EE 235, Winter 2018, Homework 9: Amplitude Modulation and Sampling
(Due Friday March 2, 2018 via Canvas Submission)
Write down *ALL* steps for full credit

HW9 Topics:

- Sampling Theorem and Aliasing
- Modulation and Demodulation

HW9 Course Learning Goals Satisfied:

- Goal 5: Understand how modulation and sampling affects the frequency components of the input signal.
- Goal 6: Use and understand standard EE terminology associated with filtering and LTI systems (e.g. LPF, HPF, etc.)

HW9 References:

- OWN Sections 7.1 - 7.3, 8.1 - 8.2

HW9 Problems (Total = 134 pts):

1. *Review.* (20 pts)

   (a) *System Properties.* To get you ready for Ch.9 in which we will introduce another way to test system properties, let us review

   i. (4 pts) Suppose a system is described by the following I/O relationship:

   $$y(t) = \int_{t}^{\infty} x(\tau - 1)d\tau$$

   Without transforming this system into another system description (e.g. $h(t)$ or $H(j\omega)$), is this system causal? Is this system stable? Justify your answers.

   Solution.
   CAUSALITY TEST: check for past and present inputs
   Here, the input is being evaluated from $x(t-1)$ to $x(\infty)$. Because input is being evaluated at $x(\infty)$, a future input, this system is $\boxed{\textit{NOT CAUSAL}}$

   STABILITY TEST: check that a bounded input produces a bounded output
   Assume: $|x(t)| \leq M_1 < \infty$
   Find $|y(t)|$: $|y(t)| = \left| \int_{t}^{\infty} x(\tau-1)d\tau \right| \leq \int_{t}^{\infty} |x(\tau-1)|\, d\tau \leq \int_{t}^{\infty} M_1 d\tau \to \infty$
   Output is not bounded, so this system is $\boxed{\textit{NOT STABLE}}$

   ii. (4 pts) Suppose an LTI system is described by the following impulse response:

   $$h(t) = (\delta(t) + 2\delta(t-1)) * (\delta(t) - \delta(t-2))$$

   Without transforming this system into another system description (e.g. I/O or $H(j\omega)$), is this system causal? Is this system stable? Justify your answers.

   Solution.
   Simplifying $h(t)$ first using echo property $h(t) = \delta(t) + 2\delta(t-1) - \delta(t-2) - 2\delta(t-3)$

   CASUALITY TEST: check that $h(t) = 0$ for $t < 0$
   All unit impulses are located at $t \geq 0$, so this system $\boxed{\textit{IS CAUSAL}}$

STABILITY TEST: check that $h(t)$ is absolutely integrable

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} |\delta(t) + 2\delta(t-1) - \delta(t-2) - 2\delta(t-3)|dt$$

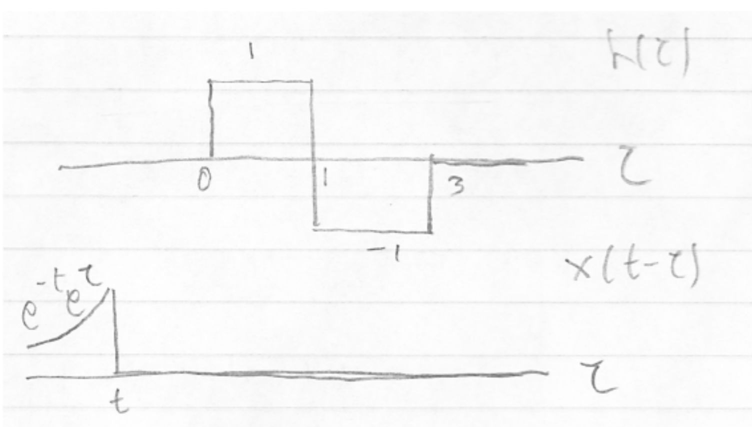
$$\int_{-\infty}^{\infty} |h(t)|dt \leq \int_{-\infty}^{\infty} [\delta(t) + 2\delta(t-1) + \delta(t-2) + 2\delta(t-3)]\, dt = 1 + 2 + 1 + 2 = 6 < \infty$$

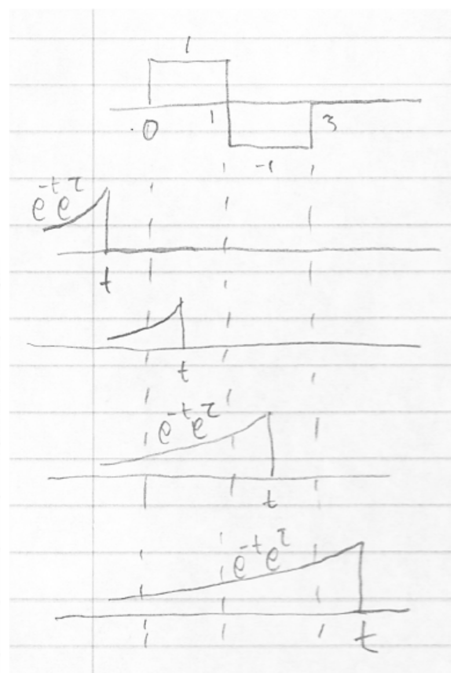
Therefore, this system $\boxed{\textit{IS STABLE}}$

(b) *LTI System Output.* To get you ready for Ch.9, in which we will introduce another way to find the output of an LTI system, let us review the three different ways we have learned so far to evaluate the output.

i. (4 pts) $x(t) = e^{-t}u(t)$ and $h(t) = u(t) - 2u(t-1) + u(t-3)$. Using convolution in the time domain, find $y(t)$. Show that $y(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t}, & 0 < t < 1 \\ 2e^{-(t-1)} - e^{-t} - 1, & 1 < t < 3 \\ 2e^{-(t-1)} - e^{-t} - e^{-(t-3)}, & t > 3 \end{cases}$

Solution.





$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

When $t < 0$: No overlap $\Rightarrow y(t) = 0$

When $0 < t < 1$: $y(t) = \int_0^t e^{-t}e^{\tau}(1)d\tau = e^{-t}(e^t - 1) = 1 - e^{-t}$

When $1 < t < 3$: $y(t) = \int_0^1 e^{-t}e^{\tau}(1)d\tau + \int_1^t e^{-t}e^{\tau}(-1)d\tau = e^{-t}(e^1 - 1) - e^{-t}(e^t - e^1) = e^{-t+1} - e^{-t} - 1 + e^{-t+1} = 2e^{-t+1} - e^{-t} - 1$

When $t > 3$: $y(t) = \int_0^1 e^{-t}e^{\tau}(1)d\tau + \int_1^3 e^{-t}e^{\tau}(-1)d\tau = e^{-t}(e^1 - 1) - e^{-t}(e^3 - e^1) = e^{-t+1} - e^{-t} - e^{-t+3} + e^{-t+1} = 2e^{-t+1} - e^{-t} - e^{-t+3}$

ii. (4 pts) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\frac{\pi}{2})$ and $h(t) = 4sinc(8t)e^{j(6t)}$. Using Fourier Series, find $y(t)$.
Hint: You will need to find $H(j\omega)$ to solve this problem.
Show that $y(t) = 1 + e^{j4t} + e^{j8t} + e^{j12t}$.

