Seoul National University 2024 Winter Mentoring
Introduction to Category Theory
Problem Set

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Last Update: February 12, 2024

Contents

Foreword

Remark. This problem set was created as a supplementary material for the 8-lecture course *Introduction to Category Theory* held from January 9, 2024, to February 8, 2024, as a part of the Seoul National University 2024 Winter Mentoring program. The objective of the course was to offer a category theoretic viewpoint on undergraduate mathematics, and to introduce basic concepts of homological algebra.

As this course does *not* delve into rigorous set theory, we will overlook set theoretical details in the problems below. In other words, we will treat all the 'instances' as if they were sets. If you require precision on such matters, you may assume that the given indeterminate categories are *small* (meaning the objects form a set, and morphisms between each pair of objects form a set), or *locally small* (meaning morphisms between each pair of objects form a set) if needed.

For most of the problems below, if a specific category is mentioned, it will be extremely difficult (and might be impossible) to solve the problem relying solely on concepts of category theory. Many of these problems are mere restatements of results covered in other courses, and you are allowed to use *any theorem from standard undergraduate mathematics textbooks* to solve them unless otherwise stated. Preliminary courses opened in SNU required for solving each problem will be (loosely) indicated at the beginning of the problem statement according to the following list. Also, exceptionally hard problems will be asterisked.

- [S]: Sets and Mathematical Logic
- [L]: Linear Algebra
- [A]: Modern Algebra
- [T]: Introduction to Topology
- [D]: Introduction to Differential Geometry
- [C]: Complex Function Theory
- [RA]: Real Analysis
- [AT]: Algebraic Topology
- [CA]: Commutative Algebra
- [AG]: Algebraic Geometry

Acknowledgement. I would like to express my gratitude to everyone who played a part in completing this problem set. Big thanks to my academic companions, Jeonghyun Ahn and Minchan Kang, for our fruitful discussions about the course outline and these problems. I am also grateful to my anonymous friend who really enjoys set theory; he diligently examined this problem set, catching typos and ensuring mathematical precision. A huge shoutout to many other friends who shared their thoughts, even though I cannot list everyone individually. Your contributions made this course material much better. Also, special thanks to the 41th student council for College of Natural Sciences at Seoul National University, and all the mentees who joined the course, for their support in making this course happen.

Lecture 1: Category and Universality

Category

Problem 1. Verify that the following examples are categories. All the composition laws are given by composition of maps.

- The category $\underline{\text{Set}}$ of sets and set maps.
- [L] The category $\underline{\text{Vec}}_k$ of vector spaces over a field k and linear transformations.
- [A] The category $\underline{\text{Grp}}$ of groups and group homomorphisms.
- [A] The category $\underline{\text{Ab}}$ of abelian groups and group homomorphisms.
- [A] The category $\underline{\text{Ring}}$ of rings with unity and ring homomorphisms (preserving unity).
- [A] The category $\underline{\text{CRing}}$ of commutative rings with unity and ring homomorphisms.
- [A] The category $\underline{\text{Field}}$ of fields and field homomorphisms.
- [A] The category $\underline{\text{Field}}_p$ of fields of characteristic p and field homomorphisms.
- [T] The category $\underline{\text{Top}}$ of topological spaces and continuous maps.
- [D] The category $\underline{\text{Man}}$ of smooth manifolds and smooth maps.
- [C] The category $\underline{\text{Hol}}$ of open subsets of $\mathbb{C}$ and holomorphic maps.
- [S] The category $\underline{\text{Poset}}$ of partially ordered sets and order-preserving maps.

Problem 2. Verify that the following examples are categories. All the composition laws are given by composition of maps.

- The category $\underline{\text{Set}}_*$ of pointed sets and base-preserving set maps. More precisely, the objects of $\underline{\text{Set}}_*$ are the pairs (X, x) where X is a set with a base point $x \in X$, and morphisms $(X, x) \to (Y, y)$ are set maps $X \to Y$ that sends x to y.
- [T] The category $\underline{\text{Top}}_*$ of pointed topological spaces and base-preserving continuous maps. More precisely, the objects of $\underline{\text{Top}}_*$ are the pairs (X, x) where X is a topological space with a base point $x \in X$, and morphisms $(X, x) \to (Y, y)$ are continuous maps $X \to Y$ that sends x to y.
- [T] The category $\underline{\text{Open}}(X)$ of open subsets of a topological space X and inclusions. More precisely, the objects of $\underline{\text{Open}}(X)$ are open subsets $U \subseteq X$, and there is the unique morphism $\iota_{UV} : U \to V$ that is the inclusion map if $U \subseteq V$, otherwise there is no morphism $U \to V$.

Categorical Concept

Problem 3. *Morphisms of a category need not be "maps."* Verify that the following examples are categories.

- *Every set is a category.* Given a set S, with an abuse of notation, let S be a category with $\text{Obj}(S) = S$ and the only morphisms are the required identities.
- [A] *Every group is a category.* Given a group G, with an abuse of notation, let G be a category with one object $\bullet$ and $\text{Mor}(\bullet, \bullet) = G$, where the composition law is given by group operation.
- [S] *Every poset is a category.* Given a poset $(P, \leq)$, with an abuse of notation, let P be a category with $\text{Obj}(P) = P$, while there exists a unique morphism $x \to y$ iff $x \leq y$. The composition law

is given in the obvious way. Note that $\underline{\mathrm{Open}}(X)$ in the previous problem is a special case of this construction—the topology of X is partially ordered by inclusion.

- [L] Given a commutative ring R, let $\underline{\mathrm{Mat}}_R$ be the category of matrices, whose objects are natural numbers, and morphisms $n \to m$ are $m \times n$ R-matrices, with composition the usual matrix multiplication.
- With an abuse of notation, let $\mathbb{N}$ be a category with one object $\bullet$ and $\mathrm{Mor}(\bullet, \bullet) = \mathbb{N}$, where the composition law is given by addition.

Problem 4. *Categorical concepts versus set theoretic concepts.* If we are working on a *concrete category*—that is, roughly speaking, a category where every object has an "underlying set"—then we can consider set theoretic properties of morphisms, such as injectivity and surjectivity.

(1) Prove the following.

  (a) Every injective morphism is monic.

  (b) Every surjective morphism is epic.

(2) However, the set theoretic concepts are not identical to categorical concepts. In each part, provide an example of a morphism between objects in a concrete category with the given conditions.

  (a) A monomorphism that is not injective.

  (b) An epimorphism that is not surjective. (Hint: $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in $\underline{\mathrm{Ring}}$)

Problem 5. (1) Prove that every isomorphism is monic and epic.

(2) Find a category with a morphism that is monic and epic, but not an isomorphism.

Problem 6. Show that in $\underline{\mathrm{Set}}$, $\underline{\mathrm{Vec}}_k$, and $\underline{\mathrm{Ab}}$, (1) every monomorphism is injective, (2) every epimorphism is surjective, and (3) every morphism that is monic and epic is an isomorphism.

Universal Object

Problem 7. Check the following.

- The emptyset $\emptyset$ is the unique initial object in $\underline{\mathrm{Set}}$. A singleton 1 is a terminal object in $\underline{\mathrm{Set}}$.
- [L] The zero space (0) is a zero object in $\underline{\mathrm{Vec}}_k$.
- [A] The trivial group $\{e\}$ is a zero object in $\underline{\mathrm{Grp}}$ and $\underline{\mathrm{Ab}}$.
- [A] The integer ring $\mathbb{Z}$ is an initial object in $\underline{\mathrm{Ring}}$ and $\underline{\mathrm{CRing}}$. The zero ring $\{0\}$ is a terminal object in $\underline{\mathrm{Ring}}$ and $\underline{\mathrm{CRing}}$.
- [A] There are no initial or terminal objects in $\underline{\mathrm{Field}}$.
- [A] The prime fields $\mathbb{Q}$, $\mathbb{F}_p$ are initial objects in $\underline{\mathrm{Field}}_p$. There is no terminal object in $\underline{\mathrm{Field}}_p$.

Problem 8. Show the following.

- Let $\{S_i\}_{i\in I}$ be an arbitrary family of sets. Then the cartesian product $\prod S_i$ and the disjoint union $\coprod S_i$ are a product and a coproduct of the family in $\underline{\mathrm{Set}}$, respectively.
- [A] Let $\{G_i\}_{i\in I}$ be an arbitrary family of groups. Then the cartesian product $\prod G_i$ and the free product $\ast\, G_i$ are a product and a coproduct of the family in $\underline{\mathrm{Grp}}$, respectively.

- [A] Let $\{A_i\}_{i\in I}$ be an arbitrary family of abelian groups. Then the cartesian product $\prod A_i$ and the direct sum $\bigoplus A_i$ are a product and a coproduct of the family in $\underline{\mathrm{Ab}}$, respectively. Also note that if I is finite, then $\bigoplus A_i$ and $\prod A_i$ coincides.
- [A] Let $\{R_i\}_{i\in I}$ be an arbitrary family of commutative rings with unity. Then the cartesian product $\prod R_i$ and the tensor product $\bigotimes R_i$ are a product and a coproduct of the family in $\underline{\mathrm{CRing}}$, respectively. (Hint: For the coproduct, prove the finite case first, and make a proper definition of $\bigotimes R_i$ for an infinite family $\{R_i\}_{i\in I}$.)
- [T] Let $\{X_i\}_{i\in I}$ be an arbitrary family of topological spaces. Then the product space $\prod X_i$ and the disjoint union $\coprod X_i$ are a product and a coproduct of the family in $\underline{\mathrm{Top}}$, respectively.
- [T] Let $\{(X_i, x_i)\}_{i\in I}$ be an arbitrary family of pointed topological spaces. Then the product space $(\prod X_i, (x_i))$ and the wedge sum $(\bigvee X_i, x) = (\coprod_i X_i/(x_i \sim x_j)_{i,j}, x)$ are a product and a coproduct of the family in $\underline{\mathrm{Top}}_*$, respectively.
- [S] Let $\{x_i\}_{i\in I}$ be an arbitrary family of elements in a poset P, regarded as a category. Then the meet $\bigwedge x_i$ and the join $\bigvee x_i$ are a product and a coproduct of the family in P (if they exist), respectively.

Problem 9. Let $\underline{\mathrm{C}}$ be a category, and Z be an object of $\underline{\mathrm{C}}$.

(1) The *overcategory* $\underline{\mathrm{C}}/Z$ *of* $\underline{\mathrm{C}}$ *over* Z is the category whose objects are morphisms $X \xrightarrow{f} Z$ in $\underline{\mathrm{C}}$, and the morphisms from $X \xrightarrow{f} Z$ to $Y \xrightarrow{g} Z$ are the morphisms $X \xrightarrow{h} Y$ in $\underline{\mathrm{C}}$ which makes the following diagram commutative.

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Verify that $\underline{\mathrm{C}}/Z$ is a category, and $\mathrm{id}_Z : Z \to Z$ is a terminal object in $\underline{\mathrm{C}}/Z$.

(2) The *undercategory* $Z/\underline{\mathrm{C}}$ *of* $\underline{\mathrm{C}}$ *under* Z is the category whose objects are morphisms $Z \xrightarrow{f} X$ in $\underline{\mathrm{C}}$, and the morphisms from $Z \xrightarrow{f} X$ to $Z \xrightarrow{g} Y$ are the morphisms $X \xrightarrow{h} Y$ in $\underline{\mathrm{C}}$ which makes the following diagram commutative.

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "h"] & & Y \end{tikzcd}$$

Verify that $Z/\underline{\mathrm{C}}$ is a category, and $\mathrm{id}_Z : Z \to Z$ is an initial object in $Z/\underline{\mathrm{C}}$.

Problem 10. Let $\underline{\mathrm{C}}$ be a category, and Z be an object of C.

(1) Let $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ be objects of the overcategory $\underline{\mathrm{C}}/Z$. The *fibered product* (or, *pullback*) of f and g is the product in $\underline{\mathrm{C}}/Z$ and is denoted by $X \times_Z Y$, with its canonical morphisms $X \times_Z Y \xrightarrow{p_1} X$ and $X \times_Z Y \xrightarrow{p_2} Y$.

Show that fibered products exist when $\underline{\mathrm{C}}$ is the category $\underline{\mathrm{Ab}}$ of abelian groups. (Hint: Define

$X \times_Z Y$ as the subgroup of $X \oplus Y$ consisting of elements (x, y) such that $f(x) = g(y)$.)

$$
\begin{tikzcd}
T \arrow[dr, dotted, "\exists! h"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}
$$

(2) Let $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$ be objects of the undercategory $Z/\underline{\mathrm{C}}$. The *fibered coproduct* (or, *pushout*) of f and g is the coproduct in $Z/\underline{\mathrm{C}}$ and is denoted by $X +_Z Y$, with its canonical morphisms $X \xrightarrow{i_1} X +_Z Y$ and $Y \xrightarrow{i_2} X +_Z Y$.
Show that fibered coproducts exist when $\underline{\mathrm{C}}$ is the category $\underline{\mathrm{Ab}}$ of abelian groups. (Hint: Define $X +_Z Y$ as the factor group of $X \oplus Y$ obtained by identifying the elements sharing the common preimages of $\iota_1 \circ f$ and $\iota_2 \circ g$.)

$$
\begin{tikzcd}
T & & \\
& X +_Z Y \arrow[ul, dotted, "\exists! h"'] & Y \arrow[l, "i_2"'] \arrow[ull, bend right, "j_2"'] \\
& X \arrow[u, "i_1"] \arrow[uul, bend left, "j_1"] & Z \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

Problem 11 (The isomorphism theorems of group theory)**.** [A] Let G be a group, and N be its normal subgroup.

