Pricing Option theory

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This is to certify that this minor thesis entitled "Pricing Option theory" submitted towards the partial fulfillment of the Mathematics PhD degree program at the Indian Institute of Science Education and Research Pune, represents work carried out by Jatin Majithia under the supervision of Dr. Anindya Goswami.

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1 Introduction

This paper examines the price process assumption of the Black-Scholes equation for pricing options.The material is organized into seven primary sections. The first section deals with some basics on option theory. In second section, we will introduce several key results in probability theory, including (most importantly) Itôs Lemma, which is used in the derivation of Black-Scholes equation. Third section represents the theory behind developing appropriate model for stock prices. Fourth section deals with the assumptions which Black Scholes made while deriving the pricing formula. Fifth section deals with derivation of Black Scholes equation. In sixth section, we look at put call parity. Finally, we conclude the paper by looking into some applications of B-S equation in Bonds and warrant valuations.

2 Notations

S : Price of underlying asset

K : Strike Price

t : Initial Time

T : Exercise time

f : Call Option Price

u : Put Option Price

r : Interest rate

v : Standard Deviation

3 Option Theory

Definition: An **option** is a contract that provides the holder with the right to either buy or sell a specified quantity of an underlying asset at a fixed price at or before the expiration date of the contract.

Since it is a right and not an obligation, the holder can choose not to exercise the right and allow the option to expire.

Here we focus on mainly two types of options:

1. Call options (right to buy)
2. Put options (right to sell)

Call Option: A call option gives the buyer of the option the right to buy the underlying asset at a fixed price (strike price or K) at any time, prior to the expiration date of the option. The buyer pays a price for this right.
At expiration:

1. If the value of the underlying asset (S) $\geq$ Strike Price(K)
   Buyer makes the difference: S - K
2. If the value of the underlying asset (S) $\leq$ Strike Price (K)
   Buyer does not exercise

More generally,

1. the value of a call increases as the value of the underlying asset increases.
2. the value of a call decreases as the value of the underlying asset decreases.

Put Option: A put option gives the buyer of the option the right to sell the underlying asset at a fixed price at any time, prior to the expiration date of the option. The buyer pays a price for this right.

At expiration:

1. If the value of the underlying asset (S) $\leq$ Strike Price(K)
   Buyer makes the difference: K-S

2. If the value of the underlying asset (S) $\geq$ Strike Price (K)
   Buyer does not exercise

More generally,

1. the value of a put decreases as the value of the underlying asset increases.

2. the value of a put increases as the value of the underlying asset decreases.

Options are broadly classified into 2 types viz.

1. American options

2. European options.

An option that can be exercised at any time prior to its expiration is termed as **American option**.
An option that can be exercised only at expiration is termed as **European option**.

4 Stocastic Process

Any variable whose value changes over time in an uncertain way is said to follow a stochastic process. Stochastic processes can be classified as discrete time or continuous time processes. A discrete-time stochastic process is one where the value of the variable can change only at certain fixed points in time, whereas a continuous-time stochastic process is one where changes can take place at any time. Stochastic processes can also be classified as continuous variable or discrete variable. In a continuous-variable process, the underlying variable can take any value within a certain range, whereas in a discrete-variable process, only certain discrete values are possible.

We will be dealing only with a continuous-variable, continuous-time stochastic process. Note that, in practice, we do not observe stock prices following continuous-variable, continuous-time processes. Stock prices are restricted to discrete values and changes can be observed only when the exchange is open. Nevertheless, the continuous-variable, continuous-time process proves to be a useful model for many purposes.

4.1 Markov Property: A Markov process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future.

4.2 Wiener Processes: It is a particular type of Markov process with a mean change of zero and a variance rate of 1.0 per unit time and at any instance follows a Guassian distribution.

