Advanced Statistical Methods for Modeling and Finance: Solutions for CTQs

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A

1 CTQ 1

For a roll of a pair of fair 6-sided dice, the sample space is defined as $\Omega = \{(i,j), 1 \leq i,j \leq 6\}$ and we have equiprobability. Then, the probability of any event equals the number of favorable cases divided by the number of possible cases, which is $card(\Omega) = 6 \times 6 = 36$

For each expression of X, let's find the the probabilities $\mathbb{P}(X < 10)$, $\mathbb{P}(X > 40)$ and $\mathbb{P}$('X is a perfect square').

[3]:
```
import numpy as np
import math
```

[30]:
```
def solver(X):
    a=0
    print()
    for i in range(6):
        for j in range(6):
            if X[i,j]<10:
                a+=1
                print((i+1,j+1),' ; ',end='')
    print('\nNumber of favorable cases:',a)
    print('************************************')
    a=0
    for i in range(6):
        for j in range(6):
            if X[i,j]>40:
                a+=1
                print((i+1,j+1),' ; ',end='')
    print('\nNumber of favorable cases:',a)
    print('************************************')
    a=0
    for i in range(6):
        for j in range(6):
            if math.isqrt(X[i,j])**2==X[i,j]:
                a+=1
                print((i+1,j+1),' ; ',end='')
    print('\nNumber of favorable cases:',a)

X=np.zeros((6,6),dtype=int)
```

1.1 X(i,j)=i²+j²

[31]:
```
for i in range(6):
    for j in range(6):
        X[i,j]=(i+1)**2+(j+1)**2
print(X)
solver(X)
```

```
[[ 2  5 10 17 26 37]
 [ 5  8 13 20 29 40]
 [10 13 18 25 34 45]
 [17 20 25 32 41 52]
 [26 29 34 41 50 61]
 [37 40 45 52 61 72]]

(1, 1)  ; (1, 2)  ; (2, 1)  ; (2, 2)  ;
Number of favorable cases: 4
************************************
(3, 6)  ; (4, 5)  ; (4, 6)  ; (5, 4)  ; (5, 5)  ; (5, 6)  ; (6, 3)  ; (6, 4)  ;
(6, 5)  ; (6, 6)  ;
Number of favorable cases: 10
************************************
(3, 4)  ; (4, 3)  ;
Number of favorable cases: 2
```

1.1.1 Probability 1: $\mathbb{P}(X < 10)$

$\mathbb{P}(X < 10)=\frac{4}{36}=\frac{1}{9}$

1.1.2 Probability 2: $\mathbb{P}(X > 40)$

$\mathbb{P}(X > 40)=\frac{10}{36}=\frac{5}{18}$

1.1.3 Probability 3: The probability that X is a perfect square

$\mathbb{P}(\text{"X is a perfect square"})=\frac{2}{36}=\frac{1}{18}$

1.2 X(i,j)=i²+j

[32]:
```python
for i in range(6):
    for j in range(6):
        X[i,j]=(i+1)**2+(j+1)
print(X)
solver(X)
```

```
[[ 2  3  4  5  6  7]
 [ 5  6  7  8  9 10]
 [10 11 12 13 14 15]
 [17 18 19 20 21 22]
 [26 27 28 29 30 31]
 [37 38 39 40 41 42]]

(1, 1)  ; (1, 2)  ; (1, 3)  ; (1, 4)  ; (1, 5)  ; (1, 6)  ; (2, 1)  ; (2, 2)  ;
(2, 3)  ; (2, 4)  ; (2, 5)  ;
Number of favorable cases: 11
************************************
(6, 5)  ; (6, 6)  ;
```

```
Number of favorable cases: 2
************************************
(1, 3)  ; (2, 5)  ;
Number of favorable cases: 2
```

1.2.1 Probability 1: $\mathbb{P}(X < 10)$

$\mathbb{P}(X < 10)=\frac{11}{36}$

1.2.2 Probability 2: $\mathbb{P}(X > 40)$

$\mathbb{P}(X > 40)=\frac{2}{36}=\frac{1}{9}$

1.2.3 Probability 3: The probability that X is a perfect square

$\mathbb{P}(\text{"X is a perfect square"})=\frac{2}{36}=\frac{1}{18}$

1.3 X(i,j)=i+j²

[33]:
```
for i in range(6):
    for j in range(6):
        X[i,j]=(i+1)+(j+1)**2
print(X)
solver(X)
```

```
[[ 2  5 10 17 26 37]
 [ 3  6 11 18 27 38]
 [ 4  7 12 19 28 39]
 [ 5  8 13 20 29 40]
 [ 6  9 14 21 30 41]
 [ 7 10 15 22 31 42]]

(1, 1)  ; (1, 2)  ; (2, 1)  ; (2, 2)  ; (3, 1)  ; (3, 2)  ; (4, 1)  ; (4, 2)  ;
(5, 1)  ; (5, 2)  ; (6, 1)  ;
Number of favorable cases: 11
************************************
(5, 6)  ; (6, 6)  ;
Number of favorable cases: 2
************************************
(3, 1)  ; (5, 2)  ;
Number of favorable cases: 2
```

