Advanced Statistical Methods for Modeling and Finance: Solutions for exercises

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1 Exercise 1

Let us consider an experience of rolling two(balanced) dice.

Let the event E such that

$$E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

Let us supppose that the dice are cast $N = 400$ times and let us find the number of times we should expect to get E occured(at least once).

Let X be the random variable which associates 1 if the event E happens and 0 else. $X \sim \mathcal{B}(p = \mathbb{P}(E))$

Let Y be the random variable that associates, after N independent rolls of two dice, the number of of trials to get E occured is just N repetitions of the previous Bernoulli experiment, independently and in the same conditions.

Then $Y \sim \mathcal{B}(N = 400, p = \mathbb{P}(E))$

Let n the number of times we should expect to get the event E occured. Then $n = E(Y) = Np$

$$\begin{aligned} N &= 400 \\ p &= \mathbb{P}(E) \\ &= \frac{6}{36} \\ &= \frac{1}{6} \end{aligned}$$

$$n = 400 \times \frac{1}{6} \approx 66.67 \approx 67$$

Conclusion

We should expect to get the event E occured(at least once) after 67 throws.

2 Exercise 2

Let us consider the ame that consists in flipping a coin twice. There are 4 possibles outcomes. There are:

(T, T); (H, H); (T, H) and (H, T)

Where T means tails and H means head

- Let A= "Both flips got head".

  $\mathbb{P}(A) = \frac{1}{4}$

- Let us compute the expected earn after playing 100 times

  - We win 100 DHS if we get 2 heads(with the probability $\frac{1}{4}$)
  - We lose 60 DHS ie we win $100DHS$ if we get 2 tails(with the probability $\frac{1}{4}$)
  - We win nothing ie O DH otherwise(with the probability $\frac{1}{2}$)

  The expected amount to earn for a game if we do not consider the bet is

  $$\begin{aligned} E_1 &= 100 \times \frac{1}{4} - 60 \times \frac{1}{4} + 0 \times \frac{1}{2} \\ &= 25 - 15 + 0 \\ &= 10 \end{aligned}$$

  By considering the bet, the expected amount to earn for a game is.

  $E_2 = E_1 - 20 = 10 - 20 = -10$

  And then We should expect to lose 10 DHS for just a game.

  Let

  $$\begin{aligned} E &= E_1 \times 100 - 20 \\ &= 10 \times 100 - 20 \\ &= 1000 - 20 \\ &= 980 \end{aligned}$$

  Then after playing the game 100 times, if we suppose that we bet the 20 DHS just once, we will epect to earn 980 DHS.

3 Exercise 3

Let us construct a random variable for exercise 2

Let $\Omega = \{(T,T);(H,H);(T,H);(H,T)\}$

Where T means tails and H means head

Let us consider the random variable Y defined from Ω to the set of possible earns which is $\{100, -60, 0\}$

4 Exercise 4

Let X be a random variable such that $\mathbb{P}[(a,b)] = b-a, 0<a<b<1$

4.1 Let us find the PDF f of X

Firstly, let us find the **CDF** F of X

Let $t \in \mathbb{R}$

$F(t) = \mathbb{P}(X \leq t)$

- Let $t \leq 0$

$$\begin{aligned}
\mathbb{P}(X \leq t) &= \mathbb{P}(X \leq t < 0) \\
&= \mathbb{P}(X < 0) \\
&= \mathbb{P}(X \in]-\infty; 0]) \\
&= 0
\end{aligned}$$

- Let $t \in]0;1[$

$$\begin{aligned}
\mathbb{P}(X \leq t) &= \mathbb{P}(X \in]0;t[) \\
&= \mathbb{P}[(0,t)] \\
&= t - 0 \\
&= t
\end{aligned}$$

