1. $\xi \sim R[0,\theta]$ Выборка объёма n

Оценки параметра θ:

$\tilde{\theta}_1 = 2\bar{x}$ $\tilde{\theta}_2 = X_{min}$ $\tilde{\theta}_3 = X_{max}$ $\tilde{\theta}_4 = X_1 + \frac{\sum_{k=2}^{n} X_k}{n-1}$

$p(x) = \frac{1}{\theta}\{(0,\theta)\}$ $M[\xi] = \frac{\theta}{2}$ $D[\xi] = \frac{\theta^2}{12}$

$\boxed{a}$ а) $\tilde{\theta}_1 = 2\bar{x}$ $M[\tilde{\theta}_1] = \theta \Rightarrow$ не смещ.

$D[\tilde{\theta}_1] = D\left[\frac{2}{n}\sum_{i=1}^{n} X_i\right] = \frac{4}{n^2}\sum_{i=1}^{n} D[X_i] = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$

$\Rightarrow$ состоятельная

б) $\tilde{\theta}_2 = X_{min}$ $\Phi(y) = 1-(1-F(y))^n$

$\varphi(y) = n\cdot(1-F(y))^{n-1}\cdot p(y) = n\left(1-\frac{y}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}\{(0,\theta)\}$

$M[\tilde{\theta}_2] = \int_0^\theta n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{y}{\theta}dy = \left\{\begin{matrix} t = 1-\frac{y}{\theta} \\ dt = -\frac{dy}{\theta}\end{matrix}\right\} =$

$= -\int_1^0 n t^{n-1}(1-t)\theta dt = \int_0^1 n t^{n-1}\theta dt - \int_0^1 n t^n \theta dt =$

$= \theta\left(1-\frac{n}{n+1}\right) = \frac{\theta}{n+1} \Rightarrow$ смещ.

$\tilde{\theta}_2' = (n+1)X_{min}$

$D[\tilde{\theta}_2'] = (n+1)^2 D[\tilde{\theta}_2]$

$M[\tilde{\theta}_2^2] = \int_0^\theta n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{y^2}{\theta}dy = \left\{\begin{matrix} t = 1-y/\theta \\ dt = \frac{-dy}{\theta}\end{matrix}\right\} =$

$= -\int_1^0 n t^{n-1}(1-t)^2\theta^2 dt = \int_0^1 n\theta^2 t^{n+1}dt - 2\int_0^1 n\theta^2 t^n dt +$

$$+\int_0^1 n\theta^2 t^{n-1}dt = \theta^2\left[\frac{n}{n+2} - 2\frac{n}{n+1} + 1\right] =$$

$$= \theta^2\left[\frac{n(n+1) - 2n(n+2) + (n+1)(n+2)}{(n+1)(n+2)}\right] = \theta^2\left[\frac{2}{(n+1)(n+2)}\right] =$$

$$= \frac{2\theta^2}{(n+1)(n+2)}$$

$$D[\tilde{\theta}_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \frac{\theta^2(2n+2-n-2)}{(n+1)^2(n+2)} = \frac{n\theta^2}{(n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$

$$D[\tilde{\theta}_2'] = \frac{n\theta^2}{n+2} \underset{n\to\infty}{\not\to} 0$$

$\tilde{\theta}_2'$ по опр.

$$\forall\theta>0 \quad \forall\varepsilon>0 \qquad P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$$P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \geq P(\tilde{\theta}_2' \geq \theta+\varepsilon) = P((n+1)X_{min} \geq \theta+\varepsilon) =$$

$$= P\left(X_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) = 1 - P\left(X_{min} < \frac{\theta+\varepsilon}{n+1}\right) =$$

$$= 1 - \left(1 - \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n\right) = \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \to e^{-\frac{(\theta+\varepsilon)}{\theta}} > 0 \Rightarrow$$

$\Rightarrow$ не явл. сост.

$\tilde{\theta}_2$ по опр.

$$P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) = P(\tilde{\theta}_2 \leq \theta - \varepsilon) + \cancel{P(\tilde{\theta}_2 \geq \theta+\varepsilon)}^{0} =$$

$$= P(X_{min} < \theta - \varepsilon) = \underset{\varepsilon<\theta}{\Phi(\theta-\varepsilon) =} 1 - \left(1 - \frac{\theta-\varepsilon}{\theta}\right)^n =$$

$$= 1 - \left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 1 \Rightarrow$$ не явл. сост.

