Seoul National University

Peer Seminar

Determinantal Point Process

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Contents

A determinantal point process is a random configuration of particles whose correlation functions are given by determinantal formulas with a fixed kernel. While these processes initially appeared in random matrix theory in the early 1960s, it was first looked into by Macchi in 1975 [9] as a distinct class, particularly in the context of how fermions distribute in a thermal equilibrium. The term 'determinantal' was introduced by Borodin and Olshanski in 2000 [4] within their study on first natural examples of processes non-Hermitian kernels, and now it became the standard terminology. These processes naturally arise in diverse fields including random matrix theory, physics, and combinatorics. Their determinantal structure provides them a rich algebraic nature.

We offer a first introduction to the theory of determinantal point processes. In Section 1, we establish the basic framework for determinantal point processes on a discrete state space, focusing particularly on characterizing and sampling these processes. Expanding upon these concepts to the continuous case in Section 2, we provide specific examples of determinantal point processes found in various fields in Section 3.

1 Discrete Determinantal Point Processes

1.1 Basic Definitions

For the sake of simplicity, let us firstly consider a finite state space $\mathfrak{X}$ equipped with the counting measure μ. A (simple) *point process* on $\mathfrak{X}$ is a random subset $X \subseteq \mathfrak{X}$. The elements of X are often referred as *particles*.

The law of a point process can be described by its correlation functions.

Definition 1.1 (Correlation function)**.** The *correlation function* of a point process X on $\mathfrak{X}$ is the function ρ defined by

$$\rho(A) = \mathbb{P}[A \subseteq X]$$

where A ranges over subsets of $\mathfrak{X}$. The *n-point correlation function* of X is the function ρ_n given by

$$\rho_n(x_1, \dots, x_n) = \rho(\{x_1, \dots, x_n\})$$

where $x_1, \dots, x_n$ are distinct points in $\mathfrak{X}$.

Definition 1.2 (Determinantal point process)**.** A *determinantal point process* X on $\mathfrak{X}$ is is a point process with a $|\mathfrak{X}| \times |\mathfrak{X}|$ matrix K such that the correlation functions of X are given by

$$\rho_n(x_1, \dots, x_n) = \det[K(x_j, x_l)]_{j,l=1}^n$$

for all $1 \le n \le N$ and distinct $x_1, \dots, x_n \in \mathfrak{X}$. The matrix K is called the *correlation kernel* for X.

Much like Gaussian processes, where Wick's formula encapsulates all information within their covariance, we may understand determinantal point processes as systems where all the data is encoded in their correlation kernels.

Remark 1.3. The correlation kernel of a determinantal point process is *not* unique, since the *gauge transformations* of the form

$$K(x, y) \longmapsto \frac{f(x)}{f(y)} K(x, y)$$

with a nonvanishing $f : \mathfrak{X} \to \mathbb{C}$ does not change the law of the process.

1.2 Determinantal Projection Processes

A specific class of determinantal point processes arises from projection operators on $L^2(\mathfrak{X})$. This is crucial for both theory-building and modeling.

Definition 1.4 (Determinantal projection process)**.** Let H be an N-dimensional subspace of $L^2(\mathfrak{X})$. A *determinantal projection process* X on $\mathfrak{X}$ associated to H is a determinantal point process whose kernel $K = K_H$ defines a projection operator $\mathcal{K}_H : L^2(\mathfrak{X}) \to H$, or equivalently,

$$K_H(x, y) = \sum_{j=1}^{N} \phi_j(x)\overline{\phi_j(y)}$$

where $\{\phi_1, \ldots, \phi_N\}$ is an orthonormal basis for H.

Definition 1.5 (Reproducing Property)**.** A $|\mathfrak{X}| \times |\mathfrak{X}|$ matrix K is a *reproducing kernel* on $\mathfrak{X}$ with respect to μ if

$$\int_{\mathfrak{X}} K(x, u)K(u, y)d\mu(u) = K(x, y)$$

for all $x, y \in \mathfrak{X}$.

