2 Planar System Modeling and Control

2.1 Model

2.1.1 Model Description

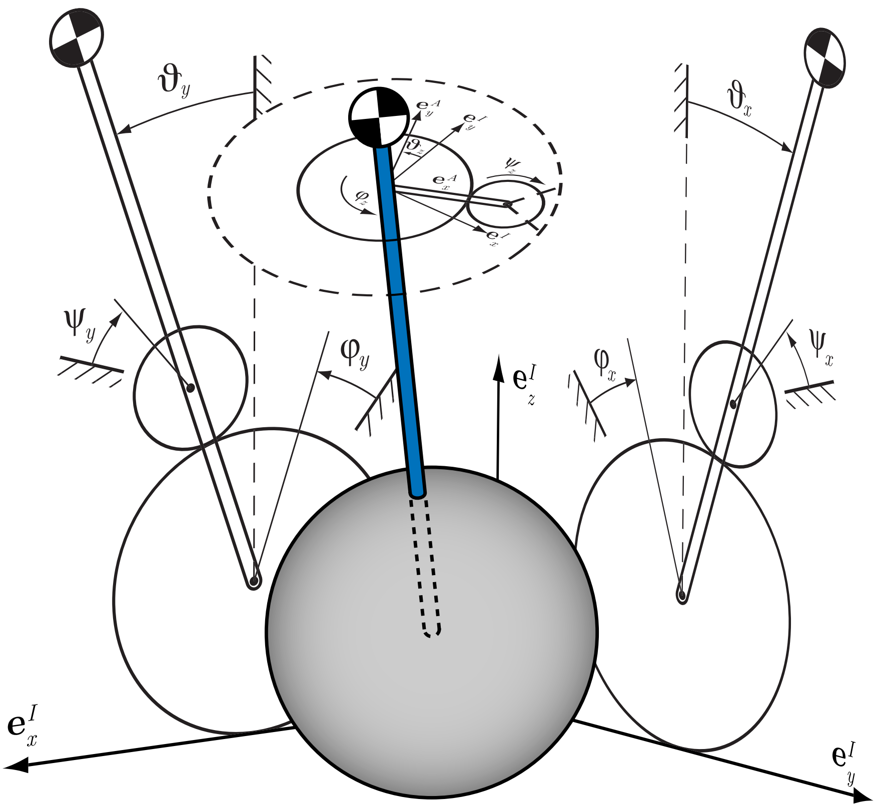

Figure 2.1: The three planar models, each model consists of the ball, the body and the actuating wheel

The three-dimensional system is divided into three independent planar models (fig: 2.1), where each can be described with the following two Degrees of Freedom (DoF):

- 1 DoF for the rotation/translation of the ball
- 1 DoF for the rotation of the body

The propulsion system, which is composed of three omniwheels, is modeled as one virtual actuating wheel in each plane which drives the ball. That particular wheel does of course not have the same position and speed as the omniwheels in the real system. Therefore, conversions are needed (see section 2.3).

The states of the system are the orientation angles and angle rates of the body and the position and speed of the ball. Since all these states are measured by the IMU and the motor encoders, the system outputs are equal to the states. The inputs are the torques generated by the motors.

2.1.2 Assumptions

In the planar system modeling, the system is treated as three independent planar models. Therefore no coupling effects between these three models are taken into account. Two of them (the ones in the xz- and yz-plane) are identical and very similar to an inverted pendulum, as shown on the left side of figure 2.2. The third one (in the xy-plane) describes the rotation around the z-axis in the body fixed reference frame, often described as the rotation around the ballbot's own axis (right side of figure 2.2).
The system is assumed to consist of three rigid bodies: the ball, the actuating wheel and the body. The three motors and omniwheels are therefore simplified as one actuating wheel that fits in a plane.
Additionally, the following assumptions are made:

No slip The contact points between the ball and the ground and between the wheels and the ball are assumed to be free of slippage. The system construction is optimized in order to minimize possible slipping. Thus, the torques applied have to be limited within the range that does not cause slippage in order to ensure the validity of the model.

No friction Even though friction occurs in the system, it is neglected except for the rotation of the ball on the ground around the z-axis, because this is the friction with the biggest impact. Non-continuous friction is hard to represent analytically and would generate equations much more complex than without friction. Also, the precise measuring of friction is difficult.

No deformation Because of its synthetic coating, deformation appears at the contact points of the ball. This has to be neglected in order to keep complexity at a manageable level.

