and a formal test for nonstationarity of the intraday volatility curve are developed in Sections 3.2–3.4. The asymptotic behavior of the estimator in a nonstationary setting is analyzed in Section 3.5. Feasible inference methods are provided in Section 4. A finite-sample bias-variance tradeoff analysis is presented in Section 5. Section 6 contains our empirical analysis. Auxiliary results, a simulation study and all proofs are provided in a Supplementary Appendix.

2. Setup and Assumptions

We assume all processes and random variables are defined on a common filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}(t))_{0\leq t\leq T}, P\right)$, with the log price process $X(t)$ of the financial asset of interest being governed by an Itô semimartingale of the form,

$$X(t) = X(0) + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dW(s) + \int_0^t \int_{\mathbb{R}} x\,\nu(ds,dx), \tag{1}$$

where $\mu(t)$ and $\sigma(t)$ are adapted càdlàg processes, $W(t)$ is a standard Brownian motion, ν is an integer-valued random measure counting the jumps in X with compensator $\chi(t)\,dt \otimes F(dx)$, $\chi(t)$ is an adapted càglàd process and F is a measure on $\mathbb{R}$. The stochastic volatility $\sigma(t)$ follows another Itô semimartingale,

$$\sigma^2(t) = \sigma^2(0) + \int_0^t \tilde{\mu}(s)\,ds + \int_0^t \breve{\sigma}(s)\,dW(s) + \int_0^t \tilde{\sigma}(s)\,d\widetilde{W}(s) + \int_0^t \int_{\mathbb{R}} x\,\tilde{\nu}(ds,dx), \tag{2}$$

where $\tilde{\mu}(t)$, $\tilde{\sigma}(t)$ and $\breve{\sigma}(t)$ are adapted càdlàg processes, $\widetilde{W}(t)$ is a standard Brownian motion independent from $W(t)$. Moreover, $\tilde{\nu}$ is the counting jump measure of σ^2 with compensator $\tilde{\chi}(t)\,dt \otimes \tilde{F}(dx)$, where $\tilde{\chi}(t)$ is an adapted càglàd process, and $\tilde{F}$ is a measure on $\mathbb{R}$.

Most models used in applied work are covered by the volatility specification in (2), with the notable exception of volatility models driven by fractional Brownian motion and/or infinite activity jumps. In the Supplementary Appendix, we extend our analysis to cover such much more general volatility specifications.

Our focus is on the calendar effect in volatility. We assume that,

$$E\left(\sigma^2(t)\right) = g\left(t - \lfloor t \rfloor\right),$$

for some positive bounded function g on $[0,1]$ with $g(0) = g(1)$. Hence, the volatility process may be nonstationary due to "calendar" effects. Our goal is to estimate,

$$f(\kappa) = \frac{g(\kappa)}{\int_0^1 g(u)du}, \quad \textit{for } \kappa \in [0,1], \tag{3}$$

which we henceforth refer to as the volatility calendar effect. We define $\eta := \int_0^1 g(u)du$ for later use. The standard approach of modeling calendar effects in volatility is through a decomposition, $\sigma^2(t) = c(t - \lfloor t \rfloor)\,\breve{\sigma}^2(t)$, where $c(t - \lfloor t \rfloor)$ is a deterministic function and $\breve{\sigma}^2(t)$ a stationary process. However, this rules out time variation in the calendar effect, which is documented in Andersen, Thyrsgaard, and Todorov (2019). Our general setup accommodates such time variation.

We now turn to our assumptions, starting with one concerning the existence of moments.

Assumption I. (i) The drift term $\mu(t)$ satisfies $E\left|\mu(t) - \mu(s)\right|^2 \leq C\left|t-s\right|$, for any $s, t \in [0,\infty)$ and some positive constant C that does not depend on s and t.
(ii) $\sup_{t\in\mathbb{R}_+} Ee^{|\mu(t)|} + \sup_{t\in\mathbb{R}_+} Ee^{|\sigma(t)|} + \sup_{t\in\mathbb{R}_+} Ee^{|\chi(t)|} < \infty$. Moreover, $F(\mathbb{R}) < \infty$, $\tilde{F}(\mathbb{R}) < \infty$, $\int_{\mathbb{R}} |x|^2 \tilde{F}(dx) < \infty$, and,

$$\sup_{t\in\mathbb{R}_+} E\left|\tilde{\mu}(t)\right|^8 + \sup_{t\in\mathbb{R}_+} E\left|\breve{\sigma}(t)\right|^8 + \sup_{t\in\mathbb{R}_+} E\left|\tilde{\sigma}(t)\right|^8 + \sup_{t\in\mathbb{R}_+} E\left|\tilde{\chi}(t)\right|^8 < \infty.$$