Lemma 1.6. *Let K be a reproducing kernel on $\mathfrak{X}$ with respect to μ with total mass*

$$d = \int_{\mathfrak{X}} K(x, x)d\mu(x).$$

Then for $n \geq 2$,

$$\int_{\mathfrak{X}} \det[K(x_j, x_l)]_{j,l=1}^{n} d\mu(x_n) = (d - n + 1) \det[K(x_j, x_l)]_{j,l=1}^{n-1}.$$

In particular,

$$\int_{\mathfrak{X}^n} \det[K(x_j, x_l)]_{j,l=1}^{n} d\mu(x_1) \ldots d\mu(x_n) = (d - n + 1)(d - n + 2) \ldots (d - 1)d.$$

Proof. Consider the cofactor expansion. □

Lemma 1.7. *Let $K = K_H$ be the projection kernel associated to an N-dimensional subspace of $L^2(\mathfrak{X})$. Then, K is a reproducing kernel on $\mathfrak{X}$ with respect to μ with total mass N.*

Proof. Pick an orthonormal basis $\{\phi_1, \ldots, \phi_N\}$ for H. Then, for each $x, y \in \mathfrak{X}$,

$$\begin{aligned}
\int_{\mathfrak{X}} K(x, u)K(u, y)d\mu(u) &= \int_{\mathfrak{X}} \sum_{j=1}^{N} \sum_{l=1}^{N} \phi_j(x)\overline{\phi_j(u)}\phi_l(u)\overline{\phi_l(y)}d\mu(u) \\
&= \sum_{j=1}^{N} \sum_{l=1}^{N} \phi_j(x)\overline{\phi_l(y)}\delta_{j,l} \\
&= K(x, y),
\end{aligned}$$

so K is a reproducing kernel. The total mass of K is

$$\int_{\mathfrak{X}} K(x, x)d\mu(x) = \int_{\mathfrak{X}} \sum_{j=1}^{N} |\phi_j(x)|^2 d\mu(x) = N.$$

□

Theorem 1.8. *Let H be an N-dimensional subspace of $L^2(\mathfrak{X})$. Then, there exists a determinantal projection process X on $\mathfrak{X}$ associated to H, whose number of particles is almost surely N.*

Proof. Let $\{\phi_1, \ldots, \phi_N\}$ be an orthonormal basis of H, and consider

$$K(x, y) = \sum_{j=1}^{N} \phi_j(x)\overline{\phi_j(y)}.$$

Then $[K(x_j, x_l)]_{j,l=1}^n = AA^*$ where $A = [\phi_j(x_l)]_{1 \leq j \leq N, 1 \leq l \leq n}$, so $\det[K(x_j, x_l)]_{j,l=1}^n \geq 0$ for all n. Also,

$$\int_{\mathfrak{X}^N} \det[K(x_j, x_l)]_{j,l=1}^N d\mu(x_1) \ldots d\mu(x_N) = N!,$$

so $\frac{1}{N!}\det[K(x_j, x_l)]_{j,l=1}^N$ is a probability density on $\mathfrak{X}^N$. Looking at the unlabeled points, we get a point process X with N-point correlation function ρ_N. The compatibility with lower correlation functions follows from the previous lemmas. $\square$

Alternatively, we can realize a determinantal projection process through the algorithm outlined below. This not only reveals the recursive nature of the determinantal projection processes, but also proves to be particularly useful in practical modeling.

Proposition 1.9. *Let H be an N-dimensional subspace of $L^2(\mathfrak{X})$. Then, the points $(X_1, \ldots, X_N)$ sampled by the following algorithm is distributed as the uniform random ordering of the determinantal projection process on $\mathfrak{X}$ associated to H.*

Algorithm 1 Recursive construction of a determinantal projection process

```
H_N ← H
n ← N
while n ≠ 0 do
    Sample X_n ∼ (1/n) μ_{H_n} where μ_{H_n}(x) = ‖K_{H_n} 1_x‖² μ(x)
    H_{n−1} ← H_n ∩ ⟨1_{X_n}⟩^⊥
    n ← n − 1
end while
```

Proof. Fix N distinct points $x_1, \ldots, x_N$ in $\mathfrak{X}$. Let $\psi_j = \mathcal{K}_H \mathbb{1}_{x_j}$. Note that $\mathcal{K}_{H_j}\mathbb{1}_{x_j} = \mathcal{K}_{H_j}\psi_j$. Hence,

$$\begin{aligned}
\mathbb{P}[X_j = x_j \ \forall 1 \leq j \leq N] &= \prod_{j=1}^{N} \frac{1}{j} \|\mathcal{K}_{H_j}\psi_j\|^2 \\
&= \frac{1}{N!} \operatorname{Vol}_H(\psi_1 \wedge \cdots \wedge \psi_N)^2 \\
&= \frac{1}{N!} \det\left[\int_{\mathfrak{X}} \psi_j \overline{\psi_l} d\mu\right]_{j,l=1}^N \\
&= \frac{1}{N!} \det[K(x_j, x_l)]_{j,l=1}^N.
\end{aligned}$$

$\square$

1.3 Processes with Hermitian Kernels

Since the correlation functions must lie in $[0, 1]$, not every kernel K defines a determinantal point process. Nevertheless, we can construct a large class of determinantal point process by taking a mixture of determinantal projection processes. In fact, this method precisely covers all determinantal point processes with Hermitian kernels.

