Supplementary Appendix to "Intraday Periodic Volatility Curves"

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Abstract

This document consists of three parts. Section A presents a Monte Carlo study. Section B contains additional theoretical results that compliment and extend the ones in the main text. Proofs of the theorems and corollaries in the main article as well as those in Section B of this Appendix are provided in Section C.

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Appendix A Monte Carlo Simulations

This section explores the performance of the general calendar-effect estimator (5) through simulation experiments. The simulation setting is described in Section A.1, while Sections A.2, A.3 and A.4 are devoted to finite-sample analysis of the functional theory in Theorem 6, the test for nonstationarity in the intraday volatility curve over time, and the pointwise theory in Corollary 7, respectively.

A.1 The Simulation Setting

The log-price process X, the volatility process σ^2, the calendar effect f, and the stationary component $\breve{\sigma}^2$ of the volatility process σ^2 are given, respectively, by,

$$\begin{cases} X(t) \;=\; X(0) \,+\, \int_0^t \sigma(s)\, dW(s) \,+\, \sum_{j=1}^{\widetilde{N}(t)} Z_j, \qquad \sigma^2(t) \;=\; f\left(t - \lfloor t \rfloor\right) \breve{\sigma}^2(t), \\ f(t) \;=\; 3\left(t - \frac{1}{2}\right)^2 \,+\, \frac{3}{4}, \qquad t \in [0,1], \qquad \breve{\sigma}^2(t) = \breve{\sigma}_1^2(t) \,+\, \breve{\sigma}_2^2(t), \\ \breve{\sigma}_1^2(t) \;=\; \breve{\sigma}_1^2(0) \,+\, \int_0^t \lambda\left(\tilde{\eta} - \breve{\sigma}_1^2(s)\right) ds \,+\, \int_0^t \xi\, \breve{\sigma}_1(s)\, d\widetilde{W}(s), \\ \breve{\sigma}_2^2(t) \;=\; \exp(-\tilde{\lambda} t)\, \breve{\sigma}_2^2(0) \,+\, \int_0^t \exp\{-\tilde{\lambda}(t-s)\}\, dz(\tilde{\lambda} s), \end{cases} \tag{A.1}$$

where $\widetilde{N}(t)$ is a homogeneous Poisson process with constant intensity λ_J, $\{Z_j\}_{j \geq 1}$ is an iid sequence of $N(0, \sigma_J^2)$ distributed random variables, the quadratic covariation is given by $[W, \widetilde{W}](t) = \rho\, t$, and z is a nonnegative increasing Lévy process such that the (stationary) marginal distribution of $\breve{\sigma}_2^2$ is $\Gamma(\nu_{\mathrm{OU}},\, 1/\alpha_{\mathrm{OU}})$. In the simulation, we exploit the specification provided by [1], fixing the model parameters as follows,

$$(X(0), \lambda, \tilde{\eta}, \xi, \lambda_J, \sigma_J, \rho, \tilde{\lambda}, \nu_{\mathrm{OU}}, \alpha_{\mathrm{OU}}) = (1, 4, 0.4068, 1.8, 0.19, 0.9654, -0.5, 0.6930, 1, 0.1).$$

Throughout, we set $n = 2{,}730$, corresponding to a sampling frequency of 30 seconds across 22.75 hours, mimicking the trading day for the e-mini S&P 500 futures in our empirical analysis. For each simulation trial, we generate a series of 1,500-day thirty-second prices. The following results are based on 1,000 trajectories with $T \leq 1{,}500$. In truncating the price jumps, we employ the time-varying threshold $u_n = 3\sqrt{BV_i \wedge RV_i}\,\Delta^{3/8}$ with,

$$BV_i = \frac{\pi}{2}\sum_{j=2}^{n}\left|\Delta_{i,j-1}^n X\right|\left|\Delta_{i,j}^n X\right| \qquad \text{and} \qquad RV_i = \sum_{j=1}^{n}\left(\Delta_{i,j}^n X\right)^2.$$

A.2 Finite-Sample Evidence for Functional Inference

This section provides a simulation experiment to explore the workings of the feasible (functional) central limit theorem in the L^2 metric, i.e., Corollary 3 and Theorem 6. Figure 1 depicts the empirical distribution of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ for $\ell = 10$ and $T = 1{,}500$ based on 1,000 trials. Because one can not explicitly evaluate the integral $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$, we approximate the integral using a Riemann sum with the interval $[0, 1]$ partitioned into 100 equidistant subintervals.

We compare the empirical distribution, obtained as indicated above, with the limiting distribution of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ (and $\mathcal{Z}$ in Corollary 3). As discussed in Section 4, the distribution of $\mathcal{Z}$ may be approximated by that of $\widehat{\mathcal{Z}}$ in equation (21), with the latter obtained through Monte Carlo simulation. This involves computing eigenvalues of the limiting covariance matrix estimator $(\widehat{C}(\kappa_i, \kappa_j))_{1\leq i,j\leq 100}$ with the entries defined in equation (18). In this study, we use the average limiting covariance matrix estimates over 1,000 trajectories rather than relying on a single trajectory to compute the eigenvalues associated with equation (21) and, consistent with the properties of the limiting variable, we retain only the terms featuring positive eigenvalues. Figure 1 also displays the limiting distribution of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ obtained in this manner for $L_n = 7$ (recall, L_n is defined below equation (18)). Note that, in Theorem 6, we require $L_n \asymp n^{\varrho}$ for a strictly positive ϱ satisfying equation (20). In our simulations, $\varpi = 3/8$. If one takes $b \approx 9/10$ and $c \approx 1/2$ for $T = 1{,}500$, $\ell = 10$, and $n = 2{,}730$, then condition (20) reduces to $\varrho < 1/4$. For simplicity, we implement $L_n = \lfloor \min\{T^{1/2}, n^{1/4}\} \rfloor$ in all our numerical illustrations, implying $L_n = 7$ for T =1,500 and $n = 2{,}730$. Figure 1 demonstrates that the limiting distribution (red curve) approximates the empirical distribution (histogram) of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ quite well, corroborating the theory developed in Corollary 3 and Theorem 6.

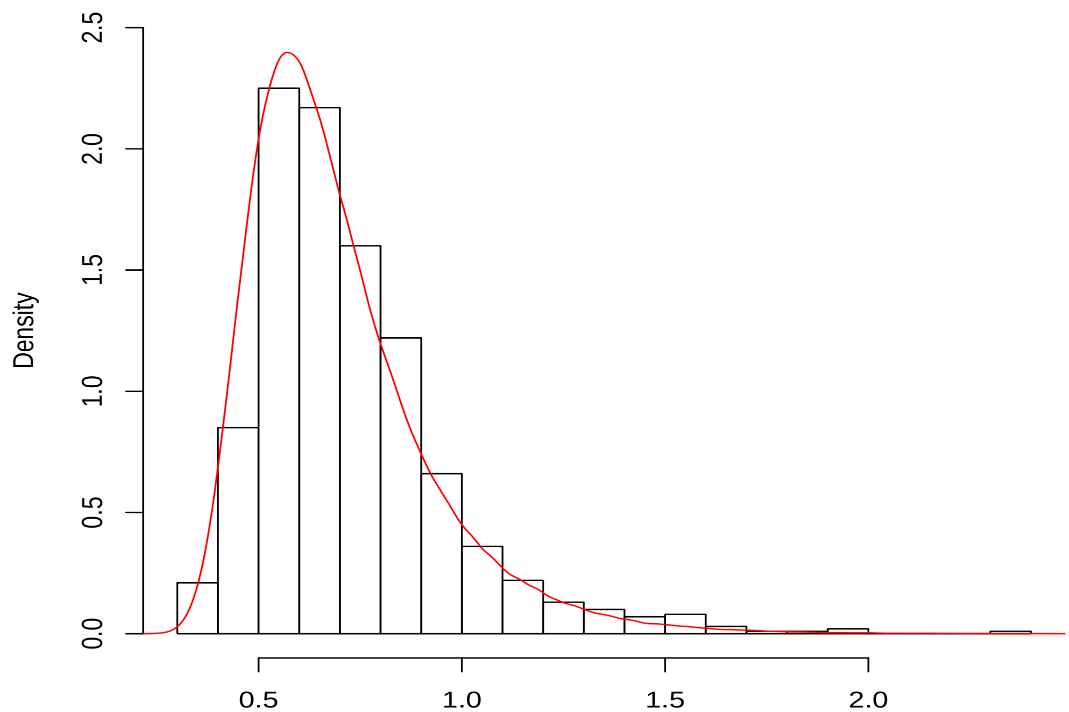

Figure 1. Approximating the distribution of the limiting variable $\mathcal{Z}$. The histogram represents the empirical distribution of $T \parallel \widehat{f}(\kappa) - f(\kappa) \parallel^2$ with $\ell = 10$ based on 1,000 simulation trials. The red curve indicates the density of $\widehat{\mathcal{Z}}$ in equation (21) obtained through Monte Carlo, as described in the main text.

A.3 Finite-Sample Test Performance

In this section, we seek to investigate the performance of our test (10) for a shift in the functional characterizing the average intraday volatility pattern for two non-overlapping periods P and P'. To ensure we capture the test performance within an empirically relevant setting, we calibrate the calendar effect to volatility curves obtained for two subsamples in our empirical study of the e-mini S&P 500 futures contract in Section 6. Specifically, we rely on the estimated volatility curves for periods covering 2005–2010 and 2015–2020, with the number of trading days equaling $T = 991$ and $T' = 1248$, respectively. We generate 1,000 simulated samples from model (A.1), but with the intraday volatility pattern modified to equal one of those estimated for the above subsamples. The shape of the curves is provided by the far left and right displays in the bottom panel of Figure 1 in Section 6.2.

The formal setup under the null hypothesis is $H_0 : f_P = f_{\mathrm{P}'} = f_{05-10}$, and it takes the form $H_A : f_{\mathrm{P}} = f_{05-10}$ and $f_{\mathrm{P}'} = f_{15-20}$ under the alternative, where f_{05-10} and f_{15-20} denote the estimated calendar effect functions for the e-mini futures over 2005-2010 and 2015-2020. That is, under the null hypothesis, we generate samples P and P' that all incorporate the volatility curve for 2005–2010. We then compute the test statistic, $T \parallel \widehat{f}_{\mathrm{P}}(\kappa) - \widehat{f}_{\mathrm{P}'}(\kappa) \parallel^2$, using a Riemann sum over the same grid employed for generating the volatility curve (with $\ell = 10$), i.e., the interval $[0, 1]$ is partitioned into 100 equal subintervals. We perform feasible inference following the procedure outlined in Section 4. Here again we use the average limiting covariance matrix estimates over 1,000 trajectories rather than relying on a single trajectory to compute the eigenvalues associated with equation (22) and, consistent with the properties of the limiting variable, we retain only the terms featuring positive eigenvalues.

