Cartesian Product of Sets

Ordered Pair: An ordered pair consists of two objects or elements in a given fixed order.

For example, if A and B are any two sets, then by an ordered pair of elements, we mean a pair (a, b) in that order, where $a \in A$, $b \in B$.

Note: An ordered pair is not a set consisting two elements. The ordering of the two elements in an ordered pair is important and the two elements need not be distinct.

Example: The position of a point in a two dimensional plane in cartesian coordinates is represented by an ordered pair. Accordingly, the ordered pairs $(1, 3), (2, 4), (2, 3)$ and $(3, 2)$ represents different points in a plane.

Equality of Ordered Pairs: Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff $a_1 = a_2$ and $b_1 = b_2$.

i.e $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

From the definition, it is clear that $(1, 2) \neq (2, 1)$ and $(1, 1) \neq (2, 2)$.

Example: Find the values of a and b, if $(3a-2, b+3) = (2a-1, 3)$

Solution: By the definition of equality of ordered pairs, we have

$$(3a-2, b+3) = (2a-1, 3)$$

$$\Leftrightarrow 3a-2 = 2a-1 \text{ and } b+3 = 3$$

$$\Leftrightarrow a = 1 \text{ and } b = 0.$$

Cartesian Product of Sets: Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.

Thus $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

Example: If $A = \{a, b\}$ and $B = \{1, 2, 3\}$, find $A \times B$, $B \times A$, $A \times A$, $B \times B$ and $(A \times B) \cap (B \times A)$.

Solution: We have $A = \{a, b\}$, $B = \{1, 2, 3\}$.

So,
$$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$$
$$B \times A = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$$
$$A \times A = \{(a,a), (a,b), (b,a), (b,b)\}$$
$$B \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

$$\Rightarrow (A \times B) \cap (B \times A) = \phi.$$

Graphical Representation of Cartesian Product of Sets:

Let A and B be any two non-empty sets. To represent graphically, we draw two mutually perpendicular lines, one horizontal and other vertical. On the horizontal line, we represent the elements of set A and on the vertical line, the elements of B.

If $a \in A$, $b \in B$, we draw a vertical line through a and a horizontal line through b. These two lines will meet in a point which will denote the ordered pair (a, b). In this manner we mark points corresponding to each ordered pair in $A \times B$. The set of points so obtained represents $A \times B$ graphically as illustrated in the following example.

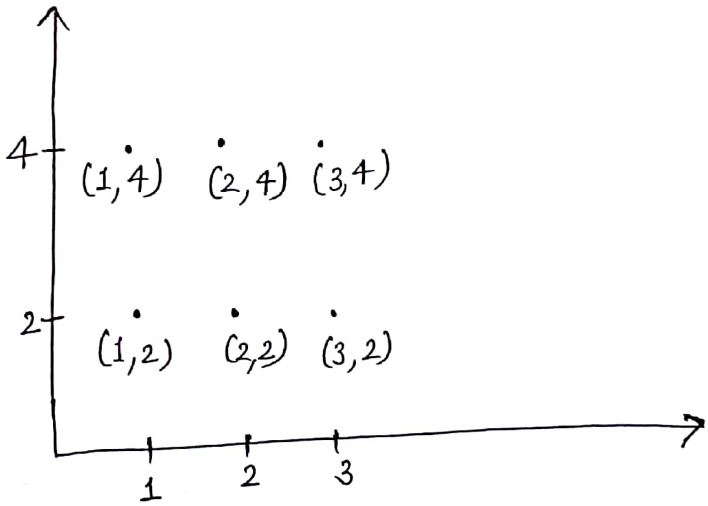
Example: If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, find $A \times B$ and show it graphically.

Solution: Clearly, $A \times B = \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)\}$

To represent $A \times B$ graphically, we draw two $\perp$ lines OX and OY as shown in the figure.

Now, we represent the set A by three points on OX and the set B by two points on OY.

The set $A \times B$ is represented by six points as shown in figure.



Note: (i) If A and B are finite sets, then

$$n(A \times B) = n(A) \cdot n(B).$$

(ii) If either A or B is an infinite set, then $A \times B$ is an infinite set.

(iii) $A \times B = \phi \Leftrightarrow A = \phi$ or $B = \phi$.

