Seoul National University

Peer Seminar

$\mathcal{M}_2 \cong \mathbb{A}^2$

Younghun Jo

Department: Mathematical Sciences

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Contents

In this article, we cover Milnor's multiplier spectrum construction [2] yielding an isomorphism between the moduli space of quadratic rational functions and the affine plane. Theoretical background is based on [4, Section 4].

Throughout this article, let K be a field of characteristic 0, and $\overline{K}$ be its algebraic closure.

1 The Moduli Spaces of Rational Functions and Dynamical Systems

In this section, we review general theory of rational functions.

A *rational function* $\phi(z) \in K(z)$ is a quotient of polynomials

$$\phi(z) = \frac{F(z)}{G(z)} = \frac{a_0 + a_1 z + \cdots + a_d z^d}{b_0 + b_1 z + \cdots + b_d z^d}$$

with no common factors, and let the *degree of* ϕ be

$$\deg \phi = \max\{\deg F, \deg G\}.$$

Or equivalently, we consider a degree d rational function $\phi(z) \in K(z)$ as a rational map

$$\phi = [F : G] : \mathbb{P}^1 \to \mathbb{P}^1$$

defined over K, where $F, G \in K[X, Y]$ are homogeneous polynomials of degree d with no common factors. One of the main concern of arithmetic dynamics is the dynamics of ϕ, namely the behavior of points in $\mathbb{P}^1$ under iterates of ϕ. We will mostly deal with the case when $d \geq 2$.

The dynamics of a rational map ϕ remains unchanged under a change of variables on $\mathbb{P}^1$. For a *linear fractional transformation* (or *Möbius transformation*)

$$f(z) = \frac{az + b}{cz + d} \in \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2,$$

the *linear conjugate of* ϕ *by* f is the map

$$\phi^f = f^{-1} \circ \phi \circ f.$$

We call the conjugacy class of ϕ the *dynamical system associated to* ϕ, and denote by $[\phi]$.

When we study the dynamics of rational maps, it is natural to consider the moduli space of all rational functions. The space Rat_d of rational maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d is an affine variety defined over $\mathbb{Q}$ with the natural identification

$$\begin{array}{rcl} \{[\mathbf{a} : \mathbf{b}] \in \mathbb{P}^{2d+1} : \mathrm{Res}(F_{\mathbf{a}}, F_{\mathbf{b}}) \neq 0\} & \longrightarrow & \mathrm{Rat}_d \\ [\mathbf{a} : \mathbf{b}] & \longmapsto & \phi = [F_{\mathbf{a}} : F_{\mathbf{b}}] \end{array}$$

where

$$F_{\mathbf{a}}(X, Y) = a_0 X^d + a_1 X^{d-1} Y + \cdots + a_d Y^d$$

for a $(d+1)$-tuple $\mathbf{a} = (a_0, \ldots, a_d)$.

Note that the ring of regular functions of Rat_d is given by

$$\mathbb{Q}[\mathrm{Rat}_d] = (\mathbb{Q}[\mathbf{a}, \mathbf{b}]_{\mathrm{Res}(F_{\mathbf{a}}, F_{\mathbf{b}})})_0,$$

that is, the set of degree 0 elements in the localization of the graded algebra $\mathbb{Q}[\mathbf{a}, \mathbf{b}]$ at the multiplicative set generated by $\mathrm{Res}(F_{\mathbf{a}}, F_{\mathbf{b}})$. Also note that Rat_d is a *fine moduli space* of rational maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d. More precisely, the space Rat_d represents the functor from schemes to sets given by

$$S \longmapsto \Big\{ S\text{-morphisms } \phi : \mathbb{P}^1_S \to \mathbb{P}^1_S \text{ with } \phi^* \mathcal{O}_{\mathbb{P}^1_S}(1) \cong \mathcal{O}_{\mathbb{P}^1_S}(d) \Big\}.$$

This can be easily proved by an affine-local argument.

The linear conjugation induces an algebraic group action of PGL_2 on Rat_d. The *moduli space* $\mathcal{M}_d$ *of dynamical systems of degree* d *on* $\mathbb{P}^1$ is defined as the quotient space

$$\mathcal{M}_d = \mathrm{Rat}_d \,/\, \mathrm{PGL}_2,$$

the abstract quotient space with no additional structure. The canonical map is denoted by $[\,\cdot\,] : \mathrm{Rat}_d \to \mathcal{M}_d$.