Table 1 reports empirical rejection rates for the test under the null hypothesis at significance levels 1%, 5% and 10%. The top row of the table shows that the test is well sized.

Under the alternative hypothesis, the data are generated with different underlying volatility curves. Hence, the universal rejections reported in the second row of Table 1 reflect high power of the test in detecting the discrepancy between the two functions governing the respective intraday volatility patterns.

Table 1. Test size and power. The null hypothesis is $H_0 : f_P = f_{P'} = f_{05-10}$, and the alternative is $H_A : f_{\mathrm{P}} = f_{05-10}$ and $f_{\mathrm{P}'} = f_{15-20}$. The test statistic is $T \parallel \widehat{f}_{\mathrm{P}}(\kappa) - \widehat{f}_{\mathrm{P}'}(\kappa) \parallel^2$, whose realization is computed using a Riemann sum over the same grid points employed in generating the calendar effects ($\ell = 10$). The limiting distribution of the test statistic under the null is approximated as described in Section 4. The table reports rejection rates for the test at significance levels 1%, 5% and 10% using 1000 trials.

Significance level	1%	5%	10%
Size under H_0	0.009	0.059	0.116
Power under H_A	1.000	1.000	1.000

A.4 Finite-Sample Evidence for Pointwise Inference

Figure 2 illustrates the pointwise feasible central limit theorem of Corollary 7, where we depict the empirical distribution of the standardized $\widehat{f}(\kappa)$ and associated Normal Q-Q plot for different values of κ for $\ell = 10$, n =2,730, and T = 1,500. The data are generated from model (A.1). Note that $\widehat{f}(\kappa)$ is standardized according to Corollary 7 with $L_n = 7$. It is apparent that the limiting distribution approximates the empirical distribution well.

We next explore how different values for ℓ and T affect the performance of the calendar-effect estimator in finite samples. Without loss of generality, we fix $\kappa = 0.2$. Table 2 reports the finite-sample bias, standard deviation (StDev), and root mean squared error (RMSE)

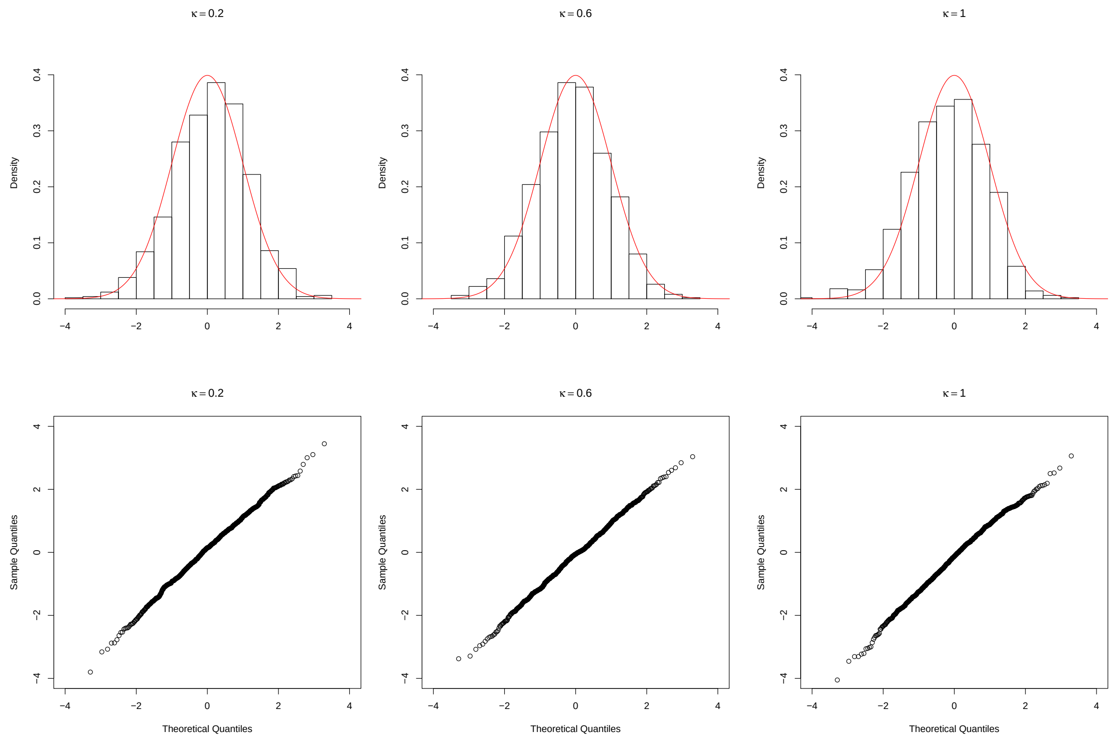

Figure 2. Empirical distribution of the standardized calendar-effect estimator. The empirical distribution of the standardized calendar-effect estimator $\widehat{f}(\kappa)$ and associated Normal Q-Q plots for different values of κ and $\ell = 10$ with data generated from model (A.1) by 1,000 simulation trials. $\widehat{f}(\kappa)$ is standardized according to Corollary 7 with $L_n = 7$.

of the estimator $\widehat{f}(\kappa)$ across different combinations of ℓ and T based on 1,000 replications. Two main conclusions emerge. First, when ℓ is fixed, larger T leads to a smaller standard deviation and root mean squared error. Note that the convergence rate of the calendar-effect estimator is $\sqrt{T}$. Second, for fixed T, the rows with ℓ ranging from 5 to 30 show that a large value of ℓ leads to a larger bias and smaller standard deviation. This finite-sample bias-variance tradeoff is also evident from the associated RMSE values. This is in line with our theoretical analysis in Section 5.

Table 2. Finite-sample performance of the calendar-effect estimator. Finite-sample statistics for the $\widehat{f}(\kappa)$ estimator with $\kappa = 0.2$. The data are generated from model (A.1) with $n = 2730$ over 1,000 trials. "Bias", "StDev" and "RMSE" refer to the bias, standard deviation and root mean squared error. The true value of $f(0.2)$ is 1.02.

ℓ	$T = 500$			$T = 1000$			$T = 1500$		
	Bias	StDev	RMSE	Bias	StDev	RMSE	Bias	StDev	RMSE
1	-0.0036	0.0873	0.0873	-0.0034	0.0630	0.0631	-0.0006	0.0497	0.0497
5	0.0022	0.0461	0.0461	0.0010	0.0325	0.0325	0.0016	0.0270	0.0270
10	0.0042	0.0368	0.0370	0.0023	0.0268	0.0268	0.0025	0.0222	0.0223
15	0.0047	0.0338	0.0341	0.0038	0.0247	0.0250	0.0042	0.0204	0.0208
20	0.0070	0.0316	0.0324	0.0060	0.0229	0.0236	0.0061	0.0189	0.0199
25	0.0084	0.0304	0.0315	0.0076	0.0221	0.0233	0.0078	0.0182	0.0198
30	0.0100	0.0300	0.0316	0.0093	0.0217	0.0236	0.0094	0.0179	0.0202

Appendix B Additional Theoretical Results

B.1 Accommodating rough volatility and infinite jump activity

This section provides an extension to our functional CLT for $\widehat{f}(\kappa)$ accommodating more general volatility and price jump settings. For the price process, we retain the setup of the main text, except that we replace the finite activity jump condition $F(\mathbb{R}) < \infty$ with,

$$\int_{\mathbb{R}} (|x|^r \vee |x|) F(dx) < \infty, \quad \text{for some } r \in [0, 1]. \tag{B.1}$$

We then define the jump component of the price process as follows,

$$X^J(t) \ := \ X(t) - X^c(t) \ = \int_0^t \int_{\mathbb{R}} x\nu(ds, dx), \tag{B.2}$$

where X^c is the continuous part of the latent price process X, given by,

$$X^c(t) \ := \ X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s).$$

Condition (B.1) allows for $X^J(t)$ to be of infinite activity, with the parameter r controlling the concentration of small jumps.

Turning to the latent volatility process, we impose the following generic assumption,

$$E|\sigma^2(t) - \sigma^2(s)|^2 \leq C|t-s|^{2H}, \tag{B.3}$$

for any $s, t > 0$, some $0 < H \leq 1$, and a generic constant C. If $\sigma^2(t)$ is an Itô semi-martingale, as stipulated in the main text, the above condition applies with $H = 1/2$. The assumption also allows for general volatility jump processes of infinite activity. When (B.3) holds with $H < 1/2$, our setup accommodates the so-called rough volatility models in which volatility is driven by fractional Brownian motion, see e.g., [4] and [12].

The next theorem presents the CLT for the calendar effect estimator under the above extended setup. It demonstrates explicitly how the jump activity index r and the volatility roughness index H affect the rate at which T, and hence ℓ, diverges.

Theorem 8. *Assume the same setup and assumptions as in Theorem 2, except for F being subject to condition* (B.1) *and $\sigma^2(t)$ being a rough process satisfying condition* (B.3)*. Let $T \asymp n^b$ and $\ell \asymp n^c$ for some nonnegative exponents b and c subject to the conditions,*

$$0 < b < \min\{2-2\varpi, 1+(1-r)\varpi, (4-2r)\varpi\} \quad \textit{and} \quad 1-\varpi(4-r) < c < 1-b/(2H), \tag{B.4}$$

where $0 < \varpi < 1/2$. *Then, as* $n \to \infty$,

$$\sqrt{T}\left(\widehat{f}(\kappa) - f(\kappa)\right) \xrightarrow{d} \mathcal{G}_{\mathcal{K}} \quad \textit{in} \quad \mathcal{L}^2,$$

where $\mathcal{G}_{\mathcal{K}}$ *is an* $\mathcal{L}^2$*-valued zero-mean Gaussian process with covariance operator* $\mathcal{K}$ *defined through the kernel* $C(\kappa, \kappa')$ *in equation* (8) *as follows,*

$$\mathcal{K}\, y(\kappa') \;=\; \int_{[0,1]} C(\kappa,\kappa')\, y(\kappa)\, d\kappa, \qquad \forall y \in \mathcal{L}^2\,.$$

We close this section by investigating how the price jump activity index r and volatility roughness index H affect the bias-variance tradeoff for ℓ. The difference between Theorems 2 and 8 is that the feasible regions of b and c given by condition (9) is replaced with that given by condition (B.4). The ℓ-related terms II and III in Section 5 now have orders,

$$\text{term II} = O_P\left(\frac{1}{n^{(b+c)/2}}\right) \quad \text{and} \quad \text{term III} = O_P\left(\frac{1}{n^{(1-c)H}}\right).$$

We now provide the optimal choice of c, c_{opt}, for each configuration of ϖ, r, H and b, which minimizes the order of the sum of terms II and III. It is readily established that, for each configuration of ϖ, r and H, the feasible values of b are given by the interval $(0, b_U)$, where,

$$b_U := \min\{2-2\varpi, 1+(1-r)\varpi, (4-2r)\varpi, 2H\};$$

and the feasible values of c consist of the interval (c_L, c_U), where.

$$c_L := \max\{1-\varpi(4-r), 0\} \quad \text{and} \quad c_U := 1 - \frac{b}{2H}.$$

posed uniform confidence band $\widehat{B}_{E_c}$ via simulations. The performance of $\widehat{B}_{E_c}$ is quite robust for $50 \leq J \leq 100$, so we provide results for $J = 50$ only. Based on 1000 trials, the simultaneous coverage rate of $\widehat{B}_{E_c}$ with 95% nominal level is 100%. By contrast, the confidence bounds, constructed using the pointwise theory in Corollary 7 with a 95% nominal level, only have a simultaneous coverage rate of 2%. The left panel of Figure 3 displays pointwise coverage rates for both the uniform confidence band and the pointwise confidence bounds. We also provide the calendar effect function estimate along with 95% confidence bounds for a particular simulation trial in the right panel of Figure 3. It illustrates the moderately wider width of the uniform confidence band $\widehat{B}_{E_c}$ relative to the pointwise bounds.

