Pricing European options with a log Student's t-distribution: A Gosset formula

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ABSTRACT

The distributions of returns for stocks are not well described by a normal probability density function (pdf). Student's t-distributions, which have fat tails, are known to fit the distributions of the returns. We present pricing of European call or put options using a log Student's t-distribution, which we call a Gosset approach in honour of W.S. Gosset, the author behind the *nom de plume* Student. The approach that we present can be used to price European options using other distributions and yields the Black–Scholes formula for returns described by a normal pdf.

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1. Introduction

In the fall of 2008, many investors and analysts witnessed multiple "once-in-a-lifetime" events in a single week, with disastrous consequences in many cases for the financial health of their portfolios. The once-in-a-lifetime designation of the events arises from calculation of the probabilities of events based on normal statistics. The returns were more than several sigma beyond the expected return, and by normal statistics, thought to be impossible.

The difficulty lies in the use of a normal pdf to describe returns. It is known that returns have "fat tails" [1,2], but for mathematical convenience and perhaps force of habit, a normal pdf is often applied. The celebrated Black–Scholes formula for pricing European options is based on several assumptions, one of which is that the returns are described by Brownian motion [3–5]. The underlying pdf for Brownian motion is a normal pdf.

In this paper we price European options using a log Student's t-distribution. We call these formulae of the prices of European options Gosset formulae, in honour of the author behind the *nom de plume* Student. We present evidence that stock returns are fit by a Student's t-distribution, in agreement with known results [6–9]. We then demonstrate the "fat tails" of the t-distribution. One of the difficulties with the fat tails is that one of the integrals, which is required to price an option, diverges. We present two similar approaches to handle successfully the divergence, and hence find a price for the option. These approaches can be used to price options using pdfs other than the t-distribution. However, we restrict our attention to the t-distribution and compare prices for options using the Gosset and Black–Scholes formulae. In general, the Gosset formula yields prices for options that are higher than the prices found from the Black–Scholes formula. This is not unexpected, as the Gosset formula gives greater weight to events in the tails.

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Heston [10] obtained a closed form solution for the price of options with stochastic volatility by using characteristic functions to solve partial differential equations (pde). Stochastic volatility leads to distributions with fat tails. The solution in the Heston approach is obtained by numerical computation of an integral. Numerical computation of an integral to find the price of an option is no different than finding the price of an option with the Black–Scholes formula, which requires numerical integration of the error function. Bouchaud and Sornette [11] obtained prices for options by a risk minimisation procedure that can be applied to more than just the stochastic process of uncorrelated Gaussian noise. The Black–Scholes formula for pricing options assumes Brownian motion, i.e., assumes uncorrelated Gaussian noise. Bouchaud and Sornette assumed that the average gain or loss should be zero and used this as their starting point to obtain prices by minimisation of risk. They considered the pricing of a European call option for a Lévy process and found that the cost of the option is infinite. They also recognized that truncation keeps the cost finite, but discounted the option price as being too sensitive to the level of truncation. Pinn [12] used the approach of Bouchaud and Sornette to study hedging and pricing of options using a Student's t-distribution. Pinn assumed, among other things, that the price of the stock was linearly and not exponentially related to the returns. This linear assumption keeps the integrals finite but also limits the applicability of the results to small changes. Lim et al. [13] studied the pricing of currency options using a generalized Student's t-distribution. The generalized t-distribution they employed was a t-distribution multiplied by a function of the form $\exp(-t^4)$. The exponential part of the generalized Student's t-distribution keeps the integrals needed to price the options finite. Lim et al. [13] assumed that the price of the option could be written as the expectation of the present value of the pay-off of the option. McCauley et al. [14] considered the pricing of options using distributions with fat tails. They used a Green's function approach to solve a pde and confirmed that the price of a call option is infinite with fat-tailed distributions. They considered using truncation to keep the integral finite, but felt that the option price would be too sensitive to the cut-off, meaning that essentially any option price could be predicted. Moriconi [15] studied hedging when the distribution for returns followed a Student's t-distribution multiplied by a wide Gaussian. The wide Gaussian effectively truncates the distribution and hence keeps the integrals that are required to price option finite. Moriconi solved a generalized Black–Scholes pde to obtain the price of a call.

In this paper we consider the pricing of European options using a Student's t-distribution. We find the price of the options using the arbitrage theorem and results from martingales. We argue that truncation or capping is physical and leads to realistic pricing. The equation that we obtain for the price of a European option is obtained without explicit solution of a pde [10,12,14,15] or functional minimisation [11]. The starting point for the pricing of the European options is the arbitrage theorem [16–18].

2. Arbitrage theorem

Consider an experiment that returns $r_i(e_j)$ for a bet of unity on outcome e_i of the N possible outcomes e_j of the experiment. In short, the arbitrage theorem states that either the mean return as calculated over the probability of the events e_j, $E\{r_i(e_j)\}$, is zero or there is a betting scheme that leads to a sure win. The betting scheme may allow for negative, zero, and positive bets. Puts and calls allow for negative, zero, and positive bets.

Let e_i be N disjoint events of which one and only one must occur. Let p_i be the probability of occurrence of event e_i and $\mathbf{c} = (c_1, c_2, \ldots, c_N)$ be a vector of bets on the N outcomes. The bets c_i may be zero, positive, or negative. The gain for a wager of c_i on the ith outcome given that event e_j occurred is $g_i(e_j) = c_i \times r_i(e_j)$. Note that in a typical bet, $r_i(e_j) = -1$ if $i \neq j$; the player forfeits the wager c_i if event i does not occur. The mean gain for a wager on the ith event is $\bar{g}_i = c_i \times \bar{r}_i$ and is the expectation over the e_j, which is obtained as $\sum_j p_j \times g_i(e_j) = c_i \times \sum_j p_j \times r_i(e_j)$.

Following the example of de Finetti [16], we recognize the $\bar{g}_i$ as N equations in N unknowns c_i. To ensure that no arbitrage exists, the average gain $\bar{g}_i = 0$. This is only possible for arbitrary c_i if $\bar{r}_i = 0$. If $\bar{r}_i \neq 0$, then the c_i can be selected to yield, on average, $\bar{g}_i > 0$. Thus to ensure no sure win, $\bar{r}_i = E\{r_i(e_j)\} = 0$. That this condition must hold for all i is a statement that there are no lucky numbers or values for assets. In the context of pricing an option on an asset, the events e_i could be the final price of the asset falling in a range.

Ross [18] shows how to price an option using the arbitrage theorem, assuming that the theorem can be generalized to handle functions rather than N discrete outcomes. If the cost of an option is C, then by the principle of no-arbitrage and the arbitrage theorem $C = E\{\rho_i\}$, where the return for a European call option is $\rho_i = (S_i - K, 0)^+$ and the return for a European put option is $\rho_i = (K - S_i, 0)^+$. The notation $(K - S, 0)^+$ means the maximum value of $(K - S)$ and 0, with K being the strike price and S the value of the asset.

Ross [18] obtains the Black–Scholes formula by assuming log normal statistics for the value of the asset and making some assumptions regarding the time value of the option. It is not necessary to solve the Black–Scholes partial differential equation to obtain the Black–Scholes formula for the price of a European option [18,19,10,14,15,12].