More precisely, a variable z follows a Wiener process if it has the following two properties:

Property 1. The change Δz_t during a small period of time Δt is

$$\Delta z_t = \varepsilon \Delta t \tag{1}$$

where ε is a random variable drawing from a standard normal distribution, $N(0, 1)$.

Property 2. The values of Δz_t for any two different short intervals of time Δt are independent.

It follows from the first property that Δz_t itself has a normal distribution with

$$\text{mean of } \Delta z_t = 0$$
$$\text{standard deviation of } \Delta z_t = \sqrt{\Delta} t$$
$$\text{Variance of } \Delta z_t = \Delta t$$

4.3 Generalized Wiener Process:

The basic Wiener process, Δz_t, that has been developed so far has a drift rate of zero and a variance rate of 1.0. The drift rate of zero means that the expected value of z at any future time is equal to its current value. The variance rate of 1.0 means that the variance of the change in z in a time interval of length T equals T. A generalized Wiener process for a variable x_t can be defined in terms of Δz_t as follows:

$$dx_t = adt + bdz_t, \quad x(0) = x_0 \tag{2}$$

where a and b are constants.

To understand equation (2), it is useful to consider the two components on the right-hand side separately. The adt term implies that x has an expected drift rate of a per unit of time. Without the bdz_t term, the equation is

$$dx_t = adt$$

Integrating with respect to time, we get

$$x = x_0 + at$$

where x_0 is the value of x at time zero. In a period of time of length T, the value of x increases by an amount aT.

The bdz_t term on the right-hand side of equation (2) can be regarded as adding noise or variability to the path followed by x. The amount of this noise or variability is b times a Wiener process. A Wiener process has a standard deviation of 1.0. It follows that b times a Wiener process has a standard deviation of b. In a small time interval Δt, the change Δx_t in the value of x is given by equations (1) and (2) as

$$\Delta x_t = a\Delta t + b\varepsilon\sqrt{\Delta} t$$

where, as before, ε is a random variable drawing from a standard normal distribution.
Thus, Δx_t has a normal distribution with:

$$\text{mean of } \Delta x_t = a\Delta t$$

$$\text{standard deviation of } \Delta x_t = b\sqrt{\Delta}t$$

$$\text{variance of } \Delta x = b^2\Delta t$$

Similar arguments to those given for a Wiener process show that the change in the value of x_t in any time interval T is normally distributed with:

$$\text{mean of change in } x_t = aT$$

$$\text{standard deviation of change in } x_t = b\sqrt{T}$$

$$\text{variance of change in } x_t = b^2T$$

Thus, the generalized Wiener process given in equation (2) has an expected drift rate (i.e., average drift per unit of time) of a and a variance rate (i.e., variance per unit of time) of b^2.

4.4 ltô's Process:

A further generalized type of stochastic process can be defined. This is known as an ltô's process. This is a generalized Wiener process in which the parameters a and b are functions of the value of the underlying variable x and time t.

Algebraically, an Itô's process can be written as

$$dx_t = a(x,t)dt + b(x,t)dz_t \tag{3}$$

Both the expected drift rate and variance rate of an Ito process are liable to change over time. In a small time interval between t and $t + \Delta t$, the variable changes from x_t to $x_t + \Delta x_t$, where

$$\Delta x_t = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta}t$$

This relationship involves a small approximation. It assumes that the drift and variance rate of x remain constant, equal to $a(x,t)$ and $b^2(x,t)$, respectively, during the time interval between t and t + Δ t.

4.5 The Process for stock price

The model proposed above did not suited stock price movement because in Itô's process if we replace variable x by value of stock price S and if we try to solve the differential equation (3) in some interval then value of stock might come to negative i.e.,

Observe, solution to equation (3) will be:

$$S_T = S_0(aT + bz_t)$$

And this value might be negative but which never happens in stock price, so the above process was modified and it got modified into:

$$dS = \mu S_t dt + \sigma S_t dz_t$$

where, we define,
Expected return $(\mu) = \frac{a(S,t)}{S}$
Volatility $(\sigma) = \frac{b(S,t)}{S}$

And when we try to solve above modified differential equation, we get solution as:

$$S_T = S_0 \exp\left((\mu - \frac{\sigma^2}{2})T - \sigma z_T\right)$$

which is never negative. Since, S_0 is positive and exponential is also positive. To arrive at this solution we will take help of Itôs lemma. Thus the above model is most widely used model of stock price behavior.

