Gaussian Kernel Approximation using Box-filtered Mip-maps

by *Tianchen Xu* (TenshinX@outlook.com)

Our goal is to express the Gaussian-blur convolution for each pixel p, using the sum of weighted samples $s_{\text{Box}}(\ell)$, where sample $s_{\text{Box}}(\ell)$ is box-filtered according to the mip-level ℓ and thus a function of ℓ:

$$p_{\text{Gaussian}} = \iint_{-\infty}^{\infty} G(x,y)s(x,y)\,dx\,dy = \int_{0}^{\infty} w_G(\ell)s_{\text{Box}}(\ell)\,d\ell \tag{1}$$

In Eqn. (1), $s(x, y)$ denotes the original sample at neighborhood of p with position offset (x, y), $w_G(\ell)$ represents the weight of box-filtered sample at level ℓ, and $G(x, y)$ is the Gaussian function.

In terms of the box-filtered sample $s_{\text{Box}}(\ell)$ at level ℓ, we assume that it is approximately equivalent to the sum of the original samples $s(x, y)$ within the box boundaries, denoted by the convolution:

$$s_{\text{Box}}(\ell) = \iint_{-\infty}^{\infty} B(\ell,x,y)s(x,y)\,dx\,dy \tag{2}$$

Since $B(\ell, x, y)$ denotes the weight of $s(x, y)$, it is evaluated as $1/N$ within the box and 0 out of the box, where the box size is $N = 2^{\ell} \times 2^{\ell}$ at mip-level ℓ. Also, $B(\ell, x, y)$ can be regarded as part of *Haar* wavelet basis. Therefore,

$$B(\ell,x,y) = \begin{cases} \dfrac{1}{N} = 4^{-\ell}, & 2|x| \le 2^{\ell} \wedge 2|y| \le 2^{\ell} \\ 0, & \text{otherwise} \end{cases}$$

Then, we substitute Eqn. (2) into Eqn. (1), and then:

$$\begin{aligned} p_{\text{Gaussian}} &= \iint_{-\infty}^{\infty} G(x,y)s(x,y)\,dx\,dy = \int_{0}^{\infty} w_G(\ell)s_{\text{Box}}(\ell)\,d\ell \\ \Rightarrow \iint_{-\infty}^{\infty} G(x,y)s(x,y)\,dx\,dy &= \int_{0}^{\infty} w_G(\ell)\iint_{-\infty}^{\infty} B(\ell,x,y)s(x,y)\,dx\,dy\,d\ell \\ &= \iint_{-\infty}^{\infty}\int_{0}^{\infty} w_G(\ell)B(\ell,x,y)s(x,y)\,d\ell\,dxdy \\ \Rightarrow G(x,y)s(x,y) &= \int_{0}^{\infty} w_G(\ell)B(\ell,x,y)s(x,y)\,d\ell = s(x,y)\int_{0}^{\infty} w_G(\ell)B(\ell,x,y)\,d\ell \\ \therefore G(x,y) &= \int_{0}^{\infty} w_G(\ell)B(\ell,x,y)\,d\ell \end{aligned} \tag{3}$$

That means, if the Gaussian function were expressed by the sum of weighted box functions, our goal would be achieved.

As we expect to express the Gaussian function using the mip-level, the box function is firstly modified to fit the Gaussian function approximately. Here, we use a circle to fit the box, with minimized area difference:

$$B(\ell,x,y) \approx \begin{cases} 4^{-\ell}, & \pi(x^2+y^2) \le 4^{\ell} \\ 0, & \text{otherwise} \end{cases}$$

