Cohomology Structure of Fixed Point Sets

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2024 Spring Algebraic Topology II
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- Based on [Bredon] VII.1 and VII.6.
- I will drop some generality for brevity; for instance, there are relative versions of the contents I cover. Also, I might lie a lot.

Notation

Throughout, we assume the following.

- Every topological space is assumed to be Hausdorff.
- G is a compact topological abelian group, and k is a field.
  Especially, $(G, k) = (\mathbb{Z}_p, \mathbb{Z}_p)$ or $(S^1, \mathbb{Q})$.
- X is a finitistic paracompact G-space.
  (e.g. compact space, finite dimensional CW-complex)
- $F = X^G$ is the fixed point set.

We use the Čech cohomology $\check{H}^*$. If we are working with CW-complexes, then the Čech cohomology coincides with the singular cohomology, and also with the sheaf cohomology.

Goal. Compute the cohomology ring $\check{H}^*(F; k)$.

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Goal. Compute the cohomology ring $\check{H}^*(F; k)$.

1. The orbit spaces are Hausdorff if G is compact. Other nice conditions hold.
   We assume abelianness for convenience.
2. A *G-space* is a space equipped with a continuous G-action. The morphisms between G-spaces are the *equivariant maps*, that is, continuous maps that commute with G-action.
3. We will not explain what does *finitistic* mean.
   (Every open cover has a finite order refinement. The order of an open cover is the maximal number of members with nonempty intersection.)
4. A space is *paracompact* if every open cover has a locally finite refinement.

Overview

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Fiber Bundle

Let F be a G-space with faithful action.
A *G-bundle over B with fiber F* is a continuous map $p : E \to B$ together with a maximal *G-atlas* $\{(U_\alpha, \varphi_\alpha)\}_\alpha$, that is,

- $\{U_\alpha\}_\alpha$ is an open cover for B.
- $\mathrm{proj} = p \circ \varphi_\alpha : F \times U_\alpha \xrightarrow{\sim} p^{-1}(U_\alpha) \to U_\alpha$ for each α.
- For each α, β, there is a *transition map* $\theta_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that

$$\varphi_\beta(x, b) = \varphi_\alpha(x \cdot \theta_{\alpha\beta}(b), b).$$

There is a unique G-action on E that is trivial on B and makes each chart φ_α equivariant.

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There is a unique G-action on E that is trivial on B and makes each chart φ_α equivariant.

1. That is, $\ker(G \to \mathrm{Homeo}(F))$ is trivial.

Universal Principal G-Bundle

A *principal G-bundle* is a G-bundle with fiber G equipped with G-action by translation.
Let $p : E_G \to B_G$ be the *universal principal G-bundle*.

- $G = \mathbb{Z}_2$: $p : S^\infty \to \mathbb{R}P^\infty$.

$$\check{H}^*(B_G; \mathbb{Z}_2) \cong \mathbb{Z}_2[t], \qquad |t| = 1.$$

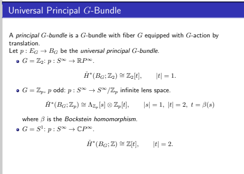
- $G = \mathbb{Z}_p$, p odd: $p : S^\infty \to S^\infty/\mathbb{Z}_p$ infinite lens space.

$$\check{H}^*(B_G; \mathbb{Z}_p) \cong \Lambda_{\mathbb{Z}_p}[s] \otimes \mathbb{Z}_p[t], \qquad |s| = 1,\ |t| = 2,\ t = \beta(s)$$

  where β is the *Bockstein homomorphism*.

