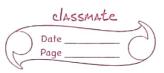


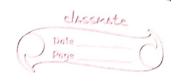
	Solutions - Assignment 6 - 2022111024-Q1.ipynb
Q1.	Prove that XTX is invertible if and only if the edumns of X axe linearly independent.
	linearly independent.
Soln.	Hence, we are to prove two things:
	(⇒) X ^T X is insertible if the columns of X are linearly independent.
	(=) The columns of X are linearly independent if the XTX is invertible; those prove the if and only if statement given.
	invertible, those prove the if and only if statement given
	, , , , , , , , , , , , , , , , , , , ,
(⇒)	Forward implication proof:
	X has linearly independent columns. To show, X TX is invertible.
	We know, when & all columns of X are linearly independent
	We know, when & all columns of X are linearly independent, X\vec{y}=\vec{b} A\vec{x}=\vec{b} has a solution \to \vec{v} (:n independent nectors of \\ \[\frac{n \times n}{n \times n} \text{ with n columns span the n-dimensional \\ \text{Vector opace, so a solution \to the will a vice.}
	nxn dimensi X with n alumns span the n-dimensional
	The second of th
	$x\vec{y} = \vec{b}$ \vec{x} since
	Thus, $A\overrightarrow{x} = \overrightarrow{b}$ has a solution \overrightarrow{x} if x has Linearly independent columns. x^{-1} x^{-1} x^{-1} x^{-1}
	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
	For X to be invertible, there must exect A = I AAT = ATAI.
	Let lei, ez, en 3 be the column of the identity motion.
	$XX^{-1}AA^{-1} = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix}$
	For X to be invertible, there must exist A = s.t. AA = A A = X. I. Let lei, ei, en 3 be the columns of the identity matrix. XX-1AA = [ê, ê2ên]
	We can show make multiplication as a min of the melon
	mulliplications such that,
	XA [a* a; an] - [xa* x1; x1] - [i a da
	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$



	$X\overrightarrow{y} = \overrightarrow{b}$
	: At - B always has a soln we are solutale and it of
	: $A\vec{x} = \vec{b}$ always has a soln, we can calculate each \vec{a}_i^* from the equation $X A \vec{a}_i^* = \vec{e}_i \forall i \in \{1, 2,, n\}$ and we can guarantee a soln to it will exist.
1	a soln lo it will exist.
1	Co po 11- NO 10 WIND .
	Thus chies 1/2 to
	Thus, solving of ai* (ie ?1,2,n) we get a can create a matrix I whose solumn nectors are the solutions to
(0 (P) 1)	masor of whose whem nectors are the solutions to
(SW354)	av* for each & column i such that
	VA TI
	$\begin{array}{c c} X \not\triangleq a_1^* a_2^* & a_3^* = I \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ \end{array}$
4 - 14 1	: A & soln to "ai" for each ai always exists, we can always
	find such as a matrix giving us thus to
	": A & soln to "ai" for each at always exists, we can always find such as a matrix, giving us "A". Thus " is invertible.
Thir	we ken $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$ $A^T B^T = (BA)^T$
	$A^{T} \left(A^{T} \right)^{T} = \left(A^{T} A \right)^{T} = I^{T} = I. \text{and} I^{T} = I.$
	So, A always has an inverse (AT) - given by
C7 +	(AT) Whenever A has an inverse A.
	Thus, AT is investible if A is investible
XIVO	Land Comment of the contract of the comment of the comment
- ACE	"X is invertible, X" is invertible. Now, padut of 2 invertible matrices is always invertible as an be showed by,
11/2	matrices is always investible as an be showed by,
708	THINK TO THE WAS IN THE THE THE THE PART OF THE PERTY OF
Ilm	C=AB, FA-1s.t. AA+ = A-1A=I, BB+=B-1B=I.
	Multiplying Boon both sides, bulliplying A on both sides
	CB-1 = ABB-1 A'C = A-1/B-A
	=) CB-1 = AI = A => A-1C = IB = B
	Lullipling A' on both sides, Mulliplying & B' onboth sides. C B'A' = AA' = I. B'A' C = B'B = I
	The Value of Alexander of States
1 811	:. $J B^{-1} A^{-1} = C^{-1} s.t. CC^{!} c^{-1} C = T$
	thouse vity
1	Hence, X'X' is invertible. X has linearly independent columns > XX is invertible.
1	A has kineary marganoune correlation = 111
	•

