EE 235, Winter 2018, Homework 4: LTI Systems and Convolution
Due Wednesday January 24, 2018 via Canvas Submission
Write down *ALL* steps for full credit

HW4 Topics:

- LTI Systems and Impulse Response
- Echo Property of Convolution
- Convolution Integral

HW4 Course Learning Goals Satisfied:

- Goal 3: Perform convolutions for arbitrary and closed-form continuous-time signals.
- Goal 4: Analyze LTI systems given different system representations (including input-output equations and impulse response).
- Goal 6: Use and understand standard EE terminology associated with LTI systems (e.g. impulse response, step response).

HW4 References: OWN Sections 2.2

HW4 Problems (Total = 72 pts):

1. *Review*

   (a) *Complex Numbers I.* In Ch.3, we will need to be able to perform integration of complex exponential functions. Here are some practice problems to help you review that.

   i. (2 pts) Evaluate: $\frac{1}{\pi}\int_0^{\pi} e^{-jk2t}dt$ where k is an integer not equal to 0.
   Use your results from HW2 Problem 1 to reduce your answer.
   Show that this integral evaluates to 0

   Solution. $\frac{1}{\pi}\int_0^{\pi} e^{-jk2t}dt = \frac{1}{jk2\pi}[e^{-jk2\pi} - 1]$
   Since we can rewrite $e^{-jk2\pi} = e^{-j2\pi k} = 1$ *for all integers* $k \neq 0$*:*
   $\frac{1}{-jk2\pi}[e^{-jk2\pi} - 1] = \frac{1}{-jk2\pi}[1 - 1] = \boxed{0}$

   ii. (4 pts) Evaluate: $j(2-k)\cdot\int_0^{\pi} e^{j2t}e^{-jkt}dt$ where k is an integer.
   Use your results from HW2 Problem 1 to reduce your answer.
   Hint: The answer will have two parts, one when k is even and when k is odd.

   Solution. *First, combine the two exponentials into one:*
   $j(2-k)\cdot\int_0^{\pi} e^{j(2-k)t}dt$
   Then, performing the integration we get:
   $\frac{j(2-k)}{j(2-k)}[e^{j(2-k)\pi} - 1] = e^{j2\pi}e^{-j\pi k} - 1$
   Since $e^{j2\pi} = 1$ *and* $e^{-j\pi k} = (-1)^k$*, we have:* $(-1)^k - 1 = \boxed{\begin{cases} -2, & k = odd \\ 0, & k = even \end{cases}}$

   (b) *Signal Properties.*
   Let $x(t) = \cos(\frac{\pi}{2}t) + 4\cos(\frac{3\pi}{2}t) + \cos(3t)$.

   i. (4 pts) Is $x(t)$ periodic? If so, find the fundamental period T_o.
   ii. (4 pts) Is $x(t)$ even or odd?

Solution. *$x(t)$ is a sum of sinusoidal functions so should be periodic only if T_o exists:*
$T_0 = LCM(\frac{2\pi}{\frac{\pi}{2}}, \frac{2\pi}{\frac{3\pi}{2}}, \frac{2\pi}{3}) = LCM(4, \frac{4}{3}, \frac{2\pi}{3})$, *which does not exist.*
Therefore, $x(t)$ is not periodic.

Even / Odd:
$x(t) = \cos(\frac{\pi}{2}t) + 4\cos(\frac{3\pi}{2}t) + \cos(3t)$
$x(-t) = \cos(\frac{-\pi}{2}t) + 4\cos(\frac{-3\pi}{2}t) + \cos(-3t)$
$x(-t) = \cos(\frac{\pi}{2}t) + 4\cos(\frac{3\pi}{2}t) + \cos(3t)$
$\rightarrow x(t) = x(-t)$
Therefore, $x(t)$ is an even function.

(c) *System Properties.*

i. Consider the system described by $y(t) = x(e^t)$. For this system, state if the following properties hold. Justify your assertion in all cases, giving a proof if "true" and a counter-example if "false".

(1 pt) Linearity.

Solution. *Yes. Let $y(t) = \mathcal{T}(x(t))$. Then $\mathcal{T}(\alpha x_1(t) + \beta x_2(t)) = \alpha x_1(e^t) + \beta x_2(e^t) = \alpha\mathcal{T}(x_1(t)) + \beta\mathcal{T}(x_2(t))$.*

(1 pt) Time-invariance.