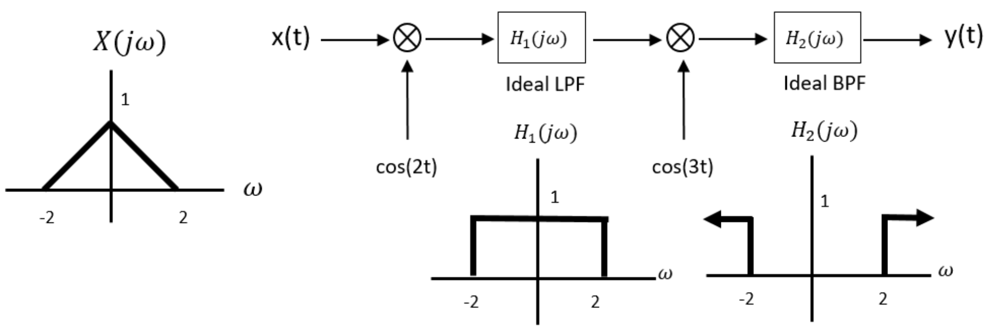
Solution.
Find a_k: Use pair $\sum \delta(t - kT) \leftrightarrow w_o = \frac{2\pi}{T}$, $a_k = \frac{1}{T}$
$T = \frac{\pi}{2} \rightarrow w_o = \frac{2\pi}{\pi/2} = 4$, $a_k = \frac{1}{T} = \frac{1}{\pi/2} = \frac{2}{\pi}$
Find $H(jw)$: $h(t) = h_1(t)e^{jkt}$, $h_1(t) = 4sinc(8t)$, $H(jw) = H_1(j(w-6))$, $H_1(jw) = 4[\frac{\pi}{8}rect(\frac{w}{16})]$
$\therefore H(jw) = \frac{\pi}{2}rect(\frac{w-6}{16})$
Find b_k: $b_k = a_kH(jkw_o) = a_kH(jk4) = \frac{\pi}{2}H(jk4)$
Only b_0, b_1, b_2, b_3 pass through
$b_0 = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$
$b_1 = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$
$b_2 = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$
$b_3 = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$
Therefore, $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_o t} = \sum_{k=-\infty}^{\infty} b_k e^{jk4t} = 1 + e^{j4t} + e^{j8t} + e^{j12t}$

iii. (4 pts) $x(t) = e^{-3t}u(t) - 2te^{-3t}u(t)$ and $h(t) = 4e^{-t}u(t) - 2e^{-2t}u(t)$. Using Fourier Transforms, find $y(t)$.

Solution.
$X(jw) = \frac{1}{3+jw} - \frac{2}{(3+jw)^2} = \frac{3+jw-2}{(3+jw)^2} = \frac{1+jw}{(3+jw)^2}$
$H(jw) = \frac{4}{1+jw} - \frac{2}{2+jw} = \frac{8+4jw-2-2jw}{(1+jw)(2+jw)} = \frac{6+2jw}{(1+jw)(2+jw)}$
$Y(jw) = \frac{1+jw}{(3+jw)^2} \cdot \frac{6+2jw}{(1+jw)(2+jw)} = \frac{2}{(3+jw)(2+jw)} = \frac{A}{3+jw} + \frac{B}{2+jw}$
$A = \frac{2}{2+jw}|_{jw=-3} = \frac{2}{2-3} = -2$
$B = \frac{2}{3+jw}|_{jw=-2} = \frac{2}{3-2} = 2$
$y(t) = -2e^{-3t}u(t) + 2e^{-2t}u(t)$

2. *Modulation.* (16 pts) The purpose of these problems is to get you comfortable walking through a block diagram consisting of modulation (by cosine) systems.

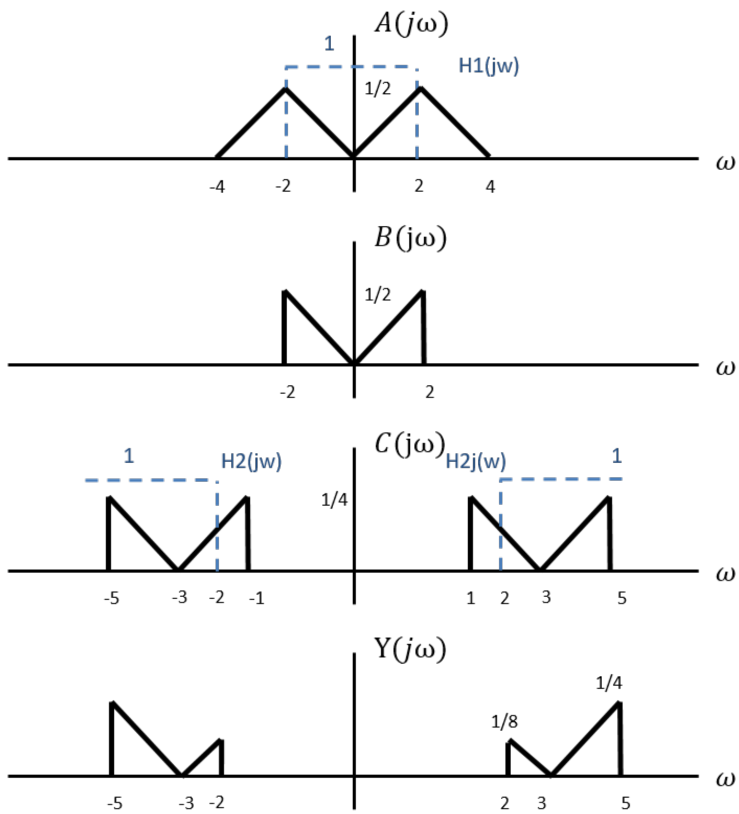
(a) (4 pts) Consider a signal $x(t)$ with Fourier Transform $X(j\omega)$ that is input to a system consisting of LTI and modulation systems as shown below:



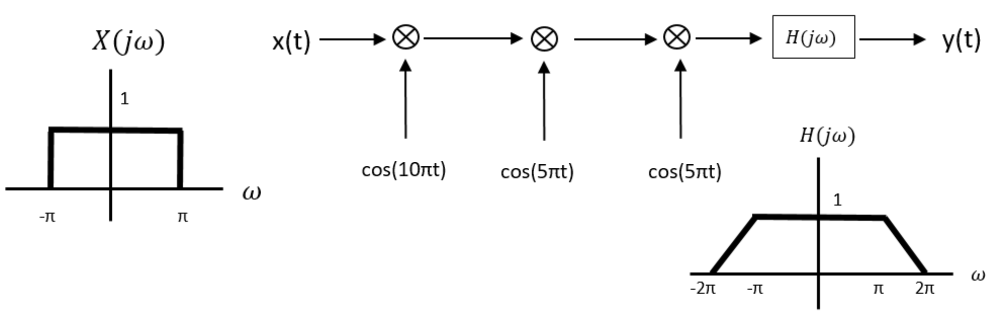
Sketch the output Fourier Transform $Y(j\omega)$.
Hint: The output will consist of two pieces with an amplitude of $\frac{1}{4}$.