We accept as a starting point that the arbitrage theorem can be generalized from N discrete outcomes to hold for functions. We consider this as a reasonable starting point. The probability functions and values of assets can always be quantized to have N discrete ranges. We price European options assuming log Student's t-distributions, the arbitrage theorem, and results from martingale theory. The results from the martingale theory allow determination of the risk neutral price of the option.

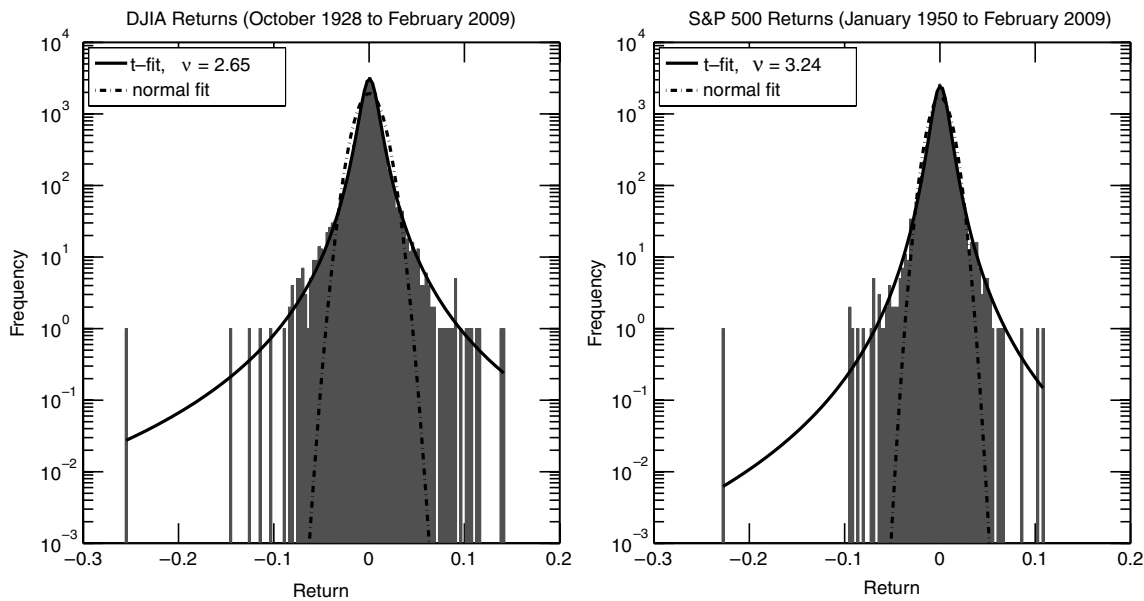First we demonstrate that the log Student's t-distribution is a reasonable distribution to use to price an option.

Fig. 1. Fit of Student's *t*-distributions and normal distributions to returns for the DJIA and S&P 500.

Table 1
Fit parameters, critical parameters, and simple descriptive statistics.

Parameter	DJIA		S&P 500	
	Normal	*t*	Normal	*t*
Location μ	$(1.679 \pm 1.6) \times 10^{-4}$	$(4.023 \pm 1.1) \times 10^{-4}$	$(2.660 \pm 1.5) \times 10^{-4}$	$(4.270 \pm 1.2) \times 10^{-4}$
Scale σ	$(1.162 \pm 0.011) \times 10^{-2}$	$(1.157 \pm 0.011) \times 10^{-2}$	$(9.55 \pm 0.11) \times 10^{-3}$	$(8.71 \pm 0.19) \times 10^{-3}$
Shape ν		2.65 ± 0.11		3.24 ± 0.19
$|x_c(10^{-4})|$	0.043	0.26	0.036	0.19
$|x_c(10^{-3})|$	0.036	0.12	0.030	0.089
$|x_c(10^{-2})|$	0.027	0.053	0.022	0.040
Count	20,186		14,870	
$\bar{r}$	1.679×10^{-4}		2.660×10^{-4}	
s	0.01157		0.00951	

3. Student's *t*-distribution

The "fat-tail" property was one of the earliest observations of asset returns, with published reports dating back to the early 1960s [1,2]. When the Black–Scholes formula was developed in the early 1970s the shortcomings of the underlying normal assumption were identified and attempts were made to incorporate kurtosis; see for example [20] and the references therein. Also in the early 1970s the Student's *t*-distribution was first used to describe asset returns [6]. There has been considerable work since 1974 comparing the *t*-distribution and other distributions to empirical data to determine which distribution best fits observations; see for example [8] and the references therein. The *t*-distribution has been found to provide realistic fits to observed asset returns (particularly equities and equity indices) and in many cases provides better fits than other distributions [8]. It is interesting to note that the distribution of returns over quiet periods has been found to be exponential in nature [21] and that the distribution of returns has been found to evolve in time [22,15]. Fat tails tend to be observed over short intervals of time and these fat-tailed distributions tend over periods of months to years to normal distributions in the central region of the distribution of returns. A semilog plot of the cumulative distribution function for the Student's *t*-distribution with $\nu = 3$ degrees of freedom is approximately linear over the central portion of the pdf and deviates above the linear relation of the central portion. This is consistent with the data and plots of Silva et al. [21].

In Fig. 1 we show fits to the observed frequency of occurrence of daily return data for the Dow Jones Industrial Average (DJIA) (left) and the S&P 500 (right) equity indices. Each plot shows the data with a best fit *t*-distribution and normal distribution. The *t*-distribution fits the observed distribution well, particularly in the tails. We have observed similar quality fits for the *t*-distribution to the frequency of occurrence of daily returns for individual equities.

The fit parameters with the associated uncertainties, some critical values, and sample descriptive statistics, such as the count of data points *N*, sample mean $\bar{r}$, and sample standard deviation *s* are given in Table 1. The critical values $x_c(q)$ are the returns which solve $P\{\xi \le x_c\} = q$. Given that there are 21,186 data points for the DJIA, it is expected, using the best fit *t*-distribution, that only 2 returns will fall below -0.26. This is in agreement with the data shown in Fig. 1.
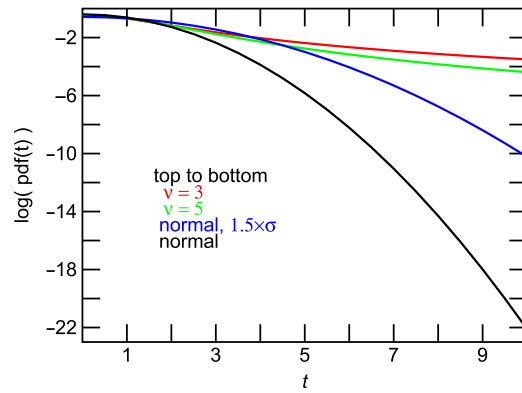
Fig. 2. Comparison of Student's t-distributions and normal pdfs.