(1) Let $\underline{\mathrm{C}}_{G/N}$ be the category of group homomorphisms from G annihilating N. More precisely, the objects of $\underline{\mathrm{C}}_{G/N}$ are group homomorphisms $G \xrightarrow{f} H$ such that $f(N) = \{e\}$, and morphisms from $G \xrightarrow{f} H$ to $G \xrightarrow{g} K$ are the group homomorphisms $H \xrightarrow{h} K$ that makes the following diagram commutative.

$$
\begin{tikzcd}
& G \arrow[dl, "f"'] \arrow[dr, "g"] & \\
H \arrow[rr, "h"] & & K
\end{tikzcd}
$$

Verify that $\underline{\mathrm{C}}_{G/N}$ is indeed a category, and the natural projection $p : G \to G/N$ is an initial object of $\underline{\mathrm{C}}_{G/N}$.

(2) Use only universality of natural projection to prove the following isomorphism theorems of group theory.

- For normal subgroups N, K of G with $N \subseteq K$, $(G/N)/(K/N) \cong G/K$.
- For subgroups H, N of G with N normal, $HN/N \cong H/(H \cap N)$.

(3) State and prove the isomorphism theorems for vector spaces, rings, and modules in the same manner.

Problem 12. [A] Let A be a commutative ring. The *ring of polynomials* $A[X]$ *in* X *over* A can be defined as the set of finitely supported functions $f : \mathbb{N} \to A$, regarded as a commutative ring under the pointwise addition and the convolution multiplication.

(1) Show that $A[X_1][X_2]$ is canonically isomorphic to $A[X_2][X_1]$. Note that this guarantees well-definedness of $A[X_1, \ldots, X_n]$.

(2) Let $S = \{x_1, \ldots, x_n\}$ be a finite set. Consider the category $\underline{\mathrm{C}}$ of n-pointed commutative rings. More precisely, the objects of $\underline{\mathrm{C}}$ are set maps $f : S \to A$ with $A \in \mathrm{Obj}(\underline{\mathrm{CRing}})$, and morphisms from $S \xrightarrow{f} A$ to $S \xrightarrow{g} B$ are the ring homomorphisms $A \xrightarrow{\phi} B$ that makes the following diagram commutative.

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "\phi"] & & B \end{tikzcd}$$

Show that $S \xrightarrow{\iota} \mathbb{Z}[X_1, \ldots, X_n]$ with $\iota(x_i) = X_i$, $i = 1, \ldots, n$ is an initial object in $\underline{\mathrm{C}}$. The ring $\mathbb{Z}[X_1, \ldots, X_n]$ is called the *free commutative ring on* n *variables.*

Problem 13 (Tensor product)**.** Let R be a ring.

(1) Let $M = M_R$ be a module-R, and $N = {}_RN$ be an R-module.
Let $\underline{\mathrm{C}}$ be a category of bilinear maps from $M \times N$. More precisely, the objects of $\underline{\mathrm{C}}$ are set maps $M \times N \xrightarrow{f} G$, where A is an abelian group, satisfying

$$\begin{aligned} f(m+m', n) &= f(m,n) + f(m', n) \\ f(m, n+n') &= f(m,n) + f(m,n') \\ f(mr, n) &= f(m, rn) \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$. The morphisms $(M \times N \xrightarrow{f} A) \to (M \times N \xrightarrow{g} A')$ are group homomorphisms $A \xrightarrow{\phi} A'$ making the following diagram commutative.

$$\begin{tikzcd} & M \times N \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "\phi"] & & A' \end{tikzcd}$$

Show that $\underline{\mathrm{C}}$ is indeed a category with an initial object $M \times N \to M \otimes_R N$. We call the abelian group $M \otimes_R N$ the *tensor product* of M and N over R. (Hint: Consider the free abelian group generated by $m \otimes n$, $m \in M$, $n \in N$, and construct $M \otimes_R N$ as an appropriate quotient of this abelian group. See Hungerford IV.5.)

(2) Suppose that R is commutative. Show that $M \otimes_R N$ is an R-module under the scalar product given by $r(m \otimes n) = (mr) \otimes n = m \otimes (rn)$.

(3) Let M and N be commutative rings, also viewed as $\mathbb{Z}$-modules. Show that $M \otimes_{\mathbb{Z}} N$ is a commutative ring under the product given by $(m \otimes n)(m' \otimes n') = (mm') \otimes (nn')$.

Appendix: Categorical Arguments

Problem 14. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in a category $\underline{\mathrm{C}}$. Show the following. Compare these with their set theoretical analogues.

(1) If f and g are monic, then so is gf.
(2) If gf is monic, then so is f.

Dually,

(3) If f and g are epic, then so is gf.
(4) If gf is epic, then so is g.

Problem 15. Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in a category $\underline{\mathrm{C}}$, and suppose that $P = X \times_Z Y$ joined with $p_1 : P \to X$ and $p_2 : P \to Y$ is their pullback. Show that if g is monic, then so is p_1. State and prove its dual assertion.

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, hook, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Problem 16. Let the following commutative diagram be a diagram in a category $\underline{\mathrm{C}}$. Suppose that each square in the diagram is a pullback, and $i : Z' \to Z$ is monic. Show that the outer rectangle is a pullback.

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & Q \arrow[r, "q_2"] \arrow[d, "q_1"] & Z' \arrow[d, hook, "i"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Lecture 2: Functor and Naturality

Functor

Problem 17. Show that the following examples are functors (in the canonical way).

- Identity functor $\mathrm{Id} : \underline{\mathrm{C}} \to \underline{\mathrm{C}}$
- Forgetful functor $\mathrm{Forget} : \underline{\mathrm{Top}} \to \underline{\mathrm{Set}},\ \underline{\mathrm{Grp}} \to \underline{\mathrm{Set}},\ \underline{\mathrm{Ring}} \to \underline{\mathrm{Grp}},\ \dots$
- The (covariant) power set $2^{(-)} : \underline{\mathrm{Set}} \to \underline{\mathrm{Set}}$
- The (contravariant) power set $2^{(-)} : \underline{\mathrm{Set}}^{\mathbf{op}} \to \underline{\mathrm{Set}}$
- [L] The dual $(-)^* : \underline{\mathrm{Vec}}_k^{\mathbf{op}} \to \underline{\mathrm{Vec}}_k$ (given by $V \mapsto V^* = \mathrm{Hom}_k(V, k)$)
- [L] The double dual $(-)^{**} : \underline{\mathrm{Vec}}_k \to \underline{\mathrm{Vec}}_k$
- [L] The nth general linear group $\mathrm{GL}_n : \underline{\mathrm{CRing}} \to \underline{\mathrm{Grp}}$
- [A] The group of units $(-)^\times : \underline{\mathrm{CRing}} \to \underline{\mathrm{Grp}}$
- [A] The abelianization $(-)^{\mathrm{ab}} : \underline{\mathrm{Grp}} \to \underline{\mathrm{Ab}}$ (given by $G \mapsto G^{\mathrm{ab}} = G/[G, G]$)
- [A] The polynomial ring $(-)[X] : \underline{\mathrm{Ring}} \to \underline{\mathrm{Ring}}$
- [A] The commutator subgroup $[-, -] : \underline{\mathrm{Grp}} \to \underline{\mathrm{Grp}}$
- [AT] The fundamental group $\pi_1 : \underline{\mathrm{Top}}_* \to \underline{\mathrm{Grp}}$

Problem 18. Let $\underline{\mathrm{C}}$ be a category, and X be an object in $\underline{\mathrm{C}}$. Show that $\mathrm{Hom}_{\underline{\mathrm{C}}}(X, -) : \underline{\mathrm{C}} \to \underline{\mathrm{Set}}$ and $\mathrm{Hom}_{\underline{\mathrm{C}}}(-, X) : \underline{\mathrm{C}}^{\mathbf{op}} \to \underline{\mathrm{Set}}$ are functors in the canonical way. These functors are called the *hom functors*.

Problem 19. (1) What is a functor between groups, regarded as one-object categories?

(2) What is a functor between posets, regarded as categories?

(3) What is a functor from a group G to $\underline{\mathrm{Set}}$?

(4) What is a functor from a group G to $\underline{\mathrm{Vec}}_k$?

Problem 20. Establish the following isomorphisms of categories.

(1) $\underline{\mathrm{Ab}} \cong \mathbb{Z}\text{-}\underline{\mathrm{Mod}}$

(2) $\underline{\mathrm{Vec}}_k \cong k\text{-}\underline{\mathrm{Mod}}$ for a field k

Problem 21. Let R be a ring.

(1) Define the *opposite ring* $R^{\mathbf{op}}$ of R as the ring structure on R with the identical addition and the multiplication $*$ given by

$$a * b = b \cdot a$$

where $\cdot$ is the primary multiplication on R. Check that $R^{\mathbf{op}}$ is indeed a ring. In particular, note that if R is commutative, then $R^{\mathbf{op}} \cong R$.

(2) Let $M = {}_R M$ be an R-module. Define $M_{R^{\mathbf{op}}}$ as the module-$R^{\mathbf{op}}$ structure on M with the identical addition and the scalar product $*$ given by

$$m * r = r \cdot m$$

for $r \in R^{\mathbf{op}}$ and $m \in M$ where $\cdot$ is the primary scalar product on M. Show that $M_{R^{\mathbf{op}}}$ is indeed a module-$R^{\mathbf{op}}$. In particular, note that if R is commutative, then an R-module is also a module-R in the natural way.

(3) Show that $(-)_{R^{\mathbf{op}}} : R\text{-}\underline{\text{Mod}} \to \underline{\text{Mod}}\text{-}R^{\mathbf{op}}$ is a functor in the canonical way. Also show that it is an isomorphism between categories. In particular, note that if R is commutative, then the categories R-$\underline{\text{Mod}}$ and $\underline{\text{Mod}}$-R are isomorphic in the canonical way.

Problem 22. Let X be a set. Show that $- \times X : \underline{\text{Set}} \to \underline{\text{Set}}$ is a functor in the canonical way.

Problem 23. Let R be a commutative ring, and M be an R-module. Show that $- \otimes_R M : R\text{-}\underline{\text{Mod}} \to R\text{-}\underline{\text{Mod}}$ is a functor in the canonical way.

Problem 24. Show that there is no functor $\underline{\text{Grp}} \to \underline{\text{Ab}}$ that sends each group to its center. (Hint: Consider $S_2 \to S_3 \to S_2$, the symmetric groups.)

Problem 25. [A] Let $\underline{\text{Dom}}_m$ be the category of integral domains and injective ring homomorphisms. Show that the field of fractions $\text{Frac} : \underline{\text{Dom}}_m \to \underline{\text{Field}}$ is a functor in the canonical way.

Problem 26. [T] Let $\underline{\text{Met}}$ be the category of metric spaces and metric-preserving maps. Denote its full subcategory of complete metric spaces by $\underline{\text{cMet}}$. Verify that the completion $\overline{(-)} : \underline{\text{Met}} \to \underline{\text{cMet}}$ is a functor.

Problem 27 (Homotopy invariance of π_1)**.** [AT] Let X and Y be topological spaces. Recall that two continuous maps $f, g : X \to Y$ are homotopic if and only if there is a homotopy taking f to g, that is, a continuous map $H : X \times [0,1] \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$. Similarly, if $f, g : (X,x) \to (Y,y)$ are base-preserving continuous maps, then a homotopy taking f to g is a continuous map $H : X \times [0,1] \to Y$ such that $H(-,0) = f$, $H(-,1) = g$, and $H(x,-) = y$.

(1) Check that being (pointed) homotopic is an equivalence relation.
(2) Let $\underline{\text{Htpy}}$ be the category of topological spaces and homotopy classes of continuous maps. Similarly, let $\underline{\text{Htpy}}_*$ be the category of pointed topological spaces and homotopy classes of base-preserving continuous maps. Show that $\underline{\text{Htpy}}$ and $\underline{\text{Htpy}}_*$ are indeed categories.
(3) Show that there are functors $\underline{\text{Top}} \to \underline{\text{Htpy}}$ and $\underline{\text{Top}}_* \to \underline{\text{Htpy}}_*$ sending each object to itself and each morphism to its homotopy class.
(4) Show that the fundamental group $\pi_1 : \underline{\text{Top}}_* \to \underline{\text{Grp}}$ factors through the functor $\underline{\text{Top}}_* \to \underline{\text{Htpy}}_*$ to define a functor $\pi_1 : \underline{\text{Htpy}}_* \to \underline{\text{Grp}}$.

