Computation of optimal hedging and cash flow in a Markov modulated market

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Notation & Abbreviation

$\langle S, R \rangle$	*: Covariation process of the processes S and R*
$\mathcal{F}$	*: Standard sigma algebra*
$(X_t)_{0 \leq t \leq \infty}$	*: Stochastic process*
X_t	*: Random variable corr. to process at time t*
$\mathbb{F}$	*: Standard filtration generated by a process*
$E[X_t]$	*: Expectation of* X_t
$E[X_t \mid \mathcal{F}_s]$	*: Conditional expectation of* X_t*, given the information till time s*

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Chapter 1

Theory of Pricing Derivatives

1.1 Introduction to Derivatives

1. A *derivative* is a financial instrument, whose value is determined by the performance of an underlying financial entity (equities, commodities).

2. An *option* is a derivative which gives the buyer the *right* but not the *liability* to buy or sell a certain amount of the underlying at a predetermined price (called the *Strike Price*).

3. *Holder* of an option is the person who has bought the option and *writer* of the option is the one who has sold it. A *call option* is one which gives the right to buy and a *put option* is one which gives the right to sell the underlying.

4. The two major types of options available in the market are the American Option and the European Option.

5. *American Option* gives the buyer the right to exercise their option at any date *before* the date of expiration.

6. The *European Option* on the other hand gives the buyer the right to exercise their option *exactly on* the date of expiration.[1]

7. *Claim* is a $\mathcal{F}_T$ measurable random variable. T being the expiration time, and $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$ being the filtration[2] generated[3] by the price process of the underlying. European call option is an example of a claim. It is nothing but the difference between the strike price and the spot price of the underlying at time T, in case the strike price is lower than the terminal price, and zero otherwise. If H is the claim and S_T is the spot price of the underlying at time T and K is the strike price then $H = (S_T - K)^+$.

8. *Portfolio* is the collection of different assets (risk-less as well as risky) owned at a time. It is mathematically written as a vector whose components are the different amounts of the multiple stocks and bonds owned at the time. For simplicity we just take portfolio having a stock component and a bond component.

9. The writer of the option is obligated to provide the holder with the agreed amount of the underlying at the strike price (if the holder desires so), in order to do that, the writer has to *replicate* the *claim* at the expiration date. For that, the writer has to trade in the market in a way, such as to minimise the risk associated with volatility of the price of underlying. A risk minimising strategy consists of a sequence of *hedging portfolios* for each time t till expiration.

[1] In this report we are dealing with the European Call Option and from now on unless otherwise stated *option* refers to this particular variant only.

[2] A.1 on page 17

[3] 4 on page 17

10. *Delta* of an option is the measure of change in option price with respect to the value of underlying asset. Delta is represented as $\frac{\partial \varphi}{\partial S}$ of option value w.r.t the price of the underlying security. *Delta-Neutral* portfolio is the one in which value of the portfolio remains unchanged with small changes in the underlying.

11. *Delta hedging* is a trading strategy of setting and keeping the delta of the portfolio as close to zero as possible. For example buying a stock and placing it in a put option would cancel deltas of the both to some extent, producing a delta neutral portfolio, as risky part of both these investments is similar.

1.2 No Arbitrage Pricing

Arbitrage is a transaction which involves earning risk free profit with zero investment. Mathematically it is represented as :

$$\begin{aligned} &P\{V_t \geq 0\} = 1 && \text{, for all } t \\ &P\{V_T > 0\} > 0 && \text{, for expiration } T \end{aligned}$$

where P is the probability measure and V_t is the value of the trading portfolio at time t and $V_0 = 0$. In modelling a market we assume that arbitrage opportunities do not exist. This is in line with our expectations, as in a real market, no such opportunities can exist for a considerable time, because if such an opportunity does arise in a market, *arbitrageurs*[4] will swoop in, and due to this excessive pressure, the market will rearrange the prices towards greater equilibrium and the arbitrage opportunity will be destroyed. Thus arbitrage opportunities self destruct in a large market.
Whenever we consider any derivative, the assumption of no-arbitrage provides an important condition in construction of fair prices.

1.3 Uncertainty and Incompleteness

In a trivial case of efficient market, where there is no randomness, any amount of asset can be bought or sold. Suppose price is not unique in such a market, for example if 1 asset costs Rs. 10 and 12 assets cost Rs 100, one can buy 12 assets together and sell them one by one to make a profit of Rs. 20 without risk. This is an arbitrage opportunity. Therefore in an arbitrage free efficient market, prices are unique and thus any claim can be attained by selling appropriate number(possibly fractional) of assets.
A market is *complete* if any contingent claim (which is a random variable on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ where $\{\mathcal{F}_t\}$ is the usual filtration generated by the asset price processes) can be generated by a self financing dynamic portfolio strategy based on the underlying assets. Or in other words, a claim H can be represented as a stochastic integral of asset processes. In an incomplete market, every claim in not necessarily attainable by a self financing strategy. Demand and Supply of assets do not always match. As a result there is a price constraint w.r.t the quantity. For example, if 1 asset costs Rs.10, one dozen assets may cost Rs.100 instead of Rs.120. This is not an arbitrage opportunity, as selling a dozen assets at a time reduces the risk of them not getting sold and thus the cost is reduced. Also we may not be able to buy 1000 assets due to non-availability in the market i.e. price would be infinite. Above suggests existence of multiple no arbitrage prices of the same product in an incomplete market which arises due to some unhedgable uncertainty.

1.4 Notion of Cash Flow and Self Financing Strategies

Let S_t be the price of a risky asset and B_t of a risk-less asset. Consider a portfolio consisting ξ_t number of first asset and ε_t number of second asset. Value V_t of the portfolio is:

$$V_t = \xi_t S_t + \varepsilon_t B_t$$

[4]people who search for and exploit arbitrage opportunities

Hence,

$$\begin{aligned} dV_t =&(\xi_t dS_t + \varepsilon_t dB_t) && \text{— term1} \\ &+(S_t d\xi_t + B_t d\varepsilon_t) && \text{— term2} \\ &+(d\langle S,\xi\rangle_t + d\langle B,\varepsilon\rangle_t) && \text{— term3} \end{aligned}$$

The LHS is the infinitesimal change in portfolio. $Term1$ in RHS represents infinitesimal gain from the market. $Term2$ looks like instantaneous cash flow at time t. $Term3$ has no such physical meaning. Change in the portfolio value can be because of the gain from the market or the external cash flow. Therefore $dV_t - term1 = term2 + term3$ should be the external cash flow. $Term2$ represents additional assets bought or sold in dt therefore it can be mistaken to be the additional cash flow. We would see at the end of this section that $term2$ does not actually represent the cash flow.

Illustration using the BSM model

Consider delta hedging (ξ, ε) in BSM market. Therefore $V_t = \varphi(t, S_t), \xi_t = \varphi_s(t, S_t)$ where φ solves BSM PDE with $dS_t = S_t(\mu dt + \sigma dW_t)$ and $dB_t = rB_t dt, \quad S_0 > 0, B_0 = 1$

$$\begin{aligned} d\xi_t =& d\varphi_s(t, S_t) \\ =&\varphi_{ts}(t, S_t)dt + \varphi_{ss}(t, S_t) + \frac{1}{2}\varphi_{sss}(t, S_t)d\langle S\rangle_t \\ =&(\varphi_{ts}(t, S_t) + \mu S_t\varphi_{ss}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \varphi sss(t, S_t))dt + \varphi_{ss}(t, S_t)\sigma S_t dW_t \end{aligned}$$

$$\begin{aligned} \therefore d\langle S,\xi\rangle_t =&(S_t\sigma\varphi_{ss}(t, S_t)\sigma S_t d\langle W\rangle_t) \\ =&\sigma^2 S_t^2\varphi_{ss}(t, S_t)dt. \end{aligned}$$

B_t is of finite variation hence $d\langle B,\varepsilon\rangle_t = 0 \therefore term3 = \sigma^2 S_t^2 \varphi_{ss}(t, S_t)dt$ The coefficient $\sigma^2 S_t^2\varphi_{ss}(t, S_t)$ is strictly positive and maximum near the strike price.
Now we prove that $Term2 + Term3 = 0$ for BSM Model with Delta Hedging.

$$\begin{aligned} dV_t =&\varphi_s(t, S_t)dS_t + \frac{\partial\varphi}{\partial t}dt + \frac{1}{2}\varphi_{ss}(t, S_t)d\langle S\rangle_t \\ =&\varphi_s(t, S_t)dS_t + \left(\frac{\partial\varphi}{\partial t} + \frac{\sigma^2 S_t^2}{2}\varphi_{ss}(t, S_t)\right)dt \\ =&\varphi_s(t, S_t)dS_t + (r\varphi(t, S_t) - rS_t\varphi_s(t, S_t))dt && \text{.... from BS PDE} \\ =&\varphi_s(t, S_t)dS_t + \frac{1}{B_t}(\varphi(t, S_t) - S_t\varphi_s(t, S_t))dB_t \\ =&\varphi_s(t, S_t)dS_t + \varepsilon_t dB_t = term1 \\ dV_t =&term1 \\ i.e. term2 + term3 =&0. \\ \int_0^T (S_t d\xi_t + B_t d\varepsilon_t) =& -\int_0^T term3 \\ =& -\int_0^T \sigma^2 S_t^2\varphi_{ss}(t, S_t)dt \\ <&0. \end{aligned}$$

If the LHS of the above expression represents external cash flow then that is shown to be negative.

