Spatial Statistics Notation

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The response, $Y(\boldsymbol{s})$, is a random variable at location(s) $\boldsymbol{s}$, where $\boldsymbol{s}$ varies continuously over the space. Typically, $\boldsymbol{s} \in \mathcal{R}^2$ or $\boldsymbol{s} \in \mathcal{D}$, where $\mathcal{D} \subset \mathcal{R}^2$, such as Montana.

$Y(\boldsymbol{s})$ is a stochastic process, or a collection of random variables. In spatial settings, $Y(\boldsymbol{s})$ is referred to as a *spatial process.*

Assume
$$\underline{Y(\boldsymbol{s})}|\mu, \underline{\theta} \sim MVN_n(\underline{\mu}, \Sigma(\underline{\theta})),$$
where $\underline{\theta}$ is a vector of parameters that determine the variance. By specifying a structured form for the covariance function, few parameters are necessary

A common covariance is the exponential covariance, where
$$Cov(Y(\boldsymbol{s}_1), Y(\boldsymbol{s}_2)) = \sigma^2 \exp(-d_{12}/\phi),$$

Spatial models can include a *nugget* effect to account point level variability. This can be thought of as measurement error, or randomness inherent in the process.

Hence for a single point,
$$Cov(Y(\boldsymbol{s}_i), Y(\boldsymbol{s}_i)) = \sigma^2 \exp(-d_{ii}/\phi) + \tau^2,$$

We can build on this notation by specifying

$$\underline{Y(\boldsymbol{s})} = \underline{\mu}(\boldsymbol{s}) + \underline{\omega}(\boldsymbol{s}) + \underline{\epsilon}(\boldsymbol{s}),$$
where $\underline{\mu}(\boldsymbol{s}) = X(\boldsymbol{s})\underline{\beta}$, $\underline{\omega}(\boldsymbol{s}) \sim N(\underline{0}, \sigma^2 H(\phi))$, and $\underline{\epsilon}(\boldsymbol{s}) \sim N(\underline{0}, \tau^2 I)$

Given that we assume

$$\boldsymbol{Y}|\mu, \underline{\theta} \sim MVN_n(\underline{\mu}, \Sigma(\underline{\theta})),$$

it is also reasonable to assume that the response at a set of unobserved locations, $\underline{Y}(\boldsymbol{s}^*)$ also follows a multivariate normal distribution.

this results in an infinite dimensional normal distribution, which is also know as a Gaussian Process (GP)

