EE 235, Winter 2018
Homework 3: Continuous-Time Systems
(Due Friday January 19, 2018 by 12:30pm via Canvas Submission)
Write down *ALL* steps for full credit

HW3 Topics:

- System Properties: C, S, I, TI

HW3 Course Learning Goals Satisfied:

- Goal 1: Describe signals in different domains (time).
- Goal 2: Understand the implications of different system properties and how to test for them.

HW3 References:

- OWN Sections 1.4, 1.5, 1.6

HW3 Problems (Total = 106 pts):

1. *Review*

   (a) *Complex Numbers.* (2 pts) In Ch.3, you will need to be comfortable working with equations with complex numbers and terms. Here is one problem for you to practice your skills. Consider the following equation where s is complex:

   $$H(s) = 3e^{2s}$$

   Using $H(s)$ above, evaluate and simplify as much as possible the following equation $y(t) = \frac{1}{2}H(j5)e^{j5t} + \frac{1}{2}H(-j5)e^{-j5t}$. *Hint: Use Euler's formulas.*

   Solution. $y(t) = \frac{1}{2}H(j5)e^{j5t} + \frac{1}{2}H(-j5)e^{-j5t}$
   $y(t) = \frac{1}{2}(3e^{2(j5)})e^{j5t} + \frac{1}{2}(3e^{2(-j5)})e^{-j5t}$
   $y(t) = 3[\frac{1}{2}e^{j(5t+10)} + \frac{1}{2}e^{-j(5t+10)}]$
   $\boxed{y(t) = 3\cos(5t + 10)}$

   (b) *Integration* We will use this problem to continue practice doing multivariable integration. In Ch.2, you will need to evaluate a so-called running integral where the upper bound is a variable as shown below:

   $$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

   Because t is a variable and changing values, the answer for $s(t)$ will also be a function of t – that is, for different regions of t, you will get a different equation for $s(t)$. See Supplementary Notes for an example. Evaluate the integral above for the following cases:

   i. (2 pts) $h(\tau) = \begin{cases} 3, & \tau > -1 \\ 0, & \tau < -1 \end{cases}$

   Show that $s(t) = \begin{cases} 0, & t < -1 \\ 3(t+1), & t > -1 \end{cases}$

   Solution.
   Case 1, $t < -1$: Note in this case $h(\tau) = 0$, so we will use that value for the integral
   $s(t) = \int_{-\infty}^{t} 0 d\tau = 0$
   Case 2, $t > -1$: Note in this case, the running integral needs to take into account two cases for $h(\tau)$ so we will need to split up the integral and integrate from $[-\infty, -1]$ and $[-1, t]$
   $s(t) = \int_{-\infty}^{-1} 0 d\tau + \int_{-1}^{t} 3 d\tau = 3(t+1)$

   In summary, we have: $\boxed{s(t) = \begin{cases} 0, & t < -1 \\ 3(t+1), & t > -1 \end{cases}}$

ii. (4 pts) $h(\tau) = \begin{cases} 0, & \tau > 0 \\ e^{2\tau}, & \tau < 0 \end{cases}$

Solution.
Case 1, $t < 0$: Note in this case $h(\tau) = e^{2\tau}$, so we will use that value for the integral $s(t) = \int_{-\infty}^{t} e^{2\tau} d\tau = \frac{1}{2} e^{2t}$
Case 2, $t > 0$: Note in this case, the running integral needs to take into account two cases for $h(\tau)$ so we will need to split up the integral and integrate from $[-\infty, 0]$ and $[0, t]$ $s(t) = \int_{-\infty}^{0} e^{2\tau} d\tau + \int_{-1}^{t} 0 d\tau = \frac{1}{2}[e^0 - e^{-\infty}] + 0 = \frac{1}{2}$

In summary, we have: $\boxed{s(t) = \begin{cases} \frac{1}{2} e^{2t}, & t < 0 \\ \frac{1}{2}, & t > 0 \end{cases}}$

(c) *Periodicity and Even/Odd.* (4 pts)
Let $z(t) = \sin^3(4t) + e^{-j6t}$. Find the odd $z_0(t)$ part of $z(t)$. Is the odd part of $z(t)$ periodic?

