Metastability of Contact Processes

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Prob. gp. reading seminar
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Thank you for introducing me.
I would like begin by thanking everyone attending here, and also the seminar organizers for giving me the chance to present my talk here.

My talk will be divided into roughly 3 parts.

1. provide some generalities & backgrounds for the theory of contact process
2. introduce our current status of knowledge on metastability of contact processes on finite graphs.

. present my result on this topic, and describe the methodology if time allows.

I. General Theory

II. Metastability on Finite Graphs.

III. Main Result

I. General Theory

· interacting particle system first suggested by Harris in 1974.

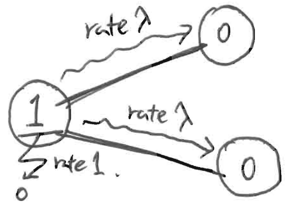
· process that models the spread of an infection on a network.

· $G=(V,E)$ locally finite graph. (connected)

· $\lambda>0$ infection rate.

· $\eta=(\eta_t)_{t\geq 0} \in \{0,1\}^V$ (0: healthy, 1: infected)

$\left[\text{"="} \{x\in V : \eta(x)=1\} \subseteq V \right]$



"Each infected site recovers w/ rate 1, and transmits infection to each adjacent site w/ rate λ."

. Think in this way: indep. clock at each site & (directed) edge.

· For $F:\{0,1\}^V \to \mathbb{R}$.

$$\mathcal{L}F(\eta) = \sum_{x\in\eta}\left[[F(\eta\setminus\{x\})-F(\eta)] + \lambda\cdot\sum_{\substack{y\notin\eta\\ y\sim x}}[F(\eta\cup\{y\})-F(\eta)]\right]$$

Note · If $\eta=\emptyset$, then the infection "extincts."

· If η_0 is infinite, then the process never extincts w/ prob 1.

infinite volume phase transition

Thm (Harris '74)

On $\mathbb{Z}^d$, $\exists \lambda_c=\lambda_c(\mathbb{Z}^d)\in(0,\infty)$ s.t.

$$\mathbb{P}_0[\eta \text{ survives forever}] \begin{cases} =0 & \text{if } \lambda<\lambda_c \\ >0 & \text{if } \lambda>\lambda_c \end{cases}$$

pf sketch

• Extinction for small λ

: Degree is bdd by $D=2^d$.

$$\frac{\text{total rate of transmission}}{\text{" of recovery}} \leq D\cdot\lambda.$$

$\Rightarrow$ If $\lambda<\frac{1}{D}$, then the process extincts a.s.

• Survival for large λ

: Comparison w/ an oriented percolation. ☒

cf. **Thm** (Bezuidenhout–Grimmett '90)

On $\mathbb{Z}^d$, $\mathbb{P}_0[\eta \text{ survives forever}]=0$ if $\lambda=\lambda_c$.

Idea: comparison w/ oriented percolation, but in a more delicate way of construction.
"Bezuidenhout–Grimmett Renormalization"

II. Metastability on Finite Graphs.

Note If G is finite, then $\mathbb{P}[\eta \text{ eventually extincts}]=1$.

$\because$ MC w/ an absorbing state $\emptyset$.

· $\tau_G=\inf\{t : \eta_t=\emptyset\}$ extinction time.

[We will assume $\eta_0=V$ all-infected state]

Main Concern:

Fix a sequence of graphs $(G_N)_{N\geq1}$ and study the growth of τ_{G_N}.

finite volume phase transition.

Thm (CGOV '84, S '85, DS '88, DL '88, M '93, M '99)

On $\mathbb{Z}_N^d$ (w/ free boundary),

(a) $\tau_{\mathbb{Z}_N^d} \sim \log|\mathbb{Z}_N^d|$ if $\lambda<\lambda_c(\mathbb{Z}^d)$

(b) $\tau_{\mathbb{Z}_N^d} \sim \exp[c_\lambda\cdot|\mathbb{Z}_N^d|]$ if $\lambda>\lambda_c(\mathbb{Z}^d)$

metastability

Overview of (b).

I. A variation of Bezuidenhout-Grimmett renormalization $\longrightarrow$ exp. growth: $c_\lambda>0$

II. Coupling lemmas.

$\Downarrow$

III. Superadditivity $\rightsquigarrow$ existence of c_λ.

Thm (Mountford–Mourrat–Yao–Valesin '16, Schapira–Valesin '17)

Suppose $\lambda > \lambda_c(\mathbb{Z})$.

(1) $\forall D>0$, $\exists c = c(\lambda, D)$ s.t.

$$\mathbb{E}\tau_G \geq \exp[c \cdot |G|]$$

$\forall G$ w/ degrees $\leq D$.