Natural Transformation

Problem 28. [L] Show that the determinant $\det : \text{GL}_n \to (-)^\times$ is a natural transformation from the nth general linear group $\text{GL}_n : \underline{\text{CRing}} \to \underline{\text{Grp}}$ to the group of units $(-)^\times : \underline{\text{CRing}} \to \underline{\text{Grp}}$, with components $\det_R : \text{GL}_n(R) \to R^\times$.

Problem 29. [L] Show that the evaluation map $\text{ev} : \text{id} \to (-)^{**}$ is a natural isomorphism from the identity $\text{id} : \underline{\text{Vec}}_k \to \underline{\text{Vec}}_k$ to the double dual $(-)^{**} : \underline{\text{Vec}}_k \to \underline{\text{Vec}}_k$, with components $\text{ev}_V : V \to V^{**}$ given by $v \mapsto (f \mapsto f(v))$

Problem 30 (The fundamental theorem of linear algebra)**.** [L] Let k be a field. Show that the category $\underline{\text{FVec}}_k$ of finite dimensional vector spaces over k is equivalent to the category $\underline{\text{Mat}}_k$ of k-matrices.

Problem 31. What is a natural transformation between two functors from a group G to $\underline{\text{Set}}$, where G is regarded as a one-object category?

Problem 32. Given categories $\underline{\text{C}}$ and $\underline{\text{D}}$, let $\underline{\text{Fun}}(\underline{\text{C}}, \underline{\text{D}})$ be a category whose objects are functors from $\underline{\text{C}}$ to $\underline{\text{D}}$ and morphisms are natural transformations. Verify that $\underline{\text{Fun}}(\underline{\text{C}}, \underline{\text{D}})$ is a well-defined category.

Problem 33 (Natural projection)**.** [A] In this problem, we assert that a natural projection is indeed natural.

(1) Let $\underline{\text{C}}$ be a category of normal subgroups. More precisely, the objects of $\underline{\text{C}}$ are pairs (G, N) of a group G and its normal subgroup N, and the morphisms $(G_1, N_1) \to (G_2, N_2)$ are pairs $(G_1 \xrightarrow{g} G_2, N_1 \xrightarrow{n} N_2)$ of group homomorphisms that make the following diagram commutative, where the hooked arrows are inclusions.

$$\begin{tikzcd} N_1 \arrow[r, hook] \arrow[d, "n"'] & G_1 \arrow[d, "g"] \\ N_2 \arrow[r, hook] & G_2 \end{tikzcd}$$

Verify that $\underline{\text{C}}$ is indeed a category.

(2) Show that $\mathbf{G} : \underline{\text{C}} \to \underline{\text{Grp}}$ assigning (G, N) the group G is a functor in the canonical way.

(3) Show that $\mathbf{G/N} : \underline{\text{C}} \to \underline{\text{Grp}}$ assigning (G, N) the group G/N is a functor in the canonical way.

(4) Show that the natural projection $\pi : \mathbf{G} \to \mathbf{G/N}$ is a natural transformation with components $\pi_{(G,N)} : G \to G/N$.

(5) Formulate and solve the analogies of this problem for vector spaces and modules.

Problem 34. [L] In this problem, we look into the dualizing functor on the category of vector spaces.

(1) Let $\underline{\text{FVec}}_k$ be the category of finite dimensional vector spaces over a field k. Verify that the dual $(-)^* : \underline{\text{FVec}}_k^{\mathbf{op}} \to \underline{\text{FVec}}_k$ given by $V \mapsto \text{Hom}_k(V, k)$ is a functor. Show that V is isomorphic to V^* as a vector space. (Note that these two are not naturally isomorphic, since constructing such an isomorphism requires a choice of a basis.)

(2) *In contrast, a finite dimensional inner product space is naturally isomorphic to its dual.* Let $\underline{\text{FVecB}}_k$ be the category of finite dimensional vector spaces over a field k equipped with a nondegenerate bilinear form. More precisely, the objects of $\underline{\text{FVecB}}_k$ are the pairs (V, B) of a finite dimensional k-vector space V and a nondegenerate bilinear form $B : V \times V \to k$ on V, and the morphisms are the B-preserving linear transformations.

Show that $\varphi : \text{id} \to (-)^*$ with components $\varphi_V : V \to V^*$ given by $v \mapsto B(-, v)$ defines a 'natural isomorphism' between the identity $\text{id} : \underline{\text{FVecB}}_k \to \underline{\text{FVecB}}_k$ and the dual $(-)^* : \underline{\text{FVecB}}_k^{\mathbf{op}} \to \underline{\text{FVecB}}_k$. (Note that our previous definition of a natural transformation does not fit into this case... Make a precise definition of a 'natural transformation' from a covariant functor $\underline{\text{C}} \to \underline{\text{D}}$ to a contravariant functor $\underline{\text{C}}^{\mathbf{op}} \to \underline{\text{D}}$, which is a special case of the concept so called a *dinatural transformation.*)

(3) Also show that $\varphi^* \circ \varphi = \mathrm{ev}$, where $\varphi^* : (-)^* \to (-)^{**}$ is a natural transformation with components $\varphi_V^* : V^* \to V^{**}$ (Make a precise definition.), and ev is the evaluation map defined in the previous problem, restricted to $\underline{\mathrm{FVecB}}_k$. (Now, read "선형대수와 군" chapter 13 & 14 again.)

Problem 35 (Product-hom adjunction)**.** (1) Let X, Y, Z be sets. Prove the following isomorphism between sets.

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

(2) Prove that the above isomorphism is natural in each place X, Y, Z. (What does this mean?)

Problem 36 (Tensor-hom adjunction)**.** Let R be a commutative ring.

(1) Let M, N, L be R-modules. Prove the following isomorphism between R-modules.

$$\mathrm{Hom}_R(M \otimes_R N, L) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L))$$

(2) Prove that the above isomorphism is natural in each place M, N, L. (What does this mean?)

Problem 37 (The fundamental theorem of Galois theory)**.** [A] Let K be a finite Galois field extension of E, and $G = \mathrm{Gal}(K/E)$ be the corresponding Galois group.

(1) Let $\underline{\mathrm{Orb}}_G$ be the *orbit category of* G. More precisely, the objects of $\underline{\mathrm{Orb}}_G$ are the subgroups H of G, which we identify with the quotients G/H regarded as left G-sets. The morphisms from G/H_1 to G/H_2 are the G-equivariant maps $G/H_1 \xrightarrow{f} G/H_2$, that is, set maps compatible with G-actions. Verify that $\underline{\mathrm{Orb}}_G$ is indeed a category.

(2) Let $\underline{\mathrm{Field}}_E^K$ be the category of intermediate fields between K and E, whose objects are the intermediate fields $E \subseteq F \subseteq K$, and the morphisms are field homomorphisms that fix E. Verify that $\underline{\mathrm{Field}}_E^K$ is indeed a category.

(3) Show that the functor $\Phi : \underline{\mathrm{Orb}}_G^{\mathbf{op}} \to \underline{\mathrm{Field}}_E^K$ given by $H \mapsto K^H$ is an isomorphism of categories. Also show that the same holds even if we consider the full subcategories that consist of normal subgroups of G and Galois intermediate fields, respectively.

Problem 38 (The fundamental theorem of covering spaces)**.** [AT] Let (X, x) be a connected and locally simply connected pointed space, and $G = \pi_1(X, x)$ be the corresponding fundamental group, regarded as an one-object category.

(1) Let $\underline{\mathrm{Cov}}(X, x)$ be the category of coverings of (X, x). More precisely, the objects of $\underline{\mathrm{Cov}}(X, x)$ are base-point preserving continuous maps $p : (E, \widetilde{x}) \to (X, x)$ such that for each $y \in X$ there exists a neighborhood $U \subseteq X$ of y such that $p^{-1}(U)$ is a disjoint union of open sets each of which is homeomorphic to U via p. The morphisms between two coverings $p : (E, \widetilde{x}) \to (X, x)$ and $q : (F, \overline{x}) \to (X, x)$ are base-point preserving continuous maps $f : (E, \widetilde{x}) \to (F, \overline{x})$ which makes the following diagram commutative.

$$\begin{tikzcd} (E, \widetilde{x}) \arrow[rr, "f"] \arrow[dr, "p"'] & & (F, \overline{x}) \arrow[dl, "q"] \\ & (X, x) & \end{tikzcd}$$

Show that $\underline{\mathrm{Cov}}(X, x)$ is indeed a category.

(2) Consider the category $\underline{\mathrm{Set}}^G$ of G-sets. Show that the group action on the fiber $\mathrm{Fib} : \underline{\mathrm{Cov}}(X,x) \to \underline{\mathrm{Set}}^G$ given by $p \mapsto p^{-1}(x)$ is an equivalence of categories.
(3) Show that the same holds if we consider the full subcategory of $\underline{\mathrm{Cov}}(X,x)$ consisting of path-connected coverings of X and the opposite orbit category $\underline{\mathrm{Orb}}_G^{\mathbf{op}}$ of G.
(4) Show that the same holds even if we consider the full subcategory of $\underline{\mathrm{Cov}}(X,x)$ consisting of path-connected normal coverings of X and the full subcategory of $\underline{\mathrm{Orb}}_G^{\mathbf{op}}$ consisting of normal subgroups of G.
(5) In particular, the full subcategory of $\underline{\mathrm{Cov}}(X,x)$ consisting of path-connected covers of X contains an initial object corresponding to a terminal object G in $\underline{\mathrm{Orb}}_G$, namely the universal covering space of (X,x). When $X = S^1$, the unit circle, what is the universal covering space of (S^1,x)?

Problem 39* (Presheaves and Sheaves)**.** [AG] Let X be a topological space, and $\underline{\mathrm{A}}$ be a *nice* category (such as $\underline{\mathrm{Set}}$, $\underline{\mathrm{Vec}}_k$, $\underline{\mathrm{Ab}}$, $\underline{\mathrm{CRing}}$, R-$\underline{\mathrm{Mod}}$, …). (There exists certain wide class of $\underline{\mathrm{A}}$ in which these definitions make sense, but working on general setting is pretty complicated. We will just work with specific examples.)

(1) An $\underline{\mathrm{A}}$-*presheaf on* X is a functor $F : \underline{\mathrm{Open}}(X)^{\mathbf{op}} \to \underline{\mathrm{A}}$. Check that the category $\underline{\mathrm{PSh}}(X,\underline{\mathrm{A}}) = \underline{\mathrm{A}}^{\underline{\mathrm{Open}}(X)^{\mathbf{op}}}$ of $\underline{\mathrm{A}}$-presheaves on X is indeed a category in the canonical way.
(2) An $\underline{\mathrm{A}}$-presheaf F is called an $\underline{\mathrm{A}}$-*sheaf on* X if for every open subset U of X and its open covering $\{U_\lambda\}_{\lambda\in\Lambda}$, the following holds:
   - (Locality) For all $s,t \in F(U)$, if $s|_{U_\lambda} = t|_{U_\lambda}$ for all $\lambda \in \Lambda$, then $s = t$. (We use the notation $s|_{U_\lambda} = s \circ F\imath_{U_\lambda \cap U, U}$.)
   - (Gluing) Given a compatible family $s_\lambda \in F(U_\lambda)$, $\lambda \in \Lambda$, that is, $s_\lambda|_{U_\lambda\cap U_\mu} = s_\mu|_{U_\lambda\cap U_\mu}$ for all $\lambda,\mu \in \Lambda$, there exists $s \in F(U)$ such that $s_\lambda = s|_{U_\lambda}$ for all $\lambda \in \Lambda$.

   Check that the full subcategory $\underline{\mathrm{Sh}}(X,\underline{\mathrm{A}})$ of sheaves of $\underline{\mathrm{PSh}}(X,\underline{\mathrm{A}})$ is a category.
(3) Let Y be a topological space. Verify that $C_{X,Y} : \underline{\mathrm{Open}}(X)^{\mathbf{op}} \to \underline{\mathrm{Set}}$ that maps $U \subseteq X$ to the set $C_{X,Y}(U) = C(U,Y)$ of continuous maps from U to Y is a sheaf on X. It is called the *sheaf of continuous maps.*
(4) State and justify the notion of *sheaf of smooth maps* between smooth manifolds in the similar way.
(5) Let $B : \underline{\mathrm{Open}}(\mathbb{R})^{\mathbf{op}} \to \underline{\mathrm{Set}}$ be the presheaf of bounded continuous real maps on $\mathbb{R}$, that is, $B(U) = C_b(U)$ for all open $U \subseteq \mathbb{R}$. Show that B is not a sheaf on $\mathbb{R}$.

Problem 40 (The Riesz representation theorem)**.** [RA] Let $\underline{\mathrm{lcHaus}}$ be the category of locally compact Hausdorff spaces and continuous maps, and let $\underline{\mathrm{Ban}}$ be the category of real Banach spaces and continuous linear maps.