Solution. *No. Let $y(t) = \mathcal{T}(x(t))$. Then $\mathcal{T}(x(t - t_0)) = x(e^t - t_0) \neq x(e^{t-t_0})$.*

(1 pt) Stability.

Solution. *Yes. Let $|x(t)| \leq B$ for all values of t. Then $|y(t)| = |x(e^t)| \leq B$ as well.*

(1 pt) Causality.

Solution. *No. The output at $t = t_0$, $y(t_0) = x(e^{t_0})$, which depends on the value of the input at time e^{t_0}, which for $t_0 > 0$ is bigger than t_0.*

ii. Consider the system described by $y(t) = \text{Odd}(x(t))$. For this system, state if the following properties hold. Justify your assertion in all cases, giving a proof if "true" and a counter-example if "false".

(1 pt) Linearity.

Solution. *Yes. Let $y(t) = \mathcal{T}(x(t))$. Then $\mathcal{T}(\alpha x_1(t) + \beta x_2(t)) = Odd(\alpha x_1(t) + \beta x_2(t)) = \alpha Odd(x_1(t)) + \beta Odd(x_2(t))$.*

(1 pt) Time-invariance.

Solution. *No. Let $y(t) = \mathcal{T}(x(t))$. Then $\mathcal{T}(x(t-t_0)) = \frac{1}{2}(x(t-t_0) - x(-t-t_0)) \neq y(t-t_0) = \frac{1}{2}(x(t-t_0) - x(-t+t_0))$.*

(1 pt) Causality.

Solution. *No, because at time $t = -1$, the output $y(-1) = \frac{1}{2}(x(-1) - x(1))$, which depends on the future time $t = 1$.*

(1 pt) Stability.

Solution. *Yes. If* $|x(t)| \leq B$ *for all values of* t*, then* $|y(t)| = \frac{1}{2}|x(t) - x(-t)| \leq \frac{1}{2}(|x(t)| + |x(-t)|) \leq B$.

(d) *Unit Impulse and Unit Step.*

i. (4 pts) Signal $x(t)$ is described below. Write an expression for $x(t)$ in terms of the step function $u(t)$.

$$x(t) = \begin{cases} 1 & \text{if } -2 \leq t < -1 \quad \text{or} \quad t > 5 \\ 2 & \text{if } 1 < t \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Hint: Plot the function first, then try to express each piece in terms of step functions.

Solution. *Plot of* $x(t)$ *is shown in Figure 1. Each "pulse" can be expressed as a difference*

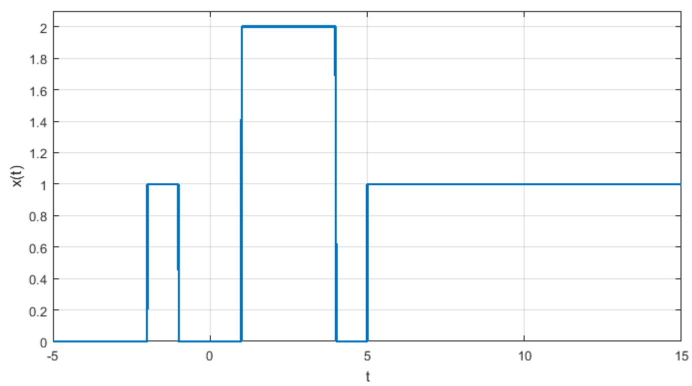


Figure 1: Plot of piecewise function $x(t)$

of step functions, or a single step function. In particular:

- *Pulse between -2 and -1 can be written as* $u(t+2) - u(t+1)$
- *Pulse between 1 and 4 can be written as* $2(u(t-1) - u(t-4))$
- *Pulse from 5 onwards can be written as* $u(t-5)$

Combining these, we get: $x(t) = u(t+2) - u(t+1) + 2(u(t-1) - u(t-4)) + u(t-5)$

ii. (4 pts) Evaluate the integral $\int_{t-4}^{\infty} \delta(\tau) d\tau$.