Solution. $z_0(t) = \dfrac{z(t) - z(-t)}{2} = \sin^3(4t) + j\sin(6t)$
$T_1 = \dfrac{2\pi}{4} = \dfrac{pi}{2}$
$T_2 = \dfrac{2\pi}{6} = \dfrac{pi}{3}$
$T_0 = LCM(T_1, T_2) = LCM(\dfrac{\pi}{2}, \dfrac{\pi}{3}) = \pi$

2. *Energy and Power*

(a) (2 pts) $x(t) = e^{-at}u(-t), a > 0$. Evaluate E_∞ and P_∞. Is $x(t)$ an energy signal or a power signal? Show that $x(t)$ is not an energy signal and not a power signal.

Solution.
$u(-t) = \begin{cases} 1 & t \leq 0 \\ 0 & t > 0 \end{cases}$
$E_\infty = \int_{-\infty}^{0} x(t)^2 dt$
$E_\infty = \int_{-\infty}^{0} e^{-2at} dt$
$E_\infty = -\frac{1}{2a} e^{-2at}|_{-\infty}^{0}$
$E_\infty = \infty$
So, $x(t)$ is not an energy signal.
$P_\infty = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)^2 dt$
$P_\infty = \lim_{T\to\infty} \frac{1}{T} \left(-\frac{1}{2a} e^{-2at}\right)|_{-T/2}^{0}$
$P_\infty = \lim_{T\to\infty} \frac{e^{aT}-1}{2aT} = \frac{e^{\infty}-1}{\infty} = \infty$
Therefore, $x(t)$ is not a power signal

(b) (2 pts) Find the energy E_∞ and power P_∞ of the signal $x(t) = \cos^2(2t)$. Is this an energy signal or a power signal? Show that $x(t)$ has infinite energy but finite power with $P_\infty = \frac{3}{8}$, so hence it is a power signal .
Hint: The power of a periodic signal can be calculated from the power in one period.

Solution.
$E_\infty = \int_{-\infty}^{\infty} |f(t)^2| dt = \int_{-\infty}^{\infty} \cos^4(2t) dt$
Since the area under $\cos^4(2t)$ is infinite, this is not an energy signal.
The power of a periodic signal can be calculated from the power in one period:
$P_\infty = \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)^2| dt$
If it is a periodic signal

$P_\infty = \lim_{kT\to+\infty} \frac{1}{kT} \int_{-kT/2}^{kT/2} |f(t)^2| dt$
$= \lim_{kT\to+\infty} \frac{1}{kT} k \int_{-T/2}^{T/2} |f(t)^2| dt$
$= \lim_{kT\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)^2| dt$
$= \frac{1}{T} \int_{-T/2}^{T/2} |f(t)^2| dt$
So, the power of this signal is
$P_\infty = \frac{2}{2\pi} \int_{-\pi/2}^{\pi/2} \cos^4(2t) dt$
$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{3}{8} + \frac{1}{2}\cos(4t) + \frac{1}{8}\cos(8t) dt$
$= \frac{1}{\pi} (\frac{3}{8} t + \frac{1}{8}\sin(4t) + \frac{1}{64}\sin(8))|_{t=-\pi/2}^{t=\pi/2}$
$= \frac{1}{\pi} \frac{3\pi}{8} = \frac{3}{8}$
Because $0 < P_\infty < \infty$*, this is a power signal.*

(c) (4 pts) Find the energy E_∞ and power P_∞ of the signal $x(t)$ is defined as below. Is this an energy signal or a power signal?

$$x(t) = \begin{cases} t+1 & -1 < t < 0 \\ -t+1 & 0 \le t < 1 \\ 0 & otherwise \end{cases}$$

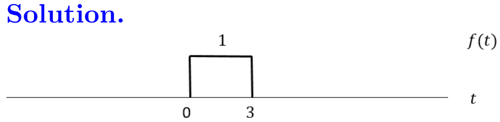
Solution.
$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-1}^{0} (t+1)^2 dt + \int_0^1 (-t+1)^2 dt = \int_{-1}^{0} t^2+2t+1 dt + \int_0^1 t^2-2t+1 dt = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$.
Hence, the energy of $x(t)$ *is* $\frac{2}{3}$.
$P_\infty = \lim_{T\to+\infty} \frac{1}{T} \times \frac{2}{3} = 0$
Hence this is an energy signal.