1.3.1 Probability 1: $\mathbb{P}(X < 10)$

$\mathbb{P}(X < 10)=\frac{11}{36}$

1.3.2 Probability 2: $\mathbb{P}(X > 40)$

$\mathbb{P}(X > 40)=\frac{2}{36}=\frac{1}{9}$

1.3.3 Probability 3: The probability that X is a perfect square

$\mathbb{P}$("X is a perfect square")$=\frac{2}{36}=\frac{1}{18}$

1.4 X(i,j)=(i+j)²

[34]:
```python
for i in range(6):
    for j in range(6):
        X[i,j]=(i+j+2)**2
print(X)
solver(X)
```

```
[[  4   9  16  25  36  49]
 [  9  16  25  36  49  64]
 [ 16  25  36  49  64  81]
 [ 25  36  49  64  81 100]
 [ 36  49  64  81 100 121]
 [ 49  64  81 100 121 144]]

(1, 1)  ; (1, 2)  ; (2, 1)  ;
Number of favorable cases: 3
************************************
(1, 6)  ; (2, 5)  ; (2, 6)  ; (3, 4)  ; (3, 5)  ; (3, 6)  ; (4, 3)  ; (4, 4)  ;
(4, 5)  ; (4, 6)  ; (5, 2)  ; (5, 3)  ; (5, 4)  ; (5, 5)  ; (5, 6)  ; (6, 1)  ;
(6, 2)  ; (6, 3)  ; (6, 4)  ; (6, 5)  ; (6, 6)  ;
Number of favorable cases: 21
************************************
(1, 1)  ; (1, 2)  ; (1, 3)  ; (1, 4)  ; (1, 5)  ; (1, 6)  ; (2, 1)  ; (2, 2)  ;
(2, 3)  ; (2, 4)  ; (2, 5)  ; (2, 6)  ; (3, 1)  ; (3, 2)  ; (3, 3)  ; (3, 4)  ;
(3, 5)  ; (3, 6)  ; (4, 1)  ; (4, 2)  ; (4, 3)  ; (4, 4)  ; (4, 5)  ; (4, 6)  ;
(5, 1)  ; (5, 2)  ; (5, 3)  ; (5, 4)  ; (5, 5)  ; (5, 6)  ; (6, 1)  ; (6, 2)  ;
(6, 3)  ; (6, 4)  ; (6, 5)  ; (6, 6)  ;
Number of favorable cases: 36
```

1.4.1 Probability 1: $\mathbb{P}(X < 10)$

$\mathbb{P}(X < 10)=\frac{3}{36}=\frac{1}{12}$

1.4.2 Probability 2: $\mathbb{P}(X > 40)$

$\mathbb{P}(X > 40)=\frac{21}{36}=\frac{7}{12}$

1.4.3 Probability 3: The probability that X is a perfect square

$\mathbb{P}$("X is a perfect square")$=\frac{36}{36}=1$

2 CTQ 2

2.1 Jacobian matrix: Definition

Let m and n be two non-zero natural numbers and

$$\begin{aligned} f\colon \mathbb{R}^m &\longrightarrow \mathbb{R}^n \\ x &\longmapsto f(x) = (f_i(x))_{1\leq i\leq n} \end{aligned}$$

a vectorial function which admits all its first-order partial derivates.

The Jacobian matrix of f is the matrix with n rows and m columns defined by

$$J_f(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \quad \forall x = (x_k)_{1\leq k\leq m}$$

2.2 Use of Jacobian matrix for random variables transformation(Multivariate Transformation Method)

We can use the Jacobian matrix to determine the joint PDF of a multivariate distribution Y that is a reversible function of another multivariate distribution X (i.e. X is a transformation of Y) knowing the joint **PDF** of X.

2.2.1 Theorem

Let $X_1, X_2, \ldots, X_n$ n continous random variables, $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Z} = g(\mathbf{X}) = (g_1(\mathbf{X}), g_2(\mathbf{X}), \ldots, g_n(\mathbf{X})) = (Z_1, Z_2, \ldots, Z_n)$ with g a reversible function with continous partial derivates.

Let $\mathbf{W} = g^{-1}(\mathbf{Z}) = (h_1(\mathbf{Z}), h_2(\mathbf{Z}), \ldots, h_n(\mathbf{Z})) = (W_1, W_2, \ldots, W_n)$.

The joint **PDF** of $\mathbf{Z}$ is given by:

$$f_{\mathbf{Z}}(\mathbf{z}) = f_{\mathbf{X}}(\mathbf{h}(\mathbf{z})) \cdot |\mathbf{J_h}(\mathbf{z})|$$

where:

- $f_{\mathbf{X}}$ is the joint **PDF** of $\mathbf{X}$.
- $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ is the inverse transformation function.
- $|\mathbf{J_h}|$ is the determinant of the Jacobian matrix $\mathbf{J_h}$ of the inverse transformation(inverse of the transformation g)

2.2.2 Illustration example

Let $X = (X_1, X_2)$ be a couple of idd continous random variables following the standard normal distribution.

Let another couple of continous random variables $Y = (Y_1, Y_2)$ such that

$$\begin{cases} Y_1 = 2X_1 - X_2 \\ Y_2 = -X_1 + X_2 \end{cases}$$

Let g the transformation function. We have

$g(X_1, X_2) = (2X_1 - X_2, -X_1 + X_2)$

Let us find $X = (X_1, X_2)$ such that $g(X) = Y = (Y_1, Y_2)$

$$
\begin{aligned}
g(X) = Y = (Y_1, Y_2) &\iff \begin{cases} Y_1 = 2X_1 - X_2 \\ Y_2 = -X_1 + X_2 \end{cases} \\
&\iff \begin{cases} -X_1 + X_2 = Y_2 \\ X_1 = Y_1 + Y_2 \end{cases} \\
&\iff \begin{cases} X_1 = Y_1 + Y_2 \\ X_2 = X_1 + Y_2 \end{cases} \\
&\iff \begin{cases} X_1 = Y_1 + Y_2 \\ X_2 = Y_1 + Y_2 + Y_2 \end{cases} \\
&\iff \begin{cases} X_1 = Y_1 + Y_2 \\ X_2 = Y_1 + 2Y_2 \end{cases}
\end{aligned}
$$

Then g is inversible and

$g^{-1}(X_1, X_2) = h(X_1, X_2) = (X_1 + X_2, X_1 + 2X_2) = (h_1(X_1, X_2), h_1(X_1, X_2))$