Explanation:

Let C_t be the instantaneous cash flow at time t. Then $V_t(= \xi_t S_t + \varepsilon_t B_t)$ can also be written as sum of two quantities, one is the return of the investment performed at an earlier instant $t - \Delta$ and the other one is the cash flow (ΔC_t).

$$\begin{aligned} ie. \quad V_t &= \xi_{t-\Delta} S_t + \varepsilon_{t-\Delta} B_t + \Delta C_t \\ or \quad \Delta C_t &= S_t(\xi_t - \xi_{t-\Delta}) + B_t(\varepsilon_t - \varepsilon_{t-\Delta}) \\ &\neq S_{t-\Delta}(\xi_t - \xi_{t-\Delta}) + B_{t-\Delta}(\varepsilon_t - \varepsilon_{t-\Delta}) \\ &\simeq S_t d\xi_t + B_t d\varepsilon_t = term2. \end{aligned} \tag{1.1}$$

Thus term2 is mistaken as cash flow. The above shows that the cash flow can be written as a stochastic integral resembling to term2. It has the same integrator and integrand but instead of Itô sense, should be defined by taking the right end points. The equation 1.1 leads to the discrete equation

$$V_t - V_{t-\Delta} = \xi_{t-\Delta}(S_t - S_{t-\Delta}) + \varepsilon_{t-\Delta}(B_t - B_{t-\Delta}) + \Delta C_t.$$

Hence $dC_t = term2 + term3$. Thus we arive at the following definition:

Definition. A strategy (ξ, ε) is defined to be self financing if

$$dV_t = \xi_t dS_t + \varepsilon_t dB_t. \tag{1.2}$$

If we further assume that B_t is of finite variation and continuous path, using A.3 we derive

$$\begin{aligned} B_t d\langle S^*, \xi\rangle_t =& d\langle BS^*, \xi\rangle_t + \xi_t d\langle S^*, B\rangle_t - d\langle S^*\xi, B\rangle_t \\ =& d\langle S, \xi\rangle_t. \end{aligned}$$

Hence, *term 2* + *term 3* = 0 implies

$$\begin{aligned} &\cancel{S_t d\xi_t} + B_t(dV_t^* - \xi_t dS_t^* - \cancel{S_t^* d\xi_t} - \cancel{d\langle S^*, \xi\rangle_t}) + \cancel{d\langle S, \xi\rangle_t} = 0 \\ &\therefore \quad dV_t^* = \xi_t dS_t^*. \end{aligned}$$

Further, if we drop the condition of self financing and denote $dC_t =$ *term 2*+*term 3*, then we have

$$dV_t^* - \xi_t dS_t^* = \frac{1}{B_t} dC_t.$$

Chapter 2

Black Scholes Model & its Generalisation

2.1 Preliminaries

Markov Chain

Markov chain $\{X_n\}_{n\geq 0}$ is a stochastic process with the property:

$$P[X_{n+1} = x_{n+1}|X_n = x_n, X_{n-1} = x_{n-1}, ..., X_0 = x_0] = P[X_{n+1} = x_{n+1}|X_n = x_n].$$

This property is called *Markov Property.*

Brownian Motion

$\{X(t), t \geq 0\}$ is a *Brownian motion* with drift coefficient μ :

1. $X(0) = 0$
2. $X(t)$ has stationary independent increments.
3. For every $t > 0, X(t)$ is normally distributed with mean 0 and variance t.

Brownian motion is a Markov Process.

Geometric Brownian Motion

Also known as *Exponential Brownian Motion.* If for $t \geq 0$ X_t is a Brownian Motion then Y_t defined by

$$Y_t = e^{X_t}$$

is a Geometric Brownian Motion. More generally, Y_t could be of the form given by the stochastic differential equation:

$$dY_t = Y_t(\mu dt + \sigma dX_t)$$

where $Y_0 > 0$. This SDE has a strong solution given by

$$Y_t = Y_0 \exp^{[(\mu - \frac{1}{2}\sigma^2)t + \sigma X_t]}.$$

There are a large number of independent stock traders in the market. As a result it is reasonable to assume stock prices to be random. In addition, there is a rate with which these prices increase, which depends on the value of the enterprise etc.[1] Thus stock prices can be approximated to be in Geometric Brownian Motion with drift μ and volatility σ parameters.

[1] See for example Samuelson's paper

2.2 BSM model

Description

Black Scholes Merton model of a financial market consists of risky S_t (equities or commodities) as well as risk-free assets B_t (bonds, fixed deposit etc.). The explicit assumptions involved in this model are, non existence of arbitrage opportunity, existence of a money market with risk free borrowing/lending interest rates, transactions do-not incur any fee (frictionless transactions), underlying securities do not pay any dividends and most importantly asset prices follow geometric Brownian motion with constant drift and volatility[2]. Indeed the market can be described by the following SDEs $dS_t = S_t(\mu dt + \sigma dW_t)$ and $dB_t = rB_t dt, \quad S_0 > 0, B_0 = 1$.

BSM PDE

it turns out that the price of a European call option in the BSM market model can be written as a function of time and the stock price which solves a Cauchy problem which is shown below.

$$\frac{\partial \eta}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \eta}{\partial s^2} + rs\frac{\partial \eta}{\partial s} - r\eta = 0$$

with

$$\eta(T, s) = (s - K)^+, \quad \eta(t, 0) = 0$$

where K is the strike price and T is the maturity time. Option price process is given by $\eta_t = \eta(t, S_t)$ where S_t is the stock price process which is GBM. The price $\eta(t, S_t)$ at time $t \in [0, T]$ is given by Black & Scholes [1973]

$$\eta(t, S_t) = S_t\Phi\left(\frac{\log(\frac{S_t}{K}) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) - e^{rt}K^*\Phi\left(\frac{\log(\frac{S_t}{K}) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right) \tag{2.1}$$

where r and σ are constants denoting fixed bank rate and fixed volatility coefficients respectively; $\Phi(x)$ is the CDF of standard normal distribution, $K^* = e^{-rT}K$. The Black-Scholes hedging strategy, called Delta hedging is given by

$$\Delta(t, s) = \frac{\partial \eta(t, s)}{\partial s}$$

where $\Delta(t, s)$ is the number of shares invested in stock.

2.3 Markov modulated GBM market model

Let $(\Omega, \mathcal{F}, P)$ be the underlying complete probability space. Let $\chi = \{1, 2, \dots, k\}$ be the state space of an irreducible Markov chain $\{X_t, t \geq 0\}$ with transition rule

$$P(X_{t+\delta t} = j | X_t = i) = \lambda_{ij}\delta t + o(\delta t), \quad i \neq j$$

where $\lambda_{ij} \geq 0$ for $i \neq j$; and $\lambda_{ii} = -\sum_{j\neq i}^{k} \lambda_{ij}$. Thus $\Lambda = [\lambda_{ij}]$ denotes the generating Q-matrix of the chain and $p_{ij} := \frac{\lambda_{ij}}{|\lambda_{ii}|}$ are the transition probabilities from state i to state j. We consider a market where the financial parameters, namely interest rate, drift coefficient, volatility coefficient are functions of the observed Markov chain X_t. Let $\{B_t, t \geq 0\}$ be the price of money market account at time t where, spot interest rate is $r(X_t)$ and $B_0 = 1$. We have

$$B_t = e^{\int_0^t r(X_u)du}.$$

We consider a market consisting only one stock as tradeable risky asset. The stock price process S_t solves

$$dS_t = S_t(\mu(X_{t-})dt + \sigma(X_{t-})dW_t), \ S_0 > 0 \tag{2.2}$$

[2] another assumption is that it is possible to buy/sell *any* amount of the assets (thus short selling is allowed).

where $\{W_t, t \geq 0\}$ is a standard Wiener process independent of $\{X_t, t \geq 0\}$. Let $\mathcal{F}_t$ be a filtration of $\mathcal{F}$ satisfying usual hypothesis and right continuous version of the filtration generated by X_t and S_t. Clearly the solution of above SDE is an $\mathcal{F}_t$ semimartingale with almost sure continuous paths. To price a claim H of European type in the above incomplete market, we would consider the locally risk minimizing pricing approach by Föllmer and Schweizer (see Föllmer and Schweizer [1991]). A hedging strategy is defined as a predictable process $\pi = \{\pi_t = (\xi_t, \varepsilon_t), 0 \leq t \leq T\}$ which satisfies

$$E\left[\int_0^T \xi_t^2 \sigma^2(X_t) S_t^2 dt + \left(\int_0^T |\xi_t||\mu(X_t)|dt\right)^2\right] < \infty \tag{2.3}$$

and $$E[\varepsilon_t^2] < \infty.$$

The components ξ_t and ε_t denote the amounts invested in S_t and B_t respectively at time t. An optimal strategy is the one for which the quadratic residual risk is minimized subject to a certain constraint(see Föllmer and Schweizer [1991] for details). It is shown in Föllmer and Schweizer [1991] that the existence of an optimal strategy for hedging an $\mathcal{F}_T$ measurable claim H is equivalent to the existence of Föllmer Schweizer decomposition of discounted claim $H^* := B_T^{-1}H$ in the form

$$H^* = H_0 + \int_0^T \xi_u^{H^*} dS_u^* + L_T^{H^*} \tag{2.4}$$

where $H_0 \in L^2(\Omega, \mathcal{F}_0, P), L^{H^*} = \{L_t^{H^*}\}_{0 \leq t \leq T}$ is a square integrable martingale orthogonal to the martingale part of S_t, $S_t^* := B_t^{-1}S_t$, and $\xi^{H^*} = \{\xi_t^{H^*}\}$ satisfies (2.3). Further, ξ^{H^*} appeared in the decomposition constitutes the optimal strategy. Indeed the optimal strategy $\pi = (\xi_t, \varepsilon_t)$ is given by

$$\begin{aligned}
\xi_t &:= \xi_t^{H^*} \\
V_t^* &:= H_0 + \int_0^t \xi_u^{H^*} dS_u^* + L_t^{H^*} \\
\varepsilon_t &:= V_t^* - \xi_t S_t^*
\end{aligned}$$

and $B_t V_t^*$ represents the locally risk minimizing price at t of the claim H. Hence the Föllmer Schweizer decomposition is the key thing to verify.

