Random Walks and Invariant Random Subgroups

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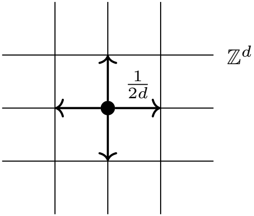
Overview

Overview

Pólya's Random Walk



Theorem (Pólya 1921)

The simple random walk on $\mathbb{Z}^d$ is recurrent if $d \le 2$, transient if $d \ge 3$.

Discrete Liouville theorem

A bounded harmonic function on $\mathbb{Z}^d$ is constant, $\forall d \ge 1$.

Random Walks on Groups

Let G be a locally compact group, $\mu \in \mathcal{P}(G)$ Radon probability measure. A *random walk* $S = (S_n)_{n\in\mathbb{N}}$ *on* G *with step distribution* μ *(*μ*-RW)* is

$$S_n = g_n \cdots g_1 \quad \text{where} \quad g_1, g_2, \ldots \overset{\text{i.i.d.}}{\sim} \mu.$$

A function $f : G \to \mathbb{R}$ is *harmonic* if $Pf = f$ where

$$Pf := \int_G f(g\,\cdot\,)\, d\mu(g).$$

Define

$$\mathrm{Har}^\infty(G,\mu) := \{f \in L^\infty(\mu) : f \text{ is harmonic}\}$$

We say (G,μ) is *Liouville* if $\mathrm{Har}^\infty(G,\mu) = \{\text{constant}\}$.

Random Walks and Group Structures

Theorem (Varopoulos 1986)

Let G be finitely generated, and $\mu \in \mathcal{P}(G)$ be symmetric, nondegenerate, and finitely supported. Then, the μ-RW is recurrent if and only if G is virtually $\{e\}$, $\mathbb{Z}$, or $\mathbb{Z}^2$.

Theorem (Rosenblatt 1981, Kaimanovich–Vershik 1983)

(G, μ) is Liouville for some $\mu \in \mathcal{P}(G)$ if and only if G is amenable.

Theorem (Choquet–Deny 1960)

Let G be abelian. Then, (G, μ) is Liouville for all nondegenerate $\mu \in \mathcal{P}(G)$.

Theorem (Frisch–Hartman–Tamuz–Vahidi-Ferdowsi 2019)

Let G be finitely generated. Then, (G, μ) is Liouville for all nondegenerate $\mu \in \mathcal{P}(G)$ if and only if G is virtually nilpotent.

Margulis's Proof of Choquet–Deny Theorem

Theorem (Choquet–Deny 1960)

Let G be abelian. Then, (G, μ) is Liouville for all nondegenerate $\mu \in \mathcal{P}(G)$.

Fix $M > 0$, let $\mathrm{Har}^{\leq M}(G, \mu) := \{f \in \mathrm{Har}^{\infty}(G, \mu) : \|f\|_{\infty} \leq M\}$.
Then, $\mathrm{Har}^{\leq M}(G, \mu)$ is G-invariant, convex, and compact.
By the Krein–Milman theorem,

$$\mathrm{Har}^{\leq M}(G, \mu) = \overline{\mathrm{co}}\big(\mathrm{ext}\big(\mathrm{Har}^{\leq M}(G, \mu)\big)\big).$$

For $f \in \mathrm{ext}\big(\mathrm{Har}^{\leq M}(G, \mu)\big)$, we have

$$f = \int_G f(g\,\cdot\,)\,d\mu(g) = \int_G f^{g^{-1}}\,d\mu(g),$$

so $f = f^{g^{-1}}$ for all $g \in \operatorname{supp}\mu$. Then f is G-invariant, so is constant. □

Recall: Poisson Formula

On a locally compact group G,

μ-Random walk
$(S_n)_{n\in\mathbb{N}}$

n-Step Transition Density
$\mathbb{P}[S_n \in dg] = p(g,n)\,dg$

$Lf = Pf - f$

$$p(g,n) = \frac{d\mu^{*n}}{dg}(g) \longleftrightarrow Lf = \int_G (f(g\,\cdot\,) - f)\,d\mu(g)$$

Recall: Poisson Formula

On the Euclidean disk $\mathbb{D}$,

Brownian motion
$(B_t)_{t\geq 0}$

Transition Density
$\mathbb{P}[B_t \in dx] = p(x,t)\,dx$

Infinitesimal Generator
$\frac{1}{2}\Delta f = \lim_{t\searrow 0} \frac{B_t f - f}{t}$

