Conditional Autoregressive Models

Gaussian Model

Suppose the full conditionals are specified as

Then using Brooks' Lemma, the joint distribution is

$$p(y_1, \ldots, y_n) \propto \exp\left(-\frac{1}{2}\boldsymbol{y}^T D^{-1}(I - B)\boldsymbol{y}\right),$$

where B is a matrix with entries b_{ij} and D is a diagonal matrix with diagonal elements $D_{ii} = \tau_i^2$.

The previous equation suggests a multivariate normal distribution, Now the full conditional distribution can be written as

$$Y_i | y_j, j \neq i \sim N\left(\sum_j w_{ij} y_j / w_{i+}, \tau^2 / w_{i+}\right)$$

Similarly the joint distribution is now

$$p(y_1, \ldots, y_n) \propto \exp\left(-\frac{1}{2\tau^2} \boldsymbol{y}^T (D_w - W) \boldsymbol{y}\right)$$

where D_w is a diagonal matrix with diagonal entries $(D_w)_{ii} = w_{i+}$

The joint distribution can also be re-written as

$$p(y_1, \ldots, y_n) \propto \exp\left(-\frac{1}{2\tau^2} \sum_{i \neq j} w_{ij} (y_i - y_j)^2\right)$$

However, both these formulations results in an improper distribution. This could be solved with a constraint, such as $Y_i = 0$.

The result is the joint distribution is improper, despite proper full conditional distributions.

IAR

The IAR cannot be used to model data directly, rather this is used a prior specification

The impropriety can be remedied by defining a parameter ρ such that

With or without ρ, $p(\boldsymbol{y})$ (or the Bayesian posterior when the CAR specification is placed on the spatial random effects) is proper.

When using ρ, the full conditional becomes

Simultaneous Autoregression Model

Rather than specifying the distribution on $\boldsymbol{Y}$, as in the CAR specification, the distribution can be specified for $\boldsymbol{\epsilon}$ which induces a distribution for $\boldsymbol{Y}$.

SAR Model

If the matrix $(I - B)$ is full rank, then

$$\boldsymbol{Y} \sim N\left(\mathbf{0}, (I - B)^{-1}\tilde{D}((I - B)^{-1})^T\right)$$

If $\tilde{D} = \sigma^2 I$, then $\boldsymbol{Y} \sim N\left(\mathbf{0}, \sigma^2 \left[(I - B)(I - B)^T\right]^{-1}\right)$

Choosing B There are two common approaches for choosing B

SAR Models are often introduced in a regression context, where the residuals ($\boldsymbol{U}$) follow a SAR model.

Let $\boldsymbol{U} = \boldsymbol{Y} - X\boldsymbol{\beta}$ and then $\boldsymbol{U} = B\boldsymbol{U} + \boldsymbol{\epsilon}$ which results in

$$\boldsymbol{Y} = B\boldsymbol{Y} + (I - B)X\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

Hence the model contains a spatial weighting of neighbors ($B\boldsymbol{Y}$) and a regression component ($(I - B)X\boldsymbol{\beta}$).