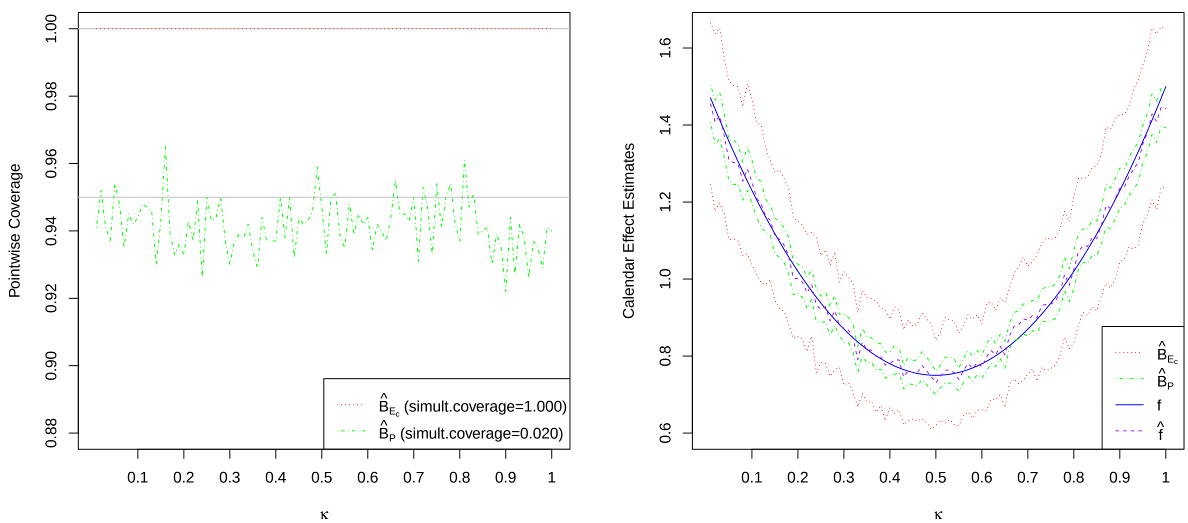

Figure 3. Comparison between $\widehat{B}_P$ and $\widehat{B}_{E_c}$ based on 1000 simulation trials. The results rely on data generated from the setup in Section A.1. $\widehat{B}_P$ indicates the pointwise 95% confidence intervals constructed using Corollary 7 with $L_n = 7$. $\widehat{B}_{E_c}$ refers to the simultaneous 95% confidence bands constructed using the theory in Section B.2. **Left panel**: Local empirical coverage rates. As indicated in the legend, the *simultaneous coverage rates* for $\widehat{B}_P$ and $\widehat{B}_{E_c}$ are 2% and 100%, respectively. **Right panel**: Calendar effect estimates together with 95% confidence bounds for a particular sample path.

An application of $\widehat{B}_{E_c}$ to the e-mini over the subsamples is presented in Figure 4. Not surprisingly, the simultaneous confidence bounds are wider than the pointwise ones. In practice, $\widehat{B}_{E_c}$ may occasionally produce negative lower bounds for real data, e.g., the subsamples covering periods 2005–2010 and 2015–2020, especially the period 2015–2020. If this is the case, we suggest using a log transformation and functional Delta method to ensure positiveness. To be precise, under mild conditions, by Theorem 2 and the functional Delta method, one would obtain

$$\sqrt{T}\left(\log\left(\widehat{f}(\kappa)\right) - \log\left(f(\kappa)\right)\right) \xrightarrow{d} \mathcal{G}_{\mathcal{K}_{\log}} \quad \text{in } \mathcal{L}^2,$$

where $\mathcal{G}_{\mathcal{K}_{\log}}$ is an $\mathcal{L}^2$-valued zero-mean Gaussian process with covariance operator $\mathcal{K}_{\log}$ that has $C(\kappa,\kappa')/(f(\kappa)f(\kappa'))$ as its kernel function. Therefore, the previous method of constructing uniform confidence bounds of $f(\kappa)$ applies straightforwardly for constructing that of $\log(f(\kappa))$. The simultaneous confidence bounds of $f(\kappa)$ then readily follows by applying the natural exponential function to both the upper and lower confidence bounds of $\log(f(\kappa))$. Figure 4 also displays simultaneous confidence bounds thus obtained for the three subsamples.

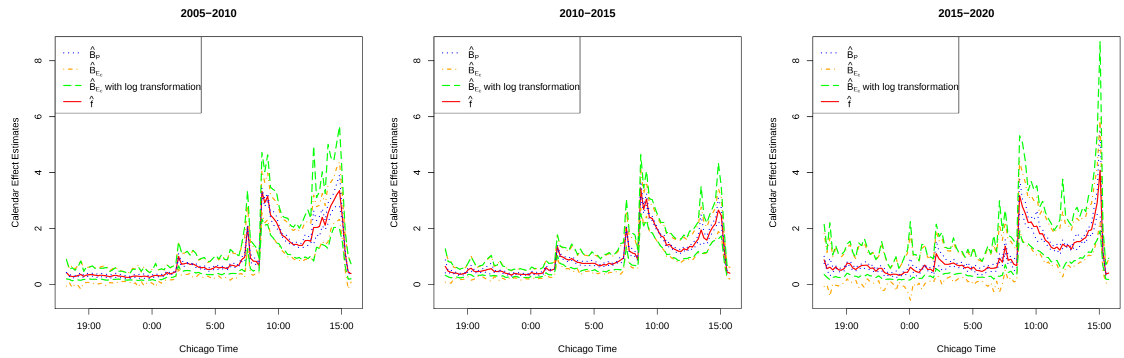

Figure 4. Intraday volatility curves for the e-mini over subsamples. $\widehat{B}_P$ indicates the pointwise 95% confidence intervals constructed using Corollary 7 with $L_n = 7$. $\widehat{B}_{E_c}$ indicates the simultaneous 95% confidence bands constructed as in Section B.2. $\widehat{B}_{E_c}$ with log transformation indicates the simultaneous confidence bands constructed based on log transformation which ensures positiveness of the lower confidence bounds.

Appendix C Proofs

Throughout this section, without further mention, we shall focus on $\kappa \in [\ell\Delta, 1]$ in the derivations of upper bounds for moments of various terms involving $\widehat{\sigma}^2_{i,\kappa}$. The same results and proofs as that for $\kappa \in [\ell\Delta, 1]$ obviously apply to the case $\kappa \in [0, \ell\Delta)$.

Recall that X^c is the continuous part of the latent price process X, defined as,

$$X^c(t) := X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s).$$

In the proofs below, we rely on the calendar-effect estimator $\widehat{f}^c(\kappa)$ for X^c given by,

$$\widehat{f}^c(\kappa) := \frac{1}{T}\sum_{i=1}^{T} \widehat{\sigma}^{2,c}_{i,\kappa} \Big/ \widehat{\eta}^c \ ,$$

where,

$$\widehat{\sigma}_{i,\kappa}^{2,c} := \frac{1}{\ell\Delta} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \left(\Delta_{i,k}^n X^c\right)^2 \quad \text{and} \quad \widehat{\eta}^c := \frac{1}{T} \sum_{i=1}^{T} \sum_{j=1}^{n} \left(\Delta_{i,j}^n X^c\right)^2.$$

Throughout the proofs, C denotes a generic positive constant and $\varsigma > 0$ is an arbitrarily small number. Both may change value from line to line.

Furthermore, we will use the following notation throughout the proofs below,

$$\begin{cases} \zeta_1(\kappa) := \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \left[\int_{t_{i,k-1}}^{t_{i,k}} \mu(t)dt\right]^2, \\ \zeta_2(\kappa) := \frac{2}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \mu(t)dt \int_{t_{i,k-1}}^{t_{i,k}} \sigma(t)dW(t), \\ \zeta_3(\kappa) := \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \left\{\left[\int_{t_{i,k-1}}^{t_{i,k}} \sigma(t)dW(t)\right]^2 - \int_{t_{i,k-1}}^{t_{i,k}} \sigma^2(t)dt\right\}, \\ \zeta_4(\kappa) := \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \left[\sigma^2(t) - \sigma^2(i-1+\kappa)\right] dt, \\ \zeta_5(\kappa) := \frac{1}{T} \sum_{i=1}^{T} \sigma^2(i-1+\kappa) - \frac{f(\kappa)}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt, \quad \text{and} \\ \zeta_6(\kappa) := f(\kappa) \left[\frac{1}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt - \widehat{\eta}^c\right], \end{cases} \tag{C.1}$$

where $\kappa \in [0,1]$ and note that, by the definition of $\widehat{\sigma}_{1,\kappa}^2$ for $\kappa \in [0, \ell\Delta)$, the inner summation variable k of the first four terms always takes values from 1 to ℓ when $\kappa \in [0, \ell\Delta)$ and the outer summation variable $i = 1$ (i.e., j_κ is fixed at ℓ in this case). Recalling the definition of $A_i(\kappa)$ in equation (7), one readily sees that $\zeta_5(\kappa) = \sum_{i=1}^{T} A_i(\kappa)/T$.

The following lemma will be repeatedly used in the proofs of Theorems 1 and 2.

Lemma 11. *Suppose that Assumption I(ii) holds. Then,*

$$E\left|\zeta_1(\kappa)\right|^m \leq \frac{C}{n^m}, \quad E\left|\zeta_2(\kappa)\right|^m \leq \frac{C}{n^{m/2}} \quad \textit{and} \quad E\left|\zeta_3(\kappa)\right|^m \leq \frac{C}{(T\ell)^{m/2}}$$

for any $m \geq 2$ and any $\kappa \in [0,1]$.