- Let $t \in]1;+\infty[$

$$\begin{aligned}
\mathbb{P}(X \leq t) &= \mathbb{P}(X \in [0;1[\cup[1;t[) \\
&= \mathbb{P}(X \in [0;1[) + \mathbb{P}(X \in [1;t[) \\
&= \mathbb{P}([0;1[) + \mathbb{P}([1;t[) \\
&= 1 + 0 \\
&= 1
\end{aligned}$$

Therefore

$$F(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } 0 < t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

And then

$$
\begin{aligned}
f(t) = F'(t) &= \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } 0 < t < 1 \\ 0 & \text{if } t \geq 1 \end{cases} \\
&= \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

4.2 Let us find $\mathbb{P}([0.2; 0.4[\cup[0.7; 0.85[)$

$$
\begin{aligned}
\mathbb{P}([0.2; 0.4[\cup[0.7; 0.85[) &= \mathbb{P}([0.2; 0.4[) + \mathbb{P}([0.7; 0.85[) \\
&= (0.4 - 0.2) + (0.85 - 0.7) \\
&= 0.2 + 0.15 \\
&= 0.2 + 0.15 \\
&= 0.35
\end{aligned}
$$

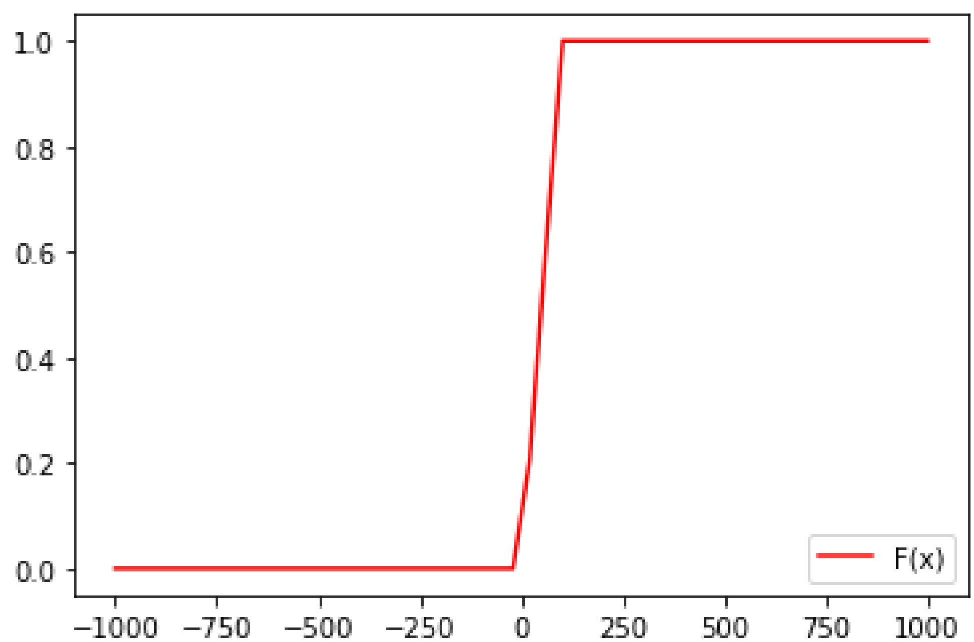
5 Exercise 5

Let us suppose a random variable X has it's **CDF** defined by:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{200} & \text{if } 0 < x < 100 \\ 1 & \text{if } x \geq 100 \end{cases}$$

5.1 The graph of F

[2]:
```python
import numpy as np
import matplotlib.pyplot as plt
F=lambda x:np.where(0<x and x<100,x/200,np.where(x<0 ,0,1))
x=np.linspace(-1000,1000).tolist()
plt.plot(x,[F(i) for i in x],label='F(x)',color='r')
plt.legend(loc='lower right')
plt.show()
```