Fast motor dynamics The power electronics control the dynamics of the motors much faster than the controller the dynamics of the stabilized system. Consequently, the motor torques are modeled to be the inputs of the system.

Ball moves only horizontally In order to keep the prerequisite for the no slip condition, vertical movement of the ball has to be neglected. Finally, this model is designed to move primarily on flat surfaces without steep inclination.

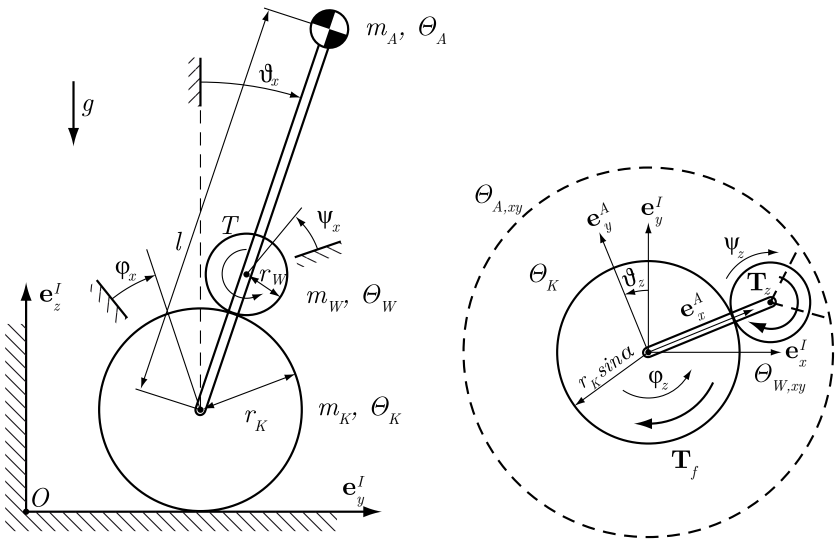

Figure 2.2: Sketch of the planar models showing the coordinates and parameters. On the left side the model for the two vertical planes and on the right side the xy-plane model is shown.

2.1.3 Coordinates

To describe the system, the coordinates shown in figure 2.2 are used. $\vartheta_{x,y,z}$ represent the orientation of the body, $\varphi_{x,y,z}$ represent the orientation of the ball and $\psi_{x,y,z}$ are the angles of the virtual actuating wheels. Using the following minimal coordinates, the system can be completely described.

$$\vec{q}_{xy} = \begin{bmatrix} \varphi_z \\ \vartheta_z \end{bmatrix} \quad \vec{q}_{yz} = \begin{bmatrix} \varphi_x \\ \vartheta_x \end{bmatrix} \quad \vec{q}_{xz} = \begin{bmatrix} \varphi_y \\ \vartheta_y \end{bmatrix} \tag{2.1}$$

2.1.4 Binding Equations

All coordinates can be expressed as functions of the minimal coordinates, which is necessary in order to derive the equations of motion. The positions of the ball, the wheel and the body can be written as follows, using the rolling constraint[1].

$$x_K = \varphi_x r_K \tag{2.2}$$

$$x_W = \varphi_x r_K + \sin\vartheta_x \cdot (r_K + r_W) \tag{2.3}$$

$$x_A = \varphi_x r_K + \sin\vartheta_x \cdot l \tag{2.4}$$

Analogous relations are valid for the y coordinates.

[1]Meaning, that for example the balls position is determined by its slip free rolling on the ground.

The rotation of the actuating wheel expressed as a function of the minimal coordinates yields the following equations.

$$\dot{\psi}_x = \frac{r_K}{r_W}\left(\dot{\varphi}_x - \dot{\vartheta}_x\right) - \dot{\vartheta}_x \tag{2.5}$$

$$\dot{\psi}_y = \frac{r_K}{r_W}\left(\dot{\varphi}_y - \dot{\vartheta}_y\right) - \dot{\vartheta}_y \tag{2.6}$$

$$\dot{\psi}_z = \frac{r_K}{r_W} \cdot \sin\alpha \cdot (\dot{\varphi}_z - \dot{\vartheta}_z) \tag{2.7}$$

The derivation can be found in the appendix B.1.

2.1.5 Forces

In order to get an idea of the forces taking effect in the system, a Newton/Euler approach is used to derive analytical equation for the forces shown in figure 2.3. The detailed equations are listed in the appendix B.3.