3. *Unit Step and Unit Impulse*

(a) Graph the following sum of unit step functions. See HW3 Supplementary Notes for extra assistance, if needed.

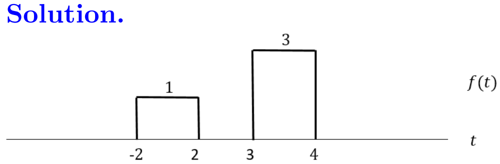
i. (2 pts) $f(t) = u(t) - u(t-3)$

Solution.



ii. (2 pts) $f(t) = u(t+2) - u(t-2) + 3u(t-3) - 3u(t-4)$

Solution.

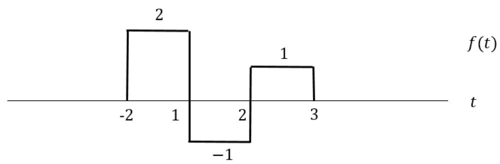


(b) Consider the following signal:

$$f(t) = -u(t-3) - 3u(t-1) + 2u(t+2) + 2u(t-2)$$

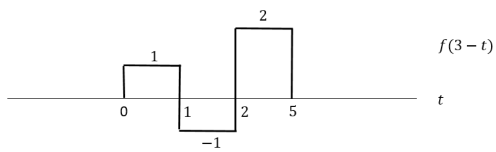
i. (2 pts) Graph $f(t)$

Solution.
Reordering terms: $f(t) = 2u(t+2) - 3u(t-1) + 2u(t-2) - u(t-3)$



ii. (2 pts) Graph $f(3-t)$

Solution.



(c) Given the following piecewise function, rewrite each function in terms of the unit step function. See HW3 Supplementary Notes for extra assistance, if needed.

i. (2 pts) $f(t) = \begin{cases} 3, & 2 < t < 4 \\ 0, & otherwise \end{cases}$

Solution.
$f(t) = \begin{cases} 3, & 2 < t < 4 \\ 0, & otherwise \end{cases} = 3 \cdot \begin{cases} 1, & 2 < t < 4 \\ 0, & otherwise \end{cases} = \boxed{3[u(t-2) - u(t-4)]}$

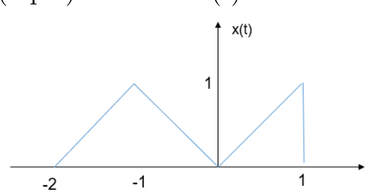
ii. (2 pts) $f(t) = \begin{cases} -2, & -3 < t < -1 \\ 4, & 5 < t < 7 \\ 0, & otherwise \end{cases}$

Solution.
$f(t) = \begin{cases} -2, & -3 < t < -1 \\ 4, & 5 < t < 7 \\ 0, & otherwise \end{cases} = -2 \cdot \begin{cases} 1, & -3 < t < -1 \\ 0, & otherwise \end{cases} + 4 \cdot \begin{cases} 1, & 5 < t < 7 \\ 0, & otherwise \end{cases} =$
$\boxed{-2[u(t+3) - u(t+1)] + 4[u(t-5) - u(t-7)]}$

(d) (2 pts) Show that the following statement is true. $\int_{-\infty}^{0} [2\delta(t+3) + 3\delta(t-1)]dt = 2$

Solution.
$\int_{-\infty}^{0} 2\delta(t+3) + 3\delta(t-1)dt = \int_{-\infty}^{0} 2\delta(t+3)dt + \int_{-\infty}^{0} 3\delta(t-1)dt = 2 + 0 = 2$

(e) (4 pts) Consider $x(t)$ below:



Simplify the following: $x(t)x(-t)[\delta(t-0.5) + \delta(t+1.5)]$.

Solution.
$x(0.5)x(-0.5)\delta(t-0.5) + x(0.5)x(-0.5)\delta(-1.5) = 0.5 \times 0.5\delta(t-0.5) + 0 = 0.25\delta(t-0.5)$.

(f) (2 pts) Evaluate $\int_{-\infty}^{\infty} \tau^2 \delta(\tau+1) d\tau$. Show that the answer is 1.