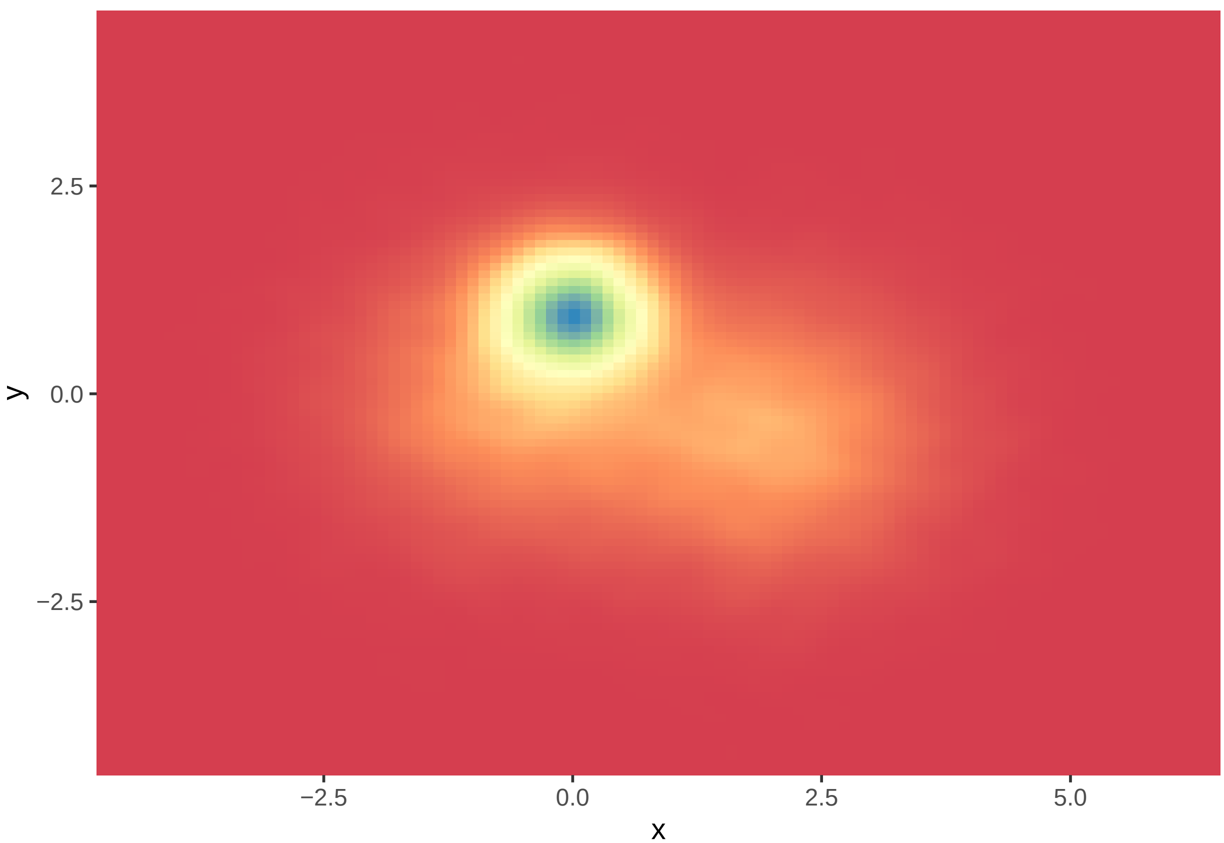
Stationarity

Assume the spatial process, $\boldsymbol{Y}(\boldsymbol{s})$ has a mean, $\underline{\mu}(\boldsymbol{s})$, and that the variance of $\boldsymbol{Y}(\boldsymbol{s})$ exists everywhere.

The process is **strictly stationary** if: for any $n \geq 1$, any set of n sites $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$ and any $\boldsymbol{h} \in \mathcal{R}^r$ (typically r =2), the distribution of $(\boldsymbol{Y}(\boldsymbol{s}_1). \ldots, \boldsymbol{Y}(\boldsymbol{s} + \boldsymbol{h}_n))$ is the same as $\boldsymbol{Y}(\boldsymbol{s} + \boldsymbol{h}_1)$

Strict Stationarity

Is this strictly stationary?