4.6 Itôs Lemma

Suppose that the value of a variable x follows the Itôs process

$$dx_t = a(x,t)dt + b(x,i)dz_t \tag{4}$$

where dz_t is a Wiener process and a and b are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2. Itô's lemma shows that a function G which is two times differentiable w.r.t x and once differentiable w.r.t t follows the process

$$dG(x_t) = \left(\left(\frac{\partial G(x_t)}{\partial x}\right) a(x_t,t) + \left(\frac{\partial G(x_t)}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial^2 G(x_t)}{\partial x^2}\right) b^2(x_t,t)\right) dt + \left(\frac{\partial G(x_t)}{\partial x}\right) b(x_t,t) dz_t \tag{5}$$

where the dz_t is the same Wiener process as in equation (1). Thus, $G(x_t)$ also follows an Itôs process. It has a drift rate of:

$$\left(\frac{\partial G(x_t)}{\partial x}\right) a(x_t, t) + \left(\frac{\partial G(x_t)}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial^2 G(x_t)}{\partial x^2}\right) b^2(x_t, t)$$

and a variance rate of:

$$\left(\frac{\partial G(x_t)}{\partial x}\right)^2 b^2(x_t, t)$$

Earlier, we argued that,

$$dS_t = \mu S_t dt + \sigma S_t dz_t \tag{6}$$

with μ and σ constant, is a reasonable model of stock price movements. From Itôs lemma, it follows that the process followed by a function $G(S_t)$ of S and t is

$$dG(S_t) = \left(\left(\frac{\partial G(S_t)}{\partial S}\right)\mu S_t + \left(\frac{\partial G(S_t)}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial^2 G(S_t)}{\partial S^2}\right)\sigma^2 S_t^2\right) dt + \left(\frac{\partial G(S_t)}{\partial S}\right)\sigma S_t dz_t \tag{7}$$

Note that both S and G are affected by the same underlying source of uncertainty, dz_t. This proves to be very important in the derivation of the Black-Scholes results.

4.6 THE LOGNORMAL PROPERTY

We now use Ito's lemma to derive the process followed by $\ln S_t$ when S follows the process in equation (3).
Define, $G(S_t) = \ln S_t$,
Since,

$$\frac{\partial G(S_t)}{\partial S_t} = \frac{1}{S_t}, \frac{\partial^2 G(S_t)}{\partial S_t^2} = \frac{-1}{S_t^2}, \frac{\partial G(S_t)}{\partial t} = 0$$

it follows from equation (7) that $G(S_t)$ satisfies

$$dG(S_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz_t \tag{8}$$

Since, μ and σ are constant, this equation indicates that $G = \ln S_t$ follows a generalized Wiener process. It has constant drift rate (μ -$\frac{\sigma^2}{2}$) and constant variance rate σ. The change in $\ln S_t$ between time zero and some future time, T, is therefore normally distributed with mean (μ -$\frac{\sigma^2}{2}$)T and variance σ^2T. This means that

$$\ln S_T - \ln S_0 \sim \phi[(\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}]$$

$$lnS_T \sim \phi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}] \tag{9}$$

where S_T is the stock price at a future time T, S_0 is the stock price at time zero, and $N(m, s)$ denotes a normal distribution with mean m and standard deviation s.