Solution.
$\int_{-\infty}^{\infty} \tau^2 \delta(\tau+1) d\tau = (-1)^2 = 1$

4. *Causal.*
Consider the following input-output relationships of a system:

(a) (2 pts) $y(t) = x(t-3) - x(3-t)$. Is the system causal? Show that the system is *not* causal.

Solution. *The system is not causal because $x(3-t)$ can depend on future values of input $x(t)$, e.g. $y(-3) = x(-6) - x(6)$.*

(b) (4 pts) $y(t) = x(t/4)$. Is the system causal?

Solution. *The system is not causal because $y(t)$ depends on future values of the input: $y(-4) = x(-1)$.*

(c) (2 pts) $y(t) = \dfrac{dx(t)}{dt}$. Is the system causal? Show that the system is causal.

Solution. *The system is causal because the derivative depends only on present values of t.*

(d) (4 pts) $y(t) = \cos(2t)x(t-7) + \sin(t)x(t-1)$. Is the system causal?

Solution. *The system is causal because $x(t-7)$ and $x(t-1)$ depend on only present or past values of t.*

(e) (4 pts) The input-output relationship is given as:
$y(t) = x(t + 3a - 2b + 4)$. Given input-output relationship, determine the values of b in terms of a that will make the system causal.

Solution. *To make the system causal, we have to make $3a - 2b + 4 \leq 0$. Hence, $3a - 2b + 4 \leq 0$ $3a + 4 \leq 2b$.* $\boxed{b \geq \dfrac{3a+4}{2}}$.

5. *Stable.*
Consider the following input-output relationships of a system:

(a) (2 pts) $y(t) = x(t)(x(t-k))$ where k is a real number. Is the system stable?
Show that the system is stable by showing that it has a constant upper bound.

Solution. *Assume $|x(t)| \leq M$ for all time t. Therefore $|y(t)| = |x(t)||x(t-k)| \leq M^2$. Since M^2 is a constant the system is stable.*

(b) (2 pts) $y(t) = \int_t^{2t} x(\tau) d\tau$. Is the system stable?
Show that the system is *not* stable by showing that it has no constant upper bound.

Solution. *Assume $|x(t)| \leq M$. Therefore $|y(t)| = |\int_t^{2t} x(\tau) d\tau| \leq |\int_t^{2t} M d\tau| = Mt. \lim_{t\to\infty}$ is not bounded, so the system is not stable.*

(c) (4 pts) $y(t) = \int_{10}^{300} x(\tau) d\tau$. Is the system stable?

Solution. *Assume $|x(t)| \leq M$ for all t. Therefore $|y(t)| = |\int_{10}^{300} x(\tau) d\tau| \leq \int_{10}^{300} M d\tau \leq 290M$. Since $290M$ is a constant the system is stable.*

(d) (4 pts) $y(t) = \frac{d}{dt}(x(t) - t^2)$. Is the system stable? Hint: find the counter-example.

Solution. *Counter example:* $x(t) = u(t)$
$y(t) = \frac{du(t)}{dt} - 2t = \delta(t) - 2t$
Input $u(t)$ *is bounded, but output* $y(t)$ *is* ∞ *when* $t = 0$*, so the system is not stable.*

6. *Invertible.*
Given the following input-output relationships of different systems find out if the system is invertible or not; if so, derive the inverse. Justify your answer in both cases.

(a) (2 pts) $y(t) = x(t/3)$. Is this system invertible?
Show that it is.

Solution. *The system represented by* $y(t) = x(t/3)$ *is invertible, because we can recover* $x(t)$ *given* $y(t)$ *as follows: let* $t' = t/3$*. Then* $x(t') = y(3t')$*.*

(b) (2 pts) $y(t) = \begin{cases} x^3(t) & t > 2 \\ 0 & otherwise \end{cases}$. Is this system invertible?
Show that it is NOT invertible.

Solution. *The system represented by* $y(t) = \begin{cases} x^3(t) & t > 2 \\ 0 & otherwise \end{cases}$ *is* not *invertible. This is because for* $t \leq 2$*, we can't recover* $x(t)$ *from zero.*

(c) (4 pts) $y(t) = tx(t)$. Is this system invertible?