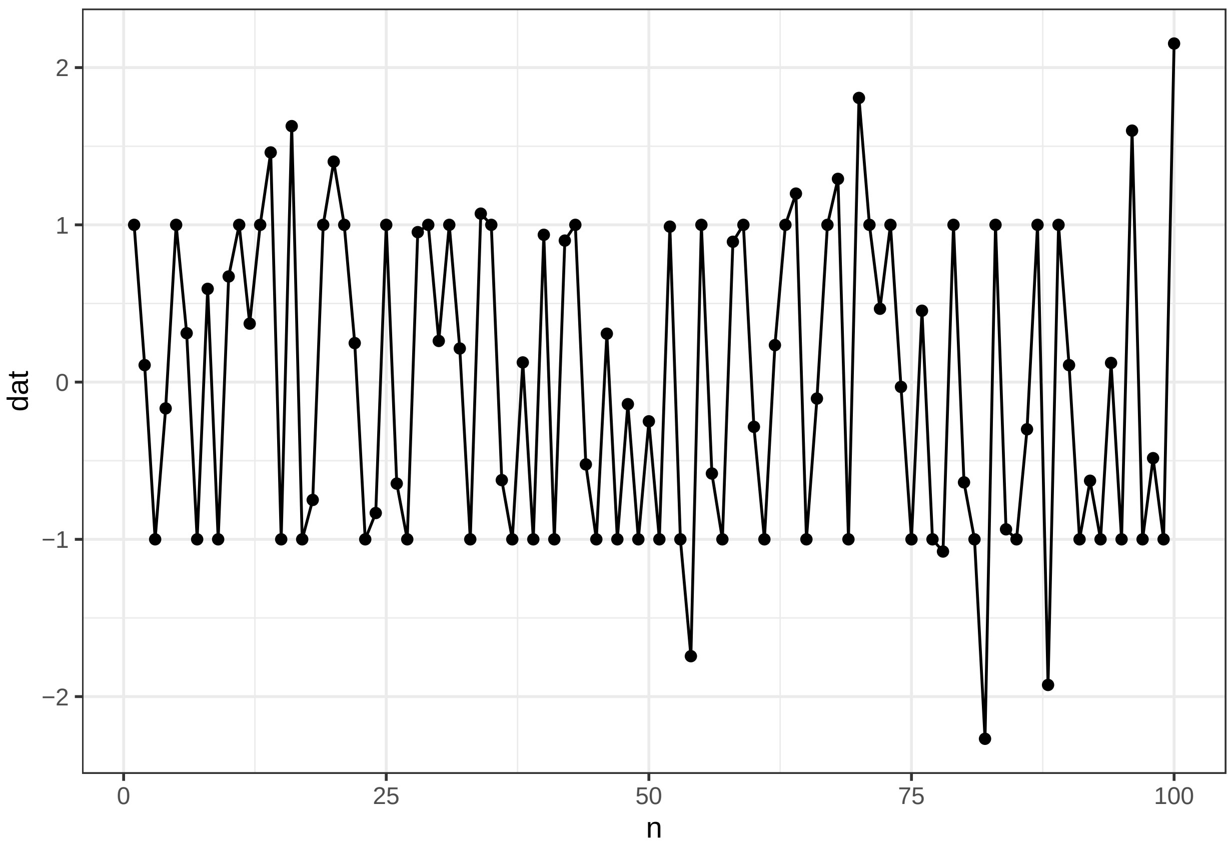
Weak Stationarity

Weak stationarity, or second-order stationarity of a spatial process, requires a constant mean and covariance that is a function of $\boldsymbol{h}$, $Cov(\boldsymbol{Y}(\boldsymbol{s}), \boldsymbol{Y}(\boldsymbol{s}+\boldsymbol{h})) = C(\boldsymbol{h})$, where $C(\boldsymbol{h})$ is a covariance function that only requires the separation vector $\boldsymbol{h}$.

Typically with spatial models, the process is assumed to be mean zero as covariates explain the mean structure. So second-order stationarity is primarily focused on the covariance structure.

Weak Stationarity

Is this weakly stationary?



Intrinsic Stationarity

A third kind of stationarity, known as intrinsic stationarity, describes the behavior of differences in the spatial process, rather than the data.

Intrinsic stationarity assumes $E[\boldsymbol{Y}(\boldsymbol{s}+\boldsymbol{h}) - \boldsymbol{Y}(\boldsymbol{s})] = 0$ then define

$$E[\boldsymbol{Y}(\boldsymbol{s}+\boldsymbol{h}) - \boldsymbol{Y}(\boldsymbol{s})]^2 = Var(\boldsymbol{Y}(\boldsymbol{s}+\boldsymbol{h}) - \boldsymbol{Y}(\boldsymbol{s})) = 2\gamma(\boldsymbol{h}),$$

which only works, and satisfies intrinsic stationarity, if the equation only depends on $\boldsymbol{h}$.

Variograms

Intrinsic stationary justifies the use of variograms $2\gamma(\boldsymbol{h})$

Variograms are often used to visualize spatial patterns:

- If the distance between points is small, the variogram is expected to be small

- As the distance between points increases, the variogram increases

There is a mathematical link between the covariance function $C(\boldsymbol{h})$ and the variogram $2\gamma(\boldsymbol{h})$.

Isotropy

If the variogram $2\gamma(\boldsymbol{h})$ depends only on the length of $\boldsymbol{h}$, $||\boldsymbol{h}||$, and not the direction, then the variogram is isotropic.

If the direction of $\boldsymbol{h}$ impacts the variogram, then the variogram is anisotropic.

