

Structured Recursive Separator Decompositions for Planar Graphs in Linear Time*

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Abstract

Given a planar graph G on n vertices and an integer $r < n$, an r -division of G with *few holes* is a decomposition of G into $O(n/r)$ regions of size at most r such that each region contains at most a constant number of faces that are not faces of G (also called *holes*), and such that, for each region, the total number of vertices on these faces is $O(\sqrt{r})$.

We provide a linear-time algorithm for computing r -divisions with few holes. In fact, our algorithm computes a structure, called *decomposition tree*, which represents a recursive decomposition of G that includes r -divisions for essentially all values of r . In particular, given an increasing sequence $\mathbf{r} = (r_1, r_2, \dots)$, our algorithm can produce a *recursive \mathbf{r} -division* with few holes in linear time.

r -divisions with few holes have been used in efficient algorithms to compute shortest paths, minimum cuts, and maximum flows. Our linear-time algorithm improves upon the decomposition algorithm used in the state-of-the-art algorithm for minimum *st*-cut (Italiano, Nussbaum, Sankowski, and Wulff-Nilsen, STOC 2011), removing one of the bottlenecks in the overall running time of their algorithm (analogously for minimum cut in planar and bounded-genus graphs).

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1 Introduction

Separators decompose a graph in a *balanced* way into two subgraphs with a limited number of vertices in common. Separators are often used in efficient algorithms using a divide-and-conquer strategy [LT80, Chu90, Shm97, RH00]. Graphs with small recursive separators include planar graphs [Ung51, LT79, Dji82, Mil86, GM90, ST96, DV97], bounded-genus graphs [Dji85, GHT84, Kel06], minor-free graphs [And86, AST90, PRS94, RW09, BLR10, KR10, WN11], and graphs with bounded tree-width [Hal76, RS86, AP89]. Furthermore, for graphs of all these classes, separators can be found efficiently, often in linear time. For planar graphs, experimental results demonstrate that separator algorithms are practical [ADGM06, HSW⁺09, FEMPS13].

Perhaps the most influential result of this kind is the linear-time algorithm of Lipton and Tarjan [LT79] for finding a separator of size $O(\sqrt{n})$ in a planar graph with n vertices. Consider the result of using this algorithm recursively until each separated subgraph has size at most some specified limit r . It is easy to show that $O(n/r)$ subgraphs result, and that the average number of boundary vertices per subgraph is $O(\sqrt{r})$. Frederickson [Fre87] showed that, with additional care, one can ensure that each subgraph has $O(\sqrt{r})$ boundary vertices; he calls such a decomposition an r -*division*, and he refers to the subgraphs as *regions*. The running time of Frederickson’s algorithm is $O(n \log r + (n/\sqrt{r}) \log n)$.

Decompositions of such kind have been used in many efficient algorithms for planar graphs, such as those for computing shortest paths [Fre87, HKRS97], maximum flow [JV82], polygon triangulation [Goo95], and I/O-efficient algorithms [MZ08].

For some of these applications, Frederickson’s algorithm for r -divisions is too slow. Goodrich [Goo95] gave a linear-time algorithm that achieves $O(\sqrt{r})$ boundary size for planar graphs (see [ABT04] for the I/O model). Even this was not enough for the linear-time shortest-path algorithm of Henzinger, Klein, Rao, and Subramanian [HKRS97], which requires recursive applications of an algorithm for r -divisions (with roughly $\log^* n$ levels of recursion). They addressed this by showing that a linear-time $O(\sqrt{n})$ separator algorithm can be used to obtain a sublinear-time separator algorithm with a worse (but still sublinear) boundary-size guarantee.

However, for some of the more involved algorithms working with planar embedded graphs, it is essential that the regions of the division are topologically “nice” in that the boundary of each region consists of a constant number of faces, also called *holes*. Such a division can be found by using small *cycle* separators (Miller [Mil86]) instead of just small separators, and by incorporating iterations in which the graph is separated according to the number of holes.

Such r -divisions with a constant number of holes were first used in algorithms of Klein and Subramanian [KS98, Sub95], and subsequently in many other algorithms [FR06, CR10, Cab12, MWN10, INSWN11, LS11, KKS11, EFN12, MS12].¹

Prior to this work, computing an r -division with a constant number of holes per region required time $O(n \log r + (n/\sqrt{r}) \log n)$ [INSWN11]. Such r -divisions were one of the time bottlenecks in the state-of-the-art algorithms for minimum st -cut and maximum st -flow [INSWN11] and minimum cut in undirected planar graphs [LS11] and bounded-genus graphs [EFN12]. Whether such an r -division can be computed in linear time was an open problem until the current work. For example, Cabello [Cab12] remarks that “*it is unclear if the algorithm of Goodrich [Goo95] can also be modified to use the cycle-separator, and thus obtain a linear-time construction of r -decompositions with a few holes.*”

Contributions

We provide a linear-time algorithm for computing r -divisions with few holes for any triangulated biconnected plane graph G and any r (Theorem 1).

In fact, the algorithm produces a *decomposition tree* of G (Theorem 2), which is a tree that naturally represents a recursive decomposition of G by cycle separators, and from which one can read off, in linear time, a recursive r -division with few holes, for any exponentially increasing sequence $\mathbf{r} = r_1, r_2, \dots$ (Theorem 3).

Our linear-time algorithm improves upon the $O(n \log r + (n/\sqrt{r}) \log n)$ -time algorithm of Italiano, Nussbaum, Sankowski, and Wulff-Nilsen [INSWN11], removing one of the time bottlenecks of the state-of-the-art algorithms for minimum st -cut and maximum st -flow [INSWN11], as well as minimum cut [LS11, EFN12] (all in undirected graphs).

¹Cabello [Cab12] requires only that the *average* number of holes per region be small.

Concurrent with and independent of our result, Arge, van Walderveen, and Zeh [AvWZ13] also gave an algorithm that produces an r -division with few holes in linear time. The focus of their paper is I/O-efficient algorithms. Their approach differs from ours in that their algorithm requires the value of r as input and uses it in a way that makes it unsuitable for producing recursive r -divisions in linear time. Such a linear time algorithm seems to be crucial for obtaining linear-time algorithms using r -divisions (see [HKRS97]).

Techniques

The overall approach of our algorithm builds on that of Goodrich [Goo95], which is based on Lipton-Tarjan vertex separators [LT79]. However, our approach must handle the additional complexities of finding cycle separators (see Miller [Mil86]), which involves maintaining spanning trees in both the input graph and its planar dual, and of bounding the number of holes. We manage both primal and dual trees simultaneously by using a primal spanning tree that follows a dual breadth-first-search tree (Section 4.2). The levels of the dual BFS tree define connected components, which we maintain in a *Component Tree* (Section 4.1). This component tree internally captures the structural connectivity of dual BFS components.

Outline

Precise definitions of r -divisions, holes, recursive divisions, and decomposition trees, as well as formal statements of our results are in Section 2.5 and Section 2.6. We give a high-level description of the r -division algorithm and prove its correctness in Section 3.

A key ingredient of our algorithm is a cycle separator algorithm (Section 4) that can be implemented efficiently by using auxiliary data structures (based on *dynamic trees* and *Euler-tour trees*, see Section 2.8). In a straightforward computation of an r -division, the cycle separator at each recursive level can be found in linear time and the graph can be cut in linear time into two regions along the cycle. In order to achieve *overall* linear time, we aim at *sublinear* time bounds for computing the cycle separator and cutting the graph along it. In our algorithm, we initialize the data structures once, and then, in subsequent steps of the recursion, we reuse what has been computed already (Section 5), and we call an efficient dynamic-tree implementation of our cycle separator algorithm (Section 4.5).

2 Preliminaries

2.1 Graph Notation

Let $G = (V, E)$ be a simple graph. For a subset V' of V , we denote by $\delta_G(V')$ the set of edges uv of G such that $v \in V'$ and $u \notin V'$. We refer to $\delta_G(V')$ as a *cut* in G . If V' and $V - V'$ both induce connected subgraphs of G , we say it is a *simple cut* (also known as a *bond*). In this case, V' and $V - V'$ are the vertex-sets of the two connected components of $\delta_G(V')$, which shows that the edges of $\delta_G(V')$ uniquely determine the bipartition $\{V', V - V'\}$.

A graph is called *biconnected* iff any pair of vertices is connected by at least two vertex-disjoint paths.

For a spanning tree T of G and an edge e of G not in T , the *fundamental cycle* of e with respect to T in G is the simple cycle consisting of e and the unique simple path in T between the endpoints of e .

2.2 Embeddings and Planar Graphs

We review the definitions of combinatorial embeddings. For elaboration, see also <http://planarity.org>.

Let E be a finite set, the edge-set. We define the corresponding set of *darts* to be $E \times \{\pm 1\}$. For $e \in E$, the darts of e are $(e, +1)$ and $(e, -1)$. We think of the darts of e as oriented versions of e (one for each orientation). We define the involution $\text{rev}(\cdot)$ by $\text{rev}((e, i)) = (e, -i)$. That is, $\text{rev}(d)$ is the dart with the same edge but the opposite orientation.

A graph on E is defined to be a pair (V, E) where V is a partition of the darts. Thus each element of V is a nonempty subset of darts. We refer to the elements of V as *vertices*. The endpoints of an edge e are the subsets $v, v' \in V$ containing the darts of e . The *head* of a dart d is the subset $v \in V$ containing d , and its *tail* is the head of $\text{rev}(d)$.

An *embedding* of (V, E) is a permutation π of the darts such that V is the set of orbits of π . For each orbit v , the restriction of π to that orbit is a permutation cycle. The permutation cycle for v specifies how the darts with head v are arranged around v in the embedding (in, say, counterclockwise order). We refer to the pair (π, E) as an embedded graph.

Let π^* denote the permutation $\text{rev} \circ \pi$, where \circ denotes functional composition. Then (π^*, E) is another embedded graph, the *dual* of (π, E) . (In this context, we refer to (π, E) as the *primal*.)

The *faces* of (π, E) are defined to be the vertices of (π^*, E) . Since $\text{rev} \circ (\text{rev} \circ \pi) = \pi$, the dual of the dual of (π, E) is (π, E) . Therefore the faces of (π^*, E) are the vertices of (π, E) .

We define an embedded graph (π, E) to be *planar* if $n - m + \phi = 2\kappa$, where n is the number of vertices, m is the number of edges, ϕ is the number of faces, and κ is the number of connected components. Since taking the dual swaps vertices and faces and preserves the number of connected components, the dual of a planar embedded graph is also planar.

Note that, according to our notation, we can use e to refer to an edge in the primal or the dual.

2.3 Some Properties of Planar Graphs

Fact 1 (Sparsity). *Let G be a simple planar graph. $|E(G)| \leq 3|V(G)| - 6$.*

When the exact constants are not considered, we write $O(|G|)$ to denote $O(|V(G)|) = O(|E(G)|)$.

Fact 2 (Simple-cycle/simple-cut duality [Whi32]). *A set of edges forms a simple cycle in a planar embedded graph G iff it forms a simple cut in the dual G^* .*

Since a simple cut in a graph uniquely determines a bipartition of the vertices of the graph, a simple cycle in a planar embedded graph G uniquely determines a bipartition of the faces.

Definition 1 (Encloses). *Let C be a simple cycle in a connected planar embedded graph G . The edges of C form a simple cut $\delta_{G^*}(S)$ for some set S of vertices of G^* , i.e. faces of G . Thus C uniquely determines a bipartition $\{F_0, F_1\}$ of the faces of G . Let f_∞, f be faces of G . We say C *encloses* f with respect to f_∞ if exactly one of f, f_∞ is in S . For a vertex/edge x , we say C *encloses* x (with respect to f_∞) if it encloses some face incident to x (encloses strictly if in addition x is not part of C).*

Fact 3 ([vS47]). *For any spanning tree T of G , the set of edges of G not in T form a spanning tree of G^* .*

For a spanning tree T of G , we typically use T^* to denote the spanning tree of G^* consisting of the edges not in T . We often refer to T^* as the *cotree* of T [Epp03].