$\frac{\partial h_1}{\partial x_1}(x_1, x_2) = 1$

$\frac{\partial h_1}{\partial x_2}(x_1, x_2) = 1$

$\frac{\partial h_2}{\partial x_1}(x_1, x_2) = 1$

$\frac{\partial h_2}{\partial x_2}(x_1, x_2) = 2$

Then $|J_h(x1, x2)| = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$

Let $(x_1, x_2) \in \mathbf{R}^2$

The joint **PDF** of **Y** is given by:

$$
f_Y((x_1, x_2)) = f_X(h(x_1, x_2)) \cdot |J_h(x_1, x_2)|
$$

$$
\begin{aligned}
f_X(x_1, x_2) &= f_{X_1}(x_1) \times f_{X_2}(x_2) \text{ bacause } X_1 \perp X_2 \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}} \\
&= \frac{1}{2\pi} e^{-\frac{x_1^2}{2} - \frac{x_2^2}{2}} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}(x_1^2 + x_2^2)}
\end{aligned}
$$

$h(x_1, x_2) = (x_1 + x_2, x_1 + 2x_2)$

$$\begin{aligned}
f_X(h(x_1, x_2)) &= f_X(x_1 + x_2, x_1 + 2x_2) \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}[(x_1+x_2)^2+(x_1+2x_2)^2]} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}[x_1^2+2x_1x_2+x_2^2+x_1^2+4x_1x_2+4x_2^2]} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}[x_1^2+x_1^2+2x_1x_2+4x_1x_2+x_2^2+4x_2^2]} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}[2x_1^2+6x_1x_2+5x_2^2]}
\end{aligned}$$

And then we have:

$$\begin{aligned}
f_Y(x_1, x_2) &= \frac{1}{2\pi} e^{-\frac{1}{2}[2x_1^2+6x_1x_2+5x_2^2]} \times 1 \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}(2x_1^2+6x_1x_2+5x_2^2)}
\end{aligned}$$

3 CTQ 3

3.1 Chebyshev's inequality

3.1.1 Statment

Let X be a random variable.

If $E(X)$ and $V(X)$ exist, then $\forall a > 0$, $\mathbb{P}(|X - E(X)| \geq a) \leq \frac{V(X)}{a^2}$

3.1.2 Proof

Let $Y = (X - E(X))^2$ and $a > 0$

$|X - E(X)| \geq a \iff (X - E(X))^2 \geq a^2 \iff Y \geq a^2$

Then $\mathbb{P}(|X - E(X)| \geq a) = \mathbb{P}(Y \geq a^2)$

Y is non-negative and $E(Y) = E((X - E(X))^2) = Var(X)$ exist .

Consequently, by using Markov's inequality with $c = a^2$ and $u(X) = Y$, we have

$\mathbb{P}(Y \geq a^2) \leq \frac{E(Y)}{a^2}$

Since $\mathbb{P}(|X - E(X)| \geq a) = P(Y \geq a^2)$ and $E(Y) = E((X - E(X))^2)$ we have

$\mathbb{P}(|X - E(X)| \geq a) \leq \frac{V(X)}{a^2}$

3.2

Let $h > 0$ and X be a random variable with MGF M_X defined on $]-h, h[$

- Let $0 < t < h$
  Let us prove that $\mathbb{P}(X \geq a) \leq exp(-at)M_X(t)$

$$\begin{aligned} X \geq a &\iff tX \geq ta \text{ (because t >0)} \\ &\iff e^{tX} \geq e^{ta} \end{aligned}$$

  Then we have

$$\mathbb{P}(X \geq a) = \mathbb{P}((e^{tX} \geq e^{ta})$$

  Put $U(X) = e^{tX}$ and $c = e^{ta}$.

  As $U(X) > 0$ and $c > 0$, we can apply Markov's inequality and then we obtain

$$\begin{aligned} \mathbb{P}(X \geq a) &= \mathbb{P}((e^{tX} \geq e^{ta}) \\ &\leq \frac{E(e^{tX})}{e^{ta}} \\ &= e^{-ta}M_X(t) \end{aligned}$$

  Hence $P(X \geq a) \leq e^{-ta}M_X(t)$

- Let's $-h < t < 0$
  Let us prove that $\mathbb{P}(X \leq a) \leq exp(-at)M_X(t)$

$$\begin{aligned} X \leq a &\iff tX \geq ta \text{ (because t <0)} \\ &\iff e^{tX} \geq e^{ta} \end{aligned}$$

  Then we have

$$\mathbb{P}(X \leq a) = \mathbb{P}(e^{tX} \geq e^{ta})$$

  Put $U(X) = e^{tX}$ and $c = e^{ta}$.

  As $U(X) > 0$ and $c > 0$, we can apply Markov's inequality and then we obtain

$$\begin{aligned} \mathbb{P}(X \leq a) &= \mathbb{P}(e^{tX} \geq e^{ta}) \\ &\leq \frac{E(e^{tX})}{e^{ta}} \\ &= e^{-ta}M_X(t) \end{aligned}$$

  Hence $\mathbb{P}(X \leq a) \leq e^{-ta}M_X(t)$

4 CTQ 4

Let X be a random variable having

- μ as expectation
- σ as standard deviation
- M_X as **MGF**, such that

  $M_X(t) = (\frac{2}{3} + \frac{1}{3}e^t)^9 \ \forall t \in \mathbb{R}$

4.1 Let us show that $\mathbb{P}(\mu - 2\sigma < X < \mu + 2\sigma) = \Sigma_{x=1}^{5}\binom{9}{x}(\frac{1}{3})^x(\frac{2}{3})^{9-x}$

Let F_X be the **CDF** of X. Then we have

$$\mathbb{P}((\mu - 2\sigma < X < \mu + 2\sigma) = F_X(\mu + 2\sigma) - F_X(\mu - 2\sigma)$$

Let $t \in \mathbb{R}$ We have

$$\begin{aligned} M_X(t) &= (\frac{2}{3} + \frac{1}{3}e^t)^9 \\ &= (1 - \frac{1}{3} + \frac{1}{3}e^t)^9 \\ &= (1 - p + pe^t)^n \text{ with } p = \frac{1}{3} \text{ and } n = 9 \\ &= M_Y(t) \text{ where } Y \sim \mathcal{B}(n = 9, p = \frac{1}{3}) \end{aligned}$$