Proof of Lemma 11. For term $\zeta_1(\kappa)$, we have,

$$
\begin{aligned}
E\left|\zeta_1(\kappa)\right|^m &\leq E\left[\frac{1}{T\ell\Delta^m}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left(\int_{t_{i,k-1}}^{t_{i,k}}\mu(t)dt\right)^{2m}\right] \\
&\leq \frac{\Delta^{m-1}}{T\ell}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}E\left(\mu(t)\right)^{2m}dt \leq \frac{C}{n^m},
\end{aligned}
$$

where the first two inequalities follow from Jensen's inequality, and the last inequality is implied by Assumption I(ii).

For the term $\zeta_2(\kappa)$, by Cauchy-Schwarz inequality, Jensen's inequality, Itô isometry and Assumption I(ii), we obtain,

$$
\begin{aligned}
E\left|\zeta_2(\kappa)\right|^m &\leq \frac{1}{T\ell\Delta^m}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}E\left|\int_{t_{i,k-1}}^{t_{i,k}}\mu(t)dt\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right|^m \\
&\leq \frac{1}{T\ell\Delta^m}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[E\left|\int_{t_{i,k-1}}^{t_{i,k}}\mu(t)dt\right|^{2m}E\left|\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right|^{2m}\right]^{1/2} \\
&\leq \frac{\Delta^{m/2-1}}{T\ell}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\int_{t_{i,k-1}}^{t_{i,k}}E\left|\mu(t)\right|^{2m}dt\int_{t_{i,k-1}}^{t_{i,k}}E\left(\sigma(t)\right)^{2m}dt\right]^{1/2} \leq \frac{C}{n^{m/2}}.
\end{aligned}
$$

We now deal with term $\zeta_3(\kappa)$. We first define the following continuous martingale,

$$
M_1(t) := \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left\{\left[\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\sigma(s)dW(s)\right]^2-\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\sigma^2(s)ds\right\}
$$

on the interval $[0,T]$. One readily sees $\zeta_3(\kappa)=M_1(T)$. The quadratic variation of $M_1(t)$ is takes the form,

$$[M_1, M_1](t) = \frac{4}{(T\ell\Delta)^2} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}} \left(\sigma(u) \int_{t_{i,k-1}}^{u} \sigma(s) dW(s) \right)^2 du,$$

following the method in Section 2.3.3 on page 136 of [9]. Then we obtain,

$$\begin{aligned}
E\left|\zeta_3(\kappa)\right|^m &\leq CE\left[\frac{4}{(T\ell\Delta)^2} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \left(\sigma(t)\int_{t_{i,k-1}}^{t} \sigma(s)dW(s)\right)^2 dt\right]^{m/2} \\
&\leq \frac{C}{(T\ell)^{m/2+1}\Delta^m} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} E\left[\int_{t_{i,k-1}}^{t_{i,k}} \left(\sigma(t)\int_{t_{i,k-1}}^{t} \sigma(s)dW(s)\right)^2 dt\right]^{m/2} \\
&\leq \frac{C}{(T\ell\Delta)^{m/2+1}} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} E\left|\sigma(t)\int_{t_{i,k-1}}^{t} \sigma(s)dW(s)\right|^m dt \\
&\leq \frac{C}{(T\ell\Delta)^{m/2+1}} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \left[E\left(\int_{t_{i,k-1}}^{t} \sigma^2(s)ds\right)^m E\left(\sigma^{2m}(t)\right)\right]^{1/2} dt \\
&\leq \frac{C}{(T\ell\Delta)^{m/2+1}} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} \left[\Delta^{m-1}\int_{t_{i,k-1}}^{t} E\left(\sigma^{2m}(s)\right) ds\right]^{1/2} dt \leq \frac{C}{(T\ell)^{m/2}},
\end{aligned}$$

where the second and third inequalities follow from Jensen's inequality, the fourth inequality follows from the Cauchy-Schwarz inequality and the Burkholder-Davis-Gundy inequality, and the last inequality follows from Jensen's inequality and Assumption I(ii). $\square$

In what follows, we use the shorthand notation

$$N(t) := \int_0^t \int_{\mathbb{R}} \nu(ds, dx) \quad \text{and} \quad \Delta_{i,j}^n N := \int_{t_{i,j-1}}^{t_{i,j}} \int_{\mathbb{R}} \nu(dt, dx).$$

The following two lemmas are repeatedly used in the proofs of Theorems 1, 2 and 6.

Lemma 12. *Let j be a positive integer. Suppose that $i_m \in \{1,2,\ldots,T\}$ and $k_m \in \{1,2,\ldots,n\}$ for $m \in \{1,2,\ldots,j\}$, and without loss of generality that $0 < t_{i_1,k_1} < t_{i_2,k_2} < \cdots < t_{i_j,k_j} \le T$. Then, under Assumption I(ii), we have that $E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}\right) \le C\Delta$ and,*

$$E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}1_{\{\Delta^n_{i_2,k_2}N>0\}}\cdots 1_{\{\Delta^n_{i_j,k_j}N>0\}}\right) \le C\Delta^{j-\varsigma},$$

for $j \ge 2$ and arbitrarily small $\varsigma > 0$.

Proof of Lemma 12. First note that $N(t) = \int_0^t \int_{\mathbb{R}} \nu(dt,dx)$ is a counting process with intensity $\chi(t)F(\mathbb{R})$. Then $N(t) - \int_0^t \chi(s)dsF(\mathbb{R})$ is a martingale by Assumption I(ii).

When $j = 1$, the result follows immediately from,

$$E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}\right) \le E\left(\Delta^n_{i_1,k_1}N\right) = F(\mathbb{R})E\int_{t_{i_1,k_1-1}}^{t_{i_1,k_1}} \chi(s)ds \le C\Delta.$$

When $j = 2$, we have that, for any $\omega > 1$,

$$\begin{aligned}
&E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}1_{\{\Delta^n_{i_2,k_2}N>0\}}\right) = E\left[1_{\{\Delta^n_{i_1,k_1}N>0\}}E_{t_{i_2,k_2-1}}\left(1_{\{\Delta^n_{i_2,k_2}N>0\}}\right)\right] \\
&\le CE\left[1_{\{\Delta^n_{i_1,k_1}N>0\}}E_{t_{i_2,k_2-1}}\left(\int_{t_{i_2,k_2-1}}^{t_{i_2,k_2}} \chi(s)ds\right)\right] \\
&= CE\left\{1_{\{\Delta^n_{i_1,k_1}N>0\}}\int_{t_{i_2,k_2-1}}^{t_{i_2,k_2}} \chi(s)ds\right\} \\
&\le C\left[E\left(\int_{t_{i_2,k_2-1}}^{t_{i_2,k_2}} \chi(s)ds\right)^{\omega}\right]^{1/\omega}\left[E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}\right)\right]^{1-1/\omega} \\
&\le C\left[\Delta^{\omega-1}\int_{t_{i_2,k_2-1}}^{t_{i_2,k_2}} E\chi(s)^{\omega}ds\right]^{1/\omega}\Delta^{1-1/\omega} \le C\Delta^{2-1/\omega},
\end{aligned}$$

where the second inequality follows from Hölder's inequality, the third inequality follows from Jensen's inequality, and the last inequality follows from Assumption I(ii). Therefore, we obtain,

$$E\left(1_{\{\Delta^n_{i_1,k_1}N>0\}}1_{\{\Delta^n_{i_2,k_2}N>0\}}\right)\leq C\Delta^{2-\varsigma},$$

for arbitrarily small $\varsigma>0$.

By induction, the lemma holds for any positive integer $j>2$. □

Lemma 13. *Suppose that Assumption I(ii) holds. Then we have*

$$E\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}^2_{i,\kappa}-\widehat{\sigma}^{2,c}_{i,\kappa}\right)\right|\leq C\Delta^{2\varpi}\quad \textit{and}\quad E\left|\widehat{\eta}-\widehat{\eta}^c\right|\leq C\Delta^{2\varpi}.$$

Proof of Lemma 13. First, for any $\omega>2$, we rewrite and calculate the stochastic order of $\sum_{i=1}^{T}\left(\widehat{\sigma}^2_{i,\kappa}-\widehat{\sigma}^{2,c}_{i,\kappa}\right)/T$ as follows,

$$\begin{aligned}
&E\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}^2_{i,\kappa}-\widehat{\sigma}^{2,c}_{i,\kappa}\right)\right|\\
&\leq\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\Bigg[E\left|\left(\Delta^n_{i,k}X^c\right)^2 1_{\left\{\left|\Delta^n_{i,k}X^c\right|>u_n\right\}}\right|\\
&\quad+E\left|\left(\Delta^n_{i,k}X\right)^2 1_{\left\{\left|\Delta^n_{i,k}X\right|\leq u_n\right\}}1_{\left\{\Delta^n_{i,k}N>0\right\}}\right|+E\left|\left(\Delta^n_{i,k}X^c\right)^2 1_{\left\{\Delta^n_{i,k}N>0\right\}}\right|\Bigg]\\
&\leq\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\Bigg[\left(E\left|\Delta^n_{i,k}X^c\right|^\omega\right)^{2/\omega}\left(E\left|1_{\left\{\left|\Delta^n_{i,k}X^c\right|>u_n\right\}}\right|\right)^{1-2/\omega}\\
&\quad+u_n^2P\left(\Delta^n_{i,k}N>0\right)+\left(E\left|\Delta^n_{i,k}X^c\right|^\omega\right)^{2/\omega}\left(P\left(\Delta^n_{i,k}N>0\right)\right)^{1-2/\omega}\Bigg]\\
&\leq C\left(\Delta^{(\omega-2)(1/2-\varpi)}\vee\Delta^{2\varpi}\vee\Delta^{1-2/\omega}\right),
\end{aligned}\tag{C.2}$$

where the second inequality follows from Hölder's inequality and the last inequality follows from Markov inequality, Burkholder-Davis-Gundy inequality and Lemma 12. Note that $\sup_{t\in\mathbb{R}_+} E\left(e^{|\mu(t)|}\right) + \sup_{t\in\mathbb{R}_+} E\left(e^{|\sigma(t)|}\right) < \infty$ in Assumption I(ii) implies boundedness of moments of all orders for $|\mu(t)|$ and $\sigma(t)$. This in turn allows one to apply Hölder's inequality with arbitrarily large $\omega > 0$ in obtaining (C.2), where the constant C arises from the upper bound of higher order moments of $|\mu(t)|$ and $|\sigma(t)|$ and the applications of Burkholder-Davis-Gundy inequality, the elementary inequality $e^x \geq 1 + x + x^2/2 + \ldots + x^m/m!$ for $x \geq 0$ and any integer $m \geq 1$, and Lemma 12. Therefore, one can always choose a large enough ω such that,

$$E\left|\frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right)\right| \leq C\Delta^{2\varpi}.$$

Second, by substituting n for ℓ in the arguments for deriving the result in (C.2), we obtain,

$$E\left|\widehat{\eta} - \widehat{\eta}^c\right| \leq C\Delta^{2\varpi},$$

completing the proof. □

C.1 Proof of Theorem 1

By triangle inequality, we have,

$$\| \widehat{f}(\kappa) - f(\kappa) \| \leq \| \widehat{f}^c(\kappa) - f(\kappa) \| + \| \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \| .$$

We divide the proof into two steps. We prove $\| \widehat{f}^c(\kappa) - f(\kappa) \| \xrightarrow{P} 0$ in the first step and $\| \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \| \xrightarrow{P} 0$ in the second.