Now onward we consider a particular claim i.e., a European call option on $\{S_t\}$ with strike price K and maturity time T. In this case the $\mathcal{F}_T$ measurable contingent claim H is given by

$$H = (S_T - K)^+. \tag{2.5}$$

Before stating price and hedging equations (see details in Goswami & Saini [2013]) in the above model we consider the following system of partial differential equations

$$\frac{\partial \varphi(t,s,i)}{\partial t} + \frac{1}{2}\sigma(i)^2 s^2 \frac{\partial^2 \varphi(t,s,i)}{\partial s^2} + r(i)s\frac{\partial \varphi(t,s,i)}{\partial s} + \sum_{j=1}^{k} \lambda_{ij}\varphi(t,s,j) = r(i)\varphi(t,s,i) \tag{2.6}$$

for $t < T$, $s > 0$ and $i = 1, 2 \ldots, k$ with the boundary condition

$$\varphi(T,s,i) = (s-K)^+ \; s \geq 0, \qquad \varphi(t,0,i) = 0 \;\forall t \in [0,T], i \in \chi \tag{2.7}$$

where φ is of polynomial growth. Note that if Λ is a null matrix i.e., the case when the Markov chain X_t does not transit almost surely, the equation (2.6) coincides with that of standard B-S-M model. In view of this, the above system can be considered as a generalization of Black-Scholes equation for a Markov modulated market where the extra coupling term represents the correction term arising due to the regime switching. Nevertheless, the fact, the solution of above problem gives locally risk minimizing price, needs a proof. This is given below in Theorem 2.3.1 which appears in Basak etal [2011]. In order to state the theorem certain terminologies should be defined.

It would be convenient to represent the Markov chain $\{X_t\}$ as a stochastic integral with respect to a Poisson random measure which would play an important role later.

For a Polish space $\mathcal{S}$, let $\mathcal{B}(\mathcal{S})$ denote its Borel σ-field and $\mathcal{M}(\mathcal{S})$ the set of all nonnegative integer valued σ-finite measures on $\mathcal{B}(\mathcal{S})$. Let $\mathcal{M}_\sigma(\mathcal{S})$ be the smallest σ-field on $\mathcal{M}(\mathcal{S})$ with respect to which the maps $\alpha_B : \mathcal{M}(\mathcal{S}) \to \mathbb{N}\bigcup\{\infty\}$ defined by $\alpha_B(\mu) := \mu(B)$ are measurable for all $B \in \mathcal{B}(\mathcal{S})$; $\mathcal{M}(\mathcal{S})$ is assumed to be endowed with the σ-field $\mathcal{M}_\sigma(\mathcal{S})$.
For $i \neq j \in \mathcal{X}$, let Λ_{ij} be consecutive (with respect to the lexicographic ordering on $\mathcal{X} \times \mathcal{X}$) left closed right open intervals of the real line, each having length λ_{ij}. By embedding $\mathcal{X}$ in $\mathbb{R}^k$ by identifying i with $e_i \in \mathbb{R}^k$ define a function $h : \mathcal{X} \times \mathbb{R} \to \mathbb{R}^k$ by

$$h(i, z) \quad := \quad \begin{cases} j - i & \text{if } z \in \Lambda_{ij} \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Then

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} h(X_{u-}, z)\wp(du, dz) \tag{2.9}$$

where the integration is over the interval $(0, t]$ and $\wp(dt, dz)$ is an $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R})$ -valued Poisson random measure with intensity $dtdz$; $\wp(dt, dz), X_0, W$ and S_0, defined on $(\Omega, \mathcal{F}, P)$ are independent.
$\hat{\wp}(dt, dz) := \wp(dt, dz) - dtdz$ is the compensated Poisson random measure.

Theorem 2.3.1. *Let $\{\varphi(t, s, i),\ i = 1, 2, \ldots, k\}$ denote the unique solution of the Cauchy problem (2.6), (2.7) in $C([0, T] \times \bar{\mathbb{R}} \times \chi) \cap C^{1,2}((0, T) \times \mathbb{R}_+, \chi)$. Then*
(i) $\varphi(t, S_t, X_t)$ is the risk minimizing option price at time t;
(ii) An optimal strategy $\pi^* = \{\xi_t^*, \varepsilon_t^*\}$ is given by

$$\xi_t^* = \frac{\partial\varphi(t, S_t, X_{t-})}{\partial s} \tag{2.10}$$

$$\varepsilon_t^* = V_t^* - \xi_t^* S_t^* \tag{2.11}$$

where

$$\begin{aligned} V_t^* \quad &= \quad \varphi(0, X_0, S_0) + \int_0^t \frac{\partial\varphi(u, S_u, X_{u-})}{\partial s} dS_u^* \\ &+ \int_0^t e^{-\int_0^u r(X_v)dv} \int_{\mathbb{R}} [\varphi(u, S_u, X_{u-} + h(X_{u-}, z)) - \varphi(u, S_u, X_{u-})]\hat{\wp}(du, dz). \end{aligned} \tag{2.12}$$

Proof. Let $0 \leq t \leq T$. By applying Ito's formula to $e^{-\int_0^t r(X_u)du}\varphi(t, S_t, X_t)$ under the measure P and using (2.6), (2.2) and (2.11), we obtain after suitable rearrangement of terms

$$\begin{aligned} e^{-\int_0^t r(X_u)du}\varphi(t, S_t, X_t) \quad &= \quad \varphi(0, S_0, X_0) + \int_0^t \frac{\partial\varphi(u, S_u, X_{u-})}{\partial s} dS_u^* + \int_0^t e^{-\int_0^u r(X_v)dv} \\ &\int_{\mathbb{R}} [\varphi(u, S_u, X_{u-} + h(X_{u-}, z)) - \varphi(u, S_u, X_{u-})]\hat{\wp}(du, dz). \end{aligned} \tag{2.13}$$

Letting $t \uparrow T$, we obtain

$$\begin{aligned} B_T^{-1}(S_T - K)^+ \quad &= \quad \varphi(0, S_0, X_0) + \int_0^T \frac{\partial\varphi(u, S_u, X_{u-})}{\partial s} dS_u^* + \int_0^T e^{-\int_0^u r(X_v)dv} \\ &\int_{\mathbb{R}} [\varphi(u, S_u, X_{u-} + h(X_{u-}, z)) - \varphi(u, S_u, X_{u-})]\hat{\wp}(du, dz). \end{aligned} \tag{2.14}$$

Since we already know that $B_T^{-1}(S_T - K)^+$ admits a *Föllmer-Schweizer decomposition* (2.4), we can argue (as in Föllmer and Schweizer [1991] Theorem 3.14) to conclude (i) and (ii) using the decomposition in (2.14). □
It is important to note that the Theorem 2.3.1 assumes existence and uniqueness of solution (2.6) and (2.7) which should be proved. The following theorem settles that along with some more interesting results.

Theorem 2.3.2. *(i) The following integral equation has a unique solution in the class of functions belonging to* $C([0,T]\times\overline{\mathbb{R}_+}\times\chi)\bigcap C^{1,2}((0,T)\times\mathbb{R}_+\times\chi)$

$$\varphi(t,s,i) = e^{-\lambda_i(T-t)}\eta_i(t,s) + \int_0^{T-t}\lambda_i e^{-(\lambda_i+r(i))v} \times\sum_j p_{ij}\int_0^\infty \varphi(t+v,x,j)\frac{e^{-\frac{1}{2}\left(\left(\ln(\frac{x}{s})-(r(i)-\frac{\sigma^2(i)}{2})v\right)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(i)\sqrt{v}x}dxdv \quad (2.15)$$

$$\text{with}\quad \varphi(T,s,i) = (s-K)^+,\qquad \varphi(t,0,i)=0\ \forall t\in[0,T], i\in\chi \quad (2.16)$$

where $\eta_i(t,s)$ *is the standard Black-Scholes price of European call option with fixed interest rate* $r(i)$ *and volatility* $\sigma(i)$.
(ii) Moreover, the solution $\varphi(t,s,i)$ *of* (2.15) *and* (2.16) *is the locally risk minimizing price of* H *(as in* (2.5)*) at time* t *with* $S_t=s, X_t=i$.

(iii) The cauchy problem 2.6 *and* 2.7 *has unique classical solution.*

The proof of this theorem and the following result can be found from Goswami & Saini [2013].

Optimal Hedging Strategy

Consider a function $\psi\in C([0,T]\times\overline{R_+}\times\chi)\bigcap C^{1,1}((0,T)\times R_+\times\chi)$ given in terms of φ, the unique solution of (2.15)-(2.16), by

$$\psi(t,s,i) = e^{-\lambda_i(T-t)}\frac{\partial\eta_i(t,s)}{\partial s} + \int_0^{T-t}\lambda_i e^{-(\lambda_i+r(i))v}\sum_j p_{ij}\int_0^\infty\varphi(t+v,x,j) \times\frac{e^{-\frac{1}{2}\left(\left(\ln(\frac{x}{s})-(r(i)-\frac{\sigma^2(i)}{2})v\right)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(i)^3v^{3/2}xs}\left(\ln\left(\frac{x}{s}\right)-\left(r(i)-\frac{\sigma^2(i)}{2}\right)v\right)dxdv \quad (2.17)$$

$$\text{for}\quad t\in[0,T),\quad s>0$$

$$\text{and}\quad \psi(T,s,i) = 1_{(K,\infty)}(s),\ \forall s\geq 0;\qquad \psi(t,0,i)=0\ \forall t\in[0,T], i\in\chi. \quad (2.18)$$

The processes $\xi_t := \psi(t,S_t,X_{t-})$ and $\varepsilon_t := B_t^{-1}(\varphi(t,S_t,X_{t-})-\xi_tS_t)$ comprise the optimal hedging strategy for the claim H in (2.5).

2.4 Simulation of the Market

In our model, market is assumed to switch between 3 *regimes*, this regime switching is modelled by an irreducible Markov chain, $(\Omega,\mathcal{F},P)$ being the complete probability space and $\chi=\{1,2,...,k\}$ the state space for the regimes and the transition rule being:

$$P(X_{t+\delta t}=j|X_t=i)=\lambda_{ij}\delta t+o(\delta t),\quad i\neq j.$$

As δt is a grid size, usually small, probability that the market shifts regime is very small.
We have simulated the regime switching according to this model. Using the simulated regime information, stock prices are simulated with Geometric Brownian Motion with predefined drift, volatility and risk free rate parameters for each regime. These simulated values are taken to represent the real market situation and can be used to verify accuracy and efficiency of hedging.