Then X and Y has the same MGF and consequently, they are the same and then we have:

$$\mathbb{P}(\mu - 2\sigma < X < \mu + 2\sigma) = F_Y(\mu + 2\sigma) - F_Y(\mu - 2\sigma) \text{ with } F_Y \text{ the } \textbf{CDF} \text{ of Y}$$

$$\begin{aligned} \mu &= np \\ &= 9 \times \frac{1}{3} \\ &= 3 \end{aligned}$$

$$\begin{aligned} \sigma &= \sqrt{npq} \\ &= \sqrt{np(1-p)} \\ &= \sqrt{9 \times \frac{1}{3} \times \frac{2}{3}} \\ &= \sqrt{2} \end{aligned}$$

$Y(\omega) = \{0, 1, \ldots, 9\}$ and $F_Y(y) = \mathbb{P}((Y < y) = \Sigma_{i=1}^{j}\mathbb{P}((Y = y_i) \ \forall y \in]y_j; y_{j+1}]$

$$\begin{aligned} \mu - 2\sigma &= 3 - 2\sqrt{2} \\ &\approx 0.1 \in]0; 1] =]y_1; y_2] \end{aligned}$$

Then

$$F_Y(\mu - 2\sigma) = \mathbb{P}((Y = y_1)$$

$$\begin{aligned} \mu + 2\sigma &= 3 + 2\sqrt{2} \\ &\approx 5.82 \in]5; 6] =]y_6; y_7] \end{aligned}$$

Then

$$F_Y(\mu + 2\sigma) = \Sigma_{i=1}^{6}\mathbb{P}((Y = y_i)$$

Consequently we have

$$\begin{aligned} \mathbb{P}(\mu - 2\sigma < X < \mu + 2\sigma) &= (\Sigma_{i=1}^{6}\mathbb{P}((Y = y_i)) - \mathbb{P}((Y = y_i) \\ &= \Sigma_{i=2}^{6}\mathbb{P}(Y = y_i) \end{aligned}$$

$$\begin{aligned} Y(\omega) &= \{0, 1, \ldots, 9\} \\ &= \{y_1, y_2, \ldots, y_{10}\} \end{aligned}$$

So $y_i = i - 1 \forall i \in \{1, 2, \ldots, 6\}$ and then $y_i = i - 1 \forall i \in \{2, 3, \ldots, 6\}$

Consequently, we have:

$$\mathbb{P}(\mu - 2\sigma < X < \mu + 2\sigma) = \Sigma_{i=2}^{6}\mathbb{P}((Y = i - 1)$$

Let $k = i - 1$

$i = 2 \Rightarrow k = 1$ and $i = 6 \Rightarrow k = 5$

Hence, as $\mathbb{P}(X = k) = \binom{9}{k}(\frac{1}{3})^k(\frac{2}{3})^{9-k}$, we get

$$\begin{aligned} \mathbb{P}(\mu - 2\sigma < X < \mu + 2\sigma) &= \Sigma_{k=1}^{5}\mathbb{P}(Y = k) \\ &= \Sigma_{k=1}^{5}\binom{9}{k}(\frac{1}{3})^k(\frac{2}{3})^{9-k} \end{aligned}$$

4.2 Let us compute that probability using R programming language

[19]:
```
#R code in jupyter notebook
n= 9
p= 1/3
```

```
mu= n * p  # Mean
sigma =sqrt(n * p * (1 - p))  # Standard deviation

#pbinom is the CDF
proba= pbinom(mu + 2 * sigma, size = n, prob = p) - pbinom(mu - 2 * sigma, size = n, prob = p)
proba
```

0.931565310166133

5 CTQ 5

Let X be a random variable that has a Poisson ditribution with λ as parameter

We have:

$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for k in $\mathbb{N}$

Let M_X be the **MGF** of X

5.1 Let us find M_X

Let $t \in \mathbb{R}$

$$\begin{aligned}
M_X(t) &= E(e^t X) \\
&= \Sigma_{k\in\mathbb{N}} e^{tk}\mathbb{P}(X = k) \\
&= \Sigma_{k\in\mathbb{N}} e^{tk}\frac{\lambda^k e^{-\lambda}}{k!} \\
&= \Sigma_{k\in\mathbb{N}} \frac{e^{tk}\lambda^k e^{-\lambda}}{k!} \\
&= \Sigma_{k\in\mathbb{N}} \frac{(e^t)^k\lambda^k e^{-\lambda}}{k!} \\
&= \Sigma_{k\in\mathbb{N}} \frac{(\lambda e^t)^k e^{-\lambda}}{k!}
\end{aligned}$$

$\forall x \in \mathbb{R}$, $e^x = \Sigma_{n\in\mathbb{N}} \frac{x^n}{n!}$

Then we have:

$$\begin{aligned}
M_X(t) &= e^{-\lambda} e^{\lambda e^t} \\
&= e^{-\lambda + \lambda e^t} \\
&= e^{\lambda e^t - \lambda} \\
&= e^{\lambda(e^t - 1)}
\end{aligned}$$

Hence $M_X(t) = e^{\lambda(e^t-1)}$

5.2 Put $Y = \frac{X-\lambda}{\sqrt{\lambda}}$

Let us show that when λ is large, $M_Y(t)$ goes to $e^{\frac{t^2}{2}}$ $\forall t \in \mathbb{R}$

Let $t \in \mathbb{R}$

$$\begin{aligned}
M_Y(t) &= M_{\frac{X-\lambda}{\sqrt{\lambda}}}(t) \\
&= E[exp(\frac{t}{\sqrt{\lambda}}(X-\lambda)] \\
&= E[exp(\frac{tX}{\sqrt{\lambda}} - \frac{\lambda}{\sqrt{\lambda}}t)] \\
&= E[exp(\frac{tX}{\sqrt{\lambda}} - \sqrt{\lambda}t)] \\
&= E[exp(\frac{tX}{\sqrt{\lambda}}) \times exp(-\sqrt{\lambda}t)] \\
&= exp(-\sqrt{\lambda}t)E[exp(\frac{tX}{\sqrt{\lambda}})] \text{ because } -\sqrt{\lambda}t \text{ doesn't depend on } X
\end{aligned}$$