Heat kernel
$p(x,t) = (2\pi t)^{-1} e^{-\frac{|x|^2}{2t}}$

Heat equation
$\partial_t p = \frac{1}{2}\Delta p$

Laplacian
$\frac{1}{2}\Delta = \frac{1}{2}(\partial_x^2 + \partial_y^2)$

Recall: Poisson Formula

On the Poincaré disk $\mathbb{D}$,

Brownian motion
$(B_t)_{t\geq 0}$

Transition Density
$\mathbb{P}[B_t \in dx] = p(x,t)\,dx$

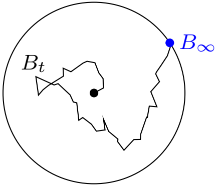
Infinitesimal Generator
$\frac{1}{2}\Delta f = \lim_{t\searrow 0} \frac{B_t f - f}{t}$

Heat kernel
$p(x,t)$

Heat equation
$\partial_t p = \frac{1}{2}\mathcal{L}p$

Laplace–Beltrami operator
$\frac{1}{2}\mathcal{L} = \frac{(1-|z|^2)^2}{8}(\partial_x^2 + \partial_y^2)$

Recall: Poisson formula



Poisson formula

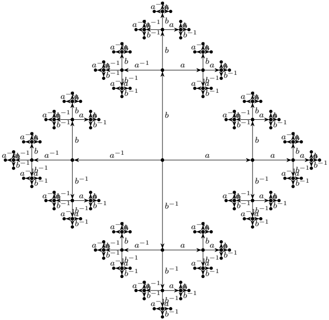
There exists an isometric isomorphism $\Phi : \mathrm{Har}^\infty(\mathbb{D}, \mathcal{L}) \to L^\infty(\partial\mathbb{D}, \nu_0)$ given by

$$\Phi f := \lim_{t\to\infty} f(B_t), \qquad \text{(MG convergence)}$$

$$\Phi^{-1}\widehat{f}(z) := \mathbb{E}^z \widehat{f}(B_\infty) = \int_{\partial\mathbb{D}} \widehat{f}\, d\nu_z.$$

Here, $P(z,\xi) := \frac{1}{2\pi}\frac{d\nu_z}{d\theta}(\xi) = \frac{1-|z|^2}{|\xi-z|^2}$ is the Poisson kernel.

Poisson Boundary



The *Poisson boundary of* (G, μ) is a measured G-space $(B(G), \nu)$ such that there exists a G-equivariant isometric isomorphism

$$\Phi : \mathrm{Har}^\infty(G, \mu) \longrightarrow L^\infty(B(G), \nu).$$

Poisson–Furstenberg Boundary Construction

Theorem (Furstenberg 1973)

Let $\mu \in \mathcal{P}(G)$ be *nice*, there exists a unique Poisson boundary of (G, μ).

Consider the trajectory space $(G^{\mathbb{N}}, \mathcal{B}(G)^{\mathbb{N}}, \mu^{\mathbb{N}}, T)$.
Define the invariant σ-algebra $\mathcal{I} = \{A \in \mathcal{B}(G)^{\mathbb{N}} : T^{-1}A = A\}$.
Then, there exists a G-equivariant linear isometry

$$\begin{aligned} \Psi : L^\infty(G^{\mathbb{N}}, \mathcal{I}, \mu^{\mathbb{N}}) &\longrightarrow \mathrm{Har}^\infty(G, \mu) \\ F &\longmapsto \Psi F(x) := \mathbb{E}^x F(S_0, S_1, \dots), \end{aligned}$$

whose inverse is

$$\lim_{n\to\infty} f(S_n) =: \Psi^{-1} f \longleftarrow f.$$

Applying the *Mackey point realization theorem*, one can construct a G-equivariant "quotient" map to the Poisson boundary:

$$\mathbf{bnd} : (G^{\mathbb{N}}, \mathcal{I}, \mu^{\mathbb{N}}) \to (B(G), \mathbb{B}(B(G)), \nu).$$

Examples of Poisson Boundary

- (G, μ) is Liouville if and only if its Poisson boundary is a point.
- The Poisson boundary of the simple RW on a free group is the boundary of its Cayley graph, which is the space of *infinite rays* in a regular tree.
- (Furstenberg 1971)
  Let $G = KAN$ be a semisimple Lie group, $\mu \ll m_G$.
  Then, the Poisson boundary of (G, μ) is $(G/P, \nu)$, where

  $$P := MAN, \qquad M := C_K(A).$$

  If μ is m_K-stationary, then ν is the only K-invariant measure on G/P, and

  $$\mathrm{Har}^\infty(K \backslash G) \cong \mathrm{Har}^\infty(G, \mu) \cong L^\infty(G/P, \nu).$$

  This recovers the classical Poisson formula:

  $$\mathrm{Har}^\infty(\mathbb{D}) = \mathrm{Har}^\infty(\mathrm{SO}(2) \backslash \mathrm{SL}_2 \mathbb{R}) \cong L^\infty(SO(2)) = L^\infty(\partial \mathbb{D}).$$

Avez Asymptotic Entropy

Assume that G is countable discrete.
The *entropy of* $\mu \in \mathcal{P}(G)$ is

$$H(\mu) := -\sum_{g\in G} \mu(g)\log\mu(g).$$

The *Avez asymptotic entropy of* (G,μ) is

$$h(G,\mu) := \lim_{n\to\infty}\frac{1}{n}H(\mu^{*n}) = \inf_{n\geq 1}\frac{1}{n}H(\mu^{*n}).$$

Here, $\frac{1}{n}H(\mu^{*n})$ can be understood as the “average information about n steps” of a μ-RW, given its terminal location S_n.