Fig. 2 is a plot of the value of the Student's t-distribution and the normal distribution as a function of the number of standard deviations from the true mean. The fat tails of the Student's t-distribution are evident. Events that are 10 standard deviations from the true value are expected approximately 10^{-2}% of the time for the Student's t-distribution with a small number of degrees of freedom, but only 10^{-18}% of the time for a normal pdf. For comparison, the values for $N(0, 1.5\sigma)$, a normal pdf with a mean of zero and a standard deviation $1.5\times$ greater than for the other curves is presented. A volatility of $1.5\times$ the measured volatility is sometimes used to price options. The enhanced volatility effectively gives the normal pdf "fatter tails". It can be observed in Fig. 2 that the $N(0, 1.5\sigma)$ curve tracks the Student's t-distribution over a greater distance from the true mean.

One possible justification for the use of Student's t-distribution to describe the daily returns for an asset is as follows. If the returns are assumed to be normally distributed with a given short term variance, the variance is taken to be a random variable, i.e., stochastic volatility [10,22,15], that varies slowly in time, and $f_t(x)$ is the pdf for the return at time some time t in the future, then $f_t(x) = \int f_t(x|\sigma_t) \times pdf_{\sigma_t}(\sigma_t)d\sigma_t$ where $f_t(x|\sigma)$ is the pdf for the return given σ and $pdf_{\sigma_t}(\sigma_t)$ is the pdf for σ at time t. Fits to the 22-day volatilities for the DJIA and for the S&P 500 show that a gamma distribution fits the reciprocal of the 22-day volatilities. A gamma distribution with shape parameter ν is chi-squared with 2ν degrees of freedom [23], and a normal distribution with a reciprocal of the variance that is chi-squared distributed is a Student's t-distribution. A Student's t-distribution for the pdf of the daily returns is consistent with the data shown in Fig. 1.

Fig. 3 is a plot of a fit of a chi-squared distribution to the frequency of occurrence of the reciprocal of the 22-day volatility for the DJIA index over the period from June 1997 to June 2009. The thin lines give the cumulative density function for the fit and for the data. These thin lines can be used to assess the quality of the fit. The data appear to be described reasonably well by a chi-squared distribution. Fits of a chi-squared distribution to the frequency of occurrence of the reciprocal of the 22-day volatility for the S&P 500 index can be found in Ref. [24]. The S&P 500 data fit a chi-squared distribution better than the fit shown in Fig. 3 Presumably this is because the S&P 500 data includes a greater number of stocks than the DJIA and thus the noise is reduced.

Bakshi et al. [25] compared the performance of several pricing models to empirical data. These models included Black–Scholes, stochastic volatility, stochastic volatility and stochastic jumps, and the stochastic jump and volatility models with stochastic risk free rate. These researchers concluded "Our empirical evidence indicates that regardless of performance yardstick, taking stochastic volatility into account is of first order importance in improving upon the Black–Scholes formula". Bakshi et al. found that incorporating stochastic volatility in a model for the price of options reduced the absolute pricing errors by 20%–70% over the Black–Scholes model. Slanina [26] examined stochastic volatility models and found their short time and asymptotic properties.

The variance of a Student's t-distribution is $\nu/(\nu - 2)$ [23]. As ν tends to infinity, the t-distribution tends to a standard normal pdf, which has a variance of one. The $\nu/(\nu - 2)$ enhancement accounts for the range of values that the variance might take. If it is known that the variance does not change in time (i.e., $pdf_{\sigma_t}(\sigma_t) = \delta(\sigma_t - \sigma_0)$ where $\delta(x)$ is a Dirac delta function), then the Gosset formula presented in this paper reduces to the Black–Scholes formula. In this respect the Gosset formula extends the Black–Scholes formula by removing the condition that the variance is constant in time. Under this interpretation, the shape parameter ν is a measure of the belief that the volatility will not change. A large ν means that one believes that the volatility will not change in time. A small ν means that one believes that there is a broad range of values that the volatility might take. If the volatility is determined as the standard deviation of N independent returns that are normally distributed with an unknown variance, then the maximum value of ν should equal $N - 1$, which is the number of degrees of freedom in a determination of a sample standard deviation. This maximum value of $\nu = N - 1$ takes into account the fact that the sample standard deviation is a random variable and has an uncertainty associated with it.

Fig. 4 is a plot of the relative areas in the tails for the same distributions presented in Fig. 2. The curves are normalised by the area in the tail of a normal pdf. Fig. 4 gives the probability of an event x for $x > t$, relative to the probability of the event x for $x > t$ for a normal pdf. The probability of an event lying >10 standard deviations from the mean is 10^{20} time mores
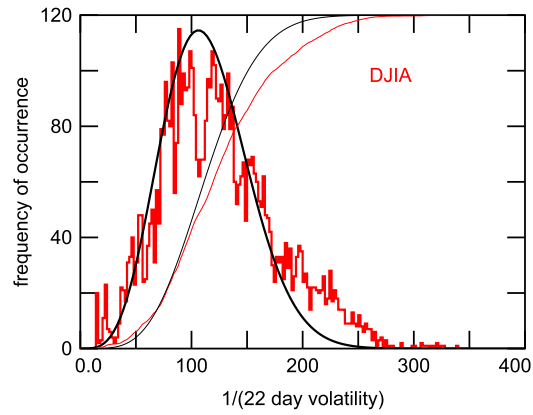
Fig. 3. Fit of a chi-squared distribution to the frequency of occurrence of the reciprocal of the 22-day volatility for the 22 years of the DJIA index.
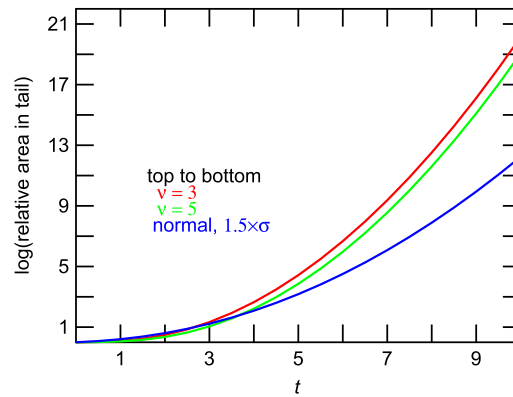
Fig. 4. Normalised areas in the tails for Student's t-distributions with 3 and 5 degrees of freedom and for a normal distribution with $1.5\times$ the volatility, each normalised by the area in the tail of a normal pdf.

likely for a process that follows a Student's t-distribution with 3 and 5 degrees of freedom than for a process that follows a normal pdf.

4. Pricing the option

Let S_t be the price of a stock at time t, $t > 0$. Let K_T be the strike price at time T, where T is the time when the option expires.

Let $S_t = A_t \exp(\sigma_t \xi)$ be the value of the stock where ξ is a random variable.

The cost of a European call option, calculated at the time of expiration T, is $C_T = E\{(S_T - K_T)^+\} = \int pdf(S_T) \times (S_T - K_T)^+ dS_T$, which is the expectation of the maximum value of $\{S_T - K_T, 0\}$. The desired quantity is C_0, which is the value of the option at time $t = 0$. The desired quantity can be obtained from the expected time value of money. If $r(t)$ is the risk free rate, then $C_0 = E\{C_T \times \exp(-\int r(t) \times dt)\} = C_T \times \exp(-r \times T)$ when the risk free rate is assumed to be time independent. This is a standard assumption in the derivation of the Black–Scholes formula [4,5].