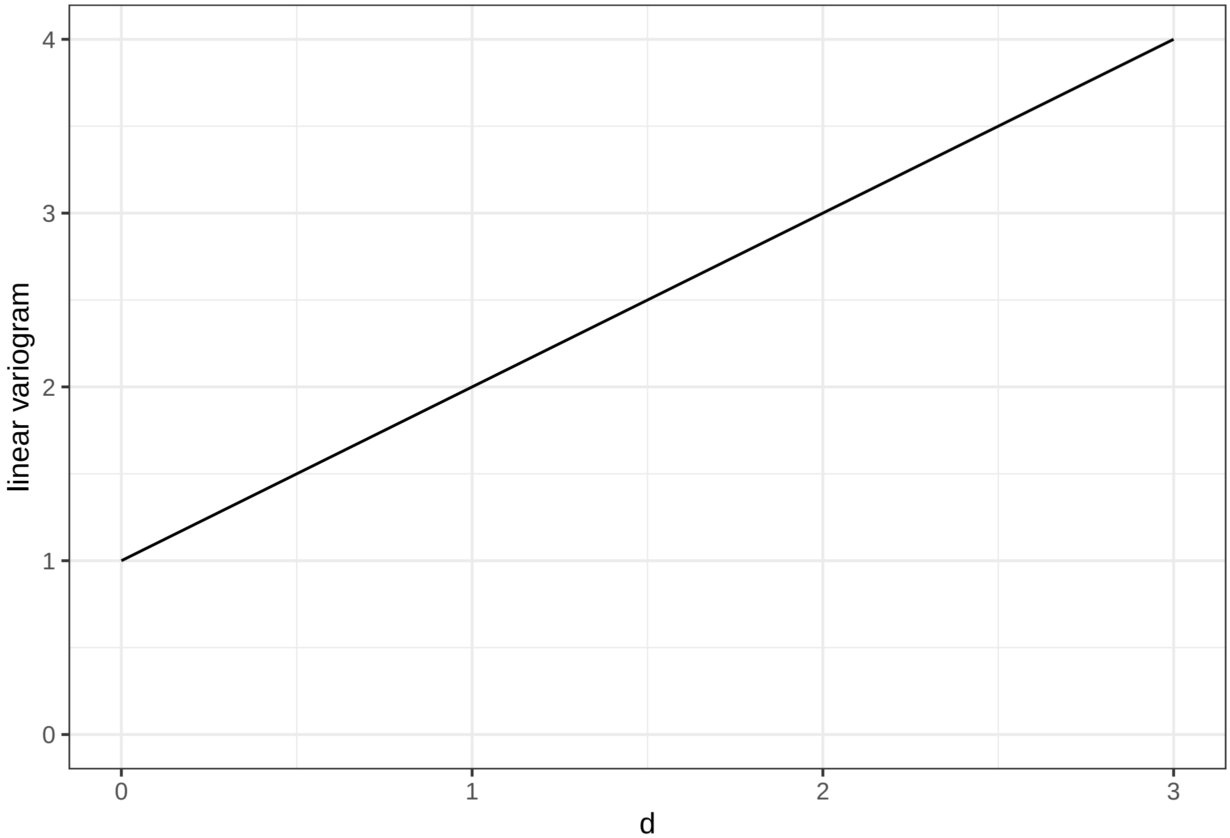
Variogram to the Covariance Function

$$\begin{aligned}
2\gamma(\boldsymbol{h}) &= Var(\boldsymbol{Y}(\boldsymbol{s}+\boldsymbol{h}) - \boldsymbol{Y}(\boldsymbol{s})) \\
&= Var(\boldsymbol{Y}(\boldsymbol{s}+\boldsymbol{h})) + Var(\boldsymbol{Y}(\boldsymbol{s})) - 2Cov(Var(\boldsymbol{Y}(\boldsymbol{s}+\boldsymbol{h}), \boldsymbol{Y}(\boldsymbol{s}))) \\
&= C(\boldsymbol{0}) + C(\boldsymbol{0}) - 2C(\boldsymbol{h}) \\
&= 2\left[C(\boldsymbol{0}) - C(\boldsymbol{h})\right]
\end{aligned}$$

Given $C()$, the variogram can easily be recovered

Linear semivariogram

$$\gamma(d) = \begin{cases} \tau^2 + \sigma^2 d & \text{if } d > 0 \\ 0 & \text{otherwise} \end{cases}$$



