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RESEARCH ARTICLE

Volterra equation for pricing and hedging in a regime switching market

Anindya Goswami[1*] and Ravi Kant Saini[2]

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*Corresponding author: Anindya Goswami, Mathematics, IISER Pune, Pune 411021, India
E-mail: anindya@iiserpune.ac.in

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Abstract: It is known that the risk minimizing price of European options in Markov-modulated market satisfies a system of coupled PDE, known as generalized B–S–M PDE. In this paper, another system of equations, which can be categorized as a Volterra integral equations of second kind, are considered. It is shown that this system of integral equations has smooth solution and the solution solves the generalized B–S–M PDE. Apart from showing existence and uniqueness of the PDE, this IE representation helps to develop a new computational method. It enables to compute the European option price and corresponding optimal hedging strategy by using quadrature method.

Keywords: Markov modulated market, locally risk minimizing option price, Black–Scholes–Merton equations, Volterra equation, quadrature method

1. Introduction

In recent years, a large amount of research is being done in the field of derivative pricing in Markov-modulated market. In such a market, floating rate of interest of a money market account, growth rates, and volatility coefficients of stock prices are taken as functions of an observable finite state continuous time Markov chain. The stock price processes are modeled as Markov-modulated geometric Brownian motions. Due to the presence of additional randomness, such regime switching

ABOUT THE AUTHORS

Many published articles on the topic of "pricing in a regime switching market" lack mathematical rigor, leading to an ambiguity. This work is accomplished with an aim to re-examine the existing literature on this topic and present a mathematically rigorous treatment of the problem. The model, discussed, has further possible extensions. The results reported in this paper also have the potential to be extended to those generalizations.

The first author of this paper works on various other topics in Applied Probability. Those include equilibrium of non-cooperative semi-Markov games under ergodic cost, portfolio optimizations and optimal control under risk sensitive cost, fluid limit in queuing networks, PDE techniques in stochastic control and differential games, stability analysis of SDE, etc.

PUBLIC INTEREST STATEMENT

In a financial market, several financial instruments are available to the investors for purchase. Evaluation of the fare price of those is, therefore, extremely important to the investors. On the other hand, since the market is subject to various uncertainties, the computation of the price and the seller's replication strategy of the instrument, solely using the received price, are often tricky.

In this paper, we consider a theoretical market model which generalizes the famous Black–Scholes–Merton model to incorporate random variability of the parameters namely interest rate, mean growth rate, and the volatility coefficient. The state-of-the-art approach suggests solving a partial differential equation to find the price and the hedging strategy. In this paper, we have shown that those can also be obtained by solving an integral equation. This finding enables to propose a new more efficient and robust numerical procedure to compute European option price and the hedging.

model leads to an incomplete market. Therefore, the option pricing is rather involved. Indeed, there are contingent claims which are not attainable by self-financing strategies. Furthermore, existence of multiple equivalent martingale measures leads to multiple no-arbitrage prices of the same contingent claim. To address this difficulty, option pricing in an incomplete market is studied by several approaches Basak, Ghosh, and Goswami (2011), Buffington and Elliott (2002), Deshpande and Ghosh (2008), DiMasi, Kabanov, and Runggaldier (1994), Guo (2002), Guo and Zhang (2004), Heath, Platen, and Schweizer (2001), Jobert and Rogers (2006), Mamon and Rodrigo (2005), Schweizer (2001), Tsoi, Yang, and Yeung (2000), etc.

To price and hedge a claim of European type in the above incomplete market, we would consider the locally risk minimizing pricing approach by Föllmer and Schweizer (1991). It is shown in Deshpande and Ghosh (2008) that the locally risk minimizing price of an option of European type can be derived from the unique solution of a Cauchy problem, where the PDE is a generalization of Black–Scholes–Merton PDE (see Deshpande & Ghosh, 2008 for details). In a recent paper by (Basak et al., 2011), an implicit stable Crank–Nicholson (C–N) scheme is developed to solve that Cauchy problem numerically. The present paper also deals with numerical computation of locally risk minimizing price but it adopts a completely different approach. In this paper, we study a system of equations which can be categorized as Volterra integral equations of second kind. It is shown that this system of integral equations has unique smooth solution and the solution solves the generalized B–S–M PDE given in Basak et al. (2011). Or in other words, the risk minimizing option price is characterized as unique solution of a system of Volterra equations. Finally, we develop a stable scheme to solve this system numerically. This finding resolves various computational challenges. First of all, it enables development of an alternative numerical approach to find the option price by using quadrature method. In principle, C–N scheme (to solve B–S–M type PDE) involves inversion and N times multiplication of a matrix of order M, where M is proportional to the space discretization (Basak et al. 2011). Therefore, $T_{PDE}(N, M)$, the corresponding computational complexity to solve the PDE is $O(NM^3)$. Here, N is the number of equi-spaced points on time horizon $[0, T]$. On the other hand, we have the following result. Let $T_{IE}(N, M)$ denote the computational complexity to solve the IE with above grid, using step-by-step quadrature method. Then, we have