(1) Verify that $\underline{\mathrm{lcHaus}}$ and $\underline{\mathrm{Ban}}$ are indeed categories.
(2) Show that the positive linear functionals on the space of compactly supported continuous real functions $C_c(-)^{*+} : \underline{\mathrm{lcHaus}} \to \underline{\mathrm{Ban}}$ is a functor. More precisely, for a locally compact Hausdorff space X, $C_c(X)^{*+}$ is the space of positive linear functionals $\Phi : C_c(X) \to \mathbb{R}$, equipped with the operator norm.
(3) Show that the space of Radon measures $M : \underline{\mathrm{lcHaus}} \to \underline{\mathrm{Ban}}$ is a functor. More precisely, for a

locally compact Hausdorff space X, $M(X)$ is the space of Radon measures μ on X, equipped with the total variation norm.

(4) Verify that the integration $\int : M \to C_c(-)^{*+}$ with components $\int_X : \mu \mapsto (f \mapsto \int_X f d\mu)$ is a natural isomorphism.

Problem 41* (Yoneda's lemma)**.** Let $\underline{\mathrm{C}}$ be a category.

(1) For an object X of $\underline{\mathrm{C}}$, denote the covariant hom functor associated to X by $h_X = \mathrm{Hom}(X, -) : \underline{\mathrm{C}} \to \underline{\mathrm{Set}}$. Consider the category $\underline{\mathrm{D}} = \underline{\mathrm{Fun}}(\underline{\mathrm{C}}, \underline{\mathrm{Set}})$ of covariant functors from $\underline{\mathrm{C}}$ to $\underline{\mathrm{Set}}$. Show that $h = (X \mapsto h_X) : \underline{\mathrm{C}}^{\mathbf{op}} \to \underline{\mathrm{D}}$ is a functor in the canonical way.

(2) Let X be an object of $\underline{\mathrm{C}}$, and $F : \underline{\mathrm{C}} \to \underline{\mathrm{D}}$ be an object in $\underline{\mathrm{D}}$. Show that there is a bijection

$$\mathrm{Mor}_{\underline{\mathrm{D}}}(h_X, F) \cong FX$$

given by $\eta \mapsto \eta_X(\mathrm{id}_X)$.

(3) Show that the above bijection is natural in both places X and F.

(4) Show that two objects X and Y of $\underline{\mathrm{C}}$ are isomorphic if and only if the functors h_X and h_Y are naturally isomorphic. (Hint: Put $F = h_Y$ into the above natural bijection.)

(5) Formulate and prove the contravariant version of this problem.

Lecture 3: Adjunction and Freeness

Adjunction

Problem 42. Given a set map $f : X \to Y$. Consider the power sets 2^X and 2^Y as poset categories ordered by inclusion.

(1) Verify that the following are functors.

(a) The direct image $f_* : 2^X \to 2^Y$ given by $f_*(U) = f(U)$

(b) The inverse image $f^{-1} : 2^Y \to 2^X$ given by $f^{-1}(U) = f^{-1}(U)$

(c) The codirect image $\forall_f : 2^X \to 2^Y$ given by $\forall_f(U) = \{y \in Y : f^{-1}(y) \subseteq U\}$

(2) Show the following.

(a) The inverse image is right adjoint to the direct image.

(b) The inverse image is left adjoint to the codirect image.

Problem 43. [T] Show that the functor $F : \underline{\text{Set}} \to \underline{\text{Top}}$ assigning the trivial topology on each set is right adjoint to the forgetful functor $\underline{\text{Top}} \to \underline{\text{Set}}$.

Problem 44. [T] Let $\underline{\text{LConn}}$ be the full subcategory of locally connected spaces in $\underline{\text{Top}}$.

(1) Define a functor $C : \underline{\text{LConn}} \to \underline{\text{Set}}$ assigning to each space X the set of its connected components in the canonical way.

(2) Show that C is left adjoint to the functor $D : \underline{\text{Set}} \to \underline{\text{LConn}}$ assigning to each set S the discrete topology on S.

Problem 45 (Product-hom adjunction)**.** Let X be a set. Show that the cartesian product functor $- \times X : \underline{\text{Set}} \to \underline{\text{Set}}$ is left adjoint to the hom functor $\text{Hom}(X, -) : \underline{\text{Set}} \to \underline{\text{Set}}$.

Problem 46 (Tensor-hom adjunction)**.** Let R be a commutative ring, and M be an R-module. Show that the tensor product functor $- \otimes_R M : R\text{-}\underline{\text{Mod}} \to R\text{-}\underline{\text{Mod}}$ is left adjoint to the hom functor $\text{Hom}_R(M, -) : R\text{-}\underline{\text{Mod}} \to R\text{-}\underline{\text{Mod}}$.

Problem 47. Consider $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$ as poset categories. Show that the floor function $\lfloor - \rfloor : \mathbb{R} \to \mathbb{Z}$ is right adjoint to the inclusion $\mathbb{Z} \to \mathbb{R}$, and the ceiling function $\lceil - \rceil : \mathbb{R} \to \mathbb{Z}$ is left adjoint to the inclusion $\mathbb{Z} \to \mathbb{R}$.

Free Functor

Problem 48. Show that each of the following example is a *free* functor, that is, a left adjoint of a forgetful (or, inclusion) functor.

- The (free) set id : $\underline{\text{Set}} \to \underline{\text{Set}}$
- [L] The k-vector space over a given basis $F : \underline{\text{Set}} \to \underline{\text{Vect}}_k$ (Hence, every vector space is free.)
- [A] The free abelian group over a given basis $F : \underline{\text{Set}} \to \underline{\text{Ab}}$
- The free R-module over a given basis $F : \underline{\text{Set}} \to R\text{-}\underline{\text{Mod}}$

- [A] The free group with given generators $F : \underline{\mathrm{Set}} \to \underline{\mathrm{Grp}}$
- [A] The abelianization $(-)^{\mathrm{ab}} : \underline{\mathrm{Grp}} \to \underline{\mathrm{Ab}}$
- [A] The field of fractions $\mathrm{Frac} : \underline{\mathrm{Dom}}_m \to \underline{\mathrm{Field}}$
- [T] The discrete topological space $F : \underline{\mathrm{Set}} \to \underline{\mathrm{Top}}$
- [T] The completion $\overline{(-)} : \underline{\mathrm{Met}} \to \underline{\mathrm{cMet}}$

Problem 49. Let S be a set.

(1) In this problem, we generalize our previous construction of free commutative rings over n variables to the infinite variables case.

The *set of monomials* $\mathcal{S}$ *over* S is the collection of set maps $S \to \mathbb{N}$ that vanish for all but finitely many elements of S. Note that $\mathcal{S}$ is endowed with a canonical multiplication structure given by

$$(X \cdot Y)(s) = X(s) + Y(s)$$

for $X, Y \in \mathcal{S}$ and $s \in S$.

The *free commutative ring* $\mathbb{Z}[S]$ *over variables in* S is the collection of set maps $\mathcal{S} \to \mathbb{Z}$ that vanish for all but finitely many elements of $\mathcal{S}$, with the pointwise addition and the *convolution multiplication*, that is,

$$(f \cdot g)(X) = \sum_{Y \cdot Z = X} f(Y)g(Z)$$

for $f, g \in \mathbb{Z}[S]$ and $X \in \mathcal{S}$, where the summation on the right-hand side ranges over the monomials $Y, Z \in \mathcal{S}$ satisfying $Y \cdot Z = X$.

Show that $\mathbb{Z}[S]$ is indeed a commutative ring.

(2) Define the category of S-pointed commutative rings. Show that $\mathbb{Z}[S]$ is an initial object in this category.

(3) Define a functor $\mathbb{Z}[-] : \underline{\mathrm{Set}} \to \underline{\mathrm{CRing}}$ in the canonical way. Show that it is a free functor, that is, a left adjoint to the forgetful functor $\underline{\mathrm{CRing}} \to \underline{\mathrm{Set}}$.

Lecture 4: Limit and Colimit

Limit and Colimit

Problem 50. Redefine the following terminologies as limits and colimits.

- Terminal objects and initial objects
- Products and coproducts
- Pullbacks and pushouts
- Kernels and cokernels
- Inverse limits and direct limits

Problem 51. Show that direct limits and inverse limits exist in the category of abelian groups, or more generally, in the category of R-modules, by explicitly constructing these. (Hint: See Lang, III.10.)

Problem 52. Show that every module is the direct limit of its finitely generated submodules.

Problem 53 (The ring of formal power series)**.** Let R be a commutative ring.

(1) The *ring* $R[[X]]$ *of formal power series in the variable* X *over* R is the collection of set maps $\mathbb{N} \to R$ with the pointwise addition and the *convolution multiplication*, that is,

$$(f \cdot g)(n) = \sum_{i+j=n} f(i)g(j)$$

for $f, g \in R[[X]]$ and $n \in \mathbb{N}$, where the summation on the right-hand side ranges over $i, j \in \mathbb{N}$ satisfying $i + j = n$. Show that $R[[X]]$ is indeed a commutative ring.

(2) Define the functor $(-)[[X]] : \underline{\mathrm{CRing}} \to \underline{\mathrm{CRing}}$ in the canonical way.

(3) We redefine $R[[X]]$ in category theoretical sense. For each $m \geq n$, consider the natural projection

$$f_n^m : R[X]/(X^m) \longrightarrow R[X]/(X^n).$$

Show that the family $((R[X]/(X^n))_n, (f_n^m)_{m \geq n})$ forms an inverse system. Define

$$R[[X]] = \varprojlim R[X]/(X^n).$$

Show that our previous construction coincides with the above inverse limit in the canonical way.

Problem 54 (The p-adic numbers)**.** Let $p \in \mathbb{Z}_{>0}$ be a prime number.

(1) The *ring* $\mathbb{Z}_p$ *of* p*-adic integers* is the set

$$\left\{ (a_n)_{n\in\mathbb{N}} \in \prod_{n\geq 0} (\mathbb{Z}/p^n\mathbb{Z}) : (a_m \bmod p^n) = a_n \ \forall m \geq n \right\}$$

with the componentwise addition and multiplication. Show that $\mathbb{Z}_p$ is indeed a commutative ring.

(2) For each $m \geq n$, consider the natural projection

$$f_n^m : \mathbb{Z}/p^m\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}.$$

Show that the family $((\mathbb{Z}/p^n\mathbb{Z})_n, (f_n^m)_{m\geq n})$ forms an inverse system, and

$$\mathbb{Z}_p \cong \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

in the canonical way.

(3) Show that $\mathbb{Z}_p$ is isomorphic to the ring of formal power series

$$\sum_{n\geq 0} b_n p^n$$

with $b_n \in \mathbb{Z}/p\mathbb{Z}$ in the canonical way.

Problem 55* (Stalk)**.** [AG] Let X be a topological space, and $\underline{\mathrm{A}}$ be a *nice* category.

(1) Let $F : \underline{\mathrm{Open}}(X)^{\mathbf{op}} \to \underline{\mathrm{A}}$ be an $\underline{\mathrm{A}}$-presheaf on X, and $x \in X$ be a point in X.
Consider the family of $F(U)$ with U ranging over open neighborhoods of x, ordered by reverse inclusion. For each $U \subseteq V$, the map $f_U^V : F(V) \to F(U)$ is given by $F\imath_{UV}$, where $\imath_{UV} : U \to V$ is the inclusion. Verify that the family $((F(U))_{x\in U}, (f_U^V)_{x\in U\subseteq V})$ forms a directed system in $\underline{\mathrm{A}}$.
The *stalk of F at x* is the directed limit $F_x = \varinjlim F(U)$.

(2) Show that $(-)_x : \underline{\mathrm{PSh}}(X, \underline{\mathrm{A}}) \to \underline{\mathrm{A}}$ is a functor in the canonical way.

(3) Note that for $U \subseteq X$ an open neighborhood of $x \in X$, there is a canonical map $F(U) \to F_x$.
Suppose that F is a sheaf. Show that for every open $U \subseteq X$, the canonical map $F(U) \to \prod_{x\in U} F_x$ is injective.

Problem 56* (Sheafification)**.** [AG] Let X be a topological space, and $\underline{\mathrm{A}}$ be a *nice* category.

(1) (Sheafification) Let F be an $\underline{\mathrm{A}}$-presheaf on X. Define an $\underline{\mathrm{A}}$-presheaf F_* on X by $F_*(U) = \prod_{x\in U} F_x$ in the canonical way. Show that F_* is a sheaf with a canonical morphism $\eta : F \to F_*$.
The *sheafification of F* is the smallest subsheaf $F^\#$ of F_* containing $U \mapsto \eta(F(U))$.

(2) Show that for each open subset U of X, $F^\#(U) \subseteq \prod_{x\in U} F_x$ consists of those $(s_x)_{x\in U}$ such that there exists an open covering $(U_\lambda)_{\lambda\in I}$ of U and $(s_\lambda)_{\lambda\in I} \in \prod_{\lambda\in I} F(U_\lambda)$ satisfying $s_\lambda|_x = s_x$ for all $x \in U_\lambda$.

