# Advanced Statistical Methods for Modeling and Finance: Solutions for exercises

Gbètoho Ezéchiel ADEDE

November 9, 2023

# 1 Exercise 1

# 2 Exercise 2

# 3 Exercise 3

# 4 Exercise 4

# 5 Exercise 5

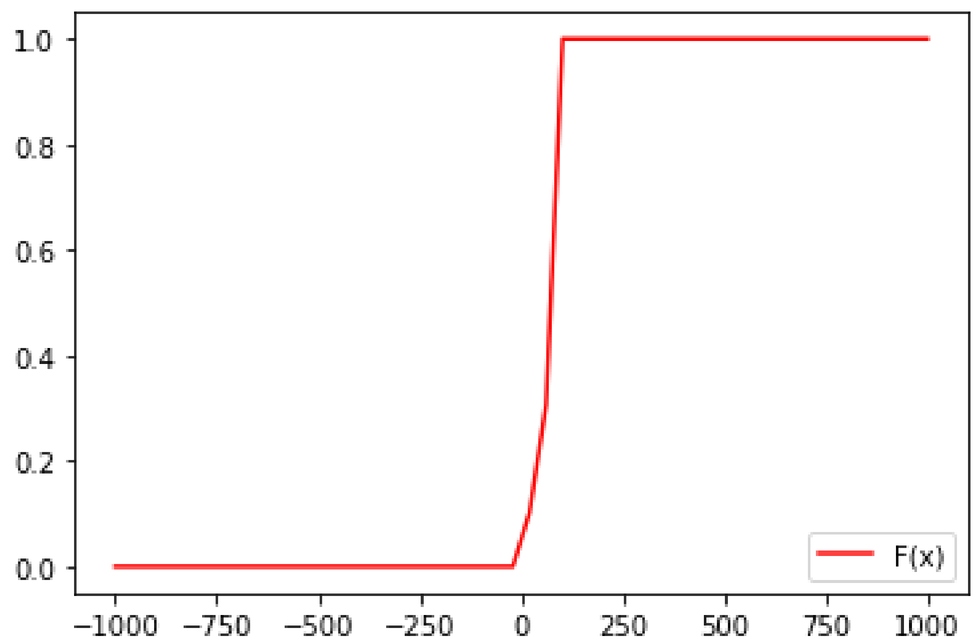
Let us suppose a random variable $X$ has it's CDF defined by:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{200} & \text{if } 0 < x < 100 \\ 1 & \text{if } x \geq 100 \end{cases}$$

## 5.1 The graph of F

[2]:

```
import numpy as np
import matplotlib.pyplot as plt
F=lambda x:np.where(0<x and x<100,x/200,np.where(x<0 ,0,1))
x=np.linspace(-1000,1000).tolist()
plt.plot(x,[F(i) for i in x],label='F(x)',color='r')
plt.legend(loc='lower right')
plt.show()
```



### 5.2 Let's compute the following probabilities

**a)** $P(-50 < X < 50)$

$$P(-50 < X < 50) = F(50) - F(-50)$$

$$\begin{aligned} 50 \in [0; 100] \Rightarrow F(50) &= \frac{50}{200} \\ &= \frac{1}{4} \end{aligned}$$

$$-50 \in ]-\infty; 0[ \Rightarrow F(-50) = 0$$

Then we have

$$\begin{aligned} P(-50 < X < 50) &= \frac{1}{4} - 0 \\ &= \frac{1}{4} \end{aligned}$$

**b)** $P(X = 0)$

$$\begin{aligned} P(X = 0) &= F(0) - \lim_{x \to 0^-} F(x) \\ &= \frac{0}{200} - \lim_{x \to 0^-} 0 \\ &= 0 \end{aligned}$$

**c)** $P(X = 100)$

$$\begin{aligned} P(X = 100) &= F(100) - \lim_{x \to 100^-} F(x) \\ &= 1 - \lim_{x \to 100^-} \frac{x}{200} \\ &= 1 - \frac{100}{200} \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

# 6 Exercise 6

Let us suppose a random variable $X$ has it's PDF defined by:

$$f(x) = \begin{cases} kx^2 & \text{if } 0 \leq x \leq 10 \\ 0 & \text{if } x > 10. \end{cases}$$

Let us find $P(7 < X < 15)$

Let us note $F$ the CDF of $X$

$$P(7 < X < 15) = F(15) - F(7)$$

$\forall x \geq 0$, we have:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dtdt \\ &= 0 + \int_{0}^{x} f(t)dt \\ &= \int_{0}^{x} f(t)dt \end{aligned}$$