$$T_{IE}(N, M) = O(N^2 M^2)$$

Secondly, we are also able to find a Volterra equation for *optimal hedging strategy*. Needless to mention, this equation can also be solved by a similar numerical method. Therefore, calculating hedging strategy becomes as easy as calculating option price. Needless to mention, solving the PDE for hedging strategy is generally much harder than solving the PDE for option price. We also study one typical example of a regime switching market and carry out computation for solving the PDE as well as the IE. The computational elapsed times are recorded for both the cases with varying M for the purpose of comparison. The elapsed time data collated in a single plot clearly shows how the proposed scheme outperforms the C–N scheme for large values of M.

This paper is organized in the following way. The Markov-modulated market model is presented in Section 2 along with the main results of the paper. We present the proofs of Theorems 2.1 and 2.2 in Section 3. In Section 4, a step-by-step quadrature method is developed to solve the IE for option price. This section also contains the proof of stability of the scheme and a detailed calculation of computational complexity. Section 5 includes performance comparison of the scheme with that in Basak et al. (2011) by considering a typical numerical example. Finally, some remarks about immediate generalization of the present work are given in Section 6.

2. Model and main result

Let $(\Omega, \mathcal{F}, P)$ be the underlying complete probability space. Let $\chi = \{1, 2, \dots, k\}$ be the state space of an irreducible Markov chain $\{X_t, t \geq 0\}$ with transition rule

$$P(X_{t+\delta t} = j | X_t = i) = \lambda_{ij} \delta t + o(\delta t), \quad i \neq j$$

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where $\lambda_{ij} \geq 0$ for $i \neq j$; and $\lambda_{ii} = -\sum_{j\neq i}^{k} \lambda_{ij}$. Thus $\Lambda = [\lambda_{ij}]$ denotes the generating Q-matrix of the chain and $p_{ij} := \frac{\lambda_{ij}}{|\lambda_{ii}|}$ are the transition probabilities from state i to state j. We consider a market where the financial parameters, namely interest rate, drift coefficient, volatility coefficient are functions of the observed Markov chain X_t. Let $\{B_t, t \geq 0\}$ be the price of money market account at time t where, spot interest rate is $r(X_t)$ and $B_0 = 1$. We have

$$B_t = e^{\int_0^t r(X_u)du}$$

We consider a market consisting only one stock as tradable risky asset. The stock price process S_t solves

$$dS_t = S_t(\mu(X_{t-})dt + \sigma(X_{t-})dW_t), \quad S_0 > 0$$

where $\{W_t, t \geq 0\}$ is a standard Wiener process independent of $\{X_t, t \geq 0\}$. Let $\mathcal{F}_t$ be a filtration of $\mathcal{F}$ satisfying usual hypothesis and right continuous version of the filtration generated by X_t and S_t. Clearly, the solution of above SDE is an $\mathcal{F}_t$ semimartingale with almost sure continuous paths. To price a claim H of European type in the above incomplete market, we would consider the locally risk minimizing pricing approach by Föllmer and Schweizer (see Föllmer & Schweizer, 1991; Heath et al., 2001). A hedging strategy is defined as a predictable process $\pi = \{\pi_t = (\xi_t, \varepsilon_t), 0 \leq t \leq T\}$ which satisfies

$$E\left[\int_0^T \xi_t^2 \sigma^2(X_t) S_t^2 dt + \left(\int_0^T |\xi_t||\mu(X_t)|dt\right)^2\right] < \infty$$
and
$$E\left[\varepsilon_t^2\right] < \infty \tag{1}$$

The components ξ_t and ε_t denote the amounts invested in S_t and B_t, respectively, at time t. An optimal strategy is the one for which the quadratic residual risk (see Föllmer & Schweizer, 1991 for details) is minimized subject to a certain constraint. It is shown in Föllmer and Schweizer (1991) that the existence of an optimal strategy for hedging an $\mathcal{F}_t$ measurable claim H is equivalent to the existence of Föllmer–Schweizer decomposition of discounted claim $H^* := B_T^{-1} H$ in the form

$$H^* = H_0 + \int_0^T \xi_u^{H^*} dS_u^* + L_T^{H^*}$$

where $H_0 \in L^2(\Omega, \mathcal{F}_0, P)$, $L^{H^*} = \{L_t^{H^*}\}_{0 \leq t \leq T}$ is a square integrable martingale orthogonal to the martingale part of S_t, $S_t^* := B_t^{-1} S_t$, and $\xi^{H^*} = \{\xi_t^{H^*}\}$ satisfies (1). Further, ξ^{H^*} appeared in the decomposition constitutes the optimal strategy. Indeed, the optimal strategy $\pi = (\xi_t, \varepsilon_t)$ is given by

$$\begin{aligned} \xi_t &:= \xi_t^{H^*} \\ V_t^* &:= H_0 + \int_0^t \xi_u^{H^*} dS_u^* + L_t^{H^*} \\ \varepsilon_t &:= V_t^* - \xi_t S_t^* \end{aligned}$$

and $B_t V_t^*$ represents the locally risk minimizing price at t of the claim H. Hence, the Föllmer–Schweizer decomposition is the key thing to verify.