Step 1. By using the notation in (C.1) and triangle inequality, we can rewrite the estimation error of $\widehat{f}^c(\kappa)$, which is built on the continuous part of X, as follows,

$$\| \widehat{f}^c(\kappa) - f(\kappa) \| = \frac{1}{\widehat{\eta}^c} \| \frac{1}{T} \sum_{i=1}^{T} \widehat{\sigma}_{i,\kappa}^{2,c} - f(\kappa)\widehat{\eta}^c \| \leq \frac{1}{\widehat{\eta}^c} \sum_{i=1}^{6} \| \zeta_i(\kappa) \| . \tag{C.3}$$

We first show that $\widehat{\eta}^c$ converges in probability to η. The difference between $\widehat{\eta}^c$ and η is decomposed as follows,

$$\widehat{\eta}^c - \eta = \left(\frac{1}{T} \sum_{i=1}^{T} \sum_{j=1}^{n} \left(\Delta_{i,j}^n X^c\right)^2 - \frac{1}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt \right) + \left(\frac{1}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt - \eta \right). \tag{C.4}$$

It follows easily from Itô's lemma and the integrability assumption for μ and σ that the first term on the right hand side of (C.4) tends to zero in probability as $n, T \to \infty$. We next deal with the second term on the right hand side of (C.4). Let E_i be conditional expectation with respect to the sigma field $\mathscr{G}_i$ (see Assumption II for definition). Then for any $\omega > 2(1+\iota)/\iota$ where ι is given in Assumption II, we have that,

$$\begin{aligned}
& E\left(\frac{1}{T} \sum_{i=1}^{T} \int_{i-1}^{i} \sigma^2(t)dt - \eta \right)^2 \\
&= \frac{2}{T^2} \sum_{i=1}^{T} \sum_{j=i+1}^{T} E\left\{ \left(\int_{i-1}^{i} \sigma^2(t)dt - \eta \right) E_i \left(\int_{j-1}^{j} \sigma^2(t)dt - \eta \right) \right\} + \frac{1}{T^2} \sum_{i=1}^{T} E\left(\int_{i-1}^{i} \sigma^2(t)dt - \eta \right)^2 \\
&= \frac{2}{T^2} \sum_{i=1}^{T} \sum_{j=i+1}^{T} \left(E\left| \int_{i-1}^{i} \sigma^2(t)dt - \eta \right|^{\omega} \right)^{1/\omega} \left(E\left| E_i \left(\int_{j-1}^{j} \sigma^2(t)dt - \eta \right) \right|^{\omega/(\omega-1)} \right)^{1-1/\omega} + \frac{C}{T} \\
&\leq \frac{C}{T^2} \sum_{i=1}^{T} \sum_{j=i+1}^{T} \alpha_{j-i}^{1-2/\omega} + \frac{C}{T} \leq \frac{C}{T},
\end{aligned}$$

where the first inequality follows from Assumptions I(ii) and II with $q = 1$, Hölder's inequality and Lemma 3.102 on page 497 of [7]. Hence, we have that the second term on the right hand side of (C.4) goes to zero in probability. Therefore, $\widehat{\eta}^c \xrightarrow{P} \eta$.

It remains to show that,

$$\sum_{i=1}^{6} \| \zeta_i(\kappa) \| \xrightarrow{P} 0.$$

To this end, we treat terms $\zeta_i(\kappa)$, $i = 1, 2, ..., 6$, one by one. For terms $\zeta_1(\kappa)$, $\zeta_2(\kappa)$ and $\zeta_3(\kappa)$, applying Lemma 11 with $m = 2$ and Jensen's inequality, we obtain,

$$E \| \zeta_i(\kappa) \| \leq \left(\int_{[0,1]} E \left| \zeta_i(\kappa) \right|^2 d\kappa \right)^{1/2} \leq C \left(\frac{1}{\sqrt{n}} \vee \frac{1}{\sqrt{T\ell}} \right),$$

for $i = 1, 2, 3$. Thus,

$$\sum_{i=1}^{3} \| \zeta_i(\kappa) \| = O_P \left(\frac{1}{\sqrt{n}} \vee \frac{1}{\sqrt{T\ell}} \right). \tag{C.5}$$

Next, we deal with the term $\zeta_4(\kappa)$. By Proposition II.1.28 and Theorem II.1.33 on pages 72-73 of [7], $\int_{\mathbb{R}} |x|^2 \tilde{F}(dx) < \infty$ and $\sup_{t \in \mathbb{R}_+} E \left| \tilde{\chi}(t) \right|^2 < \infty$ in Assumption I(ii), we have that,

$$\int_0^t \int_{\mathbb{R}} x \tilde{\nu}(ds, dx) - \int_0^t \int_{\mathbb{R}} x \tilde{\chi}(s) ds \tilde{F}(dx)$$

is a locally square integrable martingale. Then, applying Theorem I.3.17 and Proposition II.1.28 on pages 32 and 72 of [7], $\int_{\mathbb{R}} |x|^2 \tilde{F}(dx) < \infty$ and $\sup_{t \in \mathbb{R}_+} E \left| \tilde{\chi}(t) \right|^2 < \infty$ in Assumption I(ii), we obtain,

$$E \left(\int_0^t \int_{\mathbb{R}} x^2 \tilde{\nu}(ds, dx) \right) = E \left(\int_0^t \int_{\mathbb{R}} x^2 \tilde{\chi}(s) ds \tilde{F}(dx) \right).$$

Using the above results, we can bound the first moment of $\parallel \zeta_4(\kappa) \parallel$ as follows,

$$
\begin{aligned}
&E \parallel \zeta_4(\kappa) \parallel \leq \left(\int_{[0,1]} E \left|\zeta_4(\kappa)\right|^2 d\kappa \right)^{1/2} \\
\leq & \left(\int_{[0,1]} \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \int_{t_{i,j_\kappa-\ell}}^{t_{i,j_\kappa}} C \left[E \left| \int_t^{i-1+\kappa} \tilde{\mu}(s)ds \right|^2 + E \left| \int_t^{i-1+\kappa} \check{\sigma}(s)dW(s) \right|^2 \right.\right. \\
&+ E \left| \int_t^{i-1+\kappa} \tilde{\sigma}(s)d\widetilde{W}(s) \right|^2 + E \left| \int_t^{i-1+\kappa} \int_{\mathbb{R}} x\tilde{\nu}(ds,dx) - \int_t^{i-1+\kappa} \int_{\mathbb{R}} x\tilde{\chi}(s)ds\tilde{F}(dx) \right|^2 \\
&\left.\left. + E \left| \int_t^{i-1+\kappa} \int_{\mathbb{R}} x\tilde{\chi}(s)ds\tilde{F}(dx) \right|^2 \right] dt d\kappa \right)^{1/2} \\
\leq & C \left(\int_{[0,1]} \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \int_{t_{i,j_\kappa-\ell}}^{t_{i,j_\kappa}} \left[(\ell+1)\Delta \int_t^{i-1+\kappa} E\left|\tilde{\mu}(s)\right|^2 ds + \int_t^{i-1+\kappa} E\left(\check{\sigma}^2(s) + \tilde{\sigma}^2(s)\right) ds \right.\right. \\
&\left.\left. + E\left(\int_t^{i-1+\kappa} \int_{\mathbb{R}} x^2 \tilde{\chi}(s)ds\tilde{F}(dx) \right) + (\ell+1)\Delta \int_t^{i-1+\kappa} E\left(\tilde{\chi}(t)\right)^2 dt \right] dt d\kappa \right)^{1/2} \\
\leq & C\sqrt{\frac{\ell}{n}},
\end{aligned}
$$

where the first inequality follows from Jensen's inequality, the third inequality follows from the Burkholder-Davis-Gundy inequality, and the last inequality follows from Assumption I(ii). Thus,

$$
\parallel \zeta_4(\kappa) \parallel = O_P\left(\sqrt{\frac{\ell}{n}} \right).
$$

Turning next to term $\zeta_5(\kappa)$, by (7), we can rewrite this term as,

$$
\zeta_5(\kappa) = \frac{1}{T} \sum_{i=1}^{T} A_i(\kappa).
$$

It follows easily from Assumptions I(ii), II with $q = 1$ and Corollary 14.3 on page 212 of [3] that,

$$E\left|\zeta_5(\kappa)\right|^2 \leq \frac{1}{T}\sum_{h=-\infty}^{\infty}\left|\phi_{\kappa,\kappa}(h)\right| \leq \frac{C}{T}\sum_{h=-\infty}^{\infty}\alpha_{|h|}^{1-1/a-1/r}\left(E\left|A_1(\kappa)\right|^a\right)^{1/a}\left(E\left|A_{1+|h|}(\kappa)\right|^r\right)^{1/r} \leq \frac{C}{T},$$

where $a, r > 0$ and $1/a + 1/r < \iota/(1+\iota)$. Then, we immediately obtain,

$$E \parallel \zeta_5(\kappa) \parallel \leq \left(\int_{[0,1]} E\left|\zeta_5(\kappa)\right|^2 d\kappa\right)^{1/2} \leq \frac{C}{\sqrt{T}}.$$