Assumption I(i) is a weak assumption that is satisfied if the drift itself is an Itô semimartingale. In Assumption I(ii), we assume that jumps in price and volatility are of finite activity. This is imposed mainly for simplicity, but still covers a lot of popular models in applied work. This assumption is further relaxed in the Supplementary Appendix. The moment conditions in Assumption I(ii) are stronger than required, but streamline the exposition of our theoretical results. In this regard, note that $X(t)$ is the log-price. Hence, the existence of exponential moments in Assumption I(ii) is a relatively weak condition that typically is satisfied for exponentially affine models.

Our next assumption concerns stationarity and ergodicity of the volatility process.

Assumption II. For any positive integer i and $\kappa \in [0,1)$, $\sigma^2(i-1+\kappa)$ is a function (depending on κ) of $Y(i-1+\kappa)$, where $\{Y(t)\}_{t\in\mathbb{R}_+}$ is a (multivariate) Markov process, which is stationary, ergodic and α-mixing with coefficient $\alpha_s = O(s^{-q-\iota})$ for some $q > 0$, positive constant ι (which can be arbitrarily close to zero), where for $\mathscr{G}_t = \sigma\left(Y(u), u \leq t\right)$, $\mathscr{G}^t = \sigma\left(Y(u), u \geq t\right)$ and $s > 0$, we denote,

$$\alpha_s = \sup_{t\geq 0} \sup \left\{ |P(A\cap B) - P(A)P(B)| : A \in \mathscr{G}_t, B \in \mathscr{G}^{t+s} \right\}.$$

Assumption II covers a wide range of scenarios including, for instance, the case where the volatility takes the following mixture form,

$$\sigma^2(t) = f_1(t - \lfloor t \rfloor)\,\breve{\sigma}_1^2(t) + f_2(t - \lfloor t \rfloor)\,\breve{\sigma}_2^2(t),$$

for $\breve{\sigma}_1^2(t)$ and $\breve{\sigma}_2^2(t)$ being stationary processes and some deterministic functions $f_1(\kappa)$ and $f_2(\kappa)$, defined on $[0,1]$. This setup can accommodate the situation in which the calendar effect in volatility can vary over time, e.g., as a function of the current level of volatility.

3. Estimation and Inference

3.1. Estimating the Intraday Volatility Calendar Effect

We now present our estimator for the volatility calendar effect function $f(\kappa)$ defined in Equation (3). We assume that log price process $X(t)$ is discretely observed and observation times are equally spaced over $[0,T]$. In each period $[i-1,i]$, $i = 1, 2, \ldots, T$, there are $n+1$ observation times $(t_{i,j})_{0\le j\le n}$ with $i-1 \equiv t_{i,0} < t_{i,1} < \cdots < t_{i,j} < \cdots < t_{i,n} \equiv i$, and,

$$t_{i,j} = i-1+j/n \text{ with } \Delta = t_{i,j}-t_{i,j-1} = 1/n, \quad \text{for } j = 1,2,...,n.$$

Note that in the notation above, $t_{i,n} = t_{i+1,0}$. We further adopt the convention that $t_{i,k} = t_{i-1,n+k}$ for $i \geq 2$ and $-n \leq k \leq 0$. To approximate spot volatilities, using observations within a local window of some time-of-period $\kappa \in [0,1]$, we consider intervals of the form $[t_{i,j_\kappa-\ell}, t_{i,j_\kappa}]$, where ℓ is an integer and,

$$j_\kappa = \lfloor \kappa n \rfloor .$$

We define the high-frequency log returns as follows,

$$\Delta^n_{i,j}X = X(t_{i,j}) - X(t_{i,j-1}), \text{ for } \\ i = 1,2,\ldots,T, \; j = -n+1,\ldots,n \text{ and } t_{i,j-1} \geq 0.$$