Now onward we consider a particular claim i.e. a European call option on $\{S_t\}$ with strike price K and maturity time T. In this case, the $\mathcal{F}_t$ measurable contingent claim H is given by

$$H = (S_T - K)^+ \tag{2}$$

Before stating the main results we recall that in the Black–Scholes–Merton model (Black & Scholes, 1973) the $\mathcal{F}_t$ measurable claim H is attainable and the price $\eta(t, S_t)$ at time $t \in [0, T]$ is given by

$$\eta(t,S_t)=S_t\Phi\left(\frac{\log\left(\frac{S_t}{K}\right)+r(T-t)}{\sigma\sqrt{T-t}}+\frac{1}{2}\sigma\sqrt{T-t}\right)-e^{rt}K^*\Phi\left(\frac{\log\left(\frac{S_t}{K}\right)+r(T-t)}{\sigma\sqrt{T-t}}-\frac{1}{2}\sigma\sqrt{T-t}\right) \quad (3)$$

where r and σ are constants denoting fixed bank rate and fixed volatility coefficients, respectively; $\Phi(x)$ is the CDF of standard normal distribution, $K^*=e^{-rt}K$. The Black–Scholes hedging strategy, called Delta hedging is given by

$$\Delta(t,s)=\frac{\partial\eta(t,s)}{\partial s}$$

where $\Delta(t, s)$ is the number of shares invested in stock. Now the main results are given below.

THEOREM 2.1. *The following integral equation has a unique solution in the class of functions belonging to* $C([0,T]\times\overline{\mathbb{R}_+}\times\chi)\bigcap C^{1,2}((0,T)\times\mathbb{R}_+\times\chi)$ *having at most linear growth.*

$$\varphi(t,s,i)=e^{-\lambda_i(T-t)}\eta_i(t,s)+\int_0^{T-t}\lambda_i e^{-(\lambda_i+r(i))v}\times\sum_j p_{ij}\int_0^\infty\varphi(t+v,x,j)\frac{e^{-\frac{1}{2}\left(\left(\ln\left(\frac{x}{s}\right)-\left(r(i)-\frac{\sigma^2(i)}{2}\right)v\right)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(i)\sqrt{v}x}dxdv \quad (4)$$

with

$$\varphi(T,s,i)=(s-K)^+,\quad \varphi(t,0,i)=0\,\forall t\in[0,T],\quad i\in\chi \quad (5)$$

where $\lambda_i:=-\lambda_{ii}$ *and* $\eta_i(t, s)$ *is the standard Black–Scholes price of European call option with fixed interest rate* $r(i)$ *and volatility* $\sigma(i)$.

Moreover, the solution $\varphi(t, s, i)$ *of* (4) *and* (5) *is the locally risk minimizing price of H (as in* (2)*) at time t with* $S_t=s$, $X_t=i$.

THEOREM 2.2. *Consider a function* $\psi\in C([0,T]\times\overline{\mathbb{R}_+}\times\chi)\bigcap C^{1,1}((0,T)\times\mathbb{R}_+\times\chi)$ *which is given in terms of the unique solution of* (4)–(5) *in the following way*

$$\psi(t,s,i)=e^{-\lambda_i(T-t)}\frac{\partial\eta_i(t,s)}{\partial s}+\int_0^{T-t}\lambda_i e^{-(\lambda_i+r(i))v}\sum_j p_{ij}\int_0^\infty\varphi(t+v,x,j)\times\frac{e^{-\frac{1}{2}\left(\left(\ln\left(\frac{x}{s}\right)-\left(r(i)-\frac{\sigma^2(i)}{2}\right)v\right)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(i)^3v^{3/2}xs}\left(\ln\left(\frac{x}{s}\right)-\left(r(i)-\frac{\sigma^2(i)}{2}\right)v\right)dxdv \quad (6)$$

for $t\in[0,T), s>0$

and

$$\psi(T,s,i)=1_{(K,\infty)}(s)\forall s\geq 0;\quad \psi(t,0,i)=0\,\forall t\in[0,T],\quad i\in\chi \quad (7)$$

The processes $\xi_t:=\psi(t,S_t,X_{t-})$ *and* $\varepsilon_t:=B_t^{-1}(\varphi(t,S_t,X_{t-})-\xi_tS_t)$ *comprise the optimal hedging strategy for the claim H in* (2).