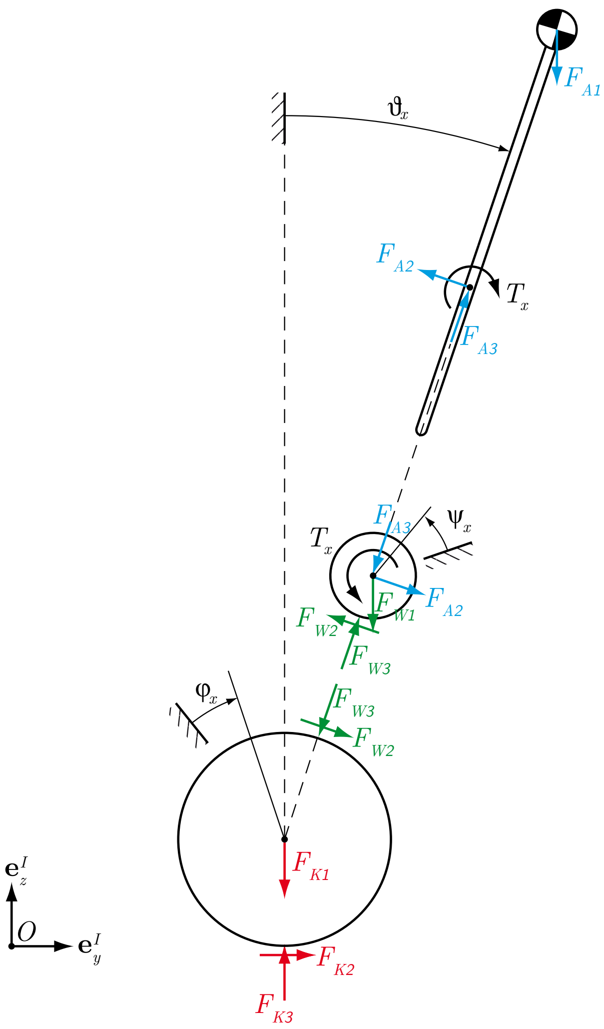

Figure 2.3: Free-body diagrams for the three parts of the model, showing all internal forces

2.1.6 Parameters

In table 2.1, the parameters describing the system are listed. Numeric values are derived by measuring or from the CAD model. Calculations of the moments of inertia are presented in section 2.3.2.

Description	Variable	Value
Mass of the ball	m_K	2.29 kg
Mass of the virtual actuating wheel	m_W	3 kg
Mass of the body	m_A	9.2 kg
Radius of the ball	r_K	0.125 m
Radius of the omniwheels	r_W	0.06 m
Radius of the body (cylinder)	r_A	0.1 m
Height of the center of gravity	l	0.339 m
Inertia of the ball	Θ_K	0.0239 kgm^2
Inertia of the actuating wheel in the yz-/xz-plane	Θ_W	0.00236 kgm^2
Inertia of the actuating wheel in the xy-plane	$\Theta_{W,xy}$	0.00945 kgm^2
Inertia of the body	Θ_A	4.76 kgm^2
Inertia of the body in the xy-plane	$\Theta_{A,xy}$	0.092 kgm^2
Gravitational acceleration	g	9.81 m/s^2
Gear ratio	i_{Gear}	26

Table 2.1: Parameters of the planar system

2.2 Dynamics

2.2.1 Approach

For the derivation of the equations of motion, the Lagrangian method was selected as the common approach for that kind of problems. The following steps are needed:

- express kinetic (T) and potential (V) energy of all rigid bodies as a function of the minimal coordinates,
- write non-potential forces as a function of the minimal coordinates,
- solve the Lagrange equation for the second derivative of the minimal coordinates.

The calculations are made for the yz-plane model, which is identical to the xz-plane model, and for the xy-plane model as well.

Since a controller for the planar model is going to be implemented on the real system, conversions have to be calculated. In order to be able to control the real system, the torques on the virtual motors have to be converted into the torques for the real motors.

Definitions

One element of the real system (omniwheel driven by a motor) is shown on the left side in figure 2.4. It actuates the ball by scrolling on a circle on the ball, characterized by the motor arrangement angle α.
The right side of figure 2.4 shows a top view on the real actuating system where the torque of each omniwheel generates a tangential force on the surface of the ball.