Solution. *Call output of first modulation system $a(t)$: $A(j\omega)$ shifts $X(j\omega)$ right and left by 2 and amplitude scaled by 1/2*
Call output of first LTI system $b(t)$: $B(j\omega) = H_1(j\omega)A(j\omega)$

Call output of second modulation system c(t): $C(j\omega)$ shifts $B(j\omega)$ right and left by 3 and amplitude scaled by another 1/2
Output of second LTI system is y(t): $Y(j\omega) = H_2(j\omega)C(j\omega)$

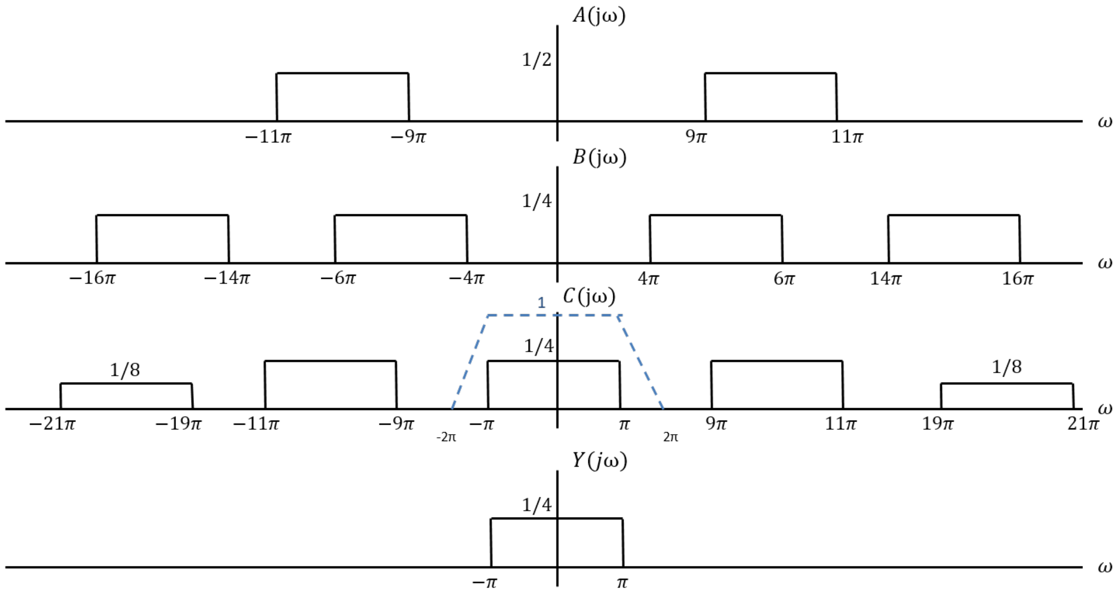


(b) (4 pts) Consider a signal $x(t)$ with Fourier Transform $X(j\omega)$ that is input to a system consisting of LTI and modulation systems as shown below:

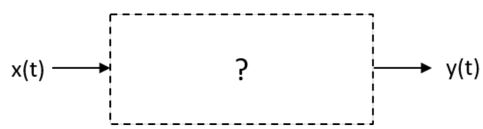


Sketch the output Fourier Transform $Y(j\omega)$.

Solution. *Call output of first modulation system a(t): $A(j\omega)$ shifts $F(\omega)$ right and left by 10π and amplitude scaled by 1/2*
Call output of second modulation system b(t): $B(j\omega)$ shifts $A(j\omega)$ right and left by 5π and amplitude scaled by another 1/2
Call output of third modulation system c(t): $C(j\omega)$ shifts $B(j\omega)$ right and left by 5π and amplitude scaled by another 1/2
Output of LTI system is y(t): $Y(j\omega) = C(j\omega)H(j\omega)$

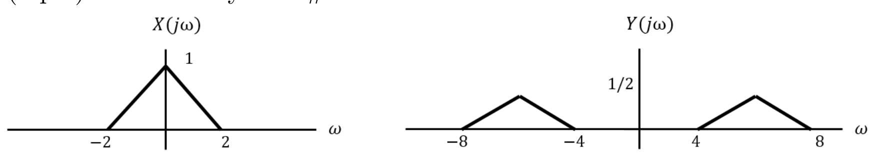


(c) Let us now tackle the problem of deducing an unknown system:

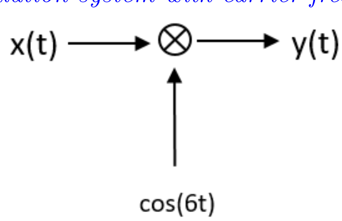


Using only LTI systems and/or modulation systems, draw the full system diagram that produces the output Fourier Transform $Y(j\omega)$ given a known input Fourier Transform $X(j\omega)$ below. If you use LTI systems, please make sure to sketch the corresponding frequency response $H(j\omega)$ that is required.

i. (4 pts) Unknown System #1:

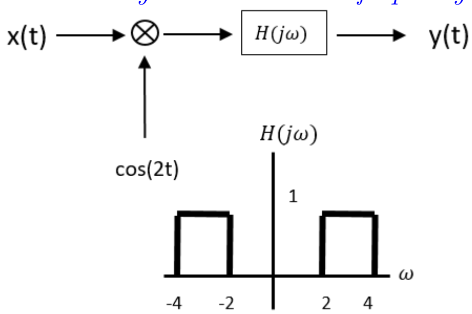


Solution. *Output is two replicas of input scaled by a half. This must be produced by a modulation system with carrier frequency* 6



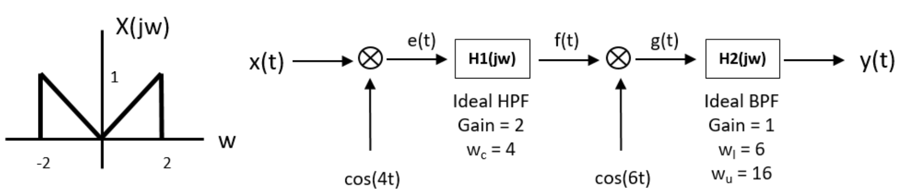
ii. (4 pts) Unknown System #2:

Solution. *Output is half of two replicas of input scaled by a half. This must be produced by a modulation system with carrier frequency 2 followed by either a high-pass or band-pass filter.*



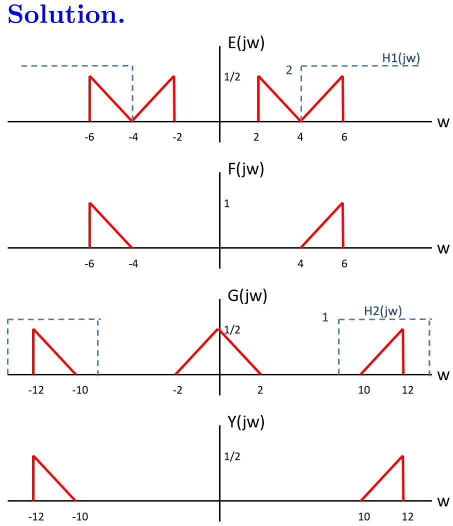
3. *Demodulation.* (26 pts) The purpose of this problem is to apply what you have learned about modulation by cosine to design a demodulation system to recover an input or obtain some other desired result.

   (a) Consider the following block diagram:



   i. (4 pts) Sketch the output spectrum $Y(j\omega)$. Precisely label all points and axes.