THEOREM 2.3. *Given a finite grid of the domain* $[0,T]\times\overline{\mathbb{R}_+}$, *let N and M be the number of discrete points on* $[0, T]$ *and* $\overline{\mathbb{R}_+}$, *respectively. Let* $T(N, M)$ *denote the computational complexity to solve* (4) *and* (5) *with above grid using step by step quadrature method. Then we have*

$$T(N,M)=O(N^2M^2) \quad (8)$$

Remark 2.1. It is interesting to note that both of the integral equations in Theorems 2.1 and 2.2, have two additive terms on right side where first terms involve functions, coming from Black–Scholes–Merton model. In particular, if the Markov chain X_t does not transit almost surely, i.e. Λ, a null matrix, then (4) and (6) give $\varphi(t,s,i)=\eta_i(t,s)$ and $\psi(t,s,i)=\frac{\partial\eta_i(t,s)}{\partial s}$ respectively. Hence the B–S–M price and hedging can be recovered from Equations 4–5 and 6–7, respectively.

3. Equations of pricing and hedging

Consider the following system of partial differential equations

$$\frac{\partial\varphi(t,s,i)}{\partial t}+\frac{1}{2}\sigma(i)^2s^2\frac{\partial^2\varphi(t,s,i)}{\partial s^2}+r(i)s\frac{\partial\varphi(t,s,i)}{\partial s}+\sum_{j=1}^{k}\lambda_{ij}\varphi(t,s,j)=r(i)\varphi(t,s,i) \tag{9}$$

for $t<T$, $s>0$ and $i=1,2,\dots,k$ with the boundary condition

$$\varphi(T,s,i)=(s-K)^+, s\geq 0,\quad \varphi(t,0,i)=0\ \forall t\in[0,T],\quad i\in\chi \tag{10}$$

where φ is of polynomial growth. Note that if Λ is a null matrix i.e. the case when the Markov chain X_t does not transit almost surely, the Equation 9 coincides with that of standard B–S–M model. In view of this, the above system can be considered as a generalization of Black–Scholes equation for a Markov-modulated market where the extra coupling term represents the correction term arising due to the regime switching. Nevertheless, the fact, the solution of above problem gives locally risk minimizing price, needs a proof. To this end, we quote the following theorem from Deshpande and Ghosh (2008).

THEOREM 3.4. *If* $\{\varphi_c(t,s,i), i=1,2,\dots,k\}$ *denotes the unique classical solution of the Cauchy problem* (9)–(10), *then*

(i) $\varphi_c(t,S_t,X_t)$ *is the locally risk minimizing price of the option H* (*as in* (2)) *at time t;*

(ii) *An optimal strategy* $\pi=(\varepsilon_t,\xi_t)$, *is given by*

$$\xi_t=\frac{\partial\varphi_c(t,S_t,X_{t-})}{\partial s},\quad \varepsilon_t=V^*_{t-}-\xi_tS^*_t$$

where

$$V^*_t=e^{-\int_0^t r(X_u)du}\varphi_c(t,S_t,X_t)$$

Proof of Theorem 2.1. We prove the first part of Theorem 2.1 primarily by constructing a smooth solution of (4)–(5). In order to do that let $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ be a complete probability space which holds a standard Brownian motion $\tilde{W}$ and a Markov process $\tilde{X}$ independent of $\tilde{W}$ such that the rate matrix of $\tilde{X}$ is the same as that of X. Let $\tilde{S}_t$ be given by

$$d\tilde{S}_t=\tilde{S}_t(r(\tilde{X}_t)dt+\sigma(\tilde{X}_t)d\tilde{W}_t),\quad \tilde{S}_0>0 \tag{11}$$

and $\tilde{\mathcal{F}}_t$ be the underlying filtration satisfying usual hypothesis. Thus, $\tilde{P}$ is risk-neutral measure for the risky asset $\tilde{S}$ given by (11). Let Y_t represent holding time i.e. the amount of time the process $\tilde{X}_t$ is at the current state after the last jump. Let the consecutive jump times be $0=T_0<T_1<T_2<\cdots$ and $n(t):=\max\{n\geq 0|T_n\leq t\}$. Hence, $T_{n(t)}=t-Y_t$. Clearly, $f(y|i):=\lambda_ie^{-\lambda_iy}$ is the conditional probability density function of holding time and $F(y|i)=1-e^{-\lambda_iy}$ is the corresponding CDF where $\lambda_i=-\lambda_{ii}$. Here, we recall the following obvious relation

$$\frac{f(y|i)}{1-F(y|i)}=\lambda_i$$

Because of Markovity of $(\tilde{S}_t, \tilde{X}_t)$, we know that there is a measurable function $\varphi: [0,T]\times[0,\infty)\times\chi\to\mathbb{R}$ such that $\varphi(t,0,i)=0$ and

$$\varphi(t,\tilde{S}_t,\tilde{X}_t)=\tilde{E}[e^{-\int_t^T r(\tilde{X}_u)du}(\tilde{S}_T-K)^+|\tilde{\mathcal{F}}_t] \tag{12}$$