Solution.

$$\begin{aligned} \int_{t-4}^{\infty} \delta(\tau) d\tau = u(\tau)|_{\tau=t-4}^{\tau=\infty} = u(\infty) - u(t-4) &= 1 - u(t-4) \\ &= \left\{ \begin{array}{lll} 1 & if & t \leq 4 \\ 0 & if & t > 4 \end{array} \right. = u(-t+4) \end{aligned}$$

2. *Linearity* Consider the following input-output relationships of systems and justify if they are linear or not.

(a) (2 pts) $y(t) = 5x(\sin^2(t)) + \cos(t^3)$. Is this system linear?
Show that it is *not* linear.

Solution. *Let the system be represented by the transformation* $\mathcal{T}$*; that is,* $y(t) = \mathcal{T}(x(t))$*. Consider the input* $x_{in}(t) = \alpha x_1(t) + \beta x_2(t)$*. Let* $y_1(t) = \mathcal{T}(x_1(t))$ *and* $y_2(t) = \mathcal{T}(x_2(t))$*. For the* *system to be linear, we must have $\mathcal{T}(x_{in}(t)) = \alpha y_1(t) + \beta y_2(t)$. We evaluate both sides of this equation and check if they are equal. First, the left-hand side:*

$$\begin{aligned}\mathcal{T}(x_{in}(t)) &= 5x_{in}(\sin^2(t)) + \cos(t^3) \\ &= 5(\alpha x_1(\sin^2 t) + \beta x_2(\sin^2(t))) + \cos(t^3).\end{aligned} \tag{1}$$

For the right-hand side, we have:

$$\begin{aligned}\alpha y_1(t) + \beta y_2(t) &= \alpha(5x_1(\sin^2(t)) + \cos(t^3)) + \beta(5x_2(\sin^2(t)) + \cos(t^3)) \\ &= 5(\alpha x_1(\sin^2(t)) + \beta x_2(\sin^2(t))) + (\alpha + \beta)\cos(t^3).\end{aligned} \tag{2}$$

For general values of α and β, we do not *have the right-hand sides of Equations 1 and 2 equal. Therefore this system is* not *linear.*

(b) (2 pts) $y(t) = \frac{d^2}{dt^2}(x(t))^2$. Is this system linear?
Show that it is *not* linear.

Solution. *Let the system be represented by the transformation $\mathcal{T}$; that is, $y(t) = \mathcal{T}(x(t))$. Consider the input $x_{in}(t) = \alpha x_1(t) + \beta x_2(t)$. Let $y_1(t) = \mathcal{T}(x_1(t))$ and $y_2(t) = \mathcal{T}(x_2(t))$. Then,*

$$\begin{aligned}\mathcal{T}(x_{in}(t)) &= \frac{d^2}{dt^2}(x_{in}^2(t)) \\ &= \frac{d^2}{dt^2}(\alpha x_1(t) + \beta x_2(t))^2 \\ &= \alpha^2 \frac{d^2}{dt^2}x_1^2(t) + \beta^2 \frac{d^2}{dt^2}x_2^2(t) + 2\alpha\beta \frac{d^2}{dt^2}(x_1(t)x_2(t)).\end{aligned} \tag{3}$$

We also have

$$\begin{aligned}\alpha y_1(t) + \beta y_2(t) &= \alpha \mathcal{T}(x_1(t)) + \beta \mathcal{T}(x_2(t)) \\ &= \alpha \frac{d^2}{dt^2}x_1^2(t) + \beta \frac{d^2}{dt^2}x_2^2(t).\end{aligned} \tag{4}$$

For general values of α and β, we do not *have Equations 3 and 4 to be equal. Therefore the system is* not *linear.*

(c) (4 pts) $y(t) = \cos(5x(t) + 3)$. Is the system linear?

Solution. *Let the system transformation be represented by $\mathcal{T}$, so we have $y(t) = \mathcal{T}(x(t))$. Then $\mathcal{T}(x(t)) = \cos(x(t))$. Let $x_{in}(t) = \alpha x_1(t) + \beta x_2(t)$. Let $y_1(t) = \mathcal{T}(x_1(t))$ and $y_2(t) = \mathcal{T}(x_2(t))$. Then:*