5.2 Let's compute the following probabilities

a)$\mathbb{P}(-50 < X < 50)$

$$\mathbb{P}(-50 < X < 50) = F(50) - F(-50)$$

$$\begin{aligned} 50 \in [0;100] \Rightarrow F(50) &= \frac{50}{200} \\ &= \frac{1}{4} \end{aligned}$$

$$-50 \in \,]-\infty;0[\, \Rightarrow F(-50) = 0$$

Then we have

$$\begin{aligned} \mathbb{P}(-50 < X < 50) &= \frac{1}{4} - 0 \\ &= \frac{1}{4} \end{aligned}$$

b)$\mathbb{P}(X = 0)$

$$\begin{aligned} \mathbb{P}(X = 0) &= F(0) - \lim_{x \to 0^-} F(x) \\ &= \frac{0}{200} - \lim_{x \to 0^-} 0 \\ &= 0 \end{aligned}$$

c)$\mathbb{P}(X = 100)$

$$\begin{aligned} \mathbb{P}(X = 100) &= F(100) - \lim_{x \to 100^-} F(x) \\ &= 1 - \lim_{x \to 100^-} \frac{x}{200} \\ &= 1 - \frac{100}{200} \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

6 Exercise 6

Let us suppose a random variable X has it's **PDF** defined by:

$$f(x) = \begin{cases} kx^2 & \text{if } 0 \leq x \leq 10 \\ 0 & \text{if } x > 10. \end{cases}$$

Let us find $\mathbb{P}(7 < X < 15)$

Let us note F the **CDF** of X

$$\mathbb{P}(7 < X < 15) = F(15) - F(7)$$

$\forall x \geq 0$, we have:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dtdt \\ &= 0 + \int_{0}^{x} f(t)dt \\ &= \int_{0}^{x} f(t)dt \end{aligned}$$

$t \in [0; x] \iff 0 \leq t \leq x$

- If $x \leq 10$, then $0 \leq t \leq x \leq 10$ ie $0 \leq t \leq 10$ and then $f(t) = 2kt^2$

  So

$$\begin{aligned} F(x) &= \int_{0}^{x} 2kt^2 dt \\ &= \frac{2}{3}k\left[t^3\right]_0^x \\ &= \frac{2}{3}kx^3 \end{aligned}$$

- If $x \geq 10$, then we have:

$$\begin{aligned}
F(x) &= \int_0^{10} f(t)dt + \int_{10}^{x} f(t)dt \\
&= \int_0^{10} f(t)dt + 0 \\
&= \int_0^{10} 2kt^2 dt \\
&= \frac{2}{3}k\left[t^3\right]_0^{10} \\
&= \frac{2}{3}10^3 k \\
&= \frac{2000k}{3}
\end{aligned}$$

$$F(x) = \frac{2}{3}k \begin{cases} x^3 & \text{if } 0 \leq x \leq 10 \\ 1000 & \text{if } x > 10. \end{cases}$$

Let us determine k.

$$\begin{aligned}
\int_{\mathbb{R}} f(t)dt &= \int_{-\infty}^{0} f(t)dt + \int_0^{10} f(t)dt + \int_{10}^{+\infty} f(t)dt \\
&= 0 + \int_0^{10} 2kt^2 dtdt + 0 \\
&= 2k[\frac{1}{3}t^3]_0^{10} \\
&= \frac{2k}{3}[t^3]_0^{10} \\
&= \frac{2k}{3}[t^3]_0^{10} \\
&= \frac{2k}{3}(10^3 - 0^3) \\
&= \frac{2k}{3}(1000) \\
&= \frac{2000k}{3}
\end{aligned}$$

f is a PDF, then $\int_{\mathbb{R}} f(t)dt = 1$

$$\begin{aligned}
\int_{\mathbb{R}} f(t)dt = 1 &\Rightarrow \frac{2000k}{3} = 1 \\
&\Rightarrow 2000k = 3 \\
&\Rightarrow k = \frac{3}{2000}
\end{aligned}$$

