RW on Groups

'24.10.2.
SNU Peer Seminar.
Younghun Jo.

§ RW on $\mathbb{Z}^d$.

Consider the integer ~~grid~~ lattice $\mathbb{Z}^d$, and the simple NNRW on it.

Thm (Pólya, 1921).
The simple RW on $\mathbb{R}^d$ is $\begin{cases} \text{recurrent} & \text{if } d \le 2. \\ \text{transient} & \text{if } d \ge 3. \end{cases}$

[pf One can show that $p^{(2n)}(0,0) \sim n^{-d/2}$.

so $E[\#\text{ of visits to } 0] = \sum p^{(2n)}(0,0) = \begin{cases} < \infty & \text{if } d = 1,2 \\ \infty & \text{if } d \ge 3. \end{cases}$ ∎]

§ RW on gps.

def (Kesten, 1959).
let Γ: cnt'ble gp. $\mu \in \mathcal{P}(\Gamma)$.
A RW on Γ w/ step distributn μ is defined by $p(g, gs) = \mu(s)$.
· μ is called symmetric if $\mu(g) = \mu(g^{-1})$ $\forall g \in \Gamma$.
· μ is called non-degenerate if $\operatorname{supp}\mu$ generates Γ.

Note · Suppose Γ: f.g., gen by S.
Define the Cayley graph of (Γ, S):
$G = (V, E)$ where $V = \Gamma$. $E = \{(g, gs) : g \in \Gamma, s \in S\}$.
Then μ-RW on Γ is ~~a~~ NNRW on G.

· $\Gamma \curvearrowright \Gamma^{\mathbb{N}}$ by$_{\text{ptwise}}$ left multiplication. [i.e., "diagonal action"]

Why are we interested in RW on gps?

- RW on gp $\longleftrightarrow$ Harmonic analysis on gp $\overset{\text{"?"}}{\longleftrightarrow}$ ~~gp characteri~~ structural property of gp.
- Related w/ RW on graphs via Cayley graph. [or more generally, Schreier graphs].
- Probabilistic viewpoint: What structures have impact on RW characteristics.
  e.g. recurrence, asymptotic behavior of transition probability.
  convergence ~~to~~ a bdy pt, harmonic ftns, ...
- Also related w/ G-actions.

§ Classification of recurrent RW on gps.

$\boxed{\text{Thm}}$ (Varopoulos 1986).

let Γ : f.g. gp.

Then, $\exists \mu \in \mathcal{P}(\Gamma)$ s.t. μ-RW on Γ is recurrent iff

$\exists H \leq \Gamma$, $[\Gamma : H] < \infty$ s.t. $H \cong \{id\}$ or $\mathbb{Z}$ or $\mathbb{Z}^2$.

§ Harmonic Functions.

<u>def</u> · $f: \Gamma \to \mathbb{R}$ is <u>μ-harmonic</u> if $f(g) = \sum_s f(gs)\mu(s)$ $\forall g \in \Gamma$.

· (Γ, μ) is <u>Liouville</u> if every bdd μ-harmonic ftn is const.

Thm (Furstenberg's conjecture ; Rosenblatt 1981, Kaimanovich – Vershik 1983)

let Γ: cnt'ble.

Then, Γ : amenable $\iff$ $\exists \mu \in \mathcal{P}(\Gamma)$ nondegen, symmetric s.t. μ-RW : Liouville.

Thm (Choquet – Deny 1960)

let Γ : abelian, $\mu \in \mathcal{P}(\mu)$ nondegen. Then, (Γ, μ) : Liouville.

<u>pf 1</u> (Margulis's proof)

Fix $M > 0$. let $\mathcal{H}ar^{\leq M}(\Gamma, \mu) := \{ f \in \mathcal{H}ar^{\infty}(\Gamma, \mu) : \|f\|_\infty \leq M \}$.

We will show that $\mathcal{H}ar^{\leq M}(\Gamma, \mu) = \{ c\mathbb{1} : c \in [-M, M] \}$.

Note : $\mathcal{H}ar^{\leq M}(\Gamma, \mu)$ is Γ-inv., cpt., convex.

By the Krein – Milman thm., $\mathcal{H}ar^{\leq M}(\Gamma, \mu) = \overline{co}\left(ext\left(\mathcal{H}ar^{\leq M}(\Gamma, \mu)\right)\right)$.

let $f \in ext\left(\mathcal{H}ar^{\leq M}(\Gamma, \mu)\right)$.

then $f(h) = \sum_k f(hk)\mu(k) = \sum_k f^{k^{-1}}(h)\mu(k)$. so $f = f^{k^{-1}}$ $\forall k \in supp\, \mu$.

hence f is Γ-inv., $f \equiv$ const. ▨

<u>pf2</u> (Probabilist's proof).

We use a coupling method.

It suffices to construct:

$\forall x, y \in \Gamma$. $\exists X, Y \sim \mu$-RW s.t. $X_0 = x$, $Y_0 = y$. $\mathbb{P}[X_n = Y_n \ \forall n \gg 0] = 1$.

let $x^{-1}y = s_1^{e_1} \cdots s_n^{e_n}$. $s_i \in supp\, \mu$.

Forget all relations bet'n s_i's.

Then construct a lazy RW just as on $\mathbb{Z}^n$. ...... ▨

§ Poisson Boundary

Recall

$$\boxed{\text{Brownian motion } (B_t)_{t\geq 0}}$$

transition density
$\mathbb{P}[B_t \in dx] = p(x,t)\,dx$.