Simulation Parameters:

- Strike price $K=1.0$

- Expiration time $T = 1$ with discretisation[3] : 751 grid points
- Three regime model with

$$(r, \mu, \sigma) = \begin{cases} (0.2, 0.2, 0.2) & \text{in regime 1} \\ (0.5, 0.6, 0.4) & \text{in regime 2} \\ (0.7, 0.8, 0.3) & \text{in regime 3} \end{cases}$$

- The transition rate matrix is

$$\Lambda = \begin{pmatrix} -1 & \frac{2}{3} & \frac{1}{3} \\ 1 & -2 & 1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{pmatrix}$$

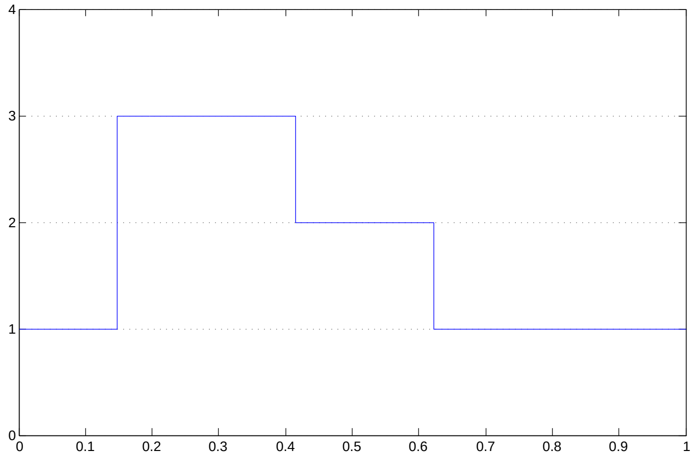

Figure 2.1: Single realisation of the regime switching Markov process generated by `"mark1.f"`

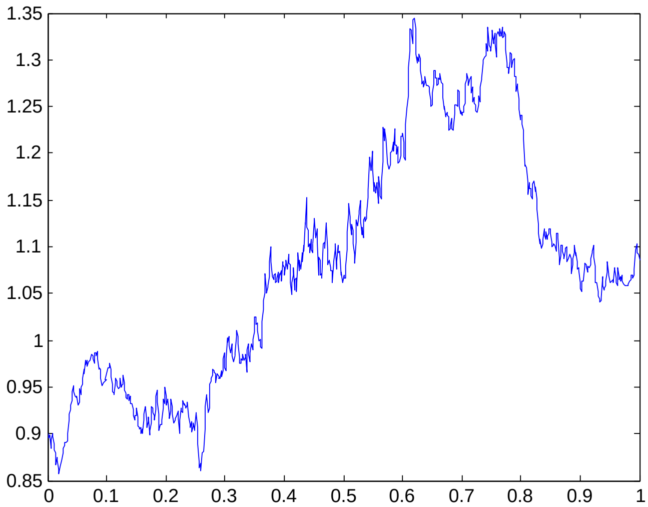

Figure 2.2: Single realisation of the Stock price process generated by `"gbm1.f"`

[3] for simulation of stock price the time grid is of 750 partitions, for calculating the raw array of option price, 15 time grid partitions are there, but for each simulation, the option price process is defined on a time grid of 750 using linear interpolation

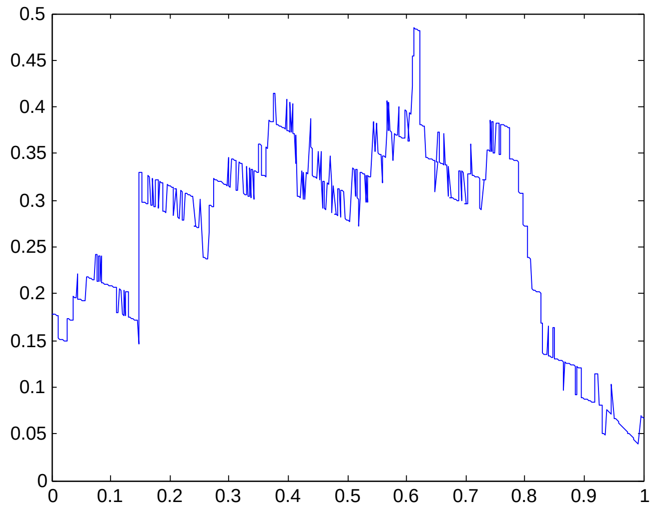

Figure 2.3: One realisation of the Option price process generated by `"calculations.f"`

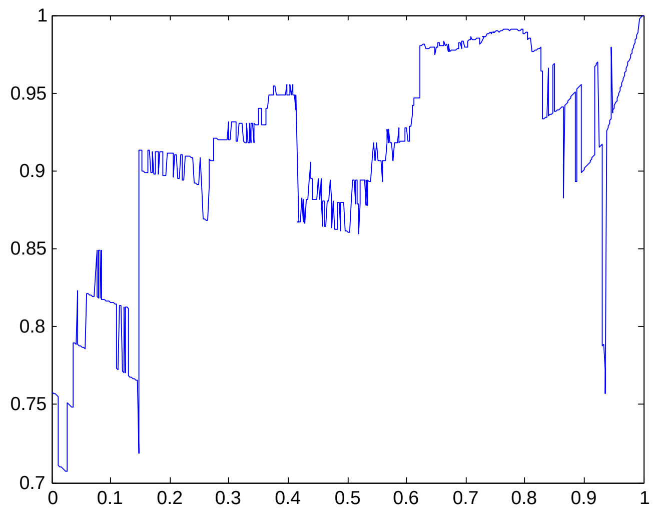

Figure 2.4: One realisation of the hedging process generated by `"calculations.f"`

2.5 Comments on Computation

Computation of prices by PDE methods has $o(NM^3)$ time complexity compared to $o(N^2M^2)$ of integral equation method. Thus for smaller step in terms of stock price, integral equation method is more efficient, but in case of decrease in time step its efficiency decreases.
For solution of PDE while using Crank-Nicholson method, boundary conditions have to be specified at both $S_t = 0$ and $S_t = S_{max}$ for some S_{max}. Therefore option price at S_{max} becomes determined and the option price graph w.r.t t and S converges to this value, inducing termination error. This can be avoided in integral equation method.
For simulation of stock price, required normal distribution is generated from uniform random number generator using Box-Muller transform. It is a close approximation to real normal distribution.

Chapter 3

Study of Cash Flow in MMGBM Model

In this Chapter we study various measures of risk in terms of cash flow. Corresponding to every such measure we enclose a numerical example to illustrate its magnitude and behavior. In order to compute the expectation numerically the market is simulated 10^4 times for each initial regime $i = 1, 2, 3$ and each initial stock price varies between 0 to 1.5 with discretisation : 50 equally-spaced grid-points.

3.1 Quadratic Residual Risk

Let H be a claim at time T in a Markov modulated market. Assume that φ denotes the locally risk minimising price function of the contract or in other words, the price at time t is $\varphi(t, S_t, X_t)$ where S_t is the stock price and X_t is the state of the Markov chain at instance t. Let $\{C_t\}_t$ denote the external cash flow process associated with the optimal hedging π, say. Then we know that the discounted value at $t = 0$ of accumulated cash flow during $[0, T]$ is given by

$$C_T^* = C_0^* + \int_0^T \frac{1}{B_t} dC_t. \tag{3.1}$$

Further, from the F-S decomposition (2.14) we have obtained

$$L_T^{H^*} = \int_0^T \frac{1}{B_t} \int_{\mathbb{R}} (\varphi(t, S_t, X_{t-} + h(X_{t-}, z)) - \varphi(t, S_t, X_{t-}))\hat{\wp}(dt, dz). \tag{3.2}$$

In view of the last equation of Chapter 1 we get the relation $L_T^{H^*} = C_T^* - C_0^*$. Hence we write

$$\begin{aligned} C_T &= C_0 + \int_0^T \int_{\mathbb{R}} (\varphi(t, S_t, X_{t-} + h(X_{t-}, z)) - \varphi(t, S_t, X_{t-}))\hat{\wp}(dt, dz) \\ &= C_0 + \sum_{t\in[0,T]} (\varphi(t, S_t, X_t) - \varphi(t, S_t, X_{t-})) - \int_0^T \sum_j \lambda_{X_{t-}j} (\varphi(t, S_t, j) - \varphi(t, S_t, X_{t-}))dt. \end{aligned} \tag{3.3}$$

using $\hat{\wp}(dt, dz) = \wp(dt, dz) - dtdz$. Hence, C_t is an RCLL process. For small Δ

$$\begin{aligned} (C_r - C_{r-\Delta})^2 &= \Big(\varphi(r, S_r, X_r) - \varphi(r, S_r, X_{r-\Delta})\Big)^2 - 2\Big(\varphi(r, S_r, X_r) - \varphi(r, S_r, X_{r-\Delta})\Big) \sum_j \lambda_{X_{r-\Delta}j} \\ &\quad \times\Big(\varphi(r, S_r, j) - \varphi(r, S_r, X_{r-\Delta})\Big)\Delta + \left(\sum_j \lambda_{X_{r-\Delta}j}\Big(\varphi(r, S_r, j) - \varphi(r, S_r, X_{r-\Delta})\Big)\right)^2 \Delta^2. \end{aligned} \tag{3.4}$$

We recall that the quadratic variation process $[C]_t$ of C_t is obtained by summing the terms as in LHS over a partition of $[0, t]$ with $\Delta \to 0$. Hence the Δ^2 terms as in the third term of RHS adds up to negligible. In the second term the coefficient of Δ converges to zero function except finitely many values of r for almost every sample path. Hence the only significant term is the first one. Hence,

$$[C]_T = \sum_{t\in[0,T]} (\varphi(t, S_t, X_t) - \varphi(t, S_t, X_{t-}))^2. \tag{3.5}$$

The quadratic residual risk is given by (Using Itô's Isometry)

$$\begin{aligned}
R_0(\pi) &= E[(C_T^* - C_0^*)^2] \\
&= E\left[\left(\int_0^T \frac{1}{B_t} dC_t\right)^2\right] \\
&= E\left[\int_0^T \frac{1}{{B_t}^2} d[C]_t\right] \\
&= E\left[\sum_{t\in[0,T]} \frac{1}{{B_t}^2}(\varphi(t, S_t, X_t) - \varphi(t, S_t, X_{t-}))^2\right] \\
&= E\left[\sum_{t\in[0,T]} (\varphi^*(t, S_t, X_t) - \varphi^*(t, S_t, X_{t-}))^2\right] \\
&= E\left[\sum_{n=1}^{n(T)} (\varphi^*(T_n, S_{T_n}, X_{T_n}) - \varphi^*(T_n, S_{T_n}, X_{T_{n-1}}))^2\right].
\end{aligned} \tag{3.6}$$

Plots & Results

R_0 vs S_0 is plotted. This plot also appears in the paper [Basak etal , 2011] for the same example but using a different approach. Needless to mention that both the curves match to each other to a great extent.