And then

$$
\begin{aligned}
F(x) &= \frac{2}{3} \times \frac{3}{2000} \begin{cases} x^3 & \text{if } 0 \leq x \leq 10 \\ 1000 & \text{if } x > 10. \end{cases} \\
&= \frac{1}{1000} \begin{cases} x^3 & \text{if } 0 \leq x \leq 10 \\ 1000 & \text{if } x > 10. \end{cases} \\
&= \begin{cases} \frac{1}{1000} x^3 & \text{if } 0 \leq x \leq 10 \\ 1 & \text{if } x > 10. \end{cases}
\end{aligned}
$$

$15 > 10$ then $F(15) = 1$

$7 \in [0; 10]$ then $F(7) = \frac{7^3}{1000} = \frac{343}{1000}$

$$
\begin{aligned}
\mathbb{P}(7 < X < 15) &= F(15) - F(7) \\
&= 1 - \frac{343}{1000} \\
&= \frac{1000 - 343}{1000} \\
&= 3 \times \frac{16}{2000} \\
&= \frac{357}{1000}
\end{aligned}
$$

7 Exercise 7

Let

- X be a random variable with f as **PDF**, such that $f(t) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}t^2); t \in \mathbb{R}$
- $Y = X^2$ another random variable.
- F_X the **CDF** of X
- F_Y the **CDF** of Y
- f_Y the **PDF** of Y

Let us show that $f_Y(y) = \frac{1}{\sqrt{2\pi y}} \exp(-\frac{y}{2}) \forall t \in \mathbb{R}$ $\forall t \in \mathbb{R}$, we have:

$$\begin{aligned}
F_Y(t) &= \mathbb{P}(Y < t) \\
&= \mathbb{P}(X^2 < t) \\
&= \mathbb{P}(Y < \sqrt{t}) \\
&= \mathbb{P}(|x| < \sqrt{t}) \text{ since } X^2 < t \Rightarrow t > 0 \\
&= \mathbb{P}(-\sqrt{t} < \sqrt{t} < \sqrt{t}) \\
&= F_X(\sqrt{t}) - F_X(-\sqrt{t})
\end{aligned}$$

$$\begin{aligned}
f_Y(t) &= F'_Y(t) \\
&= \frac{1}{2\sqrt{t}} F'_X(\sqrt{t}) - (-\frac{1}{2\sqrt{t}}) F'_X(\sqrt{t}) \\
&= \frac{1}{2\sqrt{t}} f(\sqrt{t}) - (-\frac{1}{2\sqrt{t}}) f(-\sqrt{t}) \\
&= \frac{1}{2\sqrt{t}} f(\sqrt{t}) + \frac{1}{2\sqrt{t}} f(-\sqrt{t}) \\
&= \frac{1}{2\sqrt{t}} [f(\sqrt{t}) + f(-\sqrt{t})] \\
&= \frac{1}{2\sqrt{t}} (2) (\frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}t)) \\
&= \frac{1}{\sqrt{t}} \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}t) \\
&= \frac{1}{\sqrt{2\pi y}} \exp(-\frac{t}{2})
\end{aligned}$$

8 Exercise 8

Let X be a random variable such that $Var(X)$ exists .

Let's show that $Var(X) = E(X^2) - \mu^2$ where $\mu = E(X)$

We have by definition:

$$\begin{aligned}
Var(X) &= E([X - E(X)]^2) \\
&= E(X^2 - 2XE(X) + (E(X))^2) \\
&= E(X^2 - 2\mu X + \mu^2) \\
&= E(X^2 - 2\mu X + \mu^2) \text{ with } \mu = E(X) \\
&= E(X^2) - 2\mu E(X) + \mu^2 \\
&= E(X^2) - 2\mu\mu + \mu^2 \text{ because } E(X) = \mu \\
&= E(X^2) - 2\mu^2 + \mu^2 \\
&= E(X^2) - \mu^2
\end{aligned}$$

Hence the result.

9 Exercise 9

Let X a discrete random variable with **PMF** p positive at -1, 0,1 and zero elsewhere.