$t \in [0; x] \iff 0 \leq t \leq x$

- If $x \leq 10$, then $0 \leq t \leq x \leq 10$ ie $0 \leq t \leq 10$ and then $f(t) = 2kt^2$

  So

$$\begin{aligned} F(x) &= \int_{0}^{x} 2kt^2 dt \\ &= \frac{2}{3}k \left[t^3\right]_0^x \\ &= \frac{2}{3}kx^3 \end{aligned}$$

- If $x \geq 10$, then we have:

$$\begin{aligned}
F(x) &= \int_0^{10} f(t)dt + \int_{10}^{x} f(t)dt \\
&= \int_0^{10} f(t)dt + 0 \\
&= \int_0^{10} 2kt^2 dt \\
&= \frac{2}{3}k\left[t^3\right]_0^{10} \\
&= \frac{2}{3}10^3 k \\
&= \frac{2000k}{3}
\end{aligned}$$

$$F(x) = \frac{2}{3}k \begin{cases} x^3 & \text{if } 0 \leq x \leq 10 \\ 1000 & \text{if } x > 10. \end{cases}$$

Let us determine $k$.

$$\begin{aligned}
\int_{\mathbb{R}} f(t)dt &= \int_{-\infty}^{0} f(t)dt + \int_0^{10} f(t)dt + \int_{10}^{+\infty} f(t)dt \\
&= 0 + \int_0^{10} 2kt^2 dtdt + 0 \\
&= 2k[\frac{1}{3}t^3]_0^{10} \\
&= \frac{2k}{3}[t^3]_0^{10} \\
&= \frac{2k}{3}[t^3]_0^{10} \\
&= \frac{2k}{3}(10^3 - 0^3) \\
&= \frac{2k}{3}(1000) \\
&= \frac{2000k}{3}
\end{aligned}$$

$f$ is a PDF, then $\int_{\mathbb{R}} f(t)dt = 1$

$$
\begin{aligned}
\int_{\mathbb{R}} f(t)dt = 1 &\Rightarrow \frac{2000k}{3} = 1 \\
&\Rightarrow 2000k = 3 \\
&\Rightarrow k = \frac{3}{2000}
\end{aligned}
$$

And then

$$
\begin{aligned}
F(x) &= \frac{2}{3} \times \frac{3}{2000} \begin{cases} x^3 & \text{if } 0 \leq x \leq 10 \\ 1000 & \text{if } x > 10. \end{cases} \\
&= \frac{1}{1000} \begin{cases} x^3 & \text{if } 0 \leq x \leq 10 \\ 1000 & \text{if } x > 10. \end{cases} \\
&= \begin{cases} \frac{1}{1000} x^3 & \text{if } 0 \leq x \leq 10 \\ 1 & \text{if } x > 10. \end{cases}
\end{aligned}
$$

$15 > 10$ then $F(15) = 1$

$7 \in [0; 10]$ then $F(7) = \frac{7^3}{1000} = \frac{343}{1000}$

$$
\begin{aligned}
P(7 < X < 15) &= F(15) - F(7) \\
\text{‘} &= 1 - \frac{343}{1000} \\
&= \frac{1000 - 343}{1000} \\
&= 3 \times \frac{16}{2000} \\
&= \frac{357}{1000}
\end{aligned}
$$

# 7 Exercise 7

Let

- X be a random variable with f as PDF, such that $f(t) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} t^2); t \in \mathbb{R}$
- $Y = X^2$ another random variable.
- $F_X$ the CDF of X
- $F_Y$ the CDF of Y
- $f_Y$ the PDF of Y

Let us show that $f_Y(y) = \frac{1}{\sqrt{2\pi y}} \exp(-\frac{y}{2}) \forall t \in \mathbb{R}$ $\forall t \in \mathbb{R}$, we have:

$$\begin{aligned}
F_Y(t) &= P(Y < t) \\
&= P(X^2 < t) \\
&= P(Y < \sqrt{t}) \\
&= P(|x| < \sqrt{t}) \text{ since } X^2 < t \Rightarrow t > 0 \\
&= P(-\sqrt{t} < \sqrt{t} < \sqrt{t}) \\
&= F_X(\sqrt{t}) - F_X(-\sqrt{t})
\end{aligned}$$

$$\begin{aligned}
f_Y(t) &= F'_Y(t) \\
&= \frac{1}{2\sqrt{t}} F'_X(\sqrt{t}) - (-\frac{1}{2\sqrt{t}}) F'_X(\sqrt{t}) \\
&= \frac{1}{2\sqrt{t}} f(\sqrt{t}) - (-\frac{1}{2\sqrt{t}}) f(-\sqrt{t}) \\
&= \frac{1}{2\sqrt{t}} f(\sqrt{t}) + \frac{1}{2\sqrt{t}} f(-\sqrt{t}) \\
&= \frac{1}{2\sqrt{t}} [f(\sqrt{t}) + f(-\sqrt{t})] \\
&= \frac{1}{2\sqrt{t}} (2) (\frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2} t)) \\
&= \frac{1}{\sqrt{t}} \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2} t) \\
&= \frac{1}{\sqrt{2\pi y}} \exp(-\frac{t}{2})
\end{aligned}$$

# 8 Exercise 8

Let X be a random variable such that $Var(X)$ exists .