Theorem 1.10 (Macchi [9], Soshnikov [10]). *Let K be a $|\mathfrak{X}| \times |\mathfrak{X}|$ Hermitian matrix. Then, K defines a determinantal point process on $\mathfrak{X}$ if and only if all the eigenvalues of K are in $[0, 1]$. Moreover, the number of particles in the process has the distribution of a sum of independent* Bernoulli(λ_j) *random variables, where $\lambda_j \in [0, 1]$ are the eigenvalues of K, counted with multiplicity.*

Proof. For sufficiency, consider

$$K(x, y) = \sum_{j=1}^{N} \lambda_j \phi_j(x)\overline{\phi_j(y)}$$

where $\{\phi_1, \ldots, \phi_N\}$ is an orthonormal set of eigenfunctions of K with eigenvalues $\lambda_j \in [0, 1]$. Let $I_1, \ldots, I_N$ be independent Bernoulli random variables with parameters $\lambda_1, \ldots, \lambda_N$. Define

$$K_I(x, y) = \sum_{j=1}^{N} I_j \phi_j(x)\overline{\phi_j(y)},$$

and let X_I be the determinantal point process with kernel K_I, that is, first sample $I_1, \ldots, I_N$ and then independently sample a determinantal projection process with kernel K_I. Note that X_I is a well-defined point process on $\mathfrak{X}$ by Theorem 1.8. We will show that X_I is a determinantal point process with kernel K. Note that this also proves the second assertion, since given $I_1, \ldots, I_N$, X_I has $\sum_j I_j$ particles, almost surely.

It suffices to show that

$$\mathbb{E} \det[K_I(x_j, x_l)]_{j,l=1}^{n} = \det[K(x_j, x_l)]_{j,l=1}^{n}$$

for all $n \leq N$. Let A and B be the matrices of sizes $n \times N$ and $N \times n$, respectively, with entries $A_{j,k} = I_k \phi_k(x_j)$ and $B_{k,l} = \overline{\phi_k(x_l)}$. Note that

$$[K_I(x_j, x_l)]_{j,l=1}^{n} = AB.$$

Applying the Cauchy–Binet theorem and taking expectation gives the desired identity.

For necessity, suppose that X is a determinantal point process on $\mathfrak{X}$ with Hermitian kernel K. Since the correlation functions are nonnegative, K must be positive semi-definite. Assume to the contrary that the largest eigenvalue of K is $\lambda > 1$. Consider the process Y obtained by sampling X and then keeping each particle independently at random with probability $\frac{1}{\lambda}$. Then Y is a determinantal point process with kernel $\frac{1}{\lambda}K$. By the discussion above, the number of particles in Y is a sum of independent Bernoulli random variables, and at least one of them is 1 with probability one, so Y is almost surely nonempty. However, from its construction, Y is empty with nonzero probability, a contradiction. $\square$

Remark 1.11. The construction of K_I in the above proof, together with Algorithm 1, can serve as a sampling method for determinantal point processes. This *spectral method* allows for exact sampling with a complexity of $O(|\mathfrak{X}|N^2)$, along with an additional preprocessing cost $O(|\mathfrak{X}|^3)$ for the eigendecomposition of the kernel. Since computing eigenfunctions can be computationally intensive, especially for large datasets, there have been several attempts to speed up sampling under certain constraints on K. A detailed discussion on this topic can be found in [7].

2 Continuous Determinantal Point Processes

The theory of determinantal point processes can be extended to the continuous setting. Let the state space $\mathfrak{X}$ be a locally compact separable topological space equipped with a Radon reference measure μ. For instance, one can choose $\mathfrak{X} \subseteq \mathbb{R}$ and the Lebesgue measure μ. A *point configuration* X in $\mathfrak{X}$ is a locally finite subset of $\mathfrak{X}$. The set of point configurations $\mathrm{Conf}(\mathfrak{X})$ may be endowed with the Borel σ-algebra generated by the counting functions N_A on $\mathrm{Conf}(\mathfrak{X})$ where A ranges over compact subsets of $\mathfrak{X}$. A *point process* on $\mathfrak{X}$ is a random point configuration in $\mathfrak{X}$, that is, a probability measure on $\mathrm{Conf}(\mathfrak{X})$.

