The Eyring–Kramers Law for Extinction Time of Contact Process on Stars

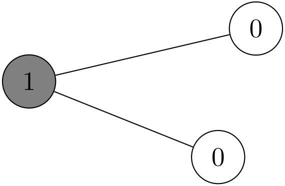
Younghun Jo
Seoul National University

2024 KMS Annual Meeting
October 25, 2024

Contact process

- $G = (V, E)$ (locally) finite connected graph
- $\lambda > 0$ *infection rate*
- Configurations of the contact process: $\eta \in \{0, 1\}^V$

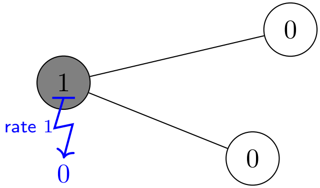
  For $x \in V$, $\eta(x) = \begin{cases} 0 & x \text{ is } \textit{healthy} \\ 1 & x \text{ is } \textit{infected} \end{cases}$
- Abuse of notation: identify η with $\{x \in V : \eta(x) = 1\}$



Contact process

- $G = (V, E)$ (locally) finite connected graph
- $\lambda > 0$ *infection rate*
- Configurations of the contact process: $\eta \in \{0, 1\}^V$

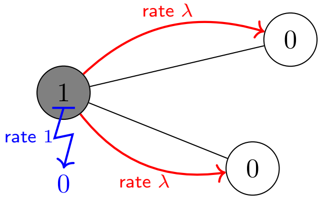
  For $x \in V$, $\eta(x) = \begin{cases} 0 & x \text{ is } \textit{healthy} \\ 1 & x \text{ is } \textit{infected} \end{cases}$
- Abuse of notation: identify η with $\{x \in V : \eta(x) = 1\}$



Contact process

- $G = (V, E)$ (locally) finite connected graph
- $\lambda > 0$ *infection rate*
- Configurations of the contact process: $\eta \in \{0,1\}^V$

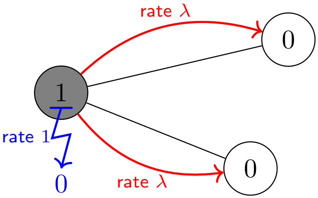
  For $x \in V$, $\eta(x) = \begin{cases} 0 & x \text{ is } \textit{healthy} \\ 1 & x \text{ is } \textit{infected} \end{cases}$
- Abuse of notation: identify η with $\{x \in V : \eta(x) = 1\}$



Contact process

- $G = (V, E)$ (locally) finite connected graph
- $\lambda > 0$ *infection rate*
- Configurations of the contact process: $\eta \in \{0, 1\}^V$

  For $x \in V$, $\eta(x) = \begin{cases} 0 & x \text{ is } \textit{healthy} \\ 1 & x \text{ is } \textit{infected} \end{cases}$
- Abuse of notation: identify η with $\{x \in V : \eta(x) = 1\}$



- The *all-healthy state* $\eta = \emptyset$ is the unique absorbing state.

Metastability of contact processes

The *extinction time* of the contact process is

$$\tau_G = \inf\{t \geq 0 : \eta_t = \emptyset\}.$$

Q. Fix a (increasing) sequence of graphs $(G_N)_{N \geq 1}$, and study the growth of τ_{G_N}.

Finite-volume phase transition for boxes ('84–'99)

On $\mathbb{Z}_N^d = [1, N]^d$ with free boundary, we have

$$\tau_{\mathbb{Z}_N^d} \sim \begin{cases} \log |\mathbb{Z}_N^d| & \text{if } \lambda < \lambda_c, \\ \exp\big(c_\lambda |\mathbb{Z}_N^d|\big) & \text{if } \lambda > \lambda_c \end{cases}$$

where $|G|$ denotes the number of vertices.

The latter case is a clear demonstration of the metastable behavior.

Metastability of contact processes

More generally, the following theorem holds.

Theorem (MMYV '16, SV '17)

Suppose that $\lambda > 0$ is sufficiently large.