- $G = S^1$: $p : S^\infty \to \mathbb{C}P^\infty$.

$$\check{H}^*(B_G; \mathbb{Z}) \cong \mathbb{Z}[t], \qquad |t| = 2.$$



1. Every compact Lie group G has a universal principal G-bundle, that is a contractible bundle on which G acts freely.
   Every principal G-bundle over a *nice* base space is a pullback of the universal principal G-bundle.
2. We will not cover the Bockstein homomorphism; see [Hatcher] 3.E. Consider SES
$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_p \longrightarrow 0\ ,$$
   then β is the connecting homomorphism of the induced LES:
$$\cdots \longrightarrow \check{H}^i(B_G; \mathbb{Z}_{p^2}) \longrightarrow \check{H}^i(B_G; \mathbb{Z}_p) \xrightarrow{\beta} \check{H}^{i+1}(B_G; \mathbb{Z}_p) \longrightarrow \cdots\ .$$
3. More generally, for H equipped with a $\mathbb{Z}_p$-action,
$$\check{H}^0(B_G; H) \cong H^{\mathbb{Z}_p}.$$

Borel Construction

The *twisted product* of G-spaces X and Y is the orbit space

$$X \times_G Y = X \times Y/G$$

of $X \times Y$ equipped with the G-action $g(x, y) = (xg^{-1}, gy)$.

The space $X_G = X \times_G E_G$ is a G-bundle over B_G with fiber X, called the *Borel construction*.
The equivariant projection $X \times E_G \to X$ induces $\varphi : X_G \to X/G$.
Then, the induced map

$$\varphi^* : \check{H}^i(X/G, F; k) \longrightarrow \check{H}^i(X_G, F_G; k)$$

is an isomorphism for all i.

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1. I am lying a bit; the precise construction goes as follows.
   The base space B_G is a CW-complex with finite N-skeleton B_G^N.
   Then E_G^N is compact and N-universal.
   Put $X_G^N = X \times_G E_G^N$, and set $X_G = \bigcup_N X_G^N = \varinjlim_N X_G^N$.
2. I am lying a bit; we must define

   $$\check{H}^i(X_G; k) = \varprojlim_N \check{H}^i(X_G^N; k).$$

   It might be different to the usual Čech cohomology groups of X_G.
3. Why do we consider the Borel construction? It is because (i) the product $X \times E_G$ is homotopically equivalent to X since E_G is contractible, and (ii) G acts freely on $X \times E_G$ since it does on E_G. It is sort of a "homotopically correct" way of defining the orbit space.
4. The proof of the last statement goes as follows: we show that each fiber $\varphi^{-1}([x])$ of $x \notin F$ is homeomorphic to B_{G_x}, so is acyclic, then apply the Vietoris–Begle mapping theorem.

Leray–Hirsch Theorem

Let $p : Y \to B$ be a G-bundle with fiber X.
A *cohomology extension of the fiber* is a graded k-module homomorphism of degree 0

$$\theta : \check{H}^*(X; k) \longrightarrow \check{H}^*(Y; k)$$

such that for all $b \in B$, the composition with the restriction to the fiber

$$\check{H}^*(X; k) \xrightarrow{\theta} \check{H}^*(Y; k) \xrightarrow{\text{res}} \check{H}^*(p^{-1}(b); k)$$

is an isomorphism.

Theorem (Leray–Hirsch)

Suppose that B is a finite CW-complex, and let θ be a cohomology extension of the fiber. Then, the map

$$\check{H}^*(B; k) \otimes_k \check{H}^*(X; k) \longrightarrow \check{H}^*(Y; k)$$

given by $\alpha \otimes \beta \mapsto p^*(\alpha) \smile \theta(\beta)$ is an $\check{H}^*(B; k)$-module isomorphism.

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given by $\alpha \otimes \beta \mapsto p^*(\alpha) \smile \theta(\beta)$ is an $\check{H}^*(B;k)$-module isomorphism.

1. I.e., θ is a section of the restriction to a fiber.
2. See also [Hatcher] 4.D.
3. It can be interpreted as a Künneth formula analogy for a fiber bundle. Note that the trivial bundle $Y = X \times B$ recovers the Künneth formula.
4. We can reduce the condition on B, but then the proof becomes much more technical.

Proof of Leray–Hirsch Theorem

Proof.