Nugget, sill, partial-sill, and range

Nugget is defined as $\gamma(d): \; d \to 0_+$

Sill is defined as $\gamma(d): \; d \to \infty$

Partial sill is defined as the sill $-$ nugget

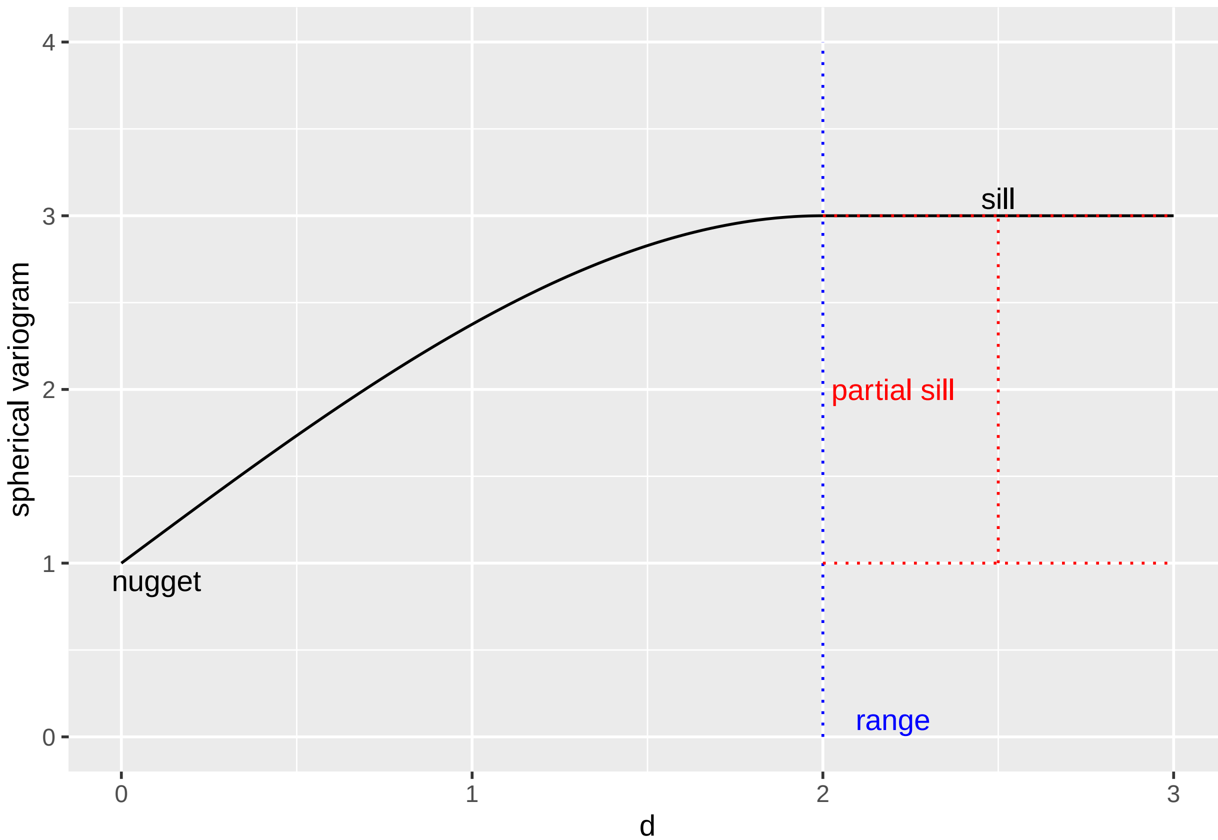
Range is defined as the first point where $\gamma(d)$ reaches the sill. Range is sometimes $(1/\phi)$ and sometimes ϕ depending on the paremeterization.

Spherical semivariogram:

$$\gamma(d) = \begin{cases} \tau^2 + \sigma^2 \ \text{ if } d \geq 1/\phi \\ \tau^2 + \sigma^2 \left[\frac{3\phi d}{2} - \frac{1}{2}(\phi d)^3 \right] \\ 0 \ \text{ otherwise} \end{cases}$$

- Sketch, or generate in R, a spherical semivariogram
- On this figure label the nugget, sill, partial sill, and range.

Spherical semivariogram: Solution



Exponential

We saw the exponential covariance earlier in class, what is the mathematical form of the covariance?

$$C(d) = \begin{cases} \tau^2 + \sigma^2 & \text{if } d = 0 \\ \sigma^2 \exp(-d/\phi) & \text{if } d > 0 \end{cases}$$

The variogram is

$$\gamma(d) = \begin{cases} \tau^2 + \sigma^2(1 - \exp(-d/\phi)) & \text{if } d > 0 \\ 0 & \text{otherwise} \end{cases}$$