holds for all $t\in[0,T]$ where $\tilde{E}$ is expectation under $\tilde{P}$. Due to irreducibility of $\tilde{X}_t$, for any fixed $\tilde{X}_0, \tilde{S}_0$, the map φ (as in (12)) is defined uniquely almost everywhere on $[0,T]\times[0,\infty)\times\chi$. Now by conditioning at transition times and using the conditional lognormal distribution of stock price process, we have

$$\begin{aligned}
\varphi(t,\tilde{S}_t,\tilde{X}_t) &= \tilde{E}\left[e^{-\int_t^T r(\tilde{X}_u)du}\left(\tilde{S}_T-K\right)^+|\tilde{S}_t,\tilde{X}_t\right]\\
&= \tilde{E}\left[\tilde{E}\left[e^{-\int_t^T r(\tilde{X}_u)du}\left(\tilde{S}_T-K\right)^+|\tilde{S}_t,\tilde{X}_t,T_{n(t)+1}\right]|\tilde{S}_t,\tilde{X}_t\right]\\
&= \tilde{P}\left(T_{n(t)+1}<T|\tilde{X}_t\right)\tilde{E}\left[e^{-\int_t^T r(\tilde{X}_u)du}\left(\tilde{S}_T-K\right)^+|\tilde{S}_t,\tilde{X}_t,\{T_{n(t)+1}<T\}\right]\\
&\quad+\int_0^{T-t}\tilde{E}\left[e^{-\int_t^T r(\tilde{X}_u)du}\left(\tilde{S}_T-K\right)^+|\tilde{S}_t,\tilde{X}_t,T_{n(t)+1}=t+v\right]\frac{f\left(t+v-T_{n(t)}|\tilde{X}_t\right)}{1-F\left(t-T_{n(t)}|\tilde{X}_t\right)}dv\\
&= e^{-\lambda_{\tilde{X}_t}(T-t)}\eta_{\tilde{X}_t}\left(t,\tilde{S}_t\right)+\int_0^{T-t}\lambda_{\tilde{X}_t}e^{-\left(\lambda_{\tilde{X}_t}+r(\tilde{X}_t)\right)v}\sum_j p_{\tilde{X}_t j}\int_0^\infty\tilde{E}\left[e^{-\int_{t+v}^T r(\tilde{X}_u)du}\left(\tilde{S}_T-K\right)^+|\tilde{S}_{t+v}=x,\right.\\
&\quad\left.\tilde{X}_{t+v}=j,\tilde{T}_{n(t)+1}=t+v\right]\frac{e^{-\frac{1}{2}\left(\left(\ln\left(\frac{x}{\tilde{S}_t}\right)-\left(r(\tilde{X}_t)-\frac{\sigma^2(\tilde{X}_t)}{2}\right)v\right)\frac{1}{\sigma(\tilde{X}_t)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(\tilde{X}_t)\sqrt{v}x}dxdv\\
&= e^{-\lambda_{\tilde{X}_t}(T-t)}\eta_{\tilde{X}_t}\left(t,\tilde{S}_t\right)+\int_0^{T-t}\lambda_{\tilde{X}_t}e^{-\left(\lambda_{\tilde{X}_t}+r(\tilde{X}_t)\right)v}\\
&\quad\times\sum_j p_{\tilde{X}_t j}\int_0^\infty\varphi(t+v,x,j)\frac{e^{-\frac{1}{2}\left(\left(\ln\left(\frac{x}{\tilde{S}_t}\right)-\left(r(\tilde{X}_t)-\frac{\sigma^2(\tilde{X}_t)}{2}\right)v\right)\frac{1}{\sigma(\tilde{X}_t)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(\tilde{X}_t)\sqrt{v}x}dxdv
\end{aligned}$$

where $\eta_i(t,s)$ is the standard Black–Scholes price of European call option with fixed interest rate $r(i)$ and volatility $\sigma(i)$. Again using irreducibility of Markov chain, we can replace $(\tilde{S}_t,\tilde{X}_t)$ by generic variable (s,x) in the above relation and thus conclude that φ is a solution of (4)–(5). The first term on the right-hand side is clearly in $C^{1,2}((0,T)\times\mathbb{R}_+\times\chi)$. The continuous differentiability in t of the second term follows from the fact that the term $\varphi(t+v,x,j)$ is multiplied by $C^1((0,\infty))$ function in v and then integrated over $v\in(0,T-t)$. Now twice continuous differentiability in s of the second term follows from direct calculation. Thus $\varphi(t,s,i)$ is in $C^{1,2}((0,T)\times\mathbb{R}_+\times\chi)$. Finally, the continuity of φ on $[0,T]\times\overline{\mathbb{R}_+}$ follows trivially. We note that the right side of (4) can be considered as the image of φ under a contraction on a suitable Banach space. Hence, uniqueness follows from Banach fixed point theorem.