If ξ is normally distributed, then S_t follows a log normal distribution and the price for the option follows the Black–Scholes formula [4,5].

If ξ follows a Student's t-distribution, then the pdf for ξ for v degrees of freedom is

$$pdf(\xi)d\xi = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})\sqrt{\pi v}} \times \frac{d\xi}{\left(1 + \frac{\xi^2}{v}\right)^{\frac{v+1}{2}}} = \Lambda(v) \times \frac{d\xi}{\left(1 + \frac{\xi^2}{v}\right)^{\frac{v+1}{2}}}, \tag{1}$$

which can be written in terms of $S_t = A_t \exp(\sigma_t \xi)$ as

$$pdf(S_t)dS_t = \frac{\Lambda(v)}{\left(1 + \frac{(\ln(S_t/A_t))^2}{\sigma_t^2 \times v}\right)^{\frac{v+1}{2}}} \times \frac{dS_t}{\sigma_t \times S_t}. \tag{2}$$

The following integrals are needed to evaluate the price of a call option at the time of expiration, C_T:

$$\int_{K_T}^{\infty} pdf(S_T) \times K_T \times dS_T = \int_{K_T}^{\infty} \frac{\Lambda(\nu) \times K_T}{\left(1 + \frac{(\ln(S_T/A_T))^2}{\sigma_T^2 \times \nu}\right)^{\frac{\nu+1}{2}}} \times \frac{dS_T}{\sigma_T \times S_T}$$

$$= \int_{\frac{\ln(K_T/A_T)}{\sigma_T}}^{\infty} \frac{\Lambda(\nu) \times K_T}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} \times d\xi ; \tag{3}$$

$$\int_{K_T}^{\infty} pdf(S_T) \times S_T \times dS_T = \int_{K_T}^{\infty} \frac{\Lambda(\nu) \times K_T}{\left(1 + \frac{(\ln(S_T/A_T))^2}{\sigma_T^2 \times \nu}\right)^{\frac{\nu+1}{2}}} \times \frac{dS_T}{\sigma_T \times S_T}$$

$$= \int_{\frac{\ln(K_T/A_T)}{\sigma_T}}^{\infty} \Lambda(\nu) \frac{A_T \exp(\sigma_T \xi)}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} \times d\xi . \tag{4}$$

Note that A_T is a function of $S_0, r, T, x_c(p)$, and Z, and that these variables are not functions of the random variable ξ. S_0 is the value of the asset at $t = 0$, $x_c(p)$ is the critical value that solves $P\{\zeta \leq x_c\} = p$, and Z is defined in the line of text following Eq. (6).

The integral in Eq. (4), which is for a call option, will be infinite for small ν; the exponential term in the numerator will dominate the denominator. Viable and physical results are obtained by restricting the upper limit of integration (i.e., truncate the pdf) or by restricting the value of the asset S_T (i.e., cap the value). It is not physical for the value of the asset S_T to approach infinity.

For a put option this problem does not directly arise as $0 \leq S_T \leq K_T$ and thus the integrals are evaluated from $-\infty$ to $\ln(K_T/A_T)/\sigma_T$. However, the average price of the asset is needed to price an option, and thus the pricing of a put option suffers also from an integral that diverges.

The integral that diverges is Eq. (4). If the value of the asset is restricted to be $\leq A_T \exp(\sigma_T x_c)$, then the offending integral, equation (4), can be rewritten as

$$\int_{K_T}^{\infty} pdf(S_T) \times S_T \times dS_T = \int_{\frac{\ln(K_T/A_T)}{\sigma_T}}^{\infty} \Lambda(\nu) \times \frac{A_T \exp(\sigma_T \xi)}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} \times d\xi$$

$$\geq \int_{\frac{\ln(K_T/A_T)}{\sigma_T}}^{x_c} \Lambda(\nu) \times \frac{A_T \exp(\sigma_T \xi)}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} \times d\xi$$

$$+ A_T \times \exp(\sigma_T x_c) \times \int_{x_c}^{\infty} \Lambda(\nu) \times \frac{d\xi}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} . \tag{5}$$

The integral in the last line of Eq. (5) is the probability $q = (1 - p)$ that the stock price $S_T = A_T \exp(\sigma_T \xi)$ is greater than some critical value $x_c(p)$, which we abbreviate to x_c. The last line in Eq. (5) gives the expected value of the cost of capping the value of the asset to $A_T \exp(\sigma_T x_c)$.

In pricing a call option, one could also truncate the pdf by setting the last line of Eq. (5) to zero. That is, one assumes that the pdf is zero for values greater than the critical value x_c. Truncation sets an upper limit on the value of the stock: $S_T \leq x_c$. It is necessary to divide the truncated pdf by p to maintain the normalisation of the truncated pdf. This follows from the definition of conditional probability: $P\{A|B\} = P\{A \cap B\}/P\{B\}$. In this case, $B := 0 \leq S_T \leq S_{\max}, A := S_T, A \cap B = S_T$ for $0 \leq S_T \leq S_{\max}$ and equals the null set otherwise. Thus $P\{S_T|0 \leq S_T \leq S_{\max}\} = P\{S_T\}/P\{0 \leq S_T \leq S_{\max}\} = P\{S_T\}/p$ if $0 \leq S_T \leq S_{\max}$ and $= 0$ otherwise.

Table 2 gives critical values and the maximum increase in stock price $= \exp(\sigma_T x_c)$ for a normal pdf and a Student's t-distribution with $\nu = 5$ degrees of freedom, volatility $\sigma_T = 0.40$, and various values of p.

The effect of the fat tails of the Student's t-distribution are evident. For a confidence level of 99.99%, the critical value for a Student's t-distribution with 5 degrees of freedom is 9.678 and the stock price has increased at most $47.992\times$, a 4799% increase. For a normal pdf, the critical value is 3.719 and there is a 99.99% chance that the stock price has increased at most $4.426\times$, a 443% increase. The other interpretation is that there is only a 0.01% chance that the stock price has increased by $>47.992\times$ under the Student's t-distribution or $>4.426\times$ under the normal distribution.

A confidence level of $p = 0.9999$ (i.e., 99.99%) seems unrealistic. This confidence level is one part in 10,000 and allows for increases of $\exp(\sigma_T x_c)$ in the price of the asset that may be physically impossible to achieve (cf. Table 3). If the time period is one year, then once in 10,000 years is too large an extrapolation to be meaningful. Even $p = 0.999$ seems unrealistic. The best way to price options may be to pick an upper bound $x_c = \ln(S_{\max})/\sigma_T$ (or confidence level p) for the price of the asset based on insight and agreement between parties, and then calculate the price using a capped distribution.

Table 2
S_t/A_t at $x_c(p)$ for volatility $= 0.4$ and selected p.

p	Student's t, $\nu = 5$		Normal	
	$x_c(p)$	$\exp(\sigma_T x_c)$	$x_c(p)$	$\exp(\sigma_T x_c)$
0.9	1.476	1.805	1.282	1.670
0.95	2.015	2.239	1.645	1.931
0.99	3.365	3.842	2.326	2.536
0.995	4.032	5.018	2.576	2.801
0.999	5.893	10.56	3.090	3.442
0.9999	9.678	47.99	3.719	4.428

Table 3
Critical values $x_c(p)$ for various values of p and ν.