в) $\tilde{\theta}_3 = X_{max} \qquad M[\tilde{\theta}_3] = \int_0^\theta n\left(\frac{z}{\theta}\right)^n dz = \frac{n}{n+1}\theta \Rightarrow$ смещ.

$$\tilde{\theta}_3' = \frac{n+1}{n} X_{max} \qquad D[\tilde{\theta}_3'] = \left(\frac{n+1}{n}\right)^2 D[\tilde{\theta}_3]$$

$$D[\hat{\theta}_3] = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\right)^2\theta^2 = \theta^2\left[\frac{n(n+1) - n^2(n+2)}{(n+1)^2(n+2)}\right] =$$

$$= \theta^2\left[\frac{n}{(n+1)^2(n+2)}\right] \xrightarrow[n\to\infty]{} 0$$

$$D[\hat{\theta}_3'] = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{сост.}$$

$\tilde{\theta}_3$ по опр. $\forall\theta>0\ \forall\varepsilon>0$

$$P(|\tilde{\theta}_3 - \theta| \ge \varepsilon) = P(X_{max} \le \theta-\varepsilon) + \overset{\nearrow 0}{P(X_{max} \ge \theta+\varepsilon)} =$$

$$= F^n(\theta-\varepsilon) \underset{0<\varepsilon<\theta}{\overset{①}{=}} \left(\frac{\theta-\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0$$

$$\underset{\varepsilon\ge\theta}{\overset{②}{=}} 0^n \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ сост.

$\tilde{\theta}_3'$ по опр. $\forall\theta>0\ \forall\varepsilon>0$

$$P(|\tilde{\theta}_3' - \theta| \ge \varepsilon) = P\left(\left(\frac{n+1}{n}\right)X_{max} \le \theta-\varepsilon\right) + P\left(\left(\frac{n+1}{n}\right)X_{max} \ge \theta+\varepsilon\right) =$$

$$= P\left(X_{max} \le \frac{n(\theta-\varepsilon)}{n+1}\right) + P\left(X_{max} \ge \frac{n(\theta+\varepsilon)}{n+1}\right) =$$

$$= F^n\left(\frac{n(\theta-\varepsilon)}{n+1}\right) + 1 - F^n\left(\frac{n(\theta+\varepsilon)}{n+1}\right) = F_1 + 1 - F_2$$

$$F_1 = \begin{cases} \theta>\varepsilon: \left(\frac{n(\theta-\varepsilon)}{(n+1)\theta}\right)^n = \left(\frac{\theta-\varepsilon}{\theta}\right)^n \cdot \left(1 - \frac{1}{n+1}\right)^n \xrightarrow[n\to\infty]{} 0 \\ \theta\le\varepsilon: 0^n \xrightarrow[n\to\infty]{} 0 \end{cases}$$

$$F_2: \quad \frac{n(\theta+\varepsilon)}{n+1} \le \theta \Rightarrow \cancel{n\theta} + n\varepsilon \le \cancel{n\theta} + \theta \Rightarrow n \le \frac{\theta}{\varepsilon} \Rightarrow$$

$= [\theta/\varepsilon]$

$$\forall\varepsilon>0\ \exists N\in\mathbb{N}: \forall n>N \hookrightarrow \frac{n(\theta+\varepsilon)}{n+1} > \theta \Rightarrow$$

$$\Rightarrow F_2 = 1^n \Rightarrow |F_2 - 1| = |1^n - 1| = 0 < \varepsilon \Rightarrow$$

$\Rightarrow F_2 \xrightarrow[n\to\infty]{} 1$

$F_1 + 1 - F_2 \xrightarrow[n\to\infty]{} 0 \Rightarrow$ сост.

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2) $\tilde{\theta}_4$ ~~$M[\tilde{\theta}_4] = \theta \Rightarrow$ не смещ.~~

~~$D[\tilde{\theta}_4] = D[X_1] + \frac{1}{(n-1)^2}\sum_{k=1}^{n} D[X_k]$~~

$M[\tilde{\theta}_4] = \theta \Rightarrow$ не смещ.

$D[\tilde{\theta}_4] = D[X_1] + \frac{1}{(n-1)^2}\sum_{k=2}^{n} D[X_k] = D[\xi] + \frac{D[\xi]}{n-1} =$

$= \frac{n}{n-1}\cdot\frac{\theta^2}{12} \not\xrightarrow[n\to\infty]{} 0$

$\tilde{\theta}_4$ по опр.

$\xi_n \xrightarrow{P} \xi,\ \eta_n \xrightarrow{P} \eta \Rightarrow \xi_n + \eta_n \xrightarrow{P} \xi + \eta$

$X_1 \xrightarrow{P} X_1$

$\frac{1}{n-1}\sum_{k=2}^{n} X_k \xrightarrow{P} M[\xi] = \frac{\theta}{2}$ (ЗБЧ Хинчина)

$\tilde{\theta}_4 \xrightarrow{P} X_1 + \frac{\theta}{2} \Rightarrow$ не сост.

$\boxed{б}$ $\tilde{\theta}_1 = 2\overline{X}$ $\tilde{\theta}'_3 = \frac{n+1}{n} X_{max}$

$D[\tilde{\theta}_1] = \frac{\theta^2}{3n}$ $D[\tilde{\theta}'_3] = \frac{\theta^2}{n(n+2)}$

$\frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3n}\ \forall\theta > 0\ \forall n > 1 \Rightarrow$

$\Rightarrow \tilde{\theta}'_3$ эффективнее, чем $\tilde{\theta}_1$