Spin Glass Theory

We do not actually understand glass!?

Spin Glass : Mathematical theory to understand magnetic substances.

e.g. magnetization? phase transition. [i.e. Curie temperature] ?

History

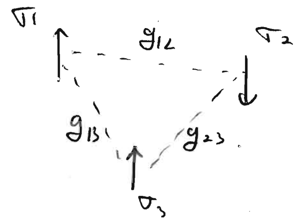
- 1953 RKKY model
- 1975 Edwards – Anderson model. : nearest neighbor interaction.
  similar to Ising model.

  Sherrington – Kirkpatrick model : <u>First solvable model</u>!!
  via "replica method".
  but flawed for low temperature.

- 1979 Parisi : Fixed the method. ⟶ 2021 Nobel Physics.
- 2000s Guerra — Talagrand : Mathematically rigorous soln
  ⟶ Talagrand : 2024 Abel.

§ SK model

So called the mean-field model.



- particles $\{1, \cdots, N\} = [N]$.
- spins $\sigma = (\sigma_1, \cdots, \sigma_N)$. $\sigma_j \in \{\pm 1\}$.
- Hamiltonian $H_N : \{\pm 1\}^N \longrightarrow \mathbb{R}$.

$$H_N(\sigma) := \underbrace{\frac{1}{\sqrt{N}}}_{\text{normalize}} \sum_{i,j} \underbrace{g_{ij}}_{\text{interaction}} \sigma_i \sigma_j . \qquad g_{ij} \sim N(0,1) \text{ iid.}$$

- inverse temperature $\beta \geq 0$.

※ double randomness.

Gibbs measure $G_{N,\beta}(\sigma) := \frac{1}{Z_N} \exp(\beta H_N(\sigma))$.

Z_N: partition number. [normalizing ⋯].

※ $\beta = 0$ $[T = \infty]$: $G_{N,\beta}$ is uniform.
$\beta = \infty$ $[T = 0]$: "$G_{N,\beta}$ is Dirac at the maximizer of H_N" roughly...

- [limity] free energy $F_N := \frac{1}{N} \log Z_N$.

※ $Z_N = \sum_{\sigma \in \{\pm 1\}^N} \exp(\beta H_N(\sigma))$

$\rightsquigarrow$ F_N reflects the typical behavior of the particles.

§ Toy model : Curie–Weiss model

We put

- $H_N(\sigma) = \frac{1}{N} \sum_{i,j} \sigma_i \sigma_j$.
- $G_{N,\beta}(\sigma) = \frac{1}{Z_N} \sum_{\sigma} \exp(\beta H_N(\sigma))$.

$\left[H_N(\sigma) = \sum_j \sigma_j m_N \right]$ mean-field interaction.

- magnetization $m_N := \frac{1}{N} \sum_j \sigma_j$.

$\Rightarrow H_N(\sigma) = N m_N^2$.

$$Z_N = \sum_{\sigma} \exp(N\beta m_N^2) = 2^N \cdot \mathbb{E} \exp(N \cdot \beta \cdot m_N^2).$$

$\boxed{\text{FACT}}$ (large deviation principle)

For $x \neq 0$, $\mathbb{P}[m_N \in (x-\varepsilon, x+\varepsilon)] \approx \exp(-N I(x))$

where $I(x) = \frac{1-x}{2} \log(1-x) + \frac{1+x}{2} \log(1+x)$.

pf: Use Stirling.

$\Rightarrow$ Roughly,

$$Z_N \approx 2^N \int_{-1}^{1} \exp(N\beta x^2) \cdot \text{"}\mathbb{P}[m_N = x]\text{"}\, dx \approx 2^N \int_{-1}^{1} e^{N\beta x^2 - N I(x)}\, dx.$$

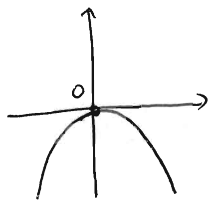
$$\Rightarrow F_N = \frac{1}{N} \log Z_N \approx \frac{1}{N} \log\left(2^N \int_{-1}^{1} e^{N(\beta x^2 - I(x))}\, dx\right).$$

$$\xrightarrow{N \to \infty} \log 2 + \max_{m \in [-1,1]} \{\beta m^2 - I(m)\} \quad \text{by Laplace's method.}$$

Put $f(m) = \log 2 + \beta m^2 - I(m)$.

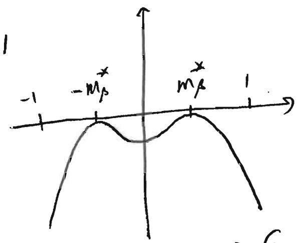
m_β^* solves $x = \tanh(\beta x)$.

- $\beta \leq 1$



$\rightsquigarrow G_{N,\beta}$ concentrated around $\{\sigma : m_N = 0\}$.