ν	$p = 0.90$	$p = 0.95$	$p = 0.99$	$p = 0.999$	$p = 0.9999$
3	1.638	2.353	4.541	10.21	22.20
4	1.533	2.132	3.747	7.173	13.03
6	1.440	1.943	3.143	5.208	8.025
40	1.303	1.684	2.423	3.307	4.094
∞	1.282	1.645	2.326	3.090	3.719

Although a confidence level $p = 0.9999$ might be unrealistic or unphysical in terms of expected increase in the stock price and extrapolation, the results presented here show that the methods allow a price for the option to be calculated. The maximum increase in the value of the asset is a function of the number of degrees of freedom and the assumed confidence level p or critical value $x_c(p)$. Table 3 presents the critical values for different values of p and degrees of freedom ν. Note that a Student's t-distribution with $\nu = \infty$ yields a normal pdf and that a t-distribution with $\nu \geq 40$ is often considered to be a normal pdf.

The variable A_T is required to price the option. This value is obtained from the assumption that an option is a fair wager (i.e., a martingale), as shown next.

5. Martingales, Doob decomposition, and risk neutral pricing

A martingale is a mathematical model for a fair bet [27]. A fair bet can be defined as one for which the conditional expectation (i.e., given all the knowledge at the time of the bet) of the payout of the bet should be zero.

A martingale is defined such that $E\{X_{n+1}|F_n\} = X_n$ for all n where F_n is the knowledge (history) up to time n and the knowledge increases as n increases. The X_n form a stochastic process. A martingale is non-anticipating because the outcome of a martingale depends only on the past and not on the future.

A **sub**martingale is defined such that $X_n \leq E\{X_{n+1}|F_n\}$ and a **super**martingale is defined such that $X_n \geq E\{X_{n+1}|F_n\}$. A submartingale represents a favourable process (the expected outcome is favourable) and a supermatingale represents an unfavourable process (like betting at a casino). Casinos offer submartingales (the patrons play supermartingales) and financiers aspire to offer or to play submartingales.

If X_n is a martingale, then $\exp(X_n)$, $\exp(-X_n)$, $|X_n|$, and X_n^2 are all submartingales.

The Doob decomposition shows that any submartingale X_n is the sum of a martingale M_n plus an increasing process, $X_n = M_n + A_n$, where $A_n \leq A_{n+1}$ and A_n depends on the knowledge to $n - 1$, F_{n-1} [27].

Let S_t be the spot price at time t for a stock. $S_t = S_0 \exp(\alpha \times t)$ where α is the expected return on a non-dividend paying stock. α is from a capital asset pricing model (CAPM) or equivalent [5]. $(\alpha - r)$ is called the risk premium and $\alpha = r + (\alpha - r)$ is the expected return on the stock, which equals the compensation for the time value of the money invested in the stock plus the compensation for the risk of holding the stock. The time value is separated out since the value of the option C_T is calculated at time T, but the value at time 0, C_0, is desired, and $C_0 = C_T \times \exp(-r \times T)$, assuming the risk free rate r is known and constant in time.

The Doob decomposition is used to create a martingale for pricing options on the stock. The drift of the stock is subtracted from the process. This procedure of subtracting the drift from the process also leads to risk neutral pricing, where one is able to price an option without explicit knowledge of the risk premium.

The average value of a stock at time $t = 0$ when we wish to price an option is S_0. The average price of the stock at some time later is $E\{S_t\} = E\{A_t \exp(\sigma_t \xi)\}$ where ξ is a random variable and the drift is contained in A_t. For a martingale (a fair wager), $E\{S_t\} = S_0 \times \exp(r \times t)$. One may use the Doob decomposition to offset the drift and require $E\{A_t \exp(\sigma_t \xi)\} = \{A'_t \exp(\sigma_t (\xi - \xi_0))\} = S_0 \times \exp(r \times t)$. This is equivalent to requiring that the pdf for S_t should be centred about the drift owing to the risk premium. The $\exp(r \times t)$ term takes into account the time value of money and allows for comparison of values at two different points in time.

It is perhaps simpler to find the normalisation $A'_t \exp(-\sigma_t \xi_0)$ that satisfies the martingale condition than to shift the pdf. For a capped distribution,

$$
\begin{aligned}
\int_0^\infty pdf(S_T) \times S_T \times dS_T &= \int_{-\infty}^\infty \Lambda(\nu) \times \frac{A'_T \exp(\sigma_T(\xi - \xi_0))}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} \times d\xi \\
&\geq A_T \times \int_{-\infty}^{x_c} \Lambda(\nu) \times \frac{\exp(\sigma_T \xi)}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} \times d\xi \\
&\quad + A_T \times \exp(\sigma_T x_c) \times \int_{x_c}^\infty \Lambda(\nu) \times \frac{d\xi}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} .
\end{aligned}
\tag{6}
$$

Let Z be the value of the integral in the middle row of Eq. (6) and let $(1 - p)$ be the value of the integral of the last line of Eq. (6), i.e., $(1 - p) = P\{\xi > x_c\}$. Eq. (6) can then be rewritten as

$$
S_0 \times \exp(r \times T) = A_T \times Z + A_T \times \exp(\sigma_T x_c) \times (1 - p) ,
\tag{7}
$$

and solved for the value of A_T:

$$
A_T = \frac{S_0 \times \exp(r \times T)}{Z + (1 - p) \times \exp(\sigma_t x_c)} .
\tag{8}
$$

For a truncated distribution where the pdf is set $= 0$ for $\xi > x_c$, $A_T = S_0 \times \exp(r \times T)/Z$ and the pdf in the integral that defines Z (middle line of Eq. (6) will have a factor of $1/p$ where $p = P\{\xi \leq x_c\}$ is the probability that $\xi \leq x_c$. This follows from the definition of conditional probability: $P\{x < \xi \leq x + dx | \xi \leq x_c\} = P\{x < \xi \leq x + dx \cap \xi \leq x_c\}/P\{\xi \leq x_c\} = pdf(\xi)dx/p$ if $\xi \leq x_c$ and zero if $\xi > x_c$.

Note that $Z = Z(\sigma, x_c(p), t, \nu)$ and that Z is not a function of the random variable ξ.

For a log normal distribution with $\mu = 0$ and $\sigma = 1$, $x_c = +\infty$, $p = 1$, $Z = \exp(\sigma_T^2/2)$, and $A_T = S_0 \times \exp(r \times T - \sigma_T^2/2)$. Thus $C_T = S_0 \times \exp(r \times T) \times N((\ln(K_T/S_0) - r \times T - \sigma_T^2/2)/\sigma_T) - K_T \times N((\ln(K_T/S_0) - r \times T + \sigma_T^2/2)/\sigma_T)$ where $N(x)$ is the cumulative distribution function (CDF) for the unit normal pdf, $N(x) = P\{X \leq x | \sigma = 1\}$.