Definition 2.1 (Correlation function)**.** For $n \geq 1$, the *n-point correlation measure* of a point process X on $\mathfrak{X}$ is the symmetric measure (if it exists) ρ_n on $\mathfrak{X}^n$ such that

$$\int_{\mathfrak{X}^n} f \rho_n = \mathbb{E}\left[\sum_{x_{j_1},\dots,x_{j_n} \in X} f(x_{j_1},\dots,x_{j_n})\right]$$

for all compactly supported bounded Borel function f on $\mathfrak{X}^n$, where the sum on the right-hand side is taken over all tuples of distinct points $x_{j_1},\dots,x_{j_n}$ in X. If ρ_n is absolutely continuous with respect to $\mu^{\otimes n}$, then its density is called the *n-point correlation function* of X, and also denoted by ρ_n.

Remark 2.2. Intuitively, when $\mathfrak{X} \subseteq \mathbb{R}$ and μ is absolutely continuous with respect to the Lebesgue measure, we have

$$\mathbb{P}[X \text{ has a particle in } [x_j, x_j + dx_j] \text{ for all } 1 \leq j \leq n] = \rho_n(x_1,\dots,x_n) d\mu(x_1)\dots d\mu(x_n).$$

Definition 2.3 (Determinantal point process)**.** A *determinantal point process* X on $\mathfrak{X}$ is a point process with a function $K : \mathfrak{X} \to \mathfrak{X} \to \mathbb{C}$ such that the correlation functions of X are given by

$$\rho_n(x_1,\dots,x_n) = \det[K(x_j,x_l)]_{j,l=1}^n$$

for all $1 \leq n \leq N$ and distinct $x_1,\dots,x_n \in \mathfrak{X}$. The function K is called the *correlation kernel* for X.

Every property we proved for the discrete case can be restated in a generalized form, and it can also be proved using essentially the same approach. We will state them without proofs. See [8] for details.

Theorem 2.4. *Let H be an N-dimensional subspace of $L^2(\mathfrak{X})$. Then, there exists a determinantal projection process X on $\mathfrak{X}$ associated to H, whose number of particles is almost surely N.*

Proposition 2.5. *Let H be an N-dimensional subspace of $L^2(\mathfrak{X})$. Then, the points $(X_1,\dots,X_N)$ sampled by Algorithm 1 is distributed as the uniform random ordering of the determinantal projection process on $\mathfrak{X}$ associated to H.*

Theorem 2.6 (Macchi [9], Soshnikov [10])**.** *Let K determine a Hermitian integral operator $\mathcal{K}$ on $L^2(\mathfrak{X})$ that is locally trace class. Then, K defines a determinantal point process on $\mathfrak{X}$ if and only if all the eigenvalues of $\mathcal{K}$ are in $[0,1]$. Moreover, the number of particles in the process has the distribution of a sum of independent* $\mathrm{Bernoulli}(\lambda_j)$ *random variables, where $\lambda_j \in [0,1]$ are the eigenvalues of $\mathcal{K}$, counted with multiplicity.*

In particular, the Macchi–Soshnikov characterization of Hermitian kerneled determinantal point processes leads to the following central limit theorem.

Theorem 2.7 (Central limit theorem for determinantal point processes; Costin–Lebowitz [6], Soshnikov [10])**.** *Let $(X_N)_{N\geq 1}$ be a sequence of determinantal point processes on $\mathfrak{X}$ whose kernels are $(K_N)_{N\geq 1}$ defining locally trace class Hermitian integral operators on $L^2(\mathfrak{X})$ with eigenvalues in $[0,1]$. Let $(D_N)_{N\geq 1}$ be a sequence of measurable subsets of $\mathfrak{X}$ such that $\mathrm{Var}(|X_N \cap D_N|) \to \infty$ as $N \to \infty$. Then, in distribution,*

$$\frac{|X_N \cap D_N| - \mathbb{E}|X_N \cap D_N|}{\sqrt{\mathrm{Var}\,|X_N \cap D_N|}} \longrightarrow N(0,1) \quad \textit{as} \quad N \to \infty.$$