- $\beta > 1$



$\rightsquigarrow G_{N,\beta}$ concentrated around $\{\sigma : m_N = \pm m_\beta^*\}$.

"phase transition".

§ SK model revisited.

- overlap $R(\sigma,\tau) := \frac{1}{N}\sum_j \sigma_j \tau_j = \frac{1}{N}\langle \sigma, \tau \rangle$.

$\Rightarrow H_N$ is a Gaussian process w/

$$\mathbb{E} H_N(\sigma) = 0$$

$$\mathbb{E}[H_N(\sigma) H_N(\tau)] = \frac{1}{N}\mathbb{E}\Big(\sum_{ij} g_{ij}\sigma_i\sigma_j\Big)\Big(\sum_{k,l} g_{kl}\tau_k\tau_l\Big)$$

$$= \frac{1}{N}\sum_{ij}\sigma_i\sigma_j\tau_i\tau_j$$

$$= N \cdot R(\sigma,\tau)^2$$

We generalize:

$H_N : \{\pm 1\}^N \longrightarrow \mathbb{R}$ Gaussian process w/

$\mathbb{E} H_N(\sigma) = 0$. $\quad \mathbb{E}[H_N(\sigma) H_N(\tau)] = N \cdot \xi(R(\sigma,\tau))$

where $\xi : \mathbb{R} \longrightarrow \mathbb{R}$. $\quad \xi(t) = \sum_p a_p t^p$. : mixed p-spin.

- $\xi(t) = t^2$: SK model.
- $\xi(t) = t^p$: pure p-spin.

Thm (Parisi's formula ; Guerra–Talagrand '06)

$$\lim_{N\to\infty} \frac{1}{N}\log Z_N = \inf_{\mu \in PM[0,1]} \mathcal{P}_\beta(\mu). \quad \text{a.s.}$$

where $\mathcal{P}_\beta(\mu) := \Phi(0,0) - \frac{\beta^2}{2}\int_0^1 t\,\xi''(t)\,\mu[0,t]\,dt + \log 2$.

and $\Phi : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ that solves the Parisi PDE:

$$\Phi_t = -\frac{\beta\,\xi''(t)}{2}\left[\Phi_{xx} + \mu[0,t]\,\Phi_x^2\right] \quad \text{w/ } \Phi(1,x) = \log\cosh x$$

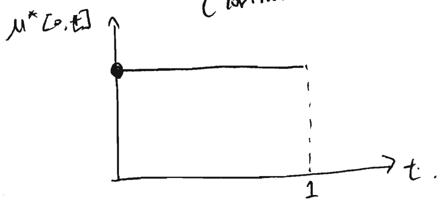
In particular, $\exists!$ minimizer μ^* of $\mathcal{P}_\beta(\mu)$ called the Parisi measure.

※ Solving a single Parisi PDE is simple,
but the hard part is that we have to solve the variational problem!

- SK model.

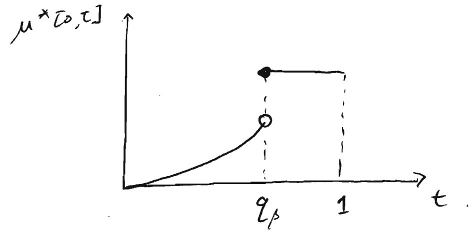
$\cdot\ \beta \le 1$.

(Talagrand '02).



Replica Symmetry (RS)

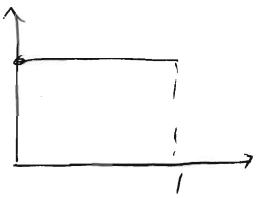
$\cdot\ \beta > 1$. (Conjectured).



Full Replica Symmetry Breaking (FRSB).

- pure p-spin.

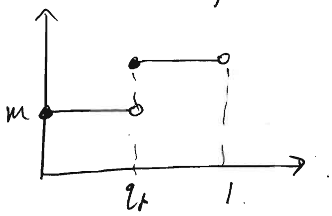
$\cdot\ \beta \le \beta_p$



RS.

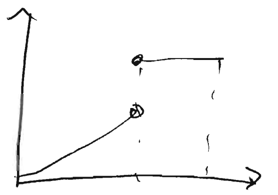
$\cdot\ \beta_p < \beta \le \beta_p'$ (Conjectured).

Gardner's phase. '84.



1-step Replica Symmetry Breaking (1RSB).

$\cdot\ \beta > \beta_p'$. (Conjectured).



FRSB.

- mixed p-spin.

$\cdot\ \beta \le \beta'$.

RS.

$\cdot\ \beta' < \beta \le \beta''$

?????!

$\cdot\ \beta > \beta''$ (Conjectured)

FRSB.

A solution to any of the above conjectures will be on the annals!