Metastability of Contact Processes

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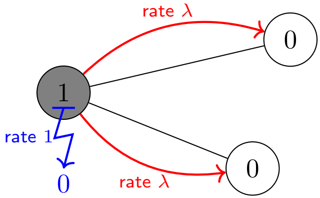
Contact process

- $G = (V, E)$ (locally) finite connected graph
- $\lambda > 0$ *infection rate*
- Configurations of the contact process: $\eta \in \{0,1\}^V$

  For $x \in V$, $\eta(x) = \begin{cases} 0 & x \text{ is } \textit{healthy} \\ 1 & x \text{ is } \textit{infected} \end{cases}$
- Abuse of notation: identify η with $\{x \in V : \eta(x) = 1\}$



- The *all-healthy state* $\eta = \emptyset$ is the unique absorbing state.

Metastability of contact processes

The *extinction time* of the contact process is

$$\tau_G = \inf\{t \geq 0 : \eta_t = \emptyset\}.$$

Q. Fix a (increasing) sequence of graphs $(G_N)_{N\geq 1}$, and study the growth of τ_{G_N}.

Finite-volume phase transition for boxes ('84–'99)

On $\mathbb{Z}_N^d = [1, N]^d$ with free boundary, we have

$$\tau_{\mathbb{Z}_N^d} \sim \begin{cases} \log |\mathbb{Z}_N^d| & \text{if } \lambda < \lambda_c, \\ \exp\big(c_\lambda |\mathbb{Z}_N^d|\big) & \text{if } \lambda > \lambda_c \end{cases}$$

where $|G|$ denotes the number of vertices.

The latter case is a clear demonstration of the metastable behavior.

Metastability of contact processes

More generally, the following theorem holds.

Theorem (MMYV '16, SV '17)

Suppose that $\lambda > 0$ is sufficiently large.

(a) For all $D > 0$, there exists $c = c(\lambda, D)$ such that

$$\mathbb{E}\tau_G \geq \exp(c|G|) \quad \text{for all } G \text{ with degrees} \leq D.$$

(b) For all $\varepsilon > 0$, there exists $c = c(\lambda, \varepsilon)$ such that

$$\mathbb{E}\tau_G \geq \exp\left(c \cdot \frac{|G|}{(\log |G|)^{1+\varepsilon}}\right) \quad \text{for all } G.$$

Precise estimate for mean extinction time

Levels of precision for mean transition time estimate:

① $\mathbb{E}\tau_N \geq \exp(cN)$ (Metastability)

② $\dfrac{1}{N} \log \mathbb{E}\tau_N \longrightarrow c$ (Large-deviation principle)

③ $\mathbb{E}\tau_N \simeq f(N)e^{cN}$ (Eyring–Kramers law)

- ① holds for graphs of uniformly bounded degree. (MMYV '16)
- ② holds for $\mathbb{Z}_N^d$ with free boundary. (Mountford '99)
  However, ② is open for even $\mathbb{Z}_N^d$ with periodic boundary.
- ② holds for a variety of random graph models. (Shapira–Valesin '21)
- ③ is open only except for two cases.
  1. The triviality: complete graph K_N
  2. Main Result: star graph S_N (**J.** '24)

Main result

Eyring–Kramers law (**J.** '24)

Let S_N be the star graph with one hub and N leaves. Then, we have

$$\mathbb{E}\tau_{S_N} \simeq \kappa_\lambda N^{-\frac{1}{1+2\lambda}} \left(\frac{(1+\lambda)^2}{1+2\lambda} \right)^N .$$

In particular, we have

$$\frac{1}{N} \log \mathbb{E}\tau_{S_N} \xrightarrow{N\to\infty} c_\lambda = 2\log(1+\lambda) - \log(1+2\lambda).$$

Main ingredients:

- *Special function theory* for precise estimation of *quasi-stationary measure*
- The *potential theoretic approach* to metastability of non-reversible processes

These methodologies have not previously been used in the study of the contact process.