(3) Show that the sheafification $(-)^\# : \underline{\mathrm{PSh}}(X, \underline{\mathrm{A}}) \to \underline{\mathrm{Sh}}(X, \underline{\mathrm{A}})$ is left adjoint to the inclusion functor $\underline{\mathrm{Sh}}(X, \underline{\mathrm{A}}) \to \underline{\mathrm{PSh}}(X, \underline{\mathrm{A}})$.

(4) Let $B : \underline{\mathrm{Open}}(\mathbb{R})^{\mathbf{op}} \to \underline{\mathrm{Set}}$ be the presheaf of bounded continuous real maps on $\mathbb{R}$. Show that $B^\# = C_{\mathbb{R},\mathbb{R}}$.

(5) Let X be locally connected, and Y be an object in $\underline{\mathrm{A}}$. Consider an $\underline{\mathrm{A}}$-presheaf on X that assigns the object Y for every open $U \subseteq X$. Its sheafification $Y_X : \underline{\mathrm{Open}}(X)^{\mathbf{op}} \to \underline{\mathrm{A}}$ is called the *constant sheaf* on X with coefficient Y. Explain that Y_X can be regarded as the sheaf of locally constant maps on X when $\underline{\mathrm{A}}$ is a concrete category.

Preservation of Limit and Colimit

Problem 57. Explain the following phenomena, from the viewpoint of the preservation of limits and colimits under adjoint functors.

- The underlying set of the product of groups is the cartesian product of their underlying sets.

- The underlying set of the product of topological spaces is the cartesian product of their underlying sets.
- The underlying set of the coproduct of topological spaces is the disjoint union of their underlying sets.
- The free R-module generated by the disjoint union $X \amalg Y$ of sets X and Y is the direct sum of the free R-modules generated by X and Y.

Problem 58. Using the preservation of limits and colimits under adjoint functors, prove the following.

- For any set map $f : A \to B$, the inverse image $f^{-1} : 2^B \to 2^A$ preserves both unions and intersections, while the direct image $f_* : 2^A \to 2^B$ only preserves unions. (Hint: Note that f^{-1} and f_* can be viewed as functors between 2^A and 2^B, regarded as posets ordered by inclusion. Recall that meet(intersection) and join(union) are product and coproduct in a poset, respectively.)
- The free group on the disjoint union $X \amalg Y$ of sets X and Y is the free product of the free groups on the sets X and Y.
- The abelianization $(G * H)^{\mathrm{ab}}$ of the free product of groups G and H is the direct sum $G^{\mathrm{ab}} \oplus H^{\mathrm{ab}}$ of the abelianizations of G and H.

Problem 59. Prove that the free group functor $F : \underline{\mathrm{Set}} \to \underline{\mathrm{Grp}}$ does not have a left adjoint.

Problem 60. Let R be a commutative ring, M be an R-module, and $(N_i)_{i \in I}$ be a family of R-modules. Using the tensor-hom adjunction, prove the following isomorphisms of R-modules.

$$\left(\bigoplus_{i \in I} N_i \right) \otimes_R M \cong \bigoplus_{i \in I} (N_i \otimes_R M)$$

$$\mathrm{Hom}_R \left(M, \prod_{i \in I} N_i \right) \cong \prod_{i \in I} \mathrm{Hom}_R(M, N_i)$$

Problem 61. Let R be a commutative ring, and M be an R-module. Using the preservation of limits and colimits under adjoint functors, show that the hom functor $\mathrm{Hom}_R(M, -) : R\text{-}\underline{\mathrm{Mod}} \to R\text{-}\underline{\mathrm{Mod}}$ is left exact, and the tensor product functor $- \otimes_R M : R\text{-}\underline{\mathrm{Mod}} \to R\text{-}\underline{\mathrm{Mod}}$ is right exact.

Problem 62*. Let X be a topological space, and $\underline{A}$ be a *nice* category. Let F be an $\underline{A}$-presheaf on X. Using the preservation of limits and colimits under adjoint functors, show that $F_x^{\#} \cong F_x$ for all $x \in X$.

Problem 63* (Hom preserves limits)**.** Let $\underline{C}$ be a category, and $X_\bullet : \underline{I} \to \underline{C}$ be a diagram.

(1) Suppose that the limit $\lim_i X_i$ exists in $\underline{C}$. Then for all objects Y of $\underline{C}$, there is a natural isomorphism

$$\mathrm{Hom} \left(Y, \lim_i X_i \right) \cong \lim_i \mathrm{Hom}(Y, X_i),$$

where on the right-hand side, we have the limit over the diagram of sets given by

$$\mathrm{Hom}(Y, X_\bullet) : \underline{I} \xrightarrow{X} \underline{C} \xrightarrow{\mathrm{Hom}(Y,-)} \underline{\mathrm{Set}}.$$

(2) Suppose that the colimit $\mathrm{colim}_i X_i$ exists in $\underline{C}$. Then for all objects Y of $\underline{C}$, there is a natural isomorphism

$$\mathrm{Hom} \left(\operatorname*{colim}_i X_i, Y \right) \cong \lim_i \mathrm{Hom}(X_i, Y),$$

where on the right-hand side, we have the limit over the diagram of sets given by

$$\operatorname{Hom}(X_\bullet, Y) : \underline{\mathrm{I}}^{\mathbf{op}} \xrightarrow{X} \underline{\mathrm{C}}^{\mathbf{op}} \xrightarrow{\operatorname{Hom}(-,Y)} \underline{\mathrm{Set}}.$$

Problem 64* (Preservation of limit)**.** In this problem, we prove the preservation of limits and colimits under adjoint functors. Suppose that a functor $F : \underline{\mathrm{D}} \to \underline{\mathrm{C}}$ is left adjoint to a functor $G : \underline{\mathrm{C}} \to \underline{\mathrm{D}}$.

(1) Suppose that $X_\bullet : \underline{\mathrm{I}} \to \underline{\mathrm{C}}$ is a diagram whose limit $\lim_i X_i$ exists in $\underline{\mathrm{C}}$. Check that for every object Y in $\underline{\mathrm{D}}$, we have a chain of natural isomorphisms

$$\begin{aligned} \operatorname{Hom}\left(Y, G\left(\lim_i X_i\right)\right) &\cong \operatorname{Hom}\left(FY, \lim_i X_i\right) \\ &\cong \lim_i \operatorname{Hom}\left(FY, X_i\right) \\ &\cong \lim_i \operatorname{Hom}\left(Y, GX_i\right) \\ &\cong \operatorname{Hom}\left(Y, \lim_i GX_i\right). \end{aligned}$$

Conclude using the Yoneda's lemma that there is a natural isomorphism

$$G\left(\lim_i X_i\right) \cong \lim_i GX_i,$$

where on the right-hand side, we have the limit in $\underline{\mathrm{D}}$ over the diagram $GX_\bullet : \underline{\mathrm{I}} \to \underline{\mathrm{D}}$.

(2) Suppose that $X_\bullet : \underline{\mathrm{I}} \to \underline{\mathrm{D}}$ is a diagram whose colimit $\operatorname{colim}_i X_i$ exists in $\underline{\mathrm{D}}$. Show that there is a natural isomorphism

$$F\left(\operatorname*{colim}_i X_i\right) \cong \operatorname*{colim}_i FX_i,$$

where on the right-hand side, we have the colimit in $\underline{\mathrm{C}}$ over the diagram $FX_\bullet : \underline{\mathrm{I}} \to \underline{\mathrm{C}}$.

Lecture 5: Classical Homology & Cohomology

Singular Homology & Cohomology

Problem 65 (Simplicial homology)**.** In this problem, we develop the notion of simplicial homology.

(1) The n-*simplex* is a convex set

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0 \text{ and } \sum_i t_i = 1\}.$$

If $v_0, \ldots, v_n$ are the vertices of an n-simplex, we denote this simplex by $[v_0, \ldots, v_n]$. Note that we keep track of the order of the vertices, i.e., our simplices are oriented. The $(n-1)$-simplices $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$ obtained by deleting one of the vertices of Δ^n is called the *faces of* Δ^n.

A topological space X has a Δ-*complex structure* if, roughly speaking, the space X is obtained by 'gluing' simplices together. More precisely, a Δ-complex structure is a collection of continuous maps $\{\sigma_\alpha : \Delta^n \to X\}_{\alpha \in I}$ satisfying:

(a) σ_α is injective on the interior $\mathring{\Delta}^n$ of Δ^n, and each point in X belongs to exactly one of the images of $\sigma_\alpha|_{\mathring{\Delta}^n}$.

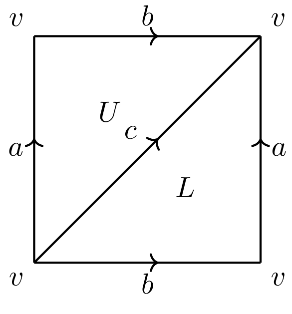
(b) Each restriction of $\sigma_\alpha : \Delta^n \to X$ to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$.

(c) A set $A \subseteq X$ is open in X if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each α.

Consider a 2-dimensional torus $X = T^2 = S^1 \times S^1$. Note that T^2 can be constructed by identifying two pairs of parallel edges of a rectangle. Let those edges a and b respectively, and let c be a diagonal edge. Also note that all four vertices of the rectangle is identified to a single vertex v in the construction. As in the figure, let U and L be triangles. Check that the collection of maps

$$\{\sigma_U, \sigma_L : \Delta^2 \to X\} \cup \{\sigma_a, \sigma_b, \sigma_c : \Delta^1 \to X\} \cup \{\sigma_v : \Delta^0 \to X\}$$

gives a Δ-complex structure of T^2.



(2) Let X be a topological space with a Δ-complex structure $\{\sigma_\alpha\}_{\alpha \in I}$. Let $\Delta_n(X)$ be the free abelian group with basis n-simplices $\sigma_\alpha : \Delta^n \to X$ on X. Elements of $\Delta_n(X)$ are called n-*chains*.

The *boundary homomorphisms of* X are the homomorphisms $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ given by

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \ldots, \widehat{v_i}, \ldots, v_n]}.$$

The sequence of abelian groups $(\Delta_n(X))_n$ with homomorphisms $(\partial_n)_n$ is called the *simplicial chain complex of* X. Elements of $\ker \partial_n$ are called *n-cycles*, and elements of $\operatorname{im} \partial_{n+1}$ are called *n-boundaries*.

Show that $\partial^2 = 0$, that is, $\partial_n \partial_{n+1} = 0$ for all n. The *nth simplicial homology group of* X is the group $H_n^\Delta(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

(3) Compute the simplicial homology groups $H_n^\Delta(T^2)$, where the torus T^2 is given with the Δ-complex structure described above.

(4) Construct a Δ-complex structure on the 2-dimensional sphere S^2, and compute its simplicial homology groups.

Problem 66 (Singular homology)**.** In this problem, we develop the notion of singular homology.

Let X be a topological space.

(1) A *singular n-simplex on* X is a continuous map $\sigma_\alpha : \Delta^n \to X$. Let $C_n(X)$ be the free abelian group with basis singular n-simplices on X. Elements of $C_n(X)$ are called *n-chains*.

The *boundary homomorphisms of* X are the homomorphisms $\partial_n : C_n(X) \to C_{n-1}(X)$ given by

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \ldots, \widehat{v_i}, \ldots, v_n]}.$$

The sequence of abelian groups $(C_n(X))_n$ with homomorphisms $(\partial_n)_n$ is called the *singular chain complex of* X. Elements of $\ker \partial_n$ are called *n-cycles*, and elements of $\operatorname{im} \partial_{n+1}$ are called *n-boundaries*.

Show that $\partial^2 = 0$, that is, $\partial_n \partial_{n+1} = 0$ for all n. The *nth singular homology group of* X is the group $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

(2) Show that if X is a single point, then $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for all $n > 0$. Note that this agrees with the obvious Δ-complex structure on X. It is known that for a Δ-complex, its singular homology groups and simplicial homology groups coincide.

(3) Show that for each n, $C_n(-)$ is a functor $\underline{\text{Top}} \to \underline{\text{Ab}}$ in the canonical way.

(4) Show that for each n, $H_n(-)$ is a functor $\underline{\text{Top}} \to \underline{\text{Ab}}$ in the canonical way.

It is known that singular homology groups are homotopy invariant, that is, $H_n(-)$ factors through the canonical functor $\underline{\text{Top}} \to \underline{\text{Htpy}}$.

Problem 67 (Singular cohomology)**.** In this problem, we develop the notion of singular cohomology.

Let X be a topological space, and A be an abelian group.

(1) Recall that $\operatorname{Hom}(-, A) : \underline{\text{Ab}}^{\mathbf{op}} \to \underline{\text{Ab}}$ is a well-defined functor.

For each n, consider the *cochain group* $C^n(X, A) = \operatorname{Hom}(C_n(X), A)$ and the *coboundary homomorphisms* $\delta^n = \operatorname{Hom}(\partial_{n+1}, A) : C^n(X, A) \to C^{n+1}(X, A)$. Elements of $C^n(X, A)$ are called *n-cochains*. The sequence of abelian groups $(C^n(X, A))_n$ with homomorphisms $(\delta^n)_n$ is called the *singular cochain complex of X with coefficients in A*. Elements of $\ker \delta^n$ are called *n-cocycles*, and elements of $\operatorname{im} \delta^{n-1}$ are called *n-coboundaries*.

