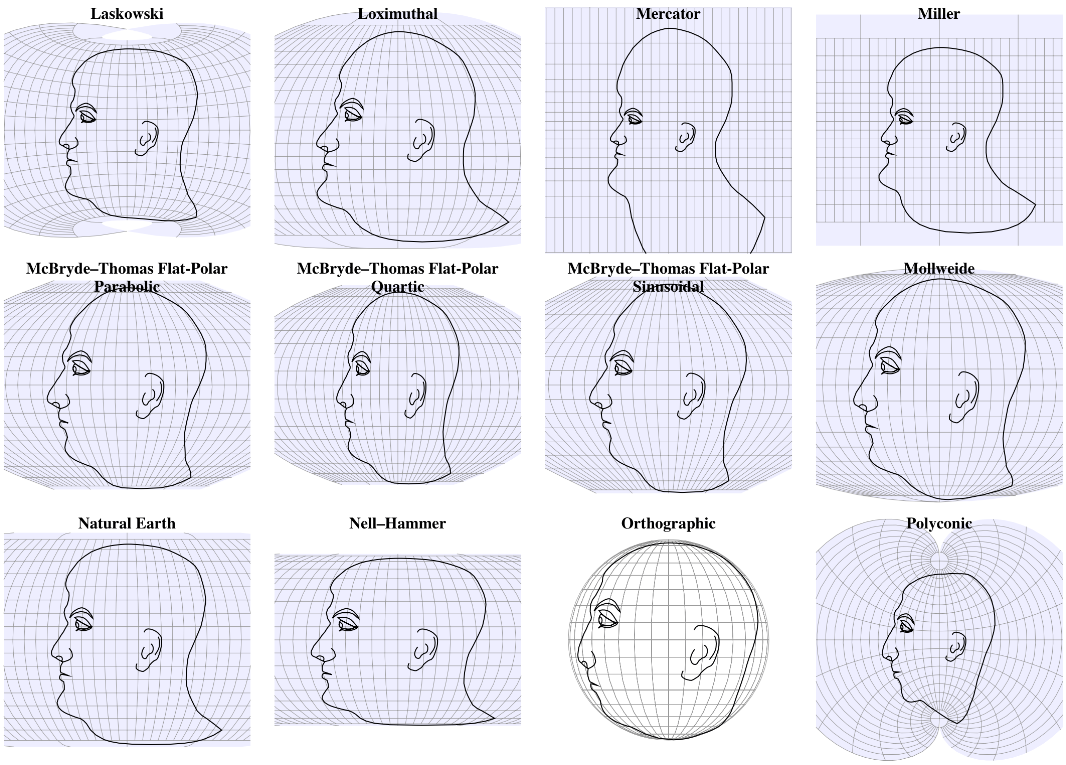
Distance and Projections

Cartography

Map Projections



Map Projections

Map projections are a representation of a surface on a plane. Specifically, functions are designed to map the geographical coordinate system (λ, ϕ) to rectangular or polar coordinates (x, y) such that:

$$x = f(\lambda, \phi), \quad y = g(\lambda, \phi),$$

where f and g are functions that define the projection.

Mercator projections

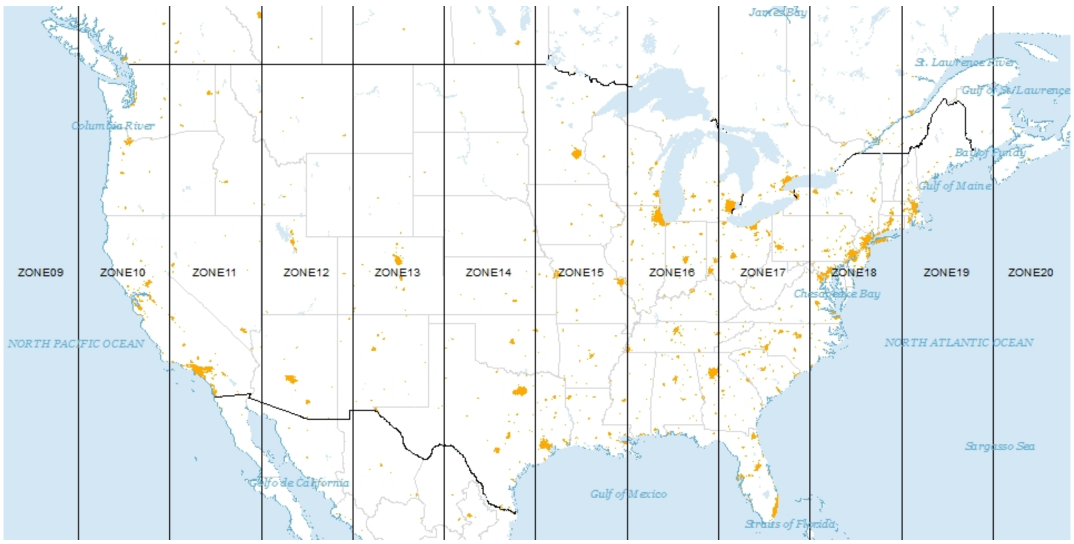
A projection type we will use in this class is the Mercator projection. The Mercator projection gives the common square maps you are used to looking at.

