Chapter 2 - MMGBM Model

2.1 Brief description and drawbacks of B-S-M model:-

The Black Scholes Model is one of the most important concepts in modern financial theory. It was developed in 1973 by Fisher Black, Robert Merton and Myron Scholes.

Assumptions

The Black–Scholes model of the market for a particular stock follow the following assumptions:

- There is no arbitrage opportunity (i.e., there is no way to make a riskless profit).
- It is possible to borrow and lend cash at a known constant risk-free interest rate.
- It is possible to buy and sell any amount, even fractional of stock (this includes short selling).
- The above transactions do not incur any fees or costs (i.e., frictionless market).
- The stock price follows a geometric Brownian motion with constant drift and volatility.
- The underlying security does not pay a dividend.

Under these assumptions Call and Put option's price is given by (part I)

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)}$$

$$d_1 = (\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)) / \sigma\sqrt{T-t}$$

$$d_2 = (\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})(T-t)) / \sigma\sqrt{T-t} = d_1 - \sigma\sqrt{T-t}$$

The price of a corresponding put option based on put-call parity is:

$$P(S,t) = Ke^{-r(T-t)} - S + C(S,t)$$

$$= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S$$

where

- $N(.)$ is the cumulative distribution function of the standard normal distribution
- T-t is the time to maturity
- S is the spot price of the underlying asset
- K is the strike price
- r is the risk free rate (annual rate, expressed in terms of continuous compounding)
- σ is the volatility of returns of the underlying asset.

Disadvantages of the Black-Scholes Model:-

- In Black-Scholes-Merton model it is assumed that
    1. The volatility is constant.
    2. The interest rate is fixed.

  These assumptions do not match reality.

- It cannot be used to accurately price options with an American-style exercise as it only calculates the option price at one point in time i.e. only at expiration. It does not consider the steps along the way where there could be the possibility of early exercise of an American option.

- It is not necessary that the stock prices always follow lognormal distribution.

Several alternative models proposed:-

- Stochastic volatility model(Appendix(1))
- Jump diffusion model etc.

2.2 MMGBM Model

Keeping the drawbacks of Black-Scholes-Merton model in mind, we model the floating interest rate r and volatility σ as Markov processes (non-constant w.r.t time).

- $\{X_t\}_{t \geq 0}$ is an observable Markov process taking values in ℵ= {1, 2,…,k} with rate matrix $\Lambda = (\lambda_{ij})$.
- X_t is modelled to present hypothetical states of the market at time t.
- Let $r(X_t)$ and $\sigma(X_t)$ be the floating interest rate and market volatility at time t.

<u>Description of the MMGBM model:</u>

Let (Ω, F, P) be the underlying complete probability space. Let χ= (1, 2) be the state space of an irreducible Markov chain $\{X_t, t \geq 0\}$ with transition rule

$$P(X_{t+h} = j \mid X_t = i) = \lambda_{ij}h + o(h),\ i \neq j \qquad (2.2.1)$$

Where $\lambda_{ij} \geq 0$ for $i \neq j$ and $\lambda_{ii} = -\sum_{j \neq i}^{k} \lambda_{ij}$. Thus $\Lambda = [\lambda_{ij}]$ denotes the generating Q-matrix of the chain and $p_{ij} = \dfrac{\lambda_{ij}}{|\lambda_{ii}|}$ are the transition probabilities from state i to state j. We consider a market where the financial parameters, namely interest rate, drift coefficient, volatility coefficient are functions of the observed Markov chain X_t. Let $\{B_t, t \geq 0\}$ be the prices of money market account at time t where, spot interest rate is $r(X_t)$ and $B_0=1$. We have

$$B_t = e\int_0^t r(X_u)du \qquad (2.2.2)$$

We consider a market consisting only one stock as tradable risky asset. The stock price process S_t solves

$$dS_t = S_t(\mu\,(X_{t^-})dt + \sigma\,(X_{t^-})dW_t),\ S_0 > 0 \qquad (2.2.3)$$

Where $\{W_t,\ t \geq 0\}$ is a standard Wiener process independent of $\{X_t,\ t\geq 0\}$.

Let $Ŧ_t$ be a filtration of Ŧ satisfying usual hypothesis and right continuous version of the filtration generated by X_t and S_t. Clearly the solution of above SDE is an $Ŧ_t$ semimartingale with almost sure continuous paths.