In view of Theorem 3.4.(i), the proof follows if φ, as above, is the unique classical solution of (9)–(10). Note that $(\tilde{S}_t,\tilde{X}_t)$ is jointly Markov with infinitesimal generator $\tilde{\mathcal{A}}$ given by

$$\tilde{\mathcal{A}}\varphi(t,s,i)=\frac{1}{2}\sigma(i)^2s^2\frac{\partial^2\varphi(t,s,i)}{\partial s^2}+r(i)s\frac{\partial\varphi(t,s,i)}{\partial s}+\sum_{j=1}^k\lambda_{ij}\varphi(t,s,i)$$

Therefore, (9) can be rewritten as $\frac{\partial\varphi}{\partial t}(t,s,i)+\tilde{\mathcal{A}}\varphi(t,s,i)=r(i)\varphi(t,s,i)$. Hence using Feynman–Kac formula, φ as in (12) is a mild solution of (9) with terminal condition (10). It is also shown above that φ is in $C([0,T]\times\mathbb{R}_+)\bigcap C^{1,2}((0,T)\times\mathbb{R}_+)$. Hence φ is a classical solution of (9)–(10) (see Proposition 3.1.2; Arendt, Batty, Hieber, & Neubrander, 2001). Uniqueness of the Cauchy problem is asserted from the stochastic representation of its solution. Hence the result follows. □

Proof of Theorem 2.2. Let us define

$$\xi_t := \frac{\partial \varphi(t, S_t, X_{t-})}{\partial s} \text{ and } \varepsilon_t := e^{-\int_0^t r(X_u)du}(\varphi(t, S_t, X_{t-}) - \xi_t S_t)$$

where φ solves (4)–(5). Using both Theorems 3.4 and 2.1 we get, $\pi := (\xi, \varepsilon)$ is an optimal strategy. The proof follows by differentiating both sides of (4) with respect to *s*. □

4. Numerical method

To solve (4)–(5), we use the step-by-step quadrature method. Let Δt and Δs be the time step and stock price step sizes, respectively. For m, m', l positive integers and $i \in \chi$, set

$$\mathcal{G}(m, m', l, i) := \frac{e^{-\frac{1}{2}\left(\left(\ln\left(\frac{m'}{m}\right) - \left(r(i) - \frac{\sigma^2(i)}{2}\right)l\Delta t\right)\frac{1}{\sigma(i)\sqrt{l\Delta t}}\right)^2}}{\sqrt{2\pi}\sigma(i)m'\Delta s\sqrt{l\Delta t}}$$

$$\varphi_m^n(i) \approx \varphi(T - n\Delta t, m\Delta s, i), \quad \varphi_0^n(i) = 0, \quad n = 0, 1, \ldots, N := \lfloor \frac{T}{\Delta t} \rfloor$$

Now we use the following quadrature rule over successive intervals $[0, n\Delta t]$ for a function ψ on this interval, we use

$$\int_0^{n\Delta t} \psi(v)dv \approx \Delta t \sum_{l=0}^{n} \omega_n(l)\psi(l\Delta t)$$

where $\omega_n(l)$ are weights to be chosen appropriately. Applying the above procedure in (4), we obtain the following set of equations

$$\begin{aligned}\varphi_m^n(i) = {} & e^{-\lambda_i n\Delta t}\eta_i(T - n\Delta t, m\Delta s) + \lambda_i \Delta t \sum_{l=1}^{n} \omega_n(l)e^{-l\Delta t(r(i)+\lambda_i)} \sum_j p_{ij}\Delta s \sum_{m'} \varphi_{m'}^{n-l}(j)\mathcal{G}(m, m', l, i) \\ & + \Delta t\omega_n(0)\lambda_i \sum p_{ij}\varphi_m^n(j)\end{aligned} \tag{13}$$

with

$$\varphi_m^0(i) = (m\Delta s - K)^+ \tag{14}$$

We choose a repeated trapezium rule by which the weights ω_n are given by

$$\omega_n(l) = \begin{cases} 1, & \text{for} \quad l = 1, 2, \ldots, n-1 \\ \frac{1}{2}, & \text{for} \quad l = 0, n \end{cases}$$

Convergence of the above scheme is obvious, the issue of stability is addressed below.

THEOREM 4.5. *Let $a := \max_\chi \lambda_i e^{-(\lambda_i + r(i))}$. For*

$$\Delta t \leq \frac{e^{-aT}}{a} \tag{15}$$

the scheme (13) is strictly stable with respect to an isolated perturbation. Moreover, the scheme displays uniformly bounded error propagation.