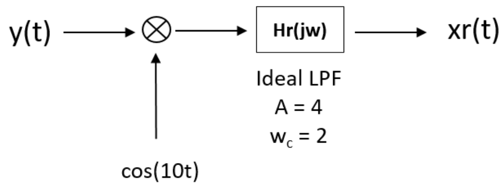
   Solution.



   ii. $X(j\omega)$ can be recovered from $Y(j\omega)$ using a single modulator followed by an ideal LPF with frequency response $H_r(j\omega)$. Answer the following questions:

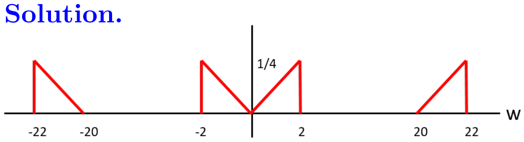
A. (4 pts) Draw the block diagram for this system. Clearly specify the modulation signal, as well as the filter gain A and cutoff frequency w_c for the ideal LPF.

Solution.



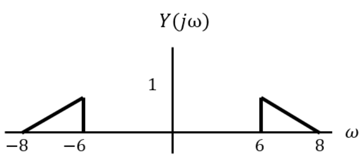
B. (2 pts) Note that several cutoff frequencies w_c are valid for (A). What range of cutoff frequencies w_c can be used to fully recover $x(t)$ from $y(t)$?

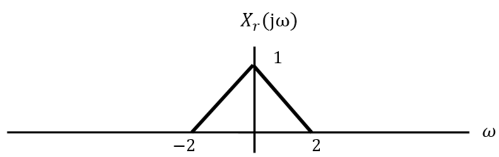
Solution.



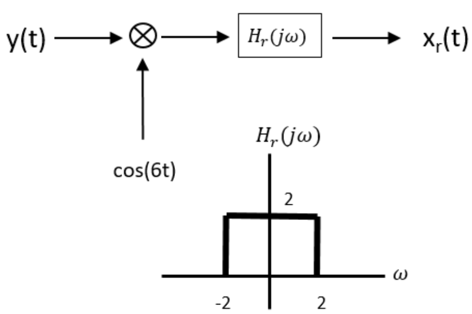
Output of modulator shown above. Therefore, the cutoff frequency can be any value in the range $2 < w_c < 20$.

(b) (4 pts) Suppose the output of a modulation system is given by $Y(j\omega)$ below. Draw the demodulation system that will recover the desired input again, shown below as $X_r(j\omega)$:



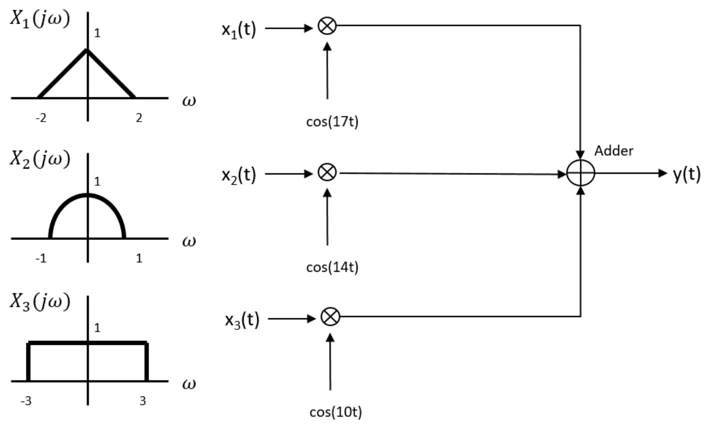


Solution. *We need to center the triangles back around* 0. *This can be accomplished by modulating by* 6. *Since modulation produces extra replicas scaled by half, we need to filter those out and just pass the signal centered around 0 using a low-pass filter. The gain must also be 2 to bring the amplitude back to 1.*



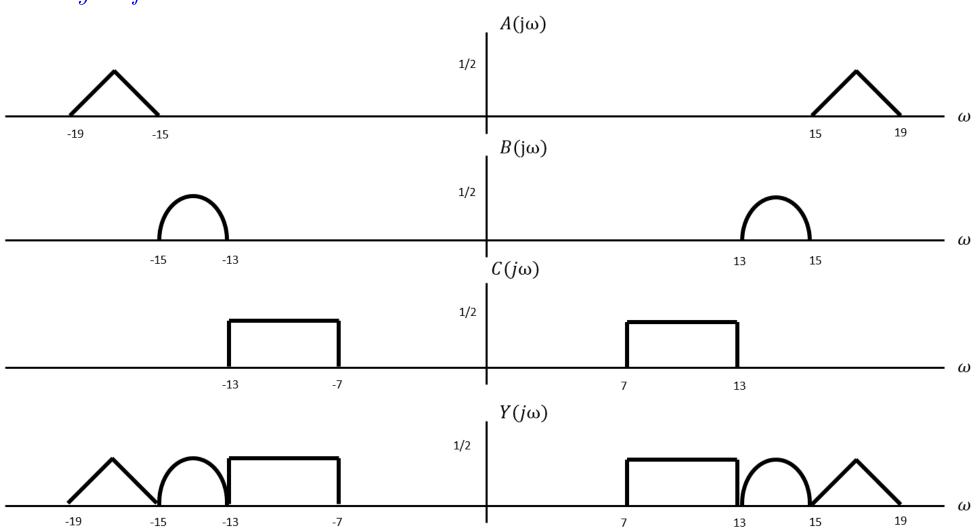
(c) We briefly learned in class that modulation is useful for different radio stations to send radio signals out simultaneously. This is a process called frequency division multiplexing, which we will practice with below.

i. (4 pts) Consider the following frequency division multiplexing system that can be used to transmit multiple signals at the same time:

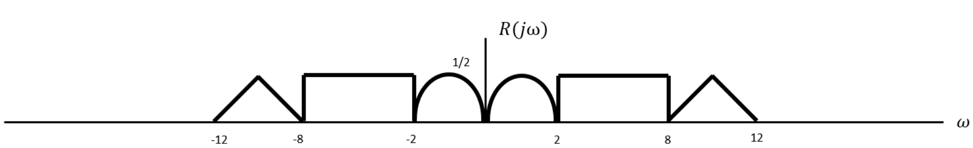


Draw the overall output Fourier Transform $Y(j\omega)$ produced by this system.

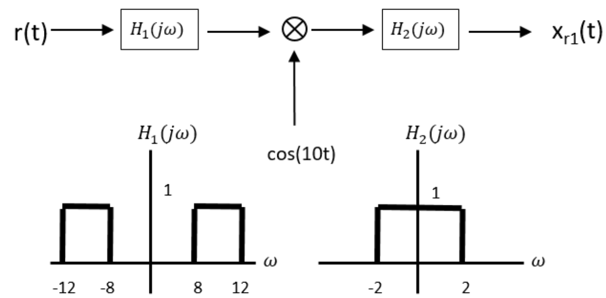
Solution. *Call output of first modulator on top $a(t)$: $A(j\omega)$ shifts right and left by 17 and amplitude scaled by half*
Call output of second modulator in middle $b(t)$: $B(j\omega)$ shifts right and left by 14 and amplitude scaled by half
Call output of third modulator on bottom $c(t)$: $C(j\omega)$ shifts right and left by 10 and amplitude scaled by half



ii. (4 pts) Suppose a receiver picks up a signal $r(t)$ whose Fourier Transform is given below:

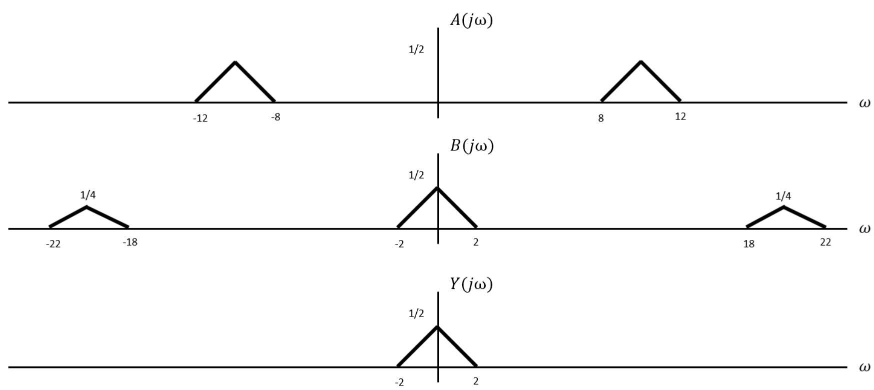


We can get back the received triangle signal $x_1(t)$ using the system below. The idea is to first filter out the desired signal and then shift it back to the center around frequency $\omega = 0$:



Sketch the output Fourier Transform $X_{r1}(j\omega)$ of this system.