Use induction on dimension and the number of cells in B.
We may assume $B = B' \cup D^n$, $B' \cap D^n = S^{n-1}$.
Tensoring the Mayer–Vietoris sequence for $B = B' \cup D^n$ with $\check{H}^*(X;k)$, we obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
\check{H}^{i-1}(S^{n-1}) \otimes \check{H}^*(X) \arrow[r] \arrow[d, "\cong"] & \check{H}^i(B) \otimes \check{H}^*(X) \arrow[r] \arrow[d] & (\check{H}^i(B') \oplus \check{H}^i(D^n)) \otimes \check{H}^*(X) \arrow[d, "\cong"] \\
\cdots \arrow[r] & \check{H}^*(Y) \arrow[r] & \cdots
\end{tikzcd}$$

where the second row is the Mayer–Vietoris sequence for the corresponding parts of Y.
The first and third arrows are isomorphisms by the induction hypothesis, hence so is the middle vertical arrow by the five lemma. □

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1. WMA by homotopy invariance.
2. Note that $\check{H}^*(X;k)$ is free since k is a field. We may reduce the condition to that k is a PID and $\check{H}^*(X;k)$ is torsion-free (so is flat).

Totally Nonhomologous to Zero

Q. When does a cohomology extension of the fiber exist?

For the trivial bundle $F_G = F \times B_G$, it clearly exists and hence

$$\check{H}^*(B_G; k) \otimes_k \check{H}^*(F; k) \xrightarrow{\sim} \check{H}^*(F_G; k).$$

We say X is *totally nonhomologous to zero* in X_G over k if the restriction

$$\check{H}^*(X_G; k) \xrightarrow{\text{res}} \check{H}^*(X; k)$$

is surjective.
Since k is a field, there exists a cohomology extension of the fiber iff X is totally nonhomologous to zero in X_G.
Hence, if X is totally nonhomologous to zero in X_G, then

$$\check{H}^*(B_G; k) \otimes_k \check{H}^*(X; k) \xrightarrow{\sim} \check{H}^*(X_G; k).$$

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Since k is a field, there exists a cohomology extension of the fiber iff X is totally nonhomologous to zero in X_G.
Hence, if X is totally nonhomologous to zero in X_G, then

$$\check{H}^*(B_G;k) \otimes_k \check{H}^*(X;k) \xrightarrow{\sim} \check{H}^*(X_G;k).$$

1. k is a field, so every k-module is free.

LES of Pairs

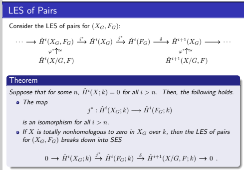
Consider the LES of pairs for (X_G, F_G):

$$\begin{tikzcd} \cdots \arrow[r] & \check{H}^i(X_G, F_G) \arrow[r, "i^*"] & \check{H}^i(X_G) \arrow[r, "j^*"] & \check{H}^i(F_G) \arrow[r, "\delta"] & \check{H}^{i+1}(X_G) \arrow[r] & \cdots \\ & \check{H}^i(X/G, F) \arrow[u, "\varphi^*"', "\cong"] & & & \check{H}^{i+1}(X/G, F) \arrow[u, "\varphi^*"', "\cong"] & \end{tikzcd}$$

Theorem

Suppose that for some n, $\check{H}^i(X; k) = 0$ for all $i > n$. Then, the following holds.

- *The map*
$$j^* : \check{H}^i(X_G; k) \longrightarrow \check{H}^i(F_G; k)$$
is an isomorphism for all $i > n$.
- *If X is totally nonhomologous to zero in X_G over k, then the LES of pairs for (X_G, F_G) breaks down into SES*
$$0 \to \check{H}^i(X_G; k) \xrightarrow{j^*} \check{H}^i(F_G; k) \xrightarrow{\delta} \check{H}^{i+1}(X/G, F; k) \to 0 \ .$$



1. That is, X is cohomologically n-dimensional.
2. I am lying a lot; the case $G = S^1$ holds if X/G is finitistic and the number of orbit types is finite.

LES of Pairs

Proof.

