Conditional Autoregressive Models

Gaussian Model

Suppose the full conditionals are specifed as

$$Y_i | y_j, j \neq i \sim N\left(\sum_j b_{ij} y_j, \tau_i^2\right)$$

Then using Brooks' Lemma, the joint distribution is

$$p(y_1, \ldots, y_n) \propto \exp\left(-\frac{1}{2}\boldsymbol{y}^T D^{-1}(I - B)\boldsymbol{y}\right),$$

where B is a matrix with entries b_{ij} and D is a diagonal matrix with diagonal elements $D_{ii} = \tau_i^2$.

The previous equation suggests a multivariate normal distribution, but $D^{-1}(I - B)$ should be symmetric.

Symmetry requires

$$\frac{b_{ij}}{\tau_i^2} = \frac{b_{ji}}{\tau_j^2}, \;\; \forall \;\; i, j$$

In general, B is not symmetric, but setting $b_{ij} = w_{ij}/w_{i+}$ and $\tau_i^2 = \tau^2/w_{i+}$ satisfies the symmetry assumptions (given that we assume W is symmetric)

Now the full conditional distribution can be written as

$$Y_i|y_j, j \neq i \sim N\left(\sum_j w_{ij}y_j/w_{i+}, \tau^2/w_{i+}\right)$$

Similarly the joint distribution is now

$$p(y_1, \dots, y_n) \propto \exp\left(-\frac{1}{2\tau^2}\boldsymbol{y}^T(D_w - W)\boldsymbol{y}\right)$$

where D_w is a diagonal matrix with diagonal entries $(D_w)_{ii} = w_{i+}$

The joint distribution can also be re-written as

$$p(y_1, \dots, y_n) \propto \exp\left(-\frac{1}{2\tau^2}\sum_{i \neq j} w_{ij}(y_i - y_j)^2\right)$$

However, both these formulations results in an improper distribution. This could be solved with a constraint, such as $Y_i = 0$.

The result is the joint distribution is improper, despite proper full conditional distributions. This model specification is often referred to as an *intrinsically autoregressive* model (IAR).

IAR

The IAR cannot be used to model data directly, rather this is used a prior specification and attached to random effects specified at the second stage of the hierarchical model.

The impropriety can be remedied by defining a parameter ρ such that $(D_w - W)$ becomes $(D_w - \rho W)$ such that this matrix is nonsingular.

The parameter ρ can be considered an extra parameter in the CAR model.

With or without ρ, $p(\boldsymbol{y})$ (or the Bayesian posterior when the CAR specification is placed on the spatial random effects) is proper.

When using ρ, the full conditional becomes

$$Y_i | y_j, j \neq i \sim N\left(\rho \sum_j w_{ij} y_j / w_{i+}, \tau^2 / w_{i+}\right)$$

Simultaneous Autoregression Model

Rather than specifying the distribution on $\boldsymbol{Y}$, as in the CAR specification, the distribution can be specified for $\boldsymbol{\epsilon}$ which induces a distribution for $\boldsymbol{Y}$.

Let $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \tilde{D})$, where $\tilde{D}$ is a diagonal matrix with elements $(\tilde{D})_{ii} = \sigma_i^2$.

Now $Y_i = \sum_j b_{ij} Y_j + \epsilon_i$ or equivalently $(I - B)\boldsymbol{Y} = \boldsymbol{\epsilon}$.

SAR Model

If the matrix $(I - B)$ is full rank, then

$$\boldsymbol{Y} \sim N\left(\boldsymbol{0}, (I - B)^{-1}\tilde{D}((I - B)^{-1})^T\right)$$

If $\tilde{D} = \sigma^2 I$, then $\boldsymbol{Y} \sim N\left(\boldsymbol{0}, \sigma^2 \left[(I - B)(I - B)^T\right]^{-1}\right)$

Choosing B There are two common approaches for choosing B

1. $B = \rho W$, where W is a contiguity matrix with entries 1 and 0. The parameter ρ is called the spatial autoregression parameter.

2. $B = \alpha \tilde{W}$ where $(\tilde{W})_{ij} = w_{ij}/w_{i+}$. The α parameter is called the spatial autocorrelation parameter.

SAR Models are often introduced in a regression context, where the residuals $(\boldsymbol{U})$ follow a SAR model.

Let $\boldsymbol{U} = \boldsymbol{Y} - X\boldsymbol{\beta}$ and then $\boldsymbol{U} = B\boldsymbol{U} + \boldsymbol{\epsilon}$ which results in

$$\boldsymbol{Y} = B\boldsymbol{Y} + (I - B)X\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

Hence the model contains a spatial weighting of neighbors $(B\boldsymbol{Y})$ and a regression component $((I - B)X\boldsymbol{\beta})$.

The SAR specification is not typically used in a GLM setting.

SAR models are well suited for maximum likelihood

Without specifying a hierarchical form, Bayesian sampling of random effects is more difficult than the CAR specification.