2.4 Separators

Definition 2. *For an assignment $W(\cdot)$ of nonnegative weights to faces, edges, and vertices of G , we say a simple cycle C is a *balanced separator* if the total weight of faces, edges, and vertices strictly enclosed by C and the total weight not enclosed are each at most $3/4$ of the total weight.*

(Traditionally, balance involves a bound of $2/3$. We use $3/4$ because it simplifies the presentation.)

One can reduce the case of face/edge/vertex weight to the case of face weight. For each vertex or edge, remove its weight and add it to an incident face. A cycle separator that is balanced with respect to the resulting face-weight assignment is balanced with respect to the original weight assignment. We may therefore assume in cycle-separator algorithms that only the faces have weight.

Let G be a planar embedded graph with face weights. Suppose that no face has more than $1/4$ of the total weight. Lipton and Tarjan [LT79] show that, if G is triangulated (every face has size at most 3) then for any spanning tree T of G , there is an edge not in T whose fundamental cycle with respect to T is a balanced separator. Goodrich [Goo95] observed that such an edge can be found by looking for an edge-separator in the cotree T^* of T .

We modify this approach slightly: let T^* be a spanning tree of the planar dual G^* of G such that T^* has maximum degree 3. Let T be the cotree of T^* , so T is a spanning tree of G . Root T^* at an arbitrary vertex of degree one or two. Let v be a leafmost vertex of T^* such that the descendants of v (including v) comprise more than $3/4$ of the weight. Let \hat{e} be the edge of T^* connecting v to a child with greatest descendant weight.

Lemma 1. *The fundamental cycle of \hat{e} with respect to T is a balanced simple cycle separator.*

Proof. Simple algebra shows that each of the two trees of $T^* \setminus \{\hat{e}\}$ comprises between $1/4$ and $3/4$ of the total weight. One of these trees consists of the faces enclosed by C , and the other consists of the faces not enclosed by C , where C is the fundamental cycle of \hat{e} with respect to T . \square

Miller [Mil86] proved the following theorem (actually, a stronger version).

Theorem (Miller [Mil86]). *For a planar triangulated biconnected graph G with weights such that the weight of each face, edge, and vertex comprises at most $2/3$ of the total weight, there is a simple cycle C of length at most $2\sqrt{2}|V(G)|$ such that the total weight strictly enclosed is at most $2/3$ and the total weight not enclosed is at most $2/3$. There is a linear-time algorithm to find such a cycle.*

In this paper, we do not use Miller’s construction. We give another construction that, with the aid of some auxiliary data structures, can be carried out in sublinear time. For simplicity of presentation, we present a construction that achieves a balance of $3/4$ instead of $2/3$, and refrain from optimizing the various constants that arise in our construction.

2.5 Divisions

Frederickson [Fre87] introduced the notion of r -divisions. Let \bar{G} be an n -vertex planar embedded graph.

Definition 3. *A region R of \bar{G} is an edge-induced subgraph of \bar{G} .*

Definition 4. *A division of \bar{G} is a collection of regions such that each edge is in at least one region. A vertex is a boundary vertex of the division if it is in more than one region. A division is an r -division if there are $O(n/r)$ regions, and each region has at most r vertices and $O(\sqrt{r})$ boundary vertices.*

Frederickson’s definition does not address the number of holes, which is not relevant in his algorithms (and in some subsequent algorithms building on his). The next definition follows the lines of that of Cabello [Cab12] but our terminology is slightly different.

Definition 5. *A natural face of a region R is a face of R that is also a face of \bar{G} . A hole of R is a face of R that is not natural.*

Definition 6. *An r -division with few holes is an r -division in which*

- *any edge of two regions is on a hole of each of them, and*
- *every region has $O(1)$ holes.*

This differs from Cabello’s definition in that his requires only that the *average* number of holes per region be $O(1)$. We use a stronger requirement because some algorithms depend on it.

Our main theorem is as follows.

Theorem 1. *For a constant s , there is a linear-time algorithm that, for any biconnected triangulated planar embedded graph \bar{G} and any $r \geq s$, outputs an r -division of \bar{G} with few holes.*

In fact, our algorithm further guarantees that all regions in the r -division are connected. This property is desirable in some applications (e.g., [LSWN12]).

2.6 Recursive Divisions and Decomposition Trees

Some algorithms, e.g. the shortest-path algorithm of Henzinger et al. [HKRS97], require that the graph be decomposed into regions that are in turn decomposed into regions, and so on. The Henzinger et al. algorithm requires roughly $\log^* n$ levels of decomposition, so it would take more than linear time to find all the different divisions independently. We describe a simple decomposition of a planar graph that allows one to obtain such recursive divisions in linear time.

Definition 7. A decomposition tree for \bar{G} is a rooted tree in which each leaf is assigned a region of \bar{G} such that each edge of \bar{G} is represented in some region. For each node x of the decomposition tree, the region R_x corresponding to x is the subgraph of \bar{G} that is the union of the regions assigned to descendants of x .

Definition 8. A decomposition tree \mathcal{T} admits an r -division with few holes if there is a set S of nodes of \mathcal{T} whose corresponding regions form an r -division of \bar{G} with few holes.

Theorem 2. For a constant s , there is a linear-time algorithm that, for any biconnected triangulated planar embedded graph G , outputs a binary decomposition tree \mathcal{T} for G that admits an r -division of G with few holes for every $r \geq s$.

Definition 9. For an exponentially increasing sequence $\mathbf{r} = (r_1, r_2, \dots)$ of numbers, a recursive \mathbf{r} -division of G with few holes is a decomposition tree for G in which, for $i = 1, 2, \dots$, the nodes at height i correspond to regions that form an r_i -division of G with few holes.

A recursive \mathbf{r} -division is exactly the kind of structure that is useful for the linear-time shortest-paths algorithm [HKRS97], although that algorithm does not require one with few holes.

Theorem 3. There is a linear-time algorithm that, given a decomposition tree satisfying the condition of Theorem 2, and given an increasing sequence \mathbf{r} , returns a recursive \mathbf{r} -division of G with few holes.

Note that Theorems 1 and 3 follow easily from Theorem 2.

2.7 A Recurrence Relation

The following lemma, whose proof is provided in the appendix, is used to analyze several quantities throughout the paper.

Lemma 2. Let $\frac{1}{2} \leq \beta < 1$, let c be a constant, and let

$$T_r(n) \leq \begin{cases} \rho n^\beta + \max_{\{\alpha_i\}} \sum_i T_r(\alpha_i n) & \text{if } n > r \\ 0 & \text{if } n \leq r \end{cases},$$

where the maximization is over $\{\alpha_i\}_{i=1}^8$ such that

1. $\alpha_i \leq 3/4 + c/\sqrt{n}$ for $i = 1, \dots, 8$, and
2. $1 \leq \sum_i \alpha_i \leq 1 + c/\sqrt{n}$.

There exists a constant s such that for every $r \geq s$, $T_r(n)$ is $O(n/r^{1-\beta})$.

2.8 Dynamic Trees and Euler-tour Trees

The dynamic-tree data structure of Sleator and Tarjan [ST83] represents rooted forests so as to support topological operations LINK and CUT (removing and adding edges to the forests, respectively), and to support assignments of weights to nodes with operations that add a number to the weights of all ancestors of a given node, and that search among the ancestors for a node with minimum or maximum weight. Each operation can be performed in $O(\log n)$ amortized time. With care, given an initial tree, a dynamic-tree representation can be constructed so that the amortization is over $O(n/\log n)$ operations.

Many extensions and variants have been developed [ST83, EIT⁺92, Fre97, ABH⁺04, AHdLT05, TW05]. For example, using the self-adjusting top trees of Tarjan and Werneck [TW05], one can search among descendants instead of ancestors, and one can represent embedded trees (adapting an idea of Eppstein et al. [EIT⁺92]).

The Euler-tour-tree data structure of Henzinger and King [HK99] represents an Euler tour of a tree using a balanced binary search tree. It supports topological LINK and CUT operations. In particular, for a vertex v of a tree, any consecutive (in the order used by the Euler tour) set of children of v can be cut in a single operation that requires a constant number of binary search tree operations, and hence takes $O(\log n)$ time. This is useful for representing embedded trees, by choosing the order in which nodes are visited by the Euler tour to be their cyclic order in the embedding. The Euler-tour tree can be decorated with additional labels to support ancestor and descendant searches in logarithmic time as well.

3 Computing a Decomposition Tree

In this section, we give a high-level description of the algorithm of Theorem 2 for computing a decomposition tree, and we prove its correctness. The input graph \bar{G} is assumed to be biconnected and triangulated. It follows that, for every region R of \bar{G} , every natural face is a triangle.

The algorithm is as follows. Given the input graph \bar{G} , the algorithm performs some preprocessing necessary for the sublinear-time cycle-separator algorithm (Algorithm 2), and calls $\text{RECURSIVEDIVIDE}(\bar{G}, 0)$. This procedure is given in Algorithm 1.

Algorithm 1: $\text{RECURSIVEDIVIDE}(R, \ell)$

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1 let  $n = |V(R)|$ 
2 if  $n \leq s$  then return a decomposition tree consisting of a leaf assigned  $R$ 
3 if  $\ell \bmod 3 = 0$  then separator chosen below to balance number of vertices
4 else if  $\ell \bmod 3 = 1$  then separator chosen to balance number of boundary vertices
5 else if  $\ell \bmod 3 = 2$  then separator chosen to balance number of holes
6 let  $R'$  be the graph obtained from  $R$  as follows: triangulate each hole by placing an artificial vertex in
   the face and connecting it via artificial edges to all occurrences of vertices on the boundary of the hole
7 find a balanced simple-cycle separator  $C$  in  $R'$  with at most  $c\sqrt{n}$  natural vertices
8 let  $F_0, F_1$  be the sets of natural faces of  $R$  enclosed and not enclosed by  $C$ , respectively
9 for  $i \in \{0, 1\}$  do
10   let  $R_i$  be the region consisting of the edges of faces in  $F_i$  and the edges of  $C$  that are in  $R$ 
11    $\mathcal{T}_i \leftarrow \text{RECURSIVEDIVIDE}(R_i, \ell + 1)$ 
12 end
13 return the decomposition tree  $\mathcal{T}$  consisting of a root with left subtree  $\mathcal{T}_0$  and right subtree  $\mathcal{T}_1$ 

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This procedure, given a connected region R with more than s edges, and given a recursion-depth parameter ℓ , first triangulates each hole of R by adding an artificial vertex and attaching it via artificial edges to each occurrence of a vertex on the boundary of the hole. Let R' be the resulting graph. See Figures 1(a), 1(b) for an illustration. The vertices and edges that are not artificial are *natural*. Triangulating in this way establishes biconnectivity of R' .

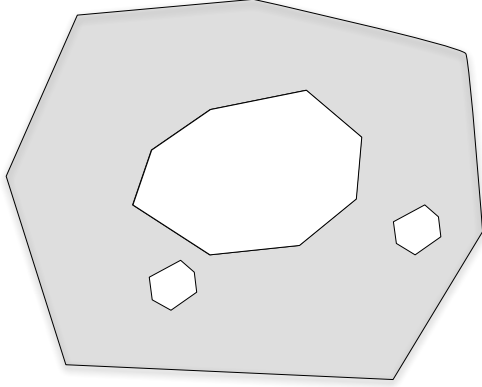
Lemma 3. R' is biconnected.

