5. $\xi \sim R[\theta, 2\theta]$ $\qquad p(x,\theta) = \frac{1}{\theta}\{(\theta,2\theta)\}$

$\vec{X}_n$ – выборка

Метод моментов

$$\alpha_1 = \int_{\theta}^{2\theta} \frac{x}{\theta}\,dx = \frac{x^2}{2\theta}\Big|_{\theta}^{2\theta} = 2\theta - \frac{\theta}{2} = \frac{3}{2}\theta$$

$$\alpha_2 = \int_{\theta}^{2\theta} \frac{x^2}{\theta}\,dx = \frac{x^3}{3\theta}\Big|_{\theta}^{2\theta} = \frac{8\theta^2}{3} - \frac{\theta^2}{3} = \frac{7}{3}\theta^2$$

$$\mu_2 = \alpha_2 - \alpha_1^2 = \frac{7}{3}\theta^2 - \frac{9}{4}\theta^2 = \frac{28-27}{12}\theta^2 = \frac{\theta^2}{12}$$

$\alpha_1 = \tilde{\alpha}_1 = \overline{x}$

$\frac{3}{2}\theta = \overline{x} \Rightarrow \tilde{\theta}_1 = \frac{2}{3}\overline{x}$

$\alpha_2 = \tilde{\alpha}_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

$M[\tilde{\theta}_1] = \frac{2}{3}M[\overline{x}] = \frac{2}{3}M[\xi] = \theta \Rightarrow$ несмещ

состоятельность:

$D[\tilde{\theta}_1] = \frac{4}{9}D[\overline{x}] = \frac{4}{9n}D[\xi] = \frac{1}{27n}\theta^2 \to 0, n\to\infty$

$\Rightarrow$ состоятельная

• ~~Регулярность модели:~~

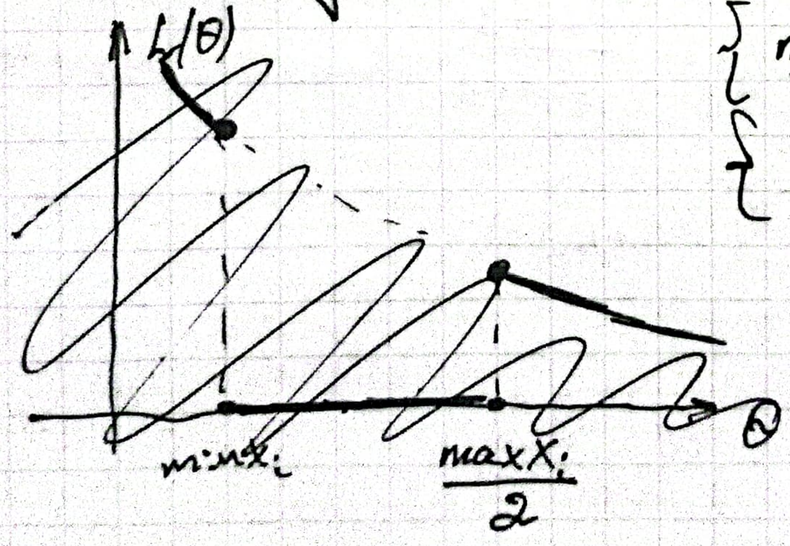
Метод максимального правдоподобия:

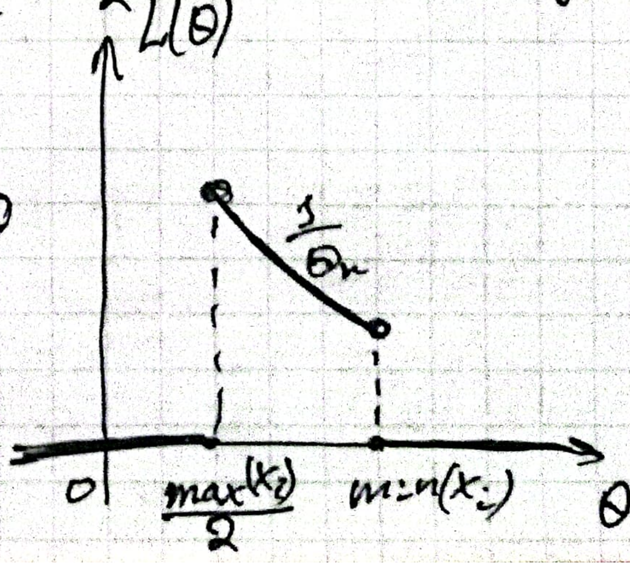
$L(\theta) = \prod_{i=1}^{n} p(x_i,\theta) = \frac{1}{\theta^n}$