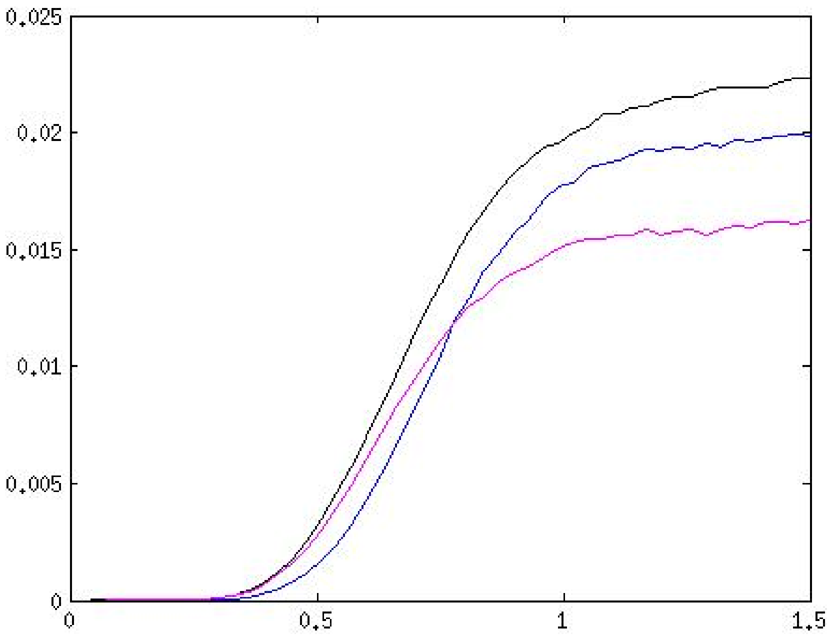

Figure 3.1: **Risk vs Initial Stock Price** Ini.R-1: Blue, Ini.R-2: Black, Ini.R-3: Magenta.

3.2 Positive Residual Risk

Positive residual risk is defined as:

$$R_0^+(\pi) = E[((C_T^* - C_0^*)^+)^2]. \tag{3.7}$$

This is always less than quadratic residual risk as it is zero when $C_T^* < C_0^*$. This functional measures the amount of cash inflow unlike the QRR which gives a measure of cash flow in both the directions. From (3.3) we further deduce

$$\begin{aligned}
C_T^* - C_0^* &= \sum_{t\in[0,T]} (\varphi^*(t,S_t,X_t) - \varphi^*(t,S_t,X_{t-})) - \int_0^T \sum_j \lambda_{X_{t-}j}(\varphi^*(t,S_t,j) - \varphi^*(t,S_t,X_{t-}))dt. \\
&= \sum_{n=1}^{n(T)} \Big\{ \varphi^*(T_n, S_{T_n}, X_{T_n}) - \varphi^*(T_n, S_{T_n}, X_{T_{n-1}}) \\
&- \sum_j \lambda_{X_{T_{n-1}}j} \int_{T_{n-1}}^{T_n} [\varphi^*(t,S_t,j) - \varphi^*(t,S_t,X_{T_{n-1}})]dt \Big\} \\
&- \sum_j \lambda_{X_{T_n}j} \int_{T_n(T)}^{T} [\varphi^*(t,S_t,j) - \varphi^*(t,S_t,X_{T_{n-1}})]dt.
\end{aligned} \tag{3.8}$$

Plots & Results

R_0^+ vs S_0 is a smooth curve having increasing risk with initial stock price.

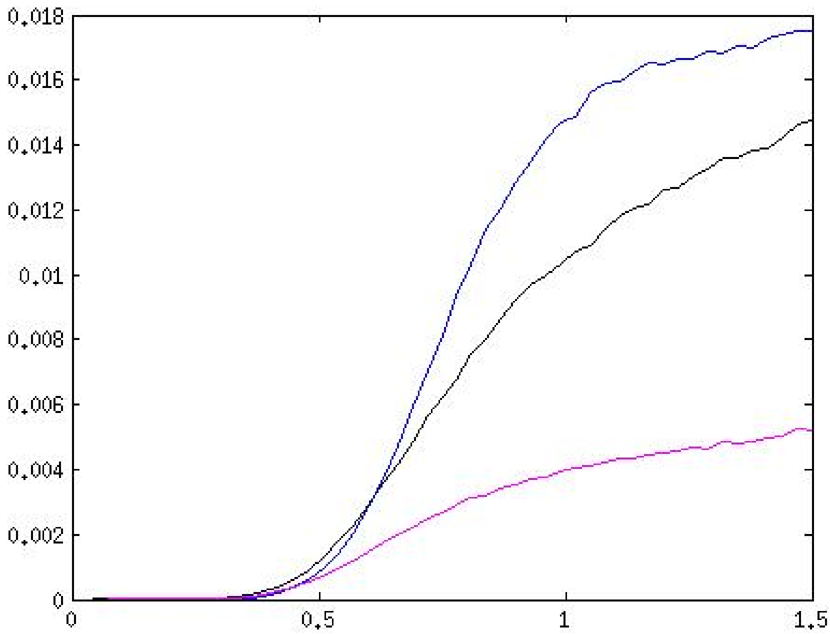

Figure 3.2: **Positive residual risk vs Initial Stock Price** Ini.R-1: Blue, Ini.R-2: Black, Ini.R-3: Magenta.

3.3 Alternative Approach (Practitioner's Approach)

We would now obtain an alternative expression of $C_T^* - C_0^*$ than that in (3.8). From the last equation of Chapter 1 we directly get

$$V_T^* = V_0^* + \int_0^T \xi_t dS_t^* + C_T^* - C_0^* \tag{3.9}$$

where V_0^* is the same as V_0 and represents the price of the option at $t = 0$. Alternatively, from 3.9 we get,

$$C_T^* - C_0^* = V_T^* - V_0 - \int_0^T \xi_t dS_t^*. \tag{3.10}$$

A practitioner trades assets at discrete time intervals. Here we assume that the trader follows the optimal hedging suggestion obtained from the continuous time model. Thus the observed cash flow for discrete trading

$$C_T^* - C_0^* = V_T^* - G_T^*$$

where

$$G_T^* = V_0 + \sum_{i=1}^{N} \xi_{t_i} \Delta S_{t_i}^*.$$

We call this variant as the *practitioner's measure* (PM) of cash flow.

Quadratic Residual Risk

Thus the PM of quadratic residual risk can be obtained as

$$PM(QRR) = E[(V_T^* - G_T^*)^2]. \tag{3.11}$$

Plots & Results

PM of QRR is plotted below.

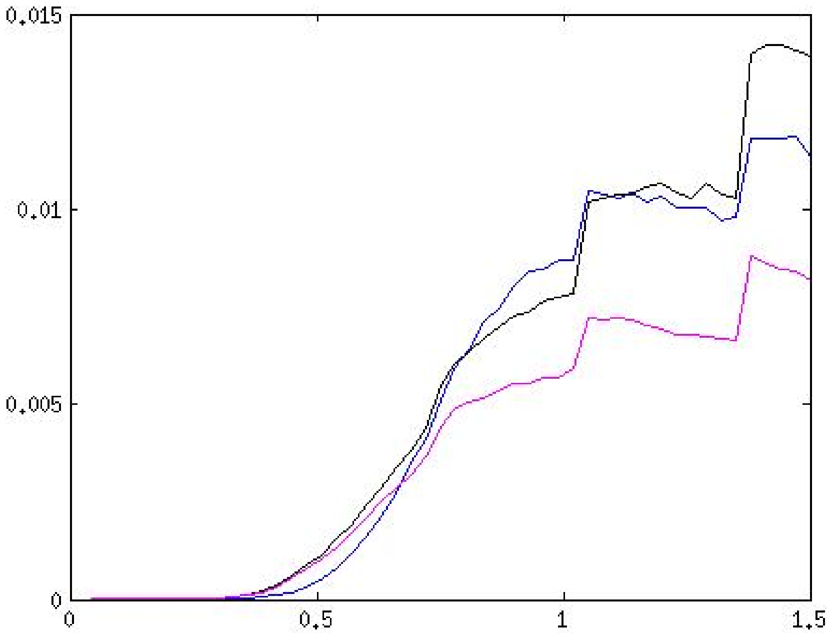

Figure 3.3: **Quadratic residual risk vs Initial Stock Price** Ini.R-1: Blue, Ini.R-2: Black, Ini.R-3: Magenta.