Equation (9) shows that $\ln S_T$ is normally distributed. A variable has a lognormal distribution if the natural logarithm of the variable is normally distributed. The model of stock price behavior we have developed so far is therefore implies that a stock's price at time T, given its price today, is lognormally distributed. The standard deviation of the logarithm of the stock price is $\sigma\sqrt{T}$. It is proportional to the square root of how far ahead we are looking.

5 Black Scholes Model

Most of the previous work on the valuation of options has been expressed in terms of warrants. For example, Sprenkle (1961), Ayres (1963), Boness (1964), Samuelson (1965), Baumol, Malkiel, and Quandt (1966), and Chen (1970) all produced valuation formulas of the same general form. Their formulas, however, were not complete, since they all involved one or more arbitrary parameters.
For example, Sprenkle's formula for the value of an option can be written as follows:

$$kxN(b_1) - k^*cN(b_2)$$
$$b_1 = \frac{\ln \frac{kx}{c} + \frac{1}{2}v^2(T-t)}{v\sqrt{(T-t)}}$$
$$b_2 = \frac{\ln \frac{kx}{c} - \frac{1}{2}v^2(T-t)}{v\sqrt{(T-t)}}$$

In this expression, x is the stock price, c is the exercise price, T is the maturity date, t is the current date, v^2 is the variance rate of the return on the stock [1], ln is the natural logarithm, and $N(b)$ is the cumulative normal density function. But k and k^* are unknown parameters. Sprenkle (1961) defines k as the ratio of the expected value of the stock price at the time the warrant matures to the current stock price, and k^* as a discount factor that depends on the risk of the stock. He tries to estimate the values of k and k^* empirically, but finds that he is unable to do so.

More typically, Samuelson (1965) has unknown parameters a and b, where a is the rate of expected return on the stock, and b is the rate of expected return on the warrant or the discount rate to be applied to the warrant [2]. He assumes that the distribution of possible values of the stock when the warrant matures is log-normal and takes the expected value of this distribution, cutting it off at the exercise price. He then discounts this expected value to the present at the rate b. Unfortunately, there seems to be no model of the pricing of securities under conditions of capital market equilibrium that

[1] The variance rate of the return on a security is the limit, as the size of the interval of measurement goes to zero, of the variance of the return over that interval divided by the length of the interval.

[2] The rate of expected return on a security is the limit, as the size of the interval of measurement goes to zero, of the expected return over that interval divided by the length of the interval.

would make this an appropriate procedure for determining the value of a warrant.

In a subsequent paper, Samuelson and Merton (1969) recognize the fact that discounting the expected value of the distribution of possible values of the warrant when it is exercised is not an appropriate procedure. They advance the theory by treating the option price as a function of the stock price. They also recognize that the discount rates are determined in part by the requirement that investors be willing to hold all of the outstanding amounts of both the stock and the option. But they do not make use of the fact that investors must hold other assets as well, so that the risk of an option or stock that affects its discount rate is only that part of the risk that cannot be diversified away. Their final formula depends on the shape of the utility function that they assume for the typical investor.

One of the concepts that we use in developing our model is expressed by Thorp and Kassouf (1967). They obtain an empirical valuation formula for warrants by fitting a curve to actual warrant prices. Then they use this formula to calculate the ratio of shares of stock to options needed to create a hedged position by going long in one security and short in the other. What they fail to pursue is the fact that in equilibrium, the expected return on such a hedged position must be equal to the return on a riskless asset. What we show below is that this equilibrium condition can be used to derive a theoretical valuation formula.

The Black-Scholes differential equation is an equation that must be satisfied by the price function of any derivative dependent on a non-dividend-paying stock .Black Scholes model involve setting up a riskless portfolio consisting of a position in the derivative and a position in the stock. In absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate, r. This leads to the Black-Scholes differential equation.

The reason a riskless portfolio can be set up is that the stock price and the derivative price are both affected by the same underlying source of uncertainty: stock price movements. In any short period of time, the price of the derivative is perfectly correlated with the price of the underlying stock. When an appropriate portfolio of the stock and the derivative is established, the gain or loss from the stock position always offsets the gain or loss from the derivative position so that the overall value of the portfolio at the end of the short period of time is known with certainty.