The following integral equation has a unique solution in the class of functions belonging to $C([0,T]\times\overline{\Re_+}\times\aleph)\bigcap C^{1,2}((0,T)\times\Re_+\times\aleph)$

$$\varphi(t,s,i) = e^{-\lambda_i(T-t)}\eta_i(t,s) + \int_0^{T-t}\lambda_i e^{-(\lambda_i+r(i))\upsilon}\times\sum_j p_{ij}\int_0^\infty \varphi(t+\upsilon,x,j)\frac{e^{-\frac{1}{2}((\ln(\frac{x}{s})-(r(i)-\frac{\sigma^2(i)}{2})\upsilon)\frac{1}{\sigma(i)\sqrt{\upsilon}})^2}}{\sqrt{2\Pi}\sigma(i)\sqrt{\upsilon}x}dxd\upsilon \qquad (2.2.4)$$

$$\varphi(T,s,i) = (s-K)^+, \qquad \varphi(t,0,i) = 0\forall t\in[0,T], i\in\aleph \qquad (2.2.5)$$

Where $\eta_i(t, s)$ is the standard Black-Scholes price of European call option with fixed interest rate r(i) and volatility σ(i).

Moreover, the solution φ (t, s, i) of (2.2.4) and (2.2.5) is the locally risk minimizing price of H at time t with S_t = s, X_t = i

(i) <u>PDE of the call option price</u>

Consider the following equation which is the generalization of Black-Scholes-Merton partial differential equation for the Markov modulated market -

$$\frac{\partial}{\partial t}\varphi(t,s,i) + r(i)s\frac{\partial}{\partial s}\varphi(t,s,i) + \frac{1}{2}\sigma^2(i)s^2\frac{\partial^2}{\partial s^2}\varphi(t,s,i) + \sum_j \lambda_{ij}\varphi(t,s,j) = r(i)\varphi(t,s,i) \qquad (2.2.6)$$

(t,s, i) ϵ $\mathcal{D} \coloneqq (0,T) \times R_+ \times \chi$ with the boundary conditions
φ(t, 0, j) = 0 for all t ϵ [0, T] and

$$\varphi(t,s,i) = (s-K)^+;\ s\epsilon(0,\infty];\quad \text{i=1, 2, 3…k} \qquad (2.2.7)$$

The equation (2.2.6) coincides with that of standard B-S-M model. In view of this, the above system can be considered as a generalization of Black- Scholes equation for a Markov modulated market where the extra coupling term represents the correction term arising due to the regime switching. Nevertheless, the fact, the solution of above problem gives locally risk minimizing price.

Chapter 3 - Mathematical preliminaries related to Implied volatility

3.1 Implicit function theorem

Let f(x, y) be a function of two variables x and y, and (a, b) be a point of its domain of definition such that

(i) f (a, b) =0

(ii) The partial derivatives f_x, f_y exist & are continuous in a certain neighbourhood of (a, b), and

(iii) f_y (a, b) $\neq$0

Then $\exists$ a rectangle (a-h, a+h; b-k, b+k) about (a, b) such that for every value of x in the interval (a-h, a+h), in the equation f(x, y) =0 determines one and only one value y=ϕ(x), lying in the interval (b-k, b+k), with following properties

(i) b= ϕ (a)

(ii) f(x, ϕ(x)) =0 for all x in (a-h, a+h)

(iii) ϕ(x) is derivable, and ϕ(x), ϕ'(x) are continuous in (a-h, a+h).

3.2 The Greeks

"The Greeks" measure the sensitivity to change of the option price under a slight change of a single parameter while holding the other parameters fixed. Formally, they are partial derivatives of the option price with respect to the independent variables.

The Greeks are not only important for the mathematical theory of finance, but for those actively involved in trading. Financial institutions will typically set limits for the Greeks that their trader cannot exceed. Delta is the most important Greek and traders will zero their delta at the end of the day.