PM of PRR

$$PM(PRR) = E[((V_T^* - G_T^*)^+)^2]$$

Plots & results

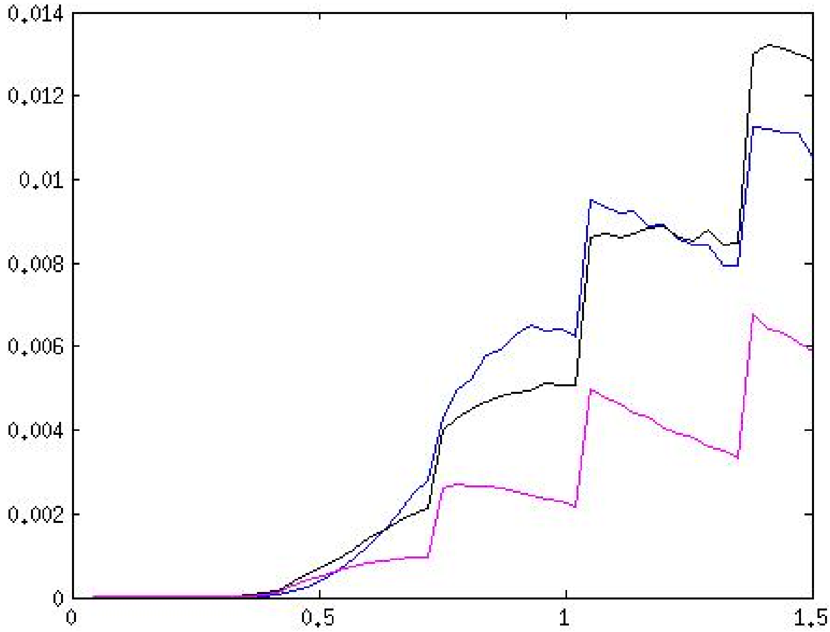

Figure 3.4: **Squared positive residual risk vs Initial Stock Price** Ini.R-1: Blue, Ini.R-2: Black, Ini.R-3: Magenta.

3.4 Comments on Computation

The graphs of QRR and PRR in practitioner's approach have sudden jumps at particular places for which we could not come up with an appropriate explanation.

Appendix A

Basics of Stochastic Calculus

A.1 Definitions & Equations

[1] We assume the existence of a complete measure space $(\Omega, \mathcal{F}, P)$ and a *filtration* $(\mathcal{F}_t)_{0\le t\le\infty}$. By a filtration we mean a family of σ-algebras $(\mathcal{F}_t)_{0\le t\le\infty}$ that is increasing, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \le t$. We denote $(\mathcal{F}_t)_{0\le t\le\infty}$ by $\mathbb{F}$.

Definition 1. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is said to satisfy the *usual hypothesis* if
(i) $\mathcal{F}_0$ contains all the P null sets of $\mathcal{F}$
(ii) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, all t, $0 \le t < \infty$; that is the filtration $\mathbb{F}$ is *right continuous.*

Definition 2. A random variable $T : \Omega \to [0,\infty]$ is a *stopping time* if the event $\{T \le t\} \in \mathcal{F}_t$, for every t, $0 \le t \le \infty$.

Definition 3. A *Stochastic Process* X on $(\Omega, \mathcal{F}, P)$ is a collection of $\mathbb{R}$ valued or $\mathbb{R}^d$ valued random variables $(X_t)_{0\le t<\infty}$. The process is said to be *adapted* if X_t is $\mathcal{F}_t$ measurable for each t.

Definition 4. A filtration $\mathbb{F}$ is called *generated* by a stochastic process $X = (X_t)$ if it is the smallest filtration to which the process is adapted. That is $\mathbb{F} = \bigcap\{\mathcal{F}_t\}_{0\le t\le\infty}$ where the intersection is over all possible filtrations to which X is adapted.

Definition 5. Two stochastic processes X and Y are said to be *modifications* if $X_t = Y_t$ a.s., each t. Two processes X and Y are said to be *indistinguishable* if a.s., for all t, $X_t = Y_t$. The functions $t \mapsto X_t(\omega)$ mapping $[0,\infty)$ into $\mathbb{R}$ are called *sample paths* of he stochastic process X.

Definition 6. A stochastic process is called *càdlàg* if it a.s. has sample paths which are right continuous, with left limits.

Definition 7. Two probability measures P and P' defined on same measurable space $(\Omega, \mathcal{F})$ are *equivalent* if

$$P(\omega) > 0 \Longleftrightarrow P'(\omega) > 0, \quad \forall \omega \in \Omega$$

Definition 8. An adapted stochastic process $S = \{S_t\}$ is *martingale* if $E[S(t+s)|\mathcal{F}_t] = S_t, \quad \forall t \ge 0$ and $s \ge 0$.

Itô differentiation chain rule

$$\begin{aligned} d(XY) &= XdY + YdX + d\langle X, Y\rangle \\ d(\underline{XY}Z) &= XYdZ + Zd(XY) + d\langle XY, Z\rangle \\ &= XYdZ + ZYdX + ZXdY + Zd\langle X, Y\rangle + d\langle XY, Z\rangle \end{aligned} \tag{A.1}$$

$$\begin{aligned} d(X\underline{YZ}) &= XdYZ + YZd(X) + d\langle X, YZ\rangle \\ &= XZdY + XYdZ + Xd\langle Y, Z\rangle + YZdX + d\langle X, YZ\rangle \end{aligned} \tag{A.2}$$

[1]The material for this section has been taken from [Phillip E. Protter , 2005]

From (A.1) and (A.2) we get,

$$Zd\langle X,Y\rangle + d\langle XY,Z\rangle = Xd\langle Y,Z\rangle + d\langle X,YZ\rangle. \tag{A.3}$$

Appendix B

Code

B.1 Overview

No.	File Name	function	Remarks	Result
1	combine.m	MATLAB program that calculates option price and hedging as a function of time and stock price.	Integral Equation Method. Numerical solution by trapezoidal rule. No. of time grids = 15	Arrays of Black-Scholes price, Option price as a solution of the integral equations and hedging as a function of time, stock price and regime.
2	iterate.f	Main program runs functions for large number of iterations.	Main program. All subroutines bellow called in this. No. of time grids = 750 in all these.	Final results
3	mark1.f	Simulation of regime switching	Discrete Markov chain. Series of Bernoulli trials used.	Array of regime at each time step.
4	gbm1.f	Simulation of stock price	Normal distribution calculated by Box-Muller algorithm.	Array of stock price at each time step
5	calculations.f	Calculates option prices and hedging for each time step. Takes simulated Markov chain of regimes and stock price as input.	Linear interpolation w.r.t. time and nearest available grid w.r.t. stock price used in-order to get approximate values from discrete grids .	Option price process, hedging process and Black-Scholes price process array w.r.t time step.
6	get.f	Extracts option price and hedging values functions from spreadsheets which are generated by combine.m .	standard	Arrays of option prices(Volterra and Black-Scholes) and hedging as a function of time, stock price and regime.
7	holding.f	Calculates holding time from the simulated regime switching data.	Calculated using discrete markov chain generated by mark1.	Array which stores time instants at which regime is changed and number of these changes.
8	error.f	Calculates QRR and PRR using two different approaches.	standard	Risk value for a particular simulation.