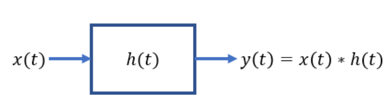
$$\begin{aligned}\mathcal{T}(x_{in}(t)) &= \cos(\alpha x_1(t) + \beta x_2(t)) \\ &= \cos(\alpha x_1(t))\cos(\beta x_2(t)) - \sin(\alpha x_1(t))\sin(\beta x_2(t)).\end{aligned} \tag{5}$$

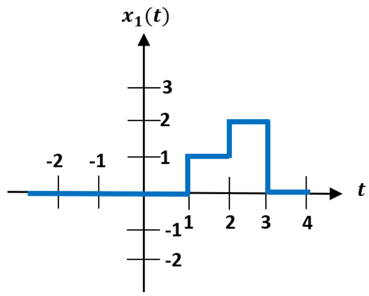
The right-hand side is

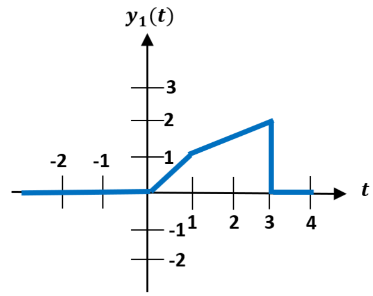
$$\alpha y_1(t) + \beta y_2(t) = \alpha \cos(x_1(t)) + \beta \cos(x_2(t)). \tag{6}$$

We do not *have the right hand sides of Equations 5 and 6 to be equal, so the system is* not *linear.*

3. *LTI Systems.*
You have an LTI system, but you don't know the system equation as shown in the figure below.

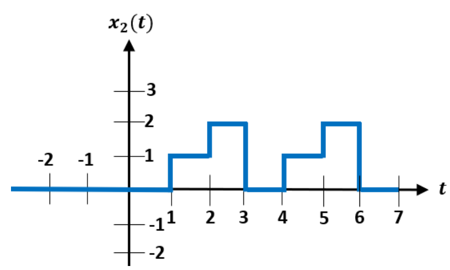


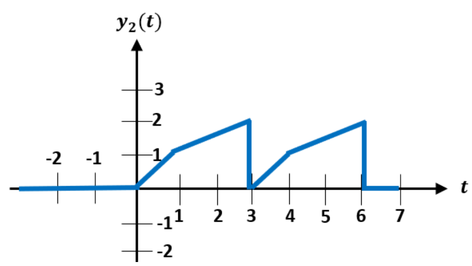




However, you know that $y_1(t)$ is the output of the system when the input is $x_1(t)$. $x_1(t)$ and $y_1(t)$ are given as below.

(a) (2 pts) Sketch $y_2(t)$ when the input is $x_2(t)$ where $x_2(t)$ is given as below. Show that sketch of $y_2(t)$ is as shown in the figure below.

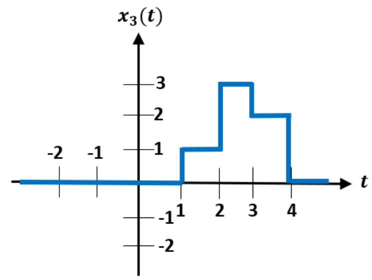


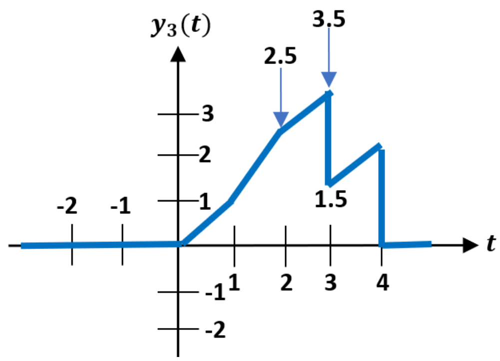


Solution. *$x_2(t)$ can be written in terms of $x_1(t)$. $x_2(t) = x_1(t)+x_1(t-3)$. Since the given system is LTI, $y_2(t) = h(x_2(t)) = h(x_1(t) + x_1(t-3)) = y_1(t) + y_1(t-3)$. The sketch of $y_2(t)$ is already shown.*

(b) (4 pts) Sketch $y_3(t)$ which is the output of the system when the input is $x_3(t)$ where $x_3(t)$ is given as below.