Proof. We first note that $\mathcal{G}(m, m', l, i)$ corresponds to a lognormal density and the holding time densities $f(\cdot|\cdot)$ are bounded. Let δ_n be an additive error in $\varphi_m^n(i)\ \forall m$ and i. Now it is easy to show that the effect of the isolated perturbation δ_n in $\varphi_m^N(i)(N := \lfloor \frac{T}{\Delta t} \rfloor)$ is additive and given by

$$\epsilon_n = a\Delta t(1 + a\Delta t)^{N-n-1}\delta_n$$

If Δt is sufficiently small and satisfies (15), we get $\epsilon_n < \delta_n$, i.e. the scheme is strictly stable with respect to an isolated perturbation. Let δ_n be bounded by a fixed constant δ. Now the total effect ϵ of the perturbation in the value $\varphi_m^N(i)$ is given by

$$\epsilon := \sum_{n=0}^{N-1} \epsilon_n < (e^{aT} - 1)\delta$$

Hence the result follows. □

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. To organize better, before computation of (13), we evaluate and store the values of known functions on the entire grid, so that those values can directly be used at later stages. Let C be the number of operations, required to accomplish that. We first estimate C. Let the constant c_η be the number of elementary operations required to evaluate η at a single entry. Similarly, let $c_{\mathcal{G}}$ and $c_{\exp}$ be the constants corresponding to the functions $\mathcal{G}$ and exponential respectively. Hence in view of (13), we obtain directly

$$C = kN(c_{\exp} + 1) + kN(c_{\exp} + 3) + kNMc_\eta + kNM^2 c_{\mathcal{G}} = O(NM^2)$$

Let $C_m^{(i)}(n)$ denote the number of additional computational operations which are required to obtain $\varphi_m^n(i)$ from (13) for fixed $n(\geq 1)$, m and i assuming that values of $\varphi_m^{n-1}(i)$ are known for all m and i. We allow $C_m^{(i)}(0)$ to represent the computational complexity of initial data at each entry. Hence, $C(n, M) := \sum_{i \in \chi, m \leq M} C_m^{(i)}(n)$ represents the total complexity at nth stage for each $n \leq N$.

It is evident from (14) that $C_m^{(i)}(0)$ is independent of i and similarly complex(c_0 say) for all m. Hence, $C(0, M) = Mc_0$.

From (13), it is not difficult to get $C_m^{(i)}(n) = 2n(k(M + 1) + 1) + 2$. Hence,

$$C(n, M) = 2[n(k(M + 1) + 1) + 1]kM$$

for all $n = 1, \dots, N$. Therefore, total number of operations i.e. $T(N, M)$ is given by

$$\begin{aligned} T(N, M) &= C + \sum_{n=0}^{N} C(n, M) \\ &= C + C(0, M) + \sum_{n=1}^{N} 2[n(k(M + 1) + 1) + 1]kM \\ &= O(N^2 M^2) \end{aligned}$$

□

Remark 4.2. In this section, we have developed a numerical scheme to compute option price using a quadrature method. It is natural to ask if this has any advantage over the one based on solving the PDE (9)–(10) using Crank–Nicholson implicit scheme. In order to compare the computational complexities, we present a brief description of the corresponding Crank–Nicholson scheme below.

To solve (9)–(10), we transform by replacing $t=T-v$ and $s=e^z$ and get a new system of PDEs

$$-\frac{\partial\varphi(v,z,i)}{\partial v}+\left(r(i)-\frac{1}{2}\sigma(i)^2\right)\frac{\partial\varphi(v,z,i)}{\partial z}+\frac{1}{2}\sigma(i)^2\frac{\partial^2\varphi(v,z,i)}{\partial z^2}+\sum_{j=1}^{k}\lambda_{ij}\varphi(v,z,j)=r(i)\varphi(v,z,i) \tag{16}$$

on the domain $(0,T)\times\mathbb{R}$ with

$$\varphi(0,z,i)=(e^z-K)^+ \tag{17}$$

Let Δt be the time mesh length and Δz be the stock mesh length in logarithmic scale. Let $N:=\left[\frac{T}{\Delta t}\right]$, z_0 a large negative number and M a large positive integer. For $n\le N, m=0,1,\ldots,M$

$$\varphi_m^n(i):=\varphi(n\Delta t, z_0+m\Delta z, i)$$

The terminal condition (17) gives

$$\varphi_m^0(i)=(e^{z_0+m\Delta z}-K)^+$$

Let $\varphi^n:=[\varphi_0^n(1),\ldots,\varphi_0^n(k),\varphi_1^n(1),\ldots,\varphi_M^n(1),\ldots,\varphi_M^n(k)]\in\mathbb{R}^{k(M+1)}$. If φ_{km+i}^n denotes the $km+i$th component of φ^n, then $\varphi_{km+i}^n=\varphi_m^n(i)$. Now the Crank–Nicholson discretization of (16) gives

$$A\varphi^{n+1}=(-2I-A)\varphi^n \tag{18}$$

where A is an appropriate block diagonal real matrix of size $k(M+1)\times k(M+1)$ (see Basak et al., 2011 for details). By repeated use of (18), the numerical solution of (16)–(17) is given by

