = -\frac{1}{3}, a_3 = a_{-3} = \frac{1}{3}, a_2 = a_{-2}^* = \frac{j}{\pi}$. Compute the average power of the signal $x(t)$.
Show that $P = \frac{4}{9} + \frac{2}{\pi^2}$

Solution.

$P = \sum_{k=-\infty}^{\infty} |a_k|^2 = \left|-\frac{1}{3}\right|^2 + \left|-\frac{1}{3}\right|^2 + \left|\frac{1}{3}\right|^2 + \left|-\frac{1}{3}\right|^2 + \left|\frac{j}{\pi}\right|^2 + \left|-\frac{j}{pi}\right|^2$

$P = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{\pi^2} + \frac{1}{\pi^2} = \boxed{\frac{4}{9} + \frac{2}{\pi^2}}$

(b) (4 pts) Suppose a signal $x(t)$ has Fourier Series representation $\omega_o = 2$ and $a_k = \begin{cases} \frac{1}{2}jk, & |k| < 3 \\ 0, & otherwise \end{cases}$

Compute the average power of the signal $x(t)$.

Solution.

The only nonzero terms in a_k are:

$a_{-2} = -j$

$a_{-1} = -\frac{1}{2}j$

$a_0 = 0$

$a_1 = \frac{1}{2}j$

$a_2 = j$

$P = \sum_{k=-\infty}^{\infty} |a_k|^2 = |-j|^2 + \left|-\frac{1}{2}j\right|^2 + \left|-\frac{1}{2}j\right|^2 + |j|^2$

$P = 1 + \frac{1}{4} + 1 + \frac{1}{4} + 1 = 2 + \frac{1}{2}\boxed{\frac{5}{2}}$

(c) (4 pts) Suppose a signal with Fourier Series representation $\omega_o = 2, a_5 = a_{-5} = -\frac{1}{3}, a_3 = a_{-3} = \frac{1}{3}, a_2 = a_{-2}^* = \frac{j}{\pi}$ is input to an LTI system with frequency response

$$H(\omega) = \begin{cases} \omega, & 0 < \omega < 8 \\ -\omega, & -8 < \omega < 0 \\ 0, & otherwise \end{cases}$$

Find the output Fourier Series coefficients b_k and the corresponding output signal $y(t)$
Show that the final output you get is $y(t) = 4\cos(6t) - \frac{8}{\pi}\sin(4t)$

Solution.

$b_k = H(jk\omega_o)a_k = H(j2k)a_k$

$b_5 = H(j10)a_5 = 0$

$b_{-5} = H(j10)a_5 = 0$

$b_3 = H(j6)a_3 = (6)(1/3) = 2$

$b_{-3} = H(-j6)a_{-3} = (-(-6))(1/3) = 2$

$b_2 = H(j4)a_2 = (4)(j/\pi) = \frac{4j}{\pi}$

$b_{-2} = H(-j4)a_{-2} = (-(-4))(-j/\pi) = -\frac{4j}{\pi}$

$\boxed{b_3 = 2, b_{-3} = 2, b_2 = \frac{4j}{\pi}, b_{-2} = -\frac{4j}{\pi}}$

Synthesize to find output in time:

$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2t} = 2e^{j6t} + 2e^{-j6t} + \frac{4j}{\pi}e^{j4t} - \frac{4j}{\pi}e^{-j4t} = 2[2\cos(6t)] + \frac{4j}{\pi}[2j\sin(4t)] = \boxed{4\cos(6t) - \frac{8}{\pi}\sin(4t)}$

(d) (4 pts) Suppose a signal with Fourier Series representation $\omega_o = 2$ and $a_k = \begin{cases} \frac{1}{2}jk, & |k| < 3 \\ 0, & otherwise \end{cases}$

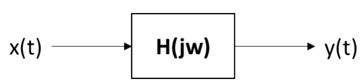
is input to an LTI system with frequency response

$$H(\omega) = \begin{cases} 2, & 10 < |\omega| < 14 \\ 0, & otherwise \end{cases}$$

Find the output Fourier Series coefficients b_k and the corresponding output signal $y(t)$

Solution.
The only nonzero terms in a_k are:
$a_{-2} = -j$
$a_{-1} = -\frac{1}{2}j$
$a_0 = 0$
$a_1 = \frac{1}{2}j$
$a_2 = j$
Calculating output coefficients:
$b_k = H(jk\omega_o)a_k = H(j2k)a_k$
$b_{-2} = H(-j4)a_{-2} = 0(-j) = 0$
$b_{-1} = H(-j2)a_{-1} = 0(-1/2) = 0$
$b_0 = H(j0)a_0 = 0(0) = 0$
$b_1 = H(j2)a_2 = 0(1/2) = 0$
$b_2 = H(j4)a_4 = 0(j) = 0$
$\boxed{b_k = 0 \text{ for all } n}$
Synthesize to find output in time:
$y(t) = \sum_{n=-\infty}^{\infty} Y_n e^{jn6t} = \boxed{0}$

(e) *Fourier Series.*
Consider the following LTI system:



Suppose the input to the system is given by: $x(t) = 2cos(4t) + je^{j2t}$.
Also, suppose the LTI system has the frequency response $H(jw) = \left\{ \begin{array}{ll} 2 & -2 < w < 10 \\ 0 & otherwise \end{array} \right.$.

i. (4 pts) Transform the input signal $x(t)$ into its Fourier Series representation by finding the fundamental frequency w_o and the Fourier Series coefficients a_k.

Solution.

$T_o = LCM(\frac{2\pi}{4}, \frac{2\pi}{2}) = \pi \rightarrow w_o = \frac{2\pi}{T_o} = \frac{2\pi}{\pi} = \boxed{2}$
$x(t) = 1e^{j4t} + 1e^{-j4t} + je^{j2t}$
$a_2 = \boxed{1}$, $a_{-2} = \boxed{1}$, $a_1 = \boxed{j}$, $a_k = \boxed{0}$ *otherwise.*

ii. (4 pts) Find the output Fourier Series coefficients b_k.

Solution.

$b_k = a_k H(jkw_o) = a_k H(jk2)$
$b_2 = (1)H(j4) = \boxed{2}$
$b_{-2} = (1)H(-j4) = \boxed{0}$
$b_1 = (j)H(j2) = \boxed{2j}$
$b_k = \boxed{0}$ *otherwise.*

iii. (4 pts) Using your results from (b), find the output signal $y(t)$.

Solution.

$$
\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jkw_o t} \\
&= 2e^{j(2)(2)t} + 2je^{j(1)(2)t} \\
&= \boxed{2e^{j4t} + 2je^{j2t}}
\end{aligned}
$$

iv. (4 pts) What is the average power of the input, which we can denote as P_{in}? What is the average power of the output, which we can denote as P_{out}?

Solution.

$$P_{in} = \sum_{k=-\infty}^{\infty} |a_k|^2 = |1|^2 + |1|^2 + |j|^2 = 1 + 1 + 1 = \boxed{3}$$

$$P_{out} = \sum_{k=-\infty}^{\infty} |b_k|^2 = |2|^2 + |2j|^2 = 4 + 4 = \boxed{8}$$

8. *Homework Self-Reflection* (10 pts) After completing your homework, go to the following link to rate your skill or concept understanding level for each item listed. Your self-reflection must be completed by the due date. All submissions are time-stamped, so please give yourself plenty of time to complete and submit your self-reflection.

`http://bit.ly/2DNFX0i`