In all cases except for $i = 1$ and $\kappa \in [0,\ell\Delta)$, the following quantities are used to approximate the spot volatility $\sigma^2(i-1+\kappa)$ and the mean of the integrated volatility, respectively,

$$\widehat{\sigma}^2_{i,\kappa} = \frac{1}{\ell\Delta}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}(\Delta^n_{i,k}X)^2\, 1_{\{|\Delta^n_{i,k}X|\le u_n\}}, \quad \widehat{\eta} = \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{n}\left(\Delta^n_{i,j}X\right)^2 1_{\{|\Delta^n_{i,j}X|\le u_n\}}, \tag{4}$$

where $u_n = \beta\Delta^{\varpi}$, for some $\varpi \in (0,1/2)$ and constant $\beta > 0$. For initialization, we simply set $\widehat{\sigma}^2_{1,\kappa} \equiv \widehat{\sigma}^2_{1,\ell\Delta}$ for $\kappa \in [0,\ell\Delta)$. Based on these quantities, we propose a general estimator of the volatility calendar effect as follows,

$$\widehat{f}(\kappa) = \frac{1}{T}\sum_{i=1}^{T}\widehat{\sigma}^2_{i,\kappa} \Big/ \widehat{\eta} . \tag{5}$$

We shall establish asymptotic theory that takes place in the Hilbert space $\mathcal{L}^2$,

$$\mathcal{L}^2 = \left\{ k : [0,1] \to \mathbb{R} \,\middle|\, \int_{[0,1]} k(u)^2 du < \infty \right\}.$$

We denote the inner product and the norm on $\mathcal{L}^2$ by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Throughout the article, we adopt the convention that $x_n \asymp y_n$ means $1/C \leq x_n/y_n \leq C$ for some positive constant C.

3.2. Consistency

We start by showing consistency of $\widehat{f}(\kappa)$. Unlike the existing literature, the consistency is established in a functional sense rather than pointwise.

Theorem 1. Assume that Assumptions I(ii) and II with $q = 1$ hold. Let $\ell \to \infty$ with $\ell\Delta \to 0$, and moreover, suppose $T \asymp n^b$ and $\ell \asymp n^c$, for some nonnegative exponents b and c, which satisfy the following condition,

$$b + c > 1 - 4\varpi , \tag{6}$$

where $0 < \varpi < 1/2$. Then, we have, as $n \to \infty$,

$$\| \widehat{f}(\kappa) - f(\kappa) \| \xrightarrow{P} 0 .$$

The lower bound condition (6) is imposed on $b+c$ to ensure that the squared difference between the spot volatility approximation with price-jump-truncation and that based on the continuous part of the price process without price-jump-truncation is asymptotically negligible. This condition is very weak and trivially satisfied, when ϖ is greater than 1/4.

3.3. Convergence in Distribution

We require additional notation to state our formal functional CLT result. First, we define,

$$A_i(\kappa) = \sigma^2(i-1+\kappa) - f(\kappa)\int_{i-1}^{i}\sigma^2(t)\,dt, \\ \kappa \in [0,1], \; i = 1,2,\ldots,T. \tag{7}$$

We then set,

$$C(\kappa,\kappa') = 1/\eta^2 \sum_{h=-\infty}^{\infty}\phi_{\kappa,\kappa'}(h), \quad \kappa,\kappa' \in [0,1], \tag{8}$$

where $\eta = \int_0^1 g(u)du$, $\phi_{\kappa,\kappa'}(h) = Cov\left(A_1(\kappa), A_{1+h}(\kappa')\right)$, if h is a nonnegative integer; and $\phi_{\kappa,\kappa'}(h) = \phi_{\kappa',\kappa}(-h)$, if h is a negative integer.

Theorem 2. Suppose $\ell \to \infty$ with $\ell\Delta \to 0$ and Assumptions I and II with $q = 3$ hold. Moreover, let $T \asymp n^b$ and $\ell \asymp n^c$, for some nonnegative exponents b and c, which satisfy the following conditions,

$$0 < b < 4\varpi \quad \text{and} \quad 1 - 4\varpi < c < 1 - b/2, \tag{9}$$

where $0 < \varpi < 1/2$. Then, as $n \to \infty$,

$$\sqrt{T}\left(\widehat{f}(\kappa) - f(\kappa)\right) \xrightarrow{d} \mathcal{G}_{\mathcal{K}} \quad \text{in} \quad \mathcal{L}^2,$$

where $\mathcal{G}_{\mathcal{K}}$ is an $\mathcal{L}^2$-valued zero-mean Gaussian process with covariance operator $\mathcal{K}$ defined through the kernel $C(\kappa,\kappa')$ in (8) as follows,

$$\mathcal{K}\,y(\kappa') = \int_{[0,1]} C(\kappa,\kappa')\,y(\kappa)\,d\kappa, \qquad \forall y \in \mathcal{L}^2 .$$

