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Question NM 25: Consider the contour integral $\oint \frac{dz}{z^4+z^3-2z^2}$, along the curve $|z| = 3$ oriented in the counterclockwise direction. If $\text{Res}[f, z_0]$ denotes the residue of $f(z)$ at the point z_0, then which of the following are TRUE?

(a) $\text{Res}[f, 0] = -\frac{1}{4}$
(b) $\text{Res}[f, 1] = \frac{1}{3}$
(c) $\text{Res}[f, -2] = -\frac{1}{12}$
(d) $\text{Res}[f, 2] = -1$

(GATE NM 2023)

Solution:

$$\frac{dz}{z^4 + z^3 - 2z^2} = \frac{dz}{z^2(z-1)(z+2)} \tag{1}$$

Poles: $z = 0, 1, -2$
Curve: $|z| = 3$, all poles inside it
Given the function $f(z) = \frac{1}{z^2(z-1)(z+2)}$, with poles at $z = 0$, $z = 1$, and $z = -2$, and considering the curve $|z| = 3$, where all poles are inside it. We want to find the residues of $f(z)$ at these poles.

The general formula for finding the residue at a pole z_0 is:

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z)\right) \tag{2}$$

where n denotes how many times the pole is repeated.

- For $z_0 = 0$, where $n = 2$, we have:

$$\text{Res}(f, 0) = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz} \left(\frac{1}{(z-1)(z+2)}\right) \tag{3}$$
$$= -\frac{1}{4} \tag{4}$$

- For $z_0 = 1$, where $n = 1$, we have:

$$\text{Res}(f, 1) = \frac{1}{(1-1)!} \lim_{z \to 1} \frac{z-1}{z^2(z+2)} \tag{5}$$
$$= \frac{1}{3} \tag{6}$$

- For $z_0 = -2$, where $n = 1$, we have:

$$\text{Res}(f, -2) = \frac{1}{(1-1)!} \lim_{z \to -2} \frac{z-(-2)}{z^2(z-1)} \tag{7}$$
$$= -\frac{1}{12} \tag{8}$$

Therefore, the correct answers are: options A, B, and C.

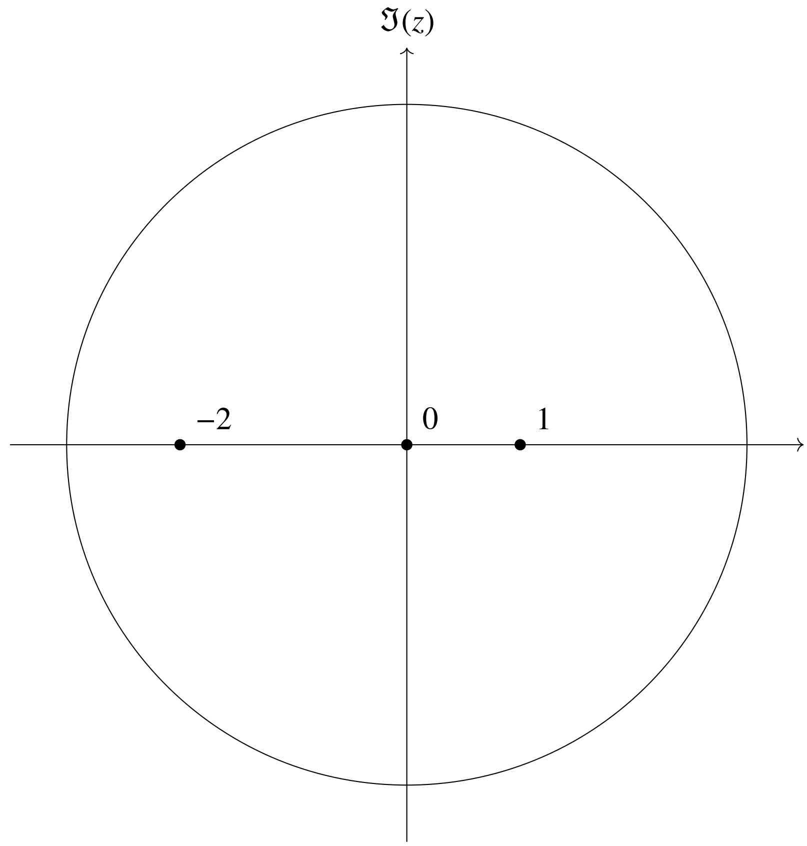

Fig. 1: Region of Convergence (ROC) for $|z| = 3$