9.1 a) Let us find $E(X)$ if $p(0) = \frac{1}{4}$

$$\begin{aligned} E(X^2) &= (-1)^2 p(-1) + (0)^2 p(0) + (1)^2 p(1) + 0 \\ &= p(-1) + p(1) \end{aligned}$$

We have

$$\begin{aligned} p(-1) + p(0) + p(1) + 0 &= p(-1) + p(0) + p(1) \\ &= p(-1) + \frac{1}{4} + p(1) \text{ since } p(0) = \frac{1}{4} \end{aligned}$$

Then

$$\begin{aligned} p(-1) + p(1) &= 1 - p(0) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

And then $E(X^2) = \frac{3}{4}$

9.2 a) Let us find $p(-1)$ and $p(1)$ if $p(0) = \frac{1}{4}$ and if $E(X) = 1$

$$\begin{aligned} E(X) &= (-1)p(-1) + (0)p(0) + (1)p(1) + 0 \\ &= p(1) - p(-1) \end{aligned}$$

$$\begin{aligned} E(X) = 1 &\Rightarrow p(1) - p(-1) = 1 \\ &\Longrightarrow p(1) = 1 + p(-1) \quad (1) \end{aligned}$$

$p(-1) + p(0) + p(1) = 1$ and $p(0) = \frac{1}{4}$

Then we get:

$$\begin{aligned} p(-1) + p(1) &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

And then $p(1) = \frac{3}{4} - P(-1)$ $\textcircled{2}$

$$
\begin{aligned}
\textcircled{1} = \textcircled{2} &\Rightarrow 1 + p(-1) = \frac{3}{4} - p(-1) \\
&\Rightarrow 2p(-1) = \frac{3}{4} - 1 \\
&\Rightarrow 2p(-1) = -\frac{1}{4} \\
&\Rightarrow p(-1) = -\frac{1}{8} \textbf{(an error???)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
p(-1) &= \frac{3}{4} - p(-1) \\
&= \frac{3}{4} + \frac{1}{8} \\
&= \frac{24 + 4}{4 \times 8} \\
&= \frac{28}{32} \\
&= \frac{7}{8}
\end{aligned}
$$

10 Exercise 10

Let X a continous random variable with **PDF** f, such that $f(x) = \frac{1}{3}$ for $-1 < x < 2$

Let M_X the **MGF** of X

Let $t \in]-1; 2[$

$$\begin{aligned}
M_X(t) &= E(e^t X) \\
&= \int_{\mathbb{R}} e^{tx} f(x)dx \\
&= \int_{-\infty}^{-1} e^{tx} f(x)dx + \int_{-1}^{2} e^{tx} f(x)dx + \int_{2}^{+\infty} e^{tx} f(x)dx \\
&= 0 + \int_{-1}^{2} \frac{1}{3} e^{tx} dx + 0 \\
&= \frac{1}{3} \int_{-1}^{2} e^{tx} dx
\end{aligned}$$

When $t \neq 0$, we have:

$$\begin{aligned}
M_X(t) &= \frac{1}{3} [\frac{1}{t} e^{tx}]_{-1}^{2} \\
&= \frac{1}{3t} [e^{tx}]_{-1}^{2} \\
&= \frac{1}{3t} (e^{2t} - e^{-t})
\end{aligned}$$

11 Exercise 11

Let $X \sim \mathcal{B}(n,p)$ and M_X the **MGF** of X

Let us show that $M_X(t) = (1-p+pe^t)^n \ \forall t \in \mathbb{R}$

Let $q = 1-p$

We have $\mathbb{P}(X=k) = \binom{n}{k}p^k q^{n-k} \forall \ k \in \{0,1,\ldots,n\}$

Let $t \in \mathbb{R}$

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \Sigma_{i=1}^{n} e^{tk}\mathbb{P}(Y=k) \\
&= \Sigma_{i=1}^{n} e^{tk}\binom{n}{k}p^k q^{n-k} \\
&= \Sigma_{i=1}^{n} p^k (e^t)^k \binom{n}{k} q^{n-k} \\
&= \Sigma_{i=1}^{n} (pe^t)^k \binom{n}{k} q^{n-k} \\
&= \Sigma_{i=1}^{n} a^k \binom{n}{k} b^{n-k} \ \textbf{with} \ a = pe^t \text{and } b = q \\
&= (a+b)^n \text{ from Newten's binomial formula} \\
&= (pe^t + 1 - p)^n \text{ since } a = pe^t \text{ and } b = q \\
&= (1-p+pe^t)^n
\end{aligned}$$

Hence the result.