Table B.1: Table of files of error calculation program

B.2 MATLAB code used to solve the IE : ‘combine.m’

```
clear
tstart=tic;
d = 1.0/(sqrt(2*pi));
f=1/sqrt(2);
T =1;    %       MATURITY
N=16;
dt=T/(N-1);
    st= 1.0; % STRIKE PRICE;
     X0=0; % LOWER LIMIT OF S
M=300;
eta= zeros(N,M,3);
u=zeros(N,M,3);
C11=zeros(N,3);
C12=zeros(1,3);
de= zeros(N,M,3);
du=zeros(N,M,3);
C21=zeros(N,3);
C22=zeros(1,3);
C3=zeros(N,3);
LN1=zeros(N,M,M,3);
LN2=zeros(N,M,M,3);
xint=zeros(N,M,3);
dx= (10*st)/M;
    % n =N*3*M;   %         Matrix dimention
     P= [0,2/3,1/3;0.5,0,0.5;1/3,2/3,0];
     lambda=[1,2,1];
     mu= [0.2 0.6 0.8];
     R=[0.2,0.5,0.7];
     SIG=[0.2,0.4,0.3];
%Black-Schole-Merton Solution
% GENERATE STANDARD NORMAL DISTRIBUTION FUNCTION
SND= 0.5+ 0.5*erf(f*(-4+0.001*(1:4000)));
%CALCULATE eta and de
for k=1:3
    rp=R(k)+(1.0/2)*SIG(k)^2;
    rm=R(k)-(1.0/2)*SIG(k)^2;
            for i=2:N
                tm=(i-1)*dt;    %tm is time to expiry:=T-t
                dn= SIG(k)*sqrt(tm);        %dn is denominator
                for j=1:M
                    s=j*dx;            %s is in (0, 10)
                    x= (log(s/st)+rm*tm)/dn;
                    if x > 4.0
                    ph= 1.0;
                      elseif x<-4.0
                             ph= 0.0;
                      elseif x<0.0
                             xx=floor(1000*(4.0+x))+1;
                             ph= SND(xx);
                      elseif  x>0.0
                             xx=floor(1000.0*(4.0 - x))+1;
                             ph= 1 - SND(xx);
```

```
                    end
                    term= st*exp(-R(k)*tm)*ph;
                    y= (log(s/st)+rp*tm)/dn;
                    if y > 4.0
                    ph= 1.0;
                      elseif y<-4.0
                             ph= 0.0;
                      elseif y<0.0
                             xx=floor(1000*(4.0+y))+1;
                             ph= SND(xx);
                      elseif  y>0.0
                             xx=floor(1000.0*(4.0 - y))+1;
                             ph= 1 - SND(xx);
                    end
                    eta(i,j,k)= s*ph - term;
                    de(i,:,k) = gradient(eta(i,:,k),dx);
                end
            end
end
%DEFINE THE INITIAL DATA \Phi^1
for j=1:M
     for i=1:3
         u(1,j,i)=max(0.0,dx*(j-1)-st);
     end
end
%DEFINE THE INITIAL DATA \Psi^1
for j=1:M
     for i=1:3
         du(1,j,i)=max(sign(dx*(j-1)-st),0);
         eta(1,j,i)=max(0.0,dx*(j-i)-st);
     end
end
%Construction of Matrix LN( v, s, x, i)
 for i=1:3
     C12(i)= d/SIG(i);
     C22(i)= d/(SIG(i)^3);
        for kk=2:N
        C11(kk,i) = lambda(i)*(exp(-(R(i)+lambda(i))*((kk-1)*dt))&
        /sqrt((kk-1)*dt));
        C21(kk,i) = lambda(i)*(exp(-(R(i)+lambda(i))*((kk-1)*dt))&
        /((kk-1)*dt)^(3/2));
        C3(kk,i) = exp(-lambda(i)*(kk-1)*dt);
        for j=1:M
        for jj=1:M
        LN1(kk,j,jj,i)=exp(-0.5*(( log(jj/j)-(R(i)-0.5*SIG(i)^2)*((kk-1)*dt))/ &
        (SIG(i)*sqrt((kk-1)*dt) ))^2);
        LN2(kk,j,jj,i)=(exp(-0.5*(( log(jj/j)-(R(i)-0.5*SIG(i)^2)*((kk-1)*dt))/&
        (SIG(i)*sqrt((kk-1)*dt) ))^2))* (log(jj/j)-(R(i)-0.5*SIG(i)^2)*((kk-1)*dt));
        end
        end
        end
end
 %CALCULATING u(t,s,i) FOR ALL t,s,i"
```

```
for k=2:N
for j=1:M
for i=1:3
    vint=0;
    for kk=2:k
        jj=1;        % INTEGRATION wrt x starts (trapizoidal rule)
        term =0;
        for ii=1:3
        term=term + u(k-kk+1,jj,ii)* P( i, ii);
        end
        xint(kk,j,i) = 0.5 * term * LN1(kk,j,jj,i)/jj;
        for jj=2:M-1
             term =0;
             for ii=1:3
                term=term + u(k-kk+1,jj,ii)* P( i, ii);
             end
        xint(kk,j,i) = xint(kk,j,i)+term*LN1(kk,j,jj,i)/jj;
        end
            jj=M;
            term =0;
            for ii=1:3
             term=term + u(k-kk+1,jj,ii)* P( i, ii);
            end
         xint(kk,j,i) = xint(kk,j,i)+0.5* term*LN1(kk,j,jj,i)/jj;
         % INTEGRATION wrt x completed
        vint = vint + xint(kk,j,i) * C11(kk,i) * dt;
    end
        u(k,j,i)=C3(k,i)*eta(k,j,i) + vint * C12(i);
end
end
end
%CALCULATING du(t,s,i) FOR ALL t,s,i"
for k=2:N
for j=1:M
for i=1:3
    vint=0;
    for kk=2:k
        jj=1;        % INTEGRATION wrt x starts (trapizoidal rule)
        term =0;
        for ii=1:3
        term=term + u(k-kk+1,jj,ii)* P( i, ii);
        end
        xint(kk,j,i) = 0.5 * term * LN2(kk,j,jj,i)/jj;
        for jj=2:M-1
             term =0;
             for ii=1:3
                term=term + u(k-kk+1,jj,ii)* P( i, ii);
             end
        xint(kk,j,i) = xint(kk,j,i)+term*LN2(kk,j,jj,i)/jj;
        end
            jj=M;
            term =0;
            for ii=1:3
```

```
             term=term + u(k-kk+1,jj,ii)* P( i, ii);
            end
         xint(kk,j,i) = xint(kk,j,i)+0.5* term*LN2(kk,j,jj,i)/jj;
         % INTEGRATION wrt x completed
        vint = vint + xint(kk,j,i) * C21(kk,i) * dt;
    end
        du(k,j,i)=C3(k,i)*de(k,j,i) + vint * C22(i)/(j*dx);
end
end
end
save('combine_result.mat');
```

B.3 Fortran code

B.3.1 Main program : 'iterate.f'

```
    program iterate
    real,parameter :: T=1.0, St=1.0
integer,parameter ::N=16, M=300, N11 = 750,num1=1000,step_s=50
real :: P(3,3),lambda(3),mu(3),R(3),SIG(3),Vu(N11+1,3),ds
integer :: ijk,ij1,ij,mark(N11+1),i,TTc,TT(20)
real,dimension(N,M,3) :: eta,u,du
real :: eta1(N11+1),u1(N11+1),du1(N11+1),ss(N11+1),err1
real,dimension(step_s) :: E1

! Transition matrix P:
P(1,1)=0.0
P(1,2)=2.0/3.0
P(1,3)=1.0/3.0
P(2,1)=1.0/2.0
P(2,2)=0.0
P(2,3)=1.0/2.0
P(3,1)=1.0/3.0
P(3,2)=2.0/3.0
P(3,3)=0.0
! Rates :
lambda=(/1.0,2.0,1.0/)
mu=(/0.2,0.6,0.8/)
R=(/0.2,0.5,0.7/)
    SIG=(/0.2,0.4,0.3/)

!Load combine data
call get(u,du,eta,M,N)
!Done loading
dx= (10*st)/M
ds=st*1.5/step_s
do 51 ijk= 1,3
print*,ijk
do 52 ij1 = 1,step_s
print*,ij1
num11=0;
E1(ij1)=0;
do 53 ij=1,num1
check = 0
```

```
call mark1(N11+1,mark,ijk,lambda,P,T)
call gbm(N11+1,ss,mu,SIG,mark,T,ij1*ds)
do 54 i=1,N11+1
if(ss(i)>St*10) then
check=1;
endif
   54 end do
if(check==1) then
    goto 53
else
num11=num11+1
end if
call calculations(N11+1,N,M,ss,eta,u,du,mark,eta1,u1,du1,dx,Vu)
call holding(TT,TTc,mark,N11+1)
call error(N11,TT,TTc,mark,ss,u1,lambda,P,R,T,err1,Vu)
E1(ij1)=E1(ij1)+err1
   53 end do
E1(ij1)=E1(ij1)/num11
   52 end do
Open (unit=1, file="MRisk"//char(ijk+48)//".xls" ,status='new')
    do 57 ij1=1,step_s
write (1,*) E1(ij1)
   57 end do
   51 end do
end program iterate
```

B.3.2 Modelling regime switching : 'mark1.f'

```
     subroutine mark1(N1,mark,start_i,lambda,P,T)
     integer :: start_i,mark(N1),i,j
 real :: lambda(3),P(3,3),T,p11
 double precision :: p1,k2,k11
 mark(1)=start_i
 dt11=T/(N1-1)
 do 21 i=2,N1
p1=1-lambda(mark(i-1))*dt11
k2=rand()
! write(9,*) 'k2= ',k2
if(k2<=p1) then
mark(i)=mark(i-1)
elseif (mark(i-1)==1) then
p11=P(1,2)
k11=rand()
if(k11<=p11) then
mark(i)=2
else
mark(i)=3
endif
elseif (mark(i-1)==2) then
p11=P(2,1)
k11=rand()
if(k11<=p11) then
mark(i)=1
```

```
else
mark(i)=3
endif
else
p11=P(3,2)
k11=rand()
if(k11<=p11) then
mark(i)=2
else
mark(i)=1
endif
endif
! write (9,*) 'mark=',i,mark(i)
   21 end do
end
```

B.3.3 Modelling stock-price process : 'gbm1.f'

```
subroutine gbm(k,stock,mu,SIG,mark,T,start_s)
    implicit none
integer:: k,mark(k),i
real :: stock(k),mu(3),SIG(3),T,start_s,W(k),dt,Nr(k-1),v1,v2,r,fac,gset,gasdev,iset
dt=T/(k-1)
!Generating normal distribution (0,dt) for dW
 do 31 i=1,k-1
      if (iset.eq.0) then
    32       v1=2.0*rand()-1.0
      v2=2.0*rand()-1.0
      r=v1**2+v2**2
      if(r.ge.1.d0)goto 32
        fac=sqrt(-2.0*log(r)/r)
        gset=v1*fac
        gasdev=v2*fac
        iset=1
      else
         gasdev=gset
        iset=0
      endif
    Nr(i)=gasdev*sqrt(dt)
   31  end do
!Generating sum (mu*dt + SIG*dW) and Stock value
 W(1)=0
 stock(1)=start_s
 do 33 i=2,k
W(i) =W(i-1)+ mu(mark(i))*dt+SIG(mark(i))*Nr(i-1)
stock(i)=start_s*exp(W(i))
   33  end do
end
```

B.3.4 Importing the arrays of solution generated by 'combine.m' : 'get.f'

```
subroutine get(u,du,eta,M,N)
    implicit none
```

```
    integer :: i,j,k,M,N
real,dimension(N,M,3) :: eta,u,du
! input u
OPEN (unit=81,FILE='u.txt',STATUS='OLD',action='read')
do 78 j=1,3
DO 75 k=1,N
       read(81,*) (u(k,i,j),i=1,M)
  75       CONTINUE
  78  continue
CLOSE(81)

! input du
OPEN (unit=82,FILE='du.txt',STATUS='OLD',action='read')
do 79 j=1,3
DO 76 k=1,N
       read(82,*) (du(k,i,j),i=1,M)
  76       CONTINUE
  79  continue
CLOSE(82)

!input eta
OPEN (unit=83,FILE='eta.txt',STATUS='OLD',action='read')
do 71 j=1,3
DO 72 k=1,N
       read(83,*) (eta(k,i,j),i=1,M)
  72       CONTINUE
  71  continue
CLOSE(83)
end
```