$M_X(y) = E[exp(yX)]$ $\forall y \in \mathbb{R}$

For $y = \frac{t}{\sqrt{\lambda}}$ we have

$M_X(\frac{t}{\sqrt{\lambda}}) = E[exp(\frac{tX}{\sqrt{\lambda}})]$

Then, since $M_X(t) = e^{\lambda(e^t-1)}$ we have

$$\begin{aligned}
M_Y(t) &= exp(-\sqrt{\lambda}t)E[exp(\frac{tX}{\sqrt{\lambda}})] \\
&= exp(-\sqrt{\lambda}t)M_X(\frac{t}{\sqrt{\lambda}}) \\
&= exp(-\sqrt{\lambda}t)exp[\lambda(exp(\frac{t}{\sqrt{\lambda}}) - 1)]
\end{aligned}$$

$$\begin{aligned}
exp(\frac{t}{\sqrt{\lambda}}) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \\
&= exp(-\sqrt{\lambda}t)exp[\lambda(exp(\frac{t}{\sqrt{\lambda}}) - 1)]
\end{aligned}$$

$$
\begin{aligned}
exp(\frac{t}{\sqrt{\lambda}}) - 1 &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(\frac{t}{\sqrt{\lambda}})^n}{n!} \\
&= 1 + [\sum_{n=1}^{\infty} \frac{t^n}{(\sqrt{\lambda})^n n!}] - 1 \\
&= \sum_{n=1}^{\infty} \frac{t^n}{(\sqrt{\lambda})^n n!}
\end{aligned}
$$

$$
\begin{aligned}
\lambda(exp(\frac{t}{\sqrt{\lambda}}) - 1) &= \lambda \sum_{n=1}^{\infty} \frac{t^n}{(\sqrt{\lambda})^n n!} \\
&= \sum_{n=1}^{\infty} \frac{\lambda t^n}{(\sqrt{\lambda})^n n!} \\
&= \sum_{n=1}^{\infty} \frac{\lambda t^n}{(\lambda^{\frac{1}{2}})^n n!} \\
&= \sum_{n=1}^{\infty} \frac{\lambda t^n}{(\lambda)^{\frac{n}{2}} n!} \\
&= \sum_{n=1}^{\infty} \frac{(\lambda)^{1-\frac{n}{2}} t^n}{n!} \\
&= \sum_{n=1}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!} \\
&= \frac{\lambda^{\frac{1}{2}} t}{1!} + \frac{\lambda^0 t^2}{2!} + \sum_{n=3}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!} \\
&= \lambda^{\frac{1}{2}} t + \frac{t^2}{2} + \sum_{n=3}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!} \\
&= \sqrt{\lambda} t + \frac{t^2}{2} + \sum_{n=3}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
exp[\lambda(exp(\frac{t}{\sqrt{\lambda}}) - 1)] &= exp[\sqrt{\lambda} t + \frac{t^2}{2} + \sum_{n=3}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!}] \\
&= exp(\sqrt{\lambda} t) \times exp(\frac{t^2}{2}) \times exp[\sum_{n=3}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!}]
\end{aligned}
$$

$$
\begin{aligned}
M_Y(t) &= exp(-\sqrt{\lambda}t)exp[\lambda(exp(\frac{t}{\sqrt{\lambda}})-1)] \\
&= exp(-\sqrt{\lambda}t) \times exp(\sqrt{\lambda}t) \times exp(\frac{t^2}{2}) \times exp[\sum_{n=3}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!}] \\
&= exp(\frac{t^2}{2}) \times exp[\sum_{n=3}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!}]
\end{aligned}
$$

Let $n \in \mathbb{N}$

$$
\begin{aligned}
n \geq 3 &\iff n > 2 \\
&\Rightarrow -n < -2 \\
&\Rightarrow 2-n < 0 \\
&\Rightarrow \frac{2-n}{2} < 0
\end{aligned}
$$

So $\lim_{\lambda \to +\infty} (\lambda)^{\frac{2-n}{2}} = 0 \ \forall n \geq 3$

Then

$\lim_{\lambda \to +\infty} \sum_{n=3}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!} = 0$

And then

$\lim_{\lambda \to +\infty} exp[\sum_{n=3}^{\infty} \frac{(\lambda)^{\frac{2-n}{2}} t^n}{n!}] = 1$

And finaly

$\lim_{\lambda \to +\infty} M_Y(t) = exp(t^2)$

Hence when λ is too large and $Y = \frac{X-\lambda}{\sqrt{\lambda}}$, $M_Y(t)$ goes to $e^{\frac{t^2}{2}}$ $\forall t \in \mathbb{R}$

