Exponential Mixing and Transfer Operators

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Exponential Mixing

Mixing

- A measure μ is *mixing* for a measure-preserving transformation $T : X \to X$ if

$$\rho(n) := \int f \circ T^n \cdot g \, d\mu - \int f \, d\mu \int g \, d\mu \xrightarrow{n \to \infty} 0 \quad \forall f, g \in L^2(X, \mu).$$

- A measure μ is *mixing* for a measure-preserving flow $\phi_t : X \to X$ if

$$\rho(t) := \int f \circ \phi_t \cdot g \, d\mu - \int f \, d\mu \int g \, d\mu \xrightarrow{t \to \infty} 0 \quad \forall f, g \in L^2(X, \mu).$$

Q. How can we show the correlation functions ρ decay *exponentially* fast?

A. By analyzing spectral properties of the transfer operators $\mathcal{L}$.

Overview

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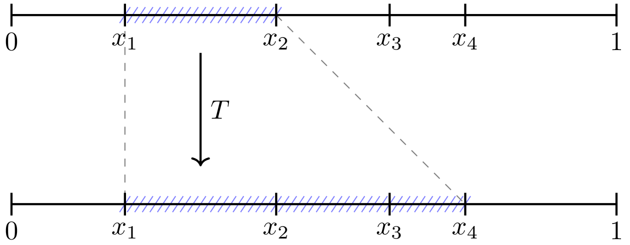
One-dimensional Expanding Map

Let $X = [0,1] = \coprod_{j=0}^{n-1}[x_j, x_{j+1}]$, and $T : X \to X$ be an *expanding map*, that is,

- T is piecewise C^∞;
- (Expanding) There exists $\beta > 1$ such that $|T'(x)| \geq \beta$ for all $x \in X$;
- (Markov) For each j, $T([x_j, x_{j+1}])$ is a union of the sub-intervals;
- (Transitivity) T has a dense orbit.

Note that we can associate a shift of finite type to T.



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Example

The doubling map $Tx = 2x \pmod 1$ is expanding with $\beta = 2$ and $x_1 = \frac{1}{2}$.

(Non-)Example

The Gauss map $Tx = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ is "almost expanding."

One-dimensional Expanding Map

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Goal (Exponential Mixing of Expanding Maps)

There exists a unique T-invariant measure $\mu \ll m$.
Moreover, there exists $0 < \theta < 1$ such that for all $f, g \in C^\infty(X)$, we have

$$\rho(n) = O(\theta^n).$$

Transfer Operator

The *transfer operator* $\mathcal{L} : L^1(m) \to L^1(m)$ is the "pushforward" of T, that is,

$$\mathcal{L}f \, dm := d(T_*(f \, m)).$$

In other words, for all $f \in L^\infty(m)$ and $g \in L^1(m)$, we have

$$\int f \circ T \cdot g \, dm = \int f \cdot \mathcal{L}g \, dm.$$

Then $\mathcal{L}$ is a contractive Markov operator (i.e. positive, integral preserving).

In case of expanding maps, we have an explicit formula:

$$\mathcal{L}f(x) = \sum_{Ty=x} \frac{1}{|T'(y)|} f(y).$$

Doeblin–Fortet Inequality

Consider the Banach space $W^{1,1}$ of absolutely continuous functions with the norm

$$\|f\|_{W^{1,1}} := \|f\|_1 + \|f'\|_1.$$

Theorem (Doeblin–Fortet 1937, Ionescu-Tulcea–Marinescu 1950, Lasota–Yorke 1972)

There exists $0 < \alpha < 1$ and $C \geq 0$ such that

$$\|\mathcal{L}^n f\|_{W^{1,1}} \leq \alpha^n \|f\|_{W^{1,1}} + C\|f\|_1 \quad \text{for all } n \geq 0,\ f \in W^{1,1}.$$

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Proof.