Suppose, for example, that at a particular point in time the relationship between a small change in the stock price, ΔS_t, and the resultant small change Δf in the price of a European call option,is given by:

$$\Delta f = 0.4\Delta S_t$$

This means that the slope of the line representing the relationship between f and S_t is 0.4. The riskless portfolio would consist of:
1. A long position i.e., buy in 0.4 share
2. A short position i.e., sell in one call option

There is one important point here is that the position in the stock and the option is riskless for only a very short period of time. (In theory, it remains riskless only for an instantaneously short period of time.) To remain riskless, it must be adjusted, or rebalanced, frequently. For example, the relationship between Δf and ΔS_t in our example might change from $\Delta f = 0.4\ \Delta S_t$ today to $\Delta f = 0.5\ \Delta s_t$ in two weeks. This would mean that, in order to maintain the riskless position, an extra 0.1 share would have to be purchased for each call option sold. It is nevertheless true that the return from the riskless portfolio in any very short period of time must be the risk-free interest rate. This is the key element in the Black-Scholes analysis and leads to their pricing formulas.

5.1 Assumptions

1. No commissions and transaction costs.
The Black-Scholes model assumes that there are no fees for buying and selling options and stocks and no barriers to trading.

2. European-style options.
The Black-Scholes model assumes European-style options which can only be exercised on the expiration date.

3. Short Selling is allowed.
A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

4. No dividends.
Another assumption is that the underlying stock does not pay dividends during the option's life. In the real world, most companies pay dividends

to their share holders.The basic Black-Scholes model was later adjusted for dividends, so there is a workaround for this.

5. Interest rates and variance rate of return is known and remains constant.

6. Returns are lognormally distributed.
This assumption suggests, returns on the underlying stock are normally distributed, which is reasonable for most assets that offer options.

7. Liquidity.
The Black-Scholes model assumes that markets are perfectly liquid and it is possible to purchase or sell any amount of stock or options or their fractions at any given time.

5.2 DERIVATION Of THE BLACK-SCHOLES DIFFERENTIAL EQUATION

The stock price process we are assuming is the one we developed in Section 3.5

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

Suppose that f is the price of a call option or other derivative contingent on S_t. Also, assume that f is some function of S and t. Hence, from Itôs lemma, we have

$$df(t, S_t) = \left(\left(\frac{\partial f(t, S_t)}{\partial S_t} \right) \mu S_t + \left(\frac{\partial f(t, S_t)}{\partial t} \right) + \frac{1}{2} \left(\frac{\partial^2 f(t, S_t)}{\partial S_t^2} \right) \sigma^2 S_t^2 \right) dt + \left(\frac{\partial f(t, S_t)}{\partial S_t} \right) \sigma S_t dz_t$$

The discrete version of above equation are:

$$\Delta S_t = \mu S_t \Delta t + \sigma S_t \Delta z_t \tag{10}$$

$$\Delta f(t, S_t) = \left(\left(\frac{\partial f(t, S_t)}{\partial S_t} \right) \mu S_t + \left(\frac{\partial f(t, S_t)}{\partial t} \right) + \frac{1}{2} \left(\frac{\partial^2 f(t, S_t)}{\partial S_t^2} \right) \sigma^2 S_t^2 \right) \Delta t + \left(\frac{\partial f(t, S_t)}{\partial S_t} \right) \sigma S_t \Delta z_t \tag{11}$$

Where ΔS_t and $\Delta f(t, S_t)$ are change in S and f in small interval Δt.

The appropriate portfolio is:

-1 : derivative (here, option)

$\frac{\partial f(t,S_t)}{\partial S_t}$: Number of Shares.