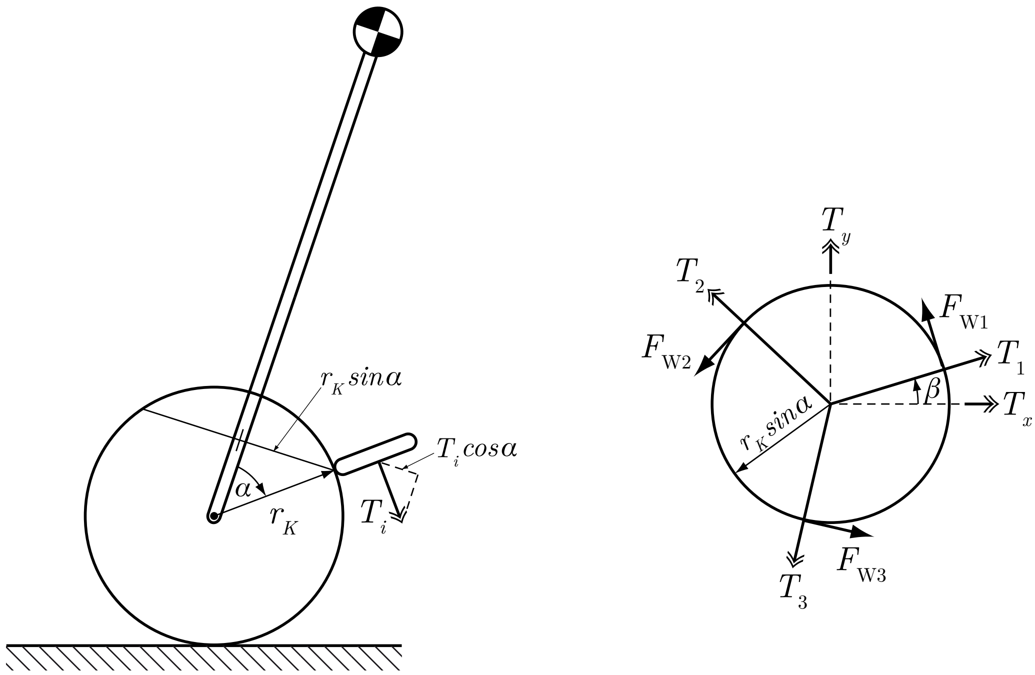

Figure 2.4: On the left side the arrangement of one real omniwheel is shown. The sketch on the right side shows the torques and tangential forces generated by the real actuating system, the dashed arrows mark the virtual torques.

$F_{W,1}$, $F_{W,2}$, $F_{W,3}$ in figure 2.4 denote the tangential forces on the ball generated by torques of the motors in the real configuration, consequently these are functions of $T_{1,2,3}$. The levers for this configuration are denoted as $r_{KW,1}$, $r_{KW,3}$, $r_{KW,3}$. For the virtual configuration, the notation $F_{W,x}$, $F_{W,y}$, $F_{W,z}$ is used for the tangential forces (as functions of $T_{x,y,z}$) and $r_{KW,x}$, $r_{KW,y}$, $r_{KW,z}$ for the levers. All those are geometrical relationships one can read out of figure 2.4. In the appendix C.1 the relationships are written out in full.

Torque on the ball

Once the relationships are given, the torque on the ball from the real and the virtual driving mechanism can be calculated. The torque is equal to the product of force and lever. Hence the torques for the real and the virtual configuration are calculated as follows.

$$T_{KW,i} = r_{KW,i} \times F_{W,i} \qquad \text{for } i = 1, 2, 3 \tag{2.28}$$

$$T_{KW,j} = r_{KW,j} \times F_{W,j} \qquad \text{for } j = x, y, z \tag{2.29}$$

Solution

Of course, the overall torque needs to be conserved:

$$T_{KW,1} + T_{KW,2} + T_{KW,3} = T_{KW,x} + T_{KW,y} + T_{KW,z} \tag{2.30}$$

Solving the relations for the real torques yields:

$$T_1 = \frac{1}{3} \cdot \left(T_z + \frac{2}{\cos\alpha} \cdot (T_x \cdot \cos\beta - T_y \cdot \sin\beta) \right) \tag{2.31}$$

$$T_2 = \frac{1}{3} \cdot \left(T_z + \frac{1}{\cos\alpha} \cdot \left(\sin\beta \cdot (-\sqrt{3}T_x + T_y) - \cos\beta \cdot (T_x + \sqrt{3}T_y) \right) \right) \tag{2.32}$$

$$T_3 = \frac{1}{3} \cdot \left(T_z + \frac{1}{\cos\alpha} \cdot \left(\sin\beta \cdot (\sqrt{3}T_x + T_y) + \cos\beta \cdot (-T_x + \sqrt{3}T_y) \right) \right) \tag{2.33}$$

Since the system of equation is linear in $T_{x,y,z}$, it is straightforward to solve for them as well (see appendix appendix C).