12 Exercise 12

Let X a random variable that has a Poisson ditribution with λ as parameter

We have:

$\mathbb{P}(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$ for k in $\mathbb{N}$

Let M_X the **MGF** of X.

12.1 Let us show that $M_X(t)=e^{\lambda(e^t-1)}\ \forall t\in\mathbb{R}$

Let $t\in\mathbb{R}$

$$
\begin{aligned}
M_X(t) &= E(e^tX)\\
&= \Sigma_{k\in\mathbb{N}}e^{tk}\mathbb{P}(X=k)\\
&= \Sigma_{k\in\mathbb{N}}e^{tk}\frac{\lambda^k e^{-\lambda}}{k!}\\
&= \Sigma_{k\in\mathbb{N}}\frac{e^{tk}\lambda^k e^{-\lambda}}{k!}\\
&= \Sigma_{k\in\mathbb{N}}\frac{(e^t)^k\lambda^k e^{-\lambda}}{k!}\\
&= \Sigma_{k\in\mathbb{N}}\frac{(\lambda e^t)^k e^{-\lambda}}{k!}
\end{aligned}
$$

$\forall x\in\mathbb{R}$, $e^x=\Sigma_{n\in\mathbb{N}}\frac{x^n}{n!}$

Then we have:

$$
\begin{aligned}
M_X(t) &= e^{-\lambda}e^{\lambda e^t}\\
&= e^{-\lambda+\lambda e^t}\\
&= e^{\lambda e^t-\lambda}\\
&= e^{\lambda(e^t-1)}
\end{aligned}
$$

Hence $M_X(t)=e^{\lambda(e^t-1)}$

12.2 Let us find it's mean and variance

Let $t\in\mathbb{R}$

- Mean $E(X)$

  $E(X)=M'_X(0)$

  $M_X(t)=e^{\lambda(e^t-1)}$

$$
\begin{aligned}
M'_X(t) &= \lambda e^t e^{\lambda(e^t-1)}\\
&= \lambda e^t M_X(t)
\end{aligned}
$$

$M_X(0) = e^{e^0-1} = e^0 = 1$

$$\begin{aligned} M'_X(0) &= \lambda e^0 M_X(0) \\ &= \lambda(1) \\ &= \lambda \end{aligned}$$

Then $E(X) = \lambda$

- Variance $Var(X)$

$var(X) = E(X^2) - (E(X))^2$ with $E(X^2) = M''_X(0)$

$$M''_X(t) = (M'_X)'(t)$$

$$M'_X(t) = \lambda e^t M_X(t)$$

$$\begin{aligned} M''_X(t) &= \lambda[e^t M_X(t) + e^t M'_X(t)] \\ &= \lambda e^t[M_X(t) + M'_X(t)] \end{aligned}$$

$$\begin{aligned} M''_X(0) &= \lambda e^0[M_X(0) + M'_X(0)] \\ &= \lambda(1)[1+\lambda] \\ &= \lambda(1+\lambda) \end{aligned}$$

Then $E(X^2) = \lambda(1+\lambda)$ and then

$$\begin{aligned} Var(X) &= \lambda(1+\lambda) - \lambda^2 \\ &= \lambda + \lambda^2 - \lambda^2] \\ &= \lambda \end{aligned}$$

13 Exercise 13

Let $X \sim \mathcal{G}(\alpha, \beta)$ and M_X the **MGF** of X and f_X it's **PDF** such that

$f_X(x) = \frac{1}{\Gamma(a)\times\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$ $\alpha > 0$; $\beta > 0$; $x > 0$