B.3.5 Calculating option price & hedging for one realisation of stock-price process : 'calculations.f'

```
     subroutine calculations(k,N,M,ss,eta,u,du,mark,eta1,u1,du1,dx,Vu)
    integer :: ii,sind,sind1,mark(k)
real :: cf,ss(k),dx,et1,et2,uu1,uu2,duu1,duu2,eta1(k),u1(k),du1(k),eta(N,M,3),u(N,M,3),du(N,M,
real :: uu11,uu21,uu12,uu22,uu13,uu23
do 41 ii=1,k-1
cf=(real(ii-1)/50.0-floor(real(ii-1)/50.0))
sind = nint(ss(ii)/dx)
if (sind>0) then
 et1=eta(16-floor((real(ii-1)/50.0)),sind,mark(ii))
     et2=eta(16-floor((real(ii-1)/50.0))-1,sind,mark(ii))
     eta1(ii) = (1-cf)*et1+(cf)*et2
     duu1= du(16-floor((real(ii-1)/50.0)),sind,mark(ii))
     duu2= du(16-floor((real(ii-1)/50.0))-1,sind,mark(ii))
     du1(ii) = (1-cf)*duu1+(cf)*duu2

     uu1= u(16-floor((real(ii-1)/50.0)),sind,mark(ii))
     uu2= u(16-floor((real(ii-1)/50.0))-1,sind,mark(ii))
     u1(ii) = (1-cf)*uu1+(cf)*uu2

     uu11= u(16-floor((real(ii-1)/50.0)),sind,1)
     uu21= u(16-floor((real(ii-1)/50.0))-1,sind,1)
```

```
     Vu(ii,1) = (1-cf)*uu11+(cf)*uu21

     uu12= u(16-floor((real(ii-1)/50.0)),sind,2)
     uu22= u(16-floor((real(ii-1)/50.0))-1,sind,2)
     Vu(ii,2) = (1-cf)*uu12+(cf)*uu22

     uu13= u(16-floor((real(ii-1)/50.0)),sind,3)
     uu23= u(16-floor((real(ii-1)/50.0))-1,sind,3)
     Vu(ii,3) = (1-cf)*uu13+(cf)*uu23


else
     eta1(ii)=0;
     u1(ii)=0;
   du1(ii)=0;
Vu(ii,1)=0;
Vu(ii,2)=0;
Vu(ii,3)=0;

endif
   41  end do
sind1=nint(ss(751)/dx)
if(sind1>0) then
    eta1(751)=eta(1,sind1,mark(751))
    u1(751)=u(1,sind1,mark(751))
    du1(751)=du(1,sind1,mark(751))
            Vu(ii,1)=u(1,sind1,1);
    Vu(ii,2)=u(1,sind1,2);
    Vu(ii,3)=u(1,sind1,3);
else
    eta1(751)=0
    u1(751)=0
    du1(751)=0
    Vu(751,1)=0;
    Vu(751,2)=0;
    Vu(751,3)=0;
endif
end
```

B.3.6 Holding time calculation : ‘holding.f’

```
subroutine holding(TT,TTc,mark,k)
    implicit none
integer :: TTc,k,mark(k),i,TT(20)
TTc=0
do 91 i=2,k
if(mark(i)/=mark(i-1)) then
TTc=TTc+1
TT(TTc)=i
end if
  91 end do
end
```

B.3.7 Error calculation: 'error.f'

```
subroutine error(N11,TT,TTc,mark,ss,u1,lambda,P,R,T,err1,Vu)
    implicit none
  integer :: tt1,tt2,N11,TT(20),TTc,mark(N11+1),ss(N11+1),kt,k,j,jj
  real :: dt,Vu(N11+1,3),IN,u1(N11+1),lambda(3),P(3,3),R(3),T,err1,term1,disc,y
  dt=T/N11
          IN=0
          tt1=0
  if(TTc==0) then
tt2=N11+1
  else
           tt2= TT(1)
  endif
          do 51 kt=1, tt2
             j = mark(1)
             y = kt*dt
             term1=0
             do 52 jj =1,3
            term1=term1+P(j,jj)*(Vu(kt,jj)-u1(kt))**2
   52        continue
             IN = IN + exp(-2.0*R(j)*y)*lambda(j)*term1
   51     continue
            disc= 1.0
          if(TTc.ge.1)then
             do 53 k=1, TTc
                disc = disc* exp(-2.0*R(mark(k))*(tt2-tt1)*dt)
                tt1 = TT(k)
if(TTc==k) then
tt2=N11+1
else
tt2=TT(k+1)
endif
                j=mark(k+1)
                  do 54 kt=1, tt2-tt1
                  y = kt*dt
                  term1=0
                  do 56 jj =1,3
                  term1 = term1 + P(j,jj) * (Vu(tt1+kt,jj)-u1(tt1+kt))**2
   56             continue
               IN=IN +disc*exp(-2.0*R(j)*y)*lambda(j)*term1
   54             continue
   53        continue
          endif
err1=IN*dt
end
```

B.3.8 Error calculation(By practitioner's approach): 'error.f'

```
subroutine error(N11,u1,du1,ss,T,mark,st,R,er1)
implicit none
integer :: N11,i,mark(N11+1)
real :: T,u1(N11+1),du1(N11+1),ss(N11+1),er1,st,Zt,ss1,ss2,Bt,R(3)
Zt=u1(1)
```

```
Bt=0
do 55 i=1,N11
        ss1=ss(i)*exp(-1*Bt)
  Bt=Bt+R(mark(i))*(T/N11)
      ss2=ss(i+1)*exp(-1*Bt)
      Zt=Zt+du1(i)*(ss2-ss1)
   55 end do
        er1=Zt-max(0.0,(ss(751)-st)*exp(-1*Bt))
end
```

Appendix C

Algorithms

iterate.f

Result: Expected Risk at each starting state and starting stock price.
num1 = number of iterations;
step_s = number of grids of starting stock price;
for $ijk = 1 \to 3$ **do**
    for $ij1 = 1 \to step\_s$ **do**
        for $ij = 1 \to num1$ **do**
            Generate states array mark by mark1();
            Generate stock array ss by gbm();
            if $ss(i) > 10 * strikefor\ somei$ **then**
                discard the iteration;
            end
            Calculations() ⇒ Prices in simulated market;
            holding() ⇒ Holdin times in simulated market;
            error() ⇒ err1;
            $E(ij1) = E(ij1) + err1$;
        end
        num11 = number of non-discarded iterations;
        E(ij1) = E(ij1)/num11;
    end
    E(ij1) ⇒ Expected Risk for starting state ijk.
end

mark1.f

Data: start_state = starting market state.
Result: mark(751) = array of simulated market states
mark(1) = start_state;
for $i = 2 \to 751$ **do**
    p1= $1 - \lambda(mark(i-1)) * dt$ `// probability that the state doesn't change`
    b = Bernoulli(p1);
    if *b == 1* **then**
        mark(i) = mark(i-1);
    else
        p1 = P(mark(i-1), j≠ mark(i-1));
        b = Bernoulli(p1);
        if *b == 1* **then**
            mark(i) = j;
        else
            mark(i) = k   where k ≠ j, mark(i-1);
        end
    end
end

gbm1.f

Data: mark(), start_s = starting stock price.
Result: ss(751) = array of simulated stock prices.
Nr = Array(750) of N(0,$\sqrt{dt}$) generated using Box-Muller transformation.;
W(1) = 0;
ss(1) = start_s;
for $i = 2 \to 751$ **do**
    W(i) = W(i-1) + μ dt + σ Nr(i-1);
    ss(i) = start_s $*e^{\mathrm{W(i)}}$;
end

calculations

Data: Black-Scholes option price, integral equation option price, hedging strategy as a function of time, stock price, market state; ss(); mark()
Result: Black-Scholes option price(eta1), integral equation option price(u1 at mark(t), Vu at all states.), hedging strategy(du1) as a function of time in simulated market.
for $t = 1 \to 750$ **do**
    sind = nearest integer(ss(t)/dx);

$$u1(i) = \left(1 - \left(\frac{(t-1)}{50} - floor\left(\frac{t-1}{50}\right)\right)\right) * u\left(16 - floor\left(\frac{t-1}{50}\right), sind, mark(t)\right) + \left(\frac{(t-1)}{50} - floor\left(\frac{t-1}{50}\right)\right) * u\left(16 - floor\left(\frac{t-1}{50}\right) - 1, sind, mark(t)\right);$$

    similar interpolation for eta1, u1, du1, Vu;
    $u1(751) = u(1, sind, mark(751))$;
    if *sind == 0* **then**
        $u1(t) = 0$;
    end
end

holding.f

Data: mark()
Result: TTc = number of state changes, TT() = time grids at which states change.
for $i = 2 \rightarrow 751$ **do**
    if *mark(i-1)* $\neq$ *mark(i)* **then**
        TTc = TTc + 1;
        TT(TTc) = i;
    end
end

Bibliography

Phillip E. Protter, *Stochastic Integration and Differential Equations, 2nd ed.*
Springer, Stochastic Modelling and Applied Probability, vol. 21
E-mail address: pep4@cornell.edu
2005

Bernt Øksendal, *Stochastic Differential Equations: An Introduction with Applications, 5th ed.*
Springer, Universitext
E-mail address: oksendal@math.uio.no
2000

Black F. and Scholes M., "The Pricing of Options and Corporate Liabilities", Journal of Political Economy, Vol. 81, No. 3 (1973), 637-654

Patrick Billingsley, *Probability and Measure, 3rd ed.*
Wiley, Wiley Series in Probability and Mathematical Statistics
2000

Gopinath Kallianpur, Rajeeva L Karandikar, *Introduction to Option Pricing Theory.*
Birkhauser,
1999

Bernt Øksendal, Agnés Sulem, *Applied Stochastic Control of Jump Diffusions.*
Springer, Universitext
E-mail address: oksendal@math.uio.no, agnes.sulem@inria.fr 2005

Sheldon M. Ross, *Stochastic Processes.*
Willey, Wiley Series in Probability and Mathematical Statistics
1996

Gopal K. Basak, Mrinal K. Ghosh & Anindya Goswami
Risk Minimizing Option Pricing for a Class of Exotic Options in a Markov-Modulated Market
Stochastic Analysis and Applications, 29:2, 259-281
2011
http://dx.doi.org/10.1080/07362994.2011.548665

Anindya Goswami, Ravi Kant Saini
Volterra equation for pricing and hedging in a regime switching market
`preprint`
2013
Can be downloaded from https://sites.google.com/site/anindyagoswami/recent

H. Föllmer, M. Schweizer
Hedging of Contingent Claims under Incomplete Information
M. H. A. Davis and R. J. Elliott (eds.), Applied Stochastic Analysis,
Stochastics Monographs, vol. 5, Gordon and Breach, London/New York, 389-414
1991