2.3.2 Inertia Calculations

The aim of this section is to find a suitable approximation for the moments of inertia of the modeled drive mechanism, which is not the same as in reality. The following approach is made by comparing the energies stored in the real omniwheels and the virtual actuating wheel respectively when driving at a constant velocity.

Approximations for sphere and body

The inertia of the ball is approximated by a hollow sphere, which is a good approximation:

$$\Theta_K = \frac{2}{3} \cdot m_K \cdot r_K^2$$

The moment of inertia of the body was initially approximated by that of a cylinder. Later, it was directly calculated in the CAD model.

B Planar System Modeling

B.1 Model for the yz-/xz-Plane

Binding Equations

One planar system has two degrees of freedom. Therefore, it is possible to write any coordinates as a function of the two minimal coordinates φ_x und ϑ_x.
The following equations are valid for the ball:

$$
\begin{aligned}
x_K &= \varphi_x r_K \\
z_K &= 0
\end{aligned}
$$

For the actuating wheel:

$$
\begin{aligned}
x_W &= x_K + \sin\vartheta_x \cdot (r_K + r_W) \\
&= \varphi_x r_K + \sin\vartheta_x \cdot (r_K + r_W) \\
z_W &= \cos\vartheta_x \cdot (r_K + r_W)
\end{aligned}
$$

And for the body:

$$
\begin{aligned}
x_A &= x_K + \sin\vartheta_x \cdot l \\
&= \varphi_x r_K + \sin\vartheta_x \cdot l \\
z_A &= \cos\vartheta_x \cdot l
\end{aligned}
$$

The binding equation for ψ_x is derived by equating the velocities (*Starrkörper-Geschwindigkeits-Formel*) in the contact point between ball and actuating wheel.

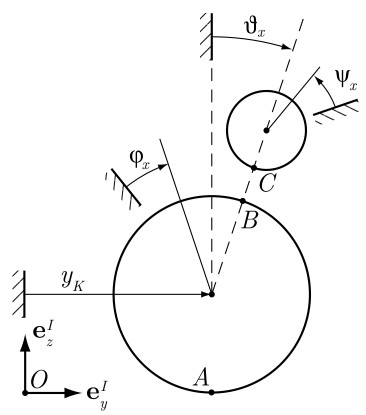

Figure B.1: Sketch of the planar model with coordinates and contact points

$$v_A = 0$$

$$v_B = \begin{bmatrix} \dot{x}_K \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\varphi}_x \\ 0 \end{bmatrix} \times \begin{bmatrix} r_K \sin\vartheta_x \\ 0 \\ r_K \cos\vartheta_x \end{bmatrix} = \begin{bmatrix} \dot{x}_K + \dot{\varphi}_x r_K \cos\vartheta_x \\ 0 \\ -r_K \dot{\varphi}_x \sin\vartheta_x \end{bmatrix}$$

$$\begin{aligned} v_C &= \begin{bmatrix} \dot{x}_K \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\vartheta}_x \\ 0 \end{bmatrix} \times \begin{bmatrix} (r_K + r_W)\sin\vartheta_x \\ 0 \\ (r_K + r_W)\cos\vartheta_x \end{bmatrix} + \begin{bmatrix} 0 \\ -\dot{\psi}_x \\ 0 \end{bmatrix} \times \begin{bmatrix} -r_W \sin\vartheta_x \\ 0 \\ -r_W \cos\vartheta_x \end{bmatrix} \\ &= \begin{bmatrix} \dot{x}_K + \dot{\vartheta}_x (r_K + r_W)\cos\vartheta_x + r_W \dot{\psi}_x \cos\vartheta_x \\ 0 \\ -\dot{\vartheta}_x (r_K + r_W)\sin\vartheta_x - r_W \dot{\psi}_x \sin\vartheta_x \end{bmatrix} \end{aligned}$$

To comply the no slip condition, the equation $v_B = v_C$ has to be true. It is then possible to equate the first and third term of v_B and v_C. Solving these equations by $\dot{\psi}_x$ yields:

$$\dot{\psi}_x = \frac{r_K}{r_W}(\dot{\varphi}_x - \dot{\vartheta}_x) - \dot{\vartheta}_x \tag{B.1}$$

Approach

For the derivation of the equations of motion the Lagrangian method was selected as the common approach for that kind of problems. The following steps are needed:

- express kinetic (T) and potential (V) energy of all rigid bodies as a function of the minimal coordinates,
- write non-potential forces as a function of the minimal coordinates,
- solve the Lagrange equation for the second derivative of the minimal coordinates.