Proof. Let u be a cut vertex of R . Let uv and uw be two consecutive (in the embedding) edges incident to u that belong to different biconnected components. Since vu, uw is a v -to- w path in R , the edge vw is not in R . Hence uv and uw belong to a face h of R that is not a triangle, hence h is a hole. After triangulation, $vx_h, x_h w$ is another vertex-disjoint path from v to w , using the artificial node x_h . \square

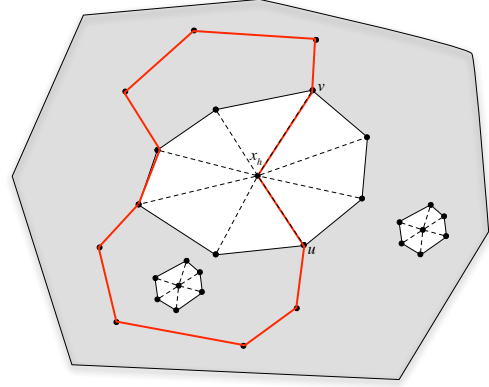
Next, the procedure uses the $\text{SIMPLECYCLESEPARATOR}$ procedure (Section 4.4) to find a simple-cycle separator C consisting of at most $c\sqrt{n}$ natural vertices, where c is a constant and $n := |V(R)|$ is the number of vertices of R (which is equivalent to the number of natural vertices of R'). Depending on the current recursion depth ℓ , the cycle separates R in a balanced way with respect to either vertices, boundary vertices, or holes.

Note that $\text{SIMPLECYCLESEPARATOR}$ is called on R' , which has more than n vertices since it also has some artificial vertices. However, we show in Lemma 6 that there are at most twelve artificial vertices, so even when using a generic algorithm for finding a simple cycle separator, the bound of $c\sqrt{n}$ still holds for some choice of c (since $n \geq s$). In fact, our procedure $\text{SIMPLECYCLESEPARATOR}$ takes into account which vertices are artificial, and returns a separator consisting of at most $4\sqrt{3n}$ natural vertices, and possibly also some artificial vertices. The number of artificial vertices on the separator does not matter in the analysis of RECURSIVEDIVIDE .

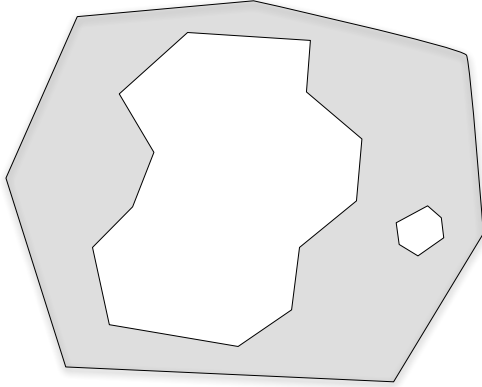
The cycle C determines a bipartition of the faces of the triangulated graph R' , which in turn induces a bipartition (F_0, F_1) of the natural faces of R . For $i \in \{0, 1\}$, let R_i be the region consisting of the edges bounding the faces in F_i , together with the edges of C that are in R (i.e. omitting the artificial edges added to triangulate the artificial faces). See Figures 1(c), 1(d) for an illustration.



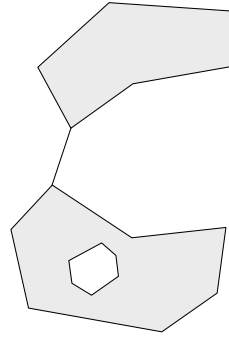
(a) A schematic diagram of a region R with four holes (white faces, one of them being the unbounded face).



(b) The graph R' and a cycle separator C (solid red). Artificial triangulation edges are dashed (triangulation edges are not shown for the unbounded hole to avoid clutter).



(c) The region R_0 consisting of the edges bounding the faces not enclosed by C together with the edges of C that belong to R . Equivalently, R_0 is the subgraph of R' not strictly enclosed by C without any artificial edges and vertices. R_0 has three holes.



(d) The region R_1 consisting of the edges bounding the faces enclosed by C together with the edges of C that belong to R . Equivalently, R_1 is the subgraph of R' enclosed by C without any artificial edges and vertices. R_1 has two holes. Note that a hole is not necessarily a simple face.

Figure 1: Illustration of triangulating a hole and separating along a cycle.

Lemma 4. *If R is connected then R_0 is connected and R_1 is connected.*

The procedure calls itself recursively on R_0 and R_1 , obtaining decomposition trees \mathcal{T}_0 and \mathcal{T}_1 , respectively. The procedure creates a new decomposition tree \mathcal{T} by creating a new root corresponding to the region R and assigning as its children the roots of \mathcal{T}_0 and \mathcal{T}_1 .

3.1 Number of Holes

The triangulation step (Line 6) divides each hole h into a collection of triangle faces. We say a hole h is *fully enclosed by C* if all these triangle faces are enclosed by C in R' .

Lemma 5. *Suppose that there are k holes that are fully enclosed by C . Then R_0 has $k + 1$ holes.*

Proof. We give an algorithmic proof. See Figure 1 for an illustration. Initialize R'_0 to be the graph obtained from R' by deleting all edges not enclosed by C . Then C is the boundary of the infinite face of R'_0 . Consider in turn each hole h of R such that a nonempty proper subset of h 's triangle faces are enclosed by C . For each such face h , C includes the artificial vertex x_h placed in h , along with two incident edges ux_h and x_hv where u and v are distinct vertices on the boundary of h . Deleting all the remaining artificial edges of h modifies the boundary of the infinite face by replacing ux_h x_hv with a subsequence of the edges forming the boundary of h . In particular, deleting these artificial edges does not create any new faces.

Finally, for each hole h that is fully enclosed by C , delete the artificial edges of h , turning h into a face of R'_0 . The resulting graph is R_0 , whose holes are the holes of R that were fully enclosed by C , together with the infinite face of R_0 . \square

If the recursion depth mod 3 is 2, Line 7 of RECURSIVEDIVIDE must select a simple cycle in R' that is balanced with respect to the number of holes. To achieve this, for each hole h of R , the algorithm assigns weight 1 to one of the triangles resulting from triangulating h in Line 6, and weight 0 to all other faces. Then the algorithm finds a cycle C that is balanced with respect to these face-weights.

Lemma 6. *For any region created by RECURSIVEDIVIDE, the number of holes is at most twelve.*

Proof. By induction on the recursion depth ℓ ,

- (A) if $\ell \bmod 3 = 0$ then R has at most ten holes;
- (B) if $\ell \bmod 3 = 1$ then R has at most eleven holes;
- (C) if $\ell \bmod 3 = 2$ then R has at most twelve holes.

For $\ell = 0$, there are no holes. Assume (C) holds for ℓ . Since the cycle separator C encloses at most $3/4$ of the weight, it fully encloses at most nine holes. By Lemma 5, R_0 has at most ten holes. The symmetric argument applies to R_1 . Thus (A) holds for $\ell + 1$.

Similarly, by Lemma 5, if (A) holds for ℓ then (B) holds for $\ell + 1$, and if (B) holds for ℓ then (C) holds for $\ell + 1$. \square

3.2 Number of Vertices and Boundary Vertices

If the recursion depth mod 3 is 0, Line 7 of RECURSIVEDIVIDE selects a simple cycle in R' that is balanced with respect to the number of natural vertices. To achieve this, for each natural vertex v , the algorithm selects an adjacent face in R' , dedicated to carry v 's weight. The weight of each face is defined to be the number of vertices for which that face was selected. Since each face in R' is a triangle, every weight is an integer between 0 and 3. A cycle C is then chosen that is balanced with respect to these face-weights.

If the recursion depth mod 3 is 1, the cycle must be balanced with respect to the number of boundary vertices. For each boundary vertex, the algorithm selects an incident face; the algorithm then proceeds as above.

In either case, the total weight enclosed by the cycle C is an upper bound on the number of vertices (natural or boundary) strictly enclosed by C . Thus at most $3/4$ of the vertices (natural or boundary) of R' are strictly enclosed by C in R' . Similarly, at most $3/4$ of the vertices are not enclosed by C in R' .

The vertices of R_0 are the natural vertices of R' enclosed by C (including the natural vertices on C , which number at most $c\sqrt{|V(R)|}$), and the vertices of R_1 are the natural vertices of R' not strictly enclosed by C . Let $n := |V(R)|$ and, for $i \in \{0, 1\}$, let $n_i := |V(R_i)|$. We obtain

$$n_0 + n_1 \leq n + c\sqrt{n}. \quad (1)$$

Moreover, if the recursion depth mod 3 is 0, then

$$\max\{n_0, n_1\} \leq \frac{3}{4}n + c\sqrt{n}. \quad (2)$$

Similarly, let b be the number of boundary vertices of R , and, for $i \in \{0, 1\}$, let b_i be the number of boundary vertices of R_i . We obtain

$$b_0 + b_1 \leq b + c\sqrt{n}. \quad (3)$$

Moreover, if the recursion depth mod 3 is 1, then

$$\max\{b_0, b_1\} \leq \frac{3}{4}b + c\sqrt{n}. \quad (4)$$

3.3 Admitting an r -division

Let N be the number of vertices in the original input graph \bar{G} . Consider the decomposition tree \mathcal{T} of \bar{G} produced by RECURSIVEDIVIDE. Each node x corresponds to a region R_x . We define $n(x) := |V(R_x)|$. In this section we show that, for any given $r \geq s$, \mathcal{T} admits an r -division of \bar{G} . We adapt the two-phase analysis of Frederickson [Fre87]. In the first phase (Lemma 8), we identify a set of $O(N/r)$ regions for which the average number of boundary vertices is $O(\sqrt{r})$. However, some of the individual regions in this set might have too many boundary vertices (since the number of vertices and boundary vertices do not necessarily decrease at the same rate). We show that each such region can be replaced with smaller regions in \mathcal{T} so that every region has $O(\sqrt{r})$ boundary vertices, and the total number of regions remains $O(N/r)$ (Lemma 9).

For a node x of \mathcal{T} and a set S of descendants of x such that no node in S is an ancestor of any other, define $L(x, S) := -n(x) + \sum_{y \in S} n(y)$. Roughly speaking, $L(x, S)$ counts the number of new boundary nodes with multiplicities when replacing x by all regions in S .

Lemma 7. *If $y \in S_1$ then $L(x, S_1) + L(y, S_2) = L(x, S_1 \cup S_2 - \{y\})$.*

Proof. Each node $z \in S_2$ contributes $n(z)$ to both sides of the equation. Since y is in S_1 , it contributes $n(y)$ to $L(x, S_1)$ (the first term on the left). It contributes $-n(y)$ to $L(y, S_2)$ (the second term on the left), and nothing to the right-hand side. Every other node $y' \in S_1$ contributes $n(y')$ to both sides. \square

If the children of x are x_0 and x_1 , Equation (1) implies

$$L(x, \{x_0, x_1\}) \leq c\sqrt{n(x)}. \quad (5)$$

Fix r and let S_r be the set of nodes y of \mathcal{T} such that y 's region has no more than r vertices but the region of y 's parent has more than r vertices. Note that no node in S_r is an ancestor of any other. Let \hat{x} be the root of \mathcal{T} .

Lemma 8 (Total Number of Boundary Vertices). *There are constants s and γ , depending on c , such that, for any $r > s$, $L(\hat{x}, S_r) \leq \frac{\gamma N}{\sqrt{r}}$.*

Proof. For each node x of \mathcal{T} , let $\ell(x)$ denote the depth of x in \mathcal{T} . For a node x of \mathcal{T} , let $S_r(x)$ denote the set of descendants y such that y 's region has no more than r vertices but the region of y 's parent has more than r vertices. For an integer $n > r$, define $B_r(n)$ to be $\max\{L(x, S_r(x)) : x \in \mathcal{T}, n(x) \leq n, \ell(x) = 0 \bmod 3\}$.