5.3 Conclusion

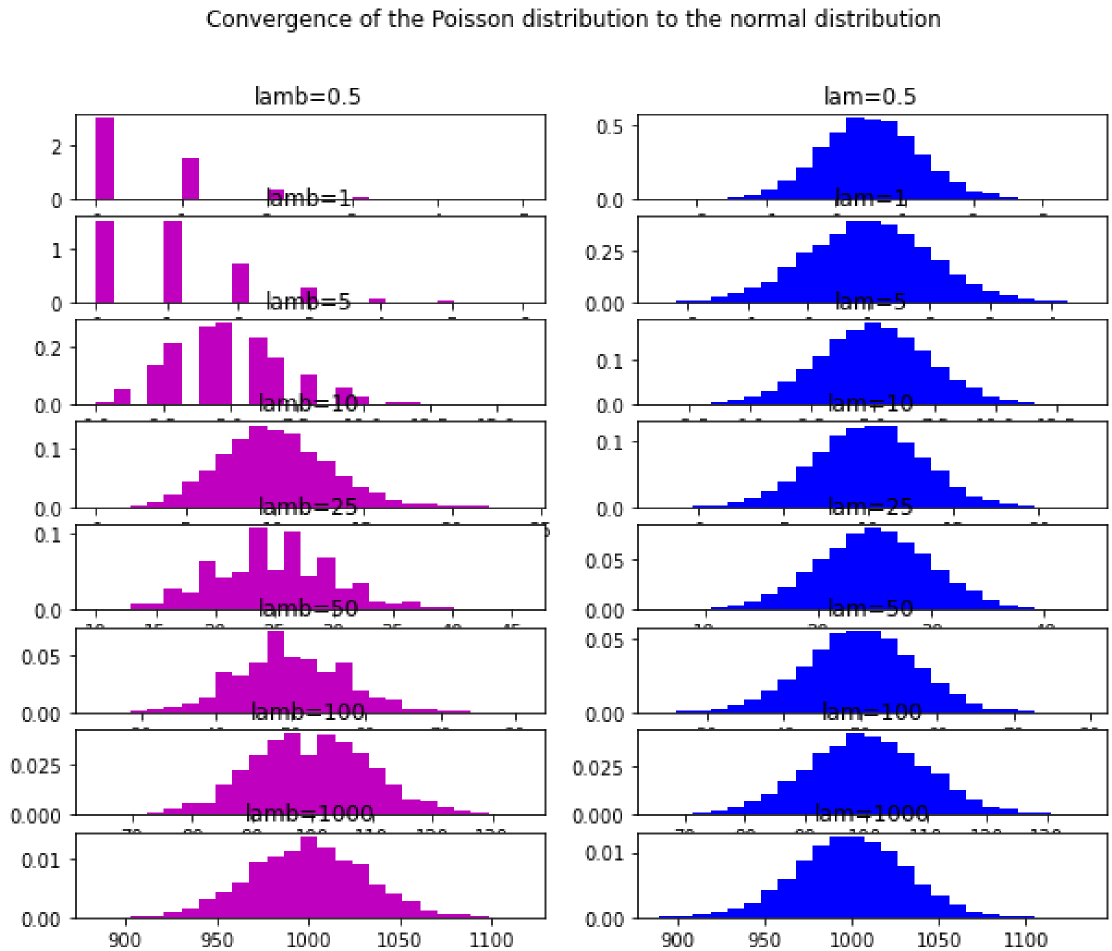
Let Z be a randow variable following the standard normal distribution. Then the **MGF** of Z is defined by $M_Z(t) = e^{\frac{t^2}{2}}$

And Then, when λ is large, $Y = \frac{X-\lambda}{\sqrt{\lambda}}$ follows the standard normal distribution and consequently $X \sim \mathcal{N}(\lambda, \sqrt{\lambda})$

5.4 Let us illustrate the convergence with a Python programm

[16]:
```
import numpy as np
import matplotlib.pyplot as plt
lambda_params = [0.5,1,5,10,25,50,100,1000]

# Générer des valeurs possibles
#X = np.linspace(-100, 100,num=100)
```

```
n=10000

#help(np.random.normal)
def func(lambda_params):
    num=1
    plt.figure(figsize=(10, 8))
    for i in range(len(lambda_params)):

        lambda_param=lambda_params[i]

        pmf_poisson = np.random.poisson(lambda_param,size=n)
        pdf_norm  = np.random.normal(loc=lambda_param,size=n,scale=np.
↪sqrt(lambda_param))

        plt.subplot(len(lambda_params),2,num)
        plt.hist(pmf_poisson,25,color='m',label='Poisson␣
↪distribution',density=True)
        plt.title('lamb='+str(lambda_param))

        plt.subplot(len(lambda_params),2,num+1)
        plt.hist(pdf_norm,25,color='blue',label='Normal␣
↪distribution',density=True)
        plt.title('lam='+str(lambda_param))
        num+=2
        #plt.legend(loc='best')

        plt.suptitle('Convergence of the Poisson distribution to the normal␣
↪distribution')
    plt.show()

func(lambda_params)
```


Convergence of the Poisson distribution to the normal distribution
lamb=0.5
lam=0.5
lamb=1
lam=1
lamb=5
lam=5
lamb=10
lam=10
lamb=25
lam=25
lamb=50
lam=50
lamb=100
lam=100
lamb=1000
lam=1000

6 CTQ 6

Let $X \sim \mathcal{U}(0,1)$ with f_X as **PDF** and F_X as **CDF**

Let $Y = e^X$ with f_Y as **PDF** and F_Y as **CDF**

Let us find F_Y

Let $t \in \mathbb{R}$

$$\begin{aligned}
F_Y(t) &= \mathbb{P}(Y \leq t) \\
&= \mathbb{P}(e^X \leq t) \\
&= \begin{cases} 0 & \text{if } t \leq 0 \\ \mathbb{P}(X \leq \ln t) & \text{else} \end{cases} \\
&= \begin{cases} 0 & \text{if } t \leq 0 \\ F_X(\ln t) & \text{else} \end{cases}
\end{aligned}$$

$$\begin{aligned}
f_Y(t) &= F'_Y(t) \\
&= \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{d}{dt}[F_X(\ln t)] & \text{else} \end{cases} \\
&= \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{1}{t} \times F'_X(\ln t) & \text{else} \end{cases} \\
&= \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{1}{t} \times f_X(\ln t) & \text{else} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\forall t > 0,\ f_X(\ln t) &= \begin{cases} 1 & \text{if } 0 \leq \ln t \leq 1 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} 1 & \text{if } 1 \leq t \leq e \\ 0 & \text{if t } \in]0;1[\cup]e;+\infty[\end{cases}
\end{aligned}$$

