Linear Algebra Primer

Matrices / Vectors A matrix is an $n \times p$ object. Matrices are often denoted by a capital letter (or Greek symbol). A few common matrices will be

$$X = \begin{pmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} \end{pmatrix}$$

or

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ddots & \sigma_n^2 \end{pmatrix}$$

Vectors are essentially one-dimension vectors and will be denoted with an underline. We will assume vectors are $q \times 1$ dimension unless noted with a transpose.

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

or

$$\underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix}$$

The transpose operator will be denoted by $\underline{y}^T = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}$ or $\underline{y}'$, both of which would result in a $1 \times n$ vector.

Matrix Multiplication The most important component in matrix multiplication is tracking dimensions.

Consider a simple case with

$$\underline{\hat{y}} = X \times \underline{\hat{\beta}},$$

where X is a 2×2 matrix, $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ and $\underline{\hat{\beta}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Then

$$\underline{\hat{y}} = \begin{bmatrix} 1 \times 3 + 2 \times 2 \\ 1 \times 3 + (-1) \times 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

In R, we use `%*%` for matrix multiplication.

```
X <- matrix(c(1,2, 1 ,-1), nrow = 2, ncol = 2, byrow = T); X
```

```
##      [,1] [,2]
## [1,]    1    2
## [2,]    1   -1
```

```
#X <- matrix(c(1, 1, 2 ,-1), nrow = 2, ncol = 2); X
beta_hat <- matrix(c(3,2),nrow =2, ncol = 1); beta_hat
```

```
##      [,1]
## [1,]    3
## [2,]    2
```

```
y_hat <- X %*% beta_hat; y_hat
```

```
##      [,1]
## [1,]    7
## [2,]    1
```

Motivating Dataset: Washington (DC) housing dataset Hopefully the connections to statistics are clear, using X and β, but let's consider a motivating dataset.

This dataset contains housing information from Washington, D.C. It was used for a STAT532 exam, so apologize in advance for any scar tissue.

```
DC <- read_csv('https://math.montana.edu/ahoegh/teaching/stat532/data/DC.csv')
```

```
## Parsed with column specification:
## cols(
##   BATHRM = col_double(),
##   HF_BATHRM = col_double(),
##   AC = col_character(),
##   BEDRM = col_double(),
##   STORIES = col_double(),
##   PRICE = col_double(),
##   CNDTN = col_character(),
##   LANDAREA = col_double(),
##   FULLADDRESS = col_character(),
##   ASSESSMENT_NBHD = col_character(),
##   WARD = col_character(),
##   QUADRANT = col_character()
## )
```

```
DC %>% group_by(WARD) %>%
  summarize(`Average Price (millions of dollars)` = mean(PRICE)/1000000, .groups = 'drop') %>%
  kable(digits = 3)
```

WARD	Average Price (millions of dollars)
Ward 1	0.879
Ward 2	1.919
Ward 3	1.294
Ward 4	0.693
Ward 5	0.592
Ward 6	0.856
Ward 7	0.321
Ward 8	0.306

```
DC %>% group_by(BEDRM) %>%
  summarize(`Average Price (millions of dollars)` = mean(PRICE)/1000000, .groups = 'drop') %>%
  kable(digits = 3)
```

BEDRM	Average Price (millions of dollars)
0	0.195
1	0.442
2	0.479
3	0.620
4	0.832
5	1.355
6	1.849
7	1.666
9	7.365