	el = PX
(L)	Backward implication proof:
Contains	purule priparation party.
7. 3. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.	X ^T X is investible. To show, columns of Xaxe linearlyidexentent.
	X X 10 Woulde . 10 /2000, Course of 1000
	To account a complace to the construction of x are limed not
A	To prove by contradiction, assume the columns of X are limst not sinearly independent: Jq.c., on is t not all of them are zero s.t. C, x, + C, x, + C, x, + C, x, = 0 where x; respresent the jth column of X, i.e. X = [x, x, xn]
-6. CO. 200	uneavy independent. : Ja, G1 Ch s. t yas an grown
	10 00 0 0 0 V - 1 1 1 1 1 1
	the jet column of X, i.e. X = 2 d2 an
	The state of the s
	Thus, we can rewrite the egn as,
17450	X C1 = 0 where I some KEV [1,n] s. s. G. f.
\$ 12	and the same of th
	Multiplying Non both sides,
	$\therefore X^{1} \times C_{1} = X^{1} \cup C_{2}$
T	Multiplying X^{T} on both sides, $X \begin{bmatrix} c_{1} \\ c_{2} \\ c_{n} \end{bmatrix} = 0 \text{ where } \exists some \ k \in \mathbb{Z}[1,n] \leq l. \ G_{\neq 0}$ Multiplying X^{T} on both sides, $X^{T} \times X \begin{bmatrix} c_{1} \\ c_{n} \end{bmatrix} = X^{T} \cdot 0$
	$\Rightarrow X^T X \begin{bmatrix} c \\ c \\ c \end{bmatrix} = 0$
we bif	6 TO SUN SURVEY ROUND L'EN JE COM LE L'EN JE
7.	:X X is invertible and fack s. E. Cx + O,
	we can say that I. a non zero veilor Esuch that (XX) = 0.
	Thus, I non-zero vector in the null space of XTX.
Minus L	La Color Col
12 61	This contracts that the fact that XTX is investible.
V	This contracts that the fact that XTX is invoctible. Thus, the columns of X are linearly independent; object the set will there or F = [c] 1.3 + (XTX) = 0
	Alam not will there or 7 3= [a7] sh NTX) =0
del cales	Ca to a Ca
	la lingarly indeas de A vi a cit a las an Ci= Oti
S	for linearly independent x; $\frac{1}{2}$ ci $\neq 0$ for an $C_i = 0 \forall i$, thus $\overline{C}_i = \overline{O}_i$ which is the only vector beloggiff to the null space of X^TX , an invertible matrix.
1. 1. 1	la fla mill source le VTX markella melle
and hands	To the number of A N, an inverse may reix.
	S C C C
T. Mina	: X ^T X is invertible > alumns of X are linearly integrate
	THE REPORT OF THE PARTY OF THE
R. 4	: XTX is innoctible \Leftrightarrow columns of X are linearly independent.
MATHE.	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