Let's show that $Var(X) = E(X^2) - \mu^2$ where $\mu = E(X)$

We have by definition:

$$\begin{aligned}
Var(X) &= E([X - E(X)]^2) \\
&= E(X^2 - 2XE(X) + (E(X))^2) \\
&= E(X^2 - 2\mu X + \mu^2) \\
&= E(X^2 - 2\mu X + \mu^2) \text{ with } \mu = E(X) \\
&= E(X^2) - 2\mu E(X) + \mu^2 \\
&= E(X^2) - 2\mu\mu + \mu^2 \text{ because } E(X) = \mu \\
&= E(X^2) - 2\mu^2 + \mu^2 \\
&= E(X^2) - \mu^2
\end{aligned}$$

Hence the result.

# 9 Exercise 9

Let X a discrete random variable with pmf f positive at -1, 0,1 and zero elsewhere.

## 9.1 a) Let us find $E(X)$ if $p(0) = \frac{1}{4}$

$$\begin{aligned} E(X^2) &= (-1)^2 p(-1) + (0)^2 p(0) + (1)^2 p(1) + 0 \\ &= p(-1) + p(1) \end{aligned}$$

We have

$$\begin{aligned} p(-1) + p(0) + p(1) + 0 &= p(-1) + p(0) + p(1) \\ &= p(-1) + \frac{1}{4} + p(1) \text{ since } p(0) = \frac{1}{4} \end{aligned}$$

Then

$$\begin{aligned} p(-1) + p(1) &= 1 - p(0) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

And then $E(X^2) = \frac{3}{4}$

## 9.2 a) Let us find $p(-1)$ and $p(1)$ if $p(0) = \frac{1}{4}$ and if $E(X) = 1$

$$\begin{aligned} E(X) &= (-1)p(-1) + (0)p(0) + (1)p(1) + 0 \\ &= p(1) - p(-1) \end{aligned}$$

$$\begin{aligned} E(X) = 1 &\Rightarrow p(1) - p(-1) = 1 \\ &\Longrightarrow p(1) = 1 + p(-1) \quad 1 \end{aligned}$$

$p(-1) + p(0) + p(1) = 1$ and $p(0) = \frac{1}{4}$

Then we get:

$$
\begin{aligned}
p(-1)+p(1) &= 1-\frac{1}{4} \\
&= \frac{3}{4}
\end{aligned}
$$

And then $p(1) = \frac{3}{4} - P(-1)$ ②

$$
\begin{aligned}
① = ② \Rightarrow 1+p(-1) &= \frac{3}{4} - p(-1) \\
\Rightarrow 2p(-1) &= \frac{3}{4} - 1 \\
\Rightarrow 2p(-1) &= -\frac{1}{4} \\
\Rightarrow p(-1) &= -\frac{1}{8}
\end{aligned}
$$

Hence

$$
\begin{aligned}
p(-1) &= \frac{3}{4} - p(-1) \\
&= \frac{3}{4} + \frac{1}{8} \\
&= \frac{24+4}{4\times 8} \\
&= \frac{28}{32} \\
&= \frac{7}{8}
\end{aligned}
$$

## 10 Exercise 10

Let X a continous random variable with pdf f, such that $f(x) = \frac{1}{3}$ for $-1 < x < 2$

Let $M_X$ the moment generating function of $X$

Let $t \in \left]-1;2\right[$

$$
\begin{aligned}
M_X(t) &= E(e^tX) \\
&= \int_{\mathbb{R}} e^{tx} f(x)dx \\
&= \int_{-\infty}^{-1} e^{tx} f(x)dx + \int_{-1}^{2} e^{tx} f(x)dx + \int_{2}^{+\infty} e^{tx} f(x)dx \\
&= 0 + \int_{-1}^{2} \frac{1}{3} e^{tx} dx + 0 \\
&= \frac{1}{3} \int_{-1}^{2} e^{tx} dx
\end{aligned}
$$