Entropy Criterion for Poisson Boundary Identification

Theorem (Kaimanovich–Vershik 1983)

Let G be countable discrete, $\mu \in \mathcal{P}(G)$ be of finite entropy.
Then, $h(G, \mu) = 0$ if and only if the Poisson boundary of (G, μ) is trivial.

Corollary

Let (X, ν) be a "candidate" for the Poisson boundary of (G, μ).
If the asymptotic entropy of the μ-RW conditioned on its "limit" in X is 0 a.s., then (X, ν) is indeed the Poisson boundary of (G, μ).

Overview

Invariant Random Subgroups

Define a compact space

$$\mathrm{Sub}(G) := \{K \leq G : K \text{ is closed}\},$$

equipped with the *Chabauty topology*. ($\mathrm{Sub}(G) \subset \{0,1\}^G$ if G countable discrete)
Here, $G \curvearrowright \mathrm{Sub}(G)$ by conjugation:

$$g \cdot K := gKg^{-1}.$$

An *invariant random subgroup (IRS)* is a G-invariant $\mu \in \mathcal{P}(\mathrm{Sub}(G))$, i.e.,

$$\mathrm{IRS}(G) := \mathcal{P}(\mathrm{Sub}(G))^G$$

equipped with the weak*-topology (so is compact).

Basic Examples of IRS

$$\mathrm{IRS}(G) = \mathcal{P}(\mathrm{Sub}(G))^G$$

- For $N \trianglelefteq G$, $\delta_N \in \mathrm{IRS}(G)$.
- Let $\Gamma \leq G$ be a lattice with fundamental domain $X = G/\Gamma$.
  Then, there exists a measurable map

  $$X \longrightarrow \mathrm{Sub}(G), \qquad x \mapsto \mathrm{Stab}_G(x).$$

  Hence, we obtain an IRS $\eta_\Gamma := (\mathrm{Stab}_G)_* m_X \in \mathrm{IRS}(G)$, supported on the closure of the conjugacy class of Γ.

$\Longrightarrow$ IRS is a natural generalization of normal subgroups and lattices.

Connection with Measure Preserving Actions

Let $G \curvearrowright (X, m)$ be a probability measure preserving (pmp) action.
Then, there exists a measurable map

$$X \longrightarrow \mathrm{Sub}(G), \qquad x \mapsto \mathrm{Stab}_G(x).$$

Hence, we obtain $(\mathrm{Stab}_G)_* m \in \mathrm{IRS}(G)$.

Theorem (Abert–Glasner–Virag 2014, Abert et al. 2017)

Every IRS of G arises as the stabilizer for a pmp G-action.

We say $\eta \in \mathrm{IRS}(G)$ is *ergodic* if it arises from an ergodic pmp G-action, or equivalently, η is an extreme point of $\mathrm{IRS}(G)$.

Stück–Zimmer Rigidity Theorem

Let G be a simple Lie group of higher rank, that is, the Cartan subalgebra of $\operatorname{Lie} G$ is of dimension ≥ 2. (e.g. $\mathrm{SL}_d \mathbb{R}$, $d \geq 3$)

Theorem (Stück–Zimmer 1994)

Let η be an ergodic IRS in G. Then, $\eta = \delta_{\{e\}}, \delta_G$, or η_Γ for some lattice $\Gamma \leq G$.

This implies:

Margulis Normal Subgroup Theorem

Let $\Gamma \leq G$ be a lattice, and $N(\neq \{e\}) \trianglelefteq \Gamma$. Then, N has finite index in Γ.