The reason behind choosing this kind of portfolio is because the number of shares that must be sold short against one option long is:

$$\frac{\partial f(t, S_t)}{\partial S_t}.$$

The holder of this portfolio is short one derivative and long an amount $\frac{\partial f}{\partial S}$ of shares. Define P as the value of the portfolio.
By definition:

$$P_t = -f(t, S_t) + \frac{\partial f(t, S_t)}{\partial S_t} S_t.$$

The change ΔP in the value of the portfolio in the time interval Δt is given by

$$\Delta P_t = -\Delta f + \frac{\partial f}{\partial S_t} \Delta S_t.$$

Substituting the terms on RHS of above equation from the expressions (9) and (10) into above equation , we get

$$\Delta P_t = \left(\frac{\partial f(t, S_t)}{\partial t} - \frac{1}{2} \frac{\partial^2 f(t, S_t)}{\partial S_t^2} \sigma^2 S_t^2 \right) \Delta t$$

Because this equation does not involve Δz_t, the portfolio must be riskless during time Δt. The assumptions listed in the preceding section imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portfolio; if it earned less, they could make a riskless profit by shorting the portfolio and buying risk-free securities. It follows that

$$\Delta P_t = r P_t \Delta t$$

where r is the risk-free interest rate. Equating above 2, we obtain:

$$\left(\frac{\partial f(t, S_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(t, S_t)}{\partial S_t^2} \sigma^2 S_t^2 \right) \Delta t = r \left(-f + \frac{\partial f(t, S_t)}{\partial S_t} \right) \Delta t$$

$$\left(\frac{\partial f(t, S_t)}{\partial t} + r S_t \frac{\partial f(t, S_t)}{\partial S_t} + \frac{1}{2} \frac{\partial^2 f(t, S_t)}{\partial S_t^2} \sigma^2 S_t^2 \right) = r f(t, S_t) \qquad (12)$$

Equation (12) is the Black-Scholes differential equation. It has many solutions, corresponding to all the different derivatives that can be defined with S as the underlying variable. The particular derivative that is obtained when the equation is solved depends on the boundary conditions that are used. These specify the values of the derivative at the boundaries of possible values of S and t.

1. In the case of a European call option, the key boundary condition is

$$f(T,S) = \max(S-K,0), \quad f(t,0) = 0$$

2. In the case of a European put option, it is:

$$f(T,S) = \max(K-S,0), \quad f(t,0) = 0$$

where K is the exercise price of the shares.

After solving equation (13) for European call option we get the following Black Scholes formula:

$$f(S,t) = SN(d_1) - Ke^{r(t-T)}N(d_2) \tag{13}$$

where,

$$d_1 = \frac{ln\frac{S}{K} + (r + \frac{1}{2}v^2)(T-t)}{v\sqrt{(}T-t)}$$

$$d_2 = \frac{ln\frac{S}{K} + (r - \frac{1}{2}v^2)(T-t)}{v\sqrt{(}T-t)}$$

and N(d) is the cumulative normal distribution function.

6 Put Call Parity

The valuation formula was derived under the assumption that the option can only be exercised at time T. Merton (1973) has shown, however, that the value of the option is always greater than the value it would have if it were exercised immediately $(S - K)$. Thus, a rational investor will not exercise a call option before maturity, and the value of an American call option is the same as the value of a European call option. There is a simple modification of the formula that will make it applicable to European put options (options to sell) as well as call options (options to buy). Writing $u(S, t)$ for the value of a put option, we see that the differential equation remains unchanged.

$$\frac{\partial u(t, S_t)}{\partial t} = ru(t, S_t) - rS_t \frac{\partial u(t, S_t)}{\partial S_t} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 u(t, S_t)}{\partial S_t \partial t}$$

The boundary condition, however, becomes:

$$u(S, T) = \begin{cases} 0, & S \geq K \\ K - S, & S < K \end{cases}$$

To get the solution to this equation with the new boundary condition, we can simply note that the difference between the value of a call and the value of a put on the same stock, if both can be exercised only at maturity, must obey the same differential equation, but with the following boundary condition:

$$f(S, T) - u(S, T) = S - K$$

The solution to the differential equation with this boundary condition is $S - Ke^{r(t-T)}$.