The first assertion relies on Smith theory, so we omit the proof.
For the second part, we will show that the map

$$i^* : \check{H}^i(X_G, F_G; k) \longrightarrow \check{H}^i(X_G; k)$$

is trivial. By the Leray–Hirsch theorem,

$$\check{H}^*(X_G; k) \cong \check{H}^*(B_G; k) \otimes_k \check{H}^*(X; k)$$

is a free $\check{H}^*(B_G; k)$-module. We can find a non-zero divisor $t \in \check{H}^*(B_G; k)$ of positive dimension. Then for each $\alpha \in \check{H}^i(X_G, F_G; k)$, we have

$$t^n i^*(\alpha) = i^*(t^n \alpha) = 0,$$

so $i^*(\alpha) = 0$. $\square$

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so $i^*(\alpha) = 0$. □

1. Smith theory is a collection of techniques to study actions of finite p-groups. We can obtain several (in)equalities on the ranks of cohomology groups.

Criterion for "Totally Nonhomologous to Zero"

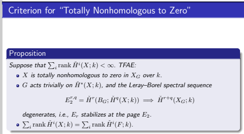
Proposition

Suppose that $\sum_i \operatorname{rank} \check{H}^i(X; k) < \infty$. *TFAE:*

- X *is totally nonhomologous to zero in* X_G *over* k.
- G *acts trivially on* $\check{H}^*(X; k)$*, and the Leray–Borel spectral sequence*

$$E_2^{r,q} = \check{H}^r(B_G; \check{H}^q(X; k)) \implies \check{H}^{r+q}(X_G; k)$$

  degenerates, i.e., E_r *stabilizes at the page* E_2.
- $\sum_i \operatorname{rank} \check{H}^i(X; k) = \sum_i \operatorname{rank} \check{H}^i(F; k)$.



1. I am lying a lot; the case $G = S^1$ holds if X/G is finitistic and the number of orbit types is finite.
2. The spectral sequence is a machinery for computing (co)homology groups. It is a 'book' with infinitely many pages, whose pages E_r are two dimensional lattices of groups with 'differentials' connecting them, and the pages stabilize at an infinite limiting page E_∞.
   We will not cover this; you can refer to [Weibel].
3. A *bigraded abelian group* E is a collection of abelian groups $\{E^{p,q}\}$.
   A *bigraded map* $D \xrightarrow{f} E$ of bidegree (a,b) is a collection of maps $f = \{D^{p,q} \xrightarrow{f^{p,q}} E^{p+a,q+b}\}$.
   A *cohomological spectral sequence* (E_r, d_r) is a collection of bigraded abelian groups E_r and bigraded maps $E_r \xrightarrow{d_r} E_r$ of degree $(r, -r+1)$ such that $d \circ d = 0$ and $E_{r+1} = \ker d_r / \operatorname{im} d_r$.
   A spectral sequence E_r *abuts* to E_∞ if $E_r^{p,q}$ stabilizes to $E_\infty^{p,q}$ for all p, q.

Poincaré Duality Space

We call X a *Poincaré duality space over* k *of formal dimension* n if:

- $\check{H}^*(X;k)$ is finitely generated;
- $\check{H}^i(X;k) = 0$ for all $i > n$ and $\check{H}^n(X;k) \cong k$; and
- the cup product pairing

$$\check{H}^i(X;k) \otimes_k \check{H}^{n-i}(X;k) \xrightarrow{\smile} \check{H}^n(X;k) \cong k$$

is nonsingular for all i.

Proposition (Poincaré duality)

A closed orientable manifold is a Poincaré duality space.

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A closed orientable manifold is a Poincaré duality space.

1. Recall [Hatcher] 3.3.

Actions on Poincaré Duality Spaces

Conjecture (Su 1964)

Let $G = \mathbb{Z}_p$ and let X be a Poincaré duality space over $\mathbb{Z}_p$. Then, each component of the fixed point set $F = X^{\mathbb{Z}_p}$ is also a Poincaré duality space over $\mathbb{Z}_p$.