Solution. *$x_3(t)$ can be written in terms of $x_1(t)$. $x_3(t) = x_1(t)+x_1(t-1)$. Since the given system is LTI, $y_3(t) = h(x_3(t)) = h(x_1(t) + x_1(t-1)) = y_1(t) + y_1(t-1)$. The sketch of $y_3(t)$ is given as below.*





4. *Impulse Response.*

(a) Consider the LTI system with input-output relationship given by $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2) d\tau$.

i. (2 pts) Derive the impulse response of this system.

ii. (1 pt) Also sketch the impulse response, labeling all the key points.

Solution. *To find the impulse response, we choose $x(t) = \delta(t)$. This gives the impulse response*

$$\begin{aligned} h(t) &= \int_{\tau=-\infty}^{\infty} e^{-(t-\tau)} \delta(\tau - 2) d\tau \\ &= \begin{cases} e^{-(t-2)} & \text{if } t \geq 2 \\ 0 & \text{otherwise} \end{cases} \\ &= e^{-(t-2)} u(t-2). \end{aligned}$$

We sketch this in Figure **??**.

(b) i. (2 pts) The impulse response of a system is given by $h(t) = \delta(t+1) + \delta(t-1)$. What is the system's input-output relationship relating output $y(t)$ to input $x(t)$?

ii. (1 pt) Based on this input-output relationship, state if the system is causal or not.
Hint: The relationship between $x(t)$, $y(t)$, and $h(t)$ is given by convolution: $y(t) = x(t) * h(t)$.

Solution. *By the definition of convolution, we have $y(t) = h(t) * x(t)$. Applying the echo property, we get $y(t) = x(t+1) + x(t-1)$. Since the output at time t depends on the future input, the system is* not *causal.*

(c) (2pts) You got an awesome bluetooth speaker as Christmas present. Unfortunately your cat played with it, and it's now damaged. The speaker plays music amplified by whatever volume

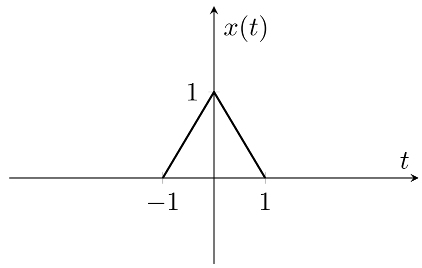


Figure 2: Input signal for Problem 5, part a

you choose; but in *addition* to this, you can hear distortion in the form of music *from two seconds ago*, *amplified* at a tenth of the selected volume. Assuming the speaker to be a simple LTI system, what is the input-output relationship? Express this in terms of the selected volume, v.

(2 pts) Also compute the impulse response of the system from this relationship.
Hint: The output is a *sum* of two signals, one good, one distorted. What is another word for 'amplified' in the language of signal operations you learnt in class? Note that the good signal only has amplification. The distorted signal, on the other hand, has amplification (at a different level!), but also, it is from some *time before the current time* - so what transformation was applied to the input to generate this distortion?

Solution. *Let the input music be $x(t)$. Then the 'good' output is $vx(t)$. The distortion is from two seconds ago, and at a tenth the volume, so it is $\frac{v}{10}x(t-2)$. Therefore the total signal is $y(t) = vx(t) + \frac{v}{10}x(t-2)$. The corresponding impulse response is $h(t) = v\delta(t) + \frac{v}{10}\delta(t-2)$.*

(d) (2 pts) The input-output relationship of an RL circuit (which is an LTI system) is given by $y(t) = \frac{R}{L}\int_{\tau=-\infty}^{t} x(\tau)e^{-\frac{R}{L}(t-\tau)}d\tau$, where R and L are the circuit resistance and inductance, respectively. Compute the impulse response of the system.

Solution. *The impulse response is obtained by plugging in $x(t) = \delta(t)$ in the input-output relationship. This is given by:*

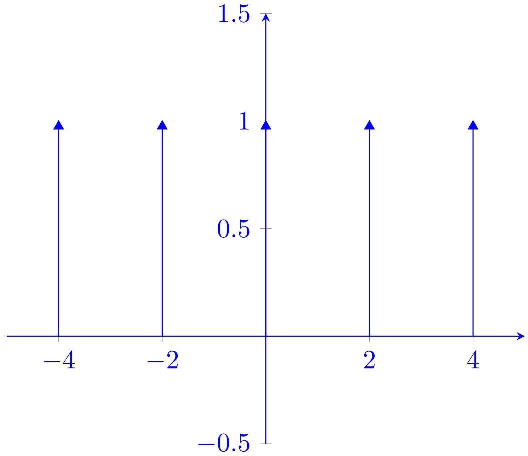
$$\begin{aligned} h(t) &= \frac{R}{L}\int_{\tau=-\infty}^{t} \delta(\tau)e^{-\frac{R}{L}(t-\tau)}d\tau \\ &= \begin{cases} \frac{R}{L}e^{-\frac{R}{L}t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{R}{L}e^{-\frac{R}{L}t}u(t). \end{aligned}$$