Therefore,

$$\parallel \zeta_5(\kappa) \parallel = O_P\left(\frac{1}{\sqrt{T}}\right).$$

Finally, for the last term $\zeta_6(\kappa)$, it follows from boundedness of $f(\kappa)$ as defined in (3) and exactly the same arguments as in calculating the upper bounds of terms $\zeta_1(\kappa)$, $\zeta_2(\kappa)$ and $\zeta_3(\kappa)$, that,

$$\parallel \zeta_6(\kappa) \parallel = O_P\left(\frac{1}{\sqrt{n}}\right).$$

To sum up, we have,

$$\sum_{i=1}^{6} \parallel \zeta_i(\kappa) \parallel \xrightarrow{P} 0.$$

Step 2. We now consider the difference between estimators $\widehat{f}(\kappa)$ and $\widehat{f}^c(\kappa)$ which are built based on X and X^c, respectively. By triangle inequality, we have,

$$\begin{aligned}\parallel \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \parallel &= \parallel \frac{1}{\widehat{\eta}T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^2 - \frac{1}{\widehat{\eta}^c T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^{2,c} \parallel \\ &\leq \frac{1}{\widehat{\eta}} \parallel \frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right) \parallel + \frac{\left|\widehat{\eta}^c - \widehat{\eta}\right|}{\widehat{\eta}\widehat{\eta}^c} \parallel \frac{1}{T}\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^{2,c} \parallel . \qquad \text{(C.6)}\end{aligned}$$

It then follows easily from (C.4) and Lemma 13 that,

$$\widehat{\eta}-\widehat{\eta}^c \xrightarrow{P} 0, \quad \frac{1}{\widehat{\eta}} \xrightarrow{P} \frac{1}{\eta}, \quad \text{and hence } \frac{\widehat{\eta}^c-\widehat{\eta}}{\widehat{\eta}} \xrightarrow{P} 0. \tag{C.7}$$

Recall that $\widehat{f}^c(\kappa)=\sum_{i=1}^{T}\widehat{\sigma}_{i,\kappa}^{2,c}/(\widehat{\eta}^c T)$ and $\| \widehat{f}^c(\kappa)-f(\kappa) \| \xrightarrow{P} 0$ by Step 1, we have thus proved that the second term on the right hand side of (C.6) goes to zero in probability.

It remains to calculate the stochastic order of $\| \frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^{2}-\widehat{\sigma}_{i,\kappa}^{2,c}\right) \|$. On the one hand, define,

$$\mathcal{S}:=\left\{(i,i',k,k')|i\neq i' \text{ or } k\neq k',\ \text{where } i,i'=1,2,...,T \text{ and } k,k'=j_\kappa-\ell+1,...,j_\kappa\right\},$$

we have that, for any $\omega>4$,

$$\begin{aligned}
&\frac{1}{(T\ell\Delta)^2}E\left\{\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^n X\right)^2 1_{\left\{\left|\Delta_{i,k}^n X\right|\leq u_n\right\}}-\left(\Delta_{i,k}^n X^c\right)^2\right]1_{\left\{\Delta_{i,k}^n N=0\right\}}\right\}^2\\
&\leq \frac{1}{(T\ell\Delta)^2}\sum_{(i,i',k,k')\in\mathcal{S}}E\left(\left(\Delta_{i,k}^n X^c\right)^2\left(\Delta_{i',k'}^n X^c\right)^2 1_{\left\{\left|\Delta_{i,k}^n X^c\right|>u_n\right\}}1_{\left\{\left|\Delta_{i',k'}^n X^c\right|>u_n\right\}}\right)\\
&\quad+\frac{1}{(T\ell\Delta)^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}E\left[\left(\Delta_{i,k}^n X^c\right)^4 1_{\left\{\left|\Delta_{i,k}^n X^c\right|>u_n\right\}}\right]\\
&\leq \frac{1}{(T\ell\Delta)^2}\sum_{(i,i',k,k')\in\mathcal{S}}\left[E\left|\Delta_{i,k}^n X^c\right|^\omega\right]^{2/\omega}E\left[\left|\Delta_{i',k'}^n X^c\right|^\omega\right]^{2/\omega}\\
&\quad\times\left[E\left(1_{\left\{\left|\Delta_{i,k}^n X^c\right|>u_n\right\}}\right)\right]^{(1-4/\omega)/2}\left[E\left(1_{\left\{\left|\Delta_{i',k'}^n X^c\right|>u_n\right\}}\right)\right]^{(1-4/\omega)/2}\\
&\quad+\frac{1}{(T\ell\Delta)^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[E\left|\Delta_{i,k}^n X^c\right|^\omega\right]^{4/\omega}\left[E\left(1_{\left\{\left|\Delta_{i,k}^n X^c\right|>u_n\right\}}\right)\right]^{1-4/\omega}\\
&\leq C\Delta^{(\omega-4)(1/2-\varpi)},
\end{aligned} \tag{C.8}$$

where the second and third inequalities follow by applying Hölder's, Burkholder-Davis-Gundy and Markov inequalities. On the other hand, we have that, for any $\omega > 4$,

$$\frac{1}{(T\ell\Delta)^2} E\left\{\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[\left(\Delta_{i,k}^n X\right)^2 1_{\left\{\left|\Delta_{i,k}^n X\right|\le u_n\right\}} - \left(\Delta_{i,k}^n X^c\right)^2\right] 1_{\left\{\Delta_{i,k}^n N>0\right\}}\right\}^2$$

$$\le \frac{1}{(T\ell\Delta)^2}\sum_{(i,i',k,k')\in\mathcal{S}} E\left[u_n^4 1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}} + u_n^2\left(\Delta_{i',k'}^n X^c\right)^2 1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\right.$$

$$\left. + u_n^2\left(\Delta_{i,k}^n X^c\right)^2 1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}} + \left(\Delta_{i,k}^n X^c\right)^2\left(\Delta_{i',k'}^n X^c\right)^2 1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\right]$$

$$+ \frac{1}{(T\ell\Delta)^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa} E\left|\left(\Delta_{i,k}^n X\right)^2 1_{\left\{\left|\Delta_{i,k}^n X\right|\le u_n\right\}} - \left(\Delta_{i,k}^n X^c\right)^2\right|^2 1_{\left\{\Delta_{i,k}^n N>0\right\}}$$

$$\le \frac{1}{(T\ell\Delta)^2}\sum_{(i,i',k,k')\in\mathcal{S}}\left[u_n^4 E\left(1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\right) + u_n^2\left[E\left|\Delta_{i',k'}^n X^c\right|^\omega\right]^{2/\omega}\right.$$

$$\times\left[E\big(1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\big)\right]^{1-2/\omega} + u_n^2\left[E\left|\Delta_{i,k}^n X^c\right|^\omega\right]^{2/\omega}\left[E\big(1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\big)\right]^{1-2/\omega}$$

$$\left. + \left[E\left|\Delta_{i,k}^n X^c\right|^\omega\right]^{2/\omega}\left[E\left|\Delta_{i',k'}^n X^c\right|^\omega\right]^{2/\omega}\left(E\big(1_{\left\{\Delta_{i,k}^n N>0\right\}} 1_{\left\{\Delta_{i',k'}^n N>0\right\}}\big)\right)^{1-4/\omega}\right]$$

$$+ \frac{1}{(T\ell\Delta)^2}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\left[u_n^4 E\left(\Delta_{i,k}^n N\right) + \left(E\left|\Delta_{i,k}^n X^c\right|^\omega\right)^{4/\omega}\left[E\left(\Delta_{i,k}^n N\right)\right]^{1-4/\omega}\right]$$

$$\le \left(\Delta^{4\varpi-\varsigma} \vee \Delta^{2\varpi-4/\omega+1-\varsigma} \vee \Delta^{2-8/\omega-\varsigma} \vee \frac{\Delta^{4\varpi-1}}{T\ell} \vee \frac{\Delta^{1-4/\omega}}{T\ell}\right), \tag{C.9}$$

where the second inequality follows from Hölder's inequality and the last inequality follows from Burkholder-Davis-Gundy inequality and Lemma 12 for arbitrarily small $\varsigma > 0$. Based on the above results, we have that, for any $\omega > 4$ and arbitrarily small $\varsigma > 0$,

$$E \parallel \frac{1}{T}\sum_{i=1}^{T}\left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c}\right) \parallel$$

$$\leq \left(\int_{[0,1]} \frac{C}{(T\ell\Delta)^2} E \left\{ \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \left[\left(\Delta_{i,k}^n X\right)^2 1_{\{|\Delta_{i,k}^n X| \leq u_n\}} - \left(\Delta_{i,k}^n X^c\right)^2 \right] 1_{\{\Delta_{i,k}^n N = 0\}} \right\}^2 d\kappa \right)^{1/2}$$

$$+ \left(\int_{[0,1]} \frac{C}{(T\ell\Delta)^2} E \left\{ \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \left[\left(\Delta_{i,k}^n X\right)^2 1_{\{|\Delta_{i,k}^n X| \leq u_n\}} - \left(\Delta_{i,k}^n X^c\right)^2 \right] 1_{\{\Delta_{i,k}^n N > 0\}} \right\}^2 d\kappa \right)^{1/2}$$

$$\leq C \left(\Delta^{(\omega-4)(1/2-\varpi)} \vee \Delta^{4\varpi-\varsigma} \vee \Delta^{2\varpi-4/\omega+1-\varsigma} \vee \Delta^{2-8/\omega-\varsigma} \vee \frac{\Delta^{4\varpi-1}}{T\ell} \vee \frac{\Delta^{1-4/\omega}}{T\ell} \right)^{1/2}.$$

Because $\sup_{t\in\mathbb{R}_+} E\left(e^{|\mu(t)|}\right) + \sup_{t\in\mathbb{R}_+} E\left(e^{|\sigma(t)|}\right) < \infty$ in Assumption I(ii) and by the same arguments as that immediately following (C.2), one can always choose a large enough ω and a small enough ς such that,

$$E \parallel \frac{1}{T} \sum_{i=1}^{T} \left(\widehat{\sigma}_{i,\kappa}^2 - \widehat{\sigma}_{i,\kappa}^{2,c} \right) \parallel \leq C \left(\Delta^{4\varpi-\varsigma} \vee \frac{\Delta^{4\varpi-1}}{T\ell} \right)^{1/2}.$$

Therefore, by letting ς be sufficiently close to zero, we have that,

$$\parallel \widehat{f}(\kappa) - \widehat{f}^c(\kappa) \parallel = o_P(1),$$

under the conditions $0 < \varpi < 1/2$ and $b + c > 1 - 4\varpi$ in Theorem 1.

Combining Step 1 and Step 2 leads to $\parallel \widehat{f}(\kappa) - f(\kappa) \parallel \xrightarrow{P} 0$, completing the proof.