Proof. By previous theorem, $|X_N \cap D_N|$ has the distribution of a sum of independent Bernoulli random variables whose parameters are eigenvalues of the operators associated to K_N restricted to D_N. Hence, the assertion follows from the Lindeberg–Feller central limit theorem. □

3 Specific Examples

3.1 Random Matrix Models

Example 3.1 (Random unitary ensemble)**.** Consider an $N \times N$ random unitary ensemble whose spectrum $\lambda = \{\lambda_1, \ldots, \lambda_N\} \subseteq \mathbb{R}$ has probability distribution

$$d\mathbb{P}[\lambda] = \frac{1}{Z_N} \prod_{j<l} (\lambda_j - \lambda_l)^2 \prod_{j=1}^{N} \exp(-V(\lambda_j))d\mu(\lambda_j)$$

where μ is the Lebesgue measure on $\mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$ is a given function with growth condition

$$V(x) - \log\bigl(x^2 + 1\bigr) \longrightarrow \infty \quad \text{as} \quad x \to \pm\infty.$$

Then, λ is an N-point determinantal projection process associated to the subspace H of $L^2(\mathbb{R}, \mu)$ spanned by $x^j e^{-\frac{1}{2}V(x)}$ for $0 \leq j \leq N-1$. In particular, for the GUE case $V(x) = \frac{1}{2}x^2$, one can find an orthonormal basis for H of the form $\{\phi_j(x) = p_j(x)e^{-\frac{x^2}{4}}\}_{0\leq j\leq N-1}$ where each $p_j(x)$ is (a constant multiple of) the Hermite polynomial of degree j.

Example 3.2 (Random normal matrix ensemble)**.** Consider the $N \times N$ random normal matrix ensemble whose spectrum $z = \{z_1, \ldots, z_N\} \subseteq \mathbb{C}$ has probability distribution

$$d\mathbb{P}[z] = \frac{1}{Z_N} \prod_{j<l} |z_j - z_l|^2 \prod_{j=1}^{N} \exp(-Q(z_j))dA(z_j)$$

where $dA(z) = d^2z/\pi$ is the area measure and $Q : \mathbb{C} \to \mathbb{R}$ is a given function with enough growth condition at ∞. Then, z is an N-point determinantal projection process associated to the subspace H of $L^2(\mathbb{C}, A)$ spanned by $z^j e^{-\frac{1}{2}Q(z)}$ for $0 \leq j \leq N-1$. In particular, for the GinUE case $Q(z) = |z|^2$, one can find an orthonormal basis for H of the form $\{\phi_j(z) = p_j(z)e^{-\frac{|z|^2}{2}}\}_{0\leq j\leq N-1}$ where $p_j(z) = z^j/\sqrt{j!}$.

3.2 Uniform Spanning Trees

Example 3.3 (Uniform spanning tree)**.** Let $G = (V, E)$ be a finite undirected connected graph, and T be uniformly chosen from the set of spanning trees of G. Note that we may interpret $L^2(E)$ as the set of flows on G by fixing orientations of all the edges and identifying

$$L^2(E) = \{\phi : E \to \mathbb{R} : \phi(vw) = -\phi(wv)\ \forall v, w \in V\}.$$

Then, the edges of T is an $(|V|-1)$-point determinantal projection process associated to the subspace H of $L^2(E)$ spanned by

$$\sum_{w \in V} (\mathbb{1}_{vw} - \mathbb{1}_{wv})$$

where v varies over the vertices of V.

3.3 Hermitian Kernels

Example 3.4 (Sine process)**.** The *sine process* $X_{\sin}$ is the determinantal point process on $\mathbb{R}$ with kernel

$$K_{\sin}(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}.$$

The existence of sine process is guaranteed by the Macchi–Soshnikov theorem. Indeed, it is a determinantal projection process associated to the Paley-Wiener space, the space of functions in $L^2(\mathbb{R})$ with Fourier transform supported on $[-\pi, \pi]$. The sine process appears as the bulk scaling limit of the spectrums of random unitary ensembles.

Example 3.5 (Airy point process)**.** The *Airy point process* X_{Ai} is the determinantal point process on $\mathbb{R}$ with kernel

$$K_{Ai}(x, y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}$$

where $Ai(x)$ is the classical Airy function. The existence of sine process is guaranteed by the Macchi–Soshnikov theorem. The Airy point process appears as the soft-edge scaling limit of the spectrums of random unitary ensembles, and the last particle in the Airy point process follows the Tracy-Widom distribution.

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