$C_0 = C_T \times \exp(-r \times T) = S_0 \times N(d_1) - \exp(-r \times T) \times K \times N(d_2)$ where $d_1 = (\ln(S_0/K_T) + r \times T + \sigma_T^2/2)/\sigma_T$ and $d_2 = d_1 - \sigma_T$. This is the standard form for the Black–Scholes formula for a European call [4,5]. Note that $K_0 = K_T \exp(-r \times T)$ and the expressions can be rewritten in terms of $K_0 : C_0 = S_0 \times N(d_1) - K_0 \times N(d_2)$ with $d_1 = (\ln(S_0/K_0) + \sigma_T^2/2)/\sigma_T$ and $d_2 = d_1 - \sigma_T$.

Thus the value for A_T appears correct and it should be possible to price an option assuming a Student's t-distribution (or any other distribution) using the approach and equations developed in this manuscript.

The price at the time of expiration for a European call, C_T, using a Student's t-distribution and capping the value of the asset at a critical value x_c is

$$
\begin{aligned}
C_T &= \Lambda(\nu) \times \int_{\frac{\ln(K_T/A_T)}{\sigma_T}}^{x_c} \frac{A_T \exp(\sigma_T \xi) - K_T}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} \times d\xi \\
&\quad + \Lambda(\nu) \times (A_T \exp(\sigma_T x_c) - K_T) \times \int_{x_c}^\infty \frac{d\xi}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} .
\end{aligned}
\tag{9}
$$

For a capped asset, A_T is defined by Eq. (8).

For a truncated t-distribution, the price of a call C_T is obtained by dropping the last line of Eq. (9), dividing the integral in the first line of Eq. (9) by $p(x_c) = P\{\xi \leq x_c\}$, and using $A_T = S_0 \times \exp(r \times T)/Z$. It is helpful to remember that Z is defined by some of the integral on the first line of Eq. (9) (see Eqs. (7) and (6) for the definition of Z) and thus is also divided by a factor of $p(x_c)$ for a truncated pdf. The value of the integral in the last line of Eq. (9) equals $p(x_c)$, which we shorten to p.

The price for a European put option at time T is P_T and equals

$$
P_T = \Lambda(\nu) \times \int_{-\infty}^{\frac{\ln(K_T/A_T)}{\sigma_T}} \frac{K_T - A_T \exp(\sigma_T \xi)}{\left(1 + \frac{\xi^2}{\nu}\right)^{\frac{\nu+1}{2}}} \times d\xi ,
\tag{10}
$$

for both a capped asset and a truncated pdf, where the truncation occurs at $x_c > \ln(K_T/A_T)/\sigma_T$. The value of Z is different for the capped and truncated approaches, and this makes A_T different for the two approaches.

In pricing a put, one could also truncate the distribution (i.e., set the lower limit of integration to a critical value x_p and set the pdf to zero for values $<x_p$) or use a floor at the critical value (rather than a ceiling as was done for a call). The values

of Z and A_T would be modified in a straightforward manner to accommodate the floor or truncation. In this paper we have chosen to use the full range of the integral in Eq. (10). There are six possibilities to calculate an option: two choices for the call (cap or truncate—one of which must be chosen) and three choices for the put (floor, truncate, or nothing). Our goal is to present the approach to pricing an option using a Student's t-distribution. In this regard, it is necessary to truncate or to cap the distribution at the upper end. We leave discussion of the merits of a floor and truncation to future work.

We consider a formula for the price of a European option that is obtained by using either a maximum value for the asset or a truncated Student's t-distribution to be a Gosset formula.

There is little difference in philosophy between options priced with a capped value or with a truncated pdf. In both approaches the return is assumed to be always less than a critical value. With the capped asset approach, more weight is applied to the maximum value of the asset than for the truncated pdf approach. The truncated pdf is more compact to write as there is not a second, additive term. There is, however, a difference in risk between the two approaches. The truncated pdf approach assumes that the value of the asset will never exceed the critical value. There is no guarantee that the value will not exceed the critical value.

The approach of capping the payout of a benefit is common in insurance. Capping the payout reduces the cost to purchase the benefit. Capping the payout removes the risk associated with the benefit exceeding the critical value. It is known, by design, that the payout will never exceed the cap but it is accepted that there is a non-zero probability that the expense will exceed the benefit.

Capping the value of the asset underlying a call option removes the risk associated with the value of the asset exceeding a critical value (assuming that the buyer of the call option agrees to the condition that the payout not exceed the cap value), and thus reduces the cost of the option. Capping the value is one approach to keep the integrals, which are needed to price an option, finite. The capped Gosset formula would be the proper formula to use to price a European option for an asset that is capped in value.

Put–call parity holds regardless of the underlying pdf. Consider $E\{S_T - K_T\} = E\{(S_T - K_T)^+ - (K_T - S_T)^+\} = C_T - P_T$ where $(x)^+ = max\{x, 0\}$, C_T is the cash value of a call evaluated at time T, and P_T is the cash value of a put evaluated at time T.

For a martingale (a fair wager), $E\{S_t\} = S_0 \times \exp(r \times t)$, and thus $E\{S_T - K_T\} = S_0 \times \exp(r \times T) - K_T = C_T - P_T$. From the time value of money, $x_t = x_0 \times \exp(r \times t)$, then $C_0 - P_0 = S_0 - K_T \times \exp(-r \times T)$ or $C_0 - P_0 = S_0 - K_0$.

One of the confusing features of pricing an option are the time values. Typically the option is priced at time $= 0$ for a stock price at time $= 0$ and a strike price at time $= T$. One is usually given S, σ, and r implicitly for time $= 0$ (i.e., S_0, and $r = r(0)$), the strike price K implicitly for time T, (i.e., $K = K_T$), and asked implicitly for C and P at time $= 0$ (i.e., C_0 and P_0). The volatility σ is evaluated at time $= 0$, and is propagated to time $= T$ using diffusion: $\sigma_T = \sigma \times \sqrt{T}$. For diffusion, the variance increases linearly with time. σ_T is used in the pricing of the options. For pricing an option with the Gosset formula, it is necessary to specify the shape parameter ν, which is a statement of belief about the range of values that the volatility might take at time $= T$.

6. Comparison of prices from Gosset and Black–Scholes formulae

Fig. 5 is comprised of plots of the price of a call option calculated using the Gosset formula minus the price of the same option calculated using the Black–Scholes formula, as a function of the number of degrees of freedom assumed for the Student's t-distribution. A capped distribution was used for the Gosset formula and confidence levels p of 0.99, 0.999, and 0.9999 were used. For small numbers of the degrees of freedom ν and $p = 0.9999$, the Gosset formula predicts a price difference $> \$5$ between the values predicted by the Gosset and Black–Scholes formulae. For the calculations, a risk free rate $r = 3\%$, $S_0 = \$50.00$, $K_T = \$49.00$, $\sigma_T = 0.3$, and $T = 1.0$ year were assumed. The Black–Scholes formula predicts a price of $\$7.12$ for a call option with these parameters. The Student's t-distribution is considered to be equivalent to the normal distribution for the number of degrees of freedom $\nu > 40$. The differences between the prices obtained from the Gosset formula and the Black–Scholes formula are between $\$0.06$ and $\$0.11$ for 40 degrees of freedom for confidence levels of 0.99, 0.999, and 0.9999.