The convergence rate for $\widehat{f}(\kappa)$ is naturally determined by the time span of the data. Condition (9) imposes restrictions on the asymptotic order of n relative to T. In evaluating the severity of this condition, we recall that ϖ – determining the truncation threshold for removing price jumps – is optimally set to a value close to 1/2, see e.g., Figueroa-López and Mancini (2019). In this case, b may take any value in $(0,2)$, implying Theorem 2 imposes very weak restrictions on the rate at which we sample intraday relative to the time span of the data. For example, n may grow at a rate slower or faster than T. This robustness to the size of n and T is contrary to related joint in-fill and long-span asymptotic results, which require $T/n \to 0$, see e.g., Todorov (2009). We discuss the optimal choice of ℓ in Section 5.

The covariance kernel $C(\kappa,\kappa')$ includes autocovariances of all lags in order to accommodate the time-series persistence in volatility. We note, however, that the asymptotic variance of $\widehat{f}(\kappa)$ is determined by that of sample averages of $A_i(\kappa)$, which only depend on *changes* in volatility over unit time intervals where,

$$\widehat{\sigma}^{2,c}_{i,\kappa} := \frac{1}{\ell\Delta}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(\Delta^n_{i,k}X^c\right)^2 \quad \text{and} \quad \widehat{\eta}^c := \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{n}\left(\Delta^n_{i,j}X^c\right)^2.$$

Throughout the proofs, C denotes a generic positive constant and $\varsigma > 0$ is an arbitrarily small number. Both may change value from line to line.

Furthermore, we will use the following notation throughout the proofs below,

$$\begin{cases} \zeta_1(\kappa) := \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\int_{t_{i,k-1}}^{t_{i,k}}\mu(t)dt\right]^2, \\ \zeta_2(\kappa) := \frac{2}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\mu(t)dt\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t), \\ \zeta_3(\kappa) := \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left\{\left[\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right]^2 - \int_{t_{i,k-1}}^{t_{i,k}}\sigma^2(t)dt\right\}, \\ \zeta_4(\kappa) := \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\sigma^2(t) - \sigma^2(i-1+\kappa)\right]dt, \\ \zeta_5(\kappa) := \frac{1}{T}\sum_{i=1}^{T}\sigma^2(i-1+\kappa) - \frac{f(\kappa)}{T}\sum_{i=1}^{T}\int_{i-1}^{i}\sigma^2(t)dt, \quad \text{and} \\ \zeta_6(\kappa) := f(\kappa)\left[\frac{1}{T}\sum_{i=1}^{T}\int_{i-1}^{i}\sigma^2(t)dt - \widehat{\eta}^c\right], \end{cases} \tag{C.1}$$

where $\kappa \in [0,1]$ and note that, by the definition of $\widehat{\sigma}^2_{1,\kappa}$ for $\kappa \in [0,\ell\Delta)$, the inner summation variable k of the first four terms always takes values from 1 to ℓ when $\kappa \in [0,\ell\Delta)$ and the outer summation variable $i = 1$ (i.e., j_κ is fixed at ℓ in this case). Recalling the definition of $A_i(\kappa)$ in equation (7), one readily sees that $\zeta_5(\kappa) = \sum_{i=1}^{T}A_i(\kappa)/T$.

The following lemma will be repeatedly used in the proofs of Theorems 1 and 2.

Lemma 11. *Suppose that Assumption I(ii) holds. Then,*

$$E\left|\zeta_1(\kappa)\right|^m \leq \frac{C}{n^m}, \quad E\left|\zeta_2(\kappa)\right|^m \leq \frac{C}{n^{m/2}} \quad \textit{and} \quad E\left|\zeta_3(\kappa)\right|^m \leq \frac{C}{(T\ell)^{m/2}}$$

for any $m \geq 2$ and any $\kappa \in [0,1]$.