Let $X = G/\Gamma$. Define $\eta \in \mathrm{IRS}(G)$ by picking a random conjugate of N by m_X. Then $G \curvearrowright (\mathrm{Sub}(G), \eta)$ is a factor of an ergodic system $G \curvearrowright (X, m_X)$, so η is ergodic. Thus N is a lattice, hence of finite index in Γ. $\square$

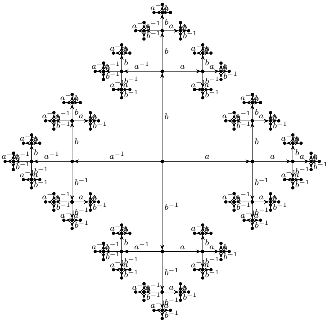
Overview

Recall: Cayley Graphs

Let G be a group generated by $S \subseteq G$.
The *Cayley graph of* (G, S) is an edge-colored directed graph $\mathrm{Cay}(G, S)$:

- The vertices are elements $g \in G$.
- The edges are (g, sg) where $g \in G$, $s \in S$.

RW on groups can be viewed as RW on Cayley graphs.

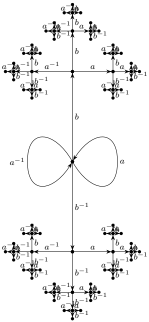


Schreier Graphs

Let G be a group generated by $S \subseteq G$, and X be a G-set.
The *Schreier graph of* (X, G, S) is an edge-colored directed graph $\mathrm{Sch}(X, G, S)$:

- The vertices are points $x \in X$.
- The edges are (x, sx) where $x \in X$, $s \in S$.

RW on G-sets can be viewed as RW on Schreier graphs.



Random Walks on G-Sets

Let X be a G-set, $\mu \in \mathcal{P}(G)$.
A *random walk* $S = (S_n x)_{n\in\mathbb{N}}$ *on* X *with step distribution* μ *(*μ*-RW)* is

$$S_n = g_n \cdots g_1 \qquad \text{where} \qquad g_1, g_2, \ldots \overset{\text{i.i.d.}}{\sim} \mu, \; x \in X.$$

However, our tools for RW on groups do *not* generalize to RW on G-sets:

Fact

Every $2d$-regular graph is isomorphic to a Schreier graph.

Non-example (Benjamini–Kozma 2010)

There exists a $2d$-regular graph of positive asymptotic entropy that is Liouville.
In particular, the entropy criterion does *not* work for general Schreier graphs.

Random Walks on Random Coset Spaces

To study RW on G-sets, we consider a "random" G-set X.
Especially, we study *random coset spaces* $X = G/K$ where $K \sim \eta \in \mathrm{IRS}(G)$.

For $\mu \in \mathcal{P}$, we consider the μ-RW $(S_n K)_{n\geq 0}$ on G/K:

$$S_n = g_n \cdots g_1 \qquad \text{where} \qquad g_1, g_2, \ldots \overset{\text{i.i.d.}}{\sim} \mu.$$

Poisson Bundle

The Poisson–Furstenberg boundary construction can be done analogously:

Consider the trajectory bundle over $(\mathrm{Sub}(G), \eta)$:

$$\widetilde{\mathrm{Sub}(G)} := \{(K; g_1K, g_2g_1K, \dots) : K \in \mathrm{Sub}(G), g_j \in G\}$$

where each fiber is equipped with the random walk measure on G/H.
Applying the Mackey point realization theorem, we obtain the *Poisson bundle*

$$(\widetilde{\mathrm{Sub}(G)}, \mathbb{P}_\eta) \xrightarrow{\mathbf{bnd}} (B(\mathrm{Sub}(G)), \nu_\eta) \xrightarrow{\pi} (\mathrm{Sub}(G), \eta),$$

whose fiber over K is the Poisson boundary $(B(G/K), \nu_H)$.

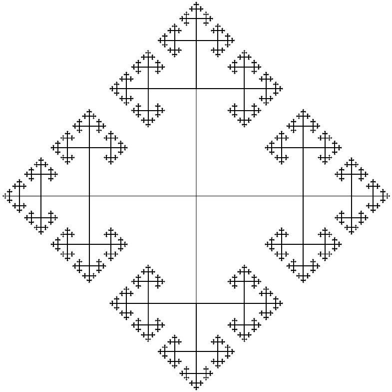
Entropy Criterion over IRS

Assume that G is countable discrete.
Define the *asymptotic entropy of μ-RW on η-coset space* as

$$\begin{aligned} h_\eta(G,\mu) &:= \lim_{n\to\infty} \frac{1}{n} \int_{\mathrm{Sub}(G)} H(S_n K)\, d\eta(K) \\ &= \inf_{n\geq 1} \frac{1}{n} \int_{\mathrm{Sub}(G)} H(S_n K)\, d\eta(K). \end{aligned}$$

Theorem (Bowen 2014)

Let $\mu \in \mathcal{P}(G)$ be of finite entropy, $\eta \in \mathrm{IRS}(G)$.
Then, $h_\eta(G,\mu) = 0$ if and only if the Poisson boundary of the μ-RW $(S_n K)_{n\in\mathbb{N}}$ is trivial for η-a.e. K.



Thank you!