3. $p(x) = \begin{cases} 1/\theta \cdot e^{-x/\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \theta > 0$ $\qquad n = 3$

$F(x) = \begin{cases} 1 - e^{-x/\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$\tilde{\theta}_1 = \overline{X} \qquad \tilde{\theta}_2 = X_{(2)}$

$$M[\xi] = \int_0^{+\infty} \frac{x}{\theta} \cdot e^{-x/\theta} dx = \theta \int_0^{+\infty} t e^{-t} dt = \theta$$

$$M[\xi^2] = \int_0^{+\infty} \frac{x^2}{\theta} \cdot e^{-x/\theta} dx = \theta^2 \int_0^{+\infty} t^2 e^{-t} dt = 2\theta^2$$

$$D[\xi] = M[\xi^2] - M^2[\xi] = 2\theta^2 - \theta^2 = \theta^2$$

$\boxed{a}$ a) $\tilde{\theta}_1 = \overline{X}$

$$M\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] = \frac{1}{n} \sum_{i=1}^{n} M[X_i] = M[\xi] = \theta \Rightarrow \text{не смещ.}$$

б) $\tilde{\theta}_2 = X_{(2)}$ $\quad \varphi(y) = n \cdot \frac{1}{\theta} C_{n-1}^{1} \left(1 - (1 - e^{-y/\theta})\right)^{n-2} \left(1 - e^{-y/\theta}\right) =$

$$= \frac{n(n-1)}{\theta} \left[e^{-\frac{y}{\theta}(n-1)} - e^{-\frac{y}{\theta} n}\right]$$

$$M[\tilde{\theta}_2] = \int_0^{+\infty} \frac{n(n-1)y}{\theta} \left[e^{-\frac{y}{\theta}(n-1)} - e^{-\frac{y}{\theta} n}\right] dy =$$

$$= n \frac{\theta}{n-1} \int_0^{+\infty} t e^{-t} dt - (n-1) \frac{\theta}{n} \int_0^{+\infty} t e^{-t} dt =$$

$$= \frac{n^2 - (n-1)^2}{n(n-1)} \theta = \frac{2n-1}{n(n-1)} \theta \qquad \hat{\theta}_2 = \frac{n(n-1)}{2n-1} X_{(2)}$$

$\boxed{B}$

$$M[\tilde{\theta}_2^2] = \int_0^{+\infty} \frac{n(n-1)y^2}{\theta}\left[e^{-\frac{y}{\theta}(n-1)} - e^{-\frac{y}{\theta}n}\right]dy =$$

$$= n\frac{\theta^2}{(n-1)^2}\int_0^{+\infty} t^2 e^{-t}dt - (n-1)\frac{\theta^2}{n^2}\int_0^{+\infty} t^2 e^{-t}dt =$$

$$= 2\theta^2\left[\frac{n^3-(n-1)^3}{n^2(n-1)^2}\right] = 2\theta^2\left[\frac{3n^2-3n+1}{n^2(n-1)^2}\right]$$

$$D[\tilde{\theta}_2] = M[\tilde{\theta}_2^2] - M^2[\theta_2] = 2\theta^2\left[\frac{3n^2-3n+1}{n^2(n-1)^2}\right] -$$

$$- \theta^2\left[\frac{4n^2-4n+1}{n^2(n-1)^2}\right] = \theta^2\left[\frac{2n^2-2n+1}{n^2(n-1)^2}\right]$$

$$\tilde{\theta}_2' = \frac{n(n-1)}{2n-1}\tilde{\theta}_2 \qquad M[\tilde{\theta}_2'] = \theta$$

$$D[\tilde{\theta}_2'] = \frac{n^2(n-1)^2}{(2n-1)^2}D[\tilde{\theta}_2] = \theta^2\left[\frac{2n^2-2n+1}{4n^2-4n+1}\right]$$

$$D[\tilde{\theta}_1] = D\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n^2}n\cdot D[\xi] = \frac{\theta^2}{n}$$

$$D[\tilde{\theta}_1] < D[\tilde{\theta}_2'] \quad \forall n > 1$$

$$D[\tilde{\theta}_1]\Big|_{n=3} = \frac{\theta^2}{3} \qquad D[\tilde{\theta}_2']\Big|_{n=3} = \frac{13}{25}\theta^2$$