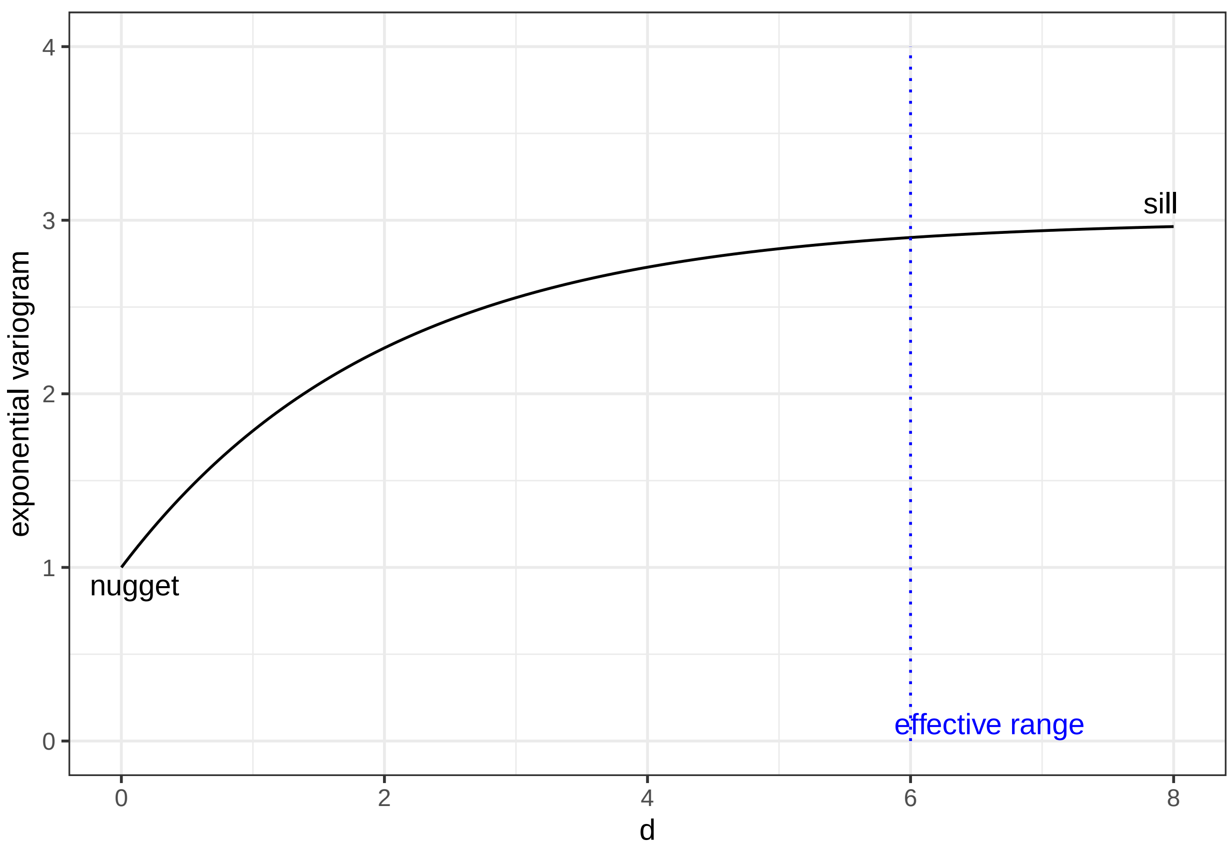
The sill is only reached asymptotically when

$$\lim_{d \to \infty} exp(-d/\phi) \to 0$$

Thus, the range, defined as distance where semivariogram reaches the sill is ∞

The *effective range* is defined as the distance where there is *effectively* no spatial structure. Generally this is determined when the correlation is .05.

$$\begin{aligned} \exp(-d_o/\phi) &= .05 \\ d_0/\phi &= -\log(.05) \\ d_0 &\approx 3\phi \end{aligned}$$



More Semivariograms: Equations

Gaussian:

$$\gamma(d) = \begin{cases} \tau^2 + \sigma^2(1 - \exp(-\phi^2 d^2)) & \text{if } d > 0 \\ 0 & \text{otherwise} \end{cases}$$

Powered Exponential:

$$\gamma(d) = \begin{cases} \tau^2 + \sigma^2(1 - \exp(-|\phi d|^p)) & \text{if } d > 0 \\ 0 & \text{otherwise} \end{cases}$$

Matérn:

$$\gamma(d) = \begin{cases} \tau^2 + \sigma^2 \left[1 - \frac{(2\sqrt{\nu} d\phi)^\nu}{2^{\nu-1}\Gamma(\nu)} K_\nu(2\sqrt{\nu} d\phi)\right] & \text{if } d > 0 \\ 0 & \text{otherwise} \end{cases},$$

where K_ν is a modified Bessel function and $\Gamma()$ is a Gamma function.