And then

$$\begin{aligned}
f_X(t) &= \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{1}{t} & \text{if } 1 \leq t \leq e \\ 0 & \text{if t } \in]0;1[\cup]e;+\infty[\end{cases} \\
&= \begin{cases} \frac{1}{t} & \text{if } 1 \leq t \leq e \\ 0 & \text{else} \end{cases}
\end{aligned}$$

7 CTQ 7

Let X be a randon variable having as **MGF** the function defined by $M_X(t) = e^{3t+8t^2}$ $\forall t \in \mathbb{R}$

Let us find $\mathbb{P}((-1 < X < 9)$

$$\begin{aligned} M_X(t) &= e^{3t+8t^2} \\ &= e^{\mu t + \sigma \frac{t^2}{2}} \text{ with } \mu = 3 \text{ and } \frac{\sigma^2}{2} = 8 \text{ ie } \sigma^2 = 16 \text{ and then } \sigma = 4 \end{aligned}$$

Then $X \sim \mathcal{N}(\mu, \sigma)$ and then $Z = \frac{X-\mu}{\sigma}$ follows the standard normal distribution and we have $X = \mu + \sigma Z$

$$\begin{aligned} \mathbb{P}(-1 < X < 9) &= \mathbb{P}(-1 < \mu + \sigma Z < 9) \\ &= \mathbb{P}(\frac{-1-\mu}{\sigma} < Z < \frac{9-\mu}{\sigma}) \\ &= \mathbb{P}(\frac{-1-3}{4} < Z < \frac{9-3}{4}) \\ &= \mathbb{P}(-1 < Z < \frac{6}{4}) \\ &= \mathbb{P}(-1 < Z < \frac{3}{2}) \\ &= \Phi(\frac{3}{2}) - \Phi(-1) \\ &= \Phi(\frac{3}{2}) - (1 - \Phi(1)) \\ &= \Phi(\frac{3}{2}) - 1 + \Phi(1) \\ &= \Phi(1) + \Phi(\frac{3}{2}) - 1 \end{aligned}$$

$$\begin{aligned} \Phi(1) &= \Phi(1.0 + 0.00) \\ &= 0.8413 \end{aligned}$$

$$\begin{aligned} \Phi(\frac{3}{2}) &= \Phi(1.5) \\ &= \Phi(1.5 + 0.00) \\ &= 0.9332 \end{aligned}$$

$$\begin{aligned} \mathbb{P}(-1 < X < 9) &= 0.8413 + 0.9332 - 1 \\ &= 0.7745 \end{aligned}$$

8 CTQ 8

Let us choose **Weibull distribution**.

Let X be a random varible.

X is said to follow the Weibulll distribution with parameters k and λ when it's PDF is defined by

$$f_X(t) = \begin{cases} \frac{k}{\lambda}(\frac{t}{\lambda})^{k-1} exp(-\frac{t}{\lambda})^k & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

- $k > 0$ is the **shape** parameter
- $\lambda > 0$ is the **scale** of the distribution

8.1 MGF of Weibull distribution

Let $t \in \mathbb{R}$ and M_X the mgf of X

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= \int_{-\infty}^{+\infty} e^{tx} f(x)dx \\ &= 0 + \int_0^{+\infty} e^{tx}(\frac{k}{\lambda})(\frac{x}{\lambda})^{k-1} e^{(-\frac{x}{\lambda})^k} dx \\ &= \int_0^{+\infty} e^{tx}(\frac{k}{\lambda})(\frac{x}{\lambda})^{k-1} e^{(-\frac{x}{\lambda})^k} dx \end{aligned}$$

Put $u = \frac{x}{\lambda}$. Then $x = \lambda u$ and $dx = \lambda du$.

And then we have

$$\begin{aligned} M_X(t) &= \int_0^{+\infty} e^{\lambda tu}(\frac{k}{\lambda}) u^{k-1} e^{(-u)^k}(\lambda) du \\ &= \int_0^{+\infty} e^{\lambda tu}(k) u^{k-1} e^{(-u)^k} du \end{aligned}$$

Put $x = u^k$. Then $dx = u^{k-1}du$ and $u = x^{\frac{1}{k}}$.

And then we have

$$M_X(t) = \int_0^{+\infty} e^{\lambda t(x)^{\frac{1}{k}}} e^{-x} dx$$

$\forall x \in [0;\infty[$, we have:

$$\begin{aligned} e^{\lambda t(x)^{\frac{1}{k}}} &= \sum_{n=0}^{+\infty} \frac{[\lambda t(x)^{\frac{1}{k}}]^n}{n!} \\ &= \sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!}(x)^{\frac{n}{k}} \end{aligned}$$

So we have

$$\begin{aligned} M_X(t) &= \int_0^{+\infty} [\sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!}(x)^{\frac{n}{k}}]e^{-x}dx \\ &= \int_0^{+\infty} (\sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!}(x)^{\frac{n}{k}}e^{-x})dx \\ &= \sum_{n=0}^{+\infty} (\int_0^{+\infty} \frac{(\lambda t)^n}{n!}(x)^{\frac{n}{k}}e^{-x})dx \\ &= \sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!} \int_0^{+\infty} (x)^{\frac{n}{k}}e^{-x}dx \end{aligned}$$

The Gamma function is defined by $\Gamma(y) = \int_0^{+\infty} x^{y-1}e^{-x}dx \ \forall y \in \mathbb{R}$

$y - 1 = \frac{n}{k} \iff y = 1 + \frac{n}{k}$

So $\int_0^{+\infty}(x)^{\frac{n}{k}}e^{-x}dx = \Gamma(1 + \frac{n}{k})$ and then

$$\begin{aligned} M_X(t) &= \sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!}\Gamma(1 + \frac{n}{k}) \\ &= \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!}\Gamma(1 + \frac{n}{k})t^n \end{aligned}$$