13.1 Let us show that $M_X(t) = \frac{1}{(1-\beta t)^\alpha}$ if $t > \frac{1}{\beta}$

Let $t \in \mathbb{R}$

$$\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \int_{-\infty}^{+\infty} e^{tx} f_X(x) dx \\
&= \int_{0}^{+\infty} e^{tx} \frac{1}{\Gamma(a)\times\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx \\
&= \int_{0}^{+\infty} \frac{1}{\Gamma(a)\times\beta^\alpha} x^{\alpha-1} e^{tx-\frac{x}{\beta}} dx \\
&= \int_{0}^{+\infty} \frac{1}{\Gamma(a)\times\beta^\alpha} x^{\alpha-1} e^{(t-\frac{1}{\beta})x} dx \\
&= \int_{0}^{+\infty} \frac{1}{\Gamma(a)\times\beta^\alpha} x^{\alpha-1} e^{-(\frac{1}{\beta}-1)x} dx
\end{aligned}$$

Put $y = (\frac{1}{\beta} - 1)x$. Then we have

$$\begin{cases}
x = \frac{1}{\frac{1}{\beta}-t} y = \frac{\beta}{1-\beta t} y \\
dx = \frac{1}{\frac{1}{\beta}-t} y = \frac{\beta}{1-\beta t} dy
\end{cases}$$

And then we get

$$
\begin{aligned}
M_X(t) &= \int_0^{+\infty} \frac{1}{\Gamma(a) \times \beta^\alpha} x^{\alpha-1} e^{-(\frac{1}{\beta}-1)x} dx \\
&= \int_0^{+\infty} \frac{1}{\Gamma(a) \times \beta^\alpha} (\frac{\beta}{1-\beta t} y)^{\alpha-1} e^{-y} \frac{\beta}{1-\beta t} dy \\
&= \int_0^{+\infty} \frac{1}{\Gamma(a) \times \beta^\alpha} (\frac{\beta}{1-\beta t})^{\alpha-1} \frac{\beta}{1-\beta t} y^{\alpha-1} e^{-y} dy \\
&= \int_0^{+\infty} \frac{1}{\Gamma(a) \times \beta^\alpha} (\frac{\beta}{1-\beta t})^{\alpha} y^{\alpha-1} e^{-y} dy \\
&= \int_0^{+\infty} \frac{\beta^\alpha}{\Gamma(a) \times \beta^\alpha (1-\beta t)^\alpha} y^{\alpha-1} e^{-y} dy \\
&= \int_0^{+\infty} \frac{1}{\Gamma(a) \times (1-\beta t)^\alpha} y^{\alpha-1} e^{-y} dy \\
&= \frac{1}{\Gamma(a) \times (1-\beta t)^\alpha} \int_0^{+\infty} y^{\alpha-1} e^{-y} dy \\
&= \frac{1}{\Gamma(a) \times (1-\beta t)^\alpha} \Gamma(a) \\
&= \frac{1}{(1-\beta t)^\alpha}
\end{aligned}
$$

13.2 Mean and variance of Gamma distribution

- Mean $E(X)$

  $E(X) = M'_X(0)$

$$
\begin{aligned}
M_X(t) &= \frac{1}{(1-\beta t)^\alpha} \\
&= (1-\beta t)^{-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
M'_X(t) &= -\alpha(-\beta)(1-\beta t)^{-\alpha-1} \\
&= \alpha\beta(1-\beta t)^{-\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
M'_X(0) &= \alpha\beta(1-0)^{-\alpha-1} \\
&= \alpha\beta(1)^{-\alpha-1} \\
&= \alpha\beta \times 1 \\
&= \alpha\beta
\end{aligned}
$$