$\boxed{C}$

1) $p(x) = \begin{cases} \frac{1}{\theta}e^{-x/\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ • Регулярность модели

непр. дифф. по θ на

$\Theta = (0, +\infty)$

2) $\int\limits_0^{+\infty}\left(\frac{1}{\theta}\cdot e^{-x/\theta}\cdot\frac{x}{\theta^2} - \frac{1}{\theta^2}e^{-x/\theta}\right)dx = \frac{1}{\theta} + \frac{1}{\theta}(0-1) = 0 =$

$= \frac{\partial}{\partial\theta}(1) = \frac{\partial}{\partial\theta}\left(\int\limits_0^{+\infty}\frac{1}{\theta}e^{-x/\theta}dx\right)$

3) $\ln\left(\frac{1}{\theta}\cdot e^{-x/\theta}\right) = -\frac{x}{\theta} - \ln\theta$

$\left(\frac{\partial\ln p}{\partial\theta}\right)^2 = \left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2$

$I(\theta) = M\left[\left(\frac{\partial\ln p}{\partial\theta}\right)^2\right] = \int\limits_0^{+\infty}\left(\frac{x^2}{\theta^4} - 2\frac{x}{\theta^3} + \frac{1}{\theta^2}\right)\frac{1}{\theta}e^{-x/\theta}dx =$

$= \frac{1}{\theta^2}\left[\int\limits_0^{+\infty}t^2e^{-t}dt - 2\int\limits_0^{+\infty}te^{-t}dt + \int\limits_0^{+\infty}e^{-t}dt\right] =$

$= \frac{1}{\theta^2}\left[2 - 2\cdot 1 + 1\right] = \frac{1}{\theta^2}$ – непр, >0 на $\Theta = (0,+\infty)$

Значит, модель регулярна

- Регулярность оценки

а) $\tilde{\theta}_1$ – не смещ. оценка θ, $D[\tilde{\theta}_1] = \frac{\theta^2}{n}$ – – огр. на $\forall$ компакте из Θ по θ $\Rightarrow$ $\tilde{\theta}_1$ регулярна

б) $\tilde{\theta}_2$ – не смещ. оценка θ, $D[\tilde{\theta}_2] = \theta^2\left[\frac{2n^2-2n+1}{4n^2-4n+1}\right]$ – – огр на $\forall$ компакте из Θ по θ $\Rightarrow$ $\tilde{\theta}_2$ регулярна

• Нер-во Крамéра-Рао

а) Модель регулярна, оценка $\tilde{\theta}_1$ регулярна

$g(\theta)=\theta$ - дифф. $\Rightarrow$ $\forall \theta \in \Theta$

$$D[\tilde{\theta}_1] \geq \frac{g'^2(\theta)}{nI(\theta)} = \frac{1}{3 \cdot \frac{1}{\theta^2}} = \frac{\theta^2}{3}$$

Т.к. $\frac{\theta^2}{3} = D[\tilde{\theta}_1]$, то $\tilde{\theta}_1$ - эффективная

б) Модель регулярна, оценка $\tilde{\theta}_2'$ регулярна

$g(\theta)=\theta$ - дифф. $\Rightarrow$ $\forall \theta \in \Theta$

$$D[\tilde{\theta}_2'] = \frac{g'^2(\theta)}{nI(\theta)} = \frac{\theta^2}{3}$$

Т.к. $\frac{\theta^2}{3} \neq D[\tilde{\theta}_2']$, то $\tilde{\theta}_2'$ - не эффективная

+ Т.к. $\tilde{\theta}_1$ - эффективная, то и $\tilde{\theta}_2' \neq \tilde{\theta}_1 \Rightarrow \tilde{\theta}_2'$ - не эффективна.