Relations :

Let A and B denote the sets of all male and female members in the royal family of Dasrath's kingdom. Clearly,

$$A = \{\text{Dasrath, Ram, Laxman, Shatrughan, Bharat}\}$$

and $B = \{$ Kaushalya, Kaikai, Sumitra, Sita, Urmila, Shruthkirti, Mandvi $\}$.

If we write R for the relation "was husband of", then the fact that Dasrath was husband of Kaushalya, Kaikai and Sumitra, Ram was husband of Sita, Laxman was husband of Urmila, Bharat was husband of Mandvi and Shatrughan was husband of Shruthkirti can be represented as :

Dasrath R Kaushlya, Dasrath R Kaikai, Dasrath R Sumitra, Ram R Sita, Laxman R Urmila, Bharat R Mandvi and Shatrughan R Shruthkirti.

Now, if we omit the letter R between the pairs of names and write them as ordered pairs, then the above fact can also be written as a set R of ordered pairs, where

R = {(Dasrath, Kaushalya), (Dasrath, Kaikai), (Dasrath, Sumitra), (Ram, Sita), (Laxman, Urmila), (Bharat, Mandvi), (Shatrughan, Shruthkirti)}.

$\Rightarrow$ $R \subseteq A \times B$.

Thus, we see that the relation "was husband of" from set A to set B gives rise to a subset R of $A \times B$

such that $(x, y) \in R$ iff xRy.

Keeping this example in mind, we may define a relation as follows.

Relation: A relation R from a non-empty set A to a non-empty set B is a subset of $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$.

Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

If $(a, b) \in R$, then we write aRb which is read as 'a is related to b by the relation R'.

If $(a, b) \notin R$, then we write $a \not R b$ and we say that a is not related to b by the relation R.

Example: Consider the sets $P = \{a, b, c\}$,

$Q = \{$Ali, Bhanu, Binoy, Chandra, Divya$\}$.

Then $P \times Q = \{$(a, Ali), (a, Bhanu), (a, Binoy), (a, Chandra), (a, Divya), (b, Ali), (b, Bhanu), (b, Binoy), (b, Chandra), (b, Divya), (c, Ali), (c, Bhanu), (c, Binoy), (c, Chandra), (c, Divya)$\}$.

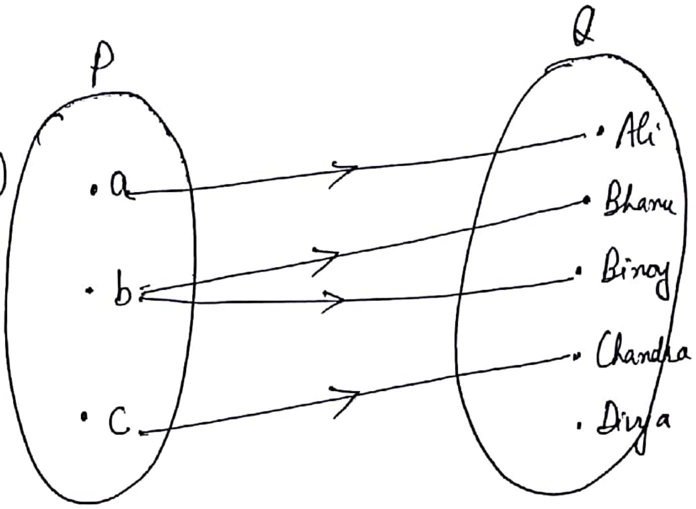
Introduce the relation

$R = \{(x, y) :$ x is the first letter of the name y, $x \in P, y \in Q\}$.

Then

$R = \{(a, \text{Ali}), (b, \text{Bhanu}), (b, \text{Binoy}), (c, \text{Chandra})\}$.

A visual representation of this relation R (called an arrow diagram) is shown in the figure.



Example: If $A = \{a, b, c, d\}$, $B = \{p, q, r, s\}$, then which of the following are relations from A to B? Give reasons for your answer.

(a) $R_1 = \{(a, p), (b, r), (c, s)\}$ (b) $R_2 = \{(q, b), (c, s), (d, r)\}$.

(c) $R_3 = \{(a, p), (a, q), (d, p), (c, r), (b, r)\}$

(d) $R_4 = \{(a, p), (q, a), (b, s), (s, b)\}$.

Solution: (a) Clearly, $R_1 \subseteq A \times B$. So R_1 is a relation from A to B.

(b) Since $(q, b) \in R_2$ but $(q, b) \notin A \times B$. So $R_2 \not\subseteq A \times B$.

Thus R_2 is not a relation from A to B.

(c) Clearly, $R_3 \subseteq A \times B$. So R_3 is a relation from $A \times B$.

(d) R_4 is not a relation from A to B, because (q, a) and (s, b) are elements of R_4 but (q, a) and (s, b) are not in $A \times B$. As such $R_4 \not\subseteq A \times B$.

Domain and Range of a Relation: The set of first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R, while the set of second elements in the relation R from the set A to the set B is called the range of the relation R.