(a) For all $D > 0$, there exists $c = c(\lambda, D)$ such that

$$\mathbb{E}\tau_G \geq \exp(c|G|) \quad \text{for all } G \text{ with degrees} \leq D.$$

(b) For all $\varepsilon > 0$, there exists $c = c(\lambda, \varepsilon)$ such that

$$\mathbb{E}\tau_G \geq \exp\left(c \cdot \frac{|G|}{(\log |G|)^{1+\varepsilon}}\right) \quad \text{for all } G.$$

Precise estimate for mean extinction time

Levels of precision for mean transition time estimate:

1. $\mathbb{E}\tau_N \geq \exp(cN)$ (Metastability)
2. $\dfrac{1}{N} \log \mathbb{E}\tau_N \longrightarrow c$ (Large-deviation principle)
3. $\mathbb{E}\tau_N \simeq f(N)e^{cN}$ (Eyring–Kramers law)

Precise estimate for mean extinction time

Levels of precision for mean transition time estimate:

1. $\mathbb{E}\tau_N \geq \exp(cN)$ (Metastability)
2. $\dfrac{1}{N}\log \mathbb{E}\tau_N \longrightarrow c$ (Large-deviation principle)
3. $\mathbb{E}\tau_N \simeq f(N)e^{cN}$ (Eyring–Kramers law)

- 1 holds for graphs of uniformly bounded degree. (MMYV '16)

Precise estimate for mean extinction time

Levels of precision for mean transition time estimate:

1. $\mathbb{E}\tau_N \geq \exp(cN)$ (Metastability)
2. $\dfrac{1}{N}\log \mathbb{E}\tau_N \longrightarrow c$ (Large-deviation principle)
3. $\mathbb{E}\tau_N \simeq f(N)e^{cN}$ (Eyring–Kramers law)

- 1 holds for graphs of uniformly bounded degree. (MMYV '16)
- 2 holds for $\mathbb{Z}_N^d$ with free boundary. (Mountford '99)
  However, 2 is open for even $\mathbb{Z}_N^d$ with periodic boundary.
- 2 holds for a variety of random graph models. (Shapira–Valesin '21)

Precise estimate for mean extinction time

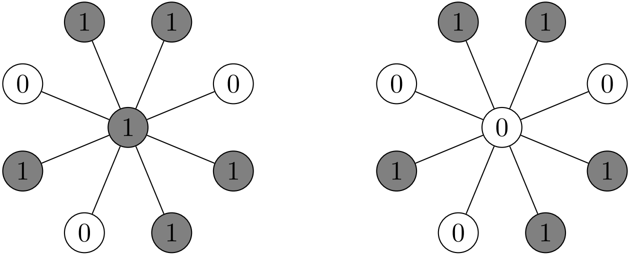
Levels of precision for mean transition time estimate:

1. $\mathbb{E}\tau_N \geq \exp(cN)$ (Metastability)
2. $\dfrac{1}{N}\log \mathbb{E}\tau_N \longrightarrow c$ (Large-deviation principle)
3. $\mathbb{E}\tau_N \simeq f(N)e^{cN}$ (Eyring–Kramers law)

- 1 holds for graphs of uniformly bounded degree. (MMYV '16)
- 2 holds for $\mathbb{Z}_N^d$ with free boundary. (Mountford '99)
  However, 2 is open for even $\mathbb{Z}_N^d$ with periodic boundary.
- 2 holds for a variety of random graph models. (Shapira–Valesin '21)
- 3 is open only except for two cases.
  1. The triviality: complete graph K_N
  2. Main Result: star graph S_N (**J.** '24)

Contact process on stars

Let S_N be the star graph with one hub and N leaves.

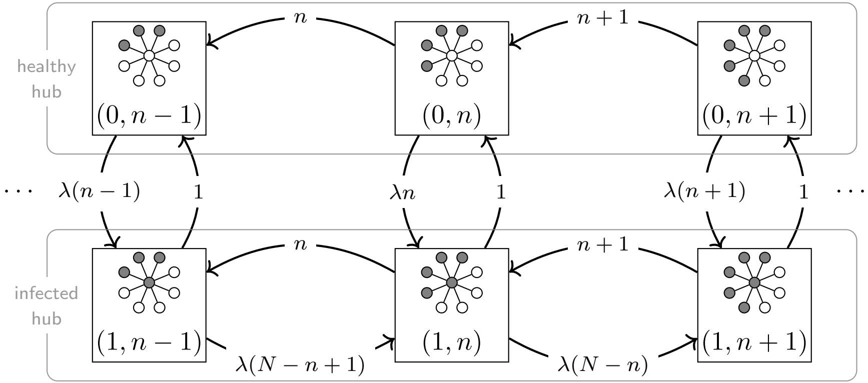


Why do we study star graphs?