Check that $\delta^2 = 0$, that is, $\delta^n \delta^{n-1} = 0$ for all n. The *nth singular cohomology group of X with coefficients in A* is the group $H^n(X, A) = \ker \delta^n / \operatorname{im} \delta^{n-1}$.

(2) Show that for each n, $C^n(-, A)$ is a functor $\underline{\text{Top}}^{\mathbf{op}} \to \underline{\text{Ab}}$ in the canonical way.

(3) Show that for each n, $H^n(-, A)$ is a functor $\underline{\mathrm{Top}}^{\mathbf{op}} \to \underline{\mathrm{Ab}}$ in the canonical way.

(4) Let $A = R$ be a commutative ring. Check that $\mathrm{Hom}(-, R) : \underline{\mathrm{Ab}}^{\mathbf{op}} \to R\text{-}\underline{\mathrm{Mod}}$ is a well-defined functor. Show that for each n, $H^n(-, R)$ is a functor $\underline{\mathrm{Top}}^{\mathbf{op}} \to R\text{-}\underline{\mathrm{Mod}}$ in the canonical way.

de Rham Cohomology

Problem 68 (de Rham cohomology, Euclidean case)**.** [D] In this problem, we develop the notion of de Rham cohomology for open subsets of an Euclidean space.

Let U be an open subset of $\mathbb{R}^N$, and $x_1, \dots, x_N \in C^\infty(\mathbb{R}^N)$ be its standard coordinates.

(1) Consider the coordinate differentials $dx_1, \dots, dx_N$. Define Ω^* as an $\mathbb{R}$-algebra generated by $\{dx_1, \dots, dx_N\}$, with multiplication $\wedge$ called the *wedge product* that satisfies

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

for all i, j. Write $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for a tuple $I = (i_1, \dots, i_k)$. Show that as $\mathbb{R}$-vector spaces,

$$\Omega^* = \bigoplus_{k=0}^{N} \Omega^k$$

where Ω^k is a $\mathbb{R}$-vector space with basis

$$\{dx_I : I = (i_1, \dots, i_k),\ 1 \le i_1 < i_2 < \cdots < i_k \le N\}.$$

(2) Define

$$\Omega^*(U) = C^\infty(U) \otimes_{\mathbb{R}} \Omega^* = \left\{ \sum_I f_I dx_I : f_I \in C^\infty(U) \right\}.$$

Note that for $\mathbb{R}$-vector spaces $\Omega^k(U) = C^\infty(U) \otimes_{\mathbb{R}} \Omega^k$, we have

$$\Omega^*(U) = \bigoplus_{k=0}^{N} \Omega^k(U).$$

Elements of $\Omega^k(U)$ are called the *differential k-forms on U*.

(3) The *exterior derivatives* are the unique $\mathbb{R}$-linear transformations $d = d^k : \Omega^k(U) \to \Omega^{k+1}(U)$ satisfying the following.

(a) For $f \in C^\infty(U)$, $df = \sum_{i=0}^{N} \frac{\partial f}{\partial x_i} dx_i$.

(b) For $\omega = \sum_I f_I dx_I$, $d\omega = \sum_I df_I \wedge dx_I$.

The sequence of $\mathbb{R}$-vector spaces $(\Omega^k(U))_k$ with morphisms $(d^k)_k$ is called the *de Rham cochain complex of U*. Elements of $\ker d^k$ are called *k-cocycles*, and elements of $\operatorname{im} d^{k-1}$ are called *k-coboundaries*.

Check that $d^2 = 0$, that is, $d^k d^{k-1} = 0$ for all k. The *nth de Rham cohomology group of U* is the $\mathbb{R}$-vector space $H^k_{\mathrm{dR}}(U) = \ker d^k / \operatorname{im} d^{k-1}$.

(4) Let $\mathfrak{X}(U)$ be the $\mathbb{R}$-module of smooth vector fields on U. Establish isomorphisms between $\mathbb{R}$-modules that makes the following diagram commutative.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(U) \arrow[r, "d"] \arrow[d, "\wr"] & \Omega^1(U) \arrow[r, "d"] \arrow[d, "\wr"] & \Omega^2(U) \arrow[r, "d"] \arrow[d, "\wr"] & \Omega^3(U) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & C^\infty(U) \arrow[r, "\mathrm{grad}"] & \mathfrak{X}(U) \arrow[r, "\mathrm{curl}"] & \mathfrak{X}(U) \arrow[r, "\mathrm{div}"] & C^\infty(U) \arrow[r] & 0
\end{tikzcd}$$

(5) Let $\underline{\text{Sm}}$ be the category of open subsets of Euclidean spaces and smooth maps. More precisely, the objects of $\underline{\text{Sm}}$ are open subsets $U \subseteq \mathbb{R}^N$ for some N, and morphisms $U \to V$ are smooth maps.
Let $f = (f_1, \ldots, f_M) : U \subseteq \mathbb{R}^N \to V \subseteq \mathbb{R}^M$ be a smooth map between open subsets of Euclidean spaces. Denote the standard coordinates of $\mathbb{R}^N$ by $\mathbb{R}^M$ by x_i's and y_j's, respectively. For a differential form $\omega = \sum_J g_J dy_J$ on V, the *pullback of* ω is a differential form on U given by

$$f^*\omega = \sum_J (g_J \circ f) df_{j_1} \wedge \cdots \wedge df_{j_k}$$

where $J = (j_1, \ldots, j_k)$.
Show that for each $0 \le k \le N$, $\Omega^k(-)$ is a functor $\underline{\text{Sm}}^{\mathbf{op}} \to \underline{\text{Vect}_{\mathbb{R}}}$, where a morphism $U \xrightarrow{f} V$ is mapped to $\Omega^k(V) \xrightarrow{f^*} \Omega^k(U)$.

(6) Show that for each k, $H^k_{\mathrm{dR}}(-)$ is a functor $\underline{\text{Sm}}^{\mathbf{op}} \to \underline{\text{Vect}_{\mathbb{R}}}$ in the canonical way. (Hint: Show that pullbacks commute with exterior derivatives, i.e., $df^* = f^*d$.)

Problem 69 (de Rham's theorem, Euclidean case)**.** In this problem, we look into a special case of de Rham's theorem.

(1) Let U be an open subset of $\mathbb{R}^N$. In the definition of singular cohomology groups of U, replace the n-simplices $\sigma_\alpha : \Delta^n \to U$ by the *smooth n-simplices* $\sigma_\alpha : \Delta^n \to U$, that is, continuous maps that extends to a smooth map on a neighborhood of $\Delta^n \subset \mathbb{R}^{n+1}$. Check that this leads to a definition of a well-defined functor $H^n_\infty(-, \mathbb{R}) : \underline{\text{Sm}}^{\mathbf{op}} \to \underline{\text{Vect}_{\mathbb{R}}}$, called the *smooth singular cohomology groups*. It is known that for open sets $U \subseteq \mathbb{R}^N$, its smooth singular cohomology groups coincide with its singular cohomology groups.

(2) Recall Stokes' theorem: for a smooth $(n+1)$-simplex σ on U and a differential n-form ω on U, we have

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega.$$

For an open subset U of $\mathbb{R}^N$, define $I_U : \Omega^n(U) \to C^n(U, \mathbb{R})$ by $\omega \mapsto (\sigma \mapsto \int_\sigma \omega)$. Using Stokes' theorem, show that this induces a morphism $I_U : H^n_{\mathrm{dR}}(U) \to H^n_\infty(U, \mathbb{R})$, which defines a natural transformation $I : H^n_{\mathrm{dR}}(-) \to H^n_\infty(-, \mathbb{R})$. Also prove that I_U is an isomorphism when U is an open ball.
Indeed, one can prove that $I : H^n_{\mathrm{dR}}(-) \to H^n_\infty(-, \mathbb{R})$ is a natural isomorphism. Moreover, one can do the same thing for cohomology groups of smooth manifolds, i.e., there is a natural isomorphism between the functors $H^n_{\mathrm{dR}}(-) : \underline{\text{Man}}^{\mathbf{op}} \to \underline{\text{Vect}_{\mathbb{R}}}$ and $H^n_\infty(-, \mathbb{R}) : \underline{\text{Man}}^{\mathbf{op}} \to \underline{\text{Vect}_{\mathbb{R}}}$.

Chain Complex and Homology

Problem 70 (The category of chain complexes in R-$\underline{\text{Mod}}$)**.** Let R be a commutative ring.

(1) Let $\underline{\text{Ch}}(R\text{-}\underline{\text{Mod}})$ be a category of chain complexes of R-modules. More precisely, the objects $C_\bullet = (C_\bullet, d)$ of $\underline{\text{Ch}}(R\text{-}\underline{\text{Mod}})$ are sequences $(C_n)_{n\in\mathbb{Z}}$ of R-modules together with R-module homomorphisms $(d_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$ that satisfy $d^2 = 0$, that is, $d_{n-1}d_n = 0$ for all n. The maps d_n are called the *boundary maps*. The morphisms $C_\bullet \xrightarrow{f_\bullet} D_\bullet$ in $\underline{\text{Ch}}(R\text{-}\underline{\text{Mod}})$, which are called *chain maps*, are collections $f_n : C_n \to D_n$ of R-module homomorphisms that satisfy $df = fd$, that is, $d_n f_n = f_{n-1} d_n$ for all n. Verify that $\underline{\text{Ch}}(R\text{-}\underline{\text{Mod}})$ is indeed a category.

(2) Show that the singular chain complex $C_\bullet(-)$ is a functor $\underline{\text{Top}} \to \underline{\text{Ch}}(\underline{\text{Ab}})$.

(3) Given a chain complex $C_\bullet = (C_\bullet, d)$, the *nth homology group of* $C_\bullet$ is the R-module $H_n(C_\bullet) = \ker d_n / \operatorname{im} d_{n+1}$. Show that for each n, $H_n(-)$ is a functor $\underline{\text{Ch}}(R\text{-}\underline{\text{Mod}}) \to R\text{-}\underline{\text{Mod}}$.

Problem 71 (The category of cochain complexes in R-$\underline{\text{Mod}}$)**.** In this problem, we define a dual notion of the chain complex, so called the cochain complex.

Let R be a commutative ring.

(1) Let $\underline{\text{CoCh}}(R\text{-}\underline{\text{Mod}})$ be a category of cochain complexes of R-modules. More precisely, the objects $C^\bullet = (C^\bullet, d)$ of $\underline{\text{CoCh}}(R\text{-}\underline{\text{Mod}})$ are sequences $(C^n)_{n\in\mathbb{Z}}$ of R-modules together with R-module homomorphisms $(d^n : C^n \to C^{n+1})_{n\in\mathbb{Z}}$ that satisfy $d^2 = 0$, that is, $d^n d^{n-1} = 0$ for all n. The maps d^n are called the *coboundary maps*. The morphisms $C^\bullet \xrightarrow{f^\bullet} D^\bullet$ in $\underline{\text{CoCh}}(R\text{-}\underline{\text{Mod}})$, which are called *cochain maps*, are collections $f^n : C^n \to D^n$ of R-module homomorphisms that satisfy $df = fd$, that is, $d^n f^n = f^{n+1} d^n$ for all n. Verify that $\underline{\text{Ch}}(R\text{-}\underline{\text{Mod}})$ is indeed a category.

(2) Show that the singular cochain complex $C^\bullet(-, R)$ is a functor $\underline{\text{Top}}^{\mathbf{op}} \to \underline{\text{CoCh}}(R\text{-}\underline{\text{Mod}})$. Also show that the de Rham cochain complex $\Omega^\bullet(-)$ is a functor $\underline{\text{Sm}}^{\mathbf{op}} \to \underline{\text{CoCh}}(\underline{\text{Vect}_{\mathbb{R}}})$.

(3) Given a cochain complex $C^\bullet = (C^\bullet, d)$, the *nth cohomology group of* $C^\bullet$ is the R-module $H^n(C^\bullet) = \ker d^n / \operatorname{im} d^{n-1}$. Show that for each n, $H^n(-)$ is a functor $\underline{\text{CoCh}}(R\text{-}\underline{\text{Mod}}) \to R\text{-}\underline{\text{Mod}}$.

Lecture 6: Abelian Category

Abelian Category

Problem 72. Let R be a ring.

(1) With an abuse of notation, let R be a preadditive category with one object $\bullet$ and $\mathrm{Mor}(\bullet, \bullet) = R$, where the composition law is given by multiplication, and abelian group structure inherited from R. Verify that R is indeed a preadditive category.
(2) Check that the opposite ring $R^{\mathbf{op}}$ of R coincides with the opposite category $R^{\mathbf{op}}$ of R.
(3) Show that the category of additive functors $R \to \underline{\mathrm{Ab}}$ is isomorphic to the category R-$\underline{\mathrm{Mod}}$ of R-modules.
(4) Show that the category of (contravariant) additive functors $R^{\mathbf{op}} \to \underline{\mathrm{Ab}}$ is isomorphic to the category $\underline{\mathrm{Mod}}$-R of modules-R, or equivalently, the category $R^{\mathbf{op}}$-$\underline{\mathrm{Mod}}$ of $R^{\mathbf{op}}$-modules.