Thus,

$$\text{Dom}(R) = \{a : (a, b) \in R\} \text{ and } \text{Range}(R) = \{b : (a, b) \in R\}.$$

The whole set 'B' is called the codomain of the relation R.

Note that $\boxed{\text{Range} \subseteq \text{Codomain}}$.

Example: If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ and let $R = \{(1, 8), (3, 6), (5, 2), (1, 4)\}$ be a relation from A to B. Then find Dom(R) and Range(R)?

Solution: $\text{Dom}(R) = \{1, 3, 5\}$ and $\text{Range}(R) = \{8, 6, 2, 4\}$.

Remarks: (i) A relation may be represented algebraically either by Roster method or by the set builder method.

(ii) An arrow diagram is a visual representation of a relation.

Example: Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation R from A to A by

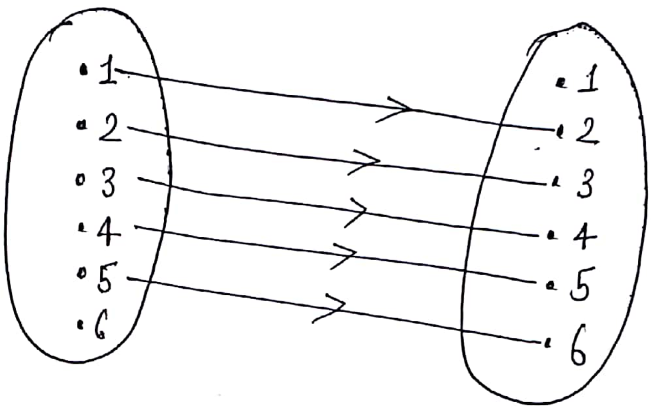
$$R = \{(x, y) : y = x + 1\}.$$

(i) Depict this relation by an arrow diagram.
(ii) Write down the domain, co-domain and range of R.

Solution:- (i) By the definition of relation,

$$R = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}$$

The corresponding arrow diagram is shown in the following figure.



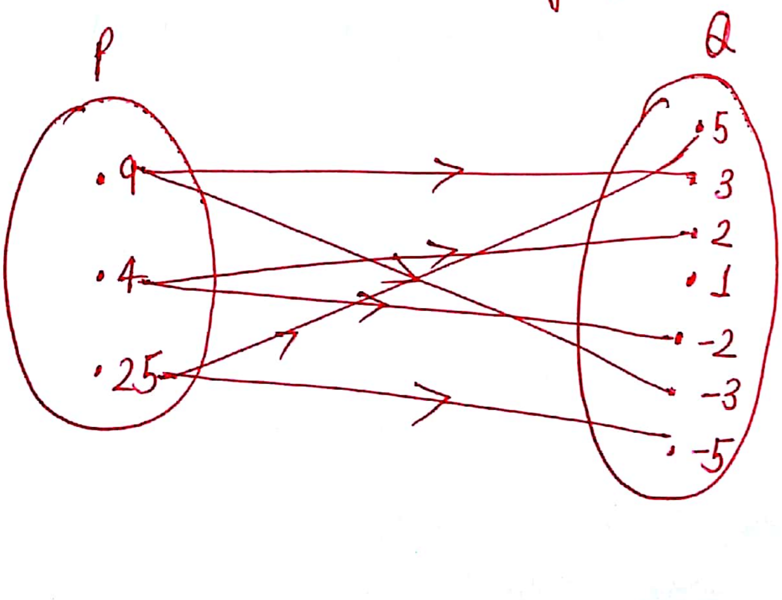
(ii) Domain $= \{1, 2, 3, 4, 5\}$.

Range $= \{2, 3, 4, 5, 6\}$.

and Codomain $= \{1, 2, 3, 4, 5, 6\}$.

Example: The following figure shows a relation between the sets P and Q. Write this relation
(i) in set-builder form.
(ii) in roster form.

What is its domain and range?



Solution: We observe that the relation R is "x is the square of y".

(i) In set-builder form,

$$R = \{(x, y) : x \text{ is the square of } y,\ x \in P,\ y \in Q\}.$$

(ii) In roster form,

$$R = \{(9, 3), (9, -3), (4, 2), (4, -2), (25, 5), (25, -5)\}.$$

The domain of this relation is $\{4, 9, 25\}$.

The range of this relation is $\{-2, 2, -3, 3, -5, 5\}$.

Note that the element 1 is not related to any element in set P.

The set Q is the codomain of this relation.

Important Note: The total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$.

If $n(A) = p$, $n(B) = q$, then

$$n(A \times B) = pq \quad \text{and}$$

the total number of relations is 2^{pq}.

Example: Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find the total number of relations from A to B.

Solution: We have

$$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}.$$

Since $n(A\times B)=4$, the number of subsets of $A\times B$ is 2^4.
Therefore, the number of relations from A to B will be

$2^4=16$.

Remark:- A relation R from A to A is also stated as a relation on A.