Infinitesimal generator
$\frac{1}{2}\Delta f = \lim_{t \searrow 0} \frac{B_t f - f}{t}$ $\quad \forall f \in D(\frac{1}{2}\Delta) \underset{\text{dense}}{\subseteq} C_0(\mathbb{R}^d)$

$$\boxed{\text{Heat kernel } \; p(x,t) = (2\pi t)^{-d/2} e^{-|x|^2/2t}} \longleftrightarrow \boxed{\text{Laplacian } \; \frac{1}{2}\Delta = \frac{1}{2}\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}}$$

Heat eqn (Kolmogorov backward eqn).
$$\partial_t p = \frac{1}{2}\Delta p.$$

- (Dirichlet problem).
  Given $f: \mathbb{D} \to \mathbb{R}$ bdd harmonic i.e., $\frac{1}{2}\Delta f = 0$.
  $(f(B_t))_{t\geq 0}$ is a bdd Martingale,
  so by MG convergence, $f(B_{t\wedge \tau_{\partial\mathbb{D}}}) \xrightarrow{t\to\infty} \exists\, \hat{f} \in L^\infty(\partial\mathbb{D})$.
  Conversely, given $\hat{f} \in L^\infty(\partial\mathbb{D})$.
  $f(x) := \mathbb{E}^x \hat{f}(B_{\tau_{\partial\mathbb{D}}})$ defines $f \in \mathcal{H}ar^\infty(\mathbb{D})$.
  $\therefore \mathcal{H}ar^\infty(\mathbb{D}) \cong L^\infty(\partial\mathbb{D})$.

- (Poisson kernel).
  This time, we consider the Poincaré disk $\mathbb{D} \cong \mathbb{H}^2$ w/ $ds^2 = \frac{4}{(1-|z|^2)^2}(dx^2+dy^2)$.
  $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad SL_2\mathbb{R}/SO(2)$.
  let $\mathcal{L} = \frac{(1-|z|^2)^2}{4}(\partial_x^2 + \partial_y^2)$ Laplace-Beltrami operator.
  (X_t) : diffusion on $\mathbb{D}$ assoc. to $\mathcal{L}$.
  Then, similarly, $\mathcal{H}ar^\infty(\mathbb{D}) \xrightleftharpoons{\sim} L^\infty(\partial\mathbb{D})$.
  $$f(z) = \int_{\partial\mathbb{D}} P(r, \theta-\phi)\, \hat{f}(e^{i\phi}) \frac{d\phi}{2\pi} \longleftrightarrow \hat{f}$$
  where $P(r,\theta) = \mathcal{R}e\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)$ Poisson kernel.

let $\Gamma^{\mathbb{N}}$: trajectory space w/ Borel σ-alg. $\mathcal{F}$.

$S : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ left shift.

$\mathcal{I} = \{ A \in \mathcal{F} : S^{-1}A = A \}$ invariant σ-alg.

$\mathcal{T} = \bigcap_{n \geq 1} \sigma(X_n, X_{n+1}, \cdots)$ tail σ-alg.

Recall (Derriennic 0-1 law)
$\mathcal{I}$ and $\mathcal{T}$ only differ by null sets. (when RW is aperiodic).

Prop The map $\Phi : L^\infty(\Gamma^{\mathbb{N}}, \mathcal{I}, \mathbb{P}) \longrightarrow \mathrm{Har}^\infty(\Gamma, \mu)$.

$$\hat{f} \longmapsto f(g) = \mathbb{E}^g \hat{f}(X_0, X_1, \cdots)$$

$$\hat{f}(X_n) := \lim_{n\to\infty} f(X_n) \xleftarrow{\text{MG conv.}} f$$

is a lin. isometry.

def The Poisson-Furstenberg boundary is $(\Pi, \mathcal{B}) =$ "$\Gamma^{\mathbb{N}} / \mathcal{T}$" (meas'ble sp.)

where $\mathcal{B}$ separate pts of Π, w/ msrs ν_g, $g \in \Gamma$ s.t.

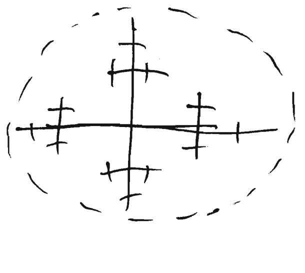
$\forall f \in \mathrm{Har}^\infty(\Gamma, \mu)$, $\exists \hat{f} \in L^\infty(\Pi, \nu)$ s.t. $f(g) = \int_\Pi \hat{f} \, d\nu_g$.

($\because \lim f(X_n)$ is $\mathcal{I}$-msr'ble).

Remark. (Γ, μ) : Liouville $\Longleftrightarrow$ $\mathcal{I}$: trivial $\Longleftrightarrow$ $\mathcal{T}$: trivial $\Longleftrightarrow$ Π is a pt.

Example $\Gamma = F_2 = \langle a, a^{-1}, b, b^{-1} \rangle$, μ simple.

e.g. tail event : X_n starts w/ a $\forall n \gg 0$.



Π = infinite words.

§ Entropy

def . $H(\mu) := -\sum_g \mu(g) \log \mu(g)$.

· The ~Avez asymptotic entropy~ is

$$h_\mu := \lim_{n\to\infty} \frac{1}{n} H(X_n) \overset{\text{Fekete}}{=} \inf_n \frac{1}{n} H(X_n).$$

$\boxed{\text{Thm}}$ (Avez – Derriennic – Kaimanovich – Vershik 1983).

Suppose $H(\mu) < \infty$.

Then, $h_\mu = 0$ iff $\mathcal{J}$: trivial.

Example On reg. trees, $H(X_n) \sim n$, so $h_\mu > 0$.