The lemma is a consequence of the following recurrence, which is justified below.

$$B_r(n) \leq 7c\sqrt{n} + \max_{\{\alpha_i\}} \sum_i B_r(\alpha_i n), \quad (6)$$

where the max is over $\alpha_1, \dots, \alpha_8$ between 0 and $\frac{3}{4} + \frac{c}{\sqrt{n}}$ such that

$$\sum_{i=1}^8 \alpha_i \leq 1 + 7c/\sqrt{n} \quad (7)$$

and

$$\sum_{i=1}^8 \alpha_i \geq 1. \quad (8)$$

Let x be a node such that $r < |n(x)| \leq n$. Let y_1, \dots, y_k be the rootmost descendants y of x such that $|n(y)| \leq r$ or $\ell(y) \bmod 3 = 0$. Note that $k \leq 8$. Repeated application of Lemma 7 and Equation (5) yield

$$L(x, \{y_1, \dots, y_k\}) \leq (k-1)c\sqrt{n}, \quad (9)$$

which implies that

$$\sum_{i=1}^k n(y_i) \leq n + (k-1)c\sqrt{n}. \quad (10)$$

For $1 \leq i \leq k$, define α_i so that $\alpha_i n = n(y_i)$. For $k < i \leq 8$, define $\alpha_i = 0$. Equation (10) then implies Equation (7). Since each vertex of R_x occurs in at least one R_{y_i} , Equation (8) holds. Let x_0, x_1 be the children of x . For each $1 \leq i \leq k$, R_{y_i} is a subgraph of R_{x_0} or R_{x_1} , so $\alpha_i n \leq \frac{3}{4}n + c\sqrt{n}$, which implies that $\alpha_i \leq \frac{3}{4} + c/\sqrt{n}$.

By definition of $B_r(\cdot)$, $L(y_i, S_r(y_i)) \leq B_r(\alpha_i n)$, so we have

$$\sum_{i=1}^8 L(y_i, S_r(y_i)) \leq \sum_{i=1}^8 B_r(\alpha_i n). \quad (11)$$

By Lemma 7,

$$L(x, S_r(x)) \leq L(x, \{y_1, \dots, y_k\}) + \sum_{i=1}^k L(y_i, S_r(y_i)),$$

so Equation (11) and Equation (9) together justify the recurrence relation (6). By Lemma 2, this recurrence relation yields that $B_r(n)$ is $O(n/\sqrt{r})$. \square

Lemma 8 implies that $\sum_{x \in S_r} n(x)$ is $O(N)$, since the regions in S_r are disjoint *except* for boundary vertices, of which there are at most $O(N/\sqrt{r})$. For each parent y of a node $x \in S_r$, the corresponding region R_y has more than r vertices, so the number of such parents is $O(N/r)$, so $|S_r|$ is $O(N/r)$. Let c' be a constant to be determined. For a node x , let $S'_r(x)$ denote the set of rootmost descendants y of x (where x is a descendant of itself) such that R_y has at most $c'\sqrt{r}$ boundary vertices. Let $S'_r = \bigcup_{x \in S_r} \{S'_r(x)\}$.

Lemma 9. *The regions $\{R_y : y \in S'_r\}$ form an r -division with a constant number of holes per region.*

Proof. It follows from the definition of S'_r that each region in this set has at most r vertices and at most $c'\sqrt{r}$ boundary vertices. It follows by induction from Lemma 4 that each of these regions is connected. It follows from Lemma 6 that each region has at most twelve holes. It remains to show that $|S'_r|$ is $O(N/r)$.

For a node x of \mathcal{T} , let $b(x)$ denote the number of boundary vertices of R_x . Lemma 8 implies that

$$\sum_{x \in S_r} b(x) \leq \frac{N}{\sqrt{r}}. \quad (12)$$

We claim that, for every node $x \in S_r$,

$$|S'_r(x)| \leq \max\{1, \frac{b(x)}{c\sqrt{r}} - 12\} \quad (13)$$

Summing over $x \in S_r$ and using Equation (12) then proves that $|S'_r|$ is $O(N/r)$.

We set $c' = 40c$. Proof of Equation (13) is by induction. If $b(x) \leq c'\sqrt{r}$ then $S'_r(x) = \{x\}$, so the claim holds. Assume therefore that $b(x) > c'\sqrt{r}$. Let y_1, \dots, y_k be the rootmost descendants y of x such that $b(y) \leq c'\sqrt{r}$ or $\ell(y) - \ell(x) = 3$, ordered such that $|S'_r(y_1)| \geq |S'_r(y_2)| \geq \dots \geq |S'_r(y_k)|$. Let q be the cardinality of $\{i : |S'_r(y_i)| > 1\}$.

Case 0: $q = 0$. In this case, $|S'_r(x)| \leq 8$, and $\frac{b(x)}{c\sqrt{r}} - 12 > 40 - 12 \geq 8$.

Case 1: $q = 1$. For some ancestor y of y_1 that is a descendant of x , the separator chosen for R_y is balanced in terms of boundary vertices. It follows that $b(y_1) \leq \frac{3}{4}b(x) + 3c\sqrt{r}$. By the inductive hypothesis, $|S'_r(y_1)| \leq \frac{b(y_1)}{c\sqrt{r}} - 12$, so

$$\begin{aligned} |S'_r(x)| &= |S'_r(y_1)| + k - 1 \\ &\leq \frac{b(y_1)}{c\sqrt{r}} - 12 + 7 \\ &\leq \frac{\frac{3}{4}b(x) + 3c\sqrt{r}}{c\sqrt{r}} - 12 + 7 \\ &\leq \frac{b(x)}{c\sqrt{r}} - \frac{b(x)}{4c\sqrt{r}} + 3 - 12 + 7 \\ &\leq \frac{b(x)}{c\sqrt{r}} - 12, \end{aligned}$$

because $b(x) > c'\sqrt{r} = 40c\sqrt{r}$.

Case 2: $q = 2$. In this case, $b(y_1) + b(y_2) \leq b(x) + 6c\sqrt{r}$. Using the inductive hypothesis on y_1 and y_2 , we have

$$\begin{aligned} |S'_r(x)| &= |S'_r(y_1)| + |S'_r(y_2)| + k - 2 \\ &\leq \frac{b(y_1)}{c\sqrt{r}} - 12 + \frac{b(y_2)}{c\sqrt{r}} - 12 - 6 \\ &\leq \frac{b(x) + 6c\sqrt{r}}{c\sqrt{r}} - 12 - 12 - 6 \\ &\leq \frac{b(x)}{c\sqrt{r}} - 12. \end{aligned}$$

Case 3: $q > 2$. This case is similar to Case 2. □

4 A Simple-Cycle Separator Algorithm

In this section we present our cycle separator algorithm. As a one-shot algorithm, the input is a simple biconnected graph G with m edges and face weights such that no face weighs more than $3/4$ the total weight and no face consists of more than 3 edges. The algorithm outputs a simple cycle C in G , such that neither the total weight strictly enclosed by C nor the total weight not enclosed by C exceeds $3/4$ of the total weight. The length of C is guaranteed to be at most $4\sqrt{|E(G)|}$. Since G is a simple graph, this implies a bound of $4\sqrt{3|V(G)|}$ on the length of the cycle. Better constants can be achieved, at the cost of complicating the algorithm and the analysis, which we avoid for the sake of simplicity and ease of presentation.

In a similar manner, we aim for $3/4$ -balance to simplify the presentation. A balance of $2/3$ can be achieved. Note that handling face-weights can be used to handle vertex-weights and edge-weights as well; simply assign the weight of every vertex and every edge to an arbitrary incident face. This works as long as the resulting face weights satisfy the requirement that no single face weighs more than $3/4$ of the total weight.

The cycle separator algorithm consists of a preprocessing step, which runs in linear time and computes certain auxiliary data structures used by the main procedure, `SIMPLECYCLESEPARATOR`, which performs the computation. These data structures can be represented so that `SIMPLECYCLESEPARATOR` takes sublinear time. One auxiliary data structure is a tree \mathcal{K} , which is called the *component tree*. The tree \mathcal{K} captures the structural connectivity of dual BFS components of G . The dual BFS components satisfy a certain disjointness

property (see Lemma 11). The other auxiliary data structures are a spanning tree T of G and its cotree T^* . The spanning tree T satisfies a certain monotonicity property (see Lemma 12). The disjointness and monotonicity properties guarantee that the length of the cycle separator output by SIMPLECYCLESEPARATOR is $4\sqrt{|E(G)|}$.

Our algorithm for constructing the decomposition tree \mathcal{T} invokes SIMPLECYCLESEPARATOR multiple times, on various regions of \bar{G} . It computes the auxiliary data structures \mathcal{K} , T , and T^* once, for the input graph \bar{G} , and efficiently updates their representation when separating a region into two regions. However, the disjointness and monotonicity properties mentioned above are slightly weaker (see Invariant 1 and Invariant 2); they only apply to natural edges (disjointness) and to natural vertices (monotonicity). One implication is that the number of natural vertices on the cycle separator produced by SIMPLECYCLESEPARATOR is bounded in terms of the (squared root of the) number of natural vertices in the input graph. However, the cycle may also consist of some artificial vertices. As argued in Section 3, artificial nodes on C do not affect the analysis.

4.1 Levels and Level Components

We define levels with respect to an arbitrarily chosen face f_∞ , which we designate as the infinite face.

Definition 10. The level $\ell^F(f)$ of a face f is the minimum number of edges on a f_∞ -to- f -path in G^* . We use L_i^F to denote the faces having level i , and we use $L_{\geq i}^F$ to denote the set of faces f having level at least i .

Definition 11. For an integer $i \geq 0$, a connected component of the subgraph of G^* induced by $L_{\geq i}^F$ is called a level- i component, or, for unspecified i , a level component. We use $K_{\geq i}$ to denote the set of level- i components. A level- i component K is said to have level i , and we denote its level by $\ell^K(K)$. A non-root level component is a level component whose level is not zero. The set of vertices of G^* (faces of G) belonging to K is denoted $F(K)$.

Note that we use K (not K^*) to denote a level component even though it is a connected component of a subgraph of the planar dual. K should be thought of as a set of faces. Thus we can refer to it as a subgraph of G^* or of G . In the former case K is the subgraph of G^* induced by the faces in the set K . In the latter case K is the subgraph of G induced by the edges that belong to faces in the set K .

Lemma 10. For any non-root level component, the subgraph of G^* consisting of faces not in $F(K)$ is connected.

Corollary 1. For any non-root level component K , the edges of $\delta_{G^*}(F(K))$ form a simple cycle in the primal G .

In view of Corollary 1, for any non-root level component K , we use $X(K)$ to denote the simple cycle in the primal G consisting of the edges of $\delta_{G^*}(F(K))$. We refer to $X(K)$ as the *bounding cycle* of K since, when viewed as a subgraph of G , K is exactly the subgraph enclosed by $X(K)$.

Lemma 11. Let K and K' be two distinct components. $X(K)$ and $X(K')$ are edge-disjoint.

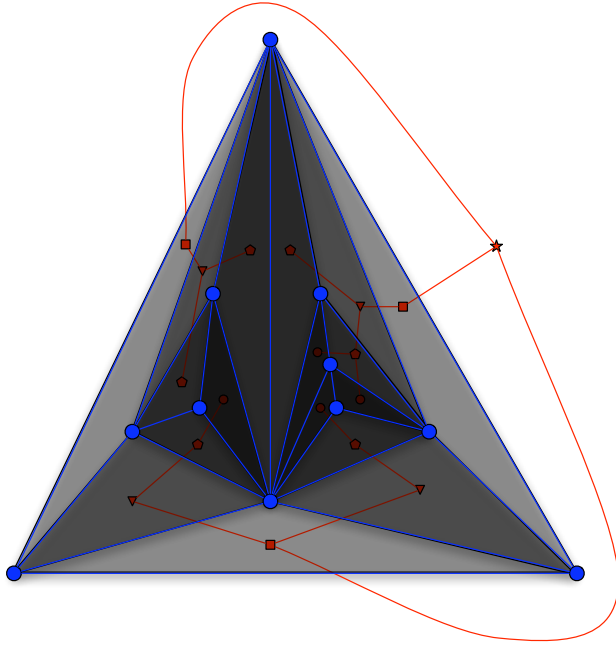
Proof. Let i be the level of K . The edges of $X(K)$ are edges of $\delta_{G^*}(F(K))$. Therefore, as edges of G^* , they have one endpoint in K and one in a level- $(i-1)$ face. If an edge of $\delta_{G^*}(F(K'))$ has an endpoint in K then $K \neq K'$ implies the level of K' is at least $i+1$, so it cannot be an edge of $X(K)$. If, on the other hand, an edge of $\delta_{G^*}(F(K'))$ has an endpoint in level $i-1$ then $K \neq K'$ implies the other endpoint is not in K . \square

The following definition is illustrated in Figure 2.