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Illustrative Examples

Example 1 Find x and y, if $(x+3, 5) = (6, 2x+y)$.

Solution: By the definition of equality of ordered pairs

$$(x+3, 5) = (6, 2x+y) \Rightarrow x+3=6 \text{ and } 5 = 2x+y$$
$$\Rightarrow x=3 \text{ and } 5 = 2x+y$$
$$\Rightarrow x=3, \quad 5 = 6+y$$
$$\Rightarrow x=3 \text{ and } y=-1.$$

Example 2 If $A=\{1, 3, 5, 6\}$ and $B=\{2, 4\}$, find $A \times B$ and $B \times A$.

Solution: $A \times B = \{(1,2), (1,4), (3,2), (3,4), (5,2), (5,4), (6,2), (6,4)\}$

and $B \times A = \{(2,1), (2,3), (2,5), (2,6), (4,1), (4,3), (4,5), (4,6)\}$

Example 3 If $A=\{1, 2, 3\}$, $B=\{3,4\}$ and $C=\{1, 3, 5\}$, find

(i) $A \times (B \cup C)$ (ii) $A \times (B \cap C)$ (iii) $(A \times B) \cap (B \times C)$.

Solution: (i) We have $B \cup C = \{1,3,4,5\}$.

$$\therefore A \times (B \cup C) = \{1, 2, 3\} \times \{1, 3, 4, 5\}$$
$$= \{(1,1), (1,3), (1,4), (1,5), (2,1), (2,3), (2,4), (2,5), (3,1), (3,3), (3,4), (3,5)\}$$

(ii) We have $B \cap C = \{3\}$.

$$A \times (B \cap C) = \{1, 2, 3\} \times \{3\} = \{(1,3), (2,3), (3,3)\}.$$

(iii) We have

$$A \times B = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\},$$

and

$A \times C = \{(1,1), (1,3), (1,5), (2,1), (2,3), (2,5), (3,1), (3,3), (3,5)\}$

$\therefore (A \times B) \cap (A \times C) = \{(1,3), (2,3), (3,3)\}$.

Example: If $A \times B = \{(a,1), (b,3), (a,3), (b,1), (a,2), (b,2)\}$, find A and B.

Solution: A is the set of all first entries in ordered pairs in $A \times B$ and B is the set of all second entries in ordered pairs in $A \times B$.

$\therefore A = \{a, b\}$ and $B = \{1, 2, 3\}$.

Example: Let A and B be two sets such that $A \times B$ consists of 6 elements. If three elements of $A \times B$ are $(1,4), (2,6), (3,6)$. Find $A \times B$ and $B \times A$.

Solution: Since $(1,4)$, $(2,6)$ and $(3,6)$ are elements of $A \times B$.

$\Rightarrow$ 1, 2, 3 are elements of A and 4, 6 are elements of B.

$\Rightarrow$ $A \times B$ has 6 elements.

$\Rightarrow$ $A = \{1, 2, 3\}$ and $B = \{4, 6\}$.

Hence, $A \times B = \{1, 2, 3\} \times \{4, 6\} = \{(1,4), (1,6), (2,4), (2,6), (3,4), (3,6)\}$.

and $B \times A = \{4, 6\} \times \{1, 2, 3\} = \{(4,1), (4,2), (4,3), (6,1), (6,2), (6,3)\}$.

Example: If $A \times B = \{(a,1), (a,5), (a,2), (b,2), (b,5), (b,1)\}$, find $B \times A$.

Solution: $B \times A$ can be obtained from $A \times B$ by interchanging the entries ~~as~~ in $A \times B$.

$$\therefore B \times A = \{(1,a), (5,a), (2,a), (2,b), (5,b), (1,b)\}.$$

Example: The cartesian product $A \times A$ has 9 elements among which are found $(-1,0)$ and $(0,1)$. Find the set A and the remaining elements of $A \times A$.

Solution: Since $(-1,0) \in A \times A$ and $(0,1) \in A \times A$.

Therefore, $(-1,0) \in A \times A \Rightarrow -1, 0 \in A$

and $(0,1) \in A \times A \Rightarrow 0, 1 \in A$.

$\therefore -1, 0, 1 \in A$.

$\Rightarrow$ A has exactly three elements. Hence $A = \{-1, 0, 1\}$.