The calculations are made for the yz-plane model, which is identical to the xz-plane model, and for the xy-plane model as well.

Energies

Ball:

$$T_K = \frac{1}{2} m_K (r_K \dot{\varphi}_x)^2 + \frac{1}{2}\Theta_K \dot{\varphi}_x^2$$
$$V_K = 0$$

B.2 Model for the xy-Plane

For the xy-plane model, the same approach is used. The complete derivation is given as follows.

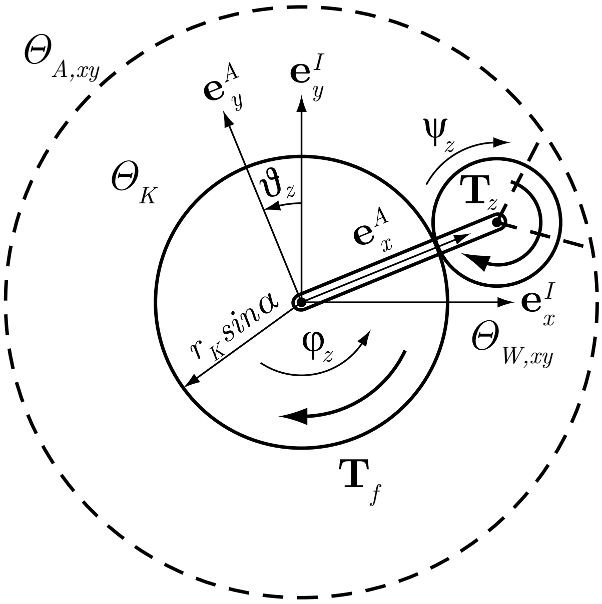

Figure B.2: Planar model for the xy-plane

Minimal coordinates

The same minimal coordinates are used (see figure B.2).

$$\vec{q} = \begin{bmatrix} \varphi_z \\ \vartheta_z \end{bmatrix}$$

Energies

Ball:

$$T_K = \frac{1}{2}\Theta_K \dot{\varphi}_z^2$$

Actuating wheel (Note: $\Theta_{W,xy}$ contains the inertias of motors and wheels around ψ_z).

$$T_W = \frac{1}{2}\Theta_{W,xy}\dot{\psi}_z^2$$

Body (Note: $\Theta_{A,xy}$ contains the inertias of motors and wheels around ϑ_z).

$$T_A = \frac{1}{2}\Theta_{A,xy}\dot{\vartheta}_z^2$$

Binding equation

$$\dot{\psi}_z = \frac{r_K}{r_W} \cdot \sin\alpha \cdot (\dot{\varphi}_z - \dot{\vartheta}_z) \tag{B.4}$$

Non-potential forces

From the actuators:

$$T_1 = T_z$$
$$J_1 = \begin{bmatrix} \frac{r_K}{r_W} \cdot \sin\alpha \\ -\frac{r_K}{r_W} \cdot \sin\alpha \end{bmatrix}$$

Counter torque:

$$T_2 = -T_z$$
$$J_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Friction between ball and ground:

$$T_3 = -T_f$$
$$J_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

f_{NP}:

$$f_{NP} = J_1 \cdot T_1 + J_2 \cdot T_2 + J_3 \cdot T_3 = \begin{bmatrix} -T_f + \frac{r_K}{r_W} \cdot \sin\alpha \cdot T_z \\ -\frac{r_K}{r_W} \cdot \sin\alpha \cdot T_z + T_z \end{bmatrix} \tag{B.5}$$

Equation of motion

Total kinetic energy:

$$T = T_K + T_W + T_A$$

Total potential energy:

$$V = 0$$

Solving the Lagrange equation for $\ddot{\varphi}$ and $\ddot{\vartheta}$ yields:

$$\ddot{\varphi} = -\frac{(r_W^2 \Theta_{A,xy} + r_K^2 \Theta_{W,xy} \sin^2\alpha) \cdot T_f + r_K r_W \Theta_{A,xy} \sin\alpha \cdot T_z}{r_W^2 \Theta_{A,xy} \Theta_K + r_K^2 (\Theta_{A,xy} + \Theta_K) \Theta_{W,xy} \sin^2\alpha} \tag{B.6}$$

C Conversions

C.1 Torque Conversion

The planar model uses a virtual wheel to actuate the system. The real system has an actuating structure which differs strongly from the one assumed in the planar model. Since a controller for the planar model is going to be implemented on the real system, conversions have to be calculated. In order to be able to control the real system, the torques on the virtual motors have to be converted into the torques for the real motors.