5. *Convolution.*

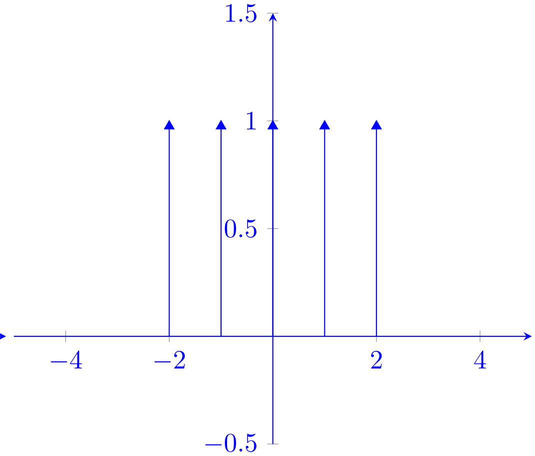
(a) (4 pts) Consider an LTI system with impulse response $h(t) = \sum_{k=-2}^{2} \delta(t-kT)$, for some positive value of T. Suppose the input to this system is the signal shown in Figure 2. Sketch the output for $T = 1$ and $T = 2$.

Solution. *In both cases, we use the echo property. For $T = 2$ and $T = 1$, the impulse responses are shown in Figure 3a and Figure 3b. The corresponding outputs are shown in Figure 4a and Figure 4.*

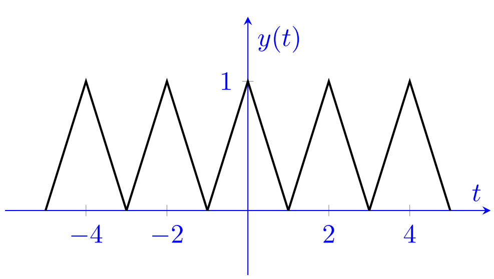
(b) (2 pts) Consider an LTI system with input $x(t) = \frac{1}{2}(u(t) - u(t-2))$ and impulse response $h(t) = 3\delta(t+1) - \delta(t-3)$.



(a) Impulse response for Problem $5a$ with $T = 2$



(b) Impulse response for Problem $5a$ with $T = 1$



(a) Output signal for Problem $5c$ with $T = 2$

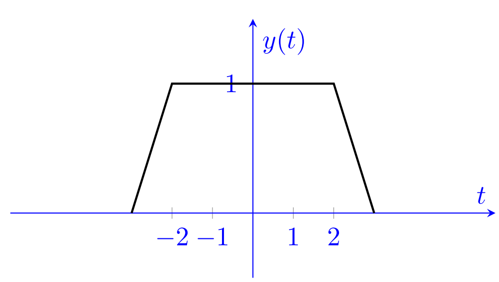


Figure 4: Output for Problem $5c$ with $T = 1$

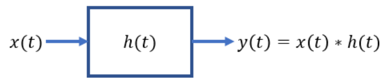
Show that $y(t) = \frac{3}{2}u(t+1) - \frac{3}{2}u(t-1) - \frac{1}{2}u(t-3) + \frac{1}{2}u(t-5)$