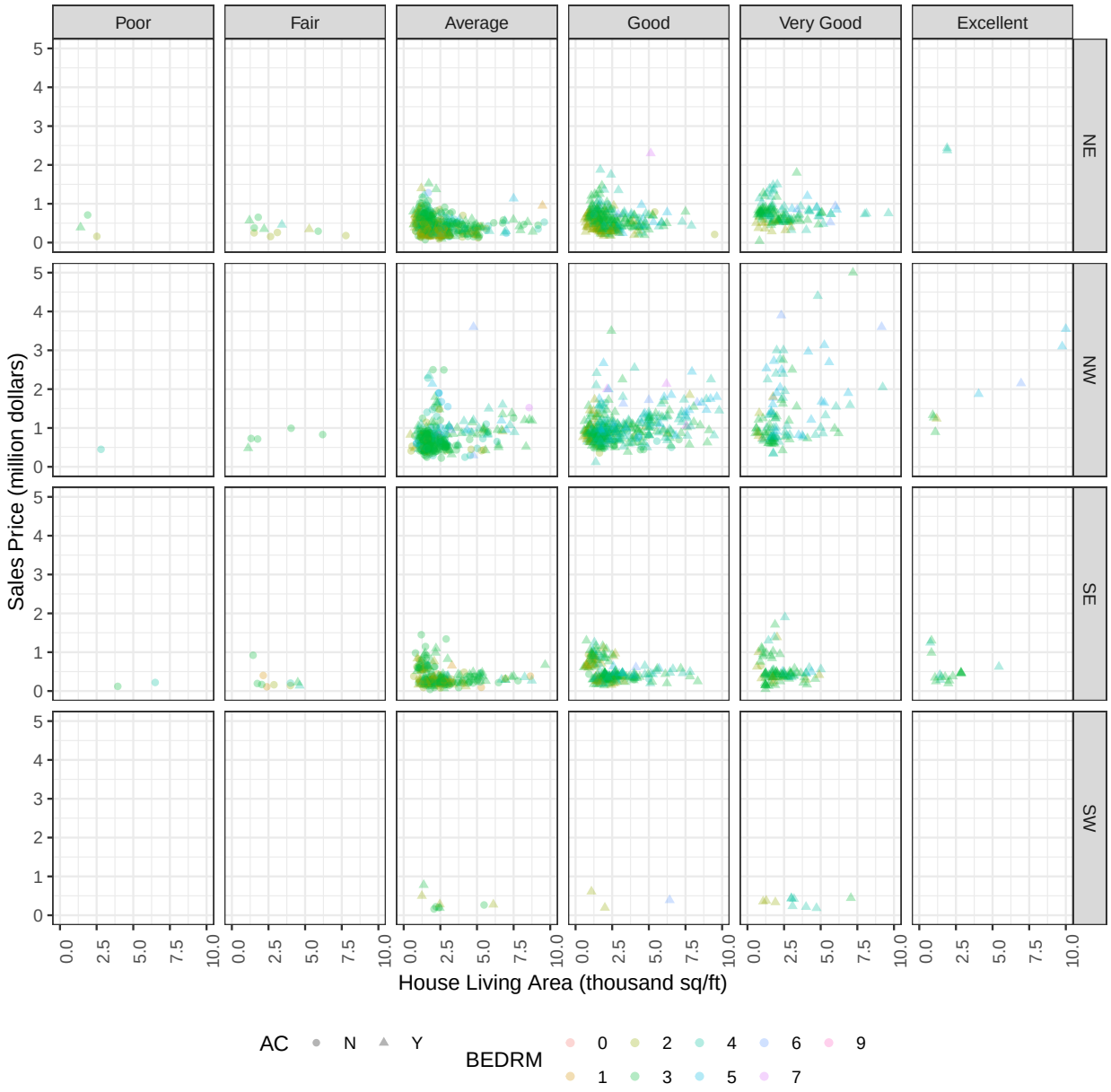


Figure 1: Washington DC Housing prices. Note the exploratory figure removes (~40) properties costing more than 5 million dollars or larger than 10,000 square feet

Regression Model

There are many factors in this dataset that can are useful to predict housing prices.

$$y_i = \beta_0 + \beta_1 * x_{SQFT,i} + \beta_2 x_{BEDRM,i} + \epsilon_i, \tag{1}$$

where y_i is the sales price of the i^{th} house, $x_{SQFT,i}$ is the living square footage of the i^{th} house, and $x_{BEDRM,i}$ is the number of bedrooms for the i^{th} house. Note this implies that we are treating bedrooms as continuous variables as opposed to categorical.

we usually write $\epsilon_i \sim N(0, \sigma^2)$. More on that soon.

In R we often write something like: `price ~ LANDUSE + BEDRM`.

Now let's write this model in matrix notation:

$$\underline{y} = X\underline{\beta} + \underline{\epsilon}, \tag{2}$$

where $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $X = \begin{bmatrix} 1 & x_{SQFT,1} & x_{BEDRM,1} \\ 1 & x_{SQFT,2} & x_{BEDRM,2} \\ \vdots & \vdots & \vdots \\ 1 & x_{SQFT,n} & x_{BEDRM,n} \end{bmatrix}$,$\underline{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$, and $\underline{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$

Now what are the implications of:

$\epsilon_i \sim N(0, \sigma^2)$ or $\underline{\epsilon} \sim N(\underline{0}, \Sigma)$, where

$$\Sigma = \sigma \times \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

These are equivalent statements and both imply that y_i and y_j are conditionally independent given X. In other words, after controlling for predictors (ward, square footage), then the price of house i gives us no additional information about price of house j.