(2) $\forall \varepsilon > 0$, $\exists c = c(\lambda, \varepsilon)$ s.t.

$$\mathbb{E}\tau_G \geq \exp\left[c \cdot \frac{|G|}{(\log |G|)^{1+\varepsilon}}\right] \quad \forall G.$$

Remark

The growth of $\mathbb{E}\tau$ is already very informative:

Prop

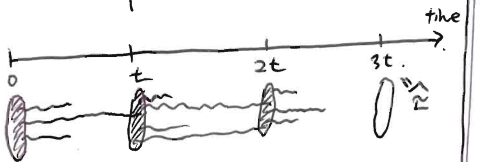
(a) (Markov) $\mathbb{P}[\tau > t] \leq \frac{\mathbb{E}\tau}{t}$

(b) ("upside-down Markov") $\mathbb{P}[\tau < t] \leq \frac{t}{\mathbb{E}\tau}$.

...tial:

...f of (b)

Let $\hat{\tau} = \inf\{ s \in t\mathbb{N} :$ The process started at time $s-t$ extincts at time $s \}$



$\Rightarrow$ · $\tau \leq \hat{\tau}$

· $\hat{\tau} \sim t \cdot \mathrm{Geom}(\mathbb{P}[\tau < t])$.

$\Rightarrow \mathbb{E}\tau \leq \mathbb{E}\hat{\tau} = \frac{t}{\mathbb{P}[\tau < t]}$. ∎

Thm (Schapira–Valesin '17)

Suppose $\lambda > \lambda_c(\mathbb{Z})$.

$$\frac{\tau_{G_N}}{\mathbb{E}\tau_{G_N}} \Rightarrow \mathrm{Exp}(1) \quad \forall (G_N) \text{ w/ } |G_N| \to \infty.$$

Let us move our concern to the more precise estimate of $\mathbb{E}\tau$.

Recall levels of estimation for $\mathbb{E}\tau$:

① $\mathbb{E}\tau \geq e^{cN}$ (metastability).

② $\frac{1}{N}\log \mathbb{E}\tau \longrightarrow \exists c$ (large deviation principle).

③ $\mathbb{E}\tau \simeq f(N)\cdot e^{cN}$ (Eyring–Kramers formula).

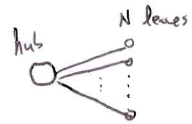
↳ $= N, N^2, e^{\sqrt{N}}, \ldots$

↳ up to a multiplicative factor of $1+o(1)$.

Returning to our topic; where are we now?

Remark

- Mostly ①.
- ②: $\exists c_\lambda$ for $\mathbb{Z}_N^d$ (w/ free boundary).
  but open for $\mathbb{Z}_N^d$ w/ periodic boundary (torus).
  [∵ We cannot prove "subadditivity"]
  $\exists c_\lambda$ for some random graph models.
  e.g.
  - supercrit. percolation on $\mathbb{Z}_N^d$.
  - superlevel set of a Gaussian free field on $\mathbb{Z}_N^d$.
  - supercrit. random geometric graph on $\mathbb{Z}_N^d$.
  - truncated Galton–Watson tree.
- ③: Open except for
  (i) triviality: K_N. complete graph. (reduces to NNRW)
  (ii) main result: S_N star graph.



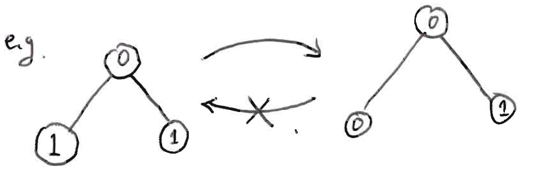
What makes this precise estimation so difficult?

Main Obstacles:

1. Spatial asymmetry of G complicates the process.
   ∵ ldp heavily relies on very specific geometric features of the underlying graph.
   · What can be a typical state of the process on $\mathbb{Z}_N^d$ w/ periodic bdy.

2. Non-reversibility.

e.g.



III. Main Result

Thm (J. '24)

On S_N,

$$\mathbb{E} Z_{S_N} \simeq K_\lambda \cdot N^{-\frac{1}{1+2\lambda}} \cdot \left(\frac{(1+\lambda)^2}{1+2\lambda}\right)^N.$$

optional.

(where $K_\lambda = \left(\frac{1+\lambda}{\lambda}\right)^{\frac{2}{1+2\lambda}} \cdot \Gamma\left(\frac{2(1+\lambda)}{1+2\lambda}\right)$.)

In particular,

$$\frac{1}{N} \log \mathbb{E} Z_{S_N} \longrightarrow C_\lambda = 2\log(1+\lambda) - \log(1+2\lambda)$$

Main Tools:

- potential theoretic method
- special ftn theory.