C.2 Proof of Theorem 2

First, we recall the definition of $A_i(\kappa_m)$,

$$A_i(\kappa_m) := \sigma^2(i - 1 + \kappa_m) - f(\kappa_m) \int_{i-1}^{i} \sigma^2(t)dt, \quad \text{for } i = 1, 2, ..., T \text{ and } m = 1, 2, ..., d,$$

where d is a positive integer. We introduce some additional notation that will be used in the proof. Denote with $E_i(\cdot)$ the conditional expectation with respect to the sigma field $\mathscr{G}_i$ (see Assumption II for definition). For each κ_m and a positive integer l, denote for $i = 1, 2, ..., T$,

$$\tilde{A}_{i,l}(\kappa_m) := \sum_{k=0}^{l-1} \left(E_i \left(A_{i+k}(\kappa_m) \right) - E_{i-1} \left(A_{i+k}(\kappa_m) \right) \right),$$

where $m = 1, 2, ..., d$ and d is a positive integer. We will show in the following that under the conditions of Theorem 2, the limit of $\tilde{A}_{i,l}(\kappa_m)$ as $l \to \infty$ exists a.s. It is denoted by,

$$\tilde{A}_{i,\infty}(\kappa_m) := \lim_{l \to \infty} \tilde{A}_{i,l}(\kappa_m). \tag{C.10}$$

Moreover, for each κ_m and a positive integer l, we define the following approximation errors,

$$R_{T,l}(\kappa_m) := \frac{1}{T} \sum_{i=1}^{T} \left(A_i(\kappa_m) - \tilde{A}_{i,l}(\kappa_m) \right) \text{ and } R_{T,\infty}(\kappa_m) := \frac{1}{T} \sum_{i=1}^{T} \left(A_i(\kappa_m) - \tilde{A}_{i,\infty}(\kappa_m) \right). \tag{C.11}$$

The following lemma is used in the proof of the limit result of Theorem 2.

Lemma 14. *Suppose that Assumptions I(ii) and II with $q = 3$ hold. Then,*

$$\left(\frac{1}{\sqrt{T}} \sum_{i=1}^{T} A_i(\kappa_1), \frac{1}{\sqrt{T}} \sum_{i=1}^{T} A_i(\kappa_2), \ldots \frac{1}{\sqrt{T}} \sum_{i=1}^{T} A_i(\kappa_d) \right)^{\top} \xrightarrow{d} \mathbf{N}_d(\mathbf{0}, \mathbf{\Lambda})$$

as $T \to \infty$, where $\mathbf{N}_d(\mathbf{0}, \mathbf{\Lambda})$ denotes the d-dimensional normal distribution with mean zero and covariance matrix $\mathbf{\Lambda}$ whose entries are given by $\Lambda_{mq} = \sum_{h=-\infty}^{\infty} \phi_{\kappa_m,\kappa_q}(h)$ for $m, q \in \{1, 2, ..., d\}$.

Proof of Lemma 14. The proof is based on approximating $A_i(\kappa_m)$ by $\tilde{A}_{i,\infty}(\kappa_m)$, where $\tilde{A}_{i,\infty}(\kappa_m)$ is defined in (C.10). Hence, the proof consists of two parts. In the first part, we show that the error due to the approximation of $A_i(\kappa_m)$ is asymptotically negligible. In the second part, we complete the proof by showing that,

$$\left(\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\tilde{A}_{i,\infty}(\kappa_1), \frac{1}{\sqrt{T}}\sum_{i=1}^{T}\tilde{A}_{i,\infty}(\kappa_2), \ldots, \frac{1}{\sqrt{T}}\sum_{i=1}^{T}\tilde{A}_{i,\infty}(\kappa_d)\right)^{\top} \xrightarrow{d} \boldsymbol{N}_d(\mathbf{0}, \boldsymbol{\Lambda}). \quad \text{(C.12)}$$

Part 1. It follows from (B.49)-(B.50) in [1] that the average difference $R_{T,l}(\kappa_m)$ between $A_i(\kappa_m)$ and $\tilde{A}_{i,l}(\kappa_m)$ has the following decomposition,

$$R_{T,l}(\kappa_m) = \frac{1}{T}\sum_{i=0}^{T-1} E_i(A_{i+l}(\kappa_m)) - \frac{1}{T}\sum_{k=1}^{l-1}\left[E_T(A_{T+k}(\kappa_m)) - E_0(A_k(\kappa_m))\right] \quad \text{(C.13)}$$

for $m = 1, 2, ..., d$. Because of Assumptions I(ii) and II with $q = 3$, and using Lemma 3.102 on page 497 of [7], we have that, for any $\omega > (3+\iota)/(2+\iota)$ where ι is given in Assumption II,

$$E\left|E_i(A_{i+k}(\kappa_m))\right| \leq C\alpha_k^{1-1/\omega}\left(E\left|A_{i+k}(\kappa_m)\right|^{\omega}\right)^{1/\omega}.$$

This further implies,

$$\begin{aligned} &E\left(\lim_{l\to\infty}\sum_{k=0}^{l-1}\left(\left|E_i(A_{i+k}(\kappa_m))\right| + \left|E_{i-1}(A_{i+k}(\kappa_m))\right|\right)\right) \\ &\leq \liminf_{l\to\infty}\sum_{k=0}^{l-1}\left(E\left|E_i(A_{i+k}(\kappa_m))\right| + E\left|E_{i-1}(A_{i+k}(\kappa_m))\right|\right) \leq C. \end{aligned}$$

I(ii). Therefore,

$$
\begin{aligned}
E\left|\zeta_2(\kappa)\right|^2 &\leq CE\left|\frac{2}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\mu(t)-\mu(t_{i,k-1})\right]dt\int_{t_{i,k-1}}^{t_{i,k}}\sigma(t)dW(t)\right|^2+E\left|M_2(T)\right|^2 \\
&\leq C\left(\frac{1}{n^{2-\varsigma}}\vee\frac{1}{nT\ell}\right)
\end{aligned}
$$

for arbitrarily small $\varsigma > 0$.

We next deal with $\zeta_4(\kappa)$. The following additional notations are needed,

$$
\begin{cases}
\zeta_{4,1}(\kappa) := -\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\int_t^{i-1+\kappa}\left(\tilde{\mu}(s)+\int_{\mathbb{R}}x\tilde{F}(dx)\tilde{\chi}(s)\right)ds\right]dt, \\
\zeta_{4,2}(\kappa) := \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\int_{t_{i,k-1}}^{t}\check{\sigma}(s)dW(s)+\int_{t_{i,k-1}}^{t}\tilde{\sigma}(s)d\widetilde{W}(s)\right. \\
\qquad\qquad\left.+\int_{t_{i,k-1}}^{t}\int_{\mathbb{R}}x\left(\tilde{\nu}(ds,dx)-\tilde{F}(dx)\tilde{\chi}(s)ds\right)\right]dt, \\
\zeta_{4,3}(\kappa) := -\frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\left[\int_{t_{i,k-1}}^{i-1+\kappa}\check{\sigma}(s)dW(s)+\int_{t_{i,k-1}}^{i-1+\kappa}\tilde{\sigma}(s)d\widetilde{W}(s)\right. \\
\qquad\qquad\left.+\int_{t_{i,k-1}}^{i-1+\kappa}\int_{\mathbb{R}}x\left(\tilde{\nu}(ds,dx)-\tilde{F}(dx)\tilde{\chi}(s)ds\right)\right]dt.
\end{cases}
\tag{C.16}
$$

We can then rewrite $\zeta_4(\kappa)$ as,

$$
\zeta_4(\kappa)=\sum_{i=1}^{3}\zeta_{4,i}(\kappa).
$$

For $\zeta_{4,1}(\kappa)$, we have,

$$
E\left|\zeta_{4,1}(\kappa)\right|^2 \leq \frac{1}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}E\left[\int_t^{i-1+\kappa}\left(\tilde{\mu}(s)+\int_{\mathbb{R}}x\tilde{F}(dx)\tilde{\chi}(s)\right)ds\right]^2dt
$$

$$\leq \frac{C}{T\ell\Delta}\sum_{i=1}^{T}\sum_{k=j_\kappa-\ell+1}^{j_\kappa}\int_{t_{i,k-1}}^{t_{i,k}}\ell\Delta\int_{t}^{i-1+\kappa}\left(E\left|\tilde{\mu}(s)\right|^2+E\left|\tilde{\chi}(s)\right|^2\right)ds \leq \frac{C\ell^2}{n^2},$$

where the first two inequalities follow from Jensen's inequality and $\int_{\mathbb{R}}|x|^2\tilde{F}(dx)<\infty$ in Assumption I(ii), and the last inequality follows from Assumption I(ii) again. For $\zeta_{4,2}(\kappa)$, by applying exactly the same technique used in the proof of Lemma 2.22 on page 144 of [9], and again Burkholder-Davis-Gundy inequality, Jensen's inequality, and Assumption I(ii), we obtain that,

$$E\left|\zeta_{4,2}(\kappa)\right|^2 \leq \frac{C}{nT\ell}.$$

Now turning to $\zeta_{4,3}(\kappa)$, we denote,

$$\begin{aligned}
M_3(t) &:= \frac{1}{T\ell}\sum_{i=1}^{T}\Bigg[\sum_{k=j_\kappa-\ell+1}^{j_\kappa-1}\Bigg(\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}(k-j_\kappa+\ell)\check{\sigma}(s)dW(s)+\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}(k-j_\kappa+\ell)\tilde{\sigma}(s)d\widetilde{W}(s)\\
&+\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}(k-j_\kappa+\ell)x\left(\tilde{\nu}(ds,dx)-\tilde{F}(dx)\tilde{\chi}(s)ds\right)\Bigg)+\int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)}\ell\check{\sigma}(s)dW(s)\\
&+\int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)}\ell\tilde{\sigma}(s)d\widetilde{W}(s)+\int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)}\ell x\left(\tilde{\nu}(ds,dx)-\tilde{F}(dx)\tilde{\chi}(s)ds\right)\Bigg],
\end{aligned}$$

which is a continuous-time martingale over the interval $[0,T]$. The quadratic variation of $M_3(t)$ is,

$$\begin{aligned}
\left[M_3,M_3\right](t) &= \frac{1}{(T\ell)^2}\sum_{i=1}^{T}\Bigg[\sum_{k=j_\kappa-\ell+1}^{j_\kappa-1}\Bigg(\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\left(\left|(k-j_\kappa+\ell)\check{\sigma}(s)\right|^2+\left|(k-j_\kappa+\ell)\tilde{\sigma}(s)\right|^2\right)ds\\
&+\int_{t\wedge t_{i,k-1}}^{t\wedge t_{i,k}}\left|(k-j_\kappa+\ell)x\right|^2\tilde{\nu}(ds,dx)\Bigg)+\int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)}\left(\left|\ell\check{\sigma}(s)\right|^2+\left|\ell\tilde{\sigma}(s)\right|^2\right)ds
\end{aligned}$$

$$+\int_{t\wedge t_{i,j_\kappa-1}}^{t\wedge(i-1+\kappa)} |\ell x|^2\,\tilde{\nu}(ds,dx)\Bigg].$$

It is easy to see that $\zeta_{4,3}(\kappa) = -M_3(T)$. Then by Burkholder-Davis-Gundy inequality, Assumption I(ii) and the arguments immediately following (C.5), we obtain that,

$$E\left|\zeta_{4,3}(\kappa)\right|^2 = E\left|M_3(T)\right|^2 \le CE\left[M_3, M_3\right](T) \le \frac{C\ell}{nT}.$$

Therefore, combining the results for terms $\zeta_{4,1}(\kappa)$, $\zeta_{4,2}(\kappa)$ and $\zeta_{4,3}(\kappa)$, we obtain that,

$$E\left|\zeta_4(\kappa)\right|^2 \le C\left[\left(\frac{\ell}{n}\right)^2 \vee \frac{\ell}{nT}\right].$$

Lastly, for term $\zeta_6(\kappa)$, by exactly the same method used in the proof of Lemma 11 for calculating the orders of $\zeta_1(\kappa)$ and $\zeta_3(\kappa)$ together with the same method used in calculating the order of $\zeta_2(\kappa)$, simply replacing ℓ with n in these derivations, we obtain that,

$$E\left|\zeta_6(\kappa)\right|^2 \le C\left(\frac{1}{n^{2-\varsigma}} \vee \frac{1}{nT}\right).$$

for arbitrarily small $\varsigma > 0$.