Alternatively, with the aid of geometric invariant theory, we may give $\mathcal{M}_d$ the structure of an algebraic variety. By some technical reasons, it is preferable to consider PSL_2-action in place of PGL_2-action. Note that PGL_2-action induces the action of PSL_2 on Rat_d. The algebraic variety $\mathcal{M}_d$ is defined as the GIT quotient

$$\mathcal{M}_d = \mathrm{Rat}_d \,//\, \mathrm{PSL}_2 = \mathrm{Spec}\, \mathbb{Q}[\mathrm{Rat}_d]^{\mathrm{PSL}_2}.$$

Adopting this definition, $\mathcal{M}_d$ is a connected integral affine scheme of dimension $2d-2$ defined over $\mathbb{Q}$, and is a *coarse moduli space* of dynamical systems of degree d on $\mathbb{P}^1$. More precisely, there exists a natural transformation from the functor from schemes to sets given by

$$S \longmapsto \Big\{ S\text{-morphisms } \phi : \mathbb{P}^1_S \to \mathbb{P}^1_S \text{ with } \phi^* \mathcal{O}_{\mathbb{P}^1_S}(1) \cong \mathcal{O}_{\mathbb{P}^1_S}(d) \Big\} \Big/ \sim$$

where the equivalence relation is given by the linear conjugation, to the hom functor into $\mathcal{M}_d$.

In particular, for $d = 2$ case, there is an explicit description for the structure of the moduli space $\mathcal{M}_2$. The main theorem of this article is the following isomorphism, firstly constructed over $\mathbb{C}$ by Milnor [2], and then generalized to schemes over $\mathbb{Z}$ by Silverman [3].

Theorem 1.1 ($\mathcal{M}_2 \cong \mathbb{A}^2$)**.** *There exists an explicit isomorphism of algebraic varieties over* $\mathbb{Z}$*:*

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2) : \mathcal{M}_2 \xrightarrow{\sim} \mathbb{A}^2,$$

which induces a bijection $\mathcal{M}_2(\mathbb{C}) \cong \mathbb{A}^2(\mathbb{C})$*.*

2 Multiplier Spectrum

In this section, construct a collection of invariants of a rational map under linear conjugation, so called the multiplier spectra of a rational map. Throughout this section, let $\phi \in \overline{K}(z)$ be a rational function of degree $d \geq 2$.

For a point P in $\mathbb{P}^1$, consider the derivative $\phi'(P)$ of ϕ at P. In general, the value of $\phi'(P)$ is not independent of a choice of coordinates. Nevertheless, if P is a fixed point of ϕ, by chain rule, for a linear fractional transformation $f \in \mathrm{PGL}_2(\overline{K})$ and $Q = f^{-1}(P)$,

$$(\phi^f)'(Q) = (f^{-1}\phi f)'(Q) = (f^{-1})'(P) \cdot \phi'(P) \cdot f'(Q) = \phi'(P).$$

Hence, roughly speaking, the *multiplier* $\lambda_P(\phi) = \phi'(P)$ *of* ϕ *at* P is coordinate-invariant if we keep track of the fixed point P under a change of coordinates. More generally, for a periodic point P of exact period n for ϕ, *the multiplier of* ϕ *at* P is defined as $\lambda_P(\phi) = (\phi^n)'(P)$.

However, considering $\lambda_P(\phi)$ itself is not appropriate, since we cannot pick out a particular fixed point (or periodic point) for a given map ϕ. Instead, we consider the whole collection of multipliers $\lambda_P(\phi)$ where P ranges over all the fixed points of ϕ, or all the periodic points of ϕ of period n. Denoting the periodic points for ϕ of period n by $\mathrm{Per}_n(\phi)$, the *n-multiplier spectrum of* ϕ is the set

$$\Lambda_n(\phi) = \{\lambda_P(\phi) : P \in \mathrm{Per}_n(\phi)\}$$

where the multipliers are counted with multiplicity. Then the multiplier spectrum $\Lambda_n(\phi)$ is PGL_2-invariant, so it defines a set-valued function $\mathcal{M}_d \to \underline{\mathrm{Set}}$. Note that a rational map ϕ of degree d has exactly $d^n + 1$ periodic points of period n counted with multiplicity.