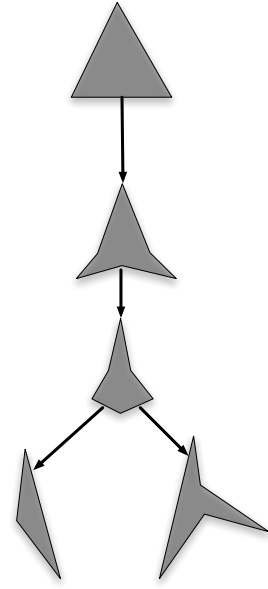
Definition 12. The component tree \mathcal{K} is the rooted tree whose nodes are the level components and in which K is an ancestor of K' if the faces of K include the faces of K' .

The root of the component tree is the unique level-0 component consisting of all of G^* .

Definition 13. An edge ff' of G^* has level i if f has level $(i-1)$ and f' has level i . We write $\ell^E(ff')$ for the level of ff' . We use L_i^E to denote the set of edges of level i .



(a) A triangulated graph (blue vertices and edges) along with a dual BFS tree (in red, faces at different BFS levels are indicated by different shapes). The (primal) components $\mathcal{K}_{\geq k}$ are indicated as shaded subgraphs. The deeper the level of a component, the darker its shade is.



(b) The corresponding component tree \mathcal{K} . Each node of the component tree corresponds to a connected component of G .

Figure 2: Illustration of the component tree \mathcal{K} .

Note that not every edge of G^* is assigned a level.

Definition 14. Let L_i^V denote the set of vertices of the primal graph G that are endpoints in the primal graph G of edges in L_i^E .

Note that a vertex of G can be an endpoint of two edges at different levels i and j , so L_i^V and L_j^V are not necessarily disjoint.

Definition 15. The level $\ell^V(v)$ of a primal vertex v is defined to be $\min_f \ell^F(f)$ over all faces f incident to v .

Note that L_i^V is *not* the set of vertices with level i .

4.2 The Primal Tree

The algorithm maintains a primal spanning tree T . We start by describing the initial value of T and its properties.

Lemma 12. *There exists a spanning tree \bar{T} such that, for any vertex u , $\ell^V(\text{parent}_{\bar{T}}(u)) < \ell^V(u)$. \bar{T} can be computed in linear time.*

Proof. For a primal vertex u with $\ell^V(u) = i$, let f be a level- i face to which u is incident (ties are broken arbitrarily, but consistently). Let v, w be the other two vertices of that face f . The parent of u in \bar{T} , denoted by $\text{parent}_{\bar{T}}(u)$, is the vertex in $\{v, w\}$ with the smaller level (again, breaking ties arbitrarily and consistently).

Since $\ell^V(u) = i$, u is not incident to a level- $(i-1)$ face. Hence v, w must be adjacent to a level- $(i-1)$ face f' . Therefore, $\ell^V(\text{parent}_{\bar{T}}(u)) \leq i-1 < \ell^V(u) = i$.

To complete the definition of \bar{T} , we choose an arbitrary vertex r incident to f_∞ to be the root of \bar{T} by assigning it to be the parent of the two remaining vertices at level $i = 0$. For convenience we set the level of the root vertex to -1 . \square

4.3 The Preprocessing Step

To compute a simple cycle separator one has to first compute the component tree \mathcal{K} , the spanning tree T , and its cotree T^* . This is done by the preprocessing step (Algorithm 2), which runs in linear time.

Algorithm 2: PREPROCESS(G)

- 1 choose an arbitrary face as f_∞
 - 2 compute face, edge, and vertex levels $\ell^F(\cdot)$, $\ell^E(\cdot)$, and $\ell^V(\cdot)$, respectively
 - 3 compute the component tree \mathcal{K}
 - 4 initialize T to be the tree \bar{T} as defined in Lemma 12
 - 5 initialize T^* to be the cotree of T
 - 6 **return** (\mathcal{K}, T, T^*)
-

The efficient implementation of RECURSIVEDIVIDE, which computes the decomposition tree \mathcal{T} in linear time, recursively separates regions of \bar{G} . It maintains the component tree \mathcal{K} , spanning tree T , and cotree T^* of the currently handled region R throughout the recursive calls. This is described in detail in Section 5. T is initialized to be the tree \bar{T} of \bar{G} . RECURSIVEDIVIDE maintains the following invariants:

Invariant 1. *Let K and K' be two distinct components. $X(K)$ and $X(K')$ do not share natural edges.*

Invariant 2. *$\ell^V(v) < \ell^V(u)$ for any two natural vertices u and v of R such that v is an ancestor of u in T .*

Since \bar{G} has only natural vertices and edges, Lemma 11 and Lemma 12 show that the invariants initially hold for $R = \bar{G}$.

4.4 Computing a Simple Cycle Separator

We first provide an intuitive description of the separator algorithm `SIMPLECYCLESEPARATOR` (Algorithm 3). Let m denote the number of natural edges in G .

1. The algorithm computes a balanced fundamental cycle \tilde{C} (Lines 3–4). If \tilde{C} consists of fewer than $4\sqrt{m}$ natural edges, then \tilde{C} is a short balanced simple-cycle separator.
2. Otherwise, the fact that \tilde{C} is long implies that it intersects components at many (more than $2\sqrt{m}$) consecutive levels of the component tree \mathcal{K} (we say that \tilde{C} intersects a component K if \tilde{C} has at least one edge in $K - X(K)$ and at least one in $G - K$). The algorithm performs a binary search procedure on the range $[l, h)$, where l, h are the minimum and maximum level $\ell^V(v)$ of a vertex $v \in \tilde{C}$, respectively. At each step of the binary search, the algorithm identifies
 - the median level $i_0 = \lfloor (l + h)/2 \rfloor$,
 - the highest-level component K_- intersected by \tilde{C} , whose level i_- is smaller than i_0 and whose bounding cycle $X(K_-)$ has few (at most \sqrt{m}) natural edges, and
 - the smallest-level component K_+ intersected by \tilde{C} , whose level i_+ is at least i_0 and whose bounding cycle $X(K_+)$ has few natural edges.

The monotonicity of the primal tree T implies that the number of natural edges of \tilde{C} between levels i_- and i_+ (that is, the number of edges of \tilde{C} in $K_- - K_+$) is at most $2\sqrt{m}$ (see Lemma 15).

3. If $W(G - K_-) > 3W/4$ we continue the binary search on the range $[l, i_-)$. Similarly, if $W(K_+) > 3W/4$ we continue the binary search on the range $[i_+, h)$.
4. Otherwise, the graph G' induced by the edges of $X(K_-)$, $X(K_+)$ and the edges of \tilde{C} in $K_- - K_+$ is a biconnected planar graph with at most $4\sqrt{m}$ natural edges, none of whose faces weighs more than $3W/4$. See Figure 3. The algorithm uses a simple greedy procedure, `GREEDYCYCLESEPARATOR`, to output a balanced simple cycle separator of G' in $O(\sqrt{m})$ time.

Algorithm 3: `SIMPLECYCLESEPARATOR`(\mathcal{G}), \mathcal{G} is the tuple (G, \mathcal{K}, T, T^*)

```

1 let  $m$  be the number of natural edges in  $G$ 
2 let  $W := W(G)$ 
3 compute  $e^*$  to be the  $3/4$ -balanced edge separator of  $T^*$ ; let  $e$  be the primal of  $e^*$ 
4 let  $\tilde{C}$  denote the fundamental cycle defined by  $e \notin T$  and  $T$ 
5 if  $\tilde{C}$  has at most  $4\sqrt{m}$  natural edges then return  $\tilde{C}$  /*  $\tilde{C}$  short enough */
6 let  $l, h$  be the minimum and maximum level  $\ell^V(v)$  of a vertex  $v \in \tilde{C}$ 
7 while  $l < h$  do /* binary search for  $i_0$  in range  $[l, h)$  */
8   set all  $i_0, i_-, i_+$  to  $l + \lfloor (h - l)/2 \rfloor$ 
9   repeat /* sequential search for level  $i_- < i_0$  with small boundary */
10    |  $i_- := i_- - 1$ ; let  $K_-$  be the unique component at level  $i_-$  that  $\tilde{C}$  intersects
11    until  $i_- < l$  or  $X(K_-)$  has at most  $\sqrt{m}$  natural edges
12    if  $W(G \setminus K_-) > 3W/4$  then  $h := i_-$ ; continue
13    repeat /* sequential search for level  $i_+ > i_0$  with small boundary */
14    |  $i_+ := i_+ + 1$ ; let  $K_+$  be the unique component of level  $i_+$  that  $\tilde{C}$  intersects
15    until  $i_+ > h$  or  $X(K_+)$  has at most  $\sqrt{m}$  natural edges
16    if  $W(K_+) > 3W/4$  then  $l := i_+$ ; continue
17    let  $G'$  be the graph induced on  $G$  by  $X(K_-) \cup X(K_+) \cup (\tilde{C} \cap (K_- - K_+))$ 
18    return GREEDYCYCLESEPARATOR( $G'$ )
19 end
```

Note: To avoid clutter, the pseudocode of `SIMPLECYCLESEPARATOR` does not handle the boundary case where the loop in line 10 terminates without finding a level K_- with $|X(K_-)| < \sqrt{m}$. In this case there is no need to shortcut \tilde{C} at a small level. Specifically, \tilde{C} is entirely enclosed by K_- , the condition in line 12 is considered false, and $X(K_-)$ is considered to be an empty set of edges. A similar statement applies to K_+ .

Lemma 13. *The components K_- and K_+ defined in Lines 10 and 14, respectively, are well defined.*

Proof. Since $l \leq i_- \leq h$, the cycle \tilde{C} must intersect some component at level i_- . The monotonicity of the tree T implies that once a rootward path leaves a component at some level it never enters any component at that level again. The lemma follows since \tilde{C} is comprised of two rootward paths plus one edge. \square

Lemma 14. *For $l \leq i \leq h$, let K_i be the unique component at level i that is intersected by \tilde{C} . If the weight not enclosed by K_- is greater than $3W/4$ then there exists a level $l \leq i_0 < i_-$ such that the weight not enclosed by any K_i with $l \leq i \leq i_0$ is at most $3W/4$, and such that the weight enclosed by any K_i with $i_0 < i \leq h$ is at most $3W/4$.*

Proof. Let K_l be the unique level- l component that encloses \tilde{C} . Since \tilde{C} is a balanced separator, the weight enclosed by K_l is at least $W/4$. Hence the weight not enclosed by K_l is at most $3W/4$. Let \tilde{K} be the component with maximum level that is intersected by \tilde{C} and whose enclosed weight is at least $W/4$. Let i_0 be the level of \tilde{K} . By the above argument $l \leq i_0$, and since the weight not enclosed by K_- is greater than $3W/4$, $i_0 < i_-$. By choice of i_0 , any K_i with $i_0 < i \leq h$ encloses at most $W/4$ weight. \square

A symmetric lemma applies to the case where the weight enclosed by K_+ is greater than $3W/4$. These lemmas show that, if l or h are updated (i.e., the binary search continues), there exists some level i_0 in the new search range for which both conditions in Lines 12 and 16 are false, and hence the binary search procedure must eventually terminate.