Hence,

$$f(S, T) - u(S, T) = S - Ke^{r(t-T)}$$

Thus the value of the European put option is

$$u(S, t) = f(S, t) - S + Ke^{r(t-T)}$$

Putting in the value of $f(S, t)$ from (13), and noting that $1 - N(d)$ is equal to $N(-d)$, we have:

$$u(S, t) = -SN(-d_1) + Ke^{-rT}N(-d_2) \tag{14}$$

In equation (14), d_1 and d_2 are defined as in equation (13). Equation (14) also gives us a relation between the value of a European call and that of a European put. We see that if an investor were to buy a call and sell a put, his returns would be exactly the same as if he bought the stock on margin, borrowing $Ke^{(r(t-T))}$ toward the price of the stock.

Merton (1973) has also shown that the value of an American put option will be greater than the value of a European put option. This is true because it is sometimes advantageous to exercise a put option before maturity, if it is possible to do so. For example, suppose the stock price falls almost to zero and that the probability that the price will exceed the exercise price before the option expires is negligible. Then it will pay to exercise the option immediately, so that the exercise price will be received sooner rather than later. The investor thus gains the interest on the exercise price for the period up to the time he would otherwise have exercised it. So far, no one has been able to obtain a formula for the value of an American put option.

7 Warrant Valuation

A warrant is an option that is a liability of a corporation. The holder of a warrant has the right to buy the corporation's stock (or other assets) on specified terms. The analysis of warrants is often much more complicated than the analysis of simple options, because:

a) The life of a warrant is typically measured in years, rather than months. Over a period of years, the variance rate of the return on the stock may be expected to change substantially.

b) The exercise price of the warrant is usually not adjusted at all for dividends. The possibility that dividends will be paid requires a modification of the valuation formula.

c) The exercise price of a warrant sometimes changes on specified dates. It may pay to exercise a warrant just before its exercise price changes. This too requires a modification of the valuation formula.

d) If the company is involved in a merger, the adjustment that is made in the terms of the warrant may change its value.

e) Sometimes the exercise price can be paid using bonds of the corporation at face value, even though they may at the time be selling at a discount. This complicates the analysis and means that early exercise may sometimes be desirable.

f) The exercise of a large number of warrants may sometimes result in a significant increase in the number of common shares outstanding.

In some cases, these complications can be treated as insignificant, and equation (13) can be used as an approximation to give an estimate of the warrant value. In other cases, some simple modifications of equation (13) will improve the approximation.

Suppose, for example, that there are warrants outstanding, which, if exercised, would double the number of shares of the company's common stock.

Let us define the equity of the company as the sum of the value of all of its warrants and the value of all of its common stock. If the warrants are exercised at maturity, the equity of the company will increase by the aggregate amount of money paid in by the warrant holders when they exercise. The warrant holders will then own half of the new equity of the company, which is equal to the old equity plus the exercise money.

Thus, at maturity, the warrant holders will either receive nothing, or half of the new equity, minus the exercise money. Thus, they will receive nothing or half of the difference between the old equity and half the exercise money. We can look at the warrants as options to buy shares in the equity rather than shares of common stock, at half the stated exercise price rather than at the full exercise price. The value of a share in the equity is defined as the sum of the value of the warrants and the value of the common stock, divided by twice the number of outstanding shares of common stock. If we take this point of view, then we will take v^2 in equation (13) to be the variance rate of the return on the company's equity, rather than the variance rate of the return on the company's common stock.

A similar modification in the parameters of equation (13) can be made if the number of shares of stock outstanding after exercise of the warrants will be other than twice the number of shares outstanding before exercise of the warrants.

