The Eyring–Kramers Law for Extinction Time of Contact Process on Stars

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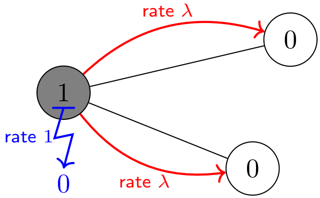
Overview

Contact process

- $G = (V, E)$ (locally) finite connected graph
- $\lambda > 0$ *infection rate*
- Configurations of the contact process: $\eta \in \{0, 1\}^V$

  For $x \in V$, $\eta(x) = \begin{cases} 0 & x \text{ is } \textit{healthy} \\ 1 & x \text{ is } \textit{infected} \end{cases}$
- Abuse of notation: identify η with $\{x \in V : \eta(x) = 1\}$



- The *all-healthy state* $\eta = \emptyset$ is the unique absorbing state.

Metastability of contact processes

The *extinction time* of the contact process is

$$\tau_G = \inf\{t \geq 0 : \eta_t = \emptyset\}.$$

Q. Fix a (increasing) sequence of graphs $(G_N)_{N\geq 1}$, and study the growth of τ_{G_N}.

Finite-volume phase transition for boxes ('84–'99)

On $\mathbb{Z}_N^d = [1, N]^d$ with free boundary, we have

$$\tau_{\mathbb{Z}_N^d} \sim \begin{cases} \log|\mathbb{Z}_N^d| & \text{if } \lambda < \lambda_c, \\ \exp\big(c_\lambda |\mathbb{Z}_N^d|\big) & \text{if } \lambda > \lambda_c \end{cases}$$

where $|G|$ denotes the number of vertices.

The latter case is a clear demonstration of the metastable behavior.

Metastability of contact processes

More generally, the following theorem holds.

Theorem (MMYV '16, SV '17)

Suppose that $\lambda > 0$ is sufficiently large.

(a) For all $D > 0$, there exists $c = c(\lambda, D)$ such that

$$\mathbb{E}\tau_G \geq \exp(c|G|) \quad \text{for all } G \text{ with degrees} \leq D.$$

(b) For all $\varepsilon > 0$, there exists $c = c(\lambda, \varepsilon)$ such that

$$\mathbb{E}\tau_G \geq \exp\left(c \cdot \frac{|G|}{(\log |G|)^{1+\varepsilon}}\right) \quad \text{for all } G.$$

Mean extinction time is already very informative

Proposition

(a) (Markov inequality) $\mathbb{P}[\tau_G > t] \leq \dfrac{\mathbb{E}\tau_G}{t}$ for all $t > 0$.

(b) ("Upside-down Markov inequality") $\mathbb{P}[\tau_G < t] \leq \dfrac{t}{\mathbb{E}\tau_G}$ for all $t > 0$.

Theorem (Schapira–Valesin '17)

Suppose that $\lambda > 0$ is sufficiently large. Then, it holds that

$$\frac{\tau_{G_N}}{\mathbb{E}\tau_{G_N}} \Longrightarrow \mathrm{Exp}(1) \qquad \textit{for all } (G_N)_{N\geq 1} \textit{ with } |G_N| \to \infty.$$

Precise estimate for mean extinction time

Levels of precision for mean transition time estimate:

① $\mathbb{E}\tau_N \geq \exp(cN)$ (Metastability)

② $\dfrac{1}{N} \log \mathbb{E}\tau_N \longrightarrow c$ (Large-deviation principle)

③ $\mathbb{E}\tau_N \simeq f(N)e^{cN}$ (Eyring–Kramers law)

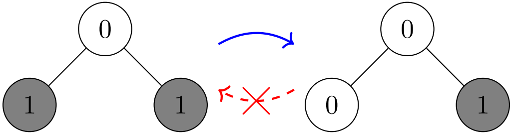
- ① holds for graphs of uniformly bounded degree. (MMYV '16)
- ② holds for $\mathbb{Z}_N^d$ with free boundary. (Mountford '99)
  However, ② is open for even $\mathbb{Z}_N^d$ with periodic boundary.
- ② holds for a variety of random graph models. (Shapira–Valesin '21)
- ③ is open only except for two cases.
  1. The triviality: complete graph K_N
  2. Main Result: star graph S_N (**J.** '24)

Main obstacles to precise estimation

1. Spatial asymmetry of the underlying graph complicates the process.

- ② heavily relies on very specific geometric features.
- What are the typical states of the process on a cycle $\mathbb{Z}_N$?