$$f(\lambda, \phi) = R\lambda, \;\; g(\lambda, \phi) = R * ln\left(tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right)$$

The Mercator projection typically includes a rectangular grid and locations are expressed in meters from the intersection of grid lines. Common terminology is to use "easting" and "northings" for the coordinates.

UTM Projections

The UTM projection divides the world into 60 vertical slices, known as zones.



Distances on the earth's surface

According to *Gauss' Theorema Eggregrium* in differential geometry, a planar map projection that preserves distances between points does not exist.

As a precursor to this class, I said "objects close in space tend to be more similar." Mathematically, we will require precise distance measurements between points.

- Euclidean distance: is the straightline distance between two points, recall the Pythagorean thereom. This works okay (with latitude and longitude) for points in a small spatial domain, such as New York City. However, the curvature of the earth causes distortions for larger areas.
- geodesic distance: is the length of the arc on the Earth's surface.

Geodesic distance

The geodesic distance in computed as $D = R\phi$, where R is the radius of the earth and ϕ is an angle (in radians) such that:

$$\cos\phi = \sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 cos(\lambda_1 - \lambda_2),$$

where θ_1 and θ_2 are the latitude measurements and λ_1 and λ_2 are the longitude measurements for two points.

Thus

$$D = R \arccos[\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 cos(\lambda_1 - \lambda_2)]$$

Chordal Distance

Another alternative is to use what is known as the chordal distance, which is equivalent to the "burrowed through the earth" distance between two points on the Earth's surface.

Let

$$\begin{aligned} x &= R\cos\theta\cos\lambda \\ y &= R\cos\theta\sin\lambda \\ z &= R\sin\theta, \end{aligned}$$

where x, y, and z form a set of Cartesian coordinates with the origin at the center of the earth, the z-axis runs between the north and south poles.

Then the chordal distance can be calculated as the Euclidean distance between two vectors $\boldsymbol{u}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{u}_1 = (x_1, y_1, z_1)$.

Distance Calculation Example

Calculate the distance between Chicago (41.8781° N, 87.6298° W) and Minneapolis (44.9778° N, 93.2650° W) using naive Euclidean, geodesic, and chordal measures. Note naive Euclidean can be computed by multiplying Euclidean distance (on radians) by R.

```
theta1 <- 41.88 * pi / 180 # in Radians
theta2 <- 44.89 * pi / 180
lambda1 <- 87.63 * pi / 180
lambda2 <- 93.22 * pi / 180
R <- 6371
```

Distance Calculation Solution

```
geodesic <- R * acos(sin(theta1)*sin(theta2) +
 cos(theta1)*cos(theta2)*cos(lambda1 - lambda2))

euc <- sqrt((theta1 - theta2)^2 + (lambda1 - lambda2)^2) * R

chord <- sqrt(
 (R * cos(theta1) * cos(lambda1) - R * cos(theta2) * cos(lambda2))^2 +
 (R * cos(theta1) * sin(lambda1) - R * cos(theta2) * sin(lambda2))^2 +
 (R * sin(theta1) - R * sin(theta2))^2)
```

The end result is that the geodesic distance is 561.99 distance and the chordal distance 561.81 are very close, but the naive Euclidean 705.96 is quite a bit different.