4. $p(x) = p\{(-1,1)\setminus\{0\}\} + q\{0\} + q\{2\}$

$$\int_{-\infty}^{+\infty} p(x)\,dx = \int_{-1}^{1} p\,dx + 2q = 2p + 2q = 1 \Rightarrow q = \frac{1}{2} - p$$

$\Theta = p$

$$p(x,\theta) = \theta\{(-1,1)\setminus\{0\}\} + \left(\frac{1}{2}-\theta\right)\{0\} + \left(\frac{1}{2}-\theta\right)\{2\}$$

$\vec{X}_n$ – выборка $\qquad \theta \in \left(0, \frac{1}{2}\right)$

Метод моментов.

$$\alpha_1 = \int_{-\infty}^{+\infty} x\,p(x,\theta)\,dx = \int_{-1}^{1} \theta x\,dx + 0\left(\frac{1}{2}-\theta\right) + 2\left(\frac{1}{2}-\theta\right) =$$

$$= 1 - 2\theta$$

$$\alpha_2 = \int_{-\infty}^{+\infty} x^2 p(x,\theta)\,dx = \int_{-1}^{1} \theta x^2 dx + 4\left(\frac{1}{2}-\theta\right) =$$

$$= \frac{2\theta}{3} + 2 - 4\theta = 2 - \frac{10}{3}\theta$$

$$\mu_2 = \alpha_2 - \alpha_1^2 = 2 - \frac{10}{3}\theta - 1 + 4\theta - 4\theta^2 =$$

$$= 1 + \frac{2}{3}\theta - 4\theta^2$$

$$\alpha_1 = \tilde{\alpha}_1 = \overline{X}$$

$$1 - 2\theta = \overline{X} \Rightarrow \tilde{\theta}_1 = \frac{1-\overline{X}}{2}$$

$$\alpha_2 = \tilde{\alpha}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$M[\tilde{\theta}] = M\left[\frac{1}{2} - \frac{\overline{X}}{2}\right] = \frac{1}{2} - \frac{1}{2}M[\xi] = \frac{1}{2} - \frac{1}{2}[1-2\theta] = \theta$$

несмещ.

состоятельность:

$$D[\tilde{\theta}_1] = D\left[\frac{1}{2} - \frac{\overline{X}}{2}\right] = \frac{1}{4}D[\overline{X}] = \frac{1}{4n}D[\xi] =$$

$$= \frac{1 + \frac{2}{3}\theta - 4\theta^2}{4n} \to 0,\ n \to \infty$$

эффективность:

• регулярность модели

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial\theta} p(x,\theta)\,dx = 0$$

$\int\limits_{-1}^{1} \theta \, 1 \, dx + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) = 2 - 2 = 0$

$I(\theta) = \int\limits_{-\infty}^{+\infty} \left(\frac{\partial \ln p(x,\theta)}{\partial \theta}\right)^2 p(x,\theta)\, dx =$

$= \int\limits_{-1}^{1} \frac{1}{\theta^2} \theta \, dx + \frac{\frac{1}{2} - \theta}{\left(\frac{1}{2} - \theta\right)^2} + \frac{\frac{1}{2} - \theta}{\left(\frac{1}{2} - \theta\right)^2} =$

$= \frac{2}{\theta} + \frac{2}{\frac{1}{2} - \theta} = \frac{(1-2\theta) + 2\theta}{\frac{1}{2}\theta - \theta^2} = \frac{2}{\theta(1-2\theta)} > 0$ и непр. на $(0, \frac{1}{2})$

$\Rightarrow$ модель регулярна

б) регулярность оценки

$D[\tilde{\theta}_1]$ огр. на $\forall$ компакте из $(0, \frac{1}{2}) \Rightarrow$ регул.

$D[\tilde{\theta}_1] \geq \frac{1}{n \frac{2}{\theta(1-2\theta)}} = \frac{\theta(1-2\theta)}{2n}$

$\frac{1 + \frac{2}{3}\theta - 4\theta^2}{4n} \geq \frac{\theta(1-2\theta)}{2n}$

~~Метод~~ Метод максимального правдоподобия:

$L(\theta) = \prod\limits_{i=1}^{n} p(x_i, \theta)$

Пусть в выборке $\vec{X}_n$ m раз встретилось значение из $\{0, 2\}$

$$L(\theta) = \theta^{n-m}\left(\frac{1}{2}-\theta\right)^{m}$$

$$\ln L(\theta) = (n-m)\ln\theta + m\ln\left(\frac{1}{2}-\theta\right)$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{n-m}{\theta} + \frac{m}{1/2-\theta}\cdot(-1) =$$

$$= \frac{(n-m)\left(\frac{1}{2}-\theta\right) - m\theta}{\theta\left(\frac{1}{2}-\theta\right)} = 0$$

$$\frac{n}{2} - n\theta - \frac{m}{2} = 0 \qquad \theta = \frac{n-m}{2n} =$$

$$= \frac{1}{2} - \frac{1}{2}\nu \Rightarrow \nu = -2\theta + 1$$

$$\frac{\partial^2 \ln L(\theta)}{\partial\theta^2} = \frac{m-n}{\theta^2} + \frac{m}{\left(\frac{1}{2}-\theta\right)^2}(-1)^3 =$$

$$= \frac{(m-n)\left(\frac{1}{2}-\theta\right)^2 - m\theta^2}{\theta^2\left(\frac{1}{2}-\theta\right)^2} = \cancel{\frac{\frac{m}{4} - m\theta + m\theta^2 - \frac{n}{4} + n\theta - n\theta^2}{\theta^2\left(\frac{1}{2}-\theta\right)^2}}$$

$$= \cancel{\frac{\frac{(m-n)}{4} + (n-m)\theta - n\theta^2}{\theta^2\left(\frac{1}{2}-\theta\right)^2}} = \frac{(\nu-1)\left(\frac{1}{2}-\theta\right)^2 - \nu\theta^2}{\theta^2\left(\frac{1}{2}-\theta\right)^2 / n} = \frac{n\theta\left(\theta-\frac{1}{2}\right)}{\theta^2\left(\frac{1}{2}-\theta\right)^2} < 0$$

$$\tilde{\theta}_2 = \frac{1}{2} - \frac{1}{2}\nu$$

$$M[\tilde{\theta}_2] = \frac{1}{2} - \frac{1}{2}M[\nu] = \frac{1}{2} - \frac{1}{2}p = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \theta\right) =$$

$$= \cancel{\frac{1}{4} + \frac{1}{2}\theta}\ \theta \Rightarrow \text{несмещ.}$$

$$D[\tilde{\theta}_2] = D\left[\frac{1}{2} - \frac{1}{2}\nu\right] = \frac{1}{4}D[\nu] = \frac{1}{4}\left(\frac{p(1-p)}{n}\right) =$$

$$= \frac{1}{4}\left(\frac{2\left(\frac{1}{2}-\theta\right)(2\theta)}{n}\right) = \frac{\theta\left(\frac{1}{2}-\theta\right)}{n} = \frac{\theta(1-2\theta)}{2n} \to 0,\ n\to\infty \Rightarrow$$

$\Rightarrow$ состоят.

$D[\tilde{\theta}_2]$ – огр. на $\forall$ комп. из $\left(0, \frac{1}{2}\right) \Rightarrow$ регулярн.

$$D[\tilde{\theta}_2] = \frac{\theta(1-2\theta)}{2n} \geq \frac{\theta(1-2\theta)}{2n} \Rightarrow \tilde{\theta}_2 \text{ эффективная.}$$

$\Rightarrow \tilde{\theta}_1$ – не эффективная.