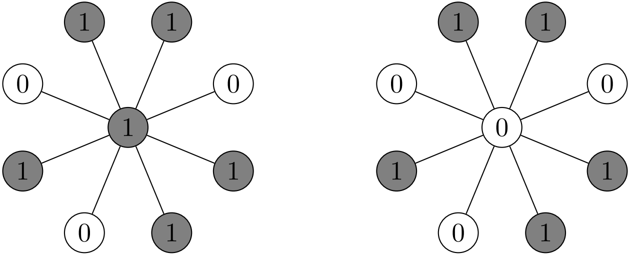
2. The process is non-reversible in general.



3. Former methodologies (e.g. percolation, coupling) rely on less precise formulas for the mean extinction time.

Contact process on stars

Let S_N be the star graph with one hub and N leaves.



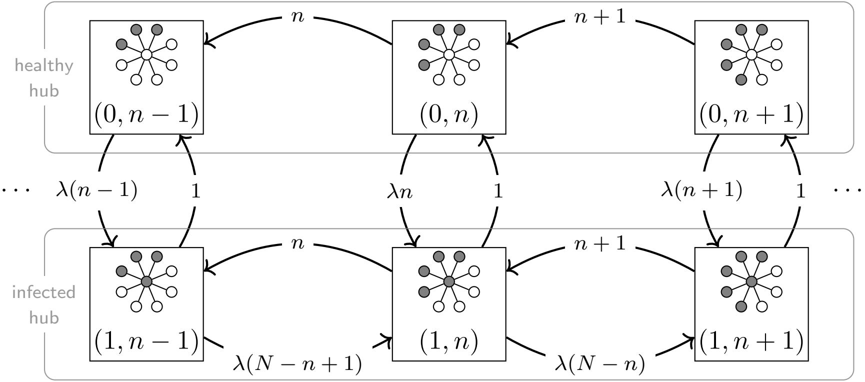
Why do we study star graphs?

- It is a natural model for studying epidemic hubs.
- It serves as a building block within larger graph structures.

Contact process on stars

All leaves are homogenous, so we reduce to a random walk on the ladder graph:

$$(o_t, n_t) = (\text{hub state}, \#\text{infected leaves}) \in \{0,1\} \times [0, N].$$



Transition rates for the contact process on a star

Main result

Eyring–Kramers law (**J.** '24)

Let $\varepsilon > 0$ be given. Then, for $x \in \{0,1\} \times [\varepsilon N, N]$, we have

$$\mathbb{E}_x \tau_{(0,0)} \simeq \kappa_\lambda N^{-\frac{1}{1+2\lambda}} \left(\frac{(1+\lambda)^2}{1+2\lambda} \right)^N .$$

In particular, we have

$$\sup_{x \in \{0,1\} \times [\varepsilon N, N]} \frac{1}{N} \log \mathbb{E}_x \tau_{(0,0)} \xrightarrow{N \to \infty} c_\lambda = 2 \log(1+\lambda) - \log(1+2\lambda).$$

Main ingredients:

- *Special function theory* for precise estimation of *quasi-stationary measure*
- The *potential theoretic approach* to metastability of non-reversible processes

These methodologies have not previously been used in the study of the contact process.

Quasi-stationary distribution

The stationary measure of the process is the Dirac mass at the all-healthy state.
To apply potential theory, we add supplementary transition rates from $\emptyset$.

- Natural choice: rate proportional to the stationary measure of the process conditioned on the non-extinction.
  $\Longrightarrow$ Hard to compute in general.
- We add a regeneration at the hub: $(0,0) \xrightarrow{\alpha} (1,0)$.
  Or equivalently, we consider the process restricted to the non-extinction.

A closed form solution for the quasi-stationary measure μ is due to (Cator–Mieghem '13).

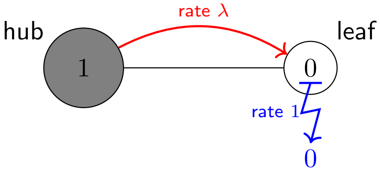
Heuristics: High infection regime

If n is large, then we may approximate that the hub is constantly infected.