Using an induction on n, it sufficies to show only for $n = 1$.
Since

$$(\mathcal{L}f)' = \sum \Big[\frac{1}{(T')^2} f' - \frac{T''}{(T')^3} f\Big] = \mathcal{L}\Big(\frac{1}{T'} f\Big) - \mathcal{L}\Big(\frac{T''}{(T')^2} f\Big),$$

we get

$$\|(\mathcal{L}f)'\|_1 \leq \Big\|\frac{1}{T'}\Big\|_\infty \|f\|_1 + \Big\|\frac{T''}{(T')^2}\Big\|_\infty \|f\|_1 \leq \frac{1}{\beta}\|f\|_1 + C'\|f\|_1. \quad \square$$

Existence of Absolutely Continuous Invariant Measures

Theorem (Doeblin–Fortet 1937, Ionescu-Tulcea–Marinescu 1950, Lasota–Yorke 1972)

There exists $0 < \alpha < 1$ and $C \geq 0$ such that

$$\|\mathcal{L}^n f\|_{W^{1,1}} \leq \alpha^n \|f\|_{W^{1,1}} + C\|f\|_1 \quad \text{for all } n \geq 0,\ f \in W^{1,1}.$$

Note that:

- The iterates $\mathcal{L}^n f$ has uniformly bounded $W^{1,1}$-norms.
- $W^{1,1}$ compactly embeds in L^1; in particular, $W^{1,1}$-balls are compact in L^1.

Consider the ergodic sum

$$h_n = \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{L}^j 1,$$

then there exists a subsequence L^1-converges to a T-invariant density $h \in L^1$. Thus, $d\mu = h\,dm$ defines a T-invariant probability measure $\mu \ll m$.

Spectral Gap

The Doeblin–Fortet inequality implies that there exists $0 < \theta < 1$ such that

$$\|\mathcal{L}^n f\|_{W^{1,1}} \leq \theta^n \|f\|_{W^{1,1}}$$

for all $f \in W^{1,1}/\mathbb{C} = \{g \in W^{1,1} : \mu(g) = 0\}$.

Hence, $\mathcal{L}$ has a *spectral gap*, i.e., we may write $\mathcal{L} = P + N$ where

- $Pg = \mu(g)$ is a rank-one projection with simple eigenvalue 1;
- $Ng = \mathcal{L}[g - \mu(g)] \in W^{1,1}/\mathbb{C}$ has a spectral radius $\leq \theta < 1$;
- $PN = NP = 0$.

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This is an analogue of the Perron–Frobenius theorem:

Markov $n \times n$ matrix A	(Markov) Transfer operator $\mathcal{L}$
• simple eigenvalue 1	• simple eigenvalue 1
• all other eigenvalues λ_j have $\lvert\lambda_j\rvert < 1$	• all other eigenvalues λ have $\lvert\lambda\rvert \leq \theta < 1$

Spectral Gap and Exponential Mixing

Since $\mathcal{L} = P + N$, we have $\mathcal{L}^n = P + N^n$.

For $f, g \in C^\infty(X)$,

$$\begin{aligned} |\rho(n)| &= \left| \int f \circ T^n \cdot g \, d\mu - \mu(f)\mu(g) \right| \\ &= \left| \int f \circ T^n \cdot [g - \mu(g)] \, d\mu \right| \\ &= \left| \int f \cdot \mathcal{L}^n [g - \mu(g)] \, d\mu \right| \\ &= \left| \int f \cdot N^n g \, d\mu \right| \\ &\leq \theta^n \|f\|_\infty \|g\|_\infty = O(\theta^n). \end{aligned}$$

Hence, μ mixes exponentially fast.

Summary

Transfer Operator

$$\mathcal{L}f(x) = \sum_{Ty=x} \frac{1}{|T'(y)|} f(y)$$

↓

Doeblin–Fortet Inequality

$$\|\mathcal{L}^n f\|_{W^{1,1}} \leq \alpha^n \|f\|_{W^{1,1}} + C\|f\|_1$$

↓

Spectral Gap $\mathcal{L} = P + N$

$P^2 = P$, $\mathrm{spr}(N) < 1$, $PN = NP = 0$

↓

Exponential Mixing

$$\rho(n) = O(\theta^n)$$

Overview

1. Discrete Case: One-dimensional Expanding Maps

2. Continuous Case: Geodesic Flows

Exponential Mixing of Geodesic Flows

Theorem (Dolgopyat 1998)

Let V be a compact surface of (variable) negative curvature, m be the Liouville probability measure on T^1V.
Then, m mixes exponentially fast for the geodesic flow $\phi_t : T^1V \to T^1V$.