Variogram Creation: How?

```
data(meuse)
meuse.small <- meuse %>% select(x, y, copper) %>% as_tibble()
meuse.small %>% head(15) %>% kable()
```

x	y	copper
181072	333611	85
181025	333558	81
181165	333537	68
181298	333484	81
181307	333330	48
181390	333260	61
181165	333370	31
181027	333363	29
181060	333231	37
181232	333168	24
181191	333115	25
181032	333031	25
180874	333339	93
180969	333252	31
181011	333161	27

Variogram Creation: Steps

1. Calculate distances between sampling locations
2. Choose grid for distance calculations
3. Calculate empirical semivariogram

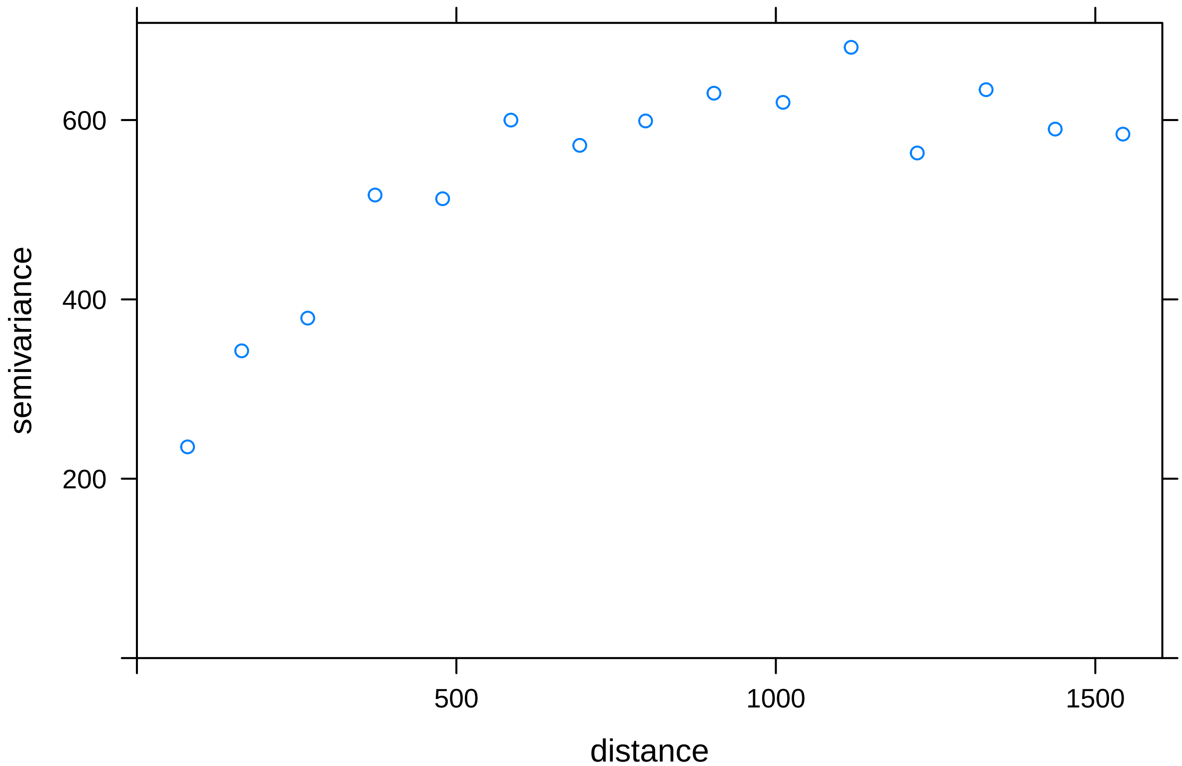
$$\hat{\gamma}(d_k) = \frac{1}{2N(d_k)} \sum_{\boldsymbol{s}_i, \boldsymbol{s}_i, \in N(d_k)} \left[\boldsymbol{Y}(\boldsymbol{s}_i) - \boldsymbol{Y}(\boldsymbol{s}_j)\right]^2,$$

where

$$N(d_k) = \{(\boldsymbol{s}_i, \boldsymbol{s}_j) : ||\boldsymbol{s}_i - \boldsymbol{s}_j|| \in I_k\}$$

and I_k is the k^{th} interval.
4. Plot the semivariogram

Variogram Creation: R function



Variogram Creation: How - Step 1

Calculate Distances between sampling locations

```
#dist(meuse.small)
dist.mat <- dist(meuse.small %>% select(x,y))
```

Variogram Creation: How - Step 2

Choose grid for distance calculations

```
cutoff <- max(dist.mat) / 3 # default maximum distance
num.bins <-  15
bin.width <- cutoff / 15
```