Fig. 6 is comprised of similar plots as for Fig. 5, except that a truncated distribution was used to calculate the price of the call option with the Gosset formula. The price of a call option obtained by using a truncated distribution is less than the price obtained by using a capped distribution. For a confidence level of 99% and $\nu > 25$, the Gosset formula predicts a value for a call option that is less than the value predicted by the Black–Scholes formula. This is not surprising given that for large ν, the Student's t-distribution approaches the normal pdf. As the two distributions become similar, any approach that excludes data in the tails will predict a lower value for the option. The difference in cost of the option between the capped asset approach and the truncated distribution approach owes to contributions from three terms. The Z term is different for the two approaches and this leads to differences in the integrands and lower limits of integration. For $p = 0.9999$, $\nu = 3$, $r = 0.03$, $\sigma_T = 0.3$, $T = 1.0$, $S_0 = \$50.00$, and $K_T = \$49.00$, the price difference, capped minus truncated, is $\$1.48$. The contribution to cost from the values above the cap (i.e., the contribution from the last line of Eq. (9)) contributes $\$3.14$ to the difference. The integrand for the truncated distribution is larger than for the capped asset. This contributes $-\$1.52$ to the difference. The lower limits of integration are slightly different for the capped asset ($Z = 1.281$ yielding $\ln(K_T/A_T)/\sigma_T = 0.6583$) and for the truncated distribution ($Z = 1.203$ yielding $\ln(K_T/A_T)/\sigma_T = 0.4488$). This area contributes $-\$0.10$, for a total difference of $\$1.52$ in the costs of the call option when calculated using the two approaches.
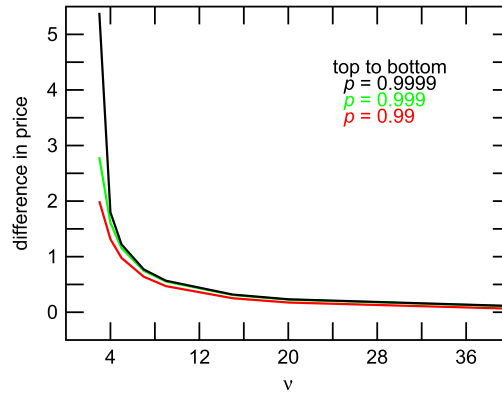
Fig. 5. Price difference, Gosset subtract Black–Scholes, as a function of ν for confidence levels of 0.99, 0.999, and 0.9999 using a capped distribution.
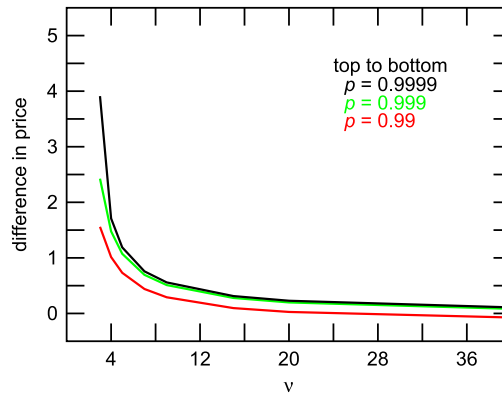

Fig. 6. Price difference, Gosset subtract Black–Scholes, as a function of ν for confidence levels of 0.99, 0.999, and 0.9999 using a truncated distribution.
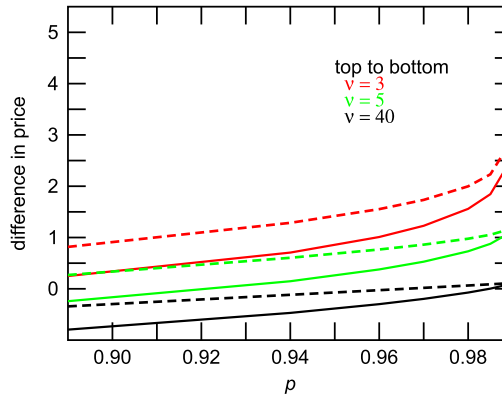

Fig. 7. Price difference, Gosset subtract Black–Scholes, of a call option as a function of the confidence level for $\nu = 3$, 5, and 40 for a truncated distribution (solid line) and a capped distribution (broken line).

Fig. 7 gives the value of a call option using the truncated and capped Gosset formulae minus the value of a call option computed using the Black–Scholes formula, as a function of the confidence level. For large numbers of degrees of freedom ν, the Gosset formula predicts a value that is less than the value predicted by the Black–Scholes formula. Again, this is not surprising as the Student's t-distribution will approach the normal pdf for large ν and any pricing mechanism that excludes area in the tails will predict a lower price.

Fig. 7 demonstrates that the price of an option for a t-distribution with a small ν increases dramatically as the confidence level increases. For a confidence level approaching 100%, the price tends to infinity since one of the integrals required to price the option diverges. Although we use the term confidence level p, it is helpful to remember that the confidence level gives a critical value, $x_c(p)$. The exponential of the critical value, $\exp(\sigma_T x_c)$, gives the value that the asset will not exceed in
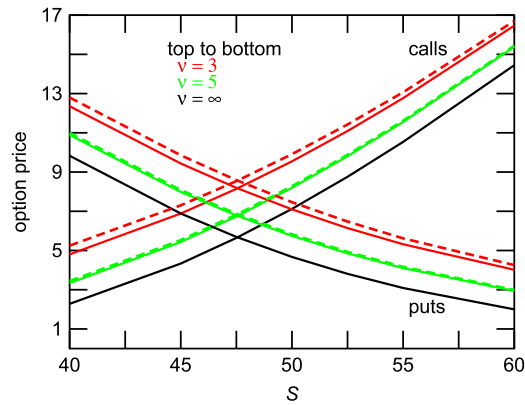
Fig. 8. Prices of put and call options for $v = 3$ and $v = 5$ using a capped Gosset formula with $p = 0.999$, and for the Black–Scholes formula.
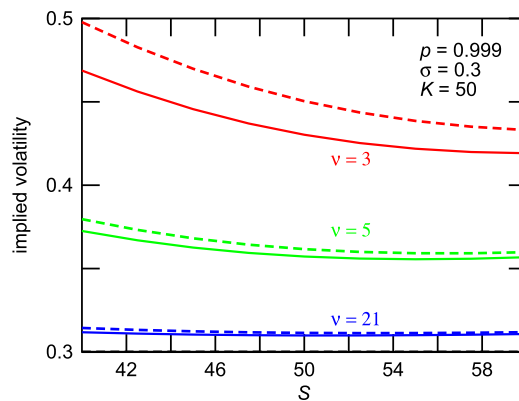
Fig. 9. Implied volatility for price calculated using a truncated (solid line) and a capped (broken line) Gosset formula with $p = 0.999$ and $v = 3$, $v = 5$, and $v = 21$.

a capped asset approach, or is hoped will not be exceeded in a truncated pdf approach. The solid line gives the difference calculated with the truncated pdf whereas the broken line gives the difference calculated using a capped pdf.