Lemma 15. *Let G be a biconnected plane graph G with m natural edges and face weights such that no face weighs more than $3/4$ the total weight and such that no face consists of more than 3 edges. The procedure `SIMPLECYCLESEPARATOR` finds a $3/4$ -balanced simple cycle separator in G with at most $4\sqrt{m}$ natural vertices.*

Proof. Since G has face size at most 3, the fundamental cycle \tilde{C} is a $3/4$ -balanced simple cycle separator. If it consists of fewer than $4\sqrt{m}$ natural edges it is returned in Line 4. Otherwise, the lemma follows from the correctness of `GREEDYCYCLESEPARATOR` (Lemma 16), provided that we show that G' is a biconnected subgraph of G with $4\sqrt{m}$ natural edges, none of whose faces weighs more than $3W/4$. We assume that both K_- and K_+ exist. The cases where one of them or both do not exist are similar. Consider the cycles $X(K_-)$, $X(K_+)$, and \tilde{C} . Since \tilde{C} intersects both K_- and K_+ , G' is biconnected. See Figure 3 for an illustration.

To establish the bound on the number of edges in G' , note that, by choice of $X(K_-)$ and $X(K_+)$, they consist of fewer than \sqrt{m} natural edges each. It remains to bound $|\tilde{C} \cap (K_- - K_+)|$, which consists of two paths in T between $X(K_+)$ and $X(K_-)$. We claim that $i_+ - i_- \leq \sqrt{m}$. To see this, observe that by definition of i_- and i_+ , for every level $i_- < i < i_+$, the bounding cycle of the unique level- i component intersected by \tilde{C} consists of more than \sqrt{m} natural edges. Since bounding cycles do not share natural edges (Invariant 1) and since there are m natural edges in G , it must be that $i_+ - i_- + 1 \leq \sqrt{m}$. Consider node u on $X(K_+)$ and let u_0, u_1, \dots, u_k be the natural vertices on the u -to-root path in T (i.e., $u_0 = u$). Let v be the first vertex that belongs to $X(K_-)$ on this path. Since u is incident to a level i_+ face, $\ell^V(u) \leq i_+$. By invariant 2, for any i , $\ell^V(u_i) \leq \ell^V(u) - i = i_+ - i$. Hence, $\ell^V(u_{\sqrt{m}}) \leq i_+ - \sqrt{m} \leq i_-$. This implies that $u_{\sqrt{m}}$ belongs to a level cycle at level at most i_- . It follows that the number of natural vertices in the path in T from u to v is at most \sqrt{m} . We therefore conclude that $|\tilde{C} \cap (K_- - K_+)|$ consists of at most $2\sqrt{m}$ natural vertices.

As for the face weights, the weight of a face f' of G' is the total weight of the faces of G that are enclosed in G by the cycle formed by the darts of f' . The faces of G' that correspond to the exterior of $X(K_-)$ and to the interior of $X(K_+)$ have weight at most $3W/4$ by the conditions in Lines 12 and 16. All of the other faces of G' are either enclosed (in G) by \tilde{C} or not enclosed by \tilde{C} . Therefore, the weight of each of these faces is at most $3W/4$ since \tilde{C} is a balanced separator. \square

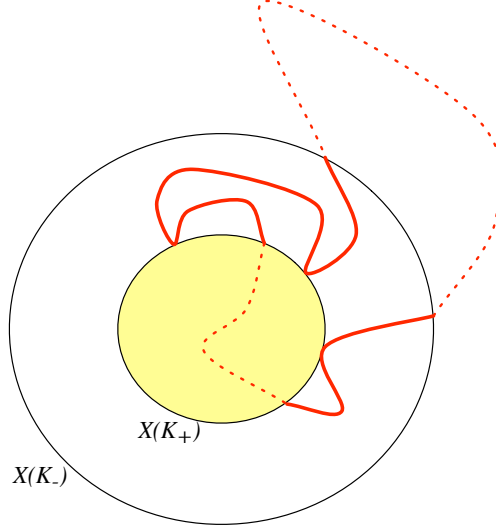


Figure 3: Illustration of the cycle separator algorithm: the fundamental cycle \tilde{C} is shown in red (solid and dashed). The level cycles $X(K_-)$ and $X(K_+)$ are indicated. The component K_+ is shaded yellow. G' is the graph consisting of the edges of \tilde{C} that belong to $K_- - K_+$ (solid red) as well as the edges of the cycles $X(K_-)$ and $X(K_+)$ (black).

Since any simple plane graph has at most three times as many edges as vertices, and since triangulating and making biconnected any plane graph G requires $O(|V(G)|)$ edges, we can restate Lemma 15 in a more general form that does not distinguish natural and artificial vertices.

Corollary 2. *Let G be a simple biconnected plane graph with face weights such that no face weighs more than $3/4$ the total weight, and such that no face consists of more than 3 edges. There exists a constant c such that the procedure SIMPLECYCLESEPARATOR finds a $3/4$ -balanced simple cycle separator in G whose length is at most $c\sqrt{|V(G)|}$.*

We next discuss the procedure GREEDYCYCLESEPARATOR that finds a simple cycle in a biconnected graph with no face weight exceeding $3W/4$.

Lemma 16. *Let G be a biconnected planar graph with m edges and face weights summing to W such that no face weighs more than $3W/4$. There exists an $O(m)$ algorithm that finds a balanced simple-cycle separator in G .*

Proof. The proof is constructive. Since G is biconnected, every face is a simple cycle. If $W(f) > W/4$ for any face f , then f is a balanced simple cycle separator. Otherwise, use the following greedy algorithm. Initialize the set S with a single arbitrary face of G . Repeatedly add a face f to S , maintaining the invariant that the edges of $\delta_{G^*}(S)$ (the boundary of S) form a simple cycle, stopping when $W(S)$ first exceeds $W/4$. At this point, the edges of $\delta_{G^*}(S)$ form a simple-cycle separator. It remains to show that, in each iteration, there exists a face to add to S while maintaining the invariant, and to show how such a face can be quickly found.

We claim that, if the boundary of S is a simple cycle, there exists a face $f \notin S$ such that the boundary of $S \cup \{f\}$ is a simple cycle. We prove the claim using an inductive argument (illustrated in Figure 4). Let e be an edge of $\delta(S)$. Let f' be the face not in S to which e belongs. Let S' be $S \cup f'$. If S' has a simple boundary, we are done. Otherwise, $\delta(S')$ consists of a set \mathcal{C} of at least two simple cycles. Consider the graph G' obtained from G by deleting the non-boundary edges of S' . Each cycle in \mathcal{C} consists of at least one edge of $\delta(S)$ and one edge of f' , and bounds a connected subgraph of G' . Let H be one such connected subgraph. Since G is biconnected, any two vertices in H are connected by two vertex-disjoint paths in G . These paths can be transformed into vertex-disjoint paths in H by rerouting along the boundary of H . Such rerouting is possible since the boundary of H is a simple cycle. Hence H is biconnected. Since H contains an edge of S , $H \cup S$ is biconnected as well. Now repeat the argument with G replaced by $S \cup H$. Since $S \cup H$ is strictly

smaller than G (it does not include at least one edge — the one in $f' \cap C'$ for some cycle $C' \in \mathcal{C}$ that is not the one bounding H), the inductive argument proves the claim.

We describe how, in each iteration, the greedy algorithm finds a face f such that the boundary of $S \cup \{f\}$ is a simple cycle. The algorithm maintains, for each face f , the number $\gamma(f)$ of consecutive maximal subpaths of $\delta(S)$ on f . (A subpath may consist of a single vertex of $\delta(S)$.) Note that $S \cup f$ has simple boundary iff $\gamma(f) = 1$. The algorithm also maintains a list of the faces f not enclosed by S that have $\gamma(f) = 1$ and at least one edge shared with S . When a face f is added to the set S , the algorithm updates $\gamma(\cdot)$ as follows. Let S' denote $S \cup f$. Since G is biconnected, f is a simple cycle. Let P be the subpath of f that consists of the boundary vertices and edges of S' that do not belong to S . Let f_1, f_2, \dots be the faces other than f incident to edges or vertices of P in clockwise order along f . Note that the f_i 's are not necessarily distinct (this is illustrated in Figure 5). Let P_i be the maximal subpath (possibly a single vertex) of P along which f_i is incident to P . For each subpath P_i , let e_i^- and e_i^+ be the edges preceding and following P_i in $\delta(S')$, respectively. If both e_i^- and e_i^+ belong to f_i , decrease $\gamma(f_i)$ by one. If neither e_i^- nor e_i^+ belongs to f_i , increase $\gamma(f_i)$ by one. Otherwise, $\gamma(f_i)$ remains unchanged. Since the work done when adding a face is proportional to the sum of degrees of the vertices of P , and since each edge or vertex belongs to P only once, when it first enters S , the total running time of this greedy algorithm is $O(m)$. \square

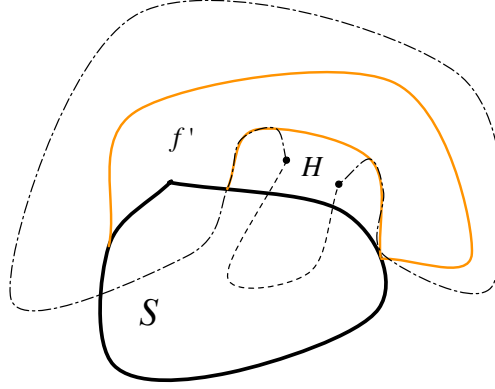


Figure 4: Illustration of the proof of the auxiliary claim in Lemma 16. The boundary of the set S is shown in solid thick black. The face f' is shown in solid orange. H is a subgraph bounded by one of the simple cycles that comprise the boundary of $S' = S \cup f'$. A pair of vertices in H is indicated, along with two vertex-disjoint paths between them in the entire graph G . These paths can be transformed into disjoint paths in H by rerouting each of them along part of the simple cycle bounding H . The subgraph $S \cup H$ is biconnected and strictly smaller than G .

4.5 Efficient Implementation

It is fairly straightforward to implement `SIMPLECYCLESEPARATOR` in linear time. In this subsection we describe the data structures that support implementing `SIMPLECYCLESEPARATOR` in sublinear time. Our preprocessing procedure initializes the tree T to be the tree \bar{T} (Lemma 12), the cotree T^* to be the spanning tree of \bar{G}^* , rooted at f_∞ whose edges are those not in T . T is represented using an Euler tour tree [HK99]. T^* is represented using a dynamic tree. It also computes the component tree \mathcal{K} and all level cycles in G . The component tree can be represented by a parent list. The level cycles are represented by splay trees [ST85].

For every edge e in \bar{G} , let $P^{\mathcal{K}}(e)$ be the path in \mathcal{K} that consists of all components K such that e has one endpoint in $X(K)$ and another endpoint strictly enclosed by $X(K)$. We maintain these paths by storing the first and last components of $P^{\mathcal{K}}(e)$ in an array entry $R(e)$. Note that R can be generated in linear time in the preprocessing step as part of the computation of the component tree.

To implement the algorithm efficiently we need to support the following operations:

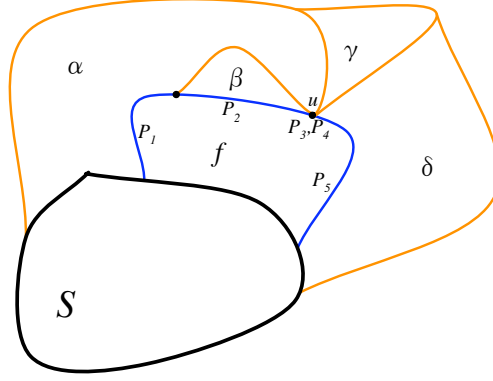


Figure 5: Illustration of the algorithm in the proof of Lemma 16. The boundary of the set S is shown in solid thick black. The path P is solid blue. The list of faces f_i is $f_1 = \alpha, f_2 = \beta, f_3 = \alpha, f_4 = \gamma, f_5 = \delta$. Note that the subpaths P_3 and P_4 both consist of the single vertex u . Before f is added to S , $\gamma(\alpha) = 1$. Since e_1^- , the edge of $\delta(S')$ preceding P_1 , belongs to α , and e_1^+ , the edge of $\delta(S')$ following P_1 , does not belong to α , P_1 does not contribute anything to $\gamma(\alpha)$. Since both e_3^- and e_3^+ do not belong to α , P_3 contributes $+1$ to $\gamma(\alpha)$. Hence, after f is added to S , $\gamma(\alpha) = 2$.