Solution. *Call output of first LTI system $a(t)$, the output of modulator $b(t)$, and then the output of the second LTI system is $y(t)$. Graphs for each section are below:*

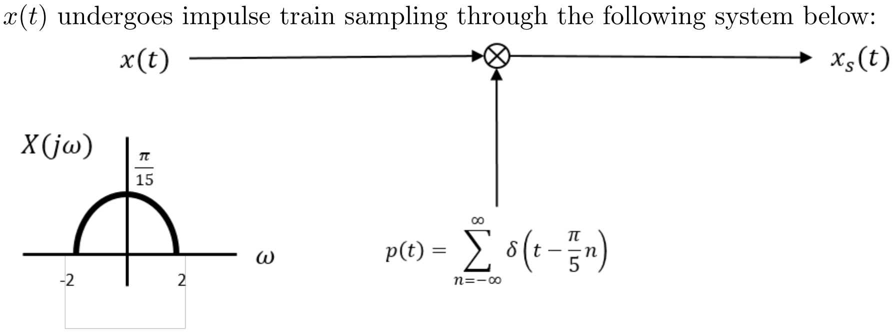


iii. (4 pts) You will notice the system has an error in the output amplitude. What change can be made to the receiver system to get the desired amplitude back up to 1? Note: there are several answers to this question.

Solution. *Option #1: Change gain of $H_1(j\omega)$ to 2*
Option #2: Modulate by $2\cos(10t)$
Option #3: Change gain of $H_2(j\omega)$ to 2

4. *Impulse Train Sampling System.* (34 pts) The purpose of this problem is to get you comfortable walking through a block diagram consisting of an impulse train sampling system and to visually understand the concept of aliasing.

   (a) $x(t)$ undergoes impulse train sampling through the following system below:



   Answer the following questions:

i. (2 pts) What is the sampling frequency ω_s used by this system? What is the equation for the output Fourier Transform $X_s(j\omega)$ in terms of $X(j\omega)$?

Show that $\omega_s = 10$ and $X_s(j\omega) = \frac{5}{\pi} \sum_{k=-\infty}^{\infty} X(j(\omega - 10k))$

Solution.

By inspection, $T_s = \frac{\pi}{5}$, *which means* $\omega_s = \frac{2\pi}{T_s} = \boxed{10}$

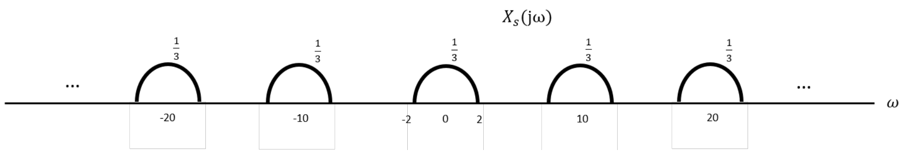
Recall from lecture: $X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$ *where* $\omega_s = \frac{2\pi}{T_s}$

Plugging our known parameters in:

$$\boxed{X_s(j\omega) = \frac{5}{\pi} \sum_{k=-\infty}^{\infty} X(j(\omega - 10k))}$$

ii. (4 pts) Using your equation from (i), sketch the output spectrum $X_s(j\omega)$ vs. ω. Make sure to label all critical points.

Solution.

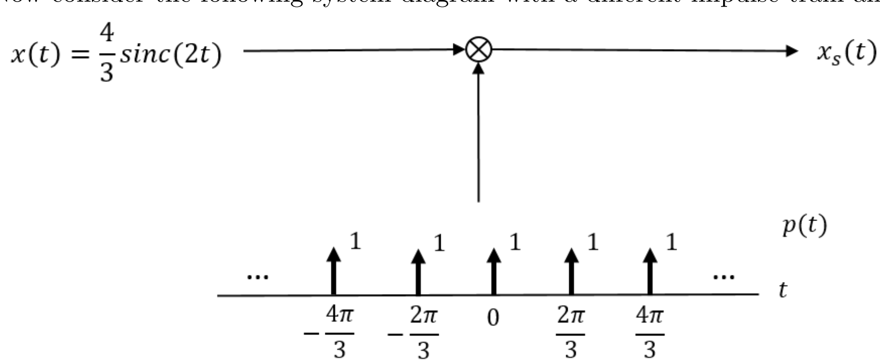


iii. (2 pts) Using your sketch from (ii) and your understanding of the concept of aliasing, explain why this is an example of sampling with no aliasing.

Solution.

There is no aliasing because the periodic replicas of $X(j\omega)$ *did not overlap in frequency, which means* $X(j\omega)$ *is fully recoverable from* $X_s(j\omega)$

(b) Now consider the following system diagram with a different impulse train and input $x(t)$:



Answer the following questions:

i. (4 pts) What is the sampling frequency f_s (not ω_s!) used by this system? What is the equation for the output Fourier Transform $X_s(j\omega)$ in terms of $X(j\omega)$?

Solution.

By inspection, $T_s = \frac{2\pi}{3}$, *which means* $\omega_s = \frac{2\pi}{T_s} = 3$ *and hence* $f_s = \frac{\omega_s}{2\pi} = \boxed{\frac{3}{2\pi}}$

Recall from lecture: $X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$ *where* $\omega_s = \frac{2\pi}{T_s}$

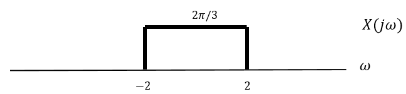
Plugging our known parameters in:

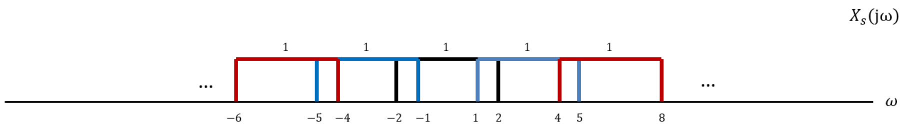
$$\boxed{X_s(j\omega) = \frac{3}{2\pi} \sum_{k=-\infty}^{\infty} X(j(\omega - 3k))}$$

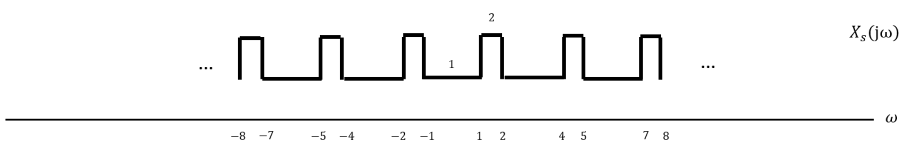
ii. (4 pts) Using your equation from (i), sketch the output spectrum $X_s(j\omega)$ vs. ω. Make sure to label all critical points.
Note: you need to first find and sketch $X(j\omega)$.