$\Longrightarrow$ Each leaf gets infected with rate λ, and recovers with rate 1.

$\Longrightarrow$ n rapidly converges to $\mathrm{Binom}(N, \frac{\lambda}{1+\lambda})$ in distribution.

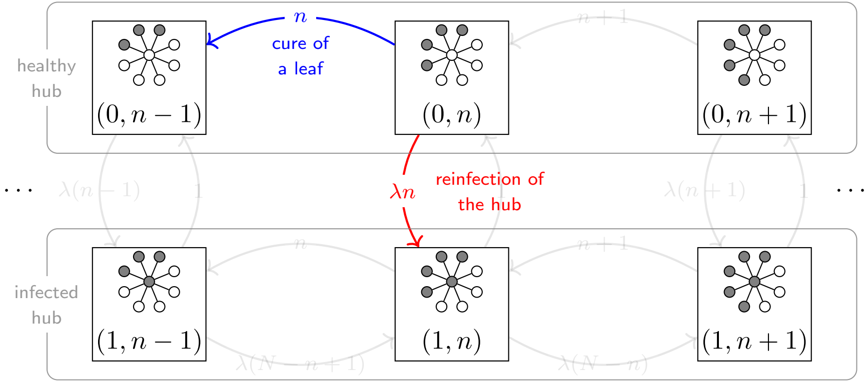
$\Longrightarrow$ $\mu(1, n) \sim \binom{N}{n}\lambda^n$.



In particular, the metastable state of the process is $n \simeq \frac{\lambda}{1+\lambda}N$.

Heuristics: Low infection regime

If n is small, the states with healthy hub are dominant in path to the extinction.

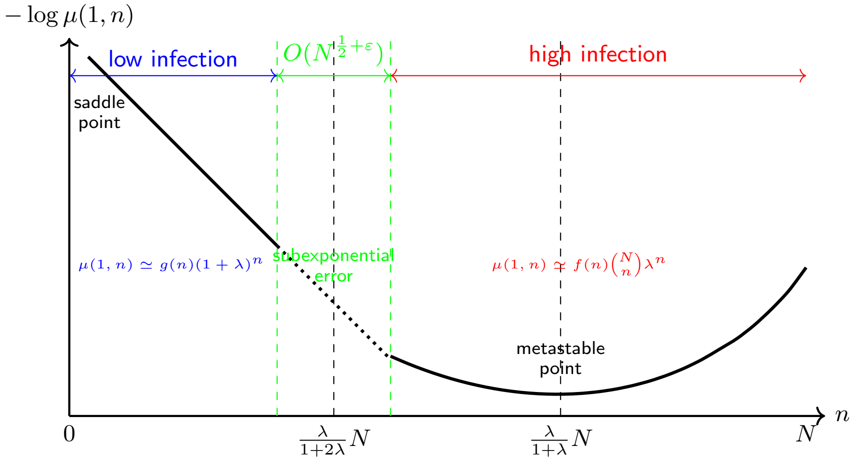


#(cured leaves before the reinfection of the hub) $\sim \mathrm{Geom}(\frac{\lambda}{1+\lambda}) \wedge n$.

$\implies \mu(1, n) \sim (1+\lambda)^n$.

Precise asymptotics for quasi-stationary measure

Using special function theory and refined Laplace's method, we compute:



Uniform asymptotic behavior of the quasi-stationary measure μ

Potential theoretic approach

A precise framework for quantifying metastability metrics (e.g. $\mathbb{E}\tau$) in terms of potential theoretic terms (e.g. capacity, equilibrium potential) was established in (BEGK '01, '04), and then extended to non-reversible settings recently.

Hitting time formula for non-reversible process

$$\mathbb{E}_x\tau_y = \frac{1}{\mathrm{cap}(x,y)}\sum_z h^{\dagger}_{x,y}(z)\mu(z).$$

Precise asymptotics for the quasi-stationary measure

$\Longrightarrow$ Good test functions/flows approximating the harmonic functions/flows

$\Longrightarrow$ Sharp estimates for capacity and potential (via variational principles)

Further topics

Q. Can we extend our methodology to other graphs, for instance, cycles $\mathbb{Z}_N$?

The potential theoretic approach is robust, but the difficult part is the computation of the quasi-stationary measure asymptotics.

Thank you!