Diagonal Matrices

The matrix we previously specified, is referred to as a diagonal matrix. This is often denoted with

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

where n is the dimension. Note this is also just shortened to I.

Correlation Matrices It turns out that I is the special case of what is referred to as a correlation matrix.

A correlation matrix is:

- is symmetric
- contains ones on the diagonal
- contains correlation terms on the off diagonal
- is positive definite (more later)

Similarly Σ is often referred to as a variance - covariance matrix (or just a covariance matrix). A covariance matrix:

- is symmetric
- contains variance terms on the diagonal
- contains covariance terms on the off diagonal
- is positive definite (more later)

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ddots & \sigma_n^2 \end{pmatrix}$$

Multivariate Normal Distribution Formally, our matrix notation has used a multivariate normal distribution.

$$\underline{y} = X\underline{\beta} + \underline{\epsilon}, \tag{3}$$

where $\underline{\epsilon} \sim N(\underline{0}, \Sigma)$, which also implies $\underline{y} \sim N(X\underline{\beta}, \Sigma)$.

Partitioned Matrices

Now consider splitting the sampling units into two partitions such that $\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix}$. Then,

$$\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \underline{\beta}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix} \right)$$

Fundamentally, there is no change to the model, we have just created "groups" by partitioning the model. Do note that Σ_{11} is an $n_1 \times n_1$ covariance matrix.

$$\Sigma_{11} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n_1} \\ \sigma_{22} & \sigma_2^2 & \cdots & \sigma_{2n_1} \\ \sigma_{31} & \sigma_{32} & \ddots & \sigma_{3n_1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n_1 1} & \sigma_{n_1 2} & \ddots & \sigma_{n_1}^2 \end{bmatrix}$$

However, while $\Sigma_{12} = \Sigma_{21}^T$, neither of these are necessarily symmetric matrices. They also do not have any variance components, but rather just covariance terms. Σ_{12} will be an $n_1 \times n_2$ matrix.

$$\Sigma_{11} = \begin{bmatrix} \sigma_{1,n_1+1} & \sigma_{1,n_1+2} & \cdots & \sigma_{1,n_1+n_2} \\ \sigma_{2,n_1+1} & \sigma_{2,n1+2} & \cdots & \sigma_{2,n_1+n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n_1,n_1+1} & \sigma_{n_1,n_1+2} & \ddots & \sigma_{n_1,n_1+n_2} \end{bmatrix}$$

Conditional Multivariate Normal

Here is where the magic happens with correlated data. Let $\underline{y_1}|\underline{Y_2} = \underline{y_2}$ be a conditional distribution for $\underline{y_1}$ given that $\underline{y_2}$ is known. Then

$$\underline{y_1}|\underline{y_2} \sim N\left(X_1\beta + \Sigma_{12}\Sigma_{22}^{-1}\left(\underline{y_2} - X_2\beta\right), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

Now let's consider a few special cases (in the context of the DC housing dataset.)

1. Let $\Sigma = \sigma^2 I$, then the batch of houses in group 1 are conditionally dependent from the houses in group 2 and

$$\underline{y_1}|\underline{y_2} \sim N\left(X_1\beta, \Sigma_{11}\right)$$

2. Otherwise, let $\Sigma = \sigma^2 H$ and we'll assume Σ_{12} has some non-zero elements. Then we have a more precise estimate of $\underline{y_1}$ as $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ will be "less than" Σ_{11} (that positive definite thing). Furthermore, the mean will shift such that highly correlated observations such as houses in close proximity (local model structure) will tend to differ from the global mean in the same fashion.

First a quick interlude about matrix inversion. The inverse of a symmetric matrix is defined such that $E \times E^{-1} = I$. We can calculate the inverse of a matrix for a 1×1 matrix, perhaps as 2×2, matrix and maybe even a 3×3 matrix. However, beyond that it is quite challenging and time consuming. Furthermore, it is also (relatively) time intensive for your computer.