Remarks.

- In case of $\kappa \equiv -1$, we may use unitary representation theory.
  However, for the variable curvature case, the transfer operator technique is the only known methodology.
- m can be generalized to Gibbs measures (e.g. Bowen–Margulis measure).
- V can be generalized to higher dimensions with negative sectional curvature, where m is the Liouville measure.
- ϕ_t can be generalized to Anosov flows.
- The result can be extended to Teichmüller flows. (Avila–Gouzel–Yoccoz 2006)

Proof Overview

1. Consider the *symbolic dynamics* to replace ϕ_t by the suspension flow.
2. Define a one-parameter family $\{\mathcal{L}_\zeta\}_{\zeta\in\mathbb{C}}$ of *transfer operators*:

$$\mathcal{L}_\zeta f(x) = \sum_{Ty=x} \frac{e^{-\zeta r(y)}}{|T'(y)|} f(y), \quad \zeta \in \mathbb{C}.$$

3. Normalize $\mathcal{L}_\zeta$ into $\widetilde{\mathcal{L}}_\zeta$ using the analogy of *Perron–Frobenius theorem*.
4. Establish *Doeblin–Fortet-type inequalities* for $\widetilde{\mathcal{L}}_\zeta$.
5. Deduce *uniform bounds for spectral radii* of $\{\mathcal{L}_\zeta\}_{\zeta\in\mathbb{C}}$.
6. Analytically extend the Laplace transformation $\widehat{\rho}$ of ρ using $\mathcal{L}_\zeta$.
7. Apply the *Paley–Wiener theorem* to imply exponential mixing.

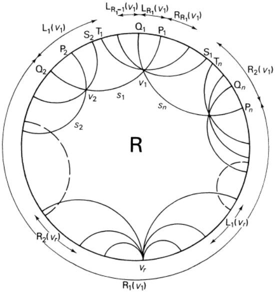
Symbolic Dynamics

We may associate ϕ_t with

- a *shift of finite type* $T : X \to X$, and
- a *return-time function* $r : X \to \mathbb{R}$.

It can be also done with general Anosov flows. (Bowen 1973, Ratner 1973)

Suppose $\kappa \equiv -1$, then $T^1V = \Gamma\backslash G$ where $G = \mathrm{SL}_2\,\mathbb{R}$ and $\Gamma \leq G$ lattice.
In this case, we may put $\partial\mathbb{H} \cong [0,1]$ in place of X.



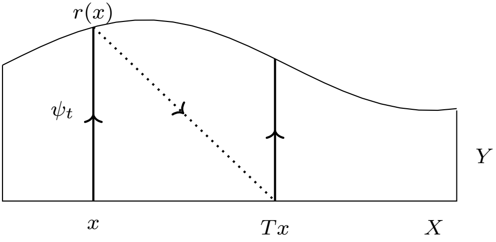
Boundary map construction for Fushsian groups (Bowen–Series 1979)

Symbolic Dynamics

Define the *suspension flow* $\psi_t : (Y, \nu) \to (Y, \nu)$ as

$$\begin{aligned} Y &:= (X \times \mathbb{R}) \big/ \big((x, u) \sim (Tx, u - r(x))\big), \\ \psi_t(x, u) &:= (x, u + t), \\ d\nu &:= \frac{1}{\int_X r \, d\mu} d\mu \times dt, \end{aligned}$$

where μ is the T-invariant probability measure on X.
Then, (ϕ_t, m) mixes exponentially fast if (ψ_t, ν) does so.



(See the monograph (Katok–Ugarcovici 2006) for details)

Paley–Weiner Theorem

Consider the Laplace transform $\widehat{\rho} : \mathbb{C} \to \mathbb{R}$ of the correlation function:

$$\widehat{\rho}(\zeta) := \int_0^\infty e^{-\zeta t} \rho(t)\, dt, \quad \zeta = \xi + i\eta \in \mathbb{C}.$$

The integral converges to an analytic function on $\mathrm{Re}(\zeta) = \xi > 0$.

Then, regularity of $\widehat{\rho}$ implies exponential decay of ρ.