The Greeks for Black–Scholes are given in closed form below. They can be obtained by straightforward differentiation of the Black–Scholes formula. Since call option price is function of many factors:

$$C=f(S, t, r, \sigma, T)$$

The Greeks for the call and put options are:

	What	Calls	Puts
Delta	$\frac{\partial C}{\partial S}$	$N(d_1)$	$-N(-d_1) = N(d_1) - 1$
Gamma	$\frac{\partial^2 C}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	
Vega	$\frac{\partial C}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	
Theta	$\frac{\partial C}{\partial t}$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$
Rho	$\frac{\partial C}{\partial r}$	$K(T-t)e^{-r(T-t)}N(d_2)$	$-K(T-t)e^{-r(T-t)}N(-d_2)$

Proof for positivity of Vega:

In the Black and Scholes model the price of an European call option on a non-dividend paying stock is

$$C = SN(d_1) - Ke^{-r\tau}N(d_2)$$

Where S is the stock's price at valuation date, K is the strike price, r is the (constant) spot rate, τ = T - t is the time to maturity, T the expiry, t the valuation date and

$$d_1 = (\log\frac{S}{K} + (r + \frac{\sigma^2}{2})\tau) / \sigma\sqrt{\tau}$$

$$d_2 = (\log\frac{S}{K} + (r - \frac{\sigma^2}{2})\tau) / \sigma\sqrt{\tau} = d_1 - \sigma\sqrt{\tau}$$

Where σ is the stock's volatility.

In order to prove the theorem we collect some common calculations in the following

Lemma 1. It holds

$$\mathbf{S\,N'(d_1) - K\,e^{-r\tau}\,N'(d_2) = 0} \qquad \mathbf{(3.2.1)}$$

Proof. First of all, we remember that

$$N'(x) = \frac{1}{\sqrt{2\Pi}} \exp^{-x^2/2} \qquad (3.2.2)$$

Statement (3.2.1) holds if and only if

S N' (d_1) = K $e^{-r\tau}$ N' (d_2) $\leftrightarrow$ (S/ K) $e^{r\tau}$ **=** **N'** (d_2) **/ N'** (d_1)

$$\leftrightarrow \log\frac{S}{K} + r\tau = \frac{d_1^2 - d_2^2}{2} \qquad \text{(using(3.2.2))}$$

Notice that the right hand side of the last condition is

$$\frac{d_1^2 - d_2^2}{2} = \frac{1}{2}(d_1 + d_2)(d_1 - d_2)$$

$$= \frac{1}{2}(2d_1 - \sigma\sqrt{\tau})\sigma\sqrt{\tau}$$

$$= \log\frac{S}{K} + (r + \frac{\sigma^2}{2})\tau - \frac{\sigma^2}{2}\tau$$

$$= \log(\frac{S}{K}) + r\tau$$

and this completes the proof of (3.2.1).

Then Vega is defined as: $\upsilon = \frac{\partial C}{\partial \sigma}$

$$= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial \sigma}$$

$$= \sqrt{\tau}SN'(d_2) + \frac{\partial d_1}{\partial \sigma}(SN'(d_1) - Ke^{-r\tau}N'(d_2)$$

$$\left(\because \frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right) = \sqrt{\tau} \text{ (Integration by parts)}$$

$$\upsilon = \frac{\partial C}{\partial \sigma} \quad = \sqrt{\tau}SN'(d_2)$$

Where N’ (d_2) is probability density function which is non zero. Hence Vega of option price is non zero.

<u>Chapter 4</u> - Implied volatility in MMGBM model

Volatility is the tendency of price to fluctuate sharply and regularly.

IMPLIED VOLATILITY:

In financial mathematics, the **implied volatility** of an option contract is that value of the volatility of the underlying instrument which, when input in an option pricing model (such as Black-Scholes) will return a theoretical value equal to the current market price of the option.

in general, the value of an option depends on an many factors of the underlying. Or mathematically:

$$C=f(S_0, t_0, \sigma, r, K, T)$$

where *C* is the theoretical value of an option, and *f* is a pricing model that depends on σ, along with other inputs.

The function *f* is monotonically increasing in σ, meaning that a higher value for volatility results in a higher theoretical value of the option. Conversely, by the implict function theorem, there can be at most one value for σ that, when applied as an input to $f(\sigma, \cdot)$, will result in a particular value for *C*. Put in other terms, assume that there is some inverse function $g = f^{-1}$, such that

$$\sigma_{\bar{C}} = g(\bar{C}, \cdot)$$

Where $\bar{C}$ is the market price for an option. The value $\sigma_{\bar{C}}$ is the volatility **implied** by the market price $\bar{C}$, or the **implied volatility**.