- Line 1 - Find the number of natural edges in G . This can be done by keeping track of the number $m_{\text{artificial}}$ of artificial triangulation edges (Lemma 17 in Section 5 shows that this takes total $O(n)$ time over all calls to `SIMPLECYCLESEPARATOR`, so we may assume $m_{\text{artificial}}$ is given). Let F be the number of faces in G . F can be found in $O(\log |E(G)|)$ amortized time by a single dynamic tree operation on T^* . Since G is triangulated, the number of natural edges is given by $3F/2 - m_{\text{artificial}}$.
- Line 2 - Find the total weight of faces in G . This is done using a single dynamic tree operation that returns the total weight of nodes (faces) in T^* .
- Line 3 - Find a balanced edge separator in T^* . This is done using a dynamic tree operation that returns a leafmost edge whose subtree has at least a certain total amount of weight ($W/4$ in our case). This operation takes $O(\log |E(G)|)$ amortized time.
- Line 4 - Find out if a path in T consists of at most a given number k of natural edges (in this case $k = 4\sqrt{|E(G)|}$). This can be done by traversing the path, and takes $O(k \log |E(G)|)$ time (the number of artificial edges on a path is bounded by the number of holes of G , which, by Lemma 6, is constant).
- Line 6 - Find least common ancestor in T . This can be done using the Euler tour representation in $O(\log |E(G)|)$ amortized time.
- Line 10, 14 - Find leafmost ancestor in T with level at most i_- . This can be done using the Euler tour representation in $O(\log |E(G)|)$ amortized time.
- Find greatest-level component to which a particular edge belongs. This information is stored in the array R .
- Line 11, 15 - Query the number of edges in a level cycle. This takes $O(\log |E(G)|)$ time since level cycles are stored as splay trees.
- Line 12, 16, 17 - Query the weight enclosed by a short simple cycle C . This can be done by summing up the weight of subtrees as follows. For each cycle edge e , if $e \in T^*$ and the rootward endpoint of e in G^* is not enclosed by C , add the total weight of the subtree of e in T^* to the sum. If $e \in T^*$ and the rootward endpoint of e in G^* is enclosed by C , subtract the total weight of the subtree of e in T^* from the sum. Since querying the total weight of a subtree can be done using a dynamic tree operation in $O(\log |E(G)|)$ amortized time, and since the length of C is $O(\sqrt{|E(G)|})$, computing the total weight enclosed by C can be done in $O(\sqrt{|E(G)|} \log |E(G)|)$ time.

This shows that, assuming that the auxiliary data structures are given in the desired representation, `SIMPLECYCLESEPARATOR` can be implemented in $O(\sqrt{|E(G)|} \log^2 |E(G)|)$ time.

5 Maintaining the Representation of Regions Efficiently

In this section we describe how to maintain the auxiliary data structures that represent the regions and are required by `SIMPLECYCLESEPARATOR` throughout the recursive calls to `RECURSIVEDIVIDE`. These include maintaining the embedding of regions, the primal spanning tree T and its monotonicity invariant (Invariant 2), the cotree T^* , the component tree, and the level cycles.

Consider an iteration of `RECURSIVEDIVIDE` in which a region R is partitioned into two regions R_0 and R_1 along a cycle separator C . Note that, due to the recursive nature of the algorithm, we may assume that first the representations for R_0 are obtained and the algorithm is invoked recursively on R_0 . After the recursive call for R_0 is completed, all changes are undone, and the representations for R_1 are obtained. To achieve linear time for computing an r -division, it is required that the update be done in sublinear time in the size of R . Essentially, the representation can be updated by performing only local operations on the vertices and edges of the separator C . This results in sublinear running time since C consists of $O(\sqrt{|R|})$ edges, and since each operation in the data structures we use takes $O(\log |R|)$ amortized time. Lemma 2 with $r = s$ shows that if the cycle can be computed in $O(|R|^\beta)$ time (for some $\beta < 1$), as established in Section 4.5, and if the representation can be updated in $O(|R|^\beta)$ time, then the total running time of the algorithm is linear. This establishes the running time claimed in Theorem 2.

5.1 Maintaining the Embedding

The embedding is represented by using a splay tree [ST85] for each vertex v of R to store the edges incident to v in cyclic order. Conceptually, R_0 is obtained from R by merging all the faces enclosed by C into a single face. Merging the faces enclosed by C corresponds to deleting all edges of R strictly enclosed by C . Similarly, R_1 is obtained from R by deleting all edges not enclosed by R . We do not perform these deletions explicitly since that would not be efficient.

We describe the procedure for R_0 . The procedure for R_1 is similar. We start by “cutting off” the edges that do not belong to R_0 . To do this we reconstruct the cyclic order of edges around the vertices of the cycle C . Consider a vertex v of C . Let π_v be the cyclic permutation of edges around v in R . There are 2 edges of C that are incident to v . These cycle edges partition π_v into 2 intervals. The edges in one interval belong to R_0 and those in the other interval do not. Using $O(1)$ cut operations, each taking $O(\log |R|)$ amortized time, we cut off the edges that do not belong to R_0 . Since the length of C is $O(\sqrt{|R|})$ (it consists of $O(\sqrt{|R|})$ natural nodes, and at most 12 artificial nodes), the total time to update all vertices of C is $O(\sqrt{|R|} \log |R|)$. Note that there is no need to explicitly delete the vertices enclosed by C from the representation because they are no longer pointed to from any vertex in R_0 .

Observe that the definition of R_0 in Line 10 of `RECURSIVEDIVIDE` and the definition of R'_0 in Line 6 of the recursive call of `RECURSIVEDIVIDE` on R_0 are equivalent to deleting all the artificial triangulation edges incident to C , and then triangulating the new artificial face of R_0 with a new artificial node and new artificial edges. This can be done using a constant number of link and cut operations per node of C , each requiring $O(\log |R|)$ amortized time. The number of triangulation edges is bounded by the number of boundary vertices of R_0 . Note, however, that even though our algorithm separates according to the number of boundary vertices every third recursive call, there is no guarantee on the ratio between the number of vertices in R_0 and the number of boundary vertices of R_0 . Instead, we bound the total time required for creating all of the triangulation edges throughout the entire execution of the recursive algorithm. Recall that for a node x of \mathcal{T} , $b(x)$ denotes the number of boundary nodes of R_x .

Lemma 17. $\sum_{x \in \mathcal{T}} b(x) \log |R_x| = O(|R_{\hat{x}}|)$, where \hat{x} is the root of \mathcal{T} (i.e., $R_{\hat{x}}$ is the input graph \bar{G}).

Proof. Consider a node x of \mathcal{T} . Let C_x denote the cycle used to separate R_x . Observe that a vertex v on C_x appears as a boundary vertex in the regions of exactly two descendants of x in each level $\ell > \ell(x)$ of \mathcal{T} , unless v appears on another cycle separator C_y for some descendant y of x . Since the depth of the subtree

of \mathcal{T} rooted at x is bounded by $c_1 \log |R_x|$ for some constant c_1 (the size of regions decreases by a constant factor every 3 levels) we have:

$$\sum_{x \in \mathcal{T}} b(x) \leq \sum_{x \in \mathcal{T}} 2c_1 |C_x| \log |R_x| \quad (14)$$

Note that if v does appear on C_y for some descendant y of x , its additional occurrences as a boundary vertex are accounted for by the contribution of C_y to the sum.

Let $T(n)$ be the maximum of $\sum_{x \in \mathcal{T}} 2c_1 |C_x| \log |R_x|$ over all n -vertex triangulated biconnected planar input graphs. Let $k, y_1, \dots, y_k, \alpha_1, \dots, \alpha_8$ be as in Lemma 8. Let $\mathcal{T}_{x, y_1, \dots, y_k}$ denote the subtree of \mathcal{T} induced by nodes that are descendants of x but not descendants of any y_i . Note that the number of internal nodes in $\mathcal{T}_{x, y_1, \dots, y_k}$ is $k - 1$, which is at most 7, and that for any internal node z of $\mathcal{T}_{x, y_1, \dots, y_k}$, $|R_z| \leq |R_x|$, and $|C_z| \leq |C_x| \leq c\sqrt{|R_x|}$. Hence,

$$T(n) \leq 2c_1 \cdot 7c\sqrt{n} \log^2 n + \max_{\{\alpha_i\}} \sum T(\alpha_i n) \quad (15)$$

Lemma 2 shows that $T(n)$ is $O(n)$. □

5.2 Representation of the Spanning Tree T and its Cotree T^*

Let T and T^* denote the primal and dual spanning trees of R , respectively, at the time the cycle separator C of R is found. In what follows we describe how to convert T into the spanning tree T_0 of R_0 . The procedure for converting T into the spanning tree T_1 of R_1 is similar.

Let h be the new artificial face of R_0 . The tree T is cut at each vertex v of h so that the only edges adjacent to v in T are those that belong to R_0 (i.e., not strictly enclosed by h). This breaks T into a forest F in $O(\sqrt{|R|} \log |R|)$ time. Let F_0 be the set of trees in F that consist of edges of R_0 . The cotree T^* is cut at the edges of h . This disconnects T^* into a forest F^* in $O(\sqrt{|R|} \log |R|)$ time. Let F_0^* be the set of trees in F^* that consist of duals of edges of R_0 .

The algorithm connects the forest F_0 into a spanning tree T_0 of R_0 while maintaining the monotonicity invariant Invariant 2).

Lemma 18. *Let D be a cycle of natural vertices in R . Let v be a vertex of D such that the parent of v in T is enclosed by D , where enclosure is defined with respect to the root of T . There exists a vertex $u \in D$ with $\ell^V(u) < \ell^V(v)$.*

Proof. Since the root of T is not enclosed by D , there must be a vertex $u \in D$ on the path in T from v to the root of T . By Invariant 2, $\ell^V(u) < \ell^V(v)$. □

Recall that, after deleting all the triangulation edges incident to h , h is retriangulated with a new artificial vertex x_h and new artificial edges vx_h for each vertex v of h . Let v_h be a vertex which minimizes $\ell^V(v)$ among all vertices of h . The algorithm attaches x_h as a child of v_h in T . It then considers every vertex v of h . If v does not have a parent in F_0 (that is, the parent of v in T was enclosed by h in R , so after cutting T , v is a root of a tree in the forest F_0) then the algorithm sets x_h to be the parent of v in F_0 by adding the artificial triangulation edge vx_h to F_0 . This operation is performed by linking the tree of F_0 whose root is v to x_h . If v does have a parent in F_0 we add vx_h to F_0^* . At the end of the process, each edge of R_0 is either in F_0 or in F_0^* , and F_0 contains no cycles. Hence F_0 and F_0^* are spanning trees of R_0 and its dual, respectively.

Lemma 19. *After the update, the tree F_0 satisfies Invariant 2.*

Proof. The only natural nodes whose parent is set by the update procedure are those on the new hole h of R_0 that do not have a parent after F was cut at the vertices of h . By Lemma 18, for each such vertex v , $\ell^V(v_h) < \ell^V(v)$. Since the algorithm sets x_h as the parent of v , and v_h as the parent of x_h , any natural vertex of R_0 that is a proper ancestor of v is an ancestor of v_h . Invariant 2 is maintained since the ancestors of v_h are not changed by the update procedure.

The number of operations required to obtain the updated representation of the tree and cotree for R_0 is proportional to the number of nodes of h . By Lemma 17, the total update cost over all recursive calls to RECURSIVEDIVIDE is $O(|\tilde{G}|)$. Obtaining the representation of the tree and cotree for R_1 is similar. □

5.3 Representation of Component Boundaries

The component tree \mathcal{K} is computed at the preprocessing step and represented by parent pointers. We do not update the component tree when RECURSIVEDIVIDE partitions the graph. We only maintain and update the level-cycles (the boundaries of components). Note that at any given recursive call \mathcal{K} correctly represents the enclosure relations between components in the current subgraph. However, \mathcal{K} also contains components that no longer belong to the current subgraph. This does not pose a problem since we only use \mathcal{K} for parent queries.