Definitions

One element of the real system (omniwheel driven by a motor) is shown on the left side in figure C.1. It actuates the ball by scrolling on a circle on the ball, characterized by the motor arrangement angle α.
The right side of figure C.1 shows a top view on the real actuating system where the torque of each omniwheel generates a tangential force on the surface of the ball.

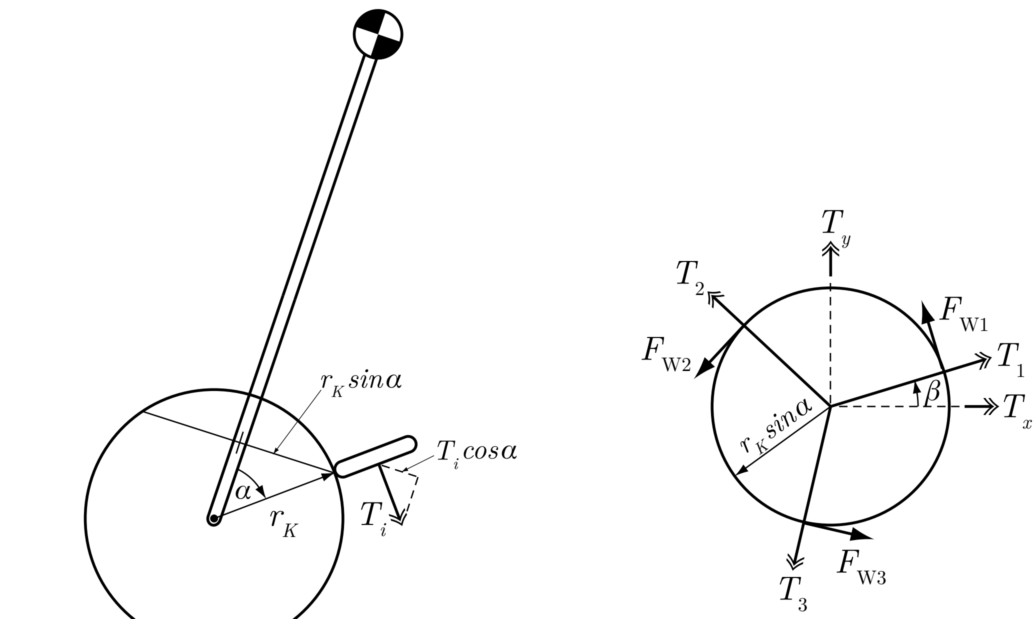

Figure C.1: Torques and tangential forces generated by the real actuating system

The following relationships are read out of figure C.1.

Tangential forces of the real configuration:

$$F_{W,1} = \frac{T_1}{r_W} \cdot \begin{bmatrix} -\sin\beta \\ \cos\beta \\ 0 \end{bmatrix} \tag{C.1}$$

$$F_{W,2} = \frac{T_2}{r_W} \cdot \begin{bmatrix} -\sin(\beta + \frac{2}{3}\pi) \\ \cos(\beta + \frac{2}{3}\pi) \\ 0 \end{bmatrix} \tag{C.2}$$

$$F_{W,3} = \frac{T_3}{r_W} \cdot \begin{bmatrix} -\sin(\beta - \frac{2}{3}\pi) \\ \cos(\beta - \frac{2}{3}\pi) \\ 0 \end{bmatrix} \tag{C.3}$$

Lever for real configuration:

$$r_{KW,1} = r_K \cdot \begin{bmatrix} \cos\beta\sin\alpha \\ \sin\beta\sin\alpha \\ \cos(\alpha \end{bmatrix} \tag{C.4}$$

$$r_{KW,2} = r_K \cdot \begin{bmatrix} \cos(\beta + \frac{2}{3}\pi)\sin(\alpha) \\ \sin(\beta + \frac{2}{3}\pi)\sin(\alpha) \\ \cos(\alpha) \end{bmatrix} \tag{C.5}$$

$$r_{KW,3} = r_K \cdot \begin{bmatrix} \cos(\beta - \frac{2}{3}\pi)\sin(\alpha) \\ \sin(\beta - \frac{2}{3}\pi)\sin(\alpha) \\ \cos(\alpha) \end{bmatrix} \tag{C.6}$$

Tangetial forces of the virtual configuration plane:

$$F_{W,x} = \frac{T_x}{r_W} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{(actually } y\text{-direction)} \tag{C.7}$$

$$F_{W,y} = \frac{T_y}{r_W} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(actually } x\text{-direction)} \tag{C.8}$$

$$F_{W,z} = \frac{T_z}{r_W} \cdot \begin{bmatrix} -\sin\beta \\ \cos\beta \\ 0 \end{bmatrix} \tag{C.9}$$