When $t \neq 0$, we have:

$$
\begin{aligned}
M_X(t) &= \frac{1}{3}[\frac{1}{t}e^{tx}]_{-1}^{2} \\
&= \frac{1}{3t}[e^{tx}]_{-1}^{2} \\
&= \frac{1}{3t}(e^{2t} - e^{-t})
\end{aligned}
$$

## 11 Exercise 11

Let $X \sim \mathcal{B}(n,p)$ and $M_X$ the MGF of $X$

Let us show that $M_X(t) = (1 - p + pe^t)^n \ \forall t \in \mathbb{R}$

Let $q = 1 - p$

We have $\mathbb{P}(X = k) = \binom{n}{k} p^k q^{n-k} \forall \ k \in \{0, 1, \ldots, n\}$

Let $t \in \mathbb{R}$

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \Sigma_{i=1}^{n} e^{tk} P(Y = k) \\
&= \Sigma_{i=1}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} \\
&= \Sigma_{i=1}^{n} p^k (e^t)^k \binom{n}{k} q^{n-k} \\
&= \Sigma_{i=1}^{n} (pe^t)^k \binom{n}{k} q^{n-k} \\
&= \Sigma_{i=1}^{n} a^k \binom{n}{k} b^{n-k} \ \textbf{with} \ a = pe^t \text{and } b = q \\
&= (a + b)^n \text{ from Newten's binomial formula} \\
&= (pe^t + 1 - p)^n \text{ since } a = pe^t \text{ and } b = q \\
&= (1 - p + pe^t)^n
\end{aligned}
$$

Hence the result.

## 12 Exercise 12

Let X a random variable that has a Poisson ditribution with $\lambda$ as parameter

We have:

$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$ for k in $\mathbb{N}$

Let $M_X$ the moment generating function of $X$ **and** $t\in\mathbb{R}$

### 12.1 Let us show that $M_X(t)=e^{\lambda(e^t-1)}$

$$
\begin{aligned}
M_X(t) &= E(e^tX)\\
&= \Sigma_{k\in\mathbb{N}} e^{tk}\mathbb{P}(X=k)\\
&= \Sigma_{k\in\mathbb{N}} e^{tk}\frac{\lambda^k e^{-\lambda}}{k!}\\
&= \Sigma_{k\in\mathbb{N}} \frac{e^{tk}\lambda^k e^{-\lambda}}{k!}\\
&= \Sigma_{k\in\mathbb{N}} \frac{(e^t)^k\lambda^k e^{-\lambda}}{k!}\\
&= \Sigma_{k\in\mathbb{N}} \frac{(\lambda e^t)^k e^{-\lambda}}{k!}
\end{aligned}
$$

$\forall x\in\mathbb{R}$, $e^x=\Sigma_{n\in\mathbb{N}}\frac{x^n}{n!}$

Then we have:

$$
\begin{aligned}
M_X(t) &= e^{-\lambda}e^{\lambda e^t}\\
&= e^{-\lambda+\lambda e^t}\\
&= e^{\lambda e^t-\lambda}\\
&= e^{\lambda(e^t-1)}
\end{aligned}
$$

Hence $M_X(t)=e^{\lambda(e^t-1)}$

### 12.2 Let us find it's mean and variance

Let $t\in\mathbb{R}$

- Mean $E(X)$

  $E(X)=M'_X(0)$

  $M_X(t)=e^{\lambda(e^t-1)}$

$$\begin{aligned} M'_X(t) &= \lambda e^t e^{\lambda(e^t-1)} \\ &= \lambda e^t M_X(t) \end{aligned}$$

$M_X(0) = e^{e^0-1} = e^0 = 1$

$$\begin{aligned} M'_X(0) &= \lambda e^0 M_X(0) \\ &= \lambda(1) \\ &= \lambda \end{aligned}$$

Then $E(X) = \lambda$

- Variance $Var(X)$

$var(X) = E(X^2) - (E(X))^2$ with $E(X^2) = M''_X(0)$

$$M''_X(t) = (M'_X)'(t)$$

$$M'_X(t) = \lambda e^t M_X(t)$$

$$\begin{aligned} M''_X(t) &= \lambda[e^t M_X(t) + e^t M'_X(t)] \\ &= \lambda e^t [M_X(t) + M'_X(t)] \end{aligned}$$

$$\begin{aligned} M''_X(0) &= \lambda e^0 [M_X(0) + M'_X(0)] \\ &= \lambda(1)[1+\lambda] \\ &= \lambda(1+\lambda) \end{aligned}$$

Then $E(X^2) = \lambda(1+\lambda)$ and then

$$\begin{aligned} Var(X) &= \lambda(1+\lambda) - \lambda^2 \\ &= \lambda + \lambda^2 - \lambda^2] \\ &= \lambda \end{aligned}$$