Solution.
Finding $X(j\omega)$: using a common pair
$x(t) = \frac{4}{3}sinc(2t) \longrightarrow X(j\omega) = \frac{4}{3}\frac{\pi}{2}rect\left(\frac{\omega}{4}\right) = \frac{2\pi}{3}rect\left(\frac{\omega}{4}\right)$

Sketching out $X_s(j\omega)$: note the amplitude of $X(j\omega)$ is scaled by $\frac{3}{2\pi}$ and hence the amplitude becomes just 1

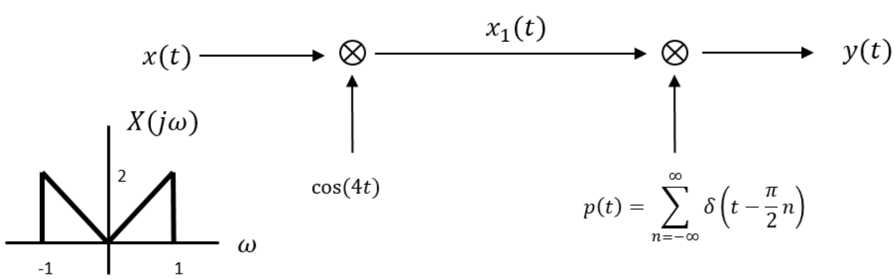






iii. (2 pts) Using your sketch from (ii), does aliasing of $x(t)$ occur? Justify your answer.
Solution.
Yes, aliasing does occur because the periodic replicas of $X(j\omega)$ overlap in frequency, which means $X(j\omega)$ is not fully recoverable from $X_s(j\omega)$

(c) Now consider the following system consisting of both modulation and impulse train sampling systems:



i. (4 pts) What is the sampling period T_s used by the impulse train sampling system? What is the equation for $X_1(j\omega)$ in terms of $X(j\omega)$? What is the equation for $Y(j\omega)$ in terms of $X_1(j\omega)$?
Solution.
By inspection $\boxed{T_s = \frac{\pi}{2}}$

$x_1(t)$ is output of modulation with $\omega_c = 4$, so we can write: $\boxed{X_1(j\omega) = \frac{1}{2}X(j(\omega - 4)) + \frac{1}{2}X(j(\omega + 4))}$

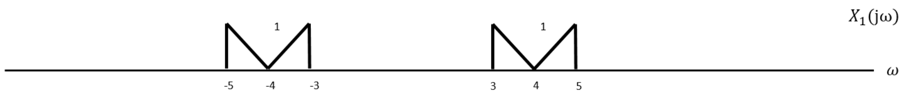
$y(t)$ is an impulse train sampling system, so we can write:

$$\boxed{Y(j\omega) = \frac{2}{\pi}\sum_{k=-\infty}^{\infty} X_1(j(\omega - 4k))}$$

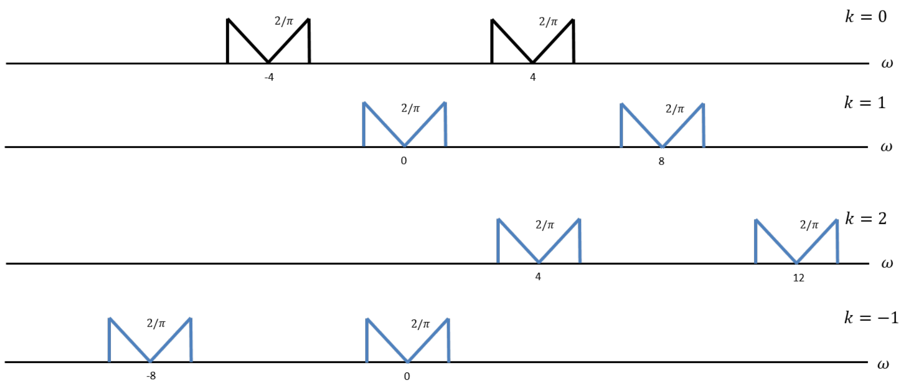
ii. (4 pts) Sketch the final output spectrum $Y(j\omega)$ vs. ω. Make sure to label all critical points.

Solution.

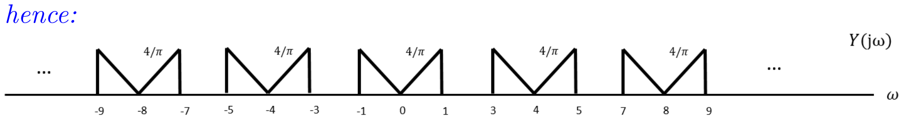
Sketching $X_1(j\omega)$ first:



Sketching $k = 0, 1, 2, -1$ terms of $Y(j\omega)$ to see a pattern:



Notice the piece will always overlap once, which doubles the amplitude from $2/pi$ to $4/pi$, so hence:

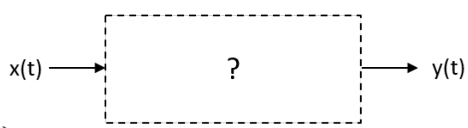


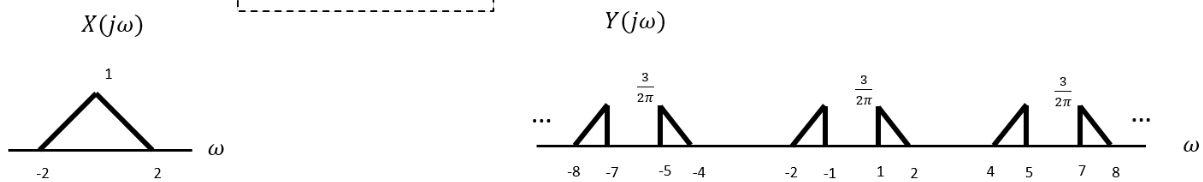
iii. (4 pts) Using your sketch, is it possible to recover the original input $x(t)$ from $y(t)$? If so, draw or describe a system diagram that will do this. Make sure to clearly specify all system blocks being used either in words or with a graph.

Solution.

Yes, it is possible. We just need to input $y(t)$ into an ideal LPF system with cutoff frequency of $w_c = 2$ and a gain of $A = \frac{\pi}{2}$.

(d) (4 pts) Using your knowledge of the behavior of impulse train sampling systems, draw the appropriate system diagram that will produce the following input-output pair. Justify your design choices.





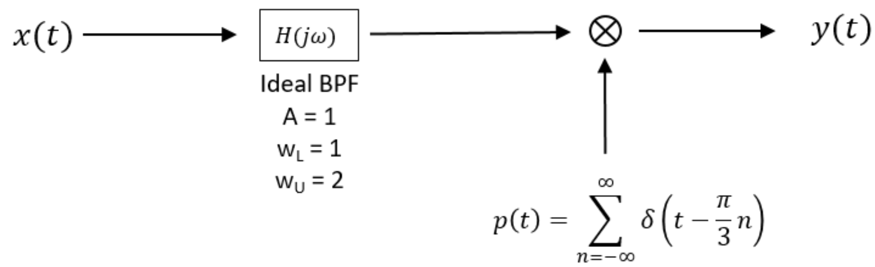
Make sure to clearly specify your systems with an equation, description, or graph.