Problem 73. Let $\underline{\mathrm{A}}$ be an additive category.

(1) Show that every kernel in $\underline{\mathrm{A}}$ is monic. In other words, for the kernel $K \xrightarrow{\ker f} X$ of a morphism $X \xrightarrow{f} Y$ in $\underline{\mathrm{A}}$, show that $\ker f$ is monic.
(2) Show that every cokernel in $\underline{\mathrm{A}}$ is epic. In other words, for the cokernel $Y \xrightarrow{\operatorname{coker} f} K$ of a morphism $X \xrightarrow{f} Y$ in $\underline{\mathrm{A}}$, show that $\operatorname{coker} f$ is epic.

Problem 74. Let f be a morphism in an abelian category $\underline{\mathrm{A}}$. Show that if f is monic and epic, then f is an isomorphism.

Problem 75. Let f be a morphism of an abelian category $\underline{\mathrm{A}}$. Recall that the image of f is defined by $\operatorname{im} f = \ker \operatorname{coker} f$. Dually, the *coimage* of f is $\operatorname{coim} f = \operatorname{coker} \ker f$. Show that the objects $\operatorname{im} f$ and $\operatorname{coim} f$ are isomorphic, that is,

$$\operatorname{im} f = \ker \operatorname{coker} f \cong \operatorname{coker} \ker f = \operatorname{coim} f.$$

Problem 76. Let $\underline{\mathrm{A}}$ be an abelian category. Show that the opposite category $\underline{\mathrm{A}}^{\mathbf{op}}$ of $\underline{\mathrm{A}}$ is also abelian.

Problem 77. Let $\underline{\mathrm{C}}$ be a category, and $\underline{\mathrm{A}}$ be an abelian category. Show that the category $\underline{\mathrm{Fun}}(\underline{\mathrm{C}}, \underline{\mathrm{A}})$ of (covariant) functors from $\underline{\mathrm{C}}$ to $\underline{\mathrm{A}}$ is also abelian.

In particular, for a topological space X, the category $\underline{\mathrm{PSh}}(X, \underline{\mathrm{A}})$ of $\underline{\mathrm{A}}$-presheaves on X is abelian.

Problem 78. Let R be a commutative ring. Show that the category $\underline{\mathrm{Ch}}(R\text{-}\underline{\mathrm{Mod}})$ of chain complexes in R-modules is abelian in the canonical way. Also show that the category $\underline{\mathrm{CoCh}}(R\text{-}\underline{\mathrm{Mod}})$ of cochain complexes in R-modules is abelian in the canonical way.

Problem 79*. Let $\underline{\mathrm{A}}$ be an abelian category. Show that the category $\underline{\mathrm{Ch}}(\underline{\mathrm{A}})$ of chain complexes in $\underline{\mathrm{A}}$ is also abelian by following the steps below.

(1) Check that $\underline{\mathrm{Ch}}(\underline{\mathrm{A}})$ is a category.
(2) Check that the hom-sets of $\underline{\mathrm{Ch}}(\underline{\mathrm{A}})$ are abelian groups under component-wise addition. The bilinearity of addition follows from bilinearity in $\underline{\mathrm{A}}$.

(3) Check that the complex $0_\bullet$ with zero objects and zero morphisms is a zero object in $\underline{\mathrm{Ch}}(\underline{\mathrm{A}})$.

(4) Let $(C_\bullet, d)$ and $(D_\bullet, \partial)$ be chain complexes in $\underline{\mathrm{A}}$. For each n, note that by universality of products in $\underline{\mathrm{A}}$, there exists a unique morphism $C_n \times D_n \to C_{n-1} \times D_{n-1}$ that makes the following diagram commutative.

$$\begin{tikzcd} C_n \arrow[d, "d"'] & C_n \times D_n \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dotted] & D_n \arrow[d, "\partial"] \\ C_{n-1} & C_{n-1} \times D_{n-1} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & D_{n-1} \end{tikzcd}$$

With respect to this morphism, show that $C_\bullet \times D_\bullet = (C_n \times D_n)_{n \in \mathbb{Z}}$ is a chain complex that is a product of $C_\bullet$ and $D_\bullet$ in $\underline{\mathrm{Ch}}(\underline{\mathrm{A}})$

(5) Let $C_\bullet \xrightarrow{f_\bullet} D_\bullet$ be a chain map. Show that the component-wise kernel $\ker f_\bullet = (\ker f_n)_{n\in\mathbb{Z}}$ is a kernel of $f_\bullet$ in $\underline{\mathrm{Ch}}(\underline{\mathrm{A}})$, and the component-wise cokernel $\operatorname{coker} f_\bullet = (\operatorname{coker} f_n)_{n\in\mathbb{Z}}$ is a cokernel of $f_\bullet$ in $\underline{\mathrm{Ch}}(\underline{\mathrm{A}})$.

(6) Show that a chain map $C_\bullet \xrightarrow{f_\bullet} D_\bullet$ is monic (resp. epic) if and only if all the components $C_n \xrightarrow{f_n} D_n$ are monic (resp. epic). Using this, conclude that every monic chain map is the kernel of its cokernel, and every epic chain map is the cokernel of its kernel.

Also show that the category $\underline{\mathrm{CoCh}}(\underline{\mathrm{A}})$ of cochain complexes in $\underline{\mathrm{A}}$ is abelian by reversing arrows.

Exactness

Problem 80. Let $\underline{\mathrm{A}}$ be an abelian category, and X be an object in $\underline{\mathrm{A}}$.

(1) Show that the covariant hom functor $\mathrm{Hom}(X, -) : \underline{\mathrm{A}} \to \underline{\mathrm{Ab}}$ is left exact.

(2) Show that the contravariant hom functor $\mathrm{Hom}(-, X) : \underline{\mathrm{A}}^{\mathbf{op}} \to \underline{\mathrm{Ab}}$ is left exact. (Hint: Note that $\mathrm{Hom}_{\underline{\mathrm{A}}}(Y, X) = \mathrm{Hom}_{\underline{\mathrm{A}}^{\mathbf{op}}}(X, Y)$.)

Problem 81* (The category of sheaves is abelian)**.** [AG] Let X be a topological space, and $\underline{\mathrm{A}}$ be a *nice* abelian category.

(1) Let F and G be $\underline{\mathrm{A}}$-sheaves on X. Show that $F \oplus G = (U \mapsto F(U) \oplus G(U))$ is indeed a sheaf, and is a direct sum of F and G as sheaves.

(2) (Sheaf kernel) Let $F \xrightarrow{\phi} G$ be a morphism between $\underline{\mathrm{A}}$-sheaves on X. Show that the kernel of ϕ is the same as the kernel of ϕ as a morphism of presheaves. In other words, show that the *sheaf kernel of* ϕ given by $\ker \phi = (U \mapsto \ker \phi_U)$ is indeed a sheaf, and is a kernel of ϕ as a morphism of sheaves.

(3) (Sheaf cokernel) Let $F \xrightarrow{\phi} G$ be a morphism between $\underline{\mathrm{A}}$-sheaves on X. Show that the cokernel of ϕ is the same as the sheafification of the cokernel of ϕ as a morphism of presheaves. In other words, show that the *sheaf cokernel of* ϕ given by $\operatorname{coker} \phi = (U \mapsto \operatorname{coker} \phi_U)^\#$ is a cokernel of ϕ as a morphism of sheaves.

(4) Explain (1), (2), and (3) from the viewpoint of the preservation of limits and colimits under adjoint functors.

(5) Show that the category $\underline{\mathrm{Sh}}(X, \underline{\mathrm{A}})$ of $\underline{\mathrm{A}}$-sheaves on X is abelian. Note that it is *not* an abelian subcategory of $\underline{\mathrm{PSh}}(X, \underline{\mathrm{A}})$, since sheaf cokernels do not match with presheaf cokernels.

(6) (Exponential sheaf sequence) Let S^1 be the unit circle. Show that the following is a short exact sequence of $\underline{\text{Ab}}$-sheaves on S^1, which is not exact as a sequence of presheaves.

$$0 \to (2\pi\mathbb{Z})_{S^1} \to C_{S^1,\mathbb{R}} \to C_{S^1,S^1} \to 0$$

Lecture 7: Homology Long Exact Sequence

Connecting Morphism

Problem 82 (The snake lemma, ker-coker sequence)**.** Let R be a commutative ring. Let the following commutative diagram of R-modules with exact rows be given.

$$\begin{tikzcd}
& M' \arrow[r, "f"] \arrow[d, "d'"] & M \arrow[r, "g"] \arrow[d, "d"] & M'' \arrow[r] \arrow[d, "d''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "f'"] & N \arrow[r, "g'"] & N''
\end{tikzcd}$$

Show that the sequence

$$\ker d' \xrightarrow{f_*} \ker f \xrightarrow{g_*} \ker d'' \xrightarrow{\delta} \operatorname{coker} d' \xrightarrow{f'_*} \operatorname{coker} d \xrightarrow{g'_*} \operatorname{coker} d''$$

in the following diagram is well-defined and exact.

$$\begin{tikzcd}
& \ker d' \arrow[r, "f_*"] \arrow[d] & \ker d \arrow[r, "g_*"] \arrow[d] & \ker d'' \arrow[d] \arrow[dddll, dashed, "\delta"] & \\
& M' \arrow[r, "f"] \arrow[d, "d'"] & M \arrow[r, "g"] \arrow[d, "d"] & M'' \arrow[r] \arrow[d, "d''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "f'"] \arrow[d] & N \arrow[r, "g'"] \arrow[d] & N'' \arrow[d] & \\
& \operatorname{coker} d' \arrow[r, "f'_*"] & \operatorname{coker} d \arrow[r, "g'_*"] & \operatorname{coker} d'' &
\end{tikzcd}$$

The morphism δ is called a *connecting morphism.*

Note that by the Freyd–Mitchell embedding theorem, the same holds for diagrams in an abelian category $\underline{\mathrm{A}}$.

Problem 83. Let $\underline{\mathrm{A}}$ be an abelian category. Show that a sequence $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ of chain complexes in $\underline{\mathrm{A}}$ is exact if and only if it is component-wisely exact, that is, the sequence $0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$ is exact for each n.

Problem 84 (Homology Long Exact Sequence)**.** Let $\underline{\mathrm{A}}$ be an abelian category. Let $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ be an exact sequence of chain complexes in $\underline{\mathrm{A}}$. Show that there are morphisms $\delta : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ such that

$$\cdots \xrightarrow{\delta} H_n(A_\bullet) \xrightarrow{f} H_n(B_\bullet) \xrightarrow{g} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \xrightarrow{f} H_{n-1}(B_\bullet) \xrightarrow{g} H_{n-1}(C_\bullet) \xrightarrow{\delta} \cdots$$

is exact. (Hint: Use the snake lemma.)

Problem 85 (The category of exact sequences)**.** Let $\underline{\mathrm{A}}$ be an abelian category.

(1) Let $\underline{\mathrm{SES}}(\underline{\mathrm{A}})$ be the category of short exact sequences in $\underline{\mathrm{A}}$. In other words, the objects of $\underline{\mathrm{SES}}(\underline{\mathrm{A}})$ are the short exact sequences in $\underline{\mathrm{A}}$, that is, exact sequences

$$C : 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $\underline{\mathrm{A}}$, and a morphism $C \xrightarrow{d} D$ is a chain map. Check that $\underline{\mathrm{SES}}(\underline{\mathrm{A}})$ is indeed a category. Furthermore, show that $\underline{\mathrm{SES}}(\underline{\mathrm{A}})$ is a preabelian category.

(2) Let $\underline{\mathrm{LES}}(\underline{\mathrm{A}})$ be the category of (long) exact sequences in $\underline{\mathrm{A}}$. In other words, $\underline{\mathrm{LES}}(\underline{\mathrm{A}})$ is the full subcategory of $\underline{\mathrm{Ch}}(\underline{\mathrm{A}})$ consisting of exact sequences. Show that $\underline{\mathrm{LES}}(\underline{\mathrm{A}})$ is a preabelian category.

(3) Prove that $\underline{\mathrm{SES}}(\underline{\mathrm{A}})$ is *not* abelian in general.