Theorem (Paley–Wiener 1934)

Suppose that $\widehat{\rho}$ extends analytically to $\mathrm{Re}(\zeta) = \xi > -\varepsilon_0$ and satisfies

$$\sup_{-\varepsilon_0 < \xi < 0} \int_{\mathbb{R}} |\widehat{\rho}(\xi + i\eta)|\, d\eta < \infty.$$

Then, for all $\varepsilon < \varepsilon_0$, we have $\rho(t) = O(e^{-\varepsilon t})$.

Transfer Operators

To deal with $\widehat{\rho}$, we define a one-parameter family of transfer operators for T:
For $\zeta = \xi + i\eta \in \mathbb{C}$, define $\mathcal{L}_\zeta : C^1(X) \to C^1(X)$ as

$$\mathcal{L}_\zeta f(x) := \mathcal{L}(e^{-\zeta r} f) = \sum_{Ty=x} \frac{e^{-\zeta r(y)}}{|T'(y)|} f(y).$$

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For $F, G : Y \to \mathbb{R}$, define $f_\zeta, g_\zeta : X \to \mathbb{R}$ as

$$f_\zeta(x) := \int_0^{r(x)} F(x,u) e^{-\zeta u} \, du, \qquad g_\zeta(x) := \int_0^{r(x)} G(x,u) e^{-\zeta u} \, du.$$

Then, one can rewrite

$$\widehat{\rho}_{F,G}(\zeta) = \frac{1}{\int r \, d\mu} \int g_{-\zeta} \cdot (1 - \mathcal{L}_\zeta)^{-1} f_\zeta \, d\mu.$$

Hence, it suffices to show $\operatorname{spr}(\mathcal{L}_\zeta) \le \theta < 1$ for $\xi \approx 0$, $\eta \to \pm\infty$.

Ruelle–Perron–Frobenius Theorem

An analogue of the Perron–Frobenius theorem holds again.

Ruelle–Perron–Frobenius Theorem

For $\mathcal{L}_\xi : C^1(X) \to C^1(X)$ with $\xi \in \mathbb{R}$, the following statements hold.

- There exists a simple eigenvalue $\lambda_\xi > 0$ of maximum modulus.
- The associated eigenfunction $h_\xi \in C^1(X)$ is strictly positive.
- The associated eigenmeasure ν_ξ is positive.

Normalize $\mathcal{L}_\zeta$, $\zeta = \xi + i\eta \in \mathbb{C}$, by letting

$$\widetilde{\mathcal{L}}_\zeta f := \frac{1}{\lambda_\xi h_\xi} \mathcal{L}_\zeta (h_\xi f)$$

so that $\widetilde{\mathcal{L}}_\xi 1 = 1$.

Dolgopyat's Inequality

Consider a modified norm on $C^1(X)$ defined as

$$\|f\|_\eta := \|f\|_\infty + \frac{1}{|\eta|}\|f'\|_\infty, \quad \eta \in \mathbb{R} \setminus \{0\}.$$

Then, from the Doeblin–Fortet inequality for $\widetilde{\mathcal{L}}_\zeta$:

$$\|(\widetilde{\mathcal{L}}_\zeta^n f)'\|_\infty \leq C\big(\alpha^n \|f'\|_\infty + |\eta|\|f\|_\infty\big),$$

we obtain

$$\|\widetilde{\mathcal{L}}_\zeta^n f\|_\eta \leq (C+1)\|f\|_\eta.$$

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Theorem (Dolgopyat 1998)

There exists $C, \beta > 0$ such that for all small ξ and large η,

$$\|\widetilde{\mathcal{L}}_\zeta^{C \log |\eta|} f\|_\eta \leq \frac{\|f\|_\eta}{|\eta|^\beta} \quad \text{for all } f \in C^1(X).$$

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This implies

$$\operatorname{spr}(\mathcal{L}_\zeta) = \limsup_{n\to\infty} \|\mathcal{L}_\zeta^n\|_\eta^{1/n} \leq \lambda_\xi \limsup_{n\to\infty} \|\widetilde{\mathcal{L}}_\zeta^n\|_\eta^{1/n} \leq \lambda_\xi e^{-\beta/C} < 1,$$

since $\lambda_\xi \approx 1$ for $\xi \approx 0$.

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Thank you!