Solution. *No, the system is not invertible. Both* $x(t) = \delta(t)$ *and* $x(t) = 2\delta(t)$ *give the same output,* $y(t) = 0$*.*

(d) (4 pts) $y(t) = \big(x(t)\big)^k$, where k is an integer. Is the system invertible?
Hint: Think of two cases, one when k is odd, second when k is even. It might help to draw a picture of $\big(x(t)\big)^k$ for a couple of small values of k to guess what the answer might be; once you guess it, you can prove invertibility or non-invertibility for each of the two cases.

Solution. *We have two cases. When* k *is even,* $x(t)$ *and* $-x(t)$ *produce the same output because* $(-1)^k = 1$ *for even value of* k*; therefore in this case, the system is* not *invertible. On the other hand, when* k *is odd, the system* is *invertible, since we can recover* x *by the formula* $x(t) = \big(y(t)\big)^{1/k}$*.*

(e) (2 pts) $y(t) = \begin{cases} x(t-7) & t > 1 \\ 8x(t) & t \leq 1 \end{cases}$. Is this system invertible or not?
Show that it is invertible.

Solution. *This* is *an invertible system, as can be seen by the following inverse function:*

$$x(t) = \begin{cases} y(t+7) & t > 1 \\ \frac{1}{8}y(t) & t \leq 1 \end{cases}.$$

7. *Time-Invariance.*
Given input-output relationship of a system, prove whether system is time-invariant.

(a) (2 pts) Consider the system $y(t) = x(5t) + \sin(x(t))$. Is this system time-invariant?

Solution. *We have,*

$$\begin{aligned} y(t - t_0) &= x(5(t - t_0)) + \sin(x(t - t_0)). \\ &= x(5t - 5t_0) + \sin(x(t - t_0)). \end{aligned} \tag{1}$$

Next,

$$\mathcal{T}(x(t - t_0)) = x(5t - t_0) + \sin(x(t - t_0)). \tag{2}$$

In general, the right hand sides of Equation 1 and 2 are not equal. Therefore this system is not *time-invariant.*

(b) Consider a system $\mathcal{T}$ with input $x(t)$ and output $y(t)$ related by:
$y(t) = x(t)\{g(t) + g(t-1)\}$

i. (2 pts) If $g(t) = 1$ for all t, show that $\mathcal{T}$ is time-invariant using the time-invariance test from lecture.

Solution. *For* $g(t) = 1$, $y(t) = x(t)\{1 + 1\} = 2x(t)$.
$y(t - t_0) = 2x(t - t_0)$
Let $x_1(t) = x(t - t_0)$
$y_1(t) = \mathcal{T}1(t) = 2x_1(t) = 2x(t - t_0) = y(t - t_0)$
The system $\mathcal{T}$ *is time invariant.*

ii. (2 pts) If $g(t) = t$, show that $\mathcal{T}$ is not time-invariant by using the time-invariance test from lecture.

Solution. *For* $g(t) = t$, $y(t) = x(t)\{2t - 1\} = (2t - 1)x(t)$.
$y(t - t_0) = (2t - 2t_0 - 1)x(t - t_0)$
Let $x_1(t) = x(t - t_0)$
$y_1(t) = Tx_1(t) = (2t - 1)x_1(t) = (2t - 1)x(t - t_0) \neq y(t = t_0)$
The system $\mathcal{T}$ *is not time invariant.*

(c) (2 pts) Consider the system $\mathcal{T}$ where $y(t) = \mathcal{T}(x(t)) = x(\sin(t))$. Is this time-invariant?

Solution. *We have,*

$$y(t - t_0) = x(\sin(t - t_0)). \tag{3}$$

Also,

$$\mathcal{T}(x(t - t_0)) = x(\sin(t) - t_0). \tag{4}$$

The right hand sides of Equations 3 and 4 are not the same, in general. So this is a time-varying system.

8. *Homework Self-Reflection* (10 pts) After completing your homework, go to the following link to rate your skill or concept understanding level for each item listed. Your self-reflection must be completed by the due date. All submissions are time-stamped, so please give yourself plenty of time to complete and submit your self-reflection.

`http://bit.ly/2CVDofA`