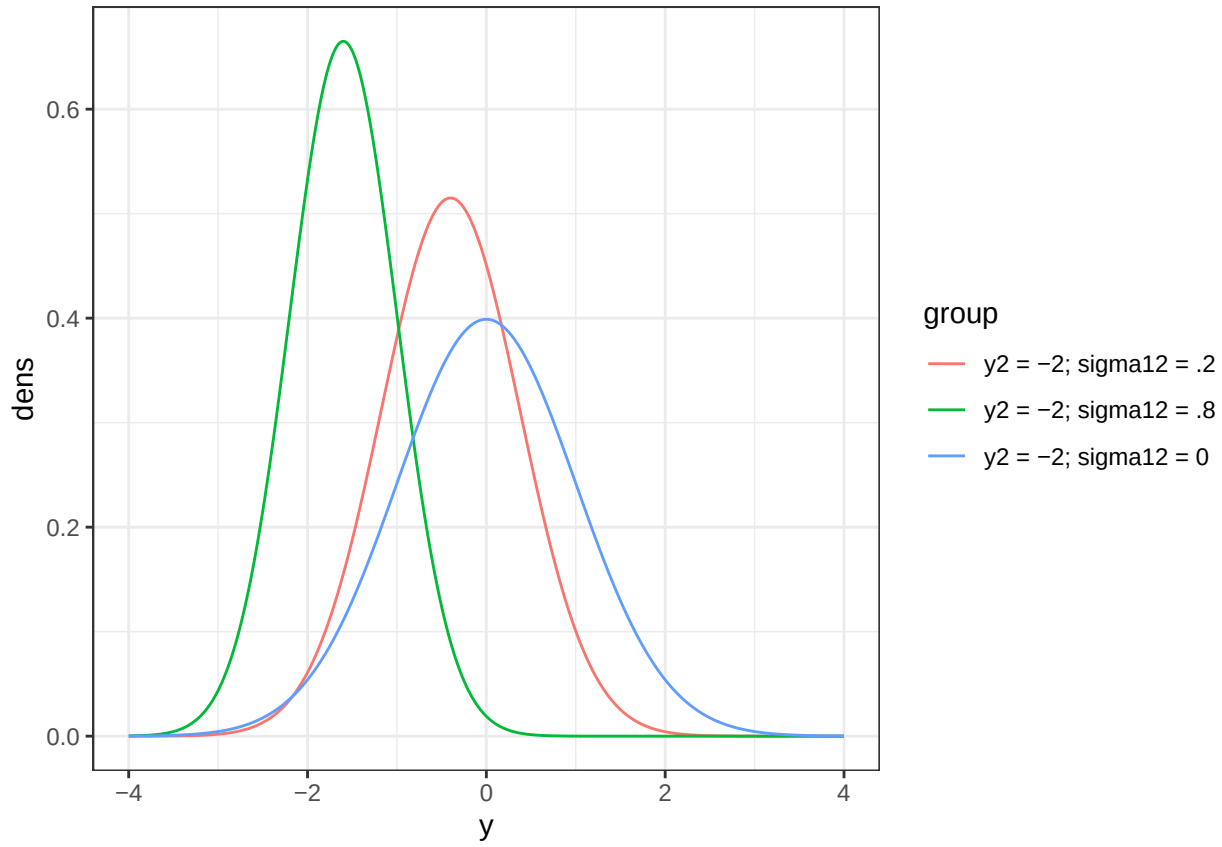
3. Let $n_1 = 1$ and $n_2 = 1$, then

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \right)$$

and

$$y_1 | y_2 \sim N\left(\mu_1 + \sigma_{12}(\sigma_2^2)^{-1}\left(y_2 - \mu_2\right), \sigma_1^2 - \sigma_{12}(\sigma_2^2)^{-1}\sigma_{21} \right)$$

Now consider an illustration for a couple simple scenarios. Let $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = 1$. Now assume $y_2 = -2$ and we compare the conditional distribution for a few values of σ_{12}.



One last note, the marginal distributions for any partition $\underline{y_1}$ are quite simple.

$$\underline{y_1} \sim N\left(X_1\beta, \Sigma_{11}\right)$$

or just

$$y_1 \sim N\left(X_1\beta, \sigma_1^2\right)$$

if y_1 is scalar.