$\left\{\frac{\max(x_i)}{2} < \theta, \min(x_i) > \theta\right\}$

$\Updownarrow$

$\left\{\frac{\max(x_i)}{2} < \theta < \min(x_i)\right\}$





$$\hat{\theta}_2 = \cancel{\max} \frac{X_{max}}{2}$$

$$M[\hat{\theta}_2] = \frac{1}{2} M[X_{max}] = \frac{1}{2} \int_{\theta}^{2\theta} \frac{nz}{\theta}\left(\frac{z}{\theta} - 1\right)^{n-1} dz = \left\{ z/\theta = t \right\}$$

$$= \frac{\theta}{2} \int_{1}^{2} nt(t-1)^{n-1} dt =$$

$$= \frac{\theta}{2}\left[t(t-1)^n \Big|_1^2 - \int_1^2 (t-1)^n dt \right] =$$

$$= \frac{\theta}{2}\left[t(t-1)^n - \frac{(t-1)^{n+1}}{n+1} \right]\Bigg|_1^2 =$$

$$\cancel{= \frac{\theta}{2}\left[2(2-1)^{n+1} - 0^{n+1}\right] = \frac{\theta}{2}} = \frac{\theta}{2}\left[2 - \frac{1^{n+1}}{n+1}\right] =$$

$$\cancel{\hat{\theta}_2' = 2X_{max}}$$

$$= \theta - \frac{\theta}{2(n+1)} = \frac{\theta(2n+1)}{2(n+1)}$$

$$\hat{\theta}_2' = \frac{2n+2}{2n+1} X_{max}/2 = \frac{n+1}{2n+1} X_{max}$$ – несмещ.

$$\cancel{D[\hat{\theta}_2'] = \left(\frac{n+1}{2n+1}\right)^2 D[X_{max}]}$$

$$D[\hat{\theta}_2] = M[\hat{\theta}_2^2] - M^2[\hat{\theta}_2]$$

$$M[\hat{\theta}_2^2] = \frac{1}{4} \int_{\theta}^{2\theta} \frac{nz^2}{\theta}\left(\frac{z}{\theta} - 1\right)^{n-1} dz = \left\{ \frac{z}{\theta} = t \right\} =$$

$$= \frac{\theta^2}{4}\int_1^2 n t^2 (t-1)^{n-1}\,dt = \left\{ \begin{matrix} f = t^2 & df = 2t \\ dg = n(t-1)^{n-1}dt & g = (t-1)^n \end{matrix} \right\} =$$

$$= \frac{\theta^2}{4}\left[t^2(t-1)^n \Big|_1^2 - 2\int_1^2 t(t-1)^n\,dt \right] =$$

$$= \frac{\theta^2}{4}\left[4 - \frac{2}{n+1}\int_1^2 (n+1)\,t\,(t-1)^n\,dt \right] =$$

$$\left\{ \begin{matrix} f = t & df = dt \\ dg = (n+1)(t-1)^n dt & g = (t-1)^{n+1} \end{matrix} \right\} =$$

$$= \frac{\theta^2}{4}\left[4 - \frac{2}{n+1}\left[t(t-1)^{n+1}\Big|_1^2 - \frac{(t-1)^{n+2}}{n+2}\Big|_1^2 \right] \right] =$$

$$= \frac{\theta^2}{4}\left[4 - \frac{2}{n+1}\left[2 - \frac{1}{n+2} \right] \right] = \frac{\theta^2}{4}\left[4 - \frac{4}{n+1} + \frac{2}{(n+1)(n+2)} \right] =$$

$$= \frac{\theta^2}{4}\left[\frac{4n^2+12n+\cancel{8} - 4n - \cancel{8} + 2}{(n+1)(n+2)} \right] = \frac{\theta^2(4n^2+8n+2)}{4(n+1)(n+2)}$$

$$D[\hat{\theta}_2] = \frac{\theta^2(4n^2+8n+2)}{4(n+1)(n+2)} - \frac{\theta^2(4n^2+4n+1)}{4(n+1)^2} =$$