Based on the above bounds for terms $\zeta_2(\kappa)$, $\zeta_4(\kappa)$ and $\zeta_6(\kappa)$, by choosing a sufficiently small $\varsigma > 0$, we can easily obtain,

$$\sqrt{T}\left(\|\ \zeta_2(\kappa)\ \| + \|\ \zeta_4(\kappa)\ \| + \|\ \zeta_6(\kappa)\ \|\right) = O_P\left(\frac{\sqrt{T}\ell}{n} \vee \sqrt{\frac{\ell}{n}}\right) = o_p(1)$$

under the conditions $c < 1 - b/2$ in (9) and $\ell\Delta \to 0$.

Step 2. The only dominant term is $\sqrt{T}\zeta_5(\kappa)/\widehat{\eta}^c$. Recall the definitions of $\zeta_5(\kappa)$, $A_i(\kappa)$,

$\tilde{A}_{i,\infty}(\kappa)$, and $R_{T,\infty}$ in (C.1), (7), (C.10) and (C.11), respectively. It follows easily from the same arguments as that used in Part 1 of the proof of Lemma 14 that,

$$\sqrt{T} \parallel R_{T,\infty}(\kappa) \parallel = \left(\int_0^1 T R_{T,\infty}(\kappa)^2 d\kappa \right)^{1/2} \xrightarrow{P} 0.$$

Because of the above result and $\widehat{\eta}^c \xrightarrow{P} \eta$ which follows from the arguments immediately following (C.4), $\sqrt{T}\zeta_5(\kappa)/\widehat{\eta}^c$ and $1/(\eta\sqrt{T}) \sum_{i=1}^T \tilde{A}_{i,\infty}(\kappa)$ have the same limiting law. From Part 2 of the proof of Lemma 14, $\{1/(\eta\sqrt{T})\tilde{A}_{i,\infty}(\kappa)\}_{i\geq 1}$ is a martingale difference array in the sense of [8]. We next check the three conditions of Theorem C of [8] which leads to the desired functional central limit theorem.

First, we have that,

$$\begin{aligned}
&\frac{1}{\eta^2 T} \sum_{i=1}^T E_{i-1}\left(\parallel \tilde{A}_{i,\infty}(\kappa) \parallel^2 \right) - \text{Trace}(\mathcal{K}) \\
&= \frac{1}{T} \sum_{i=1}^T \int_0^1 \left(1/\eta^2 E_{i-1} \tilde{A}_{i,\infty}(\kappa)^2 - C(\kappa,\kappa) \right) d\kappa \\
&= o_P(1),
\end{aligned}$$

where the last equality follows from the ergodicity Assumption II and the fact that,

$$E \left| \int_0^1 \left(1/\eta^2 E_{i-1} \tilde{A}_{i,\infty}(\kappa)^2 - C(\kappa,\kappa) \right) d\kappa \right| < \infty,$$

and $C(\kappa,\kappa)$ is defined in (8). Hence, we obtain that the first condition of Theorem C of [8], i.e.,

$$\frac{1}{\eta^2 T} \sum_{i=1}^T E_{i-1}\left(\parallel \tilde{A}_{i,\infty}(\kappa) \parallel^2 \right) \xrightarrow{P} \text{Trace}(\mathcal{K}),$$

is satisfied.

Second, we have,

$$E\left[\frac{1}{\eta^3 T^{3/2}}\sum_{i=1}^{T} E_{i-1}\left(\| \tilde{A}_{i,\infty}(\kappa) \|^3\right)\right] \leq E\left[\frac{1}{\eta^3 T^{3/2}}\sum_{i=1}^{T} E_{i-1}\int_0^1 |\tilde{A}_{i,\infty}(\kappa)|^3 d\kappa\right]$$
$$= \frac{1}{\eta^3 T^{1/2}}\int_0^1 E|\tilde{A}_{i,\infty}(\kappa)|^3 d\kappa \leq C/\sqrt{T},$$

where the last inequality follows from the result that $E|\tilde{A}_{i,\infty}(\kappa)|^3 \leq C$ as proved in Part 2 of the proof of Lemma 14. Hence, the conditional Lyapunov condition that is stronger than the second condition of Theorem C of [8], i.e.,

$$\frac{1}{\eta^3 T^{3/2}}\sum_{i=1}^{T} E_{i-1}\left(\| \tilde{A}_{i,\infty}(\kappa) \|^3\right) \xrightarrow{P} 0,$$

is satisfied.

Third, we have that, for an orthonormal basis $\{e_i\}_{i\in\mathbb{N}_+}$ in $\mathcal{L}^2$,

$$\begin{aligned}
&\frac{1}{\eta^2 T}\sum_{i=1}^{T} E_{i-1}\left(\langle \tilde{A}_{i,\infty}(\kappa), e_j\rangle\langle \tilde{A}_{i,\infty}(\kappa), e_k\rangle\right) - \langle \mathcal{K}e_j, e_k\rangle \\
&= \frac{1}{T}\sum_{i=1}^{T}\int_0^1\int_0^1 \left[\left(\frac{1}{\eta^2}E_{i-1}\tilde{A}_{i,\infty}(u)\tilde{A}_{i,\infty}(v)\right)e_j(u)e_k(v) - C(u,v)e_j(u)e_k(v)\right] dudv \\
&= o_P(1),
\end{aligned}$$

where the last equality follows from again the ergodicity Assumption II and the fact that,

$$E\left|\int_0^1\int_0^1 \left[\left(\frac{1}{\eta^2}E_{i-1}\tilde{A}_{i,\infty}(u)\tilde{A}_{i,\infty}(v)\right)e_j(u)e_k(v) - C(u,v)e_j(u)e_k(v)\right] dudv\right| < \infty.$$

Hence, it suffices to require that,

$$b < \min\{2 - 2\varpi, 1 + (1 - r)\varpi, (4 - 2r)\varpi\} \quad \text{and} \quad c > 1 - \varpi(4 - r).$$

Part II. Term $\zeta_4(\kappa)$.

Recall the definition of $\zeta_4(\kappa)$,

$$\zeta_4(\kappa) := \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} (\sigma^2(t) - \sigma^2(i - 1 + \kappa))dt.$$

Under the assumption that the volatility process $\sigma^2(t)$ is rough, we calculate the order of term $\zeta_4(\kappa)$ as follows,

$$\begin{aligned} E|\zeta_4(\kappa)|^2 &\leq E\left(\frac{1}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} (\sigma^2(t) - \sigma^2(i - 1 + \kappa))dt\right)^2 \\ &\leq \frac{1}{T\ell\Delta} \sum_{i=1}^{T} \sum_{k=j_\kappa-\ell+1}^{j_\kappa} \int_{t_{i,k-1}}^{t_{i,k}} E(\sigma^2(t) - \sigma^2(i - 1 + \kappa))^2 dt \\ &\leq C|i - 1 + \kappa - t_{i,j_\kappa-\ell}|^{2H} \leq C\left(\frac{\ell}{n}\right)^{2H}, \end{aligned}$$

where the third inequality follows from (B.3). Therefore, under the conditions (B.4), we have $\sqrt{T} \parallel \zeta_4(\kappa) \parallel = o_P(1)$.

Combining the above results and the results for the rest terms in the proof of Theorem 2 completes the proof.

References

[1] Torben G Andersen, Martin Thyrsgaard, and Viktor Todorov. Time-varying periodicity in intraday volatility. *Journal of the American Statistical Association*, 114(528):1695–1707, 2019.

[2] Hyunphil Choi and Matthew Reimherr. A geometric approach to confidence regions and bands for functional parameters. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80(1):239–260, 2018.

[3] James Davidson. *Stochastic limit theory: An introduction for econometricians*. Oxford University Press, Oxford, 1994.

[4] Jim Gatheral, Thibault Jaisson, and Mathieu Rosenbaum. Volatility is rough. *Quantitative Finance*, 18(6):933–949, 2018.

[5] Peter Hall and Christopher C Heyde. *Martingale limit theory and its application*. Academic Press, New York, 1980.

[6] Jean Jacod. Statistics and high frequency data. In *Kessler, M., Lindner, A., Sørensen, M.(Eds.), Statistical Methods for Stochastic Differential Equations*, volume 124, pages 191–310. Chapman and Hall/CRC Press New York, NY, 2012.

[7] Jean Jacod and Albert Shiryaev. *Limit theorems for stochastic processes*. Springer-Verlag, Berlin, Heidelberg, 2003.

[8] Adam Jakubowski. On limit theorems for sums of dependent hilbert space valued random variables. In *Mathematical Statistics and Probability Theory*. Springer New York, New York, 1980.

[9] Per A Mykland and Lan Zhang. The econometrics of high frequency data. In *Kessler, M., Lindner, A., Sørensen, M.(Eds.), Statistical Methods for Stochastic Differential Equations*, pages 109–190. Chapman and Hall/CRC Press New York, 2012.

[10] Matthew Reimherr. Functional regression with repeated eigenvalues. *Statistics and Probability Letters*, 107:62–70, 2015.

[11] Aad W Van Der Vaart and Jon A Wellner. *Weak Convergence and Empirical Processes*. Springer-Verlag, 1996.

[12] Xiaohu Wang, Weilin Xiao, and Jun Yu. Modeling and forecasting realized volatility with the fractional ornstein-uhlenbeck process. *Journal of Econometrics*, in press, 2021.

[13] Lan Zhang, Per A Mykland, and Yacine Aït-Sahalia. A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *Journal of the American Statistical Association*, 100(472):1394–1411, 2005.