Let

$$L = \log_2\left(\sqrt{\pi(x^2+y^2)}\right) \Rightarrow \pi(x^2+y^2) = 4^L \Rightarrow x^2+y^2 = \frac{4^L}{\pi}$$

$$\Rightarrow \left.\begin{matrix} B(\ell,x,y) = 4^{-\ell}, & \forall \ell \geq L \\ B(\ell,x,y) = 0, & \forall \ell < L \end{matrix}\right\} \Rightarrow \int_0^\infty w_G(\ell) B(\ell,x,y)\, d\ell \approx \int_L^\infty 4^{-\ell} w_G(\ell)\, d\ell$$

after we substitute the condition of the box function into Eqn. (3), and obtain a function of *L*.

$$\therefore G(x,y) \approx \int_L^\infty 4^{-\ell} w_G(\ell)\, d\ell \approx g(L) \tag{4}$$

Also, the Gaussian function can be expressed as a function of mip-level *L*:

$$G(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{4^L}{2\pi\sigma^2}\right) = g(L) \Rightarrow \begin{cases} g(\ell) = \dfrac{1}{2\pi\sigma^2}\exp\left(-\dfrac{4^\ell}{2\pi\sigma^2}\right) \\ \dfrac{dg(\ell)}{d\ell} = -\dfrac{4^\ell \ln 4}{4\pi^2\sigma^4}\exp\left(-\dfrac{4^\ell}{2\pi\sigma^2}\right) \end{cases}$$

Subsequently, *g*(*L*) can be expressed by the definite integral of its derivative:

$$g(L) = -\{0 - g(L)\} = -\left\{\lim_{\ell\to\infty} g(\ell) - g(L)\right\} = -\int_L^\infty \frac{dg(\ell)}{d\ell}\, d\ell = -\int_L^\infty -\frac{4^\ell \ln 4}{4\pi^2\sigma^4}\exp\left(-\frac{4^\ell}{2\pi\sigma^2}\right) d\ell$$

$$\therefore g(L) = \frac{\ln 4}{4\pi^2\sigma^4}\int_L^\infty 4^\ell \exp\left(-\frac{4^\ell}{2\pi\sigma^2}\right) d\ell \tag{5}$$

Combining Eqn. (4) with Eqn. (5), we compute the differentiations on the both sides:

$$\int_L^\infty 4^{-\ell} w_G(\ell)\, d\ell \approx \frac{\ln 4}{4\pi^2\sigma^4}\int_L^\infty 4^\ell \exp\left(-\frac{4^\ell}{2\pi\sigma^2}\right) d\ell \Rightarrow 4^{-\ell} w_G(\ell) \approx \frac{4^\ell \ln 4}{4\pi^2\sigma^4}\exp\left(-\frac{4^\ell}{2\pi\sigma^2}\right)$$

$$\therefore w_G(\ell) \approx \frac{16^\ell \ln 4}{4\pi^2\sigma^4}\exp\left(-\frac{4^\ell}{2\pi\sigma^2}\right)$$

In order to evaluate the blending weight between two mip levels:

$$\alpha(L) = \frac{w_G(L)}{\int_L^\infty w_G(\ell) d\ell} \approx \frac{16^L \exp\left(-\frac{4^L}{2\pi\sigma^2}\right)}{\int_L^\infty 16^\ell \exp\left(-\frac{4^\ell}{2\pi\sigma^2}\right) d\ell} = \frac{16^L \exp\left(-\frac{4^L}{2\pi\sigma^2}\right)}{-\frac{2\pi\sigma^2(4^\ell + 2\pi\sigma^2)}{\ln 4}\exp\left(-\frac{4^\ell}{2\pi\sigma^2}\right)\Big|_L^\infty}$$

$$= \frac{16^L \exp\left(-\frac{4^L}{2\pi\sigma^2}\right)}{0 + \frac{2\pi\sigma^2(4^L + 2\pi\sigma^2)}{\ln 4}\exp\left(-\frac{4^L}{2\pi\sigma^2}\right)} = \frac{16^L \ln 4}{2\pi\sigma^2(4^L + 2\pi\sigma^2)}$$

$$\therefore \alpha(L) \approx \frac{16^L \ln 4}{2\pi\sigma^2(4^L + 2\pi\sigma^2)}$$