$$\varphi^n=(-2A^{-1}-I)^n\varphi^0$$

Above scheme essentially involves inversion and multiplication of matrices of order $k(M+1)$. It is known that the computational complexity of such operation is $O(k^3M^3)$. Hence, the computational complexity of computing φ^n is $O(nk^3M^3)$. If $T(n, M)$ is the complexity of computing φ^n for $n\le N$. Then we have

$$T(N,M)=O(NM^3) \tag{19}$$

5. Numerical example and comparison

In this section, we consider an example of a Markov-modulated market with three regimes. The state space is $\mathcal{X}=\{1, 2, 3\}$. The drift coefficient, volatility, and interest rate at each regime are chosen as follows

$$(\mu(i),\sigma(i),r(i)):=\begin{cases}(0.2,0.2,0.2) & \text{if} \quad i=1\\ (0.6,0.4,0.5) & \text{if} \quad i=2\\ (0.8,0.3,0.7) & \text{if} \quad i=3\end{cases}$$

The transition rate matrix $\Lambda=(\lambda_{ij})$ is assumed to be given by

$$(\lambda_{ij})=\begin{pmatrix}-1 & 2/3 & 1/3\\ 1 & -2 & 1\\ 1/3 & 2/3 & -1\end{pmatrix}$$

For this case, we compute the price of a European call option where the strike price $K=1$ and maturity $T=1$. In order to compute numerically, we need to choose space-time discretization. For the above market, the restriction suggested by (15) is $\Delta t\le 2.45$. We consider, in particular

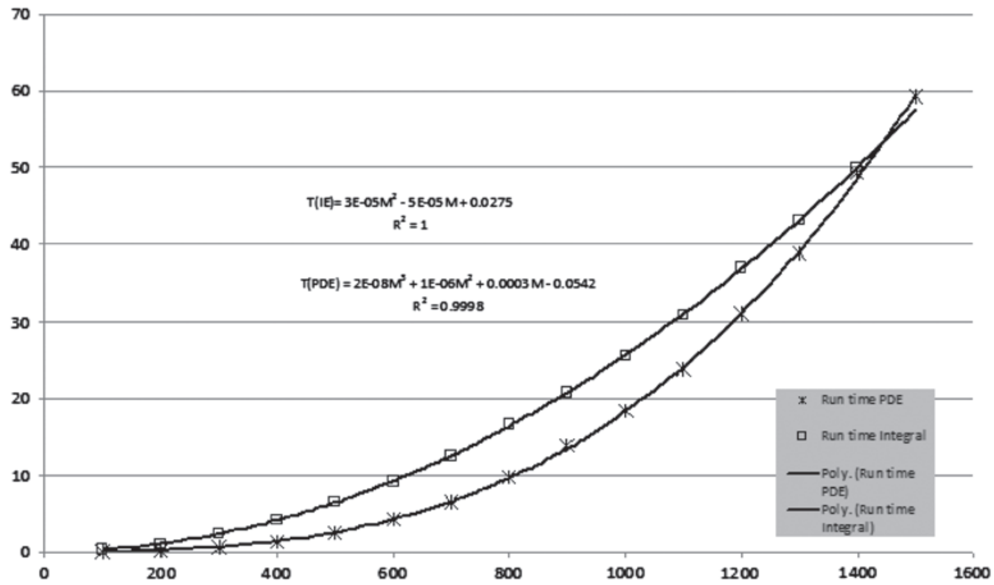

Figure 1. Run time ($\approx T(N,M)$) are plotted for both of solving PDE and IE.

$$\Delta t = 1/8, \Delta s = 10/M$$

where M is a large positive integer. We carry out computation for solving (9)–(10) as well as (4)–(5) for many different large values of M. For each M, the computational elapsed times are recorded for both the cases. In Figure 1, the elapsed time data are collated in a single plot where values of M are taken along horizontal axis and elapsed time in second is plotted along vertical axis. It shows that for a particular computing facility the proposed scheme outperforms the Crank–Nicholson scheme for large values of $M \geq 1{,}500$.

6. Conclusion

This work comprises theoretical derivations as well as numerical experiments. It also presents a self-contained proof of existence and uniqueness of generalized B–S–M PDE while proving the Theorem 2.1. It seems that the Volterra equation of optimal hedging has been studied for the first time in this paper. This paper makes it clear that such equation for hedging can also be obtained for more general semi-Markov-modulated market in the exactly similar manner. Needless to mention that this observation opens up an opportunity of practical application.

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Author details
Anindya Goswami[1]
E-mail: anindya@iiserpune.ac.in
Ravi Kant Saini[2]
E-mail: ravikdausa@gmail.com
[1] Mathematics, IISER Pune, Pune 411021, India.
[2] Mathematics, IIT Kanpur, Kanpur, India.

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