Solution. *The output has periodic replicas, so this must be some kind of impulse train sampling system.*

The replicas look like the input but with the middle section from $\omega = -1$ to $\omega = 1$ cutoff, so a BPF system is also needed with $\omega_l = 1$ and $\omega_l = 2$.

So far we can deduce that we need an ideal BPF followed by an impulse train sampling system.

Determining sampling period T_s of sampling system: notice the replicas occur every multiple of $\omega = 6$, which means $\omega_s = 6$ and $T_s = \frac{2\pi}{\omega_s} = \frac{\pi}{3}$
Notice also the amplitudes of the output are $\frac{3}{\pi}$ which is equal to $\frac{1}{2}\frac{1}{T_s}$ so that means the input to the impulse train sampling system must have an amplitude of $\frac{1}{2}$, which is consistent with the output of the BPF, so no change necessary to the filter gain.
Therefore, we can deduce the system we need is the following:



5. *Sampling Theorem and Aliasing.* (28 pts) The purpose of this problem is to apply the sampling theorem to determine how much to sample a continuous-time signal by to avoid aliasing.

(a) Every signal $x(t)$ has a so-called Nyquist rate ω_N used to determine the sampling rate ω_s needed to avoid aliasing. What is the Nyquist rate ω_N for each of the following signals below?

i. (2 pts) $x(t) = sinc(5000t) * \cos(5\pi t)$
Show that $\omega_N = 10\pi$

Solution.
Finding the bandwidth ω_B from $X(j\omega)$:
$X(j\omega) = X_1(j\omega)X_2(j\omega)$
$sinc(5000t) \longrightarrow X_1(j\omega) = \frac{\pi}{5000} rect(\frac{\omega}{10000})$
$\cos(5\pi t) \longrightarrow X_2(j\omega) = \pi\delta(\omega - 5\pi) + \pi\delta(\omega + 5\pi)$
Hence:
$X(j\omega) = \frac{\pi}{5000} rect(\frac{\omega}{10000})[\pi\delta(\omega - 5\pi) + \pi\delta(\omega + 5\pi)]$
$X(j\omega) = \frac{\pi^2}{5000} rect(\frac{\omega}{10000})\delta(\omega - 5\pi) + \frac{\pi^2}{5000} rect(\frac{\omega}{10000})\delta(\omega + 5\pi)]$
$X(j\omega) = \frac{\pi^2}{5000}\delta(\omega - 5\pi) + \frac{\pi^2}{5000}\delta(\omega + 5\pi)]$
Therefore: $\omega_B = 5\pi$ *and* $\omega_N = 2\omega_B = \boxed{10\pi}$

ii. (4 pts) $x(t) = \frac{\sin(10\pi t)\cos(10\pi t)}{10\pi t}$

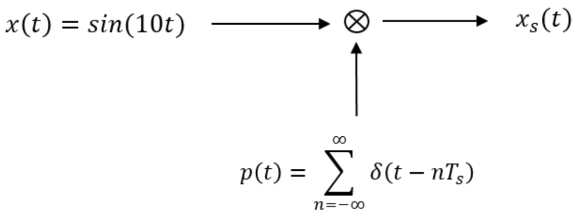
Solution. *Rewriting $x(t)$ first as a sinc: $x(t) = sinc(10\pi t)\cos(10\pi t)$*
Finding the bandwidth ω_B from $X(j\omega)$:
Letting $x_1(t) = sinc(10\pi t)$
Then $x(t) = x_1(t)cos(10\pi t) \longrightarrow X(j\omega) = \frac{1}{2}X_1(j(\omega - 10\pi)) + \frac{1}{2}X_1(j(\omega + 10\pi))$
$x_1(t) = sinc(10\pi t) \longrightarrow X_1(j\omega) = \frac{1}{10} rect(\frac{\omega}{20\pi})$
Hence:
$X(j\omega) = \frac{1}{20} rect(\frac{\omega - 10\pi}{20\pi}) + \frac{1}{20} rect(\frac{\omega + 10\pi}{20\pi})$
Therefore: $\omega_B = 20\pi$ *and* $\omega_N = 2\omega_B = \boxed{40\pi}$

iii. (2 pts) $x(t) = sinc(5000t) + \cos(5\pi t)$

Solution. *Finding the bandwidth ω_B from $X(j\omega)$:*
$X(j\omega) = X_1(j\omega) + X_2(j\omega)$
Borrowing the work from (i):
$X(j\omega) = X_1(j\omega) + X_2(j\omega) = \frac{\pi}{5000} rect(\frac{\omega}{10000}) + \pi\delta(\omega - 5\pi) + \pi\delta(\omega + 5\pi)$
Therefore: $\omega_B = 5000$ *and* $\omega_N = 2\omega_B = \boxed{10,000}$

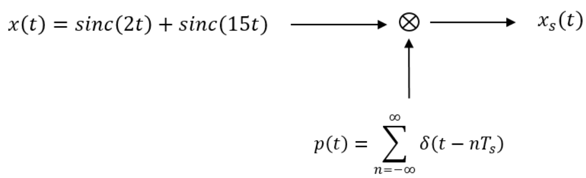
(b) Signal $x(t)$ undergoes impulse train sampling. Using the sampling theorem, what should the sampling period T_s be for each sampling system below to avoid aliasing? Justify your answer.

i. (2 pts) System #1: Show that we need $T_s < \frac{\pi}{10}$



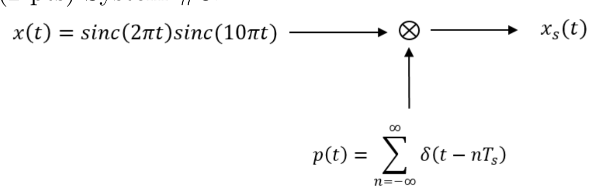
Solution. *Finding bandwidth ω_B: $X(j\omega)$ is just two spikes at $\omega = \pm 10$ so $\omega_B = 10$*
Using sampling theorem: $\omega_s = \frac{2\pi}{T_s} > 2\omega_B$ so $T_s < \frac{\pi}{\omega_B} \longrightarrow \boxed{T_s < \frac{\pi}{10}}$

ii. (2 pts) System #2:



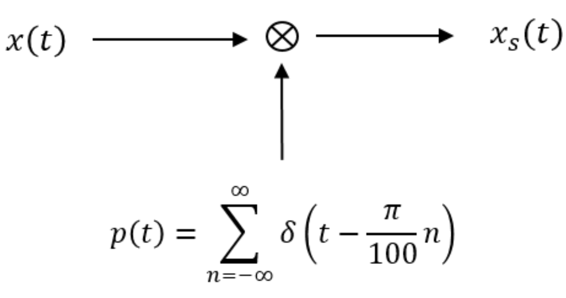
Solution. *Finding bandwidth ω_B: $X(j\omega)$ is just sum of two rects*
$\omega : [-2, 2] + [-15, 15] = [-15, 15]$ so $\omega_B = 15$
Using sampling theorem: $\omega_s = \frac{2\pi}{T_s} > 2\omega_B$ so $T_s < \frac{\pi}{\omega_B} \longrightarrow \boxed{T_s < \frac{\pi}{15}}$

iii. (2 pts) System #3:



Solution. *Finding bandwidth ω_B: $X(j\omega)$ is just convolution of two rects*
$\omega : [-2\pi, 2\pi] * [-10\pi, 10\pi] = [-12\pi, 12\pi]$ so $\omega_B = 12\pi$
Using sampling theorem: $\omega_s = \frac{2\pi}{T_s} > 2\omega_B$ so $T_s < \frac{\pi}{\omega_B} \longrightarrow \boxed{T_s < \frac{1}{12}}$

(c) Consider an impulse train sampling with a fixed sampling period as shown below:



For each input signal $x(t)$ below, what range of values should the unknown parameter ω_x satisfy so that $x(t)$ can be recovered uniquely from its samples in $x_s(t)$? Justify your answer.

i. (2 pts) $x(t) = sinc(\omega_x t)$
Show that we need $\omega_x < 100$.