For every edge e in G , let $P^{\mathcal{K}}(e)$ be the path in \mathcal{K} that consists of all components K such that e has one endpoint in $X(K)$ and another endpoint strictly enclosed by $X(K)$. The algorithm maintains these paths by storing the first and last components of $P^{\mathcal{K}}(e)$ in an array entry $\mathbf{R}(e)$. Note that \mathbf{R} is generated in linear time at the preprocessing step and is used when working with subgraphs at different recursive levels. $\mathbf{R}(e)$ need not be updated for natural edges. However, whenever a new triangulation edge e is introduced, the algorithm must store the first and last components of $P^{\mathcal{K}}(e)$ in $\mathbf{R}(e)$. The way this is done is described at the end of this subsection.

We now describe how we update the representation of the level cycles. By construction, the cycle separator C intersects at most \sqrt{m} level-cycles (at most a single cycle at each level between i_- and i_+ , see Lemma 13). These are the only level-cycles that need to be updated. We describe the procedure for R_0 . The one for R_1 is similar. As we mentioned above, these updates are first done for R_0 . When the recursive call for R_0 is completed, all changes are undone and we perform the updates for R_1 .

We first identify the components that are bisected by C . This is done by traversing the cycle C . For each edge e of C we mark each component K in $P^{\mathcal{K}}(e)$ whose level is at least i_- by the endpoint of e that belongs to $X(K)$. This can be done by starting at the last component of $P^{\mathcal{K}}(e)$ (Since C does not contain edges of any component with level greater than i_+ , the level of the last component of $P^{\mathcal{K}}(e)$ is at most i_+), and following parent pointers in the component tree \mathcal{K} until either the other end of $P^{\mathcal{K}}(e)$ is encountered or until a component of level i_- is encountered.

After traversing C , we have identified, for each marked component K , two vertices u and v on $X(K)$. We cut the cycle $X(K)$ at u and at v . Since cycle levels are represented using splay trees, this takes $O(\log |R|)$ amortized time. This results in two paths, let $X_0(K)$ be the one that consists of edges of R_0 . The endpoints u and v of $X_0(K)$ are incident to the new hole h of R_0 . We add a dummy edge uv to turn $X_0(K)$ back into a cycle. Note that dummy edge uv represents the two triangulation edges vx_h and x_hu that connect u and v in R_0 . We do not explicitly add those artificial edges to form a cycle since artificial edges (as well as the artificial node x_h) might be replaced with new ones at subsequent recursive calls. We cannot afford to update such edges when retriangulating since each triangulation edge might appear in many level cycles. To be able to associate a dummy edge with the corresponding triangulation edges, the algorithm maintains, for every natural vertex v , a list of the artificial triangulation edges incident to v . At any given time there are at most 12 items in the list. Note that, by Lemma 17, we can afford to update these lists when retriangulating holes.

Note that, even though level cycles are no longer disjoint, Invariant 1 is maintained. The reason is that level cycles may share artificial triangulation edges, but no natural edges.

We now describe how to update \mathbf{R} with new triangulation edges. For each triangulation edge that belongs to some level cycle, the algorithm stores pointers to the lowest and deepest corresponding components in \mathcal{K} . These can be identified at the time the corresponding dummy edge is inserted into the representation of level-cycle. When retriangulating a hole, every new triangulation edge inherit this information from the triangulation edge it replaced. Recall that all triangulation edges of a specific hole h are incident to a single artificial vertex x_h . Going over the edges incident to x_h in cyclic order, and knowing the level cycles to which each edge belongs, it is easy to infer the corresponding paths $P^{\mathcal{K}}(e)$, and store their endpoints in the array \mathbf{R} .

5.4 Representation of Vertex Weights

Our algorithm separates regions according to three possible balance criteria: number of natural vertices, number of boundary vertices, and number of holes. In this subsection we discuss how these quantities are represented and maintained.

As we mention in Section 3.2 and in the beginning of Section 4, vertex weights are handled by transferring the weight to some adjacent face. A cycle that is balanced with respect to the face weights is balanced with respect to the original vertex weights as well.

We define three types of face weights: **natural_weight**, **boundary_weight** and **hole_weight** such that balancing according to each weight type is equivalent to balancing according to the corresponding quantity. Initially, all face weights are initialized to zero. For each vertex v of \bar{G} , the algorithm associate with v an arbitrary adjacent face f , and increases **natural_weight**(f) by one. It maintains this vertex-face association in an array F so that, for every vertex, the associated face can be queried in constant time. Since the input graph \bar{G} is triangulated, each face has weight at most 3, so no face carries more than $3/4$ of the total **natural_weight** (recall that the cycle separator is not invoked on graphs with fewer than s vertices).

We now describe how to update the weights when separating a region R along a cycle separator C into subregions R_0 and R_1 . We describe the procedure for R_0 . The one for R_1 is similar. The algorithm checks, for each (natural) boundary vertex v of R_0 if $F[v]$ is a face of R_0 . This can be done in $O(\log |R_0|)$ time using the cotree T^* for R_0 . If $F[v]$ is not a face of R_0 , the algorithm chooses an arbitrary face f of R_0 that is adjacent to v , sets $F[v]$ to f , and increases **natural_weight**(f) by one. This guarantees that each vertex of R_0 assigns unit weight to one associated face of R_0 . Note also that for each face f of R_0 , all vertices assigning weight to f are vertices of R_0 . Hence, balancing R_0 by **natural_weight** is equivalent to balancing by the number of natural vertices of R_0 . The time to perform the update is $O(|C| \log |R|)$, which, by Lemma 17, is $O(|\bar{G}|)$ over the entire execution of the algorithm.

To update **boundary_weight**, the algorithm sets **boundary_weight**(f) to one for every new triangulation face created when triangulating R_0 . Similarly, to update **hole_weight**, the algorithm sets to one **hole_weight**(f) for one arbitrary new triangulation face of the new hole formed when cutting R along C . Again, by Lemma 17, this takes total $O(|\bar{G}|)$ time.

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A Proof of Lemma 2

Proof. We show by induction on n that, for $n \geq r$, $T_r(n) \leq \delta\gamma n/r^{1-\beta} - \gamma n^\beta$, where $\gamma > 1$ and $\delta > 1$ are constants to be determined. Note that $\delta\gamma n/r^{1-\beta} - \gamma n^\beta$ is non-negative whenever $\delta n/r^{1-\beta} \geq n^\beta$, or equivalently, whenever $n \geq \frac{r}{\delta^{1/(1-\beta)}}$.

Suppose $n > r$, and assume the lemma holds for $n' < n$. Fix values of $\alpha_1, \dots, \alpha_8$. We show that $\rho n^\beta + \sum_i T_r(\alpha_i n) \leq \delta\gamma n/r^{1-\beta} - \gamma n^\beta$. Let $\mathcal{I} = \{i : \alpha_i n > \frac{r}{\delta^{1/(1-\beta)}}\}$, and let $q = |\mathcal{I}|$.

Case 0: $q = 0$. In this case, $\alpha_i n < \frac{r}{\delta^{1/(1-\beta)}}$ for all $1 \leq i \leq 8$. Since $\sum \alpha_i \geq 1$, it must be that $r < n \leq 8r$. Hence $T_r(n) \leq \rho n^\beta$, and

$$\delta\gamma n/r^{1-\beta} - \gamma n^\beta \geq \delta\gamma r^\beta - 8\gamma r^\beta = (\delta - 8)\gamma r^\beta \quad (16)$$

Setting $\gamma > 8\rho$ we get

$$\begin{aligned} T_r(n) &\leq \rho n^\beta \\ &\leq 8\rho r^\beta \\ &\leq \gamma r^\beta \end{aligned} \quad (17)$$

Requiring $\delta \geq 9$ and combining (16) and (17) proves the inductive step in this case.

Case 1: $q \geq 1$. For every $i \notin \mathcal{I}$,

$$\alpha_i n \leq \frac{r}{\delta^{1/(1-\beta)}} \leq \frac{n}{\delta^{1/(1-\beta)}}.$$

Hence, for every $i \notin \mathcal{I}$, $\alpha_i \leq \frac{1}{\delta^{1/(1-\beta)}}$. It follows that

$$\sum_{i \in \mathcal{I}} \alpha_i \geq 1 - \frac{7}{\delta^{1/(1-\beta)}} \quad (18)$$

We claim that δ can be chosen sufficiently large so that $\sum_{i \in \mathcal{I}} \alpha_i^\beta = 1 + \epsilon$ for some $\epsilon > 0$. To see this, assume without loss of generality that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_8$, and observe that

$$\begin{aligned} \sum_{i \in \mathcal{I}} \alpha_i^\beta &\geq \alpha_1^\beta - \alpha_1 + \sum_{i \in \mathcal{I}} \alpha_i \\ &\geq \alpha_1^\beta - \alpha_1 + 1 - \frac{7}{\delta^{1/(1-\beta)}} \end{aligned} \quad (19)$$

Let s be sufficiently large so that $3/4 + c/\sqrt{n} < 1$. Since $1/8 \leq \alpha_1 \leq 3/4 + c/\sqrt{n}$, $\alpha_1^\beta - \alpha_1$ is strictly positive. Define $\epsilon' > 0$ to be $\alpha_1^\beta - \alpha_1$. Note that, by concavity of $\alpha^\beta - \alpha$, ϵ' is monotonically increasing with s . Substituting into (19) we get

$$\sum_{i \in \mathcal{I}} \alpha_i^\beta \geq 1 + \epsilon' - \frac{7}{\delta^{1/(1-\beta)}} \quad (20)$$

Define ϵ to be $\epsilon' - \frac{7}{\delta^{1/(1-\beta)}}$, and set δ sufficiently large so that $\epsilon > 0$. We have

$$\sum_{i \in \mathcal{I}} \alpha_i^\beta \geq 1 + \epsilon \quad (21)$$

For each $i \notin \mathcal{I}$, $T_r(\alpha_i n) \leq 0$. Hence

$$\begin{aligned} \rho n^\beta + \sum_i T_r(\alpha_i n) &= \rho n^\beta + \sum_{i \in \mathcal{I}} \left(\delta\gamma \alpha_i n/r^{1-\beta} - \gamma \alpha_i^\beta n^\beta \right) \\ &\leq \rho n^\beta + \delta\gamma n/r^{1-\beta} \sum_{i \in \mathcal{I}} \alpha_i - \gamma n^\beta \sum_{i \in \mathcal{I}} \alpha_i^\beta \\ &\leq \rho n^\beta + (1 + c/\sqrt{n}) \delta\gamma n/r^{1-\beta} - \gamma n^\beta (1 + \epsilon) \\ &= \delta\gamma n/r^{1-\beta} - \gamma n^\beta + \rho n^\beta + \delta\gamma c\sqrt{n}/r^{1-\beta} - \epsilon\gamma n^\beta \\ &= \delta\gamma n/r^{1-\beta} - \gamma n^\beta + n^\beta \left(\rho + \delta\gamma c n^{1/2-\beta}/r^{1-\beta} - \epsilon\gamma \right) \\ &\leq \delta\gamma n/r^{1-\beta} - \gamma n^\beta + n^\beta \left(\rho + \delta\gamma c/r^{1-\beta} - \epsilon\gamma \right) \end{aligned} \quad (22)$$

Let s be sufficiently large so that for any $r \geq s$, $\delta c/r^{1-\beta} < \epsilon/2$. Note that ϵ does depend on s , but it since it is monotonically increasing with s , there exists such sufficiently large s .

Then the right-hand side of (22) is at most

$$\delta \gamma n/r^{1-\beta} - \gamma n^\beta + n^\beta (\rho - \epsilon \gamma/2)$$

Setting $\gamma > \rho/\epsilon$ completes the proof. □