Bonds

A debt investment in which an investor loans money to an entity (corporate or governmental) that borrows the funds for a defined period of time at a fixed interest rate. Bonds are used by companies, municipalities, states and U.S. and foreign governments to finance a variety of projects and activities. Few types of bonds which we will be using below:

1. **Pure Discount Bonds**
A bond that is issued for less than its par (or face) value, or a bond currently trading for less than its par value in the secondary market.

2. **Coupon Bonds**
A coupon payment on a bond is a periodic interest payment that the bondholder receives during the time between when the bond is issued and when it matures.

It is not generally realized that corporate liabilities other than warrants may be viewed as options. Consider, for example, a company that has common stock and bonds outstanding and whose only asset is shares of common stock of a second company. Suppose that the bonds are pure discount bonds with no coupon, giving the holder the right to a fixed sum of money, if the corporation can pay it, with a maturity of 10 years. Suppose that the bonds contain no restrictions on the company except a restriction that the company cannot pay any dividends until after the bonds are paid off. Finally, suppose that the company plans to sell all the stock it holds at the end of 10 years, payoff the bond holders if possible, and pay any remaining money to the stockholders as a liquidating dividend.

Under these conditions, it is clear that the stockholders have the equivalent of an option on their company's assets. In effect, the bond holders own the company's assets, but they have given options to the stockholders to buy the assets back. The value of the common stock at the end of 10 years will be the value of the company's assets minus the face value of the bonds, or zero, whichever is greater.

Thus, the value of the common stock will be $f(S,t)$, as given by equation(13), where we take σ^2 to be the variance rate of the return on the shares held by the company, K to be the total face value of the outstanding bonds, and S to be the total value of the shares held by the company. The value of the bonds will simply be $S - f(S,t)$. By subtracting the value of the bonds given by this formula from the value they would have if there were no default risk, we can assure the discount that should be applied to the bonds due to the existence of default risk.

How dividend policy affects bonds and stocks: The corporation's dividend policy will also affect the division of its total value between the bonds and the stock.
To take an extreme example,

Suppose again that the corporation's only assets are the shares of another company, and suppose that it sells all these shares and uses the proceeds to pay a dividend to its common stockholders. Then the value of the firm will go to zero, and the value of the bonds will go to zero. The common stockholders will have stolen the company out from under the bond holders. Even for dividends of modest size, a higher dividend always favors the stockholders at the expense of the bond holders. A liberalization of dividend policy will increase the common stock price and decrease the bond price. Because of this possibility, bond indentures contain restrictions on dividend policy, and the common stockholders have an incentive to pay themselves the largest dividend allowed by the terms of the bond indenture. However, it should be noted that the size of the effect of changing dividend policy will normally be very small.

If the company has coupon bonds rather than pure discount bonds outstanding, then we can view the common stock as a compound option. The common stock is an option on an option on ... an option on the firm. After making the last interest payment, the stockholders have an option to buy the company from the bond holders for the face value of the bonds. Call this option 1. After making the next-to-the-last interest payment, but before making the last interest payment, the stockholders have an option to buy option 1 by making the last interest payment. Call this option 2. Before making the next-to-the-last interest payment, the stockholders have an option to buy option 2 by making that interest payment. This is option 3. The value of the stockholders' claim at any point in time is equal to the value of option n + 1, where n is the number of interest payments remaining in the life of the bond.

Unfortunately, these more complicated options cannot be handled by using the valuation formula (13). The valuation formula assumes that the variance rate of the return on the optioned asset is constant. But the variance of the return on an option is certainly not constant: it depends on the price of the stock and the maturity of the option. Thus the formula cannot be used, even as an approximation, to give the value of an option on an option. It is possible, however, that an analysis in the same spirit as the one that led to equation (13) would allow at least a numerical solution to the valuation of certain more complicated options.

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