Recall that the elementary symmetric polynomials on m variables are algebraically independent over any commutative ring. So considering the elementary symmetric polynomials in multipliers is equivalent to considering the set of multipliers itself. For each $1 \leq i \leq d^n + 1$, define $\sigma_i^{(n)}(\phi)$ to be the ith elementary symmetric polynomial in $\Lambda_n(\phi)$.

By symmetry, the maps $\sigma_i^{(n)}$ on $\mathcal{M}_d$ are defined over $\mathbb{Q}$ and are regular, i.e., $\sigma_i^{(n)} \in \mathbb{Q}[\mathcal{M}_d]$. (See [4, Theorem 4.50] for a detailed proof.) Hence, we obtain a morphism

$$\boldsymbol{\sigma}_{d,N} = (\sigma_i^{(n)})_{1 \leq n \leq N,\ 1 \leq i \leq d^n+1} : \mathcal{M}_d \longrightarrow \mathbb{A}^k,$$

which extracts some information of dynamical systems.

One surprising fact is that the multiplier spectrum extracts most of the information of dynamical systems.

Theorem 2.1. *Fix $d \geq 2$. Then for sufficiently large N, the map $\boldsymbol{\sigma}_{d,N}$ is finite-to-one on $\mathcal{M}_d(\mathbb{C})$ except for cetrain families of Lattès map. The number of ramification for generic points stabilizes as $N \to \infty$.*

3 The Isomorphism between $\mathcal{M}_2$ and $\mathbb{A}^2$

In this section, we look into a partial proof of Theorem 1.1, establishing the bijection $\mathcal{M}_2(\mathbb{C}) \cong \mathbb{A}^2(\mathbb{C})$.

In short, the isomorphism between $\mathcal{M}_2$ and $\mathbb{A}^2$ is given by the multiplier spectrum of fixed points, which has one redundancy coming from the Cauchy residue formula. The reason why the multiplier spectrum of fixed points sufficies for quadratic rational maps is twofold. Firstly, there are 3 fixed points of ϕ, and PGL_2-action is 3-transitive. Secondly, the Lattès maps are cubic.

We begin with the Cauchy residue formula.

Proposition 3.1. *Let $\phi(z) \in \overline{K}(z)$ be a rational map of degree $d \geq 2$. Assume that $\lambda_P(\phi) \neq 1$ for all fixed points $P \in \mathbb{P}^1(\overline{K})$ of ϕ, that is, every fixed point of ϕ is of multiplicity one. Then,*

$$\sum_{P \in \mathrm{Fix}(\phi)} \frac{1}{1 - \lambda_P(\phi)} = 1.$$

Proof. By the Lefschetz principle, we can embed a sufficiently large subfield of $\overline{K}$ into $\mathbb{C}$, so we may assume that $\overline{K} = \mathbb{C}$.

We may assume that $\phi(\infty) \neq \infty$. Put $\psi(z) = \frac{1}{\phi(z)-z}$, then the poles of ψ are exactly the fixed points of ϕ. Hence, by the Cauchy residue formula, we have

$$\begin{aligned}
0 &= \operatorname*{Res}_{z=\infty}(\psi(z)) + \sum_{P \in \mathrm{Fix}(\phi)} \operatorname*{Res}_{z=P}(\psi(z)) \\
&= -\operatorname*{Res}_{z=0}(z^{-2}\psi(z^{-1})) + \sum_{P \in \mathrm{Fix}(\phi)} \operatorname*{Res}_{z=P}(\psi(z)) \\
&= -\lim_{z \to 0} \frac{1}{z(\phi(z^{-1}) - z^{-1})} + \sum_{P \in \mathrm{Fix}(\phi)} \lim_{z \to P} \frac{z - P}{\phi(z) - z} \\
&= 1 + \sum_{P \in \mathrm{Fix}(\phi)} \frac{1}{\lambda_P(\phi) - 1}.
\end{aligned}$$

□

Let $\phi(z) \in \mathbb{C}(z)$ be a quadratic rational map. Denote its multiplier spectrum for fixed points by $\{\lambda_1, \lambda_2, \lambda_3\}$, and the associated elementary symmetric polynomials by $\sigma_1, \sigma_2, \sigma_3$. Then by above proposition, we have

$$\frac{1}{1-\lambda_1} + \frac{1}{1-\lambda_2} + \frac{1}{1-\lambda_3} = 1,$$

which implies

$$\lambda_1\lambda_2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_3 + 2 = 0, \tag{1}$$

that is, $\sigma_1 = \sigma_3 + 2$. (More precisely, the set of ϕ's with simple fixed points is dense in $\mathrm{Rat}_2(\mathbb{C})$, so we can deduce the above identity by continuity.) Indeed, this is the only redundancy between the maps $\sigma_1, \sigma_2, \sigma_3$.