Hence $E(X) = \alpha\beta$

- Variance $Var(X)$

  $Var(X) = E(X^2) - [E(X)]^2$

  $E(X^2) = M''_X(0)$

$$M'_X(t) = \alpha\beta(1-\beta t)^{-\alpha-1}$$

$$\begin{aligned} M''_X(t) &= \alpha\beta[(-\alpha-1)(-\beta)(1-\beta t)^{-\alpha-1-1}] \\ &= \alpha\beta[\beta(\alpha+1)(1-\beta t)^{-\alpha-2}] \\ &= \alpha\beta^2(\alpha+1)(1-\beta t)^{-\alpha-2} \end{aligned}$$

$$\begin{aligned} M''_X(0) &= \alpha\beta^2(\alpha+1)(1-0)^{-\alpha-2} \\ &= \alpha\beta^2(\alpha+1)(1)^{-\alpha-2} \\ &= \alpha\beta^2(\alpha+1) \times 1 \\ &= \alpha\beta^2(\alpha+1) \end{aligned}$$

Hence

$E(X^2) = \alpha\beta^2(\alpha+1)$

$Var(X) = E(X^2) - [E(X)]^2$

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= \alpha\beta^2(\alpha+1) - (\alpha\beta)^2 \\ &= \alpha\beta^2(\alpha+1) - \alpha^2\beta^2 \\ &= \beta^2[\alpha(\alpha+1) - \alpha^2] \\ &= \beta^2[\alpha^2 + \alpha - \alpha^2] \\ &= \beta^2(\alpha) \\ &= \alpha\beta^2 \end{aligned}$$

14 Exercise 14

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ such that

$$\begin{cases} \mathbb{P}(X \leq 89) = 0.90 \\ \mathbb{P}(X \leq 94) = 0.95 \end{cases}$$

Let us show that $\mu = 71.4$ and that $\sigma^2 = 189.4$

Let $Z = \frac{X-\mu}{\sigma}$. Then $X = \mu + \sigma Z$ and

$$\begin{aligned} \mathbb{P}(X \leq 89) &= \mathbb{P}(\mu + \sigma Z \leq 89) \\ &= \mathbb{P}(Z \leq \frac{89-\mu}{\sigma}) \end{aligned}$$

And similarly we have

$$\mathbb{P}(X \leq 94) = \mathbb{P}(Z \leq \frac{94-\mu}{\sigma})$$

We know that $Z \sim \mathcal{N}(0, 1)$.

Let Φ the **CDF** of Z. Then we have successively

$$\begin{cases} \mathbb{P}(X \leq 89) = 0.90 = \Phi(\frac{89-\mu}{\sigma}) \\ \mathbb{P}(X \leq 94) = 0.95 = \Phi(\frac{94-\mu}{\sigma} \end{cases}$$

$$\begin{cases} \frac{89-\mu}{\sigma} = \Phi^{-1}(0.90) ① \\ \frac{90-\mu}{\sigma} = \Phi^{-1}(0.95) ② \end{cases}$$

② − ① gives successively

$$\frac{90-\mu-89+\mu}{\sigma} = \Phi^{-1}(0.95) - \Phi^{-1}(0.90)$$

$$\frac{5}{\sigma} = \Phi^{-1}(0.95) - \Phi^{-1}(0.90)$$

$$\sigma = \frac{5}{\Phi^{-1}(0.95) - \Phi^{-1}(0.90)}$$

The relation ① gives successively

$$\begin{aligned} 89 - \mu &= \sigma \Phi^{-1}(0.90) \\ \mu &= 89 - \sigma \Phi^{-1}(0.90) \end{aligned}$$

[26]:

```python
from scipy.stats import norm
from numpy import power

p1 = 0.90
p2=0.95

sigma=5/(norm.ppf(p2)-norm.ppf(p1))
sigma_squared=power(sigma,2)
mu=89-sigma*norm.ppf(0.90)

print('sigma squared:',sigma_squared)
print('mu:',mu)
```

```
sigma squared: 189.4106020419289
mu: 71.36245122609756
```

[ ]:

Hence we have approximatively

$\mu = 71.4$ and $\sigma^2 = 189.4$