6. $$p(x) = \begin{cases} \dfrac{\theta - 1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$$

$$\int_1^{+\infty} \frac{\partial}{\partial \theta}(p(x)) = 0$$

$$\int_1^{+\infty} \frac{\partial^2}{\partial \theta^2}(p(x)) = 0$$

модель сильно регулярна

$\overline{X_n}$ - выборка

а) ОМП

$$L(\theta, \overline{X_n}) = \prod_{i=1}^{n} p(x_i, \theta) = \frac{(\theta - 1)^n}{\left(\prod_{i=1}^{n} x_i\right)^{\theta}} \{X_{min} \geq 1\}$$

$$\ln L(\theta) = n \ln(\theta - 1) - \theta \left(\sum_{i=1}^{n} \ln x_i\right)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow$$

$$\Rightarrow \theta = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-n}{(\theta - 1)^2} < 0 \Rightarrow \text{лок. max}$$

$$\tilde{\theta} = 1 + \frac{1}{\overline{\ln x}}$$

б) $$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right]$$

$$\ln p = \ln(\theta - 1) - \theta \ln x$$

$$\left(\frac{\partial \ln p}{\partial \theta}\right)^2 = \left(\frac{1}{\theta - 1} - \ln x\right)^2$$

$$I(\theta) = \int_1^{+\infty} \left(\frac{1}{(\theta - 1)^2} - 2\frac{\ln x}{\theta - 1} + \ln^2 x\right) \frac{\theta - 1}{x^{\theta}} dx =$$

$$= \int_1^{+\infty} \left(\frac{x^{-\theta}}{\theta-1} - 2\frac{\ln x}{x^{\theta}} + (\theta-1)\frac{\ln^2 x}{x^{\theta}} \right) dx =$$

$$= \frac{1}{(\theta-1)^2} - 2\int_1^{+\infty} \ln x \, d\left(\frac{-1}{(\theta-1)x^{\theta-1}}\right) + \int_1^{+\infty} \frac{(\theta-1)\ln^2 x}{x^{\theta}} dx$$

$$= \frac{1}{(\theta-1)^2} - 2\left[\ln x \cdot \frac{x^{1-\theta}}{1-\theta}\Big|_1^{+\infty} - \int_1^{+\infty} \frac{x^{-\theta}}{1-\theta} dx \right] +$$

$$+ \int_1^{+\infty} \frac{(\theta-1)\ln^2 x}{x^{\theta}} dx = \frac{-1}{(1-\theta)^2} - \int_1^{+\infty} \ln^2 x \, d\left(x^{1-\theta}\right) =$$

$$= \frac{-1}{(1-\theta)^2} - \left[\ln^2 x \cdot x^{1-\theta}\Big|_1^{+\infty} - 2\int_1^{+\infty} \frac{\ln x}{x} x^{1-\theta} dx \right] =$$

$$= \frac{-1}{(1-\theta)^2} + 2\int_1^{+\infty} \ln x \, d\left(\frac{x^{1-\theta}}{1-\theta}\right) = \frac{-1}{(1-\theta)^2} +$$

$$+ 2\left[\frac{x^{1-\theta}\ln x}{1-\theta}\Big|_1^{+\infty} - \int_1^{+\infty} \frac{x^{-\theta}}{1-\theta} dx \right] = \frac{-1}{(1-\theta)^2} +$$

$$+ 2\frac{1}{(1-\theta)^2} = \frac{1}{(1-\theta)^2}$$

$$\frac{\sqrt{n}\left(\tilde{\theta} - \theta\right)}{\tilde{\theta} - 1} \sim N(0,1)$$

$$t_1 < \sqrt{n}\ln x \left(1 + \frac{1}{\ln x} - \theta\right) < t_2$$

$$t_1 < \sqrt{n}\left((1-\theta)\ln x + 1\right) < t_2$$

$$\frac{t_1}{\sqrt{n}} - 1 < (1-\theta)\ln x < \frac{t_2}{\sqrt{n}} - 1$$

$$\frac{t_1 - \sqrt{n}}{\sqrt{n}\ln x} < 1 - \theta < \frac{t_2 - \sqrt{n}}{\sqrt{n}\ln x}$$

$$1 - \frac{t_2 - \sqrt{n}}{\sqrt{n}\ln x} < \theta < 1 - \frac{t_1 - \sqrt{n}}{\sqrt{n}\ln x}$$

б) дов. интервал для медианы

$$\frac{\partial p}{\partial \theta} = \frac{(1-\theta\ln x + \ln x)}{x^{\theta}}$$

$$\int_1^{+\infty} \frac{\partial p}{\partial \theta} dx = \int_1^{+\infty} [(1-\theta) x^{-\theta}\ln x + x^{-\theta}] dx =$$

$$= \frac{x^{-\theta+1}}{1-\theta}\Big|_1^{+\infty} + \int_1^{+\infty} \ln x \cdot d\left(x^{1-\theta}\right) =$$

$$= \frac{1}{\theta - 1} - \int_1^{+\infty} x^{-\theta} dx = 0$$

$$\frac{\partial^2 p}{\partial \theta^2} = \ln x \left(\frac{\theta\ln x - \ln x - 2}{x^{\theta}}\right)$$

$$\int_1^{+\infty} \frac{\partial^2 p}{\partial x^2} dx = \int_1^{+\infty} (\theta x^{-\theta}\ln^2 x - x^{-\theta}\ln^2 x - 2x^{-\theta}\ln x) dx$$

$$= \int_1^{+\infty} ((\theta - 1) x^{-\theta}\ln^2 x - 2x^{-\theta}\ln x) dx = 0$$

$$\int_1^{\hat{x}} \frac{\theta-1}{x^{\theta}}\,dx = \frac{-1}{\hat{x}^{\theta-1}} + 1 = \frac{1}{2}$$

$$\hat{x} = 2^{\frac{1}{\theta-1}}$$

$$g(\tilde{\theta}) = \sqrt[\tilde{\theta}-1]{2}$$

$$\sqrt{n}\,\frac{g(\tilde{\theta})-g(\theta)}{\cancel{\quad}} \rightsquigarrow N(0,1)$$

$$\sqrt{\nabla^T g(\tilde{\theta})\, I^{-1}(\tilde{\theta})\, \nabla g(\tilde{\theta})}$$

$I(\tilde{\theta})$ ~~[illegible]~~ – непр. на $(1,+\infty)$ [пункт b)]

$$\nabla g(\tilde{\theta}) = 2^{\frac{1}{\tilde{\theta}-1}} \cdot \ln(2) \cdot \frac{-1}{(\tilde{\theta}-1)^2}$$

$$\sqrt{n}\,\frac{(g(\tilde{\theta})-g(\theta))(\tilde{\theta}-1)}{2^{\frac{1}{\tilde{\theta}-1}} \cdot \ln 2} \rightsquigarrow N(0,1)$$

$$g(\tilde{\theta}) - \frac{1.96 \cdot 2^{\frac{1}{\tilde{\theta}-1}} \ln 2}{\sqrt{n}\,(\tilde{\theta}-1)} < \hat{x} < g(\tilde{\theta}) + \frac{1.96 \cdot 2^{\frac{1}{\tilde{\theta}-1}} \ln 2}{\sqrt{n}\,(\tilde{\theta}-1)}$$