Lever for the virtual configuration plane:

$$r_{KW,x} = r_{KW,y} = r_K \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{C.10}$$

$$r_{KW,z} = r_K \cdot \begin{bmatrix} \cos\beta \cdot \sin\alpha \\ \sin\beta \cdot \sin\alpha \\ 0 \end{bmatrix} \tag{C.11}$$

Torque on the ball

With given relationships the torque on the ball from the real and the virtual driving mechanism can be calculated.
Torques on the Ball from the real omniwheels:

$$T_{KW,1} = r_{KW,1} \times F_{W,1} \tag{C.12}$$

$$T_{KW,2} = r_{KW,2} \times F_{W,2} \tag{C.13}$$

$$T_{KW,3} = r_{KW,3} \times F_{W,3} \tag{C.14}$$

Torques on the Ball from the virtual omniwheels:

$$T_{KW,x} = r_{KW,x} \times F_{W,x} \tag{C.15}$$

$$T_{KW,y} = r_{KW,y} \times F_{W,y} \tag{C.16}$$

$$T_{KW,z} = r_{KW,z} \times F_{W,z} \tag{C.17}$$

Solution

Following equation describes that the two different systems need to introduce the same torque on the ball.

$$T_{KW,1} + T_{KW,2} + T_{KW,3} = T_{KW,x} + T_{KW,y} + T_{KW,z} \tag{C.18}$$

The relationship between real and virtual driving mechanism can be obtained by solving for the unknowns.

Solved for real motor torques:

$$T_1 = \underline{\frac{1}{3} \cdot \left(T_z + \frac{2}{\cos\alpha} \cdot (T_x \cdot \cos\beta - T_y \cdot \sin\beta) \right)} \tag{C.19}$$

$$T_2 = \underline{\frac{1}{3} \cdot \left(T_z + \frac{1}{\cos\alpha} \cdot \left(\sin\beta \cdot (-\sqrt{3}T_x + T_y) - \cos\beta \cdot (T_x + \sqrt{3}T_y) \right) \right)} \tag{C.20}$$

$$T_3 = \underline{\frac{1}{3} \cdot \left(T_z + \frac{1}{\cos\alpha} \cdot \left(\sin\beta \cdot (\sqrt{3}T_x + T_y) + \cos\beta \cdot (-T_x + \sqrt{3}T_y) \right) \right)} \tag{C.21}$$

$$\tag{C.22}$$

Solved for virtual motor torques:

$$T_x = \underline{\cos\alpha \cdot \left(T_1 \cdot \cos\beta - T_2 \cdot \sin(\beta + \frac{\pi}{6}) + T_3 \cdot \sin(\beta - \frac{\pi}{6}) \right)} \tag{C.23}$$

$$T_y = \underline{\cos\alpha \cdot \left(-T_1 \cdot \sin\beta - T_2 \cdot \cos(\beta + \frac{\pi}{6}) + T_3 \cdot \cos(\beta - \frac{\pi}{6}) \right)} \tag{C.24}$$

$$T_z = \underline{T_1 + T_2 + T_3} \tag{C.25}$$

C.2 Inertia Calculations

The aim of this section is to find a suitable approximation for the moments of inertia of the modeled drive mechanism, which is not the same as in reality. The following approach is made by comparing the energies stored in the real omniwheels and the virtual actuating wheel respectively, when driving at a constant velocity.

Approximations for ball and body

$$\Theta_K = \frac{2}{3} m_K r_K^2 \qquad \text{(hollow sphere)} \tag{C.26}$$

$$\Theta_A = \frac{1}{4} \cdot m_A \cdot r_A^2 + \frac{1}{2} \cdot m_A \cdot h^2 + m_A \cdot l^2 \qquad \text{(cylinder)} \tag{C.27}$$

Motors and omniwheels

The real rotor inertia of a motor is given by

$$\Theta_M = 3.33 \cdot 10^{-6}\,\mathrm{kgm}^2$$

and the real inertia of an omniwheel is calculated as:

$$\Theta_{OW} = \frac{1}{2} m_{OW} r_W^2 = 900 \cdot 10^{-6}\,\mathrm{kgm}^2$$

Since the reduction of the gear box is $i = 26$, the rotor of the motor turns 26 times faster. Therefore, a factor i^2 is added to calculate the energy ($E = 1/2 \cdot \Theta \cdot v^2$). So