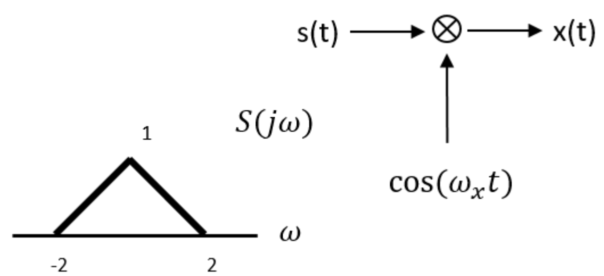
Solution. *We must satisfy the sampling theorem: $\omega_s > 2\omega_B$*

From our given system: $T_s = \frac{\pi}{100} \rightarrow \omega_s = \frac{2\pi}{\pi/100} = 200$
Finding ω_B*:* $X(j\omega)$ *is a rect in range* $[-\omega_x, \omega_x]$*. Hence,* $\omega_B = \omega_x$
Therefore, to satisfy the sampling theorem: $200 > 2\omega_x \rightarrow \boxed{\omega_x < 100}$

ii. (2 pts) $x(t) = sinc^2(\omega_x t)$

Solution. *We must satisfy the sampling theorem:* $\omega_s > 2\omega_B$
From our given system: $T_s = \frac{\pi}{100} \rightarrow \omega_s = \frac{2\pi}{\pi/100} = 200$
Finding ω_B*:* $X(j\omega)$ *is a convolution of two rects* $\omega : [-\omega_x, \omega_x] * [-\omega_x, \omega_x] = [-2\omega_x, 2\omega_x]$*.*
Hence, $\omega_B = 2\omega_x$
Therefore, to satisfy the sampling theorem: $200 > 2(2\omega_x) \rightarrow \boxed{\omega_x < 50}$

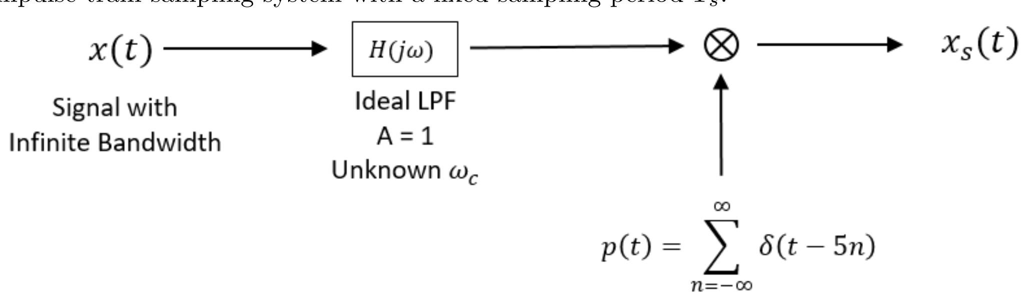
iii. (2 pts) $x(t)$ is generated from the system below:



Solution. *We must satisfy the sampling theorem:* $\omega_s > 2\omega_B$
From our given system: $T_s = \frac{\pi}{100} \rightarrow \omega_s = \frac{2\pi}{\pi/100} = 200$
Finding ω_B*:* $X(j\omega)$ *comes from a modulation system so* $\omega : [-2, 2] * [-\omega_x, \omega_x] = [-2+\omega_x, 2+\omega_x]$*. Hence,* $\omega_B = 2\omega_x$
Therefore, to satisfy the sampling theorem: $200 > 2(2 + \omega_x) \rightarrow \boxed{\omega_x < 98}$

(d) In the real-world, signals are not bandlimited in frequency ω and hence must undergo pre-filtering before sampling for perfect reconstruction. Answer the following questions for each ideal pre-filter and impulse train sampling system below:

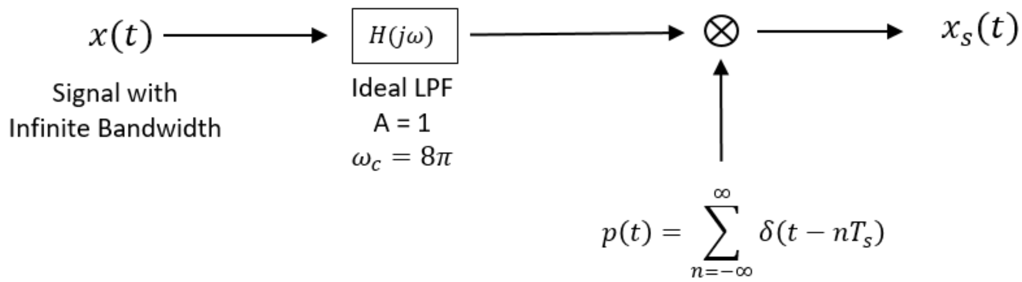
i. (4 pts) System #1: Consider an ideal pre-filter with unknown cutoff frequency ω_c and an impulse train sampling system with a fixed sampling period T_s:



What range of cutoff frequencies ω_c in the pre-filter are allowed that will prevent aliasing? Justify your answer.

Solution. *We need to again satisfy the sampling theorem in the end:* $\omega_s > 2\omega_B$
Our system has a fixed ω_s*:* $T_s = 5$ *so* $\omega_s = \frac{2\pi}{5}$
Hence, to avoid aliasing, we need: $\frac{2\pi}{5} > 2\omega_B \longrightarrow \omega_B < \frac{\pi}{5}$
The pre-filter in the end will set the bandwidth with the choice of ω_c*, so we need* $\boxed{\omega_c < \frac{\pi}{5}}$

ii. (4 pts) System #2: Consider an idea pre-filter with known cutoff frequency ω_c and an impulse train sampling system with unknown paramters:



What should the sampling period T_s of the impulse train sampling system be to avoid aliasing? Justify your answer.

Solution. *We need to again satisfy the sampling theorem in the end:* $\omega_s > 2\omega_B$
Our signal after the pre-filter has a known bandwidth set by the ideal filter: $\omega_B = \omega_c = 8\pi$
Hence, to avoid aliasing, we need: $\omega_s > 16\pi$ *Because* $\omega_s = \frac{2\pi}{Ts}$, *we need* $\frac{2\pi}{Ts} > 16\pi \longrightarrow$
$\boxed{T_s < \frac{1}{8}}$

6. *Homework Self-Reflection*
(10 pts) After completing your homework, go to the following link to rate your skill or concept understanding level for each item listed. Y our self-reflection must be completed by the due date. All submissions are time-stamped, so please give yourself plenty of time to complete and submit your self-reflection.
`http://bit.ly/2CGRqxF`