Now we prove the injectivity and surjectivity of the map $(\sigma_1, \sigma_2) : \mathcal{M}_2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$. It can be done with the following lemma, which easily follows from 3-transitivity of PGL_2-action.

Lemma 3.2 (Normal form)**.** *Let $\phi(z) \in \mathbb{C}(z)$ be a quadratic rational map.*

(a) If $\lambda_1\lambda_2 \neq 1$, then $\lambda_1, \lambda_2 \neq 1$, and ϕ is linearly conjugate to a rational map of the form

$$\frac{z^2 + \lambda_1 z}{\lambda_2 z + 1}.$$

Note that the resultant of $z^2 + \lambda_1 z$ and $\lambda_2 z + 1$ is $1 - \lambda_1\lambda_2$.

(b) If $\lambda_1\lambda_2 = 1$, then $\lambda_1 = \lambda_2 = 1$, and ϕ is linearly conjugate to a rational map of the form

$$z + \sqrt{1-\lambda_3} + \frac{1}{z}.$$

Proof. Note that the identity (1) implies

$$(\lambda_1 - 1)^2 = (\lambda_1\lambda_2 - 1)(\lambda_1\lambda_3 - 1),$$
$$(\lambda_2 - 1)^2 = (\lambda_2\lambda_3 - 1)(\lambda_2\lambda_1 - 1),$$
$$(\lambda_3 - 1)^2 = (\lambda_3\lambda_1 - 1)(\lambda_3\lambda_2 - 1).$$

(1) If $\lambda_1\lambda_2 \neq 1$, then by the above identities, λ_1 and λ_2 are not equal to 1. Hence, the fixed points associated ot λ_1 and λ_2 are distinct, so we may assume that they are 0 and ∞, respectively. Then ϕ is of the form

$$\phi(z) = \frac{a_0z^2 + a_1z}{b_1z + b_2}, \qquad a_0b_2 \neq 0.$$

We may assume that $a_0 = 1$, then $\phi'(0) = \lambda_1 = a_1/b_2$ and $\phi'(\infty) = b_1 = \lambda_2$ implies that

$$\phi(z) = \frac{z^2 + b_2\lambda_1 z}{\lambda_2 z + b_2}, \qquad b_2 \neq 0.$$

Conjugating ϕ by $f(z) = b_2z$ gives the desired result. Note that the three fixed points of ϕ are $0, \infty, \frac{1-\lambda_1}{1-\lambda_2}$.

(2) If $\lambda_1\lambda_2 \neq 1$, then by the above identities, λ_1 and λ_2 are both equal to 1. We may assume that the double (or triple) fixed point associated to λ_1 is ∞. Since $\phi'(\infty) = 1$, ϕ is of the form

$$\phi(z) = \frac{a_0z^2 + a_1z + a_2}{a_0z + b_2}, \qquad a_0 \neq 0.$$

We may assume that $a_0 = 1$. Conjugating ϕ by $f(z) = z - b_2$ gives a rational map of the form

$$\frac{z^2 + a_1'z + a_2'}{z},$$

then again conjugating by $g(z) = \sqrt{a_2'}z$ gives

$$z + a + \frac{1}{z}.$$

The third multiplier is $\phi'(-a^{-1}) = 1 - a^2 = \lambda_3$, so $a = \sqrt{1-\lambda_3}$. Note that the three fixed points of ϕ are $\infty, \infty, -\frac{1}{\sqrt{1-\lambda_3}}$. $\square$

By the above lemma, the unique dynamical system with multipliers $\lambda_1, \lambda_2, \lambda_3$ comes from one of the normal forms in the lemma, so $(\sigma_1, \sigma_2) : \mathcal{M}_2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$ is bijective.

References

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