- It is a natural model for studying epidemic hubs.
- It serves as a building block within larger graph structures.

All leaves are homogenous, so we reduce to a random walk on the ladder graph:

$$(o_t, n_t) = (\text{hub state}, \#\text{infected leaves}) \in \{0,1\} \times [0, N].$$



Transition rates for the contact process on a star

Main result

Eyring–Kramers law (**J.** '24)

Let S_N be the star graph with one hub and N leaves. Then, we have

$$\mathbb{E}\tau_{S_N} \simeq \kappa_\lambda N^{-\frac{1}{1+2\lambda}} \left(\frac{(1+\lambda)^2}{1+2\lambda} \right)^N .$$

In particular, we have

$$\frac{1}{N} \log \mathbb{E}\tau_{S_N} \xrightarrow{N\to\infty} c_\lambda = 2\log(1+\lambda) - \log(1+2\lambda).$$

Main ingredients:

- *Special function theory* for precise estimation of *quasi-stationary measure*
- The *potential theoretic approach* to metastability of non-reversible processes

These methodologies have not previously been used in the study of the contact process.

Quasi-stationary distribution

The stationary measure of the process is the Dirac mass at the all-healthy state.

To apply potential theory, we add supplementary transition rates from $\emptyset$.

Quasi-stationary distribution

The stationary measure of the process is the Dirac mass at the all-healthy state.
To apply potential theory, we add supplementary transition rates from $\emptyset$.

- Natural choice: rate proportional to the stationary measure of the process conditioned on the non-extinction.
  $\Longrightarrow$ Hard to compute in general.

Quasi-stationary distribution

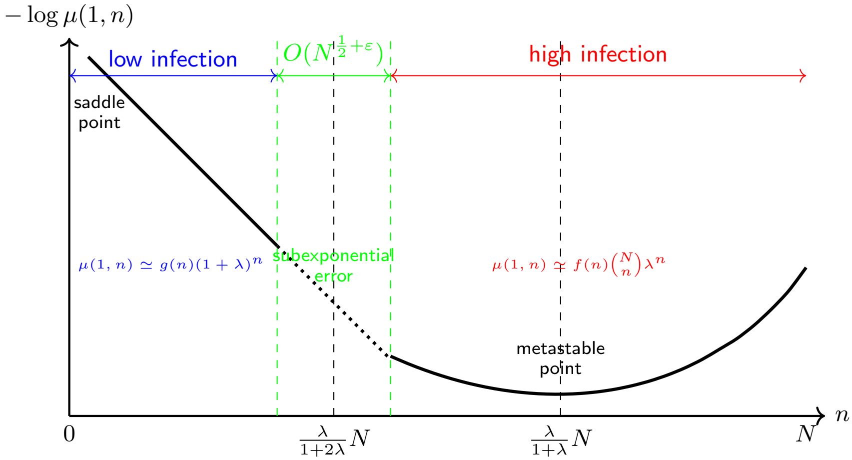
The stationary measure of the process is the Dirac mass at the all-healthy state.
To apply potential theory, we add supplementary transition rates from $\emptyset$.

- Natural choice: rate proportional to the stationary measure of the process conditioned on the non-extinction.
  $\Longrightarrow$ Hard to compute in general.
- We add a regeneration at the hub: $(0,0) \xrightarrow{\alpha} (1,0)$.
  Or equivalently, we consider the process restricted to the non-extinction.

A closed form solution for the quasi-stationary measure μ is due to (Cator–Mieghem '13).

Precise asymptotics for quasi-stationary measure

Using special function theory and refined Laplace's method, we compute:



Uniform asymptotic behavior of the quasi-stationary measure μ

Potential theoretic approach

A precise framework for quantifying metastability metrics (e.g. $\mathbb{E}\tau$) in terms of potential theoretic terms (e.g. capacity, equilibrium potential) was established in (BEGK '01, '04), and then extended to non-reversible settings recently.

Hitting time formula for non-reversible process

$$\mathbb{E}_x \tau_y = \frac{1}{\mathrm{cap}(x, y)} \sum_z h^{\dagger}_{x,y}(z) \mu(z).$$

Precise asymptotics for the quasi-stationary measure

$\Longrightarrow$ Good test functions/flows approximating the harmonic functions/flows

$\Longrightarrow$ Sharp estimates for capacity and potential (via variational principles)