Q2.	Find the solution for B if the matrix X is deemposed as X=QB using QR decomposition where Q is orthogonal of R is an upper briangulas matrix.
	X=OK using OR decomposition where O is all organil & D
1 4	is an upper brigneylas mokrix
Pro .	Commence of the second
John.	Using OR decomposition up lied Y-DD along D. Dl. Oln
000	Voirg QR desomposition, we find X=OR where Quorthogonal & Ris reporter.
	Cla know that B is given by YB-Y
	We know that B is given by XB = Y. Multiplying XT on both sides.
	puntiplying X on both sides,
	> XT (XB)= XTY > (XTX)B = XTY
) (XTX) B = XTY
	Substituting of X = QR.
	Substituting $\not\supset X = \mathbb{Q}R$, $[(\mathbb{Q}R)^{T}(\mathbb{Q}R)]B = (\mathbb{Q}R)^{T}Y$
	$\mathcal{F}[\mathcal{R}^{T}, \mathcal{Q}^{T}] \mathcal{G} = (\mathcal{R}^{T}, \mathcal{Q}^{T}) \mathcal{G} \mathcal{G} \mathcal{G} \mathcal{G} \mathcal{G} \mathcal{G} \mathcal{G} \mathcal{G}$
	(1361.3)
	$\Rightarrow \left[R^{T}(Q^{T}Q)R\right]B = R^{T}Q^{T}Y$
100	The file was again at the second of the seco
	: Q is orthogonal, $Q^TQ = I$.
	1 sun f
7	> RTIRB = RTOTY
	> RTRB=RTQTyY
	of the state of th
	: R is upper briangular, R is investible (disgonal elements renes)
	: (X) exists x. (x). X = 1
	Multiplying (K.) on both sides,
	$(R^{\tau})^{-1} \cdot R^{\tau} \cdot R^{\tau} = (R^{\tau}) \cdot R \cdot D^{\tau} \cdot Q^{\tau}$
	\Rightarrow IK/3 = IO44
	$(R^{T})^{T} exists s.l.(R^{T}) \cdot R = I$ $(R^{T})^{-1} \cdot R^{T} \cdot R \cdot B = (R^{T})^{-1} \cdot R^{T} \cdot R^{T} \cdot Y$ $\Rightarrow IR\beta = IQ^{T} \cdot Y$ $\Rightarrow R\beta = Q^{T} \cdot Y \cdot Y$
UT	RB=074 let 07=W, RB=W, we do not need to take the inverse of R to solve for B with & B=R Wome Rie an upper triangular matrix.
	101070 of R la solver for B with & 13 = R Waine
	On a whole bulenular maker.
	K & an wife war from



Inflood, if we is the wan be willen as t	los
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Instead, we solve for Bi [ie?1,...m3] using Wi [ie?1,...m3] using upper se elements of R given by Rij, [i,je?1,...m3]

Bn = zn/Rnn Bn-1 = (zn-1 - Rn-gn Pn)/Rn-1n-1

B1 = Z1 - R12 B2 - . - R1/Bn)/R11

Thus we have calculated B=

[B1]

[B2]

Bn]

Q3. Find the solution for 13 if the matrix X is decomposed its X = UDV using singular value decomposition, where U, Value orthogonal matrix.

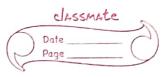
X=UDY XB=Y.

(X T X = 1) XT

On, form/Moore-Penrose equation given to us,

 $(X^T X)^{-1} X^T Y = /3$

 $\geqslant \beta = ((vDV^{T})^{T}(vDV^{T}))^{-1}(vDV^{T})^{T}Y \\
= ((vD^{U}^{T})((vDV^{T}))^{-1}((vDV^{T})^{T}Y) \\
= (vD^{T}(v^{T}v)DV^{T})^{-1}((vDV^{T})^{T}Y) \\
= (vD^{T}DV^{T})^{-1}((vDV)^{T}Y)$



L	
	= (VDTDVT)-1 (UDV)TY
	\neq Diagonal matrix D is symmetric: if has no non-diagonal elements to be inverted. Thus, $D^T = D$.
	$= (V D^2 V^{\tau})^{-1} (VDU^{\tau}) y$
	$= \left[\left(\bigvee^{T} \right)^{-1} \mathcal{D}^{-2} \bigvee^{-1} \right] \left(\bigvee \mathcal{D} \mathcal{U}^{T} \right) Y$
	$= \left(\left(V^{T} \right)^{T} D^{-2} V^{T} \right) \left(V D U^{T} \right) Y$
	Since, V is orthogonal, V is orthogonal. V'= V T. Similarly (VT) = (V) We need not bother about calculating D': it is a
	We need not bother about calculating D': it is a diagonal matrix and its inverse is given by the diagonal matrix formed the by the multiplicative inverse of each diagonal dement in D, as is give shown by, I D= [61 & 00] D'= [761 & 00] O (100-2) (100) (100) (100)
	each diagonal dement in D, as is governous by,
	3 B = (VD-2 VP) (VDUT) Y
	$\exists \mathcal{B} = (VD^{-2}(V^{\sharp}V)DU^{\dagger})Y$
	$\exists \mathcal{B} = (VD^{-2}IDU^{T})Y$
	> B = (VD-2DUT) Y > B = (VD-1XUT) Y we simply need to be brongose
_	VT and V on decomposing X and find the multiplicative
	> 3 = (VD'\TU') Y , we simply need to be transpose. V and U on decomposing X and find, the multiplicative inverses of the diagonal dements of D to form D' making it so compulation of B much faster.
_	