Solution. $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
$y(t) = \frac{1}{2}\int_{-\infty}^{\infty}(u(\tau) - u(\tau-2))(3\delta(t-\tau+1) - \delta(t-\tau-3))d\tau$
$y(t) = \frac{1}{2}\int_{-\infty}^{\infty}[3u(\tau)\delta(t+1-\tau) - 3u(\tau-2)\delta(t+1-\tau) - u(\tau)\delta(t-3-\tau) + u(\tau-2)\delta(t-3-\tau)]d\tau$
$y(t) = \frac{3}{2}\int_{-\infty}^{\infty}u(\tau)\delta(t+1-\tau)d\tau - \frac{3}{2}\int_{-\infty}^{\infty}u(\tau-2)\delta(t+1-\tau)d\tau - \frac{1}{2}\int_{-\infty}^{\infty}u(\tau)\delta(t-3-\tau)d\tau + \frac{1}{2}\int_{-\infty}^{\infty}u(\tau-2)\delta(t-3-\tau)d\tau$
$y(t) = \frac{3}{2}u(t+1) - \frac{3}{2}u(t-2+1) - \frac{1}{2}u(t-3) + \frac{1}{2}u(t-2-3)$
$y(t) = \frac{3}{2}u(t+1) - \frac{3}{2}u(t-1) - \frac{1}{2}u(t-3) + \frac{1}{2}u(t-5)$

(c) (4 pts) Consider an LTI system with input $x(t) = \delta(t) + 2\delta(t-4)$ and impulse response $h(t) = \cos(t)$. What is the output?

Solution. *Since convolution is commutative, we can swap the input and impulse response to make the calculation easy. By echo property, the output is* $y(t) = \cos(t) + 2\cos(t-4)$.

6. *Convolution II.* Consider the system as shown below.



(a) (2 pts) Consider $x(t) = e^{-(t-2)}u(t)$ and $h(t) = u(t+3)$. Find $y(t)$. Represent $y(t)$ in terms of $u(t)$. Show that $y(t) = e^2(1 - e^{-(t+3)})u(t+3)$.

Solution. $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
$= \int_{-\infty}^{\infty} e^{-(\tau-2)}u(\tau)u(t-\tau+3)d\tau$
(i) if $t < -3$*:*
$y(t) = 0$ *because of unit step functions.*
(ii) if $t \geq -3$*:*
$y(t) = \int_0^{\infty} e^{-(\tau-2)}u(t-\tau+3)d\tau = \int_0^{t+3} e^{-(\tau-2)}d\tau = e^2(1 - e^{-(t+3)})$
Hence,
$y(t) = e^2(1 - e^{-(t+3)})u(t+3)$

(b) (4 pts) $x(t) = e^t u(-4-t)$ and $h(t) = e^{-2t}u(t+2)$. Find $y(t)$ where $y(t) = x(t) * h(t)$. Represent $y(t)$ in terms of $u(t)$. Also plot $h(t-\tau)$.

Solution. $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
$y(t) = \int_{-\infty}^{\infty} e^{\tau}u(-4-\tau)e^{-2(t-\tau)}u(t-\tau+2)d\tau$

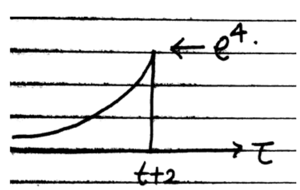
(i) if $-\infty < t < -6$
$y(t) = \int_{-\infty}^{t+2} e^{\tau}e^{-2t}e^{2\tau}d\tau$
$y(t) = e^{-2t}\int_{-\infty}^{t+2} e^{3\tau}d\tau = \frac{1}{3}e^{-2t}(e^{3(t+2)} - 0)$
$y(t) = \frac{1}{3}e^{t+6}$

(ii) if $-6 < t < \infty$
$y(t) = \int_{-\infty}^{-4} e^{\tau}e^{-2t}e^{2\tau}d\tau$
$y(t) = e^{-2t}\int_{-\infty}^{-4} e^{3\tau}d\tau = \frac{1}{3}e^{-2t}(e^{-12} - 0)$
$y(t) = \frac{1}{3}e^{-2t-12}$

$$y(t) = \begin{cases} \frac{1}{3}e^{t+6} & \textit{if } t < -6 \\ \frac{1}{3}e^{-2t-12} & \textit{if } t > -6 \end{cases}$$

$y(t) = \frac{1}{3}e^{t+6}u(-t-6) + \frac{1}{3}e^{-2t-12}u(t+6)$

$h(t-\tau)$ *is shown below.*



7. *Homework Self-Reflection* (10 pts) After completing your homework, go to the following link to rate your skill or concept understanding level for each item listed. Your self-reflection must be completed

by the due date. All submissions are time-stamped, so please give yourself plenty of time to complete and submit your self-reflection.

http://bit.ly/2mFStbQ