8.2 Mean and variance of Weibull distribution

- Mean $E(X)$

$E(X) = M'_X(0)$

$$\begin{aligned}
M_X(t) &= \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!}\Gamma(1+\frac{n}{k})t^n \\
&= \frac{\lambda^0}{0!}\Gamma(1+\frac{0}{k})t^0 + \sum_{n=1}^{+\infty} \frac{\lambda^n}{n!}\Gamma(1+\frac{n}{k})t^n \\
&= \Gamma(1) + \sum_{n=1}^{+\infty} \frac{\lambda^n}{n!}\Gamma(1+\frac{n}{k})t^n
\end{aligned}$$

$$\begin{aligned}
M'_X(t) &= 0 + \sum_{n=1}^{+\infty} \frac{\lambda^n}{n!}\Gamma(1+\frac{n}{k})nt^{n-1} \\
&= \sum_{n=1}^{+\infty} \frac{\lambda^n}{n(n-1)!}\Gamma(1+\frac{n}{k})nt^{n-1} \\
&= \sum_{n=1}^{+\infty} \frac{\lambda^n}{(n-1)!}\Gamma(1+\frac{n}{k})t^{n-1} \\
&= \frac{\lambda^1}{(1-1)!}\Gamma(1+\frac{1}{k})t^{1-1} + \sum_{n=2}^{+\infty} \frac{\lambda^n}{(n-1)!}\Gamma(1+\frac{n}{k})t^{n-1} \\
&= \lambda\Gamma(1+\frac{1}{k}) + \sum_{n=2}^{+\infty} \frac{\lambda^n}{(n-1)!}\Gamma(1+\frac{n}{k})t^{n-1}
\end{aligned}$$

$\forall n \geq 2,$ $n > 1$ **ie** $n-1 > 0$ and then we have

$$\begin{aligned}
M'_X(0) &= \lambda\Gamma(1+\frac{1}{k}) + \sum_{n=2}^{+\infty} \frac{\lambda^n}{(n-1)!}\Gamma(1+\frac{n}{k})0^{n-1} \\
&= \lambda\Gamma(1+\frac{1}{k}) + 0 \\
&= \lambda\Gamma(1+\frac{1}{k})
\end{aligned}$$

Hence $E(X) = \lambda\Gamma(1+\frac{1}{k})$

- Variance $Var(X)$

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = M''_X(0)$

$$\begin{aligned}
M'_X(t) &= \lambda\Gamma(1+\frac{1}{k}) + \sum_{n=2}^{+\infty}\frac{\lambda^n}{(n-1)!}\Gamma(1+\frac{n}{k})t^{n-1} \\
&= \lambda\Gamma(1+\frac{1}{k}) + \frac{\lambda^2}{(2-1)!}\Gamma(1+\frac{2}{k})t^{2-1} + \sum_{n=3}^{+\infty}\frac{\lambda^n}{(n-1)!}\Gamma(1+\frac{n}{k})t^{n-1} \\
&= \lambda\Gamma(1+\frac{1}{k}) + \lambda^2\Gamma(1+\frac{2}{k})t + \sum_{n=3}^{+\infty}\frac{\lambda^n}{(n-1)!}\Gamma(1+\frac{n}{k})t^{n-1}
\end{aligned}$$

$$\begin{aligned}
M''_X(t) &= 0 + \lambda^2\Gamma(1+\frac{2}{k}) + \sum_{n=3}^{+\infty}\frac{\lambda^n}{(n-1)!}\Gamma(1+\frac{n}{k})(n-1)t^{n-2} \\
&= \lambda^2\Gamma(1+\frac{2}{k}) + \sum_{n=3}^{+\infty}\frac{\lambda^n}{(n-1)(n-2)!}\Gamma(1+\frac{n}{k})(n-1)t^{n-2} \\
&= \lambda^2\Gamma(1+\frac{2}{k}) + \sum_{n=3}^{+\infty}\frac{\lambda^n}{(n-2)!}\Gamma(1+\frac{n}{k})t^{n-2}
\end{aligned}$$

$\forall n \geq 3$**,** $n > 3$ **ie** $n-3>0$ and then we have

$$\begin{aligned}
M''_X(0) &= \lambda^2\Gamma(1+\frac{2}{k}) + \sum_{n=3}^{+\infty}\frac{\lambda^n}{(n-2)!}\Gamma(1+\frac{n}{k})0^{n-2} \\
&= \lambda^2\Gamma(1+\frac{2}{k}) + 0 \\
&= \lambda^2\Gamma(1+\frac{2}{k})
\end{aligned}$$

Hence

$E(X^2) = \lambda^2\Gamma(1+\frac{2}{k})$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \lambda^2\Gamma(1+\frac{2}{k}) - [\lambda\Gamma(1+\frac{1}{k})]^2 \\
&= \lambda^2\Gamma(1+\frac{2}{k}) - \lambda^2[\Gamma(1+\frac{1}{k})]^2 \\
&= \lambda^2\Gamma(1+\frac{2}{k}) - \lambda^2\Gamma^2(1+\frac{1}{k}) \\
&= \lambda^2[\Gamma(1+\frac{2}{k}) - \Gamma^2(1+\frac{1}{k})]
\end{aligned}$$

8.3 One research paper in finance which uses Weibull distribution

8.3.1 Title

Portfolio value-at-risk with two-sided Weibull distribution: Evidence from cryptocurrency markets

8.3.2 Link to the paper

Here is a link to the paper: https://www.sciencedirect.com/science/article/pii/S1544612319312024.

8.3.3 Summary

The authors of that paper introduced a new Value-at-Risk(**VaR**) measurement model based on the two-sided Weibull distribution to assess potential losses in cryptocurrencies.

The Value-at-Risk (**VaR**) is the minimum loss expected on an investment, over a given time period and at a specific quantile level. It is one of the well-known market risk measures.

For their study, they used dataset on four major cryptocurrencies such as Bitcoin, Litcoin, Ripple and Dash, to compare the performance of this new model with ten other risk models using different evaluation methods. The empirical results showed that the model based on the Weibull distribution performed better than the other models.

9 CTQ 9

10 CTQ 10