Fig. 8 gives the prices of calls and puts as a function of the $t = 0$ price of the asset. The truncated and capped Gosset formulae for C_T and P_T, Eqs. (9) and (10), were used to calculate the prices of the options. When the option prices were used in the put–call parity formula, the difference was of the order of the precision of the calculation. This held for both the truncated and capped approaches. Fig. 8 displays prices for $v = 3$, $v = 5$, and for the Black–Scholes formula. A confidence level of 99.9%, $K_T = \$49.00$, $r = 3\%$, $\sigma_T = 0.3$, and $T = 1.0$ year were used. The curve for $v = 40$ essentially overwrites the Black–Scholes curve for the scale used. The difference between the $v = 40$ and Black–Scholes prices were less than $\$0.10$ for the range of S_0.

A comparison of the solid and broken lines of Fig. 8 shows that the truncated Gosset formula, drawn with solid lines, predicts lower prices for the options than does the capped Gosset formula, drawn with broken lines. The truncated Gosset formula ignores events in the tails and thus costs less than the capped Gosset formula, which allows for events in the tails but places a limit on the value of the events in the tails.

Figs. 9 and 10 show the implied volatility as a function of the value of the underlying asset, S, for the number of degrees of freedom, $v = 3$, 5, and 21 and for confidence levels of $p = 0.999$ and $p = 0.9999$. For both of these curves, $K = 50$. Thus the curves show "moneyness" ranging from -20% to $+20\%$. The solid lines in the figures were calculated with a truncated pdf whereas the broken lines were calculated with a capped pdf. Fig. 9 was calculated for $p = 0.999$ and Fig. 10 was calculated for $p = 0.9999$. A comparison of Figs. 9 and 10 reveals that the smile and offset increase as v decreases and that the offset and smile increase as the level of confidence p is increased. This is not unexpected since decreasing v or increasing p increases the area in the wings of the Student's t-distribution. The Black–Scholes formula requires an increase in the effective volatility to capture events in the wings, as shown in Figs. 2 and 4.

It is interesting to note that the implied volatility is roughly $1.5\times$ the scale parameter $\sigma = 0.3$ Figs. 2 and 4 compare the pdfs and areas in the tails for Student's t-distributions with $v = 3$ and $v = 5$ with normal pdfs with scale parameters $\sigma = 1$ and $\sigma = 1.5$ Figs. 2 and 4 show that the normal pdf tracks the fat-tailed Student's t-distributions over a larger range when a σ of $1.5\times$ is used.
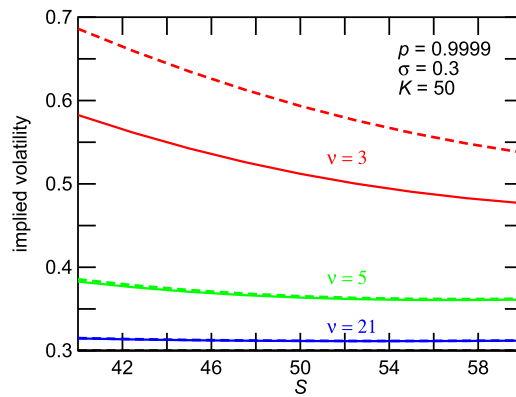
Fig. 10. Implied volatility for price calculated using a truncated (solid line) and a capped (broken line) Gosset formula with $p = 0.9999$ and $\nu = 3$, $\nu = 5$, and $\nu = 21$.

The smiles shown in Figs. 9 and 10 should be essentially symmetric, as the only asymmetry in the underlying t-distribution is the cap or truncation on large returns.

7. Conclusion

In this paper we have presented a method to price European options using a Student's t-distribution. We refer to this method as a Gosset method and the formulae for pricing options as Gosset formulae, in honour of the person who published under the *nom de plume* (or pseudonym) Student. The Gosset formula essentially extends the Black–Scholes formula by removing the constraint that the volatility be constant in time.

Student's t-distributions have "fat tails" and fit the returns from assets better than a normal pdf. The difficulty with pricing options using the Student's t-distribution is that the fat tails cause one of the integrals, which is needed to price an option, to diverge. A viable and physically reasonable value for the price of the option can be obtained by either capping the value of the underlying asset or by truncating the distribution. The value for the cap or where the truncation takes place need not be unrealistic. The required integral can be evaluated for a confidence level of 99.99%, which implies a price increase of $> 4\,000\%$ for realistic parameters (volatility $\sigma_T = 0.4$ and shape parameter $\nu = 5$) of the Student's t-distribution. An increase of 4 000% in the value of an asset would be phenomenal and perhaps unphysical.

There is little difference in philosophy between the approaches of pricing an option by capping the value of the asset or truncating the pdf. Both approaches reduce the cost of the option. However, there can be a large difference in the exposure to risk between the two approaches.

Truncation of the pdf assumes that the value of the underlying asset will never exceed the value where the truncation occurs. There is no guarantee that this value will not be exceeded.

Capping the value of the asset ensures, by design and with consent, that the risk associated with the value of the option exceeding the value of the cap does not exist. In the cap approach, it is accepted that the value of the asset might exceed the value of the cap. However, the price of the option is set assuming that the payout will not exceed the value of the cap. Provided that the option writer and buyer agree to this approach, then there is no risk associated with a large increase in the value of the asset. The buyer gives away the right to extreme returns in exchange for a lower price for the option.

If the option writer and buyer do not agree to a cap on the payout, then the approach of capping the value of the asset still holds risk that the value of the asset will exceed the cap value. For the same critical value and parameters for the pdf, the cap approach offers an increased premium for the risk as compared to the approach of truncation of the pdf. This is because the cap approach allows for the possibility that the value of the asset will exceed the value of the cap.

The Student's t-distribution has an additional parameter (the number of degrees of freedom, ν) as compared to the normal pdf. This parameter is a statement of the belief about the range of values that the volatility might take at the time that the option expires. A large value of ν is a statement that the volatility will change little in time. In the limit as ν approaches infinity, the Student's t-distribution approaches the normal distribution and hence the Gosset formula for the price of an option becomes the same as the Black–Scholes formula. The Black–Scholes formula is derived on the assumption that the volatility is constant in time and known. A small value of ν is a statement that the volatility might take one of a broad range of values. Fits to the DJIA for returns over 100 years give a value of $\nu = 2.65$. This value of $\nu = 2.65$ allows for the distribution of volatility that has been observed over the 100 years of returns. In turbulent markets, it would be wise to use a small value for the number of degrees of freedom and a large confidence level. A confidence level is required in the pricing of options using a Student's t-distribution. In calm markets, it might be acceptable to use a large ν, of order $N - 1$ where N is the number of independent samples that are used to calculate the volatility, and a smaller confidence level to match the expectations of the future and tolerance to risk. The flexibility offered by the Gosset formulae of choice of approach (cap or truncate), of the number of degrees of freedom to match the fat tails of returns, and of the critical value allow the analyst and investor additional opportunity to use their insight and circumstances to price options.

The approach used to derive the Gosset formulae can be adapted to distributions other than the Student's t-distribution. This may be of interest in situations and for returns on assets that do not appear to be well described by a Student's t-distribution.

Acknowledgement

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