Variogram Creation: How - Step 3

Calculate empirical semivariogram

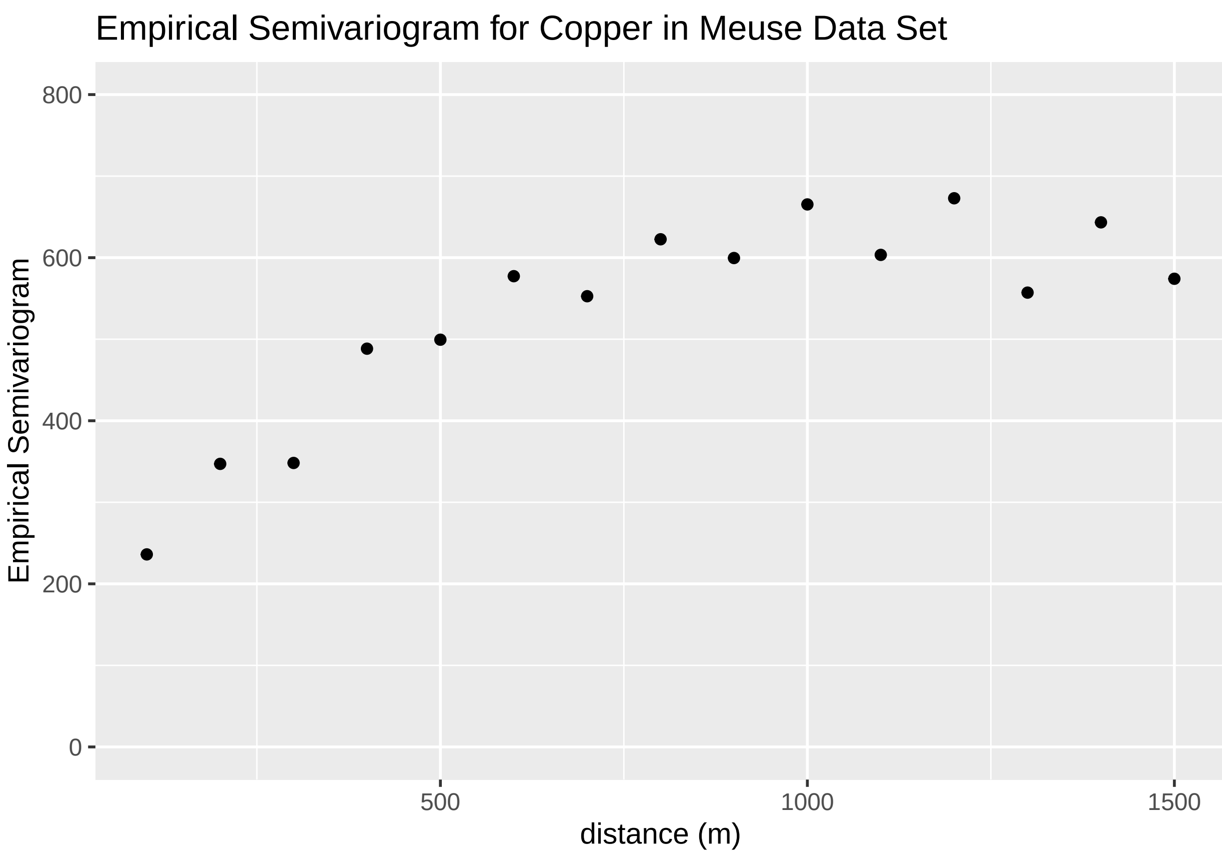
```
## `summarise()` ungrouping output (override with `.groups` argument)
```

```
## # A tibble: 15 x 2
##      bin emp.sv
##    <dbl>  <dbl>
##  1     1   236.
##  2     2   347.
##  3     3   348.
##  4     4   488.
##  5     5   499.
##  6     6   577.
##  7     7   553.
##  8     8   623.
##  9     9   600.
## 10    10   665.
## 11    11   603.
## 12    12   673.
## 13    13   557.
## 14    14   643.
## 15    15   574.
```

Variogram Creation: How - Step 4

Plot empirical semivariogram

```
## Scale for 'x' is already present. Adding another scale for 'x', which will
## replace the existing scale.
```



Variogram Fitting

Now given this empirical semivariogram, how to we choose a semivariogram (and associated covariance structure) and estimate the parameters in that function??

```
## Scale for 'x' is already present. Adding another scale for 'x', which will
## replace the existing scale.
```