$$= \frac{\theta^2}{4}\left[\frac{\cancel{4n^3}+\cancel{8n^2}+2n+\cancel{4n^2}+\cancel{8n}+\cancel{2}-\cancel{4n^3}-\cancel{4n^2}-n-\cancel{8n^2}-\cancel{8n}-\cancel{2}}{(n+1)^2(n+2)} \right] =$$

$$= \frac{n\theta^2}{4(n+1)^2(n+2)}$$

$$D[\tilde{\theta}_2'] = \left(\frac{2n+2}{2n+1}\right)^2 \frac{n\theta^2}{4(n+1)^2(n+2)} =$$

$$= \frac{n\theta^2}{(n+2)(2n+1)^2} \longrightarrow 0,\ n\to\infty \Rightarrow \text{состоят.}$$

c) $D[\tilde{\theta}_1] = \dfrac{\theta^2}{27n} \geq \dfrac{n\theta^2}{(n+2)(2n+1)^2} = D[\tilde{\theta}_2']$ при $n \geq 3$

$\tilde{\theta}_2'$ эффективнее $\tilde{\theta}_1$

d) Точный доверительный интервал (β)

$\vec{X}_n$ - выборка $\xi \sim R[\theta, 2\theta]$

$$f(\vec{X}_n, \theta) = \frac{X_{max}}{\theta} - 1$$

$$P(f < t) = P(X_{max} < \theta t + \theta) = (F(\theta t + \theta))^n =$$

$$= \left\{ F(x) = \begin{cases} 0, & x \leq \theta \\ \frac{x}{\theta} - 1, & \theta < x \leq 2\theta \\ 1, & x > 2\theta \end{cases} \right\} = \begin{cases} 0, & t \leq 0 \\ t^n, & 0 < t \leq 1 \\ 1, & t > 1 \end{cases}$$

$$t_1 = q_{\alpha/2} = \sqrt[n]{\alpha/2} = \sqrt[n]{\frac{1-\beta}{2}} \quad t_2 = q_{1-\frac{\alpha}{2}} = \sqrt[n]{1 - \frac{\alpha}{2}} = \sqrt[n]{\frac{1+\beta}{2}} \quad \alpha+\beta = 1$$

$$P\left(t_1 < \frac{X_{max}}{\theta} - 1 < t_2\right) = \beta$$

$$\Downarrow$$

$$t_1 + 1 < \frac{X_{max}}{\theta} < t_2 + 1$$

$$\Downarrow$$

$$\frac{1}{1+t_2} < \frac{\theta}{X_{max}} < \frac{1}{1+t_1}$$

$$P\left(\boxed{\frac{X_{max}}{1+\sqrt[n]{\frac{1+\beta}{2}}} < \theta < \frac{X_{max}}{1+\sqrt[n]{\frac{1-\beta}{2}}}}\right) = \beta$$

c) Асимптотический доверительный интервал (β)

ОММ $\tilde{\theta}_1 = \frac{2}{3}\overline{X} = \frac{2}{3}\tilde{\alpha}_1 = g(\tilde{\alpha}_1)$

$$g(\alpha_1) = \cancel{\frac{3}{2}}\frac{2}{3}\alpha_1 = \theta \quad \nabla g = \frac{2}{3}$$

$$K_{11} = \alpha_2 - \alpha_1^2 \cancel{= \frac{\theta^2}{12} - \frac{9}{4}\theta^2}$$

$$\tilde{\mu}_2 = \tilde{\alpha}_2 - \tilde{\alpha}_1^2 = \frac{S^2(n-1)}{n}$$

$$\frac{\sqrt{n}\,(\tilde{\theta}_1 - \theta)}{\frac{2S}{3}\sqrt{\frac{n-1}{n^2}}} = \frac{3n(\tilde{\theta}_1 - \theta)}{2S\sqrt{n-1}} \rightsquigarrow N(0,1)$$

$$P\left(t_1 < \frac{3n(\tilde{\theta}_1 - \theta)}{2S\sqrt{n-1}} < t_2\right) = \beta \qquad t_{1,2}^* = \mp 1.96$$

$$\Downarrow$$

$$\frac{2St_1\sqrt{n-1}}{3n} < \tilde{\theta}_1 - \theta < \frac{2St_2\sqrt{n-1}}{3n}$$

$$\Downarrow$$

$$P\left(\tilde{\theta}_1 - \frac{2St_2\sqrt{n-1}}{3n} < \theta < \tilde{\theta}_1 - \frac{2St_1\sqrt{n-1}}{3n}\right) = \beta$$