Problem 86 (Naturality of δ)**.** Let $\underline{\mathrm{A}}$ be an abelian category. Show that the homology long exact sequence is an additive functor $\underline{\mathrm{SES}}(\underline{\mathrm{Ch}}(\underline{\mathrm{A}})) \to \underline{\mathrm{LES}}(\underline{\mathrm{A}})$. In other words, show that for each n, the connecting morphism δ defines a natural transformation between functors $H_n(C_\bullet)$ and $H_{n-1}(A_\bullet)$ from $\underline{\mathrm{SES}}(\underline{\mathrm{Ch}}(\underline{\mathrm{A}}))$ to $\underline{\mathrm{A}}$. (Hint: Consider the following diagram, where $\mathbf{Z}_n = \ker d_n$ and $\mathbf{B}_n = \operatorname{im} d_{n+1}$.)

$$\begin{tikzcd}
 & \mathbf{Z}_n(A) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{Z}_n(B) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{Z}_n(C) \arrow[dr] \arrow[dd] \arrow[ddddddllll, dotted, "\delta"] & & & \\
 & & \mathbf{Z}_n(A') \arrow[rr] \arrow[dd] & & \mathbf{Z}_n(B') \arrow[rr] \arrow[dd] & & \mathbf{Z}_n(C') \arrow[dd] \arrow[ddddddllll, dotted, "\delta"] & & \\
0 \arrow[r] & A_n \arrow[rr, "f"] \arrow[dr] \arrow[dd, "d"] & & B_n \arrow[rr, "g"] \arrow[dr] \arrow[dd, "d"] & & C_n \arrow[rr] \arrow[dr] \arrow[dd, "d"] & & 0 & \\
0 \arrow[rr] & & A'_n \arrow[rr, "f'"] \arrow[dd, "d"] & & B'_n \arrow[rr, "g'"] \arrow[dd, "d"] & & C'_n \arrow[rr] \arrow[dd, "d"] & & 0 \\
0 \arrow[r] & A_{n-1} \arrow[rr, "f"] \arrow[dr] \arrow[dd] & & B_{n-1} \arrow[rr, "g"] \arrow[dr] \arrow[dd] & & C_{n-1} \arrow[rr] \arrow[dr] \arrow[dd] & & 0 & \\
0 \arrow[rr] & & A'_{n-1} \arrow[rr, "f'"] \arrow[dd] & & B'_{n-1} \arrow[rr, "g'"] \arrow[dd] & & C'_{n-1} \arrow[rr] \arrow[dd] & & 0 \\
 & A_{n-1}/\mathbf{B}_{n-1}(A) \arrow[rr] \arrow[dr] & & B_{n-1}/\mathbf{B}_{n-1}(B) \arrow[rr] \arrow[dr] & & C_{n-1}/\mathbf{B}_{n-1}(C) \arrow[dr] & & & \\
 & & A'_{n-1}/\mathbf{B}_{n-1}(A') \arrow[rr] & & B'_{n-1}/\mathbf{B}_{n-1}(B') \arrow[rr] & & C'_{n-1}/\mathbf{B}_{n-1}(C') & &
\end{tikzcd}$$

Appendix: Diagram Chasing

Problem 87 (The short five lemma)**.** Let the following be a commutative diagram in an abelian category $\underline{A}$. Suppose that the rows of the diagram are short exact sequences. Prove the following.

(1) If α and γ are monic, then so is β.

(2) If α and γ are epic, then so is β.

(3) In particular, if α and γ are isomorphisms, then so is β.

Try not to use the Freyd–Mitchell embedding theorem in the proof if you want.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

Problem 88 (The four lemma)**.** Let the following be a commutative diagram in an abelian category $\underline{A}$. Suppose that the rows of the diagram are exact. Prove the following.

(1) If β and δ are monic and α is epic, then γ is monic.

(2) If α and γ are epic and δ is monic, then β is epic.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D'
\end{tikzcd}$$

Problem 89 (The five lemma)**.** Let the following be a commutative diagram in an abelian category $\underline{A}$. Suppose that the rows of the diagram are exact.

Show that if β and δ are isomorphisms, α is epic, and ε is monic, then γ is an isomorphism. (Hint: Use the four lemma.)

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

Problem 90 (The 3×3 lemma)**.** Let the following be a commutative diagram in an abelian category $\underline{A}$. Suppose that the columns of the diagram are short exact sequences. Prove the following.

(1) If the bottom two rows are short exact sequences, then so is the top row.

(2) If the top two rows are short exact sequences, then so is the bottom row.

(3) If the top and bottom rows are short exact sequences, and the middle row is a chain complex, then the middle row is also exact.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Problem 91* (Elements in an abelian category)**.** Let $\underline{\mathrm{A}}$ be an abelian category. An *element* of an object X in $\underline{\mathrm{A}}$ is an equivalence class $[Z, x]$ of pairs (Z, x) where $x : Z \to X$ is a morphism in $\underline{\mathrm{A}}$, and two pairs (Z, x) and (W, y) are equivalent iff there exists epimorphisms $u : P \to Z$ and $v : P \to W$ such that $xu = yv : P \to X$. In this case, we write $x \equiv y$.

Note that:

- Every object X has a unique zero element $0 : 0 \to X$.
- Every element x in X has a negative $-x$.
- Any morphism $f : X \to Y$ carries elements x of X to elements fx of Y.

(1) Verify that the relation above is an equivalence relation.

(2) Prove the following elementary rules for diagram chasing using elements.

  (a) $f : X \to Y$ is monic iff for each element x of X, $fx \equiv 0$ implies $x \equiv 0$.

  (b) $f : X \to Y$ is monic iff for each elements x, x' of X, $fx \equiv fx'$ implies $x \equiv x'$.

  (c) $f : X \to Y$ is epic iff for each element y of Y, there exists an element x of X such that $fx \equiv y$.

  (d) $f : X \to Y$ is zero iff for all element x of X, $fx \equiv 0$.

  (e) A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact at Y iff $gf = 0$ and for every element y of Y with $gy \equiv 0$, there exists an element x of X such that $fx \equiv y$.

  (f) (Subtraction) Given $f : X \to Y$ and elements x, x' of X with $fx \equiv fx'$, there exists an element m of X such that $fm \equiv 0$. Moreover, any $g : X \to Z$ with $gx \equiv 0$ has $gx' \equiv gm$, and any $h : X \to W$ with $hx' \equiv 0$ has $hx \equiv -hm$.

(3) Using the rules above, give proofs for the four lemma, the five lemma, the snake lemma, and the 3×3 lemma in an arbitrary abelian category without using the Freyd–Mitchell embedding theorem.

Lecture 8: Derived Functor

Projective and Injective

Problem 92. Let $\underline{\mathrm{A}}$ be an abelian category.

(1) Show that if $\underline{\mathrm{A}}$ has enough projectives, then every object in $\underline{\mathrm{A}}$ has a projective resolution.

(2) Show that if $\underline{\mathrm{A}}$ has enough injectives, then every object in $\underline{\mathrm{A}}$ has an injective resolution.

Problem 93* (The horseshoe lemma)**.** Let $\underline{\mathrm{A}}$ be an abelian category. Suppose that $0 \to A \to B \to C \to 0$ be a short exact sequence in $\underline{\mathrm{A}}$, and $P_\bullet$, $Q_\bullet$ be projective resolutions of A and C, respectively. Show that there exists a projective resolution $P_\bullet \oplus Q_\bullet$ of B such that the following diagram is commutative, where $P_n \to P_n \oplus Q_n$ is the natural injection and $P_n \oplus Q_n \to Q_n$ is the natural projection.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Derived Functor

Problem 94* (Tor and Ext)**.** Let R be a commutative ring.

(1) Recall that for an R-module M, the tensor product functor $- \otimes_R M : R\text{-}\underline{\mathrm{Mod}} \to R\text{-}\underline{\mathrm{Mod}}$ is a right exact functor. For R-modules M and N, the *Tor modules of M and N* are the R-modules defined by $\mathrm{Tor}_n^R(M, N) = (L_n(- \otimes_R N))(M)$.
Show that for each n, $\mathrm{Tor}_n^R(M, N)$ defines a functor $R\text{-}\underline{\mathrm{Mod}} \to R\text{-}\underline{\mathrm{Mod}}$ for each places M and N.
(Balancing Tor) It is known that we may also define Tor modules dually, that is,
$$\mathrm{Tor}_n^R(M, N) = (L_n(- \otimes_R N))(M) \cong (L_n(M \otimes_R -))(N)$$
naturally.

(2) Recall that for an R-module M, the covariant hom functor $\mathrm{Hom}_R(M, -) : R\text{-}\underline{\mathrm{Mod}} \to R\text{-}\underline{\mathrm{Mod}}$ is a left exact functor. For R-modules M and N, the *Ext modules of M and N* are the R-modules defined by $\mathrm{Ext}_R^n(M, N) = (R^n \mathrm{Hom}_R(M, -))(N)$.
Show that for each n, $\mathrm{Ext}_R^n(M, N)$ defines a contravariant functor $R\text{-}\underline{\mathrm{Mod}}^{\mathbf{op}} \to R\text{-}\underline{\mathrm{Mod}}$ for the first place M, and a covariant functor $R\text{-}\underline{\mathrm{Mod}} \to R\text{-}\underline{\mathrm{Mod}}$ for the second place N.

(Balancing Ext) It is known that we may also define Ext modules dually, that is,

$$\operatorname{Ext}_R^n(M,N) = (R^n \operatorname{Hom}_R(M,-))(N) \cong (R^n \operatorname{Hom}_R(-,N))(M)$$

naturally. Note that we consider a projective resolution of M on the right-hand side, since $\operatorname{Hom}_R(-,N)$ is a contravariant functor, and an injective resolution in the opposite category R-$\underline{\mathrm{Mod}}^{\mathbf{op}}$ is just a projective resolution in R-$\underline{\mathrm{Mod}}$.

Problem 95* (Group cohomology)**.** [A] Let G be a group.

(1) The *group ring* $\mathbb{Z}[G]$ *over* $\mathbb{Z}$ is a free abelian group generated by elements of G, which is also a ring with multiplication linearly extended from the group law of G, that is,

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) = \sum_{g\in G}\left(\sum_{hk=g} a_h b_k\right) g,$$

with $a_g, b_g \in \mathbb{Z}$, where the second summation on the right-hand side ranges over the pair of elements $h, k \in G$ satisfying $hk = g$. Establish the well-definedness of $\mathbb{Z}[G]$.
A $\mathbb{Z}[G]$-module M is called a *G-module*, and the category of G-modules is denoted by G-$\underline{\mathrm{Mod}}$. In other words, a G-module is an abelian group M equipped with a G-action, which is compatible with the addition of M.

(2) For a G-module M, let M^G be the submodule of G-invariant elements, that is,

$$M^G = \{m \in M : gm = m \ \forall g \in G\}.$$

Show that $(-)^G$ defines a left exact functor G-$\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}}$ in the canonical way. In fact, we have $(-)^G = \operatorname{Hom}_G(\mathbb{Z}, -)$ where we interpret $\mathbb{Z}$ as a G-module equipped with trivial G-action.
The *group cohomology functors of G* are defined as the right derived functors $H^n(G,-) = R^n(-)^G = \operatorname{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},-)$.

(3) Let M be a G-module. The *group $Z^1(G,M)$ of 1-cocycles of G into M* consists of the set maps $\phi : G \to M$ satisfying

$$\phi(g+h) = \phi(g) + g\phi(h) \qquad \forall g,h \in G.$$

The *group $B^1(G,M)$ of 1-coboundaries of G into M* consists of the maps $\phi : G \to M$ of the form $\phi(g) = gm - m$ for some fixed $m \in M$.
Show that the 1st cohomology group $H^1(G,M)$ is isomorphic to the cocycles modulo coboundaries, that is, the quotient $Z^1(G,M)/B^1(G,M)$. (Hint: Note that by balancing Ext, we may compute group cohomology via constructing a projective resolution, and especially, a free resolution.)

Problem 96* (Sheaf cohomology)**.** [AG] Let X be a topological space. We denote the category of $\underline{\mathrm{Ab}}$-sheaves on X by $\underline{\mathrm{Sh}}(X, \underline{\mathrm{Ab}}) = \underline{\mathrm{Ab}}_X$. It is known that $\underline{\mathrm{Ab}}_X$ has enough injectives.

(1) Let U be an open subset of X. The *functor* $\Gamma(U,-) : \underline{\mathrm{Ab}}_X \to \underline{\mathrm{Ab}}$ *of sections over U* is defined by $F \mapsto F(U)$. Check that $\Gamma(U,-)$ is a well-defined functor.
In particular, $\Gamma(X,-)$ is called the *global sections functor*.

(2) Show that $\Gamma(U,-)$ is a left exact functor.
The *sheaf cohomology groups* are defined as the right derived functors $H^n(X,-) = R^n\Gamma(X,-)$.

Problem 97* (Sheaf-theoretic de Rham isomorphism)**.** [D] [AG] In this problem, we prove the de Rham theorem using the notion of sheaf cohomology. This reveals that under certain conditions, both singular cohomology and de Rham cohomology can be understood as special cases of sheaf cohomology.

(1) Let X be a locally contractible topological space, and R be a ring. Show that we have a natural isomorphism $H^n(X, R) \cong H^n(X, R_X)$ for each n, where the left-hand side is the singular cohomology of X with coefficients in R, and the right-hand side is the sheaf cohomology of the constant sheaf R_X on X.

(2) Let M be a paracompact smooth manifold. Show that we have a natural isomorphism $H^n_{dR}(M) \cong H^n(M, \mathbb{R}_M)$ for each n, where the left-hand side is the de Rham cohomology of M, and the right-hand side is the sheaf cohomology of the constant sheaf $\mathbb{R}_M$ on M.

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