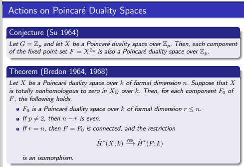
Theorem (Bredon 1964, 1968)

Let X be a Poincaré duality space over k of formal dimension n. Suppose that X is totally nonhomologous to zero in X_G over k. Then, for each component F_0 of F, the following holds.

- *F_0 is a Poincaré duality space over k of formal dimension $r \leq n$.*
- *If $p \neq 2$, then $n - r$ is even.*
- *If $r = n$, then $F = F_0$ is connected, and the restriction*

$$\check{H}^*(X;k) \xrightarrow{\text{res}} \check{H}^*(F;k)$$

is an isomorphism.



1. The conjecture is true for a closed orientable manifold X equipped with a locally smooth action of $\mathbb{Z}_p$; in which case F also is a closed orientable manifold.
2. I am lying a lot; the case $G = S^1$ holds if X/G is finitistic.
3. Rough sketch of the proof:
   Recall that $\check{H}^*(B_G) \cong \Lambda_k[s] \otimes_k k[t]$ if $k = \mathbb{Z}_p$, p odd.
   Pick $b(\neq 0) \in \check{H}^i(F_0)$.
   Since j^* is surjective in high degrees, pullback

   $$t^k \otimes b \in \check{H}^*(B_G) \otimes \check{H}^*(F) \cong \check{H}^*(F_G)$$

   along j^* for some large k, then restrict to $X_G \to X$ so that we can use the Poincaré duality for X.
   Bring back to F_0 along the cohomology extension to the fiber and j^*: these are all injective.

Example: Involutions on $S^n \times S^m$

We write $X \sim_2 Y$ if X and Y have isomorphic cohomology rings over $\mathbb{Z}_2$.
We write $X \sim_2 P^h(n)$ if

$$\check{H}^*(X;\mathbb{Z}_2) \cong \mathbb{Z}_2[a]/(a^{h+1}), \qquad |a| = n.$$

Proposition

Suppose that $G = \mathbb{Z}_2$ and $X \sim_2 S^n \times S^m$, $n \leq m$. If $F \neq \emptyset$, then one of the following possibilities occur:

- $F \sim_2 S^q \times S^r$, $\quad 0 \leq q \leq n,\ 0 \leq r \leq m$.
- $F \sim_2 P^2(q) \# P^2(q)$, $\quad n \geq q = 1, 2, 4, 8$.
- $F \sim_2 P^3(q)$, $\quad n > q$.
- $F \sim_2 \mathrm{point} \sqcup P^2(q)$, $\quad n > q$.
- $F \sim_2 S^q \sqcup S^r$, $\quad 0 \leq q \leq m,\ 0 \leq r \leq m$.
- $F \sim_2 S^q$.

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- $F \sim_2 P^3(q)$, $\quad n > q$.
- $F \sim_2 \mathrm{point} \sqcup P^2(q)$, $\quad n > q$.
- $F \sim_2 S^q \sqcup S^r$, $\quad 0 \leq q \leq m$, $0 \leq r \leq m$.
- $F \sim_2 S^q$.

1. Smith theory imposes some modularity conditions and inequalities on the ranks of the cohomology groups:

$$\chi(X) + \chi(F) = 2\chi(X/G) \equiv 0 \pmod 2,$$

and

$$\operatorname{rank} \check{H}^h_\rho(X) + \sum_{i \geq h} \operatorname{rank} \check{H}^i(F) \leq \sum_{i \geq h} \operatorname{rank} \check{H}^i(X).$$

Also note that $\chi(X) \equiv 0 \pmod 2$. These implies that $F \sim_2 S^q$ or $\operatorname{rank} \check{H}^*(F) = \operatorname{rank} \check{H}^*(X)$, in which case X is totally nonhomologous to zero in X_G, so F is a Poincaré duality space